Rigidity and lack of rigidity for solenoidal matrix fields

A. Garroni * V. Nesi*

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Abstract

We study the problem of identifying conditions under which a divergence free matrix field takes values in some prescribed sets of matrices $\mathcal{K}$. We treat in detail the case when $\mathcal{K}$ is made of two or three matrices. Our results are parallel to those on curl free matrices. In that case Ball and James showed rigidity when $\mathcal{K}$ is made of two matrices and Tartar proved lack of rigidity when $\mathcal{K}$ is made of four matrices. For our problem we prove rigidity when $\mathcal{K}$ is made of two matrices and lack of rigidity when is made of three.

We give examples when the differential constraints are yet of a different type and present some applications to composites.

1 Introduction

Consider the following problem. Given a set $\mathcal{K}$ of matrices, find mappings U in a suitable space such that DU belongs to the set $\mathcal{K}$ at each point of the reference domain. In this problem the set $\mathcal{K}$ is the *data*. The maps with gradient in $\mathcal{K}$ are the unknowns and the goal is to characterize them. Ball and James studied in detail the two-gradient problem in [3] proving, among other things, the results below. It is convenient to denote by $\mathcal{M}^{m\times n}$ the set of $m \times n$ real matrices.

Proposition 1.1 Two-gradients: partial rigidity (Ball and James, [3]). *Let $\Omega \subseteq \mathbf{R}^n$ be an open and connected set. Let $A_1, A_2 \in \mathcal{M}^{m\times n}$ and set $\mathcal{K} \equiv \{A_1, A_2\}$.*

Let $U \in W^{1,\infty}(\Omega, \mathbf{R}^m)$ satisfy

$$DU(x) = \chi_{\Omega_1}(x)A_1 + \chi_{\Omega_2}(x)A_2$$

where Ω_1 and Ω_2 are measurable disjoint subsets of Ω of positive measure with $\Omega = \Omega_1 \cup \Omega_2$. The following alternative holds:

(i) If $\operatorname{rank}(A_2 - A_1) \geq 2$, then $DU = A_1$ a.e. or $DU = A_2$ a.e.

(ii) If $\operatorname{rank}(A_2 - A_1) = 1$, then, for every $\theta \in [0, 1]$, there exist $\Omega_{1,\theta}$ with $|\Omega_{1,\theta}| = \theta|\Omega|$ and there exists $U_\theta \in W^{1,\infty}(\Omega, \mathbf{R}^m)$ such that

$$DU_\theta(x) = \chi_{\Omega_{1,\theta}}(x)A_1 + \chi_{\Omega\setminus\Omega_{1,\theta}}(x)A_2\,.$$

*Dipartimento di Matematica, Università di Roma "La Sapienza" P.le A. Moro 2, 00185 Rome, ITALY

Let us set some terminology. When $\theta \in (0,1)$ any partition $\{\Omega_{1,\theta}, \Omega\backslash\Omega_{1,\theta}\}$ for which a solution to the two-gradient problem exists, is called a "geometry". The condition $\mathrm{rank}\,(A_1 - A_2) = 1$ is called rank-one connectedness. Proposition 1.1 is then rephrased by saying that if A_1 and A_2 are not rank-one connected, no (non trivial) geometry exists such that the corresponding map U has its gradient in $\mathcal{K}$. In contrast, if A_1 and A_2 are rank-one connected than there exist non trivial geometries with the desired properties. Indeed one can easily check that simple "laminates" work (i.e. the sets $\Omega_{1,\theta}$ and $\Omega\backslash\Omega_{1,\theta}$ can be taken to be "strips", at least locally).

This result leaves out the possibility of the existence of a sequence of maps which achieves the goal in some approximate sense. Ball and James proved the following result.

Theorem 1.2 Two-gradients: full rigidity (Ball and James, [3]) *Let $\Omega \subseteq \mathbf{R}^n$ be an open and connected set. Let $A_1, A_2 \in \mathcal{M}^{m\times n}$ and set $\mathcal{K} \equiv \{A_1, A_2\}$. Let $U_j \rightharpoonup U$ in $W^{1,p}(\Omega;\mathbf{R}^m)$, with $p > 2$, and suppose that*

$$\mathrm{dist}\,(DU_j, \mathcal{K}) \to 0 \quad \textit{in measure}\,.$$

If $\mathrm{rank}\,(A_2 - A_1) \geq 2$, *then*

$$DU_j \to A_1 \quad \textit{in measure} \quad \textit{or} \quad DU_j \to A_2 \quad \textit{in measure}\,.$$

A more subtle analysis based on Young's measures shows that necessarily these non trivial microgeometries are essentially laminates in the sense that the associated Young measures are convex combinations of Dirac masses at A_1 and A_2, see [20] for details.

Later, Šverák [27], [28], proved the following analogue of Proposition 1.1 and Theorem 1.2 for the case when $\mathcal{K}$ consists of three matrices which are pairwise rank-one disconnected.

Theorem 1.3 Three-gradients: partial and full rigidity (Šverák, [28]). *Let $\Omega \subseteq \mathbf{R}^n$ be an open and connected set and let $A_1, A_2, A_3 \in \mathcal{M}^{m\times n}$. Set $\mathcal{K} \equiv \{A_1, A_2, A_3\}$ and assume that*

$$\mathrm{rank}\,(A_i - A_j) \neq 1 \quad \textit{for} \quad i \neq j\,.$$

Let $U \in W^{1,\infty}(\Omega, \mathbf{R}^m)$ satisfies

$$DU \in \mathcal{K} \quad \textit{a.e.} \quad \textit{then} \quad DU \quad \textit{is constant}\,.$$

Let $p > 2$ and let $U_j \rightharpoonup U$ in $W^{1,p}(\Omega;\mathbf{R}^m)$. If $\mathrm{dist}\,(DU_j, \mathcal{K}) \to 0$ *in measure, then $DU_j \to A_i$ in measure, for some $i \in \{1,2,3\}$.*

In contrast, when the set $\mathcal{K}$ consists of four matrices, a partial lack of rigidity holds. More precisely it is possible to construct a sequence of fields which are curl-free and whose distance from the set $\mathcal{K}$ approaches zero. This is the precise statement.

Lemma 1.4 Four-gradients: partial lack of rigidity (Tartar, [36]). *Set*

$$A_1 = -A_3 = \begin{pmatrix} -1 & 0 \\ 0 & -3 \end{pmatrix}, \quad A_2 = -A_4 = \begin{pmatrix} -3 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathcal{K}_T \equiv \{A_1, A_2, A_3, A_4\}\,.$$

There exists a sequence $U_j \in W^{1,\infty}(\Omega, \mathbf{R}^2)$ such that $\mathrm{dist}\,(DU_j, \mathcal{K}_T) \to 0$ *in measure and DU_j does not converge in measure.*

Note that $\operatorname{rank}(A_i - A_j) = 2$ for $i \neq j$, so that $\mathcal{K}$ is made of pairwise rank-one disconnected matrices.

Many other interesting results have been proved in this field. Let us quote two of them. Chlebík and Kirchheim [9] proved that for the four-gradients problem, no "exact geometry" can be found. In other words the introduction of sequences is necessary. Kirchheim and Preiss [17] proved that the five-gradient problem has instead exact solutions.

In this paper we address a different but very related problem. The issue is to replace in this scheme, the constraint of being a gradient of a mapping (i.e. to be a curl free matrix valued function) with a different linear differential constraint. We mostly consider the natural constraint that the matrix field under consideration is solenoidal (i.e. each row of the matrix valued field is divergence free), rather than irrotational, (i.e. each row is curl free).

Typically we ask the following kind of question. Given a set $\mathcal{K} \in \mathcal{M}^{m\times n}$ find a matrix valued field $B(x)$ such that

$$B(x) \in \mathcal{K} \quad \text{a.e.} \quad \text{and} \quad \operatorname{Div} B = 0\,. \tag{1.1}$$

In Section 2, we address problem (1.1), with $\mathcal{K}$ made of two elements. We prove Proposition 2.1 and Theorem 2.2 which are the analogue of Proposition 1.1 and Theorem 1.2, establishing full rigidity under the condition of rank-$(n-1)$ disconnectedness.

In Section 3, we digress from our main theme. We show that our proof of Theorem 2.2 can be slightly modified to be applied to the two-gradients problem. In that case it actually yields a result slightly stronger than Theorem 1.2. We will comment later in this section about this technical point.

In Section 4, we prove Lemma 4.1 showing partial lack of rigidity for the case of three matrices. This should be regarded as the counterpart of Lemma 1.4.

In Section 5, we consider some examples showing that in some cases, when the differential constraint is of a different type from those considered so far, even for $\mathcal{K} = \{A_1, A_2\}$ the rigidity does not depend only upon the rank of $A_2 - A_1$ but also on other details.

The aim of Section 6 is to provide various new motivations to a study which falls in the framework named $\mathcal{A}$-quasiconvexity (see, for instance, the work of Fonseca and Müller [11]) and which has its roots in the work of Tartar [35] on compensated compactness. We present some examples from the literature of composite materials outlining the very strict connection between the problems addressed in the present paper and that of bounding effective energies in both linear and non linear problems.

Let us now turn to some more technical issues concerning our proof of Theorem 2.2. We already remarked, that when applied to the case of the two gradients, our approach yields a slightly stronger result than Theorem 1.2. Indeed our proof applies with $p \in (1, \infty]$, the original one requires $p > 2$. The reason is explained in technical detail in Section 2. We give here an informal explanation. The basic idea in the proof of Theorem 1.2 is to use what are nowadays often called "minors relations". In other words one makes use of the continuity of the minors of Jacobians matrices with respect to weak convergence. In order to be useful one needs to use minors which have at least a quadratic growth. However, the latter are weakly continuous in $W^{1,p}$ only for $p > 2$. Our proof is totally different being based on a lemma of Nečas [21]. Therefore it turns out that $p > 2$ is only a technical assumption in Theorem 1.2 and it can be removed. We thank L. Ambrosio who pointed out to us that an extension of Theorem 1.2 for $p \leq 2$ could be also obtained using a regularization lemma proved by Kristensen [18].

In Section 2 we give also a different proof of the rigidity for solenoidal fields which is more parallel to that of Ball and James. It is based on the weak semicontinuity of an appropriate quadratic function proposed by Tartar [34] and for reasons similar to the one involving minors of Jacobians, it only yields a corresponding statement in $W^{1,p}$, with $p \geq 2$. In this respect it is less efficient. We nevertheless present it since it provides a very nice parallel with the approach of bounding effective moduli using an idea of Tartar [34], see also [33].

The proof of Lemma 4.1 in Section 4 is based on an explicit construction which is very similar to that used by Tartar. This natural idea actually has appeared in the work of many authors including [24], [6] and [22]. The particular example here resembles the construction due to Milton and Nesi [22] since it involves exactly three distinct matrices.

2 The two divergence free fields problem

The goal of this section is to prove the analogue of Proposition 1.1 and of Theorem 1.2 for problem (1.1) with $\mathcal{K} \equiv \{A_1, A_2\}$. We begin by recalling a result which, in this generality, is due to Nečas.

For $1 \leq p \leq +\infty$, we shall denote by $W_0^{1,p}(\Omega)$ the usual Sobolev space and by $W^{-1,q}(\Omega)$ its dual space, with $\frac{1}{p} + \frac{1}{q} = 1$. If Ω is a bounded open and connected set with Lipschitz boundary, for any $1 < q < +\infty$ the following inequalities hold

$$\|f\|_{L^p(\Omega)} \leq C \left(\|\nabla f\|_{W^{-1,p}(\Omega)} + \|f\|_{W^{-1,p}(\Omega)} \right) \tag{2.1}$$

for any $f \in L^p(\Omega)$ (see [21]).

To state our results it is convenient to introduce the following notations. For $J : \Omega \subseteq \mathbf{R}^n \to \mathcal{M}^{m\times n}$, we denote by $\operatorname{Div} J$ the operator which acts as the divergence in the sense of distribution of any row.

We are now ready to prove the analogue of Proposition 1.1 for (1.1) when $\mathcal{K} \equiv \{A_1, A_2\}$.

Proposition 2.1 *Let Ω be an open and connected set in $\mathbf{R}^n$. Let $A_1, A_2 \in \mathcal{M}^{m\times n}$, with $m \geq n$ and $\operatorname{rank}(A_1 - A_2) = n$. Let $B : \Omega \to \mathcal{M}^{m\times n}$ be a measurable function with $B \in \mathcal{K} = \{A_1, A_2\}$ and $\operatorname{Div} B = 0$. Then B is constant.*

Proof. Without loss of generality we may assume $\mathcal{K} = \{0, H\}$ with $\operatorname{rank} H = n$. Thus there exists a subset $E \subseteq \Omega$ such that $B = H\chi_E$.

Since $\operatorname{Div} H\chi_E = H\nabla\chi_E = 0$ in the sense of distributions and $\operatorname{rank} H = n$, we have $\nabla\chi_E = 0$ in the sense of distributions, which concludes the proof. ◯

We now prove the analogue of Theorem 1.2 for (1.1) and $\mathcal{K} \equiv \{A_1, A_2\}$.

Theorem 2.2 *Let Ω be a bounded open and connected set in $\mathbf{R}^n$, with Lipschitz boundary, and let $\mathcal{K} = \{A_1, A_2\} \subset \mathcal{M}^{m\times n}$, $m \geq n \geq 1$, be such that $\operatorname{rank}(A_1 - A_2) = n$.*

Let B_h be a sequence weakly convergent to B in $L^p(\Omega, \mathcal{M}^{m\times n})$, with $p > 1$, such that

$$\mathrm{Div}\, B_h \to 0 \qquad \textit{strongly in } W^{-1,p}(\Omega, \mathbf{R}^m)$$

and

$$\mathrm{dist}\,(B_h, \mathcal{K}) \to 0 \qquad \textit{in measure}\,. \tag{2.2}$$

Then

$$B_h \to A_1 \qquad \textit{or} \qquad B_h \to A_2 \qquad \textit{in measure}\,.$$

Proof. Without loss of generality we may assume that $\mathcal{K} = \{0, H\}$, with $\mathrm{rank}\, H = n$.

We want to define a projection Z_h of B_h onto $\mathcal{K} = \{0, H\}$. In order to define it uniquely, up to a set of zero Lebesgue measure, let E_h be a subset of Ω such that $|B_h(x)| < |B_h(x) - H|$ a.e. in $\Omega \setminus E_h$, and $|B_h(x) - H| \le |B_h(x)|$ a.e. in E_h, then we set

$$Z_h = H\chi_{E_h}.$$

Finally set $R_h = B_h - Z_h$. By construction, $R_h \in L^p(\Omega, \mathcal{M}^{m\times n})$ and $|R_h(x)| = \mathrm{dist}(B_h(x), K)$ a.e. in Ω. Thus R_h converges to zero strongly in $L^p(\Omega, \mathcal{M}^{m\times n})$. This implies that

$$\mathrm{Div}\, Z_h = \mathrm{Div}\, B_h - \mathrm{Div}\, R_h \to 0 \qquad \text{strongly in } W^{-1,p}(\Omega, \mathbf{R}^m)\,.$$

Moreover, since $\mathrm{Div}\, Z_h = \mathrm{Div}\,(H\chi_{E_h}) = H\nabla\chi_{E_h}$ and $\mathrm{rank}\, H = n$, we have that

$$\nabla\chi_{E_h} \to 0 \qquad \text{strongly in } W^{-1,p}(\Omega, \mathbf{R}^m)\,. \tag{2.3}$$

In particular $\nabla\chi_{E_h}$ converges to zero in the sense of distribution, thus χ_{E_h} converges weak$*$ in L^∞ to a *constant* $\theta \in [0, 1]$. We now set $f_h = \chi_{E_h} - \theta$. By (2.1), (2.3) and the fact that f_h converges to zero strongly in $W^{-1,p}(\Omega)$ we deduce that $\chi_{E_h} - \theta$ strongly converges to zero in $L^p(\Omega)$. Then either $\theta = 0$ or $\theta = 1$, hence B_h converges in measure to either 0 or H. ◯

We now give a different proof of Theorem 2.2 which is parallel to the one originally given for the case of two gradients by Ball and James in [3] (see Theorem 1.2 in the Introduction). As in the latter result, this new proof requires a stronger assumption on the integrability on the sequence B_h. In the case of gradients the proof of Ball and James uses in an essential way the weak continuity of the quadratic minors for $p > 2$. Our proof for the case of divergence free fields uses the weak lower semicontinuity of a functional suggested by Tartar. More precisely we have the following lemma.

Lemma 2.3 (Tartar, [34]) *Let B_h be a sequence converging to B_0 weakly in $L^2(\Omega, \mathcal{M}^{n\times n})$ and such that* $\mathrm{Div}\, B_h$ *is compact in $W^{-1,2}(\Omega, \mathbf{R}^n)$. Set for all $A \in \mathcal{M}^{n\times n}$*

$$F(A) = (n-1)|A|^2 - (\mathrm{tr}\, A)^2\,.$$

Then

$$\liminf_h \int_\Omega F(B_h)\varphi\, dx \ge \int_\Omega F(B_0)\varphi\, dx \qquad \forall\, \varphi \in C_0^\infty(\Omega)\,,\ \varphi \ge 0\,. \tag{2.4}$$

Alternative proof of Theorem 2.2 for $p \geq 2$. Clearly it suffices to consider the case $p = 2$. As in the proof of Theorem 2.2 we may assume $K = \{0, H\}$ and $\operatorname{rank} H = n$. Let us first consider the case $m = n$. Now left multiplication by H^{-1} leaves the problem invariant. Therefore we will assume $H = I$. As before we may assume $B_h = \chi_{E_h} I$ and that, up to a subsequence, χ_{E_h} converges weak$*$ in L^∞ to a *constant* $\theta \in [0, 1]$. Then B_h converges to θI weak$*$ in L^∞. Fix a nonnegative $\varphi \in C_0^\infty(\Omega)$. We have

$$\liminf_h \int_\Omega F(B_h)\varphi \, dx = \liminf_h F(I) \int_\Omega \chi_{E_h} \varphi \, dx = \theta F(I) \int_\Omega \varphi \, dx$$

and

$$\int_\Omega F(\theta I)\varphi \, dx = \theta^2 F(I) \int_\Omega \varphi \, dx \, .$$

Since $F(I) < 0$, by (2.4) and the two above equalities, we have

$$\theta(\theta - 1) \geq 0 \, .$$

Hence either $\theta = 0$ or $\theta = 1$.

The general case $m > n$ can be easily recovered applying the above argument to the n independent rows of the matrix H. ○

3 A refinement of the rigidity for the two gradients problem

Our techniques can be used also to give an alternative proof of the prototype result of Ball and James stated in the Introduction (Theorem 1.2), in the more general version below. We define the Curl operator. It acts on $m \times n$ matrix valued fields as the distributional curl on each row.

Theorem 3.1 *Let $\Omega \subseteq \mathbf{R}^n$ be a bounded open and connected set, with Lipschitz boundary. Let $\mathcal{K} \equiv \{A_1, A_2\} \subset \mathcal{M}^{m\times n}$, with $m \geq n \geq 1$, and let B_h be a sequence weakly convergent to B in $L^p(\Omega, \mathcal{M}^{m\times n})$, with $p > 1$, such that*

$$\text{(3.1)} \qquad \operatorname{Curl} B_h \to 0 \qquad \textit{strongly in } W^{-1,p}(\Omega, \mathcal{M}^{n\times n})^m$$

and

$$\operatorname{dist}(B_h, \mathcal{K}) \to 0 \qquad \textit{in measure}\, .$$

If $\operatorname{rank}(A_1 - A_2) > 1$, *then*

$$B_h \to A_1 \qquad \textit{or} \qquad B_h \to A_2 \qquad \textit{in measure}\, .$$

Proof. As in the case of Theorem 2.2 it is not restrictive to assume that $\mathcal{K} = \{0, H\}$, with $\operatorname{rank} H > 1$. Up to a change of variables we may also assume that the first two rows of H are given by the first two elements of the canonical basis. Moreover we may assume that there exists $E_h \subseteq \Omega$ such that

$$B_h = H\chi_{E_h} \, ,$$

thus by (3.1) we have

$$\text{(3.2)} \qquad \operatorname{Curl} H\chi_{E_h} \to 0 \qquad \text{strongly in } W^{-1,p}(\Omega, \mathcal{M}^{n\times n})^m \, .$$

With our convention on the Curl operator, the first row in (3.2) implies that $\partial_i \chi_{E_h} \to 0$ strongly in $W^{-1,p}(\Omega)$ for every $i \neq 1$, while the second row handles the case $i = 1$. As in Theorem 2.2, this implies the strong convergence of χ_{E_h} either to 0 or to 1. ○

The improved generality of the latter statement with respect to Theorem 1.2 relies on the weaker L^p-integrability requirements about the sequence B_h. We only require $p > 1$.

As already pointed out, the present version could also be obtained by Theorem 1.2, using a regularization lemma due to Kristensen (see [18], Lemma 3.3).

4 Lack of rigidity

Lemma 4.1 *Given $m \geq n \geq 3$, there exist three pairwise rank n-connected $m \times n$ matrices, A_1, A_2, A_3, and there exists a sequence $B_h \in L^\infty(\Omega, \mathcal{M}^{m\times n})$ such that setting $\mathcal{K} = \{A_1, A_2, A_3\}$, one has*

$$\text{dist}\,(B_h, \mathcal{K}) \to 0 \qquad \textit{strongly in} \qquad L^p(\Omega) \;\;, \forall\, p \geq 1\,, \tag{4.1}$$

$$\text{Div}\, B_h \to 0 \qquad \textit{strongly in} \quad W^{-1,p}(\Omega, \mathbf{R}^m) \quad , \forall\, p \geq 1 \tag{4.2}$$

and $B_h \rightharpoonup B$ in $w*$-L^∞, with $B \neq A_i$ for any $i = 1, 2, 3$.

Remark 4.2 One can actually construct a sequence B_h with the desired properties and which, in addition, is bounded in L^∞.

We will need the following algebraic lemma.

Lemma 4.3 *Let $q_i \in (0, 1)$, $i = 1, 2, 3$, be given. There exist six 3×3 diagonal matrices, A_1, A_2, A_3 and S_1, S_2, S_3, and three unit vectors e_1, e_2 and e_3, satisfying the following properties*

$$\det\,(A_i - A_j) \neq 0 \qquad \textit{if } i \neq j \tag{4.3}$$

$$(A_i - S_i)e_i = 0 \qquad \textit{for} \;\; i = 1, 2, 3 \tag{4.4}$$

$$q_iA_i + (1 - q_i)S_i = S_{i-1} \;\textit{mod}\; 3 \qquad i = 1, 2, 3\,. \tag{4.5}$$

Proof. The following is an explicit choice: $\{e_1, e_2, e_3\}$ is the canonical basis in $\mathbf{R}^3$,

$$A_1 = 0\,, \quad A_2 = \text{diag}\left\{\frac{-q_3(1-q_2)}{q_2}, \frac{q_3}{q_1+q_3-q_1q_3}, \frac{q_1+q_2-q_1q_2}{(1-q_1)q_2}\right\}\,, \quad A_3 = I\,,$$

$$S_1 = \text{diag}\left\{0, \frac{q_3}{q_1+q_3-q_1q_3}, \frac{1}{(1-q_1)}\right\}\,, \quad S_2 = \text{diag}\left\{q_3, \frac{q_3}{q_1+q_3-q_1q_3}, 1\right\}\,,$$

$$S_3 = \text{diag}\left\{0, \frac{q_3(1-q_1)}{q_1+q_3-q_1q_3}, 1\right\}\,. \tag{4.6}$$

$\bigcirc$

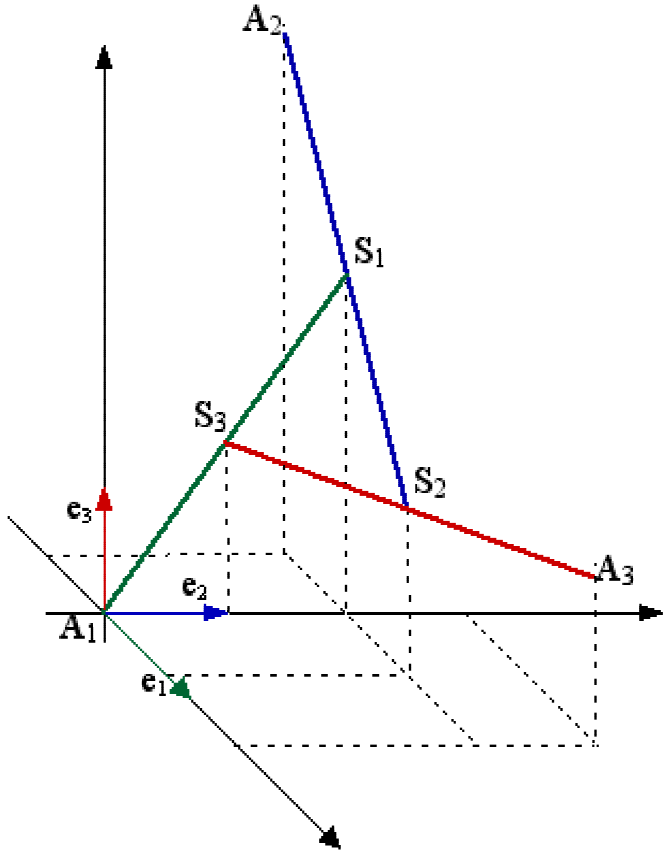

Figure 1: Representation of A_i and S_i in eigenvalues space

The previous lemma admits a geometric interpretation which can be visualized with the help of figures 1 and 2.

To prove Lemma 4.1, for the case $n = m = 3$ and $\Omega = Q = (0, 1)^3$, it is enough to follow the scheme implemented in the work of Tartar [36] (see also [22], [6], [24]). The construction is the same as in [22] using the three matrices given in Lemma 4.3. It requires a little modification and an additional technical effort in order to verify assumption (4.2). Roughly the idea is the following. For any given $\varepsilon > 0$ and $N \in \mathbf{N}$, we construct a piecewise constant matrix valued function B_ε^N making N laminations with well separated scales of oscillation $\varepsilon_k = \alpha_k(\varepsilon)$, $k = 1, ..., N$, where $\alpha_k(\varepsilon) \to 0$ as $\varepsilon \to 0$ and $\alpha_{k+1}(\varepsilon) \ll \alpha_k(\varepsilon)$. The basic construction has three laminations. We choose A_i and S_i, $i = 1, 2, 3$, satisfying (4.3), (4.4) and (4.5) and for simplicity we set $q_1 = q_2 = q_3 = \frac{1}{2}$. We begin by laminating A_1 and S_1 in "direction" e_1 (i.e. the interface is orthogonal to e_1) with oscillations of size ε_1. Namely if $\chi_i(x) = \chi_{[0,1/2)}(x_i)$, for $i = 1, 2, 3$, then

$$B_\varepsilon^1 = A_1\chi_1\left(\frac{x}{\varepsilon_1}\right) + S_1\left(1 - \chi_1\left(\frac{x}{\varepsilon_1}\right)\right). \tag{4.7}$$

Clearly $|\{B_\varepsilon^1 \notin \mathcal{K}\}| = q_1 = \frac{1}{2}$. Next we replace S_1, in the set $Q_1 = \{B_\varepsilon^1 \notin \mathcal{K}\}$, by a laminate of A_2 and S_2 in direction e_2 with oscillations at the smaller scale ε_2, i.e.

$$B_\varepsilon^2 = A_1\chi_1\left(\frac{x}{\varepsilon_1}\right) + \left(1 - \chi_1\left(\frac{x}{\varepsilon_1}\right)\right)\left[A_2\chi_2\left(\frac{x}{\varepsilon_2}\right) + S_2\left(1 - \chi_2\left(\frac{x}{\varepsilon_2}\right)\right)\right],$$

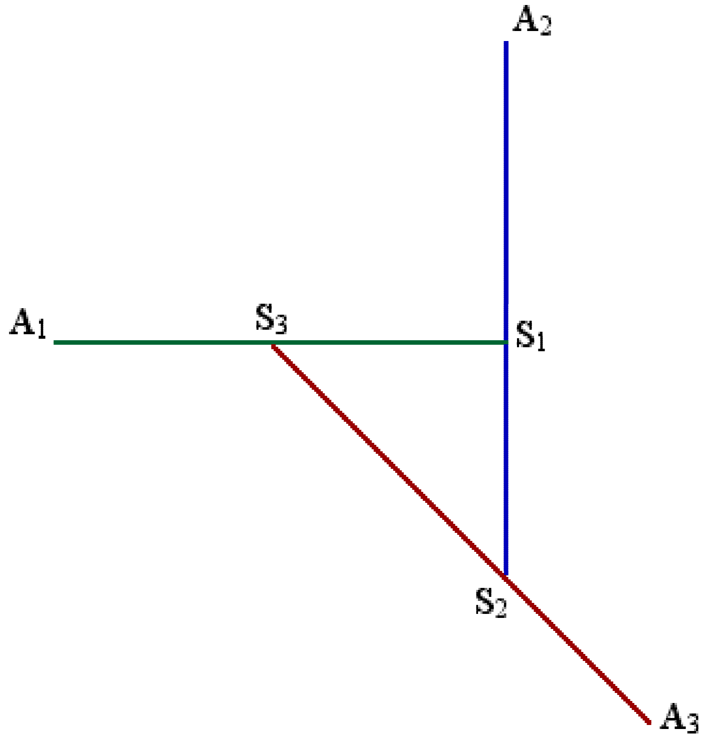

Figure 2: Projection in the A_i plane of Figure 1

and we have that the set $Q_2 = \{B^2_\varepsilon \notin \mathcal{K}\}$ has measure $q_1 q_2 = \frac{1}{4}$. We continue replacing S_2 by a laminate of A_3 and S_3 in direction e_3 with oscillations at scale ε_3 obtaining

$$B^3_\varepsilon = A_1\chi_1\left(\frac{x}{\varepsilon_1}\right) + \left(1-\chi_1\left(\frac{x}{\varepsilon_1}\right)\right)\left\{A_2\chi_2\left(\frac{x}{\varepsilon_2}\right) + \right.$$

$$\left. + \left(1-\chi_2\left(\frac{x}{\varepsilon_2}\right)\right)\left[A_3\chi_3\left(\frac{x}{\varepsilon_3}\right) + S_3\left(1-\chi_3\left(\frac{x}{\varepsilon_3}\right)\right)\right]\right\}.$$

At this point the only set, Q_3, where $B^3_\varepsilon \notin \mathcal{K}$ is occupied by S_3 and $|Q_3| = \frac{1}{8}$. Next we repeat the above three steps construction in the set Q_3 using the scales ε_{3+1}, ε_{3+2} and ε_{3+3} respectively. This leaves out the set Q_6 where again $B^6_\varepsilon \notin \mathcal{K}$ and obviously $|Q_6| = \frac{1}{8}|Q_3| = \frac{1}{2^6}$.

Iterating this procedure, for any fixed N, we construct a sequence B^N_ε which achieves the following goals: $B^N_\varepsilon \in \mathcal{K}$ up to a set of measure $\frac{1}{2^N}$ and $\operatorname{Div} B^N_\varepsilon$ converges to zero strongly in $W^{-1,2}$ as $\varepsilon \to 0$. To prove the latter property one makes use of (4.4) and (4.5), and of the fact that the scales are well separated. This can be done, for instance, using results by Allaire and Briane concerning multiple scales convergence (see [1], Theorem 3.3 and Corollary 3.4). In order to obtain a sequence satisfying (4.1) and (4.2) simultaneously one needs a fine tuning of the parameter ε in terms of the number of the iterations N. To implement the above strategy, one needs a more refined use of the multiple scales results mentioned above, leading to an infinitely many scales convergence as in the work of Briane ([4]).

The sequence constructed in this way converges weakly to S_3 and therefore does not converges to any element of $\mathcal{K}$.

The case $n \geq 3$ and $m \geq 3$ is handled by an obviuos modification of the previous scheme. We omit the details.

Remark 4.4 The result of this section arises a natural new questions. Does a Clebík Kirchheim-type theorem holds? In other words do exact solutions exist? We do not know.

5 Differential constraints of mixed type

The spirit of this section is slightly different from the previous one. Let $n \geq 2$, $m \geq 2$ and m vector fields $b_i : \Omega \subseteq \mathbf{R}^n \to \mathbf{R}^n$ be given. Suppose one has

$$\mathcal{L}_i b_i = 0\,, \qquad i = 1, \cdots, m\,,$$

where the $\mathcal{L}_i$ are linear differential operators which, for simplicity, we will assume to be either div or curl. In fact this assumption is definitely not important. The "constant rank" hypothesis seem to be the only crucial one.

We wish to address the rigidity problem in this context. We will comment at the end of Section 6 why this issue arises in a natural way. We want to start giving an example showing the occurrence of a new phenomenon.

We assume $n = m = 2$, $\mathcal{L}_1 = \text{curl}$ and $\mathcal{L}_2 = \text{div}$. In other words we are considering the case

$$\left\{ \begin{array}{l} \text{curl}\, b_1 = 0 \\ \text{div}\, b_2 = 0\,. \end{array} \right. \tag{5.1}$$

We are again interested in whether the matrix

$$B = \begin{pmatrix} b_1^1 & b_1^2 \\ b_2^1 & b_2^2 \end{pmatrix}$$

may belong to a given set of constant matrices. For the most basic example, we set $\mathcal{K} = \{A_1, A_2\}$ and $H = A_2 - A_1$. We ask the usual rigidity question in the context of exact solutions. In other words we ask whether one can find a matrix field $B(x)$ belonging to $\mathcal{K}$ almost everywhere and satisfying (5.1). Then we have:

$$\textit{lack of rigidity} \quad \Longleftrightarrow \quad \text{rank}\, H = 2 \textit{ and the rows of } H \textit{ are orthogonal}.$$

One reason why we think this result is interesting is that it contrasts with the classical examples where the rank of the matrix H is the only information needed for deciding about rigidity. The proof is an easy consequence of the known results and will be just sketched now. Indeed, since we are in two dimensions, the problem (5.1) is equivalent to the following one

$$\left\{ \begin{array}{r} \text{curl}\, b_1 = 0 \\ \text{curl}\, J b_2 = 0\,, \end{array} \right. \tag{5.2}$$

where J is a ninety degree rotation matrix. Performing a translation by a constant matrix, we see that the constraint $B(x) \in \mathcal{K}$ is the same as $b_1(x) = \chi(x) h_1$ and $b_2(x) = \chi(x) h_2$, where the constant vectors h_1 and h_2 are the rows of H. Hence, (5.2) is equivalent to

$$\left\{ \begin{array}{l} \nabla \chi = \lambda h_1 \\ \nabla \chi = \nu J h_2\,, \end{array} \right. \tag{5.3}$$

for some λ and ν in $W^{-1,2}$. Each of the two above equations can be solved, thus (5.3) holds if and only if

$$\nabla\chi = \lambda h_1 = \nu J h_2 \,. \tag{5.4}$$

Hence, if h_1 is not parallel to Jh_2, (5.2) has no exact solutions and therefore rigidity holds. Conversely, if h_1 is parallel to Jh_2, by the Ball and James result there is lack of rigidity (and the only possible geometries are laminates at least locally).

The case of approximate solutions can be handled using again Nečas' result and therefore it holds in the version with $p > 1$.

Generalizations to higher dimensions are elementary and will be omitted. Let us just note that in higher dimension the same phenomenon occurs. The rank of the matrix made by the m vectors is not in general sufficient to deciding about rigidity.

6 Link with bounds on effective moduli

One of the most challenging issues in the theory of composites, addresses the problem of minimizing the "overall" energy of some class of microgeometries for media which are made by a certain number of given phases with known density of energy. In the present section we shall try to illustrate, with the help of a few examples, how the problem of bounding the effective moduli for composites is tightly linked with the issues treated in this paper. First we will show how our rigidity results can be used to obtain that some elementary bounds are not "optimal". In the case of power-law materials this provides a new, although expected, result. The idea is that many problems of bounds in composites can be reduced to the ("approximate") solvability of differential inclusions of the type

$$\begin{cases} \mathcal{L}\,B = 0 \\ B(x) \in \mathcal{K} \quad \text{a.e.} \end{cases} \tag{6.1}$$

for a suitable choice of the differential operator $\mathcal{L}$ and of the set $\mathcal{K}$. Conversely the knowledge of an explicit bound for some class of composites, can give an answer to the question on the solvability of the "corresponding" problem of type (6.1).

Let us start with the most basic and well known example. Let $0 < \alpha_1 < \alpha_2$, $\theta \in (0,1)$, an open bounded and simply connected set Ω and a constant matrix F be given. Consider the family of functions $\alpha(x) = \alpha_1\chi(x) + \alpha_2(1-\chi(x))$ where χ varies among all possible characteristic functions of some measurable subset of Ω. One of the basic tasks is to find, for a given $\theta \in (0,1)$, the highest "overall conductivity"

$$G(F) := \sup_{\chi : \fint \chi = \theta} \left(\inf_{U - Fx \in W_0^{1,2}(\Omega;R^n)} \fint \alpha(x)|DU(x)|^2 dx \right) . \tag{6.2}$$

In most cases one is especially interested in the restriction on the "volume fraction" given by $\fint \chi = \theta$ which can be interpreted as the "cost" of using, say, the "most expensive phase" in the composite.

The latter is just the prototype problem, one of the few for which the solution is well understood. A convenient way to think of this is that the conductivity $\sigma(x)$ takes two values, $\alpha_1 I$ and $\alpha_2 I$.

For problem (6.2) one has the elementary bound

$$G(F) \le (\alpha_1\theta + \alpha_2(1-\theta))|F|^2 \,, \tag{6.3}$$

obtained taking $U = Fx$ as a test field. One may wonder whether this bound could be "attainable" or at least "optimal" for a full rank matrix F. We will say that (6.3) is attainable if there exists a geometry χ for which (6.3) holds as an equality and such that the supremum in (6.2) is achieved. It is optimal if one has the weaker result that the inequality (6.3) cannot be improved. In this very particular case the non optimality can be easily obtained using the well-known optimal bounds in homogenization [14], [34] and [19] which actually give a much stronger result (in the L^2 context). But for the sake of illustration let us pretend we did not know the linear bounds and let us see as the non optimality result can be immediately obtained as a consequence of Theorem 2.2. For simplicity consider the case $F = I$. Then assume by contradiction that the bound (6.3) is attainable. This implies that there exists a geometry χ_0 such that the corresponding minimum problem in (6.2) is achieved by the function $U = x$. Using the Euler Lagrange equation associated to the functional one obtains that

$$\mathrm{Div}\,([\alpha_1\chi_0(x) + \alpha_2(1 - \chi_0(x))]I) = 0\,. \tag{6.4}$$

It is readily seen that this contradicts Proposition 2.1, applied with $\mathcal{K} = \{\alpha_1 I, \alpha_2 I\}$.

Similarly, assuming that (6.3) is optimal rather than attainable would imply the existence of a sequence of geometries χ_h for which the corresponding minimum points in (6.2) are "close" to x. One can prove that in this case the contradiction follows by Theorem 2.2.

An obvious generalization of the problem described above is to require that $\sigma(x)$ takes values in the set $\{\alpha_i I \ : \ i = 1, ..., M\}$. In the traditional terminology one is considering a mixture of M isotropic phases. We say that the volume fractions are prescribed if the constraint $⨍\chi_i = \theta_i$ is imposed for some given positive numbers θ_i's adding up to one. As in the case of the two-phase linear problem, an argument very similar to the one we used in the proof of Theorem 2.2, can be used to deduce the non optimality of the elementary bound (6.3) for the M-phase problem when $F = I$.

Another class which has been extensively studied arises when the integrand has some p-power growth greater than one, (these are the so-called power-law materials). Choose any $p > 2$, any integer $M \geq 2$ and $\alpha(x) = \sum_{i=1}^{M} \alpha_i\chi_i(x)$. Set $U = (U^1, \cdots, U^n)$. Given the α_i and given $\theta_i = ⨍\chi_i$, we have the elementary upper bound

$$\sup_{\chi_i : ⨍\chi_i = \theta_i} \left(\inf_{U - x \in W_0^{1,p}(\Omega;R^n)} \frac{1}{p} ⨍ \alpha^{\frac{p}{2}}(x) \sum_{j=1}^{n} |\nabla U^j(x)|^p dx \right) \leq \frac{n}{p} (\sum_{i=1}^{M} \alpha_i^{\frac{p}{2}}\theta_i)\,. \tag{6.5}$$

The most challenging issue here is to improve upon the latter upper bound. When $p < 2$ one can use the results of Willis [37], Talbot and Willis [30], and Ponte-Castañeda [23], but for $p > 2$ these results do not apply.

Only a few results concerned with special assumptions on the structure of the integrand are available in which there is a (slight) improvement upon the elementary bound. These are mostly due to the very nice work by Talbot and Willis [31] and [32] and Talbot [29]. We now ask again whether (6.5) is optimal. This time we cannot invoke better bounds in general (except for $M = 2$), therefore the question is non trivial. In fact, using the same ideas as the one explained in the linear case, one can prove the following result.

Proposition 6.1 *The elementary bound (6.5) is not optimal for any $p > 1$.*

Remark 6.2 Note that, however, the bounds are attainable for $p = \infty$ in a sense explained in [12]. This is related to the so-called ideal plasticity (see [15] and [5]).

There are many variations on the theme, some with important consequences in various applications, obtained replacing the conductivity $\sigma(x) = \alpha(x)I$ by a uniformly elliptic symmetric (conductivity) matrix taking values in a prescribed set. Some of the mathematically less intricate, yet unsolved, problems arise when one is considering a material which is polycrystalline. In this case the conductivity has the form

$$\sigma(x) = \sum_{i=1}^{M} \chi_i(x) R^t(x) \mathrm{diag}(\alpha_1^{(i)}, \cdots, \alpha_n^{(i)}) R(x) \,.$$

Here the $\alpha_j^{(i)}$ are given positive numbers and the measurable matrix field of rotation R can be arbitrarily chosen.

To fix ideas consider the simplest case $M = 1$. Again one aims to find

$$G(F) := \sup_{R \in SO(n)} \left(\inf_{U - Fx \in W_0^{1,2}(\Omega; R^n)} \fint \mathrm{tr}[DU(x)\sigma(x)DU^t(x)]dx \right) . \tag{6.6}$$

The elementary bound for $F = I$, yields

$$G(I) \leq \sum_{i=1}^{n} \alpha_i \tag{6.7}$$

and it is obviously attained for any *constant* rotation matrix R_0. The interesting question, of course, is whether it can be achieved for some non constant $R_{\mathrm{opt}} \in L^\infty(\Omega, SO(n))$. If such an optimal rotation exists, then using the Euler Lagrange equations one finds that

$$\mathrm{Div}(R_{\mathrm{opt}}^t(x) \mathrm{diag}(\alpha_1, \cdots, \alpha_n) R_{\mathrm{opt}}(x)) = 0 \tag{6.8}$$

which reduces to (6.1) for $\mathcal{L} = \mathrm{Div}$ and $\mathcal{K}$ given by

$$\bigcup_{R \in SO(n)} \{R^t \mathrm{diag}(\alpha_1, \cdots, \alpha_n) R\} \,. \tag{6.9}$$

The rigidity for the latter problem depends on the dimension. It is interesting to see that in the L^2 context, the question can be answered by looking at the corresponding "G-closure" results. In dimension two (6.6) is not optimal (see [16] and [10]), while in higher dimension the bound is known to be optimal (see [25] and also [2]). This implies rigidity for problem (6.8) in dimension two and lack rigidity in higher dimension.

Finally let us mention a yet different problem arising in homogenization that can be reduced to the type of issue treated in Section 5 and therefore gives further motivations for it. To fix ideas let us consider the case of the linear M-phase problem in two dimensions. It is well known that the eigenvalues $\lambda_1 \leq \lambda_2$ of the effective conductivity matrix satisfy

$$h := \left(\fint \alpha^{-1}(x)dx \right)^{-1} \leq \lambda_1 \leq \lambda_2 \leq a := \fint \alpha(x)dx \,.$$

A more subtle and very interesting question is the following. Assume $\lambda_1 = h$, what is the range of λ_2? For two phases the answer is known: λ_2 must be a. For $M > 2$, thank to the work of Cherkaev and Gibiansky [7], it is known that the range is non trivial but the complete answer is not known (see also [13], [8]). This question can be rephrased in terms of the rigidity of (6.1) with $\mathcal{L} \equiv \{\mathrm{div}, \mathrm{curl}\}$, $\mathcal{K} \equiv \cup_{i=1}^{M}\{\mathrm{diag}(\alpha_i, \alpha_i^{-1})\}$.

The present section hopefully will provide various new motivations to the study which falls in a framework of $\mathcal{A}$-quasiconvexity (Fonseca and Müller [11]).

More generally the present paper arises new questions concerning the "relaxations" of the set $\mathcal{K}$ (see [20] and references therein) in the cases when there is lack of rigidity.

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