Asymptotic analysis for some linear eigenvalue problems via Gamma-Convergence

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Abstract

This paper is devoted to the analysis of the asymptotic behaviour when the parameter λ goes to $+\infty$ for operators of the form $-\Delta+\lambda a$ or more generally, cooperative systems operators of the form $\begin{pmatrix} -\Delta+\lambda a & -b \\ -c & -\Delta+\lambda d \end{pmatrix}$ where the potentials a and d vanish in some subregions of the domain Ω. We use the theory of Γ-convergence, even for the non-variational cooperative system, to prove that for any reasonable bounded potentials a and d those operators converge in the strong resolvent sense to the operator in the vanishing regions of the potentials, so does the spectrum. The class of potentials considered here is fairly large, substantially improving previous results, allowing in particular ones that vanish on a Cantor set, and forcing us to enlarge the class of domains to the so-called *quasi-open sets*. For the system various situations are considered applying our general result to the interplay of the vanishing regions of the potentials of both equations.

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Key words. Γ-convergence, eigenvalues, semiclassical analysis, cooperative systems.

Contents

Introduction

The aim of this paper is to ascertain the asymptotic behaviour of the spectrum of a certain class of linear eigenvalue problems of the following form

$$(L + \lambda a)w_\lambda = \sigma_k(\lambda)w_\lambda, \tag{0.1}$$

in a bounded domain $\Omega \subset \mathbb{R}^N$ when the parameter λ goes to infinity. Here L is a linear elliptic operator and a is a potential that is non negative. We denote $\sigma_k(\lambda)$ the kth-eigenvalue for the operator $A(\lambda) := L + \lambda a$ under homogeneous Dirichlet boundary conditions or Neumann-type boundary conditions and w_λ an eigenfunction associated with $\sigma_k(\lambda)$, normalized in L^2. Depending on the operator those eigenfunctions may be vectorial. We are particularly interested in the problem when the potential a may vanish in a quite large subregion of Ω as was studied before by Dancer [9], Dancer and López-Gómez [10] and [2].

In our approach we shall use the so-called *Gamma-Convergence* theory (Γ-convergence in short) to describe the behaviour of the equations asymptotically when λ tends to infinity. Relying on an idea of Ennio De Giorgi [13], the powerful tool of Γ-convergence was developed by the Italian School in the seventies in order to study the convergence of variational problems in several and quite different contexts (e.g. Buttazzo [5], De Giorgi-Franzoni [14], Spagnolo [30] and others). It rapidly became the required point of view to carry out in a skillful way, through easy to use necessary and sufficient conditions, the convergence of minimizers associated to a family of functionals. The survey of Dal Maso [8], from which we were inspired for the preliminary section of the present paper, is now a classic in the field. We refer to its bibliography for a more precise history of the theory.

Primarily, the Γ-convergence related to quadratic forms and elliptic operators (see [30] and [8] Chapter 12) will stand for our principal interest. As a first case we study the limiting behaviour of the eigenvalues and eigenfunctions for one single equation with the operator $-\Delta + \lambda a$. In this particular case we suppose that the potential a in front of the parameter λ is a non-negative function vanishing in a subdomain (maybe disconnected) Ω_0^a. In the same spirit as Buttazzo and Dal Maso [6, 7] (see also Stollmann [31] and Simon [25] for similar approaches), we cover this convergence under very general assumptions on the potentials (Theorem 30), where the potential a is a Borel function from $\overline{\Omega}$ to $\mathbb{R}^+$ satisfying

$$\sup_{x \in \overline{\Omega}} a(x) < +\infty \quad \text{and} \quad H_0^1(\Omega_0^a) = \{u \in H_0^1(\Omega); u = 0 \text{ a.e. on } \ \Omega_+^a\}.$$

For instance, the case when a is not continuous and vanishing on a dense set is supposed. It also includes the particular situation when a is any function in $C^0(\overline{\Omega})$. Under our circumstances we obtain the desired convergence, when the parameter λ goes to infinity, for the whole spectrum in Ω. Indeed, we prove that the eigenvalues always converge to the eigenvalues in the domain where the potential vanishes, denoted by $\Omega_0^a := \{x \in \bar{\Omega}\,;\, a(x) = 0\}$. In other words, within our Γ-convergence context, to the Laplacian defined in $H_0^1(\Omega_0^a)$. Since Ω_0^a might not be an open set, $H_0^1(\Omega_0^a)$ has to be understood in a general sense lying on *quasi-open sets* (see Section 1.2). That forces us to work on a wider class of sets in order to achieve our more general results. However, under some additional *stability* properties on the interior of Ω_0^a we have that $H_0^1(\Omega_0^a) = H_0^1(\text{Int}\Omega_0^a)$, obtaining the limiting problem in the classical sense on a domain.

In our setting it turns out that the convergence of the operators $-\Delta + \lambda a$ to the operator $-\Delta$ in the strong resolvent sense, reduces naturally to the Γ-convergence of the associated quadratic form, which holds here quite efficiently. It is worth mentioning that the convergence of all the eigenvalues follows quasi-automatically from this convergence. It seems that these techniques were not exploited in the past despite its efficient and elegant consequences. Indeed, the present paper provides a different perspective and maybe a more synthetic and simplified way of proving some convergence results that were established in the past in a more laborious fashion. Moreover, this allows us to improve previous results since we are imposing less restrict regularity conditions for the domains and, as mentioned before, a more general class of potentials, obtaining the convergence of the whole spectrum, not just for the first eigenvalue (cf. [1, 9, 10]).

Note that when $h := \frac{1}{\sqrt{\lambda}}$, the equation (0.1), with $A(\lambda) = -\Delta + \lambda a$, becomes $\left(-h^2\Delta + a(x)\right) w_h = \sigma_k(h) w_h$, which plays an important role in Semiclassical Analysis. It is said in Semiclassical Analysis that when h, the Planck's constant, approximates zero all quantum effects are neglected. Indeed, this transition between classical and quantum mechanics is what was coined as Semiclassical Analysis. Actually, there is a huge amount of literature on the aforementioned semi-classical problem so that is would be difficult to give some exhaustive references. However we shall quote some related work going back to Simon [26–29], Helffer and Sjöstrand [18], Stollmann [31] (see also the book of Dimassi and Sjöstrand [15] and the references therein).

Furthermore, the analysis carried out here might play a relevant role in the context of Semiclassical Analysis, at the difference that we are dealing with highly degenerate potentials that possibly vanish on a full-measure set that most of the time implies an upper bound on the first eigenvalue which is slightly different from the situation usually treated in the work mentioned just before.

In continuation we also treat the case of a cooperative system corresponding to the operator

$$A(\lambda) := \begin{pmatrix} -\Delta + \lambda a & -b \\ -c & -\Delta + \lambda d \end{pmatrix}, \qquad \text{with} \quad w_\lambda := (u_\lambda, v_\lambda). \tag{0.2}$$

The system is supposed to be cooperative in the sense that the terms outside the principal diagonal b and c are two point-wise positive functions in Ω. Moreover, as an extension from the case of one single equation the potentials a and d in front of the parameter are two non-negative functions still fulfilling our same general conditions.

As we shall see in the penultimate section, the direct method of Γ-convergence is no more pertinent for the case of an elliptic system which is not of variational type, as the one associated to the operator (0.2) when $b \neq c$. However, we are able in this situation to benefit from the Γ-convergence of the single equations to get a clear vision of the problem, which leads us to obtain at the end a substantial improvement of the known results about the non-variational system. To be more precise, let us denote

$$\Omega_0^a := \{x \in \overline{\Omega}; a(x) = 0\}, \qquad \Omega_0^d := \{x \in \overline{\Omega}; d(x) = 0\},$$

$$\Omega_0 := \{x \in \overline{\Omega}; a(x) = 0 = d(x)\},$$

the subdomains where the potentials a and d vanish. Note that the particular structural assumptions on Ω_0^a and Ω_0^d supposed in [2] were pivotal. Moreover, the situation when both potentials vanish in the same region was covered in [24] basing the proof upon the construction of an appropriate supersolution. A method that is no longer available in a general distribution as the one considered here. In this work those assumptions are eventually relaxed generalizing and improving those findings. Indeed in our result, no tight restrictions are imposed on the regularity and the spatial distribution of the subdomains Ω_0^a and Ω_0^d. We manage to identify a limiting problem for almost any reasonable bounded potentials a and d without any structural assumption on the vanishing domains (see Theorem 30). Then, with further assumptions on the structural compositions of the vanishing domains, we are able to recover some of the results that were dealt with before in the literature and give some examples when the convergence is substantially different (see our last section).

We would like to mention that all the results of this paper could be easily extended without any important changes in the proofs replacing the standard Laplace operator by any elliptic operator of the form

$$L(u) = \sum_{i,j=1}^{N} D_i(a_{ij} D_j u),$$

with coefficients $a_{ij} \in L^\infty(\Omega)$ such that $a_{ij} = a_{ji}$ and

$$c_0|\xi|^2 \leq \sum_{i,j=1}^{N} a_{ij}(x)\xi_j\xi_i \leq c_1|\xi|^2,$$

for a.e. $x \in \Omega$, for every $\xi \in \mathbb{R}^N$ and for some positive constants $0 < c_0 \leq c_1$. However, we found it clearer to simply write everything in the special case of the classical Laplacian.

The distribution of the present paper is as follows. In the first Section we give some preliminary results. In particular in Section 1.1, we recall briefly some elements from the Theory of Γ-convergence and quadratic forms that will be needed throughout the paper. We have tried through this section to make this paper self contained and accessible to nonspecialists. In Section 1.2, we give a general definition of $H_0^1(E)$ when E is any subset (not necessarily open), using the capacity and the notion of quasi-open sets. Then, we define in Section 1.3 the general class of *nice potentials* for which the convergence results hold and also recall some stability results on the space $H_0^1(\Omega)$ that will give some further information on the limiting problem when some more regularity is supposed. Some examples of such *nice potentials* are exhibited. In Section 2, we apply the Theory shown in the preliminaries to study the limiting behaviour of a single Linear Problem. We first study the asymptotic for the Dirichlet problem (Section 2.1) and then, in a similar way, deal with Neumann boundary conditions (Section 2.2). Section 3 is devoted to the case of a cooperative system. In the case when the matrix operator is symmetrical (it is then a variational problem), we can argue as for the single equation and use the Theory of Γ-convergence directly on the system to study its asymptotic. This is done in Section 3.1. Finally in Section 3.2, we get the same conclusions for a non-symmetrical system using this time, in a suitable way, the Γ-convergence of each component of the system given by the result of Section 2 about one single equation. In the last section we describe more precisely this convergence for some explicit examples.

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Some Notations :
$\mathcal{L}(H)$: the space of bounded linear operators from H to H;
$\mathscr{L}^N$: the N-dimensional Lebesgue measure;
$A \triangle B := A \backslash B \cup B \backslash A$: the symmetric difference between two sets A and B;
$A^c = \mathbb{R}^N \backslash A$: the complement of A;
$\overline{A}$: the closure of A;
Int$A$: the interior of A;
cap(E) : the capacity of E (defined in (1.4)).
H' : the topological dual of the Hilbert space H.

1 Preliminaries

1.1 Short review on Γ-convergence and quadratic forms

We recall here some results that are contained in the survey of Dal Maso [8] about Gamma-convergence, principally in Chapter 11 and Chapter 12. In what follows H denotes a Hilbert space with scalar product $\langle .,. \rangle$ and norm $\|.\|$. If $Q : H \to \overline{\mathbb{R}}$ is any quadratic form on H, the domain of Q is defined by $D(Q) := \{u \in H; Q(u) < +\infty\}$. The bilinear form associated to Q is the only bilinear form

$$B : D(Q) \times D(Q) \to \mathbb{R},$$

such that $Q(u) = B(u,u)$ for every $u \in D(Q)$. We say that Q is lower semi-continuous on H if

$$Q(u) \leq \liminf_{n \to +\infty} Q(u_n),$$

for any $u \in H$ and any sequence $\{u_n\}$ converging to u.

We denote $V := \overline{D(Q)}$. The operator A associated with Q is the linear operator A on V defined as follows : the domain $D(A)$ of A is the set of all $u \in D(Q)$ such that there exists $f \in V$ satisfying $B(u,v) = \langle f, v \rangle$, for every $v \in D(Q)$, and $A(u) = f$ for every $u \in D(A)$ (the uniqueness of such an f follows from the density of $D(Q)$ in V). This operator A is positive and symmetric. In addition, if Q is lower semi-continuous on H then A is self-adjoint on V ([8] Theorem 12.13).

Given a constant $\alpha > 0$, any quadratic form $Q : H \to \overline{\mathbb{R}}$ on H will be said to be *α-coercive* if $Q(u) \geq \alpha \|u\|^2$ for every $u \in H$. A sequence Q_n of quadratic forms are *equi-coercive* if they are α-coercive with same constant $\alpha > 0$.

If Q is *α-coercive* then the associated operator A is invertible and the inverse map A^{-1} is a bounded operator on $W := \overline{D(A)}$. If Q is positive then for every $\mu > 0$ the operator $A + \mu \mathrm{Id}$ is μ-coercive thus invertible.

Let us now describe our different notions of convergence.

Definition 1 (G-convergence)**.** *We say that a sequence A_n of α-coercive and self-adjoint operators G-converges to an α-coercive and self-adjoint operator A in the strong topology if for every $f \in H$, $A_n^{-1} P_n f$ strongly converges to $A^{-1} P f$ in H, where P_n and P are the orthogonal projections onto $W_n := \overline{D(A_n)}$ and $W := \overline{D(A)}$ respectively.*

Definition 2 (Convergence in the resolvent sense)**.** *We say that a sequence A_n of positive self-adjoint operators converges to a positive self-adjoint operator A in the strong resolvent sense if for every $\mu > 0$, $\mu \mathrm{Id} + A_n$ G-converges to $\mu \mathrm{Id} + A$ in the strong topology.*

Definition 3 (Γ-convergence)**.** *Given a sequence Q_n of quadratic forms from H into $\overline{\mathbb{R}}$ we say that Q_n Γ-converges to a quadratic form Q if for every $u \in H$ the two following properties hold :*
i) for every sequence $\{u_n\}$ that converges to u one has $Q(u) \leq \liminf_{n\to+\infty} Q_n(u_n)$;
ii) there exists $\{u_n\}$ that converges to u and such that $Q(u) \geq \limsup_{n\to+\infty} Q_n(u_n)$.

Corollary 13.7 in [8] says in particular that for a sequence of semi-continuous and equi-coercive quadratic forms, the G-convergence and the convergence in the strong resolvent sense for the associated operators are equivalent. It also shows the link with respect to Γ-convergence in this general setting (G-convergence is equivalent with Γ-convergence in the strong and weak topology).

However, we will always use the notion of Γ-convergence in a particular case where our quadratic forms admit as domain a subspace compactly embedded into H (typically, the Sobolev space H^1 into L^2). In this setting the link between Γ-convergence and convergence in the strong resolvent sense is slightly simpler. Let us define our class of quadratic forms $\mathcal{Q}_\alpha(X, Y)$.

Definition 4. *Let $X \subseteq H$ be a subspace of H, which is a Hilbert space with scalar product $\langle ., . \rangle_X$. Assume also that the imbedding of X into H is compact. Given a constant α we denote $\mathcal{Q}_\alpha(X, H)$ the class of all lower semi-continuous quadratic forms $Q : H \to [0, +\infty]$ such that $D(Q) \subseteq X$ and $Q(x) \geq \alpha \|x\|_X^2$ for every $x \in D(Q)$.*

From Theorem 13.12. in [8] we deduce the following.

Theorem 5. *Let $\alpha > 0$ and $X \subseteq H$ be as in the Definition 4. Let Q_n be a sequence of quadratic forms in the class $\mathcal{Q}_\alpha(X, H)$ and let A_n be the associated operators. Then, the following statements are equivalent:*
(a) Q_n Γ-converges to Q in the strong topology of H;
(b) A_n converges to A in the the strong resolvent sense;
(c) A_n G-converges to A.

The following Lemma coming from the book of Dunford and Schwartz ([16] Lemma XI.9.5. page 1091) will be used in the sequel.

Lemma 6. [16] *Let T_n and T be compact operators, and let $T_n \to T$ in the operator norm of $\mathcal{L}(H)$. Let $\sigma_k(T)$ be an enumeration of the non-zero eigenvalues of T, each repeated according to its multiplicity. Then, there exist enumerations $\sigma_k(T_n)$ of the non-zero eigenvalues of T_n, with repetitions according to multiplicity, such that*

$$\lim_{n\to+\infty} \sigma_k(T_n) = \sigma_k(T), \quad k \geq 1,$$

the limit being uniform in k.

We still denote P_n and P the projection of H onto $D(Q_n)$ and $D(Q)$ respectively. Now we can prove the following useful result.

Proposition 7. *Assume that we are under the same assumptions as for Theorem 5. If one of the conditions (a), (b), (c) holds, then $A_n^{-1} P_n$ converges to $A^{-1} P$ in the operator norm of $\mathcal{L}(H)$. In addition, denoting $\sigma_{k,n}$ the eigenvalues of A_n labelled in increasing order and σ_k the ones of A, we have that $\sigma_{k,n}$ converges to σ_k for every $k \geq 0$.*

Proof. Since (a), (b) and (c) are equivalent we may assume that (c) it is true. Let us denote $R_n := A_n^{-1}P_n$ and $R := A^{-1}P$. We have that

$$\|R_n - R\|_{\mathcal{L}(H)} = \sup_{\|f\|_H \leq 1} \|R_n(f) - R(f)\|_H. \tag{1.1}$$

Suppose that $f_n \in H$ is such that $\|f_n\|_H \leq 1$ and

$$\|R_n - R\|_{\mathcal{L}(H)} \leq \|R_n(f_n) - R(f_n)\|_H + \frac{1}{n}, \tag{1.2}$$

(such a sequence $\{f_n\}$ exists by the definition of the supremum in (1.1)). We can assume up to a subsequence (not relabelled) that f_n weakly converges to f in H. By (1.2), to prove the convergence of R_n for the operator norm, it is enough to prove that $R_n(f_n)$ converges strongly in H to $R(f)$. Since (c) holds and $f \in H$ we already know that $R_n(f)$ strongly converges to $R(f)$. Now

$$\|R_n(f_n) - R(f_n)\|_H \leq \|R_n(f_n) - R_n(f)\|_H + \|R_n(f) - R(f)\|_H + \|R(f) - R(f_n)\|_H. \tag{1.3}$$

On the other hand, since the Q_n are equi-coercive, we deduce from the inequality

$$\alpha\|R_n(g)\|^2 \leq Q_n(R_n(g)) = \langle A_n R_n(g), R_n(g)\rangle_H = \langle P_n(g), R_n(g)\rangle_H \leq \|g\|_{H'}\|R_n(g)\|_H,$$

that for any $g \in H'$,

$$\|R_n(g)\| \leq \frac{1}{\alpha}\|g\|_{H'}.$$

Hence, the R_n are equi-continuous on H' and passing to the limit in (1.3) we get that

$$\|R_n(f_n) - R(f_n)\|_H \to 0.$$

Finally, observe that under our assumptions, the operators $A_n^{-1}P_n$ and $A^{-1}P$ are compact (because they are bounded respectively from $\overline{D(A_n)}$ to X and $\overline{D(A)}$ to X, and because the embedding of X into H is compact). Therefore, the convergence of the eigenvalues of A_n is a direct consequence of the convergence of R_n for the operator norm together with Lemma 6. □

1.2 A general definition of $H_0^1(E)$

In order to be able to deal with very general situations, we need to define the space $H_0^1(E)$ even in the case when E may not be an open set. As we shall see in the following, the good point of view to understand the space H_0^1 is to enlarge the class of open sets and deal with *quasi-open* sets that are defined below as in [7].

For any set $E \subset \mathbb{R}^N$ we define

$$\mathrm{cap}(E) := \inf\left\{\int_{\mathbb{R}^N} |\nabla v|^2; \quad v \in H^1(\mathbb{R}^N), v \geq 1 \text{ a.e. on a neighborhood of } E\right\}, \tag{1.4}$$

as being the capacity of E (a.e. denotes almost everywhere). Now, we define the class of *quasi-open sets.*

Definition 8. *The class of quasi-open sets $\mathcal{F}(\Omega)$, with $\Omega \subset \mathbb{R}^N$ a bounded open set, is defined as the class of all the subsets A of Ω such that for every $\varepsilon > 0$ one can find an open subset Ω_ε of Ω satisfying* $\mathrm{cap}(\Omega_\varepsilon \triangle A) < \varepsilon$.

It is well known that any open set is the level set of a continuous function. Analogously, an alternative way to define quasi-open sets is to say that they are level sets of quasi-continuous functions. Here is the definition.

Definition 9. *A function $u : \Omega \to \mathbb{R}$ is* ***quasi-continuous*** *if there is a non-increasing sequence of open sets $A_n \subset \Omega$ such that $\lim_{n\to+\infty} \text{cap}(A_n) = 0$ and the restriction of u to $\Omega\backslash A_n$ is continuous for any $n \geq 0$.*

It turns out (see for instance Theorem 3.3.29 of [19] or [33]) that any $u \in H^1(\Omega)$ has a representative $\tilde{u}$ which is quasi-continuous in Ω. The following result is quoted here for sake of completeness but will not be needed.

Theorem 10 ([4] Theorem 4.1.4.)**.** *Suppose that $A \subseteq \mathbb{R}^N$. Then, the following assertions are equivalent:*
i) A is quasi-open;
ii) A is the union of a finely open set and a set of zero capacity;
iii) $A = \{u > 0\}$ for some non-negative quasi-continuous function $u \in H^1(\mathbb{R}^N)$.

We should mention that in *ii*) a finely open set means an open set for the *fine topology*, which is defined as the coarsest topology making the super-harmonic functions continuous. Let us introduce some more definitions.

Definition 11. *We say that a property holds* ***quasi-everywhere*** *(and we denote* ***q.e.****) if it holds everywhere except on a set of zero capacity.*

For any open set $\Omega \subset \mathbb{R}^N$ there is a nice characterization of $H_0^1(\Omega)$ using capacity (see for instance [4] Theorem 4.1.2. or [33]), namely we have

$$u \in H_0^1(\Omega) \Leftrightarrow (u \in H^1(\mathbb{R}^N) \text{ and } u = 0 \text{ q.e. on } \mathbb{R}^N\backslash\Omega).$$

This suggest to define, when $E \subseteq \mathbb{R}^N$ is any measurable subset (not necessarily open),

$$H_0^1(E) := \{u \in H^1(\mathbb{R}^N) \text{ such that } u = 0 \text{ q.e. on } \mathbb{R}^N\backslash E\}. \tag{1.5}$$

Note that $H_0^1(E)$ is a closed subspace of $H^1(\mathbb{R}^N)$ and inherits its Hilbert structure. Consequently, the imbedding $H_0^1(E)$ into $L^2(E)$ remains compact. Notice also that according to our definition $H_0^1(E)$ is never empty, because it always contains the function identically equal to 0. It is also clear from the definition that $H_0^1(E) = \{0\}$ when $\text{cap}(E) = 0$.

The following Proposition shows that the case when A is quasi-open is actually the standard case.

Proposition 12. ([19] Proposition 3.3.44.) *For every subset $E \subset \mathbb{R}^N$, there exists a unique quasi-open set A, such that*

$$H_0^1(E) = H_0^1(A).$$

Observe in particular that $H_0^1(E) = \{0\}$ if and only if $\text{cap}(A(E)) = 0$, where $A(E)$ is the quasi-open set given by the above proposition associated with E. This also implies that any quasi-open set $A \subseteq \mathbb{R}^N$ such that $\text{cap}(A) > 0$ satisfies $\mathscr{L}^N(A) > 0$ (because in this case $H_0^1(E) \neq \{0\}$).

Now, in order to fix everything within the framework we are working on we consider the linear problem

$$(-\Delta + \lambda a)u = f, \qquad u \in H_0^1(A), \tag{1.6}$$

such that $A \in \mathcal{F}(\Omega)$ is a quasi-open set. The operator is defined through the dual form,

$$-\Delta + \lambda a : H_0^1(A) \to H^{-1}(A),$$

$f \in H^{-1}(A)$ and $a \in L^\infty(A)$. Thus, $u \in H_0^1(A)$ is a solution of (1.6) if

$$\int_A (\nabla u \nabla \varphi + \lambda a u \varphi) = \langle f, \varphi \rangle, \quad \forall \varphi \in H_0^1(A),$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $H^{-1}(A)$ and $H_0^1(A)$.

1.3 Stability for $H_0^1(\Omega)$ and nice potentials

In this section we introduce the class of *nice potentials* that will allow us to prove some convergence results associated to a very general class of potentials a.

Let $\Omega \subset \mathbb{R}^N$ be open. For any Borel function $a : \overline{\Omega} \to \mathbb{R}^+$ we denote

$$\Omega_0^a := \{x \in \overline{\Omega}\ ;\ a(x) = 0\} \quad \text{and} \quad \Omega_+^a := \{x \in \overline{\Omega}\ ;\ a(x) > 0\} = \overline{\Omega} \backslash \Omega_0^a.$$

Notice that for a general a the subdomains Ω_0^a and Ω_+^a might not be open nor closed. However, when a is continuous Ω_0^a is closed and Ω_+^a is open. Moreover, we will always assume a to be a Borel function so that Ω_0^a and Ω_+^a are Lebesgue measurable.

Let us now define the type of potentials that fulfill our framework.

Definition 13. *Let $\Omega \subset \mathbb{R}^N$ be an open set. A Borel function $a : \overline{\Omega} \to \mathbb{R}^+$ is said to be a nice potential, and we denote $a \in \mathcal{A}(\Omega)$, if the following two properties hold :*

$$\sup_{x \in \overline{\Omega}} a(x) < +\infty, \tag{1.7}$$

$$H_0^1(\Omega_0^a) = \{u \in H_0^1(\Omega); u = 0 \text{ a.e. on } \Omega_+^a\}. \tag{1.8}$$

Notice that our *Nice potentials* are not necessarily smooth, not even continuous. We shall see, for instance in Section 1.3.2, an example of nontrivial nice potential that vanishes on a dense set. We call them *Nice* because we can prove the desired convergence result for some eigenvalue problems when the potentials lie in this class. The only kind of regularity assumption is contained in (1.8) which can be understood as a stability-type property for H_0^1. More precisions are given in the remark and section below.

Remark 14. Observe that by our general definition of H_0^1 (given by (1.5)) we always have that

$$H_0^1(\Omega_0^a) = \{u \in H_0^1(\Omega); u = 0 \text{ q.e. on } \Omega_+^a\}. \tag{1.9}$$

This means that the assumption (1.8) is lying on some very thin regularity property of Ω_+^a, depending on the given potential a. This will be studied with more details below when we will introduce the notion of *stable domains*. It is worth mentioning that when Ω_+^a is open (and this is the case when

a is continuous), the term at the right hand side of (1.9) and (1.8) are equal because of the quasi-continuity of u. This is no more the case in general if Ω_+^a is not open, as it can be shown by the following example : $a := \chi_{\mathbb{R}^2\setminus D}$ where $D \subset \mathbb{R}^2$ is the open set defined by

$$D := B(0,1)\setminus([0,1] \times \{0\}).$$

It is clear in this case that one can find a smooth function u in $H^1(\mathbb{R}^2)$ that vanishes a.e. in $\mathbb{R}^2\setminus D$ but that is not vanishing q.e. on $[0,1] \times \mathbb{R}$ since

$$\mathscr{L}^2([0,1] \times \{0\}) = 0 \quad \text{and} \quad \text{cap}([0,1] \times \{0\}) > 0,$$

and, hence, u may not belong to $H_0^1(D)$.

Subsequently, from the above remark we deduce the following consequence.

Remark 15. A Borel function $a : \overline{\Omega} \to \mathbb{R}^+$ satisfying (1.7) and such that Ω_a^+ is open, is a nice potential.

A particular and important case is when a is continuous.

Proposition 16. *Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain. Then*

$$C^0(\overline{\Omega}) \subseteq \mathcal{A}(\Omega).$$

Proof. Since any function $a \in C^0(\overline{\Omega})$ is bounded we automatically get (1.7), and (1.8) follows from Remark 15. □

As the last proposition says our definition of nice potentials will allow us to prove the convergence of some linear eigenvalue problems, for instance, when a is any continuous function. On the other hand, due to our very general setting the limiting problem could be defined on a space $H_0^1(A)$ where A is not an open set (but quasi-open). However, we would like to go further and give some regularity assumptions on a and its level sets to guarantee that the limiting problem will be actually defined on the standard space $H_0^1(\Omega')$, with Ω' an open set. For this purpose we recall some stability properties of H_0^1. A good reference for those results could be for instance [19] Chapter 3, [4] Section 4.1 or [33].

1.3.1 Nice potentials and Stable domains

Let $\Omega \subseteq \mathbb{R}^N$ be open. In this section (and only here) we assume $a \in C^0(\overline{\Omega})$. In the sequel we give a criterion on a to insure that $H_0^1(\Omega_0^a) = H_0^1(\text{Int}\Omega_0^a)$. In other words, we want to find some conditions to put on an open set Ω that implies $u \in H_0^1(\Omega)$ as soon as $u \in H^1(\mathbb{R}^N)$ and $u = 0$ a.e. in $\mathbb{R}^N\setminus\overline{\Omega}$. This is the case when Ω is *stable*.

Recall that for any open set Ω it holds

$$H_0^1(\Omega) := \{u \in H^1(\mathbb{R}^N) \text{ and } u = 0 \text{ q.e. on } \mathbb{R}^N\setminus\Omega\}.$$

Now, if $u = 0$ q.e. only on $\mathbb{R}^N\setminus\overline{\Omega}$, we cannot say a priori that $u \in H_0^1(\Omega)$. However, domains that have this property, also called *stable*, have been studied a long time ago ([4, 17, 19, 21, 22]) and the following statement summarizes the known characterization of such domains (see for instance Theorem 3.4.6. in [19]).

Theorem 17. *Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain. Then, the following properties are equivalent.*
(i) For every $v \in H^1(\mathbb{R}^N)$, $v = 0$ q.e. on $\overline{\Omega}^c$ implies $v = 0$ q.e. on Ω^c;
(ii) For any open set A, $\mathrm{cap}(A \backslash \overline{\Omega}) = \mathrm{cap}(A \backslash \Omega)$);
(iii) For any $x \in \mathbb{R}^N$ and $r > 0$, $\mathrm{cap}(B(x,r) \backslash \overline{\Omega}) = \mathrm{cap}(B(x,r) \backslash \Omega)$);
(iv) $\liminf_{r \to 0} \dfrac{\mathrm{cap}(B(x,r) \backslash \overline{\Omega})}{\mathrm{cap}(B(x,r) \backslash \Omega)} > 0$ q.e. $x \in \partial \Omega$.

Definition 18. *If Ω satisfies one of the properties $(i) - (iv)$ of Theorem 17 we will say that Ω is stable.*

Observe that being stable imposes on the domain some sort of regularity on the boundary. For instance, a Lipschitz domain is stable. More generally, a domain satisfying a *Corkscrew condition* is stable. Recall that Ω satisfies a corkscrew condition if there exists r_0 and λ such that for every $x \in \partial \Omega$ and $r \leq r_0$ one can find $y \in B(x,r)$ such that $B(y, \lambda r) \subseteq \Omega^c$.

Remark 19. Let Ω' and Ω be two open sets such that $\Omega' \subseteq \Omega \subseteq \mathbb{R}^N$. Also let $u \in H_0^1(\Omega)$ and let $\tilde{u}$ be a quasi-continuous representative of u. Then, since $\Omega \backslash \overline{\Omega'}$ is open we have that

$$(\tilde{u} = 0 \text{ q.e. on } \Omega \backslash \overline{\Omega'}) \Leftrightarrow (\tilde{u} = 0 \text{ a.e. on } \Omega \backslash \overline{\Omega'}).$$

In addition, it is clear that for any $u \in H_0^1(\Omega)$ we have that $u \chi_\Omega \in H^1(\mathbb{R}^N)$.

In our applications we will need a localized version of Theorem 17. More precisely, we assume that $\Omega' \subseteq \Omega \subseteq \mathbb{R}^N$ and Ω' is stable in $\mathbb{R}^N$. Then, we want to say that $u \in H_0^1(\Omega')$ as soon as $u \in H_0^1(\Omega)$ and vanishes outside $\overline{\Omega'}$. From Remark 19 we can state the following consequence of Theorem 17.

Corollary 20. *Let Ω', Ω be two open sets such that $\Omega' \subseteq \Omega \subseteq \mathbb{R}^N$ and assume that Ω' is stable. Then,*

$$(u \in H_0^1(\Omega) \text{ and } u = 0 \text{ a.e. on } \Omega \backslash \overline{\Omega'}) \Rightarrow u \in H_0^1(\Omega').$$

As an application we can be more precise in the case when $\mathrm{Int}(\Omega_0^a)$ is stable.

Proposition 21. *Let $\Omega \subset \mathbb{R}^N$ be bounded and let $a \in C^0(\overline{\Omega})$. If $\mathrm{Int}(\Omega_0^a)$ is stable then*

$$H_0^1(\Omega_0^a) = H_0^1(\mathrm{Int}(\Omega_0^a)).$$

The advantage of Proposition 21 is to provide a sufficient condition to insure that $H_0^1(\Omega_0^a)$, which is a priori defined by the general identity (1.5), actually coincides with the classical Sobolev space defined on open domains.

1.3.2 A nice potential vanishing on a dense set

Now, to show how general our results are we want to give explicit examples of non trivial nice potentials which are not continuous and that vanishes respectively on a dense set and a Cantor set. According to [19] exercise 3.9., we construct a compact set $K \subset [0,1]$ of empty interior with positive Lebesgue measure by setting

$$E := \bigcup_{n=2}^{+\infty} \bigcup_{k=0}^{2^n} B(k 2^{-n}, 2^{-2n}) \cap]0,1[,$$

and

$$K := [0,1] \backslash E.$$

The following properties are satisfied.

i) E is open and $\overline{E} = [0,1]$;
ii) K is compact and $\text{Int}K = \emptyset$;
iii) $\mathscr{L}^1(K \cap B(x,\varepsilon)) > 0$ for all $x \in K$ and $\varepsilon > 0$;
iv) $H^1_0(E) = \{u \in H^1_0(]0,1[); u = 0 \text{ a.e on } K\}$;
v) E is not stable.

Actually i), ii) and v) are clear from the construction of E and iii) comes from the definition of E and a computation made by summing the length of all the intervals of the form $B(k2^{-n}, 2^{-2n})$ that lie in $B(x,\varepsilon)$ when $x \in K$ (the total sum is strictly less than 2ε). Thus, only iv) may require some explanations. We know by definition that $H^1_0(E) := \{u \in H^1_0(]0,1[); u = 0 \text{ q.e on } K\}$. Now, if $u \in H^1_0(]0,1[)$ is such that $u = 0$ a.e. on K, we claim that $u = 0$ everywhere on K. Indeed, since we are in dimension 1 we know that u admits a continuous representative and, hence, the set $O := \{u > 0\}$ is open. Assume by contradiction that $O \cap K$ is not empty and pick a point $x \in O \cap K$. Then, there exists $\varepsilon > 0$ such that $B(x,\varepsilon) \subset O$. Now, using iii) we have that $\mathscr{L}^1(K \cap B(x,\varepsilon)) > 0$, so that there exists a point $y \in K \cap B(x,\varepsilon)$ such that $u(y) = 0$ and this is not possible. After this, iv) is proved.

From the latter properties we deduce the following.

Proposition 22. *Under the above notations we have that χ_K and χ_E both belong to $\mathcal{A}(]0,1[)$.*

Proof. It is clear that both χ_K and χ_E are Borel measurable and satisfy (1.7). Then, from iv) we deduce that χ_K satisfies (1.8), and since $[0,1] \backslash K$ is open we deduce by Remark 15 that χ_E also satisfies (1.8). □

Remark 23. Notice that since $H^1_0(]0,1[) \subseteq C^0(]0,1[)$ we have that

$$H^1_0(K) := \{u \in H^1_0(]0,1[); u = 0 \text{ q.e on } E\} = \{0\}.$$

1.4 A classical semi-continuity result

In the sequel we will need the following lemma which is quite standard (see for instance [8], Proposition 1.18.).

Lemma 24. *Let H be a Hilbert space and let $F : H \to \overline{\mathbb{R}}$ be a convex function. Then, F is lower-semicontinuous on H in the strong topology if and only if F is lower semicontinuous on H in the weak topology.*

Remark 25. A consequence of Lemma 24 and the convexity of $u \mapsto (\int_\Omega |\nabla u|^2)^{\frac{1}{2}}$ is that for any $\Omega \subseteq \mathbb{R}^N$ and any $\{u_n\}$ that weakly converges to u in $H^1_0(\Omega)$,

$$\int_\Omega |\nabla u|^2 \leq \liminf_{n \to +\infty} \int_\Omega |\nabla u_n|^2.$$

Now, let A be any subset of Ω and consider the quadratic form $Q : L^2(\Omega) \mapsto \bar{\mathbb{R}}$ defined by

$$Q(u) := \begin{cases} \displaystyle\int_A |\nabla u|^2, & \text{if } u \in H_0^1(A), \\ +\infty, & \text{otherwise.} \end{cases} \tag{1.10}$$

Proposition 26. *The quadratic form Q defined by* (1.10) *is semicontinuous on $L^2(\Omega)$.*

Proof. We have to prove that for any sequence $\{u_n\}$ that strongly converges to u in $L^2(\Omega)$,

$$Q(u) \leq \liminf_{n \to +\infty} Q(u_n). \tag{1.11}$$

To prove (1.11) it is enough to consider the case when $\lim Q(u_n)$ exists and is finite. In particular, in the latter situation we have $Q(u_n) \leq M$ for n big enough, say $n \geq n_0$. By definition of Q this automatically implies that $u_n \in H_0^1(A)$ for $n \geq n_0$. Moreover, since the sequence is bounded in $H_0^1(A)$ and $H_0^1(A)$ is compactly imbedded in $L^2(A)$, we can extract a subsequence $\{u_{n_k}\}$ that converges weakly in $H_0^1(A)$ and strongly in $L^2(A)$. Note that this imbedding remains in our framework of *quasi-open sets*. Thus, by uniqueness of the limit in $L^2(\Omega)$ the weak limit of $\{u_{n_k}\}$ is equal to u. Hence, by Remark 25 we have that

$$\int_A |\nabla u|^2 \leq \liminf_{k \to +\infty} Q(u_{n_k}) = \lim_{n \to +\infty} Q(u_n) < +\infty, \tag{1.12}$$

Now since $u \in H_0^1(A)$ we have that $Q(u) = \int_A |\nabla u|^2$ and (1.12) says that (1.11) holds which proves the proposition. □

2 Semiclassical analysis for highly degenerate potentials

2.1 The Dirichlet case

In this section we use the Γ-convergence theory to study the asymptotic as $\lambda \to +\infty$ for the following linear eigenvalue problem

$$-\Delta u + \lambda a u = \sigma_k(\lambda) u, \qquad u \in H_0^1(\Omega), \tag{2.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain, the potential $a \in \mathcal{A}(\Omega)$ and $\sigma_k(\lambda)$ is the kth-eigenvalue of the operator $-\Delta + \lambda a$ under homogeneous Dirichlet boundary conditions. Under the additional assumption $\|u\|_{L^2(\Omega)} = 1$, u will be called a *normalized eigenfunction* associated with the eigenvalue $\sigma_k(\lambda)$. Observe that the solution u of problem (2.1) may not be unique for $k \neq 0$. We still denote Ω_0^a the subset of Ω defined by

$$\Omega_0^a := \{x \in \overline{\Omega} \; ; \; a(x) = 0\}.$$

Moreover, we introduce the following quadratic forms on the Hilbert space $L^2(\Omega)$:

$$Q_\lambda(u) = \begin{cases} \int_\Omega (|\nabla u(x)|^2 + \lambda a(x) u^2(x)) dx, & \text{if } u \in H_0^1(\Omega), \\ +\infty, & \text{otherwise,} \end{cases}$$

and

$$Q_\infty(u) = \begin{cases} \int_\Omega |\nabla u(x)|^2 dx, & \text{if } u \in H_0^1(\Omega_0^a), \\ +\infty, & \text{otherwise.} \end{cases}$$

We denote $\mathfrak{L}_\lambda := -\Delta + \lambda a$ the operator associated with Q_λ, and $-\Delta_{\Omega_0^a}$ the one associated with Q_∞. We begin with three remarks.

Remark 27. By the Sobolev inequality Q_λ and Q_∞ are all C_N-coercive on $H_0^1(\Omega)$ with same constant C depending only on N and $|\Omega|$. In addition, Q_λ and Q_∞ are semicontinuous on $L^2(\Omega)$ (by Proposition 26). In other words, Q_λ and Q_∞ belong to $\mathcal{Q}_C(H_0^1(\Omega), L^2(\Omega))$ (see Definition 4).

Remark 28. It is not difficult to see using an integration by parts that the operator $\mathfrak{L}_\lambda$ associated with Q_λ is exactly $-\Delta + \lambda a$ in the distributional sense. Moreover, from the regularity theory of elliptic PDE, we deduce that the weak solutions of Problem (2.1) are also strong solutions and the eigenvalues of $\mathfrak{L}_\lambda$ are solving Problem (2.1) in the strong sense.

Remark 29. Observe that $C_0^\infty(\Omega) \subseteq D(\mathfrak{L}_\lambda)$. This can be easily seen using an integration by parts to obtain $B_\lambda(\phi, u) = \langle f_\lambda, u\rangle_{L^2}$ for any $\phi \in C_0^\infty(\Omega)$ and $u \in H_0^1(\Omega)$, where $f_\lambda = -\Delta\phi + \lambda a\phi$ and B_λ is the bilinear forms associated to Q_λ. Now, since $C_0^\infty(\Omega)$ is dense in $L^2(\Omega)$, we deduce that $\overline{D(\mathfrak{L}_\lambda)} = L^2(\Omega)$. This means that for every $f \in L^2(\Omega)$, $P_\lambda(f) = f$ (where P_λ is the projection onto $\overline{D(\mathfrak{L}_\lambda)}$).

The main result of this section is the following.

Theorem 30. *Assume that $a \in \mathcal{A}(\Omega)$. Then, $\mathfrak{L}_\lambda$ converges to $-\Delta_{\Omega_0^a}$ in the strong resolvent sense when $\lambda \to +\infty$. As a consequence*

$$\lim_{\lambda\to+\infty} \sigma_k(\lambda) = \sigma_k,$$

where σ_k are the (ordered) eigenvalues of the Dirichlet Laplacian $-\Delta_{\Omega_0^a}$. In addition, any sequence of normalized solution u_λ of Problem (2.1) admits a subsequence that converges strongly in $H_0^1(\Omega)$ to a normalized eigenfunction $u \in H_0^1(\Omega_0^a)$ of $-\Delta_{\Omega_0^a}$ associated to its kth-eigenvalue. If in addition, σ_k is simple, this convergence holds for the whole sequence $\{u_\lambda\}$.

Remark 31. By convention we fix $\sigma_k = +\infty$, for all $k \geq 0$, when $H_0^1(\Omega_0^a) = \{0\}$. Note that this convention is coherent with the lower bound based on the Faber-Krahn inequality in terms of $|\Omega|$ when $|\Omega|$ tends to 0 (see [2] and references therein for further details). With this convention, Theorem 30 says in particular that $\sigma_k(\lambda) \to +\infty$ if and only if $H_0^1(\Omega_0^a) = \{0\}$, and this happens if and only if $\text{cap}(A(\Omega_0^a)) = 0$, where $A(\Omega_0^a)$ is the quasi-open set associated with Ω_0^a and given by Proposition 12.

As we shall see later, Theorem 30 will be seen as a consequence of the following Γ-convergence result.

Theorem 32. *Assume that a is a nice potential. Then, Q_λ Γ-converges to Q_∞ when $\lambda \to +\infty$.*

Proof. We begin with the proof of *ii)* which is very simple. Indeed, for any $v \in L^2(\Omega)$ let us consider the identically constant sequence $\{v_\lambda\}$, with $v_\lambda = v$ for every λ, that obviously converges strongly to v in $L^2(\Omega)$. We have to prove that

$$Q_\infty(v) \geq \limsup_{\lambda\to+\infty} Q_\lambda(v_\lambda). \tag{2.2}$$

If v is not in $H_0^1(\Omega_0^a)$ then, by definition, we have that $Q_\infty(v) = +\infty$ and (2.2) trivially holds. Now, if $v \in H_0^1(\Omega_0^a)$, by definition of Ω_0^a and $H_0^1(\Omega_0^a)$ we find that $a(x)v(x) = 0$ a.e. in Ω, so that $Q_\infty(v) = Q_\lambda(v)$ and (2.2) is again satisfied, which proves condition *ii)* in full generality.

Let us now prove i). To this aim we consider a sequence $\{v_\lambda\}$ that converges strongly to v in $L^2(\Omega)$ and we have to prove that

$$Q_\infty(v) \leq \liminf_{\lambda\to+\infty} Q_\lambda(v_\lambda). \tag{2.3}$$

We may assume that

$$\liminf_{\lambda\to+\infty} Q_\lambda(v_\lambda) < +\infty, \tag{2.4}$$

otherwise (2.3) is trivial. In the latter situation we claim that $v \in H_0^1(\Omega_0^a)$. Indeed, (2.4) implies in particular that

$$\liminf_{\lambda\to+\infty} \lambda \int_\Omega a v_\lambda^2 dx < +\infty. \tag{2.5}$$

Since $\{v_\lambda\}$ converges strongly in $L^2(\Omega)$ to v, (2.5) actually implies that

$$\int_\Omega a v^2 dx = 0,$$

from which we easily deduce that $v = 0$ a.e. in Ω_+^a. Now, since a is a nice potential, from (1.8) we conclude that $v \in H_0^1(\Omega_0^a)$, and this allows us to write, using also Proposition 26,

$$Q_\infty(v) \;=\; \int_{\Omega_0^a} |\nabla v|^2 dx = \int_\Omega |\nabla v|^2 dx \leq \liminf_{\lambda\to+\infty} \int_\Omega |\nabla v_\lambda|^2 dx \leq \liminf_{\lambda\to+\infty} Q_\lambda(v_\lambda)$$

and the proof is complete. □

Theorem 32 already implies that $\mathfrak{L}_\lambda$ converges to $-\Delta_{\Omega_0^a}$ in the strong resolvent sense and the eigenvalues $\sigma_k(\lambda)$ converges to the eigenvalues σ_k respectively, as $\lambda \to +\infty$. Now to prove the strong convergence of the eigenfunctions in $H_0^1(\Omega)$ we will need some further lemmas, that will also be used later in the analysis of the system. We begin with the following result that can be seen as a modification of [1] Theorem 3.1.

Lemma 33. *Assume that $a \in \mathcal{A}(\Omega)$ and let $f \in L^2(\Omega)$. For any $\lambda > 0$ we denote v_λ the unique solution for the problem*

$$(-\Delta + \lambda a)v_\lambda = f, \quad v_\lambda \in H_0^1(\Omega),$$

and we also denote

$$\alpha(\lambda) := \int_\Omega |\nabla v_\lambda|^2 + \lambda \int_\Omega a v_\lambda^2 - 2\int_\Omega f v_\lambda. \tag{2.6}$$

Then, $\alpha \in C^1(]0, +\infty[)$,

$$\alpha'(\lambda) = \int_\Omega a v_\lambda^2,$$

and

$$\left(\sup_{\lambda>0} \alpha(\lambda) < +\infty\right) \;\Rightarrow\; \left(\liminf_{\lambda\to+\infty} \lambda \int_\Omega a v_\lambda^2 = 0\right).$$

Proof. It is well known (see e.g. [8] Proposition 12.12.) that

$$\alpha(\lambda) := \min_{v\in H_0^1(\Omega)} \left(\int_\Omega |\nabla v|^2 + \lambda \int_\Omega a v^2 - 2\int_\Omega f v\right).$$

Then, for any $\lambda, \lambda' > 0$ such that $\lambda < \lambda'$ we have

$$\begin{aligned}\alpha(\lambda) &\leq \int_\Omega |\nabla v_{\lambda'}|^2 + \lambda \int_\Omega a v_{\lambda'}^2 - 2\int_\Omega f v_{\lambda'} \\ &= \alpha(\lambda') + (\lambda - \lambda')\int_\Omega a v_{\lambda'}^2 < \alpha(\lambda'),\end{aligned}$$

for every $v'_\lambda \in H_0^1(\Omega)$. Hence, $\lambda \to \alpha(\lambda)$ is increasing and by continuity of $\lambda \to v_\lambda$ we deduce that

$$\alpha'(\lambda) = \lim_{\lambda' \to \lambda} \frac{|\alpha(\lambda) - \alpha(\lambda')|}{|\lambda - \lambda'|} = \int_\Omega a v_\lambda^2$$

and $\alpha'(\lambda)$ is continuous so that $\alpha \in C^1$. Now suppose that

$$\sup_{\lambda > 0} \alpha(\lambda) < +\infty, \tag{2.7}$$

and assume by contradiction that there exists $\lambda_0, \varepsilon > 0$ such that

$$\lambda \int_\Omega a v_\lambda^2 = \lambda \alpha'(\lambda) > \varepsilon > 0 \qquad \text{for } \lambda > \lambda_0.$$

Thus, integrating the last inequality between λ_0 and λ we deduce that

$$\alpha(\lambda) \geq \alpha(\lambda_0) + \varepsilon \ln\left(\frac{\lambda}{\lambda_0}\right)$$

which contradicts (2.7) and completes the proof of the Lemma. □

Further, we will need the following result which is a consequence of Γ-convergence and Lemma 33.

Lemma 34. *Assume that $a \in \mathcal{A}(\Omega)$ is such that $H_0^1(\Omega_0^a) \neq \{0\}$ and let $f \in L^2(\Omega)$. For $\lambda > 0$ we denote v_λ the solution of*

$$(-\Delta + \lambda a)v_\lambda = f, \quad v_\lambda \in H_0^1(\Omega), \tag{2.8}$$

and v the solution of

$$-\Delta v = f, \quad v \in H_0^1(\Omega_0^a). \tag{2.9}$$

Then, there exists a subsequence λ_k such that $\{v_{\lambda_k}\}$ converges strongly to v in $H_0^1(\Omega)$ when λ_k goes to $+\infty$.

Remark 35. The existence of the solutions for the problem (2.9) is guaranteed and understood in the sense of (1.6). Observe in particular that $f \in L^2(\Omega)$ can be considered as belonging to $H_0^1(\Omega_0^a)'$ via the bounded linear form on $H_0^1(\Omega_0^a)$ given by $u \mapsto \int_\Omega fu$.

Proof of Lemma 34. Applying Theorem 32, Theorem 5 and Proposition 7, we already know that the resolvent operator $R_\lambda := \mathfrak{L}_\lambda^{-1} P_\lambda$ converge to the resolvent $R := -\Delta_{\Omega_0^a}^{-1} P$ for the topology of operator norm in $\mathcal{L}(L^2)$. Thus, we deduce that $\{v_\lambda\}$ converges to v strongly in $L^2(\Omega)$, where v is the solution of (2.9). Hence, we are left with the proof of the strong convergence in $H_0^1(\Omega)$.

Multiplying (2.8) by v_λ and integrating by parts in Ω we get

$$\int_\Omega |\nabla v_\lambda|^2 + \int_\Omega \lambda a v_\lambda^2 = \int_\Omega f v_\lambda.$$

Moreover, since v is a solution of the problem (2.9), by our definition we also have

$$\int_\Omega |\nabla v|^2 = \langle f, v \rangle = \int_\Omega f v,$$

where $\langle .,. \rangle$ is the duality pairing between $H_0^1(\Omega_0^a)$ and its dual. On the other hand, since $\{v_\lambda\}$ is converging in $L^2(\Omega)$ we know that $\int_\Omega f v_\lambda$ converges to $\int_\Omega f v$. The latter implies that

$$\int_\Omega |\nabla v_\lambda|^2 + \int_\Omega \lambda a v_\lambda^2 = \int_\Omega f v_\lambda \to \int_\Omega f v = \int_\Omega |\nabla v|^2. \tag{2.10}$$

This means in particular that $\alpha(\lambda)$ (defined by (2.6)) is bounded and by Lemma 33 we deduce that there exists a subsequence such that $\lambda_k \int_\Omega a v_{\lambda_k}^2$ tends to 0, as $\lambda_k \to +\infty$. Therefore, (2.10) shows that

$$\int_\Omega |\nabla v_{\lambda_k}|^2 \to \int_\Omega |\nabla v|^2.$$

Extracting a further subsequence (not relabelled) me may assume that $\{v_{\lambda_k}\}$ converges weakly to v. Consequently, by the weak convergence together with the convergence of norms we obtain

$$\|v_{\lambda_k} - v\|_{H_0^1}^2 = \|v_{\lambda_k}\|_{H_0^1}^2 + \|v\|_{H_0^1}^2 - 2\langle v_{\lambda_k}, v\rangle_{H_0^1} \to 0,$$

which proves that $\{v_{\lambda_k}\}$ converges strongly to v in $H_0^1(\Omega)$. □

Remark 36. For any $f \in L^2(\Omega)$ and any $\lambda > 0$, if v_λ is the solution of (2.8) by equi-coercivity of $-\Delta + \lambda a$ on $H_0^1(\Omega)$ one can easily obtain that

$$\|v_\lambda\|_{H_0^1(\Omega)} \leq C\|f\|_{L^2(\Omega)},$$

where C does not depend on λ. Therefore, the following inequality holds

$$\|\mathfrak{L}_\lambda^{-1}(f)\|_{H_0^1(\Omega)} \leq C\|f\|_{L^2(\Omega)},$$

with C independent of λ.

Now, we prove that when f_λ converges to f we still obtain the convergence of the corresponding solutions.

Lemma 37. *Assume that $a \in \mathcal{A}(\Omega)$ and let $f_\lambda \in L^2(\Omega)$ be a sequence of functions that converges strongly to $f \in L^2(\Omega)$. Let v_λ be the solution of*

$$(-\Delta + \lambda a)v_\lambda = f_\lambda, \quad v_\lambda \in H_0^1(\Omega),$$

and v the one of

$$-\Delta v = f, \quad v \in H_0^1(\Omega_0^a).$$

Then, there exists a subsequence λ_k such that $\{v_{\lambda_k}\}$ converges strongly to v in $H_0^1(\Omega_0^a)$.

Proof. Let us write

$$\|v_\lambda - v\|_{H_0^1(\Omega)} = \|\mathfrak{L}_\lambda^{-1}(f_\lambda) - \mathfrak{L}_\lambda^{-1} P_\lambda(f)\|_{H_0^1(\Omega)} + \|\mathfrak{L}_\lambda^{-1}(f) - (-\Delta_{\Omega_0^a})^{-1} P(f)\|_{H_0^1(\Omega)}.$$

Then, by Lemma 34 we know that up to a subsequence

$$\|\mathfrak{L}_\lambda^{-1}(f) - (-\Delta_{\Omega_0^a})^{-1} P(f)\|_{H_0^1(\Omega)} \to 0.$$

Furthermore, by Remark 36 we have that

$$\|\mathfrak{L}_\lambda^{-1}(f_\lambda) - \mathfrak{L}_\lambda^{-1}(f)\| \leq C\|f_\lambda - f\|_{L^2} \to 0.$$

Therefore, the Lemma is proved. □

We are now ready to prove Theorem 30.

Proof of Theorem 30. As was said before, gathering together Theorem 32, Theorem 5 and Proposition 7 we directly get that $\mathfrak{L}_\lambda$ converges to $-\Delta_{\Omega_0^a}$ in the strong resolvent sense and the eigenvalues $\sigma_k(\lambda)$ converges to the eigenvalues σ_k respectively, as $\lambda \to +\infty$. This includes the particular case when $H_0^1(\Omega_0^a) = \{0\}$ and if this occurs, $\sigma_k(\lambda) \to +\infty$ for every $k \geq 0$. So it remains only to prove the strong convergence of the normalized eigenfunctions to the normalized eigenfunction in $H_0^1(\Omega)$.

Then, let $\{u_\lambda\} \in D(\mathfrak{L}_\lambda)$ be a sequence of eigenfunctions for $\mathfrak{L}_\lambda$ associated to $\sigma_k(\lambda)$ and such that $\|u_\lambda\|_2 = 1$. Multiplying the equation in (2.1) by u_λ and integrating by parts in Ω, we easily deduce that $\{u_\lambda\}$ is uniformly bounded in $H_0^1(\Omega)$, provided that $H_0^1(\Omega_0^a) \neq \{0\}$. Hence, we can extract a subsequence such that $\{u_{\lambda_k}\}$ converges to some v weakly in $H_0^1(\Omega)$ and strongly in $L^2(\Omega)$. By applying Lemma 37 with $f_\lambda := \sigma_k(\lambda)u_\lambda$ we deduce that v is an eigenfunction for $-\Delta_{\Omega_0^a}$ associated to σ_k, and that up to take a further subsequence the convergence of $\{u_{\lambda_k}\}$ holds strongly in $H_0^1(\Omega)$. Then, if σ_k is simple, since $\|v\|_2 = \lim_k \|u_{\lambda_k}\|_2 = 1$, we have that v is the unique normalized eigenfunction associated to the eigenvalue σ_k. In other words, in the latter case v is the unique point in the adherence of $\{u_\lambda\}$ and this proves actually that the whole sequence $\{u_\lambda\}$ converges to v strongly in $H_0^1(\Omega)$. □

2.2 The Neumann case

Here we observe that in the proofs of the above section if we replace $H_0^1(\Omega)$ by $H^1(\Omega)$ we get the same result, provided a slight modification of the definition of nice potentials. For any $a : \overline{\Omega} \to \mathbb{R}^+$ we still denote

$$\Omega_0^a := \{x \in \overline{\Omega}\ ;\ a(x) = 0\} \quad \text{and} \quad \Omega_+^a := \{x \in \overline{\Omega}\ ;\ a(x) > 0\}.$$

Definition 38. *Let $\Omega \subset \mathbb{R}^N$ be an open set. A Borel function $a : \overline{\Omega} \to \mathbb{R}^+$ is said to be a nice potential for Neumann and we denote $a \in \mathcal{A}_N(\Omega)$, if the following two properties hold.*

$$\sup_{x \in \overline{\Omega}} a(x) < +\infty, \tag{2.11}$$

$$H_0^1(\Omega_0^a) = \{u \in H^1(\Omega); u = 0 \ a.e.\ on\ \Omega_+^a\}. \tag{2.12}$$

We have a similar result as Corollary 20, provided this time that Ω' is compactly contained in Ω. The proof uses a similar argument as for Corollary 20 and has been omitted here.

Corollary 39. *Let Ω', Ω be two bounded open sets such that $\overline{\Omega'} \subseteq \Omega \subseteq \mathbb{R}^N$ and assume that Ω' is stable. Then,*

$$(u \in H^1(\Omega) \text{ and } u = 0 \ a.e.\ on\ \Omega \backslash \overline{\Omega'}) \Rightarrow u \in H_0^1(\Omega').$$

We deduce in particular the following interesting case.

Proposition 40. *Let $\Omega \subset \mathbb{R}^N$ be an bounded open set and $a \in C^0(\overline{\Omega})$ be such that $\overline{\Omega_0^a} \subset \Omega$. Then, $a \in \mathcal{A}_N(\Omega)$.*

We are now ready to study the asymptotic as $\lambda \to +\infty$ for the following Neumann eigenvalue problem

$$\begin{cases} -\Delta u + \lambda a u = \mu_k(\lambda) u, & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0, & \text{on } \partial\Omega, \end{cases} \tag{2.13}$$

where as before, $\Omega \subset \mathbb{R}^N$ is a bounded domain and the potential $a \in \mathcal{A}_N(\Omega)$. Since no regularity is assumed on Ω, the solutions for problem (2.13) have to be understood only as weak solutions, namely

$$\int_\Omega (\nabla u \cdot \nabla \varphi + \lambda a u \varphi) dx = \mu_k(\lambda) \int_\Omega u \varphi dx, \qquad \forall \varphi \in H^1(\Omega).$$

Now we consider the following quadratic form on $L^2(\Omega)$

$$\tilde{Q}_\lambda(u) = \begin{cases} \displaystyle\int_\Omega \big(|\nabla u(x)|^2 + \lambda a(x) u(x)^2\big) dx, & \text{if } u \in H^1(\Omega), \\ +\infty, & \text{otherwise,} \end{cases}$$

and Q_∞ still denotes

$$Q_\infty(u) = \begin{cases} \displaystyle\int_{\Omega_0^a} |\nabla u(x)|^2 dx, & \text{if } u \in H_0^1(\Omega_0^a), \\ +\infty, & \text{otherwise.} \end{cases}$$

The domain of $\tilde{Q}_\lambda$ is now $D(\tilde{Q}_\lambda) = H^1(\Omega)$. Moreover, they remain equi-coercive on $H^1(\Omega)$ and semicontinuous on $L^2(\Omega)$. As before we know by standard arguments that the operator $\mathfrak{L}_\lambda^N$ associated with $\tilde{Q}_\lambda$ with domain $D(\mathfrak{L}_\lambda^N) \subseteq H^1(\Omega)$ coincides with the operator $-\Delta + \lambda a$ in the distributional sense and its eigenfunctions are weak solutions for Problem (2.13). The corresponding eigenvalues $\mu_k(\lambda)$ are called the Neumann eigenvalues of $-\Delta + \lambda a$ in Ω.

Analogously, as Theorem 30 we have the following result regarding to the Neumann eigenvalues.

Theorem 41. *Assume that $a \in \mathcal{A}_N(\Omega)$. Then, $\mathfrak{L}_\lambda^N$ converges to $-\Delta_{\Omega_0^a}$ in the strong resolvent sense when $\lambda \to +\infty$. As a consequence*

$$\lim_{\lambda \to +\infty} \mu_k(\lambda) = \sigma_k,$$

where σ_k are the (ordered) eigenvalues of the Dirichlet Laplacian $-\Delta_{\Omega_0^a}$. In addition, any sequence of normalized solution u_λ of Problem (2.1) *admits a subsequence that converges strongly in $H^1(\Omega)$ to a normalized eigenfunction $u \in H_0^1(\Omega_0^a)$ of $-\Delta_{\Omega_0^a}$ associated to its kth-eigenvalue. If in addition σ_k is simple this convergence holds for the whole sequence $\{u_\lambda\}$.*

Proof. The proof is the same as for Theorem 30, using this time the property (2.12) instead of (1.8) of nice potentials in the proof of the Γ-convergence of $\tilde{Q}_\lambda$ to Q. □

Remark 42. In the same spirit as Theorem 41, in the general case when $\Omega_0^a \subseteq \Omega$, one could imagine any boundary conditions on $\partial\Omega$ and shall prove that, under suitable stability assumptions on Ω_0^a and good definition of nice potentials, the solution u_λ converges to the solution u with a Dirichlet condition on $\partial\Omega_0^a \backslash \partial\Omega$ and the original boundary conditions given on $\partial\Omega \cap \partial\Omega_0^a$.

3 A cooperative system

In this section we ascertain the limiting behaviour as $\lambda \uparrow \infty$ of the eigenvalues and its associated eigenfunctions of the linear eigenvalue problem

$$\begin{cases} (-\Delta + \lambda a)u - bv = \tau_k(\lambda) u, \\ (-\Delta + \lambda d)v - cu = \tau_k(\lambda) v, \end{cases} ; \quad (u, v) \in H_0^1(\Omega) \times H_0^1(\Omega), \tag{3.1}$$

where Ω is a bounded domain of $\mathbb{R}^N$. We assume that a, d are nice potentials (as in Definition 13). We also suppose that $b, c \in C^0(\overline{\Omega})$ and (3.1) is *strongly cooperative* in the sense that

$$b(x) > 0 \quad \text{and} \quad c(x) > 0, \quad \text{for all } x \in \bar{\Omega}.$$

We still denote

$$\Omega_0^a := \{x \in \overline{\Omega}; a(x) = 0\}, \qquad \Omega_0^d := \{x \in \overline{\Omega}; d(x) = 0\},$$

and define

$$\Omega_0 := \Omega_0^{a+d} = \Omega_0^a \cap \Omega_0^d = \{x \in \overline{\Omega}; a(x) = d(x) = 0\}.$$

Next, we denote by

$$\mathfrak{L}(V_1, V_2) := \begin{pmatrix} -\Delta + V_1 & -b \\ -c & -\Delta + V_2 \end{pmatrix}, \qquad V_1, V_2 \in L^\infty(\bar{\Omega}), \tag{3.2}$$

the differential operator involved in the linear eigenvalue problem (3.1) (with $V_1 = \lambda a$ and $V_2 = \lambda d$), which is strongly cooperative as discussed by Figueiredo & Mitidieri [12], Sweers [32], López-Gómez & Molina-Meyer [23]. The following result, which we state without proof, provides us with the existence of solutions for the systems (3.1).

Proposition 43. *For any potential $V_1, V_2 \in L^\infty(\Omega, \mathbb{R}^+)$, the operator $\mathfrak{L}(V_1, V_2)$ admits a discrete set of eigenvalues that tend to $+\infty$ and there exists at least a solution $(u, v) \in H_0^1(\Omega) \times H_0^1(\Omega)$.*

Remark 44. To prove Proposition 43, one firstly have that for a sufficiently big $\alpha > 0$ the resolvent of the operator $\mathfrak{L}(V_1, V_2) + \alpha \mathrm{Id}$ is a compact positive linear operator in $H_0^1(\Omega) \times H_0^1(\Omega)$ [32]. Then, owing to [3, Theorem VI.8] the spectrum might contain either infinitely many isolated eigenvalues or a finite number of isolated eigenvalues. Note that when the operator is self-adjoint the method shown in [20] can be used to prove that there are infinitely many eigenvalues.

Remark 45. Furthermore, due to the proof of Proposition 43 for every λ the system (3.1) admits at least a solution $(u, v) \in H_0^1(\Omega) \times H_0^1(\Omega)$.

Remark 46. It should be pointed out that the eigenvalues might be complex apart from the first one, which might be also positive. Then, according to a version for systems of a very classical result establishing the existence and dominance of the first eigenvalue (see [1, Theorem 2.1] and the references therein), for any other eigenvalue τ_k of the operator $\mathfrak{L}(V_1, V_2)$ we find that $Re\ \tau_k > \tau_0$, for any $k \geq 1$.

We also consider the limiting system

$$\begin{cases} -\Delta u - bv = \tau_k u, \\ -\Delta v - cu = \tau_k v, \end{cases} \quad ; \quad (u, v) \in H_0^1(\Omega_0^a) \times H_0^1(\Omega_0^d) \tag{3.3}$$

We say that (u, v) is a solution for the system (3.3) when each equation of the system is satisfied in the sense of (1.6).

Remark 47. Observe that from Proposition 43 we also know that the spectrum for the problem (3.3) is a discrete set of eigenvalues that tends to $+\infty$ and the resolvent of the operator $\mathfrak{L}_\infty + \alpha \mathrm{Id}$ is compact in $H_0^1(\Omega_0^a) \times H_0^1(\Omega_0^d)$, for a sufficiently big $\alpha > 0$.

We assume that $\tau_k(\lambda)$ and τ_k are labelled in an increasing order with respect to real part. The aim of this section is to prove that $\tau_k(\lambda)$ converges to τ_k (see Theorem 51 and Theorem 56). Hereafter, we will distinguish two different situations when the system is of variational type or not. The first case is in particular contained in the second one, which is more general, but we found it interesting in order to give an alternative proof of this particular situation where the limiting problem can be solved by a direct method of Γ-convergence (Section 3.1). In the general case (Section 3.2) we use a different argument, considering the system as two separated equations and using the Γ-convergence results obtained in Section 2 for one single equation to conclude. As mentioned before, the latter gives in particular a second proof of the result of Section 3.1.

In all the sequel, we deal only with the case of homogeneous Dirichlet boundary problems but as in Section 2, one can also consider some Neumann or mixed boundary condition without substantial changes in the proofs.

3.1 Variational cooperative system

In this section we assume that

$$b(x) = c(x) =: \gamma(x), \quad \forall x \in \overline{\Omega}.$$

Furthermore, in order to have coerciveness we denote

$$\alpha := \max_{x\in\overline{\Omega}} \gamma(x) \tag{3.4}$$

and we consider the following elliptic system

$$\begin{cases} (-\Delta + \lambda a + \alpha)u - \gamma v = \sigma_k^\alpha(\lambda)u, \\ (-\Delta + \lambda d + \alpha)v - \gamma u = \sigma_k^\alpha(\lambda)v, \end{cases} \quad ; \quad (u,v) \in H_0^1(\Omega)\times H_0^1(\Omega), \tag{3.5}$$

where

$$\mathcal{S}_\lambda + \alpha \mathrm{Id} = \begin{pmatrix} -\Delta + \lambda a & -\gamma \\ -\gamma & -\Delta + \lambda d \end{pmatrix} + \alpha \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{3.6}$$

and $\sigma_k^\alpha(\lambda)$ denotes the eigenvalue for the operator $\mathcal{S}_\lambda + \alpha\mathrm{Id}$ under homogeneous Dirichlet boundary conditions which we know is well defined.

Remark 48. In order to prove the convergence of eigenvalues for the operator $\mathcal{S}_\lambda$ (defined as (3.2) with $b = c$ in $\overline{\Omega}$), it is enough to prove the convergence of the eigenvalues for $\mathcal{S}_\lambda + \alpha\mathrm{Id}$. Indeed, it holds $\sigma_k^\alpha = \sigma_k + \alpha$.

Multiplying (3.5) by a test function $(\varphi,\psi) \in C_0^\infty(\Omega)\times C_0^\infty(\Omega)$, integrating in Ω and applying the formula of integration by parts yields

$$\int_\Omega \nabla u\cdot\nabla\varphi + \int_\Omega \nabla v\cdot\nabla\psi + \lambda\left[\int_\Omega au\varphi + \int_\Omega dv\psi\right] + \alpha\left[\int_\Omega u\varphi + \int_\Omega v\psi\right] \\ - \int_\Omega \gamma v\varphi - \int_\Omega \gamma u\psi - \sigma_k^\alpha(\lambda)\left[\int_\Omega u\varphi + \int_\Omega v\psi\right] = 0.$$

This suggests to introduce

$$\mathcal{I}_\lambda(u,v) := \int_\Omega \left(|\nabla u|^2 + |\nabla v|^2 + \lambda(au^2 + dv^2)\right) - 2\int_\Omega \gamma uv + \alpha\int_\Omega (u^2+v^2),$$

and define Q_λ the quadratic form on the Hilbert space $L^2(\Omega) \times L^2(\Omega)$ by

$$Q_\lambda(u, v) = \begin{cases} \mathcal{I}_\lambda(u, v), & \text{if } u, v \in H_0^1(\Omega), \\ +\infty, & \text{otherwise .} \end{cases}$$

We begin by showing that the solutions (u, v) for the system (3.5) are eigenfunctions for the operator $\mathcal{S}_\lambda + \alpha \mathrm{Id}$ defined by (3.6) and associated with Q_λ. We denote $\mathcal{S}_\infty := \mathcal{S}(0, 0)$.

Lemma 49. *Q_λ is C-coercive on $H_0^1(\Omega) \times H_0^1(\Omega)$, with constant C depending on Ω (in particular does not depend on λ). Also, the operator associated with Q_λ with domain contained in $H_0^1(\Omega) \times H_0^1(\Omega)$ is equal to $\mathcal{S}_\lambda + \alpha \mathrm{Id}$ (defined in (3.6)) in the distributional sense. In particular the eigenfunctions of $\mathcal{S}_\lambda + \alpha \mathrm{Id}$ are solutions for the system (3.5).*

Proof. The Hilbert space $H := H_0^1(\Omega) \times H_0^1(\Omega)$ is endowed with the scalar product

$$\langle (u, v), (f, g) \rangle_H := \langle u, f \rangle_{H_0^1(\Omega)} + \langle v, g \rangle_{H_0^1(\Omega)}.$$

Therefore, it is easy to see using Poincaré-Sobolev inequality that Q_λ is C-coercive for a positive constant C by the definition of α (see (3.4)).

Let $\mathcal{A}$ be the operator associated to Q_λ and B_λ its corresponding bilinear form given, for any $\hat{u} := (u_1, u_2)$, $\hat{v} := (v_1, v_2)$ in $D(Q_\lambda)$, by the formula

$$B_\lambda(\hat{u}, \hat{v}) := \langle \nabla \hat{u}, \nabla \hat{v} \rangle_{L^2 \times L^2} + \langle (\lambda a + \alpha - \sigma_k^\alpha(\lambda) - 2\gamma) \hat{u}, \hat{v} \rangle_{L^2 \times L^2}.$$

We already know by definition that $D(\mathcal{A}) \subseteq D(Q_\lambda) = H_0^1(\Omega) \times H_0^1(\Omega)$. On the other hand, for every $\hat{u}, \hat{v} \in H_0^1(\Omega) \times H_0^1(\Omega)$, an integration by parts in the distributional sense yields

$$\langle (\mathcal{S}_\lambda + \alpha \mathrm{Id})(\hat{u}), \hat{v} \rangle = B_\lambda(\hat{u}, \hat{v}), \tag{3.7}$$

which in particular for $\hat{u} \in D(\mathcal{A})$ allows us to identify $\mathcal{S}_\lambda + \alpha \mathrm{Id}$ with $\mathcal{A}$ as claimed in the statement of the Lemma. Finally, from classical regularity result for elliptic PDE we deduce that the eigenfunctions of $\mathcal{A}$ are strong solutions for the system (3.5). □

Remark 50. Observe that in the case if $b \neq c$, equality (3.7) is false, so that the present method does not extend to this situation.

Let us now introduce the quadratic form on $L^2(\Omega) \times L^2(\Omega)$,

$$Q_\infty(u, v) := \begin{cases} \displaystyle\int_\Omega \left(|\nabla u|^2 + |\nabla v|^2 \right) - 2 \int_\Omega \gamma u v + \alpha \int_\Omega (u^2 + v^2), & \text{if } (u, v) \in H_0^1(\Omega_0^a) \times H_0^1(\Omega_0^d), \\ +\infty, & \text{otherwise .} \end{cases}$$

Note that Q_λ and Q_∞ are equi-coercive on $H_0^1(\Omega)^2$ and thus belong to $\mathcal{Q}_C(H_0^1(\Omega)^2, L^2(\Omega)^2)$ for some constant C depending on Ω. Denoting σ_k the eigenvalues of $\mathcal{S}_\infty$ such that $\mathcal{S}_\infty + \alpha \mathrm{Id}$ is the operator associated with Q_∞, we prove the following result.

Theorem 51. *Assume that a and d are nice potentials and that $c = b =: \gamma$. Then, for every $k \geq 0$ we have*

$$\lim_{\lambda \to +\infty} \sigma_k(\lambda) = \sigma_k.$$

In addition, any sequence of normalized solution (u_λ, v_λ) *of Problem* (3.1) *admits a subsequence that converges strongly in* $H_0^1(\Omega) \times H_0^1(\Omega)$ *to a normalized eigenfunction* $(u, v) \in H_0^1(\Omega_0^a) \times H_0^1(\Omega_0^d)$ *of* $\mathcal{S}_\infty$ *associated to its kth-eigenvalue. If in addition* σ_k *is simple, this convergence holds for the whole sequence* $\{(u_\lambda, v_\lambda)\}$.

Owing Remark 48 and arguing as in Theorem 30, Theorem 51 will be a consequence of the following result.

Theorem 52. *Assume that* a *and* d *are nice potentials. Then,* Q_λ Γ*-converges to* Q_∞ *when* $\lambda \to +\infty$.

Proof. The proof is very similar to the proof of Theorem 32. We begin as before with the proof of *ii*) and consider for any $(u, v) \in L^2(\Omega) \times L^2(\Omega)$ the identically constant sequence $\{(u_\lambda, v_\lambda)\}$, with $(u_\lambda, v_\lambda) = (u, v)$ for every λ, that obviously converges strongly to (u, v). We have to prove that

$$Q_\infty(u, v) \geq \limsup_{\lambda \to +\infty} Q_\lambda(u_\lambda, v_\lambda). \tag{3.8}$$

If (u, v) is not in $H_0^1(\Omega_0^a) \times H_0^1(\Omega_0^d)$ then, by definition, we have that $Q_\infty(u, v) = +\infty$ and (3.8) trivially holds. On the other hand, if $(u, v) \in H_0^1(\Omega_0^a) \times H_0^1(\Omega_0^d)$ then $Q_\infty(u, v) = Q_\lambda(u, v)$, since by definition of Ω_0^a, Ω_0^d and $H_0^1(\Omega_0^a) \times H_0^1(\Omega_0^d)$ we find that $a(x)u(x) = 0$ and $d(x)v(x) = 0$ a.e. in Ω. Hence, (3.8) is again satisfied and condition *ii*) is proved in full generality.

Let us now prove *i*). So let $\{(u_\lambda, v_\lambda)\}$ be a sequence that strongly converges to (u, v) in $L^2(\Omega) \times L^2(\Omega)$, and let us prove that

$$Q_\infty(u, v) \leq \liminf_{\lambda \to +\infty} Q_\lambda(u_\lambda, v_\lambda). \tag{3.9}$$

We may assume that

$$\liminf_{\lambda \to +\infty} Q_\lambda(u_\lambda, v_\lambda) < +\infty, \tag{3.10}$$

otherwise (3.9) is trivial. In the latter situation we claim that $(u, v) \in H_0^1(\Omega_0^a) \times H_0^1(\Omega_0^d)$. Indeed, (3.10) implies in particular that

$$\liminf_{\lambda \to +\infty} (\lambda \int_\Omega a u_\lambda^2 dx + \lambda \int_\Omega d v_\lambda^2 dx) < +\infty. \tag{3.11}$$

Since $\{(u_\lambda, v_\lambda)\}$ strongly converges in $L^2(\Omega) \times L^2(\Omega)$ to (u, v), (3.11) actually implies that

$$\int_\Omega a u^2 dx = 0 \quad \text{and} \quad \int_\Omega d v^2 dx = 0,$$

from which we easily deduce that $u = 0$ a.e. in Ω_+^a and $v = 0$ a.e. in Ω_+^d. Moreover, since a and d are nice potentials we conclude that $u \in H_0^1(\Omega_0^a)$ and $v \in H_0^1(\Omega_0^d)$, and this allows us to write, using Proposition 26,

$$\begin{aligned}
Q_\infty(u, v) &= \int_\Omega |\nabla u|^2 dx + \int_\Omega |\nabla v|^2 dx - 2\int_\Omega \gamma uv dx + \alpha \int_\Omega (u^2 + v^2) \\
&\leq \liminf_{\lambda \to +\infty} \left(\int_\Omega |\nabla u|^2 dx + \int_\Omega |\nabla v|^2 dx - 2\int_\Omega \gamma uv dx + \alpha \int_\Omega (u^2 + v^2) \right) \\
&\leq \liminf_{\lambda \to +\infty} Q_\lambda(u_\lambda, v_\lambda),
\end{aligned}$$

and the proof is now complete. □

Proof of Theorem 51. The proof is similar to the one for Theorem 56 applied for the auxiliary problem (3.5). Namely the convergence of eigenvalues and of the operator in the strong resolvent sense follows from the Γ-convergence of the quadratic forms (Theorem 52) and the convergence of eigenfunctions strongly in $H_0^1(\Omega)$ follows using some analogue of Lemma 33 and Lemma 37 that are omitted here. Finally, since $\sigma_k^\alpha = \sigma_k + \alpha$ the proof is complete. □

Let us mention a further monotonicity property on the first eigenvalue $\sigma_0(\lambda)$ which is easy to prove in the variational context. In what follows $\sigma_0[A, \Omega']$ denotes the smallest eigenvalue of the operator A with domain contained in $H_0^1(\Omega')$.

Lemma 53. *Consider the cooperative system*

$$\begin{cases} (-\Delta + \lambda a)u - \gamma v = \sigma_0(\lambda)u, \\ (-\Delta + \lambda d)v - \gamma u = \sigma_0(\lambda)v, \end{cases} \quad ; \quad (u,v) \in H_0^1(\Omega) \times H_0^1(\Omega).$$

Then the real function $\lambda \mapsto \sigma_0(\lambda)$ *is increasing and bounded above by*

$$\sigma_0(\lambda) \le \min\big(\, \sigma_0[\mathcal{S}_\infty, \Omega_0],\ \sigma_0[-\Delta, \Omega^a],\ \sigma_0[-\Delta, \Omega^d]\,\big). \tag{3.12}$$

Proof. Since we are in a variational type framework, we know that $\sigma_0(\lambda)$ is obtained by minimizing the Rayleigh quotient

$$\sigma_0(\lambda) = \min\{\hat{Q}_\lambda(u,v);\ u,v \in H_0^1(\Omega) \text{ and } \|(u,v)\|_2 = 1\},$$

where $\hat{Q}_\lambda$ is the quadratic form associated with $\mathcal{S}_\lambda$. In particular, if $\lambda_1 < \lambda_2$, denoting (u_i, v_i) (for $i = 1,2$) the eigenfunctions associated to $\sigma_0(\lambda_i)$, we have that

$$\sigma_0(\lambda_1) = \hat{Q}_{\lambda_1}(u_1,v_1) \le \hat{Q}_{\lambda_1}(u_2,v_2) = \hat{Q}_{\lambda_2}(u_2,v_2) + (\lambda_1 - \lambda_2)\int_\Omega (a u_2^2 + d v_2^2) < \sigma_0(\lambda_2) \tag{3.13}$$

which proves the monotonicity in λ.

Now to prove (3.12) it suffice to compare $\hat{Q}_\lambda(u_\lambda, v_\lambda)$ with $\hat{Q}_\lambda(u,v)$ where the competitor (u,v) is successively (u_0, v_0) (an eigenfunction associated with $\sigma_0[\mathcal{S}_\infty, \Omega_0]$), $(u_a, 0)$ (where u_a is an eigenfunction associated with $\sigma_0[-\Delta, \Omega^a]$) and $(0, v_d)$ (where v_d is an eigenfunction associated with $\sigma_0[-\Delta, \Omega^d]$). □

Remark 54. Actually using (3.13) together with the fact that $\lambda \to v_\lambda$ is continuous in λ, one can prove that $\lambda \to \sigma_0(\lambda)$ is C^1 and the derivative $\sigma_0'(\lambda)$ with respect to λ is $\sigma_0'(\lambda) = \int_\Omega (a u_\lambda^2 + d v_\lambda^2)$ (compare with (3.4) in [1]).

Finally, let us say a few words about what happens with the equations in the "cross region" $\Omega_0^a \triangle \Omega_0^d$. Let us consider, for instance, the region $\Omega^a := \Omega_0^a \backslash \Omega_0^d$ where $a = 0$ but $d > 0$ (the other part will follow exchanging the role of u_λ and v_λ). In this region the two following equations hold:

$$\begin{cases} -\Delta u_\lambda - \gamma v_\lambda = \sigma_k(\lambda) u_\lambda, \\ -\Delta v_\lambda + \lambda d v_\lambda - \gamma u_\lambda = \sigma_k(\lambda) v_\lambda. \end{cases} \tag{3.14}$$

Since v_λ tends to 0 and $\sigma_k(\lambda)u_\lambda$ tends to $\sigma_k u$ in $L^2(\Omega^a)$, passing the first equation to the limit (which is justified by standard convergence results on the resolvent $-\Delta^{-1}$ on L^2) we obtain that u is the solution of the following problem

$$\begin{cases} -\Delta w = \sigma_k w, & \text{in } \Omega^a, \\ w - u \in H_0^1(\Omega^a). \end{cases} \tag{3.15}$$

Now, as far as the second equation of (3.14) is concerned, multiplying by a test function, passing to the limit and bearing in mind that $v = 0$ in Ω^a we obtain that

$$\lambda d(x) v_\lambda(x) \rightharpoonup \gamma u \quad \text{in } L^2(\Omega^a). \tag{3.16}$$

Remark 55. By the same argument as for Proposition 33, one can prove that if $\sigma_k(\lambda)$ is bounded, then

$$\liminf_{\lambda\to+\infty} \lambda\Big(\int_\Omega a u_\lambda^2 + \int_\Omega d v_\lambda^2\Big) = 0$$

which implies in particular that up to a subsequence, $\sqrt{\lambda d} v_\lambda$ converges to 0 strongly in $L^2(\Omega)$. We would like to point out that in (3.16) it is a different limit since we are considering $\lambda d(x) v_\lambda(x)$ and not $\sqrt{\lambda d(x)} v_\lambda(x)$. Actually, one can find in our last section an example of system for which $\lambda d(x) v_\lambda(x)$ do not converges to 0 in Ω^a (Proposition 61 case 2).

3.2 Non-Variational cooperative system

We are now considering the general case when b and c could be different in (3.1) and we give an alternative proof in this context. Indeed, the aim of the present section is to prove the following.

Theorem 56. *Let $\Omega \subset \mathbb{R}^N$ be an open set and assume that a and d are nice potentials. Then, we have*

$$\lim_{\lambda\to+\infty} \tau_k(\lambda) = \tau_k, \tag{3.17}$$

where $\tau_k(\lambda)$ is the kth-eigenvalue associated to the system (3.1), and τ_k is the one corresponding to (3.3). In addition, any sequence of normalized eigenfunctions $\{(u_\lambda, v_\lambda)\}$ associated with $\tau_k(\lambda)$ admits a subsequence that converges strongly in $H_0^1(\Omega)\times H_0^1(\Omega)$ to the normalized eigenfunction associated with τ_k. In particular if τ_k is simple, then the whole sequence $\{(u_\lambda, v_\lambda)\}$ converges strongly in $H_0^1(\Omega)\times H_0^1(\Omega)$ to the corresponding normalized eigenfunction.

Remark 57. As usual, here we set $\tau_k = +\infty$ for all $k \geq 0$ if $H_0^1(\Omega_0^a)\times H_0^1(\Omega_0^d) = \{0\}$.

Proof of Theorem 56. Let us first prove the convergence of the spectrum. Owing to Lemma 6, it is sufficient to prove that the inverse of the operators

$$\mathcal{L}_\lambda + \alpha \mathrm{Id} := \begin{pmatrix} -\Delta + \lambda a + \alpha & -b \\ -c & -\Delta + \lambda d + \alpha \end{pmatrix}$$

are compact and converge to the inverse of the limiting operator (denoted by $\mathcal{L}_\infty + \alpha\mathrm{Id}$) in the topology of operator norm on $L^2 \times L^2$. We will soon prove that those operators are actually invertible. Observe that we add $\alpha\mathrm{Id}$ to the operator $\mathcal{L}_\lambda$ with a positive α big enough, that will be chosen later (similarly to (3.4)) in order to guarantee the compactness of the inverse $(\mathcal{L}_\lambda + \alpha\mathrm{Id})^{-1}$. The desired convergence will follow from an idea given to us by Prof. Norman Dancer who we would like to thank for his suggestion. Firstly, let $R_\lambda^a := (-\Delta + \lambda a + \alpha)^{-1}$ and $R_\lambda^d := (-\Delta + \lambda d + \alpha)^{-1}$ be the resolvents associated with each equation of the system depending on α, which exist for any λ. Now for any f and g in L^2 one can write the system $(\mathcal{L}_\lambda + \alpha\mathrm{Id})(u_\lambda, v_\lambda) = (f, g)$ in the following form

$$\binom{u_\lambda}{v_\lambda} - A(\lambda)\binom{u_\lambda}{v_\lambda} = \binom{R_\lambda^a f}{R_\lambda^d g}, \quad \text{where} \quad A(\lambda) := \begin{pmatrix} 0 & b(-\Delta + \lambda a + \alpha)^{-1} \\ c(-\Delta + \lambda d + \alpha)^{-1} & 0 \end{pmatrix}.$$

Next, since the principal eigenvalue of the operator $\mathrm{Id} - A(\lambda)$ is positive we find that $\mathrm{Id} - A(\lambda)$ is invertible for any λ. This is true because the principal eigenvalue for that operator has the form

$$\tau_0[\mathrm{Id} - A(\lambda), \Omega] = 1 - \frac{\tau_0^\alpha(\lambda)}{\tau_0^*(\lambda)}, \quad \text{where} \quad \tau_0^* := \tau_0 \left[\begin{pmatrix} -\Delta + \lambda a + \alpha & 0 \\ 0 & -\Delta + \lambda d + \alpha \end{pmatrix}, \Omega \right],$$

and $\tau_0^\alpha(\lambda) := \tau_0[\mathcal{L}_\lambda + \alpha \mathrm{Id}, \Omega]$. Thanks to the monotonicity of the principal eigenvalue with respect to the potential we have that $\tau_0^*(\lambda) > \tau_0^\alpha(\lambda)$. Hence, $\tau_0[\mathrm{Id} - A(\lambda), \Omega] > 0$.

Then, according to that equivalence we deduce that $\mathcal{L}_\lambda + \alpha\mathrm{Id}$ is invertible and the inverse can be written as

$$(\mathcal{L}_\lambda + \alpha\mathrm{Id})^{-1} = (\mathrm{Id} - A(\lambda))^{-1}\mathcal{R}_\lambda$$

where $\mathcal{R}_\lambda$ is the matrix operator containing $(R_\lambda^a, R_\lambda^d)$ on its diagonal and 0 elsewhere. Moreover for α large enough, $(\mathcal{L}_\lambda + \alpha\mathrm{Id})^{-1}$ are all compact operators. This is a consequence of Remark 44 but it also follows from the easy a priori estimate

$$\|(u_\lambda, v_\lambda)\|_{H_0^1 \times H_0^1} \le C\|f, g\|_{L^2 \times L^2}$$

which holds when (u_λ, v_λ) is the solution of the system $(\mathcal{L}_\lambda + \alpha\mathrm{Id})(u_\lambda, v_\lambda) = (f, g)$ and provided a sufficiently big constant $\alpha > 0$.

Now, we use the fact that since a and d are nice potentials, we know by our Γ-convergence results that R_λ^a and R_λ^d converge to $R := (-\Delta)^{-1}$ in the topology of operator nom in L^2. This implies the convergence of both $\mathcal{R}_\lambda$, to the diagonal matrix $R\mathrm{Id}$, and $A(\lambda)$ to the operator

$$A := \begin{pmatrix} 0 & b(-\Delta + \alpha)^{-1} \\ c(-\Delta + \alpha)^{-1} & 0 \end{pmatrix}.$$

Therefore, since $\mathrm{Id} - A$ is invertible (for the same reason that $\mathrm{Id} - A(\lambda)$ was), passing to the inverse we deduce that $(\mathcal{L}_\lambda + \alpha\mathrm{Id})^{-1}$ converge in the topology of operator norm to $(\mathcal{L}_\infty + \alpha\mathrm{Id})^{-1}$ as $\lambda \to +\infty$. Applying Lemma 6, and since the inverses are compact operators, we obtain the convergence of the eigenvalues $\tau_k^\alpha(\lambda)$ to τ_k^α, as λ goes to infinity, for the operators $\mathcal{L}_\lambda + \alpha\mathrm{Id}$ and $\mathcal{L}_\infty + \alpha\mathrm{Id}$ respectively. Consequently, by the equalities $\tau_k^\alpha(\lambda) = \tau_k(\lambda) + \alpha$ and $\tau_k^\alpha = \tau_k + \alpha$ (3.17) holds. In particular $|\tau_k(\lambda)| \to +\infty$ if and only if $H_0^1(\Omega_0^a) \times H_0^1(\Omega_0^d) = \{0\}$. It is not restrictive now to assume that $\lim_{\lambda\to+\infty} |\tau_k(\lambda)| < +\infty$.

To conclude the proof let $\{(u_\lambda, v_\lambda)\}$ be a sequence of normalized solutions for (3.1). This implies that the $H_0^1(\Omega)$ norms are bounded. Indeed, multiplying (3.1) by (u_λ, v_λ) and integration by parts yields

$$\int_\Omega |\nabla u_\lambda|^2 + \int_\Omega |\nabla v_\lambda|^2 + \lambda \int_\Omega (a u_\lambda^2 + d v_\lambda^2) = \tau_k(\lambda) + \int_\Omega b u_\lambda v_\lambda + \int_\Omega c u_\lambda v_\lambda.$$

Then, by Hölder inequality we find that

$$\int_\Omega |\nabla u_\lambda|^2 \le K, \quad \int_\Omega |\nabla v_\lambda|^2 \le K, \quad \lambda \int_\Omega (a u_\lambda^2 + d v_\lambda^2) \le K,$$

for some positive constant K. Hence, we can extract a subsequence, again labelled by $\{(u_\lambda, v_\lambda)\}$, weakly convergent in $H_0^1(\Omega) \times H_0^1(\Omega)$ and strongly in $L^2(\Omega) \times L^2(\Omega)$ to some function (u, v). This implies that $\tau_k(\lambda)u_\lambda + bv_\lambda$ and $\tau_k(\lambda)v_\lambda + cu_\lambda$ converges strongly in $L^2(\Omega) \times L^2(\Omega)$ to $\tau_k u + bv$ and $\tau_k v + cu$ respectively. Then, using that a and d are nice potentials and applying Lemma 37 to both equations of the system (3.1), with f_λ equals $\tau_k(\lambda)u_\lambda + bv_\lambda$ and $\tau_k(\lambda)v_\lambda + cu_\lambda$ respectively, we can assure that, up to a subsequence, $\{(u_\lambda, v_\lambda)\}$ converges strongly in $H_0^1(\Omega) \times H_0^1(\Omega)$ and the limit $(u, v) \in H_0^1(\Omega_0^a) \times H_0^1(\Omega_0^d)$ is a solution of

$$\begin{cases} -\Delta u - bv = \tau_k u, \\ -\Delta v - cu = \tau_k v, \end{cases} \quad ; \quad (u, v) \in H_0^1(\Omega_0^a) \times H_0^1(\Omega_0^d). \tag{3.18}$$

Finally, if τ_k is simple, by uniqueness of the normalized solution (u, v) of the problem (3.18) associated with τ_k, the limit of the whole sequence converges to the eigenfunction associated with τ_k. □

Remark 58. Observe that to prove the convergence of the spectrum in Theorem 56, it is enough to prove first that the $|\tau_k(\lambda)|$ are bounded (and this can be obtained by monotonicity results) and then up to a subsequence, pass to the limit in the system (as we did for proving the convergence of eigenfunctions) to conclude. However, here we obtain more by proving directly the convergence of resolvents in the topology of operator norm which is a stronger result.

4 Some concrete examples

In this last section we present some explicit examples that shows how the limiting behaviour of the eigenvalues can differ from one situation to another, depending on the structural configuration of the vanishing domains of the potentials.

4.1 The case of two distinct components

Let Ω be a bounded connected and smooth domain in $\mathbb{R}^N$ and let $a \in C^0(\overline{\Omega})$ be a nice potential such that Ω_0^a (which is closed) is of the form $\Omega_0^a := \overline{\Omega_1} \cup \overline{\Omega_2}$ where Ω_1 and Ω_2 are two disjoint connected components, that for simplicity we assume to be smooth domains (at least stable). Moreover assume that the smallest Dirichlet eigenvalue in Ω_2 is strictly larger than the one in Ω_1, in other words

$$\sigma_0[-\Delta, \Omega_1] < \sigma_0[-\Delta, \Omega_2].$$

From the latter assumption we know that $\sigma_0[-\Delta, \Omega_0^a]$ is simple and

$$\sigma_0[-\Delta, \Omega_0^a] = \sigma_0[-\Delta, \Omega_1].$$

Then, according to Theorem 30 we know that the normalized solution u_λ of the eigenvalue problem

$$-\Delta u_\lambda + \lambda a u_\lambda = \sigma_0(\lambda) u_\lambda, \qquad u_\lambda \in H_0^1(\Omega),$$

(where $\sigma_0(\lambda)$ is the smallest Dirichlet eigenvalue for the operator $-\Delta + \lambda a$) strongly converges in $H_0^1(\Omega)$ to the unique normalized eigenvalue for the Dirichlet Laplacian in Ω_1 (and, thus, vanishes everywhere else). On the whole, we do not know what the limit is of u_λ in the case when $\sigma_0[-\Delta, \Omega_1] = \sigma_0[-\Delta, \Omega_2]$, although we know that the limit exists and is unique (because it is a Cauchy sequence in $H_0^1(\Omega)$ thank to [1]). However, some numerical computations[1] can give us an

[1]All the numerical simulations of this paragraph have been computed with the freeware *Freefem++* available on the website *freefem.org*. The resolution of the eigenvalue problem is based on the *Arpack++* subroutines which implements a variant of the Arnoldi process for finding eigenvalues called Implicit restarted Arnoldi method (IRAM). The 3D pictures has been made using the OpenGL-based scientific visualization software *Medit 3.0*. We thank Jimmy Lamboley for bringing to our attention the existence of the quite powerful and intuitive software *Freefem++*.

idea of what happen. Indeed, let us consider the rectangular domain $[-\pi, 2\pi] \times [-\pi, 0] \subset \mathbb{R}^2$ and the potential

$$a_1(x, y) := \max(0, \sin(x) + \sin(y) + 1),$$

that vanishes in the two disjoints balls $B^{\pm} := B(x^{\pm}, r_0)$ with

$$x^- = (-\frac{\pi}{2}, 0), \quad x^+ = (\frac{3\pi}{2}, 0) \quad \text{and } r_0 := \frac{\pi}{2}.$$

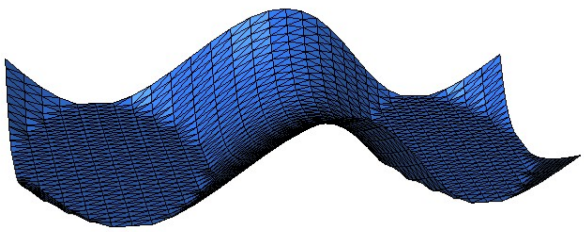

Figure 1.
$a_1(x, y) := \max(0, \sin(x) + \sin(y) + 1), \quad (x, y) \in [-\pi, 2\pi] \times [-\pi, 0].$

Since a_1 is symmetrical, it is not difficult to see that when $\lambda \to +\infty$, the normalized functions u_λ converge to $\frac{1}{2}(\phi_+ + \phi_-)$, where ϕ_+ is the state function in B^+ and ϕ_- the one in B^-.

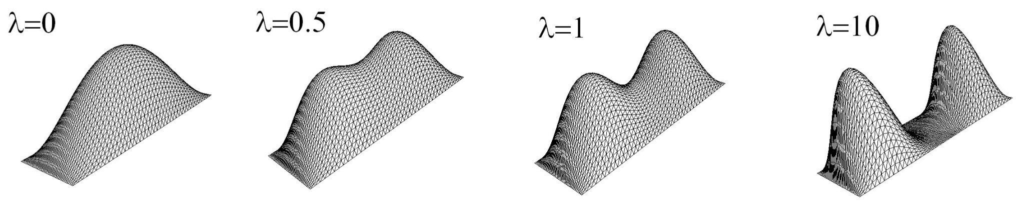

Figure 2.
$u_\lambda \to \frac{1}{2}(\phi_- + \phi_+)$ when $a := a_1$.

Now, we want to brake the symmetry of the potential. For this purpose we consider the step function

$$f(x, y) := \exp(\max((x - 2\pi + 0.3), 0)) - 1,$$

which is a continuous function equal to 0 when $x \in [-\pi, 2\pi - 0.3]$ and very positive when $x \in [2\pi - 0.3, 2\pi]$. Next we define

$$a_2(x, y) := a_1(x, y) + f(x, y).$$

By this way, a_2 still vanishes in $B^{\pm}$ but now the region around B^+ is "advantaged" compared to B^-. Then, as the numerical computations seems to show, the functions u_λ this time tend to φ^+.

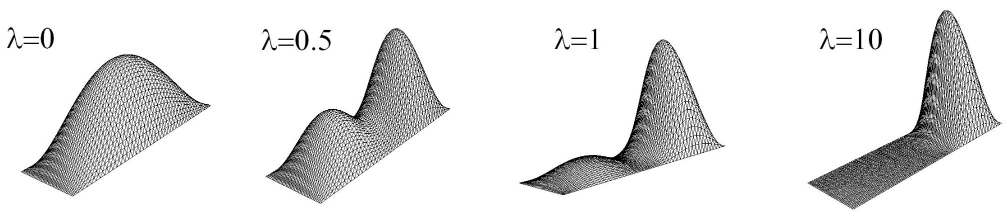

Figure 3.
$u_\lambda \to \phi_+$ when $a := a_2$.

This phenomenon seems to be linked with the so-called "Tunnel effect" in semi-classical analysis that was studied by Simon [29] and Helffer-Sjöstrand [18] in the case of a double well the potential. Our observations suggest that the "flea on the elephant" leads to the same consequences in our situation with highly degenerate potential as the ones that Simon observes in his paper [29] for the case of a double well potential. We would like to thank B. Helffer for communicate to us this remark.

4.2 The one dimensional Cantor set

Let us consider the potentials χ_E and χ_K introduced in Proposition 22. Recall that E is an open and dense set in $]0,1[$ and $K = [0,1]\backslash E$ is a Cantor set, compact of empty interior but with positive Lebesgue measure.

Let us consider first χ_E. Recall that E is an open set in $]0,1[$ which is *not* stable. However using our results one can still prove the convergence of eigenvalues. Indeed, since a is a nice potential we deduce from Theorem 30 that for each $k \geq 0$ the kth-eigenvalue $\sigma_k(\lambda)$ for the problem

$$-\Delta u + \lambda \chi_E u = \sigma_k(\lambda) u, \qquad u \in H_0^1(]0,1[),$$

has a finite limit, namely converges to the eigenvalue σ_k of the limiting problem

$$-\Delta u = \sigma_k u, \qquad u \in H_0^1(E).$$

Now for the case $a(x) := \chi_K(x)$ we also get the convergence of $\sigma_k(\lambda)$ but now by Remark 23 we know that $H_0^1(\Omega_0^a) = \{0\}$ so that $\sigma_k = +\infty$ for every $k \geq 0$. We deduce the following interesting fact.

Proposition 59. *There exists a nice potential $a \in \mathcal{A}(]0,1[)$ satisfying*

$$\mathscr{L}^1(\Omega_0^a) > 0,$$

and such that

$$\lim_{\lambda \to +\infty} \sigma_0(\lambda) = +\infty, \tag{4.1}$$

where $\sigma_0(\lambda)$ is the first Dirichlet eigenvalue of $-\Delta + \lambda a$ in $]0,1[$.

Notice that (4.1) is always true when $\mathrm{cap}(\Omega_0^a) = 0$. Proposition 59 shows in particular that the reverse implication is false in general.

4.3 Different behaviours for the limiting system

Now we want to study a particular situation for the system in order to recover some results shown in [2] and maybe extend them for slightly more general situations (cf. [11]). Let Ω be a bounded connected and smooth domain in $\mathbb{R}^N$ and let $a, d \in C^0(\overline{\Omega})$ be some nice potentials. We denote as usual the closed sets

$$\Omega_0^a := \{a = 0\}, \quad \Omega_0^d := \{d = 0\}, \quad \Omega_0 := \{a = 0 = d\},$$

and we also introduce

$$\Omega^a := \Omega_0^a \backslash \Omega_0, \quad \Omega^d := \Omega_0^d \backslash \Omega_0.$$

Let us assume that all those sets are disjoint and the following particular configuration holds:

$$\overline{\Omega^a} \cap \overline{\Omega^d} = \emptyset, \qquad \overline{\Omega^a} \cap \overline{\Omega_0} = \emptyset \quad \text{and} \quad \overline{\Omega^d} \cap \overline{\Omega_0} = \emptyset \tag{4.2}$$

For simplicity we also assume all those domains to be smooth. We will need an analogue of Lemma 53 about the monotonicity of $\tau_0(\lambda)$ with respect to λ. In our general context this property cannot be proved by the traditional min-max principle, except from the case when $b = c$. However, it will follows from an argument shown in [2] based upon a characterization of the Maximum Principle in terms of the positivity of the first eigenvalue and the existence of a positive strict supersolution (c.f. [23]).

Lemma 60. *Under the above assumptions the real function $\tau_0(\lambda)$ is continuous in λ, increasing and bounded above by*

$$\tau_0(\lambda) \leq \min\big(\, \tau_0[\mathfrak{L}_\infty, \Omega_0],\ \sigma_0[-\Delta, \Omega^a],\ \sigma_0[-\Delta, \Omega^d]\, \big).$$

Proof. The continuity and the fact that $\tau_0(\lambda)$ is increasing is a consequence from the monotonicity of the eigenvalues with respect to the potential, that was shown in [2], provided some regularity on the boundary of Ω. Moreover, in the case when the domain Ω_0, is regular, say Lipschitz, we can apply directly the monotonicity properties with respect to the domain to obtain that

$$\tau_0(\lambda) = \tau_0[\mathfrak{L}(\lambda a, \lambda d); \Omega] < \tau_0[\mathfrak{L}(\lambda a, \lambda d); \Omega_0] = \tau_0[\mathfrak{L}_\infty, \Omega_0],$$

because $a = d = 0$ in Ω_0. This provides us with an upper bound for the function $\tau_0(\lambda)$.

On the other hand, assuming now that Ω^a and Ω^d are Lipschitz and taking into account the first equation of the eigenvalue problem (3.1), and particularizing it in Ω^a, we find that

$$\begin{array}{lll} -\Delta u_\lambda - \tau_0(\lambda) u_\lambda = b v_\lambda > 0, & \text{in} & \Omega^a, \\ u_\lambda > 0, & \text{on} & \partial\Omega^a \end{array}.$$

Hence, applying the main theorem in [23] it yields

$$\tau_0(\lambda) < \sigma_0[-\Delta, \Omega^a].$$

Similarly for the second equation of (3.1), we obtain that

$$\tau_0(\lambda) < \sigma_0[-\Delta, \Omega^d],$$

and this ends the proof. □

Proposition 61. *Assume that we are under the same assumptions as above and denote*

$$\beta_0 = \min(\sigma_0[-\Delta, \Omega^d], \sigma_0[-\Delta, \Omega^a]).$$

Then, the following alternative holds :

Case 1. *if $\tau_0[\mathfrak{L}_\infty, \Omega_0] < \beta_0$ then, $\lim_{\lambda \to +\infty} \tau_0(\lambda) = \tau_0[\mathfrak{L}_\infty, \Omega_0]$ and the normalized eigenfunctions (u_λ, v_λ) of $\mathfrak{L}_\lambda$ associated to $\tau_0(\lambda)$ strongly converge in $H_0^1(\Omega) \times H_0^1(\Omega)$ to the normalized first eigenfunctions $(u, v) \in H_0^1(\Omega_0) \times H_0^1(\Omega_0)$ of $\mathfrak{L}_\infty$ in Ω_0;*

Case 2. *If* $\tau_0[\mathfrak{L}_\infty, \Omega_0] > \beta_0$ *then,* $\lim_{\lambda\to+\infty} \tau_0(\lambda) = \beta_0$ *and the normalized solutions* (u_λ, v_λ) *of* $\mathfrak{L}_\lambda$ *associated to* $\tau_0(\lambda)$ *strongly converge in* $H_0^1(\Omega) \times H_0^1(\Omega)$ *as follows*

$$\lim_{\lambda\to+\infty}(u_\lambda, v_\lambda) = \begin{cases} (u, 0), & \text{if } \sigma_0[-\Delta, \Omega^a] < \sigma_0[-\Delta, \Omega^d], \\ (0, v), & \text{if } \sigma_0[-\Delta, \Omega^a] > \sigma_0[-\Delta, \Omega^d], \end{cases}$$

where u *and* v *are respectively the eigenfunctions of* $-\Delta$ *associated to smallest Dirichlet eigenvalue in* Ω^a *and* Ω^d*.*

Proof. By Theorem 56 we have that any converging subsequence of (u_λ, v_λ) converges to some (u, v) satisfying the following limiting problem

$$\begin{cases} -\Delta u - bv = \tau_0 u, \\ -\Delta v - cu = \tau_0 v, \end{cases} \quad ; \quad (u, v) \in H_0^1(\Omega_0^a) \times H_0^1(\Omega_0^d). \tag{4.3}$$

for some $\tau_0 > 0$. Notice that under assumption (4.2), we have that

$$H_0^1(\Omega_0^a) = H_0^1(\Omega_0) \cap H_0^1(\Omega^a) \quad \text{and} \quad H_0^1(\Omega_0^d) = H_0^1(\Omega_0) \cap H_0^1(\Omega^d).$$

Therefore, from (4.3) we deduce that (u, v) satisfies in particular

$$\begin{cases} -\Delta u - bv = \tau_0 u, & (u, v) \in H_0^1(\Omega_0) \times H_0^1(\Omega_0), \\ -\Delta v - cu = \tau_0 v, & (u, v) \in H_0^1(\Omega_0) \times H_0^1(\Omega_0), \\ -\Delta u = \tau_0 u, & u \in H_0^1(\Omega^a), \\ -\Delta v = \tau_0 v, & v \in H_0^1(\Omega^d), \end{cases} \tag{4.4}$$

and the conclusion follows from a careful inspection of the compatibility of the above equations together with the inequality

$$\tau_0 := \lim_{\lambda\to+\infty} \tau_0(\lambda) \leq \min(\beta_0, \tau_0[\mathfrak{L}_\infty, \Omega_0])$$

(coming from Lemma 60) and the fact that $\|(u, v)\|_{L^2(\Omega)} = 1$. □

Remark 62. Observe that similarly to Section 4.1, in the case when $\beta_0 = \tau_0[\mathfrak{L}_\infty, \Omega_0]$ or $\tau_0 < \tau_0[\mathfrak{L}_\infty, \Omega_0]$ and $\sigma_0[-\Delta, \Omega^a] = \sigma_0[-\Delta, \Omega^d]$, the precise limit of (u_λ, v_λ) remains uncertain.

Remark 63. We would like to point out that just before the submission of this work we have been informed by E.N. Dancer of the existence of a pre-print [11] in which, independently, similar results to our Proposition 61 are obtained. We would also like to thank him for sharing his results with us.

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