MINKOWSKI CONTENT FOR REACHABLE SETS

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Abstract. In 1955, Martin Kneser showed that the Minkowski content of a compact p-rectifiable subset M of $\mathbb{R}^n$ is equal to its p-Hausdorff measure:

$$\lim_{t\to 0, t>0} \frac{\mathcal{L}^n\left(\overline{B}(M,t)\right)}{\alpha(n-p)\, t^{n-p}} = \mathcal{H}^p(M).$$

We extend his result to the reachable sets of a linear control system

$$\dot{x} = f(x)\, u,$$

and we give an interpretation in terms of Riemannian distance.

1. Introduction

The tube of radius t around a subset M of $\mathbb{R}^n$ is the set of points at distance less than t to M:

$$\overline{B}(M,t) = \left\{ x \,\middle|\, \mathrm{d}(M,x) \le t \right\}.$$

The behavior of its volume

$$\mathcal{L}^n\left(\overline{B}(M,t)\right)$$

-assuming that M is compact- as a function of t has been studied from different point of view.

Exact -polynomial- formulas are given under regularity assumptions on the set M. When it is convex, Steiner's formula holds for any positive t (see Steiner [St1840], Federer [Fed]). When M is a C^2 submanifold, Weyl's formula holds for $t \le t_0$ (Weyl [We39]). Both approaches have been unified by Herbert Federer [Fe59] with the sets of positive reach, that he defined for this purpose.

As motivations for the calculus of the volume of tubes, we should first mention the paper of Herbert Hotelling [Ho39] who gave a polynomial formula when M is a curve on a sphere, for purposes in statistics. His work apparently motivated Hermann Weyl who's paper is published right after Hotelling's paper. Since then, many applications have been given in probability and statistics.

When exact formulas are not known, the other important issue is the asymptotic behavior of the volume of the tube $\mathcal{L}^n\left(\overline{B}(M,t)\right)$. Two cases are to be looked at, $t \to +\infty$ and $t \to 0$.

The study of the asymptotic behavior at infinity gives information on the space and is of interest in the Riemmannian setting, and trivial in $\mathbb{R}^n$.

When $t \to 0$, with very little regularity (M rectifiable, i.e., M is the image of a compact subset of $\mathbb{R}^p$ by a Lipschitzian map), the asymptotic begavior is given

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by the formula on the Minkowski content (name of the left-hand side), by Martin Kneser (1955):

Theorem A. [Kn55, Satz 3], [Fed, Theorem 3.2.39] *Let M be a compact p-rectifiable subset of $\mathbb{R}^n$. Then*

$$\lim_{t\to 0, t>0} \frac{\mathcal{L}^n\left(\overline{B}(M,t)\right)}{\alpha(n-p)\, t^{n-p}} = \mathcal{H}^p(M),$$

where $\alpha(i) = \mathcal{L}^i(B_{\mathbb{R}^i}(0,1))$.

The regularity assumption on M is close to be minimal: take $M = \cup_{n\in\mathbb{N}}\{1/2^n\}\cup\{0\}$, it is countably 0-rectifiable, $\mathcal{L}^1(\overline{B}(M,\delta_n)) = \delta_n(-2\log(\delta_n)/\log 2 + 2)$ for $\delta_n = 1/2^n$, and $\limsup_{t\to 0,t>0} \frac{\mathcal{L}^1(\overline{B}(M,t))}{t} = +\infty$. See [Kn55, Satz 5] and [Fed, 3.2.40] for other counter-examples and references. However an extension has been given by Ambrosio, Fusco, Pallara [AmbFusPal, Theorem 2.104], assuming M to be countably rectifiable together with a weak regularity assumption.

The aim of our paper is to extend Kneser's result on the Minkowski content to reachable sets. Take a dynamic

$$\dot{x} = f(x)\, u,$$

with $f(x) \in GL_n(\mathbb{R})$ -f continuous-, and u being a measurable map with values in $\overline{B}(0,1)$. The reachable set at time t is the set of points that can be reached by a trajectory of the control system, at time t:

$$\mathcal{R}_f(M,t) = \left\{x \;\middle|\; \exists x_0 \in M,\ \exists u, \quad x = x(t;x_0,u)\right\}$$

In our setting (since u can take any values in $\overline{B}(0,1)$), it is also the reachable set at time less than t.

Different issues are studied on reachable sets. Regularity, estimates on the perimeter, on the volume. In fact our paper was initially motivated by Alvarez, Cardaliaguet, Monneau [AlCaMo05, 2005], on dislocation dynamics. To make it short, they realize a dislocation (a closed curve moving with normal velocity) as a reachable set of a time-dependent system

$$\dot{x} = f(t,x)\, u,$$

with $f(t,x) \in GL_n(\mathbb{R})$ and u having values in the unit ball.

The link with tubes can easily be seen taking $f(x) = \mathrm{id}$, the identity map, for every x. The trajectory $x(t;x_0,u) = x_0 + \int_0^t u$ is equal to $x_0 + t\,u$ when u is constant, and

$$\mathcal{R}_f(M,t) = \overline{B}(M,t).$$

More generally, take the minimum time function

$$\tau_f(x,y) = \inf\{t \mid \exists u,\ x(t;x,u) = y\},$$

which is a "quasi-distance" (possible infinite values) in our setting. Then

$$\mathcal{R}_f(M,t) \approx \overline{B}_{\tau_f}(M,t),$$

precisely, $\mathcal{R}_f(M,t) \subset \overline{B}_{\tau_f}(M,t) \subset \mathcal{R}_f(M,t')$ for every $t < t'$. So our question was: can we extend Kneser's result on the Minkowski content, something like

Question.

$$\lim_{t\to 0, t>0} \frac{\mathcal{L}^n\left(\overline{B}_{\tau_f}(M,t)\right)}{\alpha(n-p)\, t^{n-p}} = \mathcal{H}^p_{\tau_f}(M) \quad ?$$

With $\mathcal{H}_{\tau_f}$ denoting the Haussdorff measure associated to the metric τ_f. If we take $f(x) = \lambda \mathrm{id}$ for every x,

$$\begin{aligned}
\mathcal{R}_f(M,t) &= \overline{B}(M,\lambda t) = \overline{B}_{\tau_f}(M,t),\\
\tau_f(x,y) &= \frac{\|x-y\|}{\lambda},\\
\lim_{t\to 0, t>0} \frac{\mathcal{L}^n\left(\mathcal{R}_f(M,t)\right)}{\alpha(n-p)\, t^{n-p}} &= \lim_{t\to 0, t>0} \frac{\mathcal{L}^n\left(\overline{B}(M,\lambda t)\right)}{\alpha(n-p)\, (\lambda\, t)^{n-p}} \lambda^{n-p} = \lambda^{n-p}\mathcal{H}^p(M)
\end{aligned}$$

but

$$\mathcal{H}^p_{\tau_f}(M) = \lambda^{-p}\mathcal{H}^p(M).$$

This non-intrinsic factor λ^n prevents a direct extension like the above question. We need to adapt the Hausdorff measure, following Cannarsa and Cardaliaguet [CaCa06], to the dynamic f and we obtain the following extension:

Theorem B. *Let M be a compact p-rectifiable subset of $\mathbb{R}^n$. Then*

$$\lim_{t\to 0, t>0} \frac{\mathcal{L}^n(\mathcal{R}_f(M,t))}{\alpha(n-p)\, t^{n-p}} = \mathcal{H}^p_f(M),$$

This result can be translated in terms of Riemannian distance. Take a continuous Riemannian tensor F and its associated distance d_F. Then:

Theorem C. *Let M be a compact p-rectifiable subset of $\mathbb{R}^n$. Then*

$$\lim_{t\to 0, t>0} \frac{\mathcal{L}^n\left(\overline{B}_{\mathrm{d}_F}(M,t)\right)}{\alpha(n-p)\, t^{n-p}} = \mathcal{H}^p_{\mathbf{F}}(M),$$

with an adapted Hausdorff measure $\mathcal{H}^p_{\mathbf{F}}(M)$. This is obtained by the dictionary

$$F \leftrightarrow {}^t\!f^{-1}\, f^{-1}.$$

Associated to the dynamic f is the Riemannian tensor ${}^t\!f^{-1}\, f^{-1}$, and the reciprocal follows from Cholesky's decomposition.

Many extensions of the theories covering the formulas on the volume of tubes have been given in the Riemannian setting. Federer explains how to extend the theory of Hausdorff measures to Riemannian manifolds [Fed, paragraph 3.2.46]. Joseph H. G. Fu [Fu89] extended much of the theory of Federer on curvature measures. Steiner-Weyl formulas have been extended to the Riemannian setting, assuming regularity of the set M (see, for example Gray and Vanhecke [GrVa81]). But apparently, there has been no strict extension of Kneser's result on the Minkowski content.

Our paper is organised as follows. We present our main results in Section 2. Section 2.1 gives the setting, our main result (Theorem B) is properly stated in Section 2.2, reformulated in terms of minimum time function in Section 2.3 and in terms of Riemannian distance in Section 2.4; Section 2.5 is devoted to some remarks on the dynamic f. In Section 3, we state and prove results on the adapted Hausdorff measure and on reachable sets, that will be useful for the forthcoming proofs. Our main result is proved in Section 4, and the equivalence with the Riemannian setting in Section 5.

2. Main results

2.1. **Basic definitions.** [1]

2.1.1. *The control system.* Consider a continuous map

$$f : \Omega \to GL_n(\mathbb{R}),$$

where Ω is a non empty open subset of $\mathbb{R}^n$. For any measurable function (control)

$$u : [0, +\infty) \to \overline{B}(0, 1)$$

and initial point $x_0 \in \Omega$, consider the Cauchy problem

$$\dot{x} = f(x)\, u, \quad x(0) = x_0 \tag{2.1}$$

Denote by $X(x_0, u)$ the solution set of (2.1):

$$X(x_0, u) = \Big\{ (I, x) \,\Big|\, I \text{ is an interval containing } 0, \\ x \in C^0(I, \mathbb{R}^n),\ \forall t \in I,\ x(t) = x_0 + \int_0^t f(x(s))\, u(s) ds \Big\} \tag{2.2}$$

The solutions are clearly absolutely continuous, hence a.e. differentiable, and:

$$\dot{x}(t) = f(x(t))\, u(t) \quad \text{a.e. } t$$

In view of the classical existence theorem of Peano and Carathéodory, and of Zorn's lemma, the set $X(x_0, u)$ is non empty, contains maximal elements $(I, x(.; x_0, u))$ which all satisfy:

$$\text{for every } r < d(\mathbb{R}^n \setminus \Omega, x_0)\,,\ \frac{r}{\sup_{\overline{B}(x_0, r)} \|f\|} \in I. \tag{2.3}$$

In other words, we have an estimation (lower bound) of the time for wich the trajectories are defined. The reachable set, or attainable set, at time t, from a set M, is defined by:

$$\mathcal{R}_f(M, t) = \big\{ x \,\big|\, \exists x_0 \in M,\ \exists u \in \mathcal{L}^1_{loc}([0, +\infty), \overline{B}), \exists (I, x(.; x_0, u)) \in X(x_0, u), \\ t \in I, x = x(t; x_0, u) \big\}.$$

The reachable sets have the semi-group property, which follows from the definition: for every non negative t and τ,

$$\mathcal{R}_f(M, t + \tau) = \mathcal{R}_f(\mathcal{R}_f(M, t), \tau).$$

One easily proves that $\mathcal{R}_f(M, t)$ is compact, if M is compact and for t small enough.

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[1] For a linear map $A \in M_n(R)$, we denote its associated norm by $\|A\| = \sup_{x \in \overline{B}(0,1)} A\, x$. For any $\Omega \subset \mathbb{R}^n$ and $f : \Omega \to GL_n(\mathbb{R})$ we let

$$\|f\|_\infty = \sup_{x \in \Omega} \|f(x)\|\,.$$

$\mathcal{L}^n$ denotes the Lebesgue measure on $\mathbb{R}^n$.

2.1.2. *The adapted Hausdorff measure.* For a given map

$$f : \Omega \to GL_n(\mathbb{R}),$$

where Ω is a non empty subset of $\mathbb{R}^n$, and for a fixed integer $p \in \{1, \dots, n\}$, we will now recall the notion of the *adapted p-dimensionnal Hausdorff measure* to f, $\mathcal{H}^p_f$, that was introduced in [CaCa06] for $p = n-1$.

First of all, for any set $K \subset \Omega$ let us denote by $\mathrm{diam}_{f,p}(K)$ the (adapted) diameter of K, that is,

$$\mathrm{diam}_{f,p}(K) := \sup_{x,y \in K} |\det(f(x))|^{1/p} \left| f(x)^{-1}(x-y) \right| .$$

Then, for any set $E \subset \Omega$ and any number $\delta > 0$, let

$$\mathcal{H}^p_{f,\delta}(E) := \inf \left\{ \frac{\alpha(p)}{2^p} \sum_{i=1}^{\infty} \left(\mathrm{diam}_{f,p}(K_i) \right)^p \;\Big|\; E \subset \bigcup_{i=1}^{\infty} K_i \,,\; \mathrm{diam}_{f,p}(K_i) \le \delta \right\},$$

where $\alpha(p)$ is the volume of the unit ball of $\mathbb{R}^p$. Finally, define

$$\mathcal{H}^p_f(E) := \lim_{\delta \to 0^+} \mathcal{H}^p_{f,\delta}(E).$$

In fact this is exactly the Carathéodory's construction (see for example [Fed]) on the set Ω associated to the map $\mathrm{diam}_{f,p}$. If $E \subset \Omega$, the construction on the set $\mathbb{R}^n$ and the construction on the set Ω give the same value for $\mathcal{H}^p_\delta(E)$ and $\mathcal{H}^p(E)$, hence removing any ambiguity. It is easily seen that the above quantities $\mathcal{H}^p_{f,\delta}(E)$ and $\mathcal{H}^p_f(E)$ reduce to the usual ones, $\mathcal{H}^p_\delta(E)$ and $\mathcal{H}^p(E)$, when $f(x)$ coincides with the identity matrix. Let us notice at this point that the factor $|\det(f(x))|^{1/p}$ -when differing from 1!- in the definition of the diameter $\mathrm{diam}_{f,p}(K)$ makes the adapted Hausdorff measure heavily depend on the dimension of the ambient space, and thus somewhat non intrinsic, contrarily to the classical Hausdorff measure. We address this issue in Section 2.4.

2.1.3. *Rectifiable sets.* We take the terminology of Federer [Fed, 3.2.14]. A subset M of a metric space X is p rectifiable if and only if there exists a Lipschitzian function mapping some bounded subset of $\mathbb{R}^p$ onto M.

2.2. **Statement of the results.** Our main result, which clearly implies Theorem B in the introduction, extends Theorem A of Kneser, on the Minkowski content of a compact p-rectifiable subset M of $\mathbb{R}^n$, to a dynamic of the type presented above, for a continuous map

$$f : \Omega \to GL_n(\mathbb{R}),$$

where Ω is a non empty open subset of $\mathbb{R}^n$. Then the adapted p-Minkowski content of the set M, where the reachable sets $\mathcal{R}_f(M,t)$ replace the tubes $\overline{B}(M,t)$, is equal to the adapted p-dimensionnal Hausdorff measure to f of M.

Theorem 2.1. *Let M be a compact p-rectifiable subset of Ω. Then*

$$\lim_{t \to 0, t>0} \frac{\mathcal{L}^n(\mathcal{R}_f(M,t))}{\alpha(n-p)\, t^{n-p}} = \mathcal{H}^p_f(M),$$

where $\alpha(i) = \mathcal{L}^i(B_{\mathbb{R}^i}(0,1))$.

The proof of Theorem 2.1 is given in Section 4 . The upper bound of the Minkowski content, Section 4.2, is quite straightforward, by mean of local linear approximations. It could also be obtained by the general theory, following the lines of Federer [Fed] on the Minkowski content and on the Carathéodory construction. The lower bound, Section 4.3, is more tricky to obtain, it still uses local linear approximations, but it also requires to locally shrink the covering cubes used in the local linear approximation.

2.3. **With the minimum time function.** Associated to the control system is the minimum time function

$$\tau_f(x_1, x_2) = \inf\{t \mid \exists u,\ x(t; x_1, u) = x_2\}.$$

This is the distance induced by the control system: it is symmetric, due to the particular form of the dynamic that we consider. It also enjoys the other properties of a distance -note that it can be infinite, if Ω is not connected. The natural extension, to our control system, of the definition of an Euclidean tube around a set M, is the tube for the distance τ_f:

$$\overline{B}_{\tau_f}(M,t) = \left\{ x \;\middle|\; \inf_{y\in M} \tau_f(x,y) \le t \right\}.$$

The extension of Theorem 2.1 on the Minkowski content should then be stated in terms of tubes $\overline{B}_{\tau_f}(M,t)$.

Theorem 2.2. *Let M be a compact p-rectifiable subset of Ω. Then*

$$\lim_{t\to 0, t>0} \frac{\mathcal{L}^n\left(\overline{B}_{\tau_f}(M,t)\right)}{\alpha(n-p)\, t^{n-p}} = \mathcal{H}^p_f(M).$$

For a general control system, and every $0 \le t < t'$, the following inclusions hold:

$$\cup_{s\le t}\mathcal{R}_f(M,s) \subset \overline{B}_{\tau_f}(M,t) \subset \cup_{s\le t'}\mathcal{R}_f(M,s),$$

with possible strict inclusions, hence

$$\lim_{t\to 0, t>0} \frac{\mathcal{L}^n\left(\overline{B}_{\tau_f}(M,t)\right)}{\alpha(n-p)\, t^{n-p}} = \lim_{t\to 0, t>0} \frac{\mathcal{L}^n\left(\cup_{s\le t}\mathcal{R}_f(M,s)\right)}{\alpha(n-p)\, t^{n-p}}.$$

With the control system that we consider, because of its linear structure and because of the control space equal to the unit ball,

$$\mathcal{R}_f(M,t) = \cup_{s\le t}\mathcal{R}_f(M,s).$$

So both statements, Theorem 2.1 in terms of reachable sets, and Theorem 2.2 in terms of tubes for the minimum time function, are equivalent. Because of our interest in control, where the notion of reachable sets prevails, we have chosen a presentation in terms of reachable sets. Also, we have seen that the natural extension, to reachable sets, of the definition of a tube, would rather be the *reachable set at time less than t*:

$$\cup_{s\le t}\mathcal{R}_f(M,s).$$

Since in our case $\mathcal{R}_f(M,t) = \cup_{s\le t}\mathcal{R}_f(M,s)$, we stick to the traditional definition of reachable sets, and we keep the lighter writing $\mathcal{R}_f(M,t)$ for the convenience of the reader. Only keep this notice in mind when thinking to other possible dynamics.

2.4. **Interpretation (translation) in terms of Riemannian distance.** Theorem 2.1 can be equivalently reformulated in a Riemannian setting, in terms of Riemannian distance. We briefly recall the setting, see for example Federer [Fed, p. 281]. Let Ω be an open subset of $\mathbb{R}^n$, and

$$F : \Omega \to S_n^{++}(\mathbb{R})$$

be a continuous map such that $F(x)$ is positive and definite for every $x \in \Omega$. Corresponding to the Riemannian tensor F, we define the distance $\mathrm{d}_F(x, y)$ (possibly infinite) between two elements x and y in Ω by:

$$\mathrm{d}_F(x,y) = \inf \left\{ \int_0^1 \left({}^t\gamma'(t)F(\gamma(t))\gamma'(t)\right)^{1/2} dt \;\middle|\; \gamma : [0,1] \to \Omega, \right.$$
$$\left. \gamma(0) = x, \gamma(1) = y, \gamma \text{ is Lispschitzian} \right\}.$$

The fact that d_F satisfies the properties of a (quasi)-distance can be found for example in Federer [Fed]. Associated to d_F is the notion of tube around a subset M of Ω:

$$\overline{B}_{\mathrm{d}_F}(M,t) = \left\{ x \;\middle|\; \inf_{y\in M} \mathrm{d}_F(x,y) \le t \right\}.$$

Following Carathéodory's construction of the Hausdorff measure on the metric space (Ω, d_F), we obtain the p-dimensional Hausdorff measure $\mathcal{H}^p_{\mathrm{d}_F}$. But the result on the Minkowski content does not extend with the Hausdorff measure $\mathcal{H}^p_{\mathrm{d}_F}$. This can easily be seen by taking $F = \lambda^2 \mathrm{id}_{\mathbb{R}^n}$, $\lambda \neq 0$. Then

$$\begin{aligned}
\mathrm{d}_F(x,y) &= |\lambda|\,|x-y| \\
\overline{B}_{\mathrm{d}_F}(M,t) &= \overline{B}\left(M, \frac{t}{|\lambda|}\right) \\
\lim_{t\to 0, t>0} \frac{\mathcal{L}^n(\overline{B}_{\mathrm{d}_F}(M,t))}{\alpha(n-p)\, t^{n-p}} &= \lim_{t\to 0, t>0} \frac{\mathcal{L}^n\left(\overline{B}\left(M, \frac{t}{|\lambda|}\right)\right)}{\alpha(n-p)\,\left(\frac{t}{|\lambda|}\right)^{n-p}} |\lambda|^{p-n} \\
&= |\lambda|^{p-n}\, \mathcal{H}^p(M), \text{ with } M \text{ } p\text{-rectifiable,}
\end{aligned}$$

but

$$\mathcal{H}^p_{\mathrm{d}_F} = |\lambda|^p \mathcal{H}^p(M).$$

Since the behavior of the Minkowski content depends on the local variations of F, we have little hope of a direct change in the formula of the Minkowski content, that would yield the Hausdorff measure $\mathcal{H}^p_{\mathrm{d}_F}$.

So we introduce an adapted, non intrinsic, Hausdorff measure $\mathcal{H}^p_{\mathbf{F}}$ -with hopefully no confusion on the notations!- as in Section 2.1.2, by defining

$$\mathrm{diam}_{\mathbf{F},p}(K) := \sup_{x,y\in K} |\det(F(x))|^{-1/2p}\, \mathrm{d}_F(x,y), \tag{2.4}$$

and following Carathéodory's construction. We can now give the result on the Minkowski content in this simple Riemannian setting, which clearly implies Theorem C in the introduction:

Theorem 2.3. *Let M be a compact p-rectifiable subset of Ω. Then*

$$\lim_{t\to 0, t>0} \frac{\mathcal{L}^n\left(\overline{B}_{\mathrm{d}_F}(M,t)\right)}{\alpha(n-p)\, t^{n-p}} = \mathcal{H}^p_{\mathbf{F}}(M).$$

Theorem 2.1 and Theorem 2.3 are equivalent. A control system $\dot{x} = f(x)\,u$, with a continuous map f and a control u with values in $\overline{B}(0,1)$, induces a Riemannian tensor

$$F = {}^t\!f^{-1}\, f^{-1},$$

and the continuity of F is obvious. Conversely, given a Riemannian tensor F, Cholesky's decomposition provides a (unique) map f such that $F = {}^t\!f^{-1}\, f^{-1}$, with $f(x)$ upper triangular for every x. The elementary operations yielding Cholesky's decomposition and the continuity of $A \mapsto A^{-1}$ in $GL_n(\mathbb{R})$, allow to derive the continuity of the map f from the continuity of F.

The equivalence between Theorem 2.1 and Theorem 2.3 follows then from the following two observations. First,

$$\tau_f = \mathrm{d}_{{}^t\!f^{-1}\, f^{-1}},$$

which implies that

$$\overline{B}_{\tau_f}(M,t) = \overline{B}_{\mathrm{d}_F}(M,t),$$

with $F = {}^t\!f^{-1}\, f^{-1}$. Second, the adapted p-dimensional Hausdorff measure to f, $\mathcal{H}^p_f$, is also obtained on compact sets by Carathéodory's construction, taking

$$\operatorname{diam}_{\tau_f,p}(K) := \sup_{x,y\in K} |\det(f(x))|^{1/p}\, \tau_f(x,y), \tag{2.5}$$

which is obviously equal to $\operatorname{diam}_{\mathbf{F},p}(K)$. Thus $\mathcal{H}^p_f$, the adapted Hausdorff measure to the dynamic f, and $\mathcal{H}^p_{\mathbf{F}}$, the adapted Hausdorff measure to the Riemannian tensor $F = {}^t\!f^{-1}\, f^{-1}$, do coincide on the set M:

$$\mathcal{H}^p_{\mathbf{F}}(M) = \mathcal{H}^p_f(M).$$

These two observations imply the equivalence between Theorem 2.2 and Theorem 2.3, hence between Theorem 2.1 and Theorem 2.3.

We prove these two facts in Section 5.

2.5. **Remarks about the dynamic.**

2.5.1. *Considering more general dynamics.* The control system that we consider immediately extends the usual euclidean distance, keeping its main properties. Among others, the distance induced by the control system, namely the minimum time function is locally Lipschitz equivalent to the euclidean distance. More precisely, the associated ball is almost the linear transformation of the euclidean ball

$$f(x_0)\,\overline{B}(x_0,t).$$

This explains why the Minkowski content extends so nicely to this control system. Of course one would want to extend the Minkowski content type result to more general control system

$$\dot{x} = f(x,u).$$

With comparisons between $\mathcal{R}_f(x_0,t)$ and sets of the type $A(x_0)\,B(x_0,\tau)$, with τ depending on t, one can expect bounds on the upper and lower Minkowski contents.

But, in general, two difficulties immediately appear, already with the linear control system

$$\dot{x} = f(x)\, u.$$

One from the configuration of the control space. The other from the rank of the linear map f. In $\mathbb{R}^2$, consider, for example,

$$f = \mathrm{id}_{\mathbb{R}^2},\ u(t) \in [0,1] \times \{0\}.$$

Or

$$f = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\ u(t) \in \overline{B}(0,1).$$

In both cases, the associated dynamic is

$$\dot{x} = \begin{pmatrix} u_1 \\ 0 \end{pmatrix}, u_1(t) \in [0,1],$$

and

$$\begin{array}{rcl} \mathcal{R}_f([0,1] \times \{0\}, t) & = & [0, 1+t] \times \{0\}; \\ \mathcal{R}_f(\{0\} \times [0,1], t) & = & [0,t] \times [0,1]. \end{array}$$

This makes a general formulation already more tricky! Although one can look at a degenerate linear map f as a limit of linear maps of full rank, and consider the possible limit of the associated formulas, given by Theorem 2.1. Such a study is of deep interest, but it goes beyond the scope of our paper.

2.5.2. *Regularity of the dynamic.* The continuity of the function f is not essential to define the adapted Hausdorff measure $\mathcal{H}^p_f$. So one will wonder what happens when considering the same problem with a non continuous map f. Several concepts of solutions exist when dealing with non continuous maps. The general study of this issue goes far beyond the scope of our paper. Let us mention that, in this direction, it will be worth studying the question for a differential inclusion

$$\dot{x} \in F(x).$$

3. Preliminary results

In this section, we give lemmas on the adapted Hausdorff measure (Section 3.1) and on reachable sets (Section 3.2), which will be crucial for the proof of Theorem 2.1, but that can also be read independently from its proof.

3.1. **On the adapted Hausdorff measure.** As noted in [CaCa06], the adapted Hausdorff measure $\mathcal{H}^p_f$ can be easily estimated by the usual Hausdorff measure $\mathcal{H}^p$ as follows. Let $\Omega \subset \mathbb{R}^n$ be such that, for some constant $\mu > 0$,

$$|\det f(x)| \geq \frac{1}{\mu}, \quad \|f(x)\| \leq \mu \qquad \forall x \in \Omega\,.$$

Then,

$$\frac{1}{c(\mu)}\, \mathcal{H}^p(E) \ \leq\ \mathcal{H}^p_f(E) \ \leq\ c(\mu)\, \mathcal{H}^p(E) \tag{3.1}$$

for every set $E \subset \Omega$ and some constant $c(\mu) \geq 1$.

If f is constant, then $\mathcal{H}^p_f$ is a simple rescaling of the usual Hausdorff measure as shown by the following lemma.

Lemma 3.1. *Let $A \in GL_n(\mathbb{R})$. Then, for every set $E \subset \mathbb{R}^n$,*

$$\mathcal{H}^p_A(E) = |\det A|\, \mathcal{H}^p(A^{-1}E).$$

Proof of Lemma 3.1. Observe that, for any set $K \subset \mathbb{R}^n$,

$$\mathrm{diam}_{A,p}(K) := |\det A|^{1/p} \sup_{x,y\in K} \left|A^{-1}(x-y)\right| = |\det A|^{1/p}\mathrm{diam}(A^{-1}K)\,.$$

Therefore, for any $\delta > 0$ and any $E \subset \mathbb{R}^n$

$$\begin{aligned}
&\mathcal{H}^p_{A,\delta}(E)\\
&= |\det A| \inf\Big\{ \frac{\alpha(p)}{2^p} \sum_{i=1}^{\infty} \big(\mathrm{diam}(A^{-1}K_i)\big)^p \;\Big|\; E \subset \bigcup_{i=1}^{\infty} K_i\,,\ \mathrm{diam}(A^{-1}K_i) \le \frac{\delta}{|\det A|^{1/p}} \Big\}\\
&= |\det A| \inf\Big\{ \frac{\alpha(p)}{2^p} \sum_{i=1}^{\infty} \big(\mathrm{diam}(K_i')\big)^p \;\Big|\; A^{-1}E \subset \bigcup_{i=1}^{\infty} K_i'\,,\ \mathrm{diam}(K_i') \le \lambda\delta \Big\}\\
&= |\det A| \mathcal{H}^p_{\lambda\delta}(A^{-1}E)\,,
\end{aligned}$$

where $\lambda := 1/|\det A|^{1/p}$. The conclusion follows as $\delta \to 0^+$. ∎

We now turn to analyze the dependance of the adapted Hausdorff measure $\mathcal{H}^p_f$ on the dynamic f.

Lemma 3.2. *Let $\Omega \subset \mathbb{R}^n$ be an open domain and let $f, g : \mathbb{R}^n \to GL_n(\mathbb{R})$ be continuous maps satisfying*

$$\forall x \in \Omega \quad \begin{cases} (a) \quad \|f(x)\|\,,\ \|g(x)\| \le \mu \\ (b) \quad |\det f(x)|\,,\ |\det g(x)| \ge \frac{1}{\mu} \end{cases} \tag{3.2}$$

for some constant $\mu > 0$. Then there is a constant $c_p(\mu) > 0$ such that

$$|\mathcal{H}^p_f(E) - \mathcal{H}^p_g(E)| \le c_p(\mu)\|f-g\|_\infty \min\{\mathcal{H}^p_f(E), \mathcal{H}^p_g(E)\} \tag{3.3}$$

for every set $E \subset \Omega$.

Proof. Observe, first, that for all sets $K \subset \Omega$

$$\mathrm{diam}_{g,p}(K) \le d_p(f,g)\, \mathrm{diam}_{f,p}(K) \tag{3.4}$$

where

$$d_p(f,g) := \sup_{x\in\Omega} \frac{|\det g(x)|^{1/p}\|g(x)^{-1}f(x)\|}{|\det f(x)|^{1/p}}\,.$$

Now, let $E \subset \Omega$, fix positive numbers ε and δ, and let $\{K_i\}_{i\in\mathbb{N}}$ be a family of subsets of Ω such that $\mathrm{diam}_{f,p}(K_i) \le \delta$, $E \subset \cup_i K_i$ and

$$\mathcal{H}^p_{f,\delta}(E) > \frac{\alpha(p)}{2^p} \sum_{i=1}^{\infty} \big(\mathrm{diam}_{f,p}(K_i)\big)^p - \varepsilon\,.$$

In view of (3.4), $\operatorname{diam}_{g,p}(K_i) \le d_p(f,g)\delta =: d_p\delta$ for all $i \ge 1$, and

$$\begin{aligned}
\mathcal{H}^p_{g,d_p\delta}(E) - \mathcal{H}^p_{f,\delta}(E) &< \frac{\alpha(p)}{2^p} \sum_{i=1}^{\infty} \Big[\big(\operatorname{diam}_{g,p}(K_i)\big)^p - \big(\operatorname{diam}_{f,p}(K_i)\big)^p \Big] + \varepsilon \\
&\le \big[d_p(f,g)^p - 1\big] \frac{\alpha(p)}{2^p} \sum_{i=1}^{\infty} \big(\operatorname{diam}_{f,p}(K_i)\big)^p + \varepsilon \\
&< \big[d_p(f,g)^p - 1\big] \Big(\mathcal{H}^p_{f,\delta}(E) + \varepsilon\Big) + \varepsilon\,.
\end{aligned}$$

Thus, since ε is arbitrary,

$$\mathcal{H}^p_{g,d_p\delta}(E) - \mathcal{H}^p_{f,\delta}(E) \le \big[d_p(f,g)^p - 1\big]\, \mathcal{H}^p_{f,\delta}(E)\,,$$

whence, as $\delta \to 0^+$,

$$\mathcal{H}^p_{g}(E) - \mathcal{H}^p_{f}(E) \le \big[d_p(f,g)^p - 1\big]\, \mathcal{H}^p_{f}(E)\,. \tag{3.5}$$

Finally, let us bound the term $d_p(f,g)^p - 1$. We have

$$\begin{aligned}
&d_p(f,g)^p - 1 \\
&\quad\le \sup_{x\in\Omega} \frac{|\det g(x)| - |\det f(x)|}{|\det f(x)|}\, \|g(x)^{-1} f(x)\|^p + \sup_{x\in\Omega} \big(\|g(x)^{-1} f(x)\|^p - 1\big)\,.
\end{aligned}$$

Since

$$|a^p - b^p| \le p(a+b)^{p-1}|b-a| \qquad \forall a,b > 0\,,$$

we conclude that

$$\begin{aligned}
\|g(x)^{-1} f(x)\|^p - 1 &\le p\big(1 + \|g(x)^{-1} f(x)\|\big)^{p-1} \big|\|g(x)^{-1} f(x)\| - 1\big| \\
&\le c_p(\mu) \big|\|g(x)^{-1} f(x)\| - 1\big|\,.
\end{aligned}$$

Moreover, owing to assumption (3.2),

$$\begin{aligned}
\big|\|g(x)^{-1} f(x)\| - 1\big| &= \big|\|g(x)^{-1} f(x)\| - \|g(x)^{-1} g(x)\|\big| \\
&\le \|g(x)^{-1}\|\, \|f(x) - g(x)\| \le c_p(M)\|f-g\|_\infty\,.
\end{aligned}$$

Furthermore, noting that $\det f$ and $\det g$ are polynomials in f_{ij} and g_{ij}, respectively, we also have

$$|\det g(x)| - |\det f(x)| \le c_p(\mu)\|f-g\|_\infty\,.$$

So,

$$d_p(f,g)^p - 1 \le c_p(\mu)\|f-g\|_\infty \tag{3.6}$$

Then, by (3.5),

$$\mathcal{H}^p_{g}(E) - \mathcal{H}^p_{f}(E) \le c_p(\mu)\|f-g\|_\infty\, \mathcal{H}^p_{f}(E)\,.$$

Exchanging f and g, we obtain

$$\mathcal{H}^p_{f}(E) - \mathcal{H}^p_{g}(E) \le c_p(\mu)\|f-g\|_\infty\, \mathcal{H}^p_{g}(E)\,,$$

whence

$$|\mathcal{H}^p_{f}(E) - \mathcal{H}^p_{g}(E)| \le c_p(\mu)\|f-g\|_\infty \max\{\mathcal{H}^p_{f}(E), \mathcal{H}^p_{g}(E)\}\,.$$

The conclusion follows noting that, on account of (3.5) and (3.6),

$$\mathcal{H}^p_{g}(E) \le d_p(f,g)^p \mathcal{H}^p_{f}(E) \le \big[1 + c_p(\mu)\|f-g\|_\infty\big]\mathcal{H}^p_{f}(E)\,. \qquad \blacksquare$$

We now give a result describing an additivity property of the adapted Hausdorff measure with respect to cubic coverings. For any $x^0 \in \mathbb{R}^n$ and any $r > 0$, we denote by $\mathcal{F}(x^0, r)$ the covering of $\mathbb{R}^n$

$$\mathcal{F}(x^0, r) = \left\{ \prod_{i=1}^{n} \left[x_i^0 + k_i r,\, x_i^0 + (k_i+1)r\right] \;\Big|\; (k_1, \ldots, k_n) \in \mathbb{Z}^n \right\}$$

made of cubes with sides of length r, parallel to the coordinate planes $x_i = 0$, and vertices in the lattice generated by x_0.

Lemma 3.3. *Let $M \subset \mathbb{R}^n$ be such that $\mathcal{H}^p(M) < \infty$. Then, for any $r > 0$ there is a countable set $D \in \mathbb{R}^n$ such that, for every $x_0 \in \mathbb{R}^n \setminus D$ and $r > 0$,*

$$\mathcal{H}_f^p(M) = \sum_{K \in \mathcal{F}(x^0, r)} \mathcal{H}_f^p(M \cap K) = \sum_{K \in \mathcal{F}(x^0, r)} \mathcal{H}_f^p(M \cap \operatorname{int} K)\,.$$

Proof of Lemma 3.3. Since

$$\sum_{K \in \mathcal{F}} \mathcal{H}_f^p(M \cap \operatorname{int} K) \le \mathcal{H}_f^p(M) \le \sum_{K \in \mathcal{F}} \mathcal{H}_f^p(M \cap K)\,,$$

it suffices to show that $\mathcal{H}_f^p(M \cap \operatorname{bd} K) = 0$ for every cube $K \in \mathcal{F}(x^0, r)$. For this purpose we note that, if

$$K = \prod_{i=1}^{n} \left[x_i^0 + k_i r,\, x_i^0 + (k_i+1)r\right]$$

for some $(k_1, \ldots, k_n) \in \mathbb{Z}^n$, then

$$M \cap \operatorname{bd} K \subset \bigcup_{i=1}^{n} \left(\left(M \cap H^i_{x_i^0 + k_i r}\right) \cup \left(M \cap H^i_{x_i^0 + (k_i+1)r}\right) \right)$$

where $H^i_\lambda = \{\ x \in \mathbb{R}^n \mid x_i = \lambda\ \}$. It is thus sufficient to show that there is a countable set $D_i \subset \mathbb{R}$ such that $\mathcal{H}_f^p(M \cap H^i_\lambda) = 0$ for all $\lambda \notin D_i$, or, recalling (3.1), such that $\mathcal{H}^p(M \cap H^i_\lambda) = 0$ for all $\lambda \notin D_i$. This can be done by a classical measure theoretic argument. Indeed, let

$$D_i := \{\ \lambda \in \mathbb{R} \mid \mathcal{H}^p(M \cap H^i_\lambda) > 0\ \}\,.$$

Then, $D_i^k := \{\ \lambda \in \mathbb{R} \mid \mathcal{H}^p(M \cap H^i_\lambda) > 1/k\ \} \uparrow D_i$ as $k \to \infty$. Moreover, D_i^k is a finite set since $\mathcal{H}^p(M) \ge \sum_{\lambda \in D_i^k} \mathcal{H}^p(M \cap H^i_\lambda)$. Therefore, D_i is at most countable. This completes the proof. ■

3.2. **On reachable sets.** The following result will be of use in the proof of Theorem 2.1. It can easily be generalized by only assuming $\|f - g\|_\infty \le \varepsilon$, adapting the values in the inclusion. Note that the second inclusion amounts to an interior ball property, see [**?**] in a more general setting.

Lemma 3.4. *Let Ω be an open subset of $\mathbb{R}^n$, let f and g be two continuous maps from Ω to $GL_n(\mathbb{R})$, let $\varepsilon > 0$ such that*

$$\sup_{(x,y) \in \Omega \times \Omega} \|f(x) - g(y)\| \le \varepsilon.$$

Let M be a compact subset of Ω. Then, for every $t \ge 0$ small enough

$$\mathcal{R}_f(M, t) \subset \mathcal{R}_g(M, t) + \overline{B}(0, \varepsilon t) \subset \mathcal{R}_g\left(M, \left(1 + \varepsilon \|g^{-1}\|_\infty\right) t\right)$$

Proof of Lemma 3.4. We denote by $X(x_0,u)$ -respectively $Y(x_0,u)$- the solution set of the equation

$$\dot{x} = f(x)\,u, \quad x(0) = x_0 \text{ -resp., } \dot{y} = f(y)\,u, \quad y(0) = x_0\text{- } u(.) \in \overline{B}(0,1).$$

We noticed, see (2.3), that all the maximal elements in $X(x_0,u)$ and $Y(x_0,u)$ -the trajectories- are defined at least till a given time. Since the set M is compact, we obtain a -positive- lower bound t_M of the time till which the trajectories starting in M are defined. Assume that $t \leq t_M$, take an element $x(t;x_0,u)$ in $\mathcal{R}_f(M,t)$. Consider a maximal element $y(.;x_0,u)$ in $Y(x_0,u)$, which is thus defined till t_M. The first inclusion follows from the observation

$$\begin{aligned} |x(t;x_0,u) - y(t;x_0,u)| &= \left| \int_0^t \dot{x}(s;x_0,u) - \dot{y}(s;x_0,u) ds \right| \\ &= \left| \int_0^t \left(f(x(s;x_0,u)) - g(y(s;x_0,u)\right) u(s) ds \right| \\ &\leq \varepsilon t. \end{aligned}$$

The second inclusion is merely an interior ball property. Since the trajectories are locally Lipschitz, $\mathcal{R}_g(M,t) \subset B(M,Kt)$ for some constant K and for t small enough. Since M is at a positive distance from $\mathbb{R}^n \setminus \Omega$, take t small enough such that

$$\mathcal{R}_g(M,t) + \overline{B}(0,\varepsilon t) \subset \Omega.$$

Take a point $y_0 = y(t;x_0,u)$ in $\mathcal{R}_g(M,t)$ and an element h in $\overline{B}(0,\varepsilon t)$. The segment $[y_0, y_0+h]$ is thus contained in Ω. Let

$$T = \max \left\| g^{-1}([y_0, y_0+h]) \right\| \; |h|$$

Take the control v defined, for $0 \leq \tau \leq T$, by

$$v(\tau) = g^{-1}\left(y_0 + \frac{\tau}{T} h\right) \frac{h}{T}$$

Then $v(\tau) \in \overline{B}(0,1)$ for every τ, and the map

$$y(\tau;y_0,v) = y_0 + \frac{\tau}{T} h$$

satisfies

$$\dot{y}(\tau;y_0,v) = \frac{h}{T} = g(y(\tau;y_0,v))v(\tau)$$

Thus

$$y_0 + h = y(\tau;y_0,v) \in \mathcal{R}_g(\mathcal{R}_g(M,t),T).$$

By the semigroup property of the reachable sets, we have

$$\mathcal{R}_g(M,t) + \overline{B}(0,\varepsilon t) \subset \mathcal{R}_g(\mathcal{R}_g(M,t),T) = \mathcal{R}_g(M,t+T) \subset \mathcal{R}_g\left(M\left(1+\varepsilon \|g^{-1}\|_\infty\right) t\right) \qquad \blacksquare$$

4. Proof of Theorem 2.1

In this section, we prove Theorem 2.1 in three steps. First, if we assume that the dynamic f is constant (Section 4.1). Then, in the general case, we show (Section 4.2) the inequality:

$$\limsup_{t \to 0, t>0} \frac{\mathcal{L}^n(\mathcal{R}_f(M,t))}{\alpha(n-p)\, t^{n-p}} \leq \mathcal{H}^p_f(M)$$

Finally, in Section 4.3, we show the reverse inequality:

$$\mathcal{H}^p_f(M) \leq \liminf_{t\to 0, t>0} \frac{\mathcal{L}^n(\mathcal{R}_f(M,t))}{\alpha(n-p)\, t^{n-p}}$$

4.1. **The constant case.** Assume that the dynamic f is constant, i.e., $f(x) = A$ for every x, with $A \in GL_n(\mathbb{R})$. Notice that

$$\mathcal{R}_A(M,t) = M + tA\overline{B}(0,1).$$

then

$$\begin{aligned} \mathcal{L}^n(\mathcal{R}_A(M,t)) &= \mathcal{L}^n\left(A\left(A^{-1}M + t\overline{B}(0,1)\right)\right) \\ &= |\det A|\mathcal{L}^n\left(A^{-1}M + t\overline{B}(0,1)\right) \end{aligned}$$

Apply Kneser's Theorem A on the Minkowski content, to the set $A^{-1}M$, and Lemma 3.1 to the set M and the linear map A

$$\begin{aligned} \lim_{t\to 0, t>0} \frac{\mathcal{L}^n(\mathcal{R}_A(M,t))}{\alpha(n-p)\, t^{n-p}} &= |\det A|\, \mathcal{H}^p(A^{-1}M) \\ &= \mathcal{H}^p_A(M)\ \blacksquare \end{aligned}$$

4.2. **Upper bound.** Since the map f is uniformly continuous on a neighborhood of the compact set M, for every positive integer ν, let $r > 0$ such that

$$|x-y| \leq (1+n^{1/n})r \Rightarrow |f(x) - f(y)| \leq \frac{1}{\nu}.$$

We cover the set M by a finite family F of cubes with disjoint interiors, each one with side length r and meeting M. Assume that r is small enough to have $K + B(0,r) \subset \Omega$ for every cube $K \in F$. We write

$$M = \cup_{K\in F} M \cap K$$

and we notice that

$$\mathcal{R}_f(M,t) = \cup_{K\in F}\mathcal{R}_f(M\cap K, t).$$

Take an element x_K in each cube K. By definition of the covering F,

$$\sup_{x\in K+B(0,r)} \|f(x) - f(x_K)\| \leq \frac{1}{\nu},$$

and from Lemma 3.4,

$$\mathcal{R}_f(M\cap K, t) \subset \mathcal{R}_{f(x_K)}\left(M\cap K, \left(1 + \frac{\|f(x_K)^{-1}\|}{\nu}\right)t\right)$$

Then

$$\mathcal{L}^n(\mathcal{R}_f(M,t)) \leq \sum_{K\in F} \mathcal{L}^n\left(\mathcal{R}_{f(x_K)}\left(M\cap K, \left(1 + \frac{\|f(x_K)^{-1}\|}{\nu}\right)t\right)\right). \tag{4.1}$$

Notice that, for every cube K, in view of Section 4.1 (Theorem 2.1 in the constant case)

$$\lim_{t\to 0, t>0} \frac{\mathcal{L}^n\left(\mathcal{R}_{f(x_K)}\left(M\cap K, \left(1+\frac{\|f(x_K)^{-1}\|}{\nu}\right)t\right)\right)}{\alpha(n-p)\, t^{n-p}} = \left(1 + \frac{\|f(x_K)^{-1}\|}{\nu}\right)^{n-p} \mathcal{H}^p_{f(x_K)}(M\cap K)$$

Divide Equation (4.1) by $\alpha(n-p)\, t^{n-p}$, take the upper limit when $t \to 0$ noticing that the sum of the upper limits is greater than the upper limit of the sum, and take C to be greater than the supremum of the $\|f(x_K)^{-1}\|$,

$$\limsup_{t\to 0, t>0} \frac{\mathcal{L}^n(\mathcal{R}_f(M,t))}{\alpha(n-p)\, t^{n-p}} \leq \left(1+\frac{C}{\nu}\right)^{n-p} \sum_{K\in F} \mathcal{H}^p_{f(x_K)}(M\cap K) \tag{4.2}$$

For every cube K, $\sup_{x\in K} \|f(x) - f(x_K)\| \leq \frac{1}{\nu}$, and from Lemma 3.2, by eventually changing the constant C,

$$\mathcal{H}^p_{f(x_K)}(M\cap K) \leq \left(1+\frac{C}{\nu}\right) \mathcal{H}^p_f(M\cap K)$$

From Lemma 3.3, it is possible to choose the cubic covering such that

$$\mathcal{H}^p_f(M) = \sum_{K\in F} \mathcal{H}^p_f(M\cap K)$$

Equation (4.2) then implies

$$\limsup_{t\to 0, t>0} \frac{\mathcal{L}^n(\mathcal{R}_f(M,t))}{\alpha(n-p)\, t^{n-p}} \leq \left(1+\frac{C}{\nu}\right)^{n-p+1} \mathcal{H}^p_f(M),$$

and taking the limit when $\nu \to \infty$,

$$\limsup_{t\to 0, t>0} \frac{\mathcal{L}^n(\mathcal{R}_f(M,t))}{\alpha(n-p)\, t^{n-p}} \leq \mathcal{H}^p_f(M) \qquad \blacksquare$$

4.3. **Lower bound.** Take an integer ν and associate a covering F like in the previous section. Recall that

$$\mathcal{R}_f(M,t) = \cup_{K\in F} \mathcal{R}_f(M\cap K, t).$$

Take an integer μ, greater than ν. For every cube K of the covering F, let K_μ be the smaller cube, at distance $\frac{1}{\mu}$ from $\mathbb{R}^n \setminus K$:

$$K_\mu = \left\{ x\in\mathbb{R}^n \,\middle|\, d(x, \mathbb{R}^n\setminus K) \geq \frac{1}{\mu} \right\}.$$

Then, for every cube K,

$$\mathcal{R}_f(M\cap K_\mu, t) \subset \mathcal{R}_f(M\cap K, t).$$

Recalling that $\sup_{x\in K+B(0,r)} \|f(x) - f(x_K)\| \leq \frac{1}{\nu}$ and by Lemma 3.4, taking C to be the supremum[2] of $\|f^{-1}(M + B(0,r))\|$, for t small enough,

$$\mathcal{R}_{f(x_K)}\left(M\cap K_\mu, \left(1+\frac{C}{\nu}\right)^{-1} t\right) \subset \mathcal{R}_f(M\cap K_\mu, t)$$

We thus have the inclusion

$$\bigcup_{K\in F} \mathcal{R}_{f(x_K)}\left(M\cap K_\mu, \left(1+\frac{C}{\nu}\right)^{-1} t\right) \subset \mathcal{R}_f(M, t).$$

The union in the left-hand side is clearly disjoint for t small enough. Then

$$\mathcal{L}^n\left(\bigcup_{K\in F} \mathcal{R}_{f(x_K)}\left(M\cap K_\mu, \left(1+\frac{C}{\nu}\right)^{-1} t\right)\right) \leq \mathcal{L}^n(\mathcal{R}_f(M,t)). \tag{4.3}$$

---

[2]since the map $A \mapsto A^{-1}$ is continuous in $GL_n(\mathbb{R})$, the map f^{-1} is continuous.

Notice that, for every cube K, in view of Section 4.1 (Theorem 2.1 in the constant case)

$$\lim_{t\to 0, t>0} \frac{\mathcal{L}^n\left(\mathcal{R}_{f(x_K)}\left(M\cap K_\mu, \left(1+\frac{C}{\nu}\right)^{-1} t\right)\right)}{\alpha(n-p)\, t^{n-p}} = \left(1+\frac{C}{\nu}\right)^{p-n} \mathcal{H}^p_{f(x_K)}(M\cap K_\mu)$$

Divide Equation (4.3) by $\alpha(n-p)\, t^{n-p}$, take the lower limit when $t\to 0$ noticing that the sum of the lower limits is smaller than the lower limit of the sum,

$$\left(1+\frac{C}{\nu}\right)^{p-n} \sum_{K\in F} \mathcal{H}^p_{f(x_K)}(M\cap K_\mu) \le \liminf_{t\to 0, t>0} \frac{\mathcal{L}^n(\mathcal{R}_f(M,t))}{\alpha(n-p)\, t^{n-p}} \tag{4.4}$$

For every cube K, $\sup_{x\in K}\|f(x)-f(x_K)\| \le \frac{1}{\nu}$, and from Lemma 3.2, by eventually changing the constant C

$$\left(1+\frac{C}{\nu}\right)^{-1} \mathcal{H}^p_f(M\cap K_\mu) \le \mathcal{H}^p_{f(x_K)}(M\cap K_\mu)$$

Thus

$$\left(1+\frac{C}{\nu}\right)^{p-n-1} \sum_{K\in F} \mathcal{H}^p_f(M\cap K_\mu) \le \liminf_{t\to 0, t>0} \frac{\mathcal{L}^n(\mathcal{R}_f(M,t))}{\alpha(n-p)\, t^{n-p}}$$

Take the limit when $\mu\to\infty$, notice that $\lim_{\mu\to\infty} \mathcal{H}^p_f(M\cap K_\mu) = \mathcal{H}^p_f(M\cap \operatorname{int} K)$:

$$\left(1+\frac{C}{\nu}\right)^{p-n-1} \sum_{K\in F} \mathcal{H}^p_f(M\cap \operatorname{int} K) \le \liminf_{t\to 0, t>0} \frac{\mathcal{L}^n(\mathcal{R}_f(M,t))}{\alpha(n-p)\, t^{n-p}} \tag{4.5}$$

From Lemma 3.3, it is possible to choose the cubic covering such that

$$\mathcal{H}^p_f(M) = \sum_{K\in F} \mathcal{H}^p_f(M\cap \operatorname{int} K)$$

Equation (4.5) then implies

$$\left(1+\frac{C}{\nu}\right)^{p-n-1} \mathcal{H}^p_f(M) \le \liminf_{t\to 0, t>0} \frac{\mathcal{L}^n(\mathcal{R}_f(M,t))}{\alpha(n-p)\, t^{n-p}}$$

and taking the limit when $\nu\to\infty$,

$$\mathcal{H}^p_f(M) \le \liminf_{t\to 0, t>0} \frac{\mathcal{L}^n(\mathcal{R}_f(M,t))}{\alpha(n-p)\, t^{n-p}} \qquad \blacksquare$$

5. Proof of the equivalence of Theorem 2.1 and Theorem 2.3

We recall that the equivalence between Theorem 2.1 and Theorem 2.3 is a direct consequence of the following two facts:

$$\tau_f = \mathrm{d}_{{}^t f^{-1} f^{-1}},$$

and

$$\mathcal{H}^p_{\mathbf{F}}(M) = \mathcal{H}^p_f(M), \text{ with } F = {}^t f^{-1} f^{-1}.$$

They imply the equivalence between Theorem 2.2 and Theorem 2.3, hence between Theorem 2.1 and Theorem 2.3. We now proceed to prove this two facts in the following two subsections.

5.1. **Minimum time function and Riemannian distance.**

Lemma 5.1. *Le Ω be a nonempty open subset of $\mathbb{R}^n$, and $f : \Omega \to \mathbb{R}^n$ be a continuous map. Then:*

$$\tau_f = \mathrm{d}_{{}^t f^{-1}\, f^{-1}}.$$

Proof of Lemma 5.1. Consider two elements y and z in Ω. take a positive real number T and a control $u : [0,+\infty) \to \overline{B}(0,1)$ such that:

$$x(T; y, u) = z.$$

Then

$$\begin{aligned} \int_0^T \left({}^t\dot{x}(t;y,u)\ \left({}^t f^{-1}\ f^{-1}\right)(x(t;y,u))\,\dot{x}(t;y,u)\right)^{1/2} dt &= \int_0^t |u(t)| dt \\ &\leq T. \end{aligned}$$

In order to keep the normalization in the definition of the distance $\mathrm{d}_{{}^t f^{-1}\, f^{-1}}$, define $\gamma(t) = x(Tt; y, u)$. Thus $\mathrm{d}_{{}^t f^{-1}\, f^{-1}}(y,z) \leq T$, and by taking the infimum on T, we obtain

$$\mathrm{d}_{{}^t f^{-1}\, f^{-1}}(y,z) \leq \tau_f(y,z).$$

Now, to prove the converse inequality, take a Lipschitz continuous $\gamma : [0,1] \to \Omega$, such that $\gamma(0) = y$, $\gamma(1) = z$ and let

$$\begin{aligned} T &= \int_0^1 \left({}^t\gamma'(t)\ \left({}^t f^{-1}\ f^{-1}\right)(\gamma(t))\,\gamma'(t)\right)^{1/2} dt \\ &= \int_0^1 \left| f(\gamma(t))^{-1}\,\gamma'(t)\right| dt. \end{aligned}$$

Define $\psi : [0,1] \to \mathbb{R}_+$ by

$$\psi(t) = \int_0^t \left| f(\gamma(t))^{-1}\,\gamma'(t)\right| ds.$$

Assume that $\gamma'(t) \neq 0$ for almost every t. Then ψ is absolutely continuous, increasing, $\psi(0) = 0$, $\psi(1) = t$ and

$$\psi'(t) = \left| f(\gamma(t))^{-1}\,\gamma'(t)\right|.$$

Moreover, since $\psi'(t) \neq 0$ for a.e. t, the map ψ^{-1} is also absolutely continuous. Define:

$$\begin{aligned} x(t) &= \gamma(\psi^{-1}(t)) \qquad \text{for } t \in [0,T] \\ u(t) &= \frac{f(x(t))^{-1}\,\gamma'(\psi^{-1}(t))}{|f(x(t))^{-1}\,\gamma'(\psi^{-1}(t))|} \quad \text{for a.e. } t \in [0,T]. \end{aligned}$$

The map u is measurable and has values in $\overline{B}(0,1)$. Having in mind that $\gamma \circ \psi^{-1}$ is absolutely continuous, notice that

$$\begin{aligned} \int_0^t f(x(s))\,u(s) ds &= \int_0^t \frac{\gamma'(\psi^{-1}(s))}{|f(x(s))^{-1}\,\gamma'(\psi^{-1}(s))|} ds \\ &= \int_0^t \frac{\gamma'(\psi^{-1}(s))}{\psi'(\psi^{-1}(s))} ds \\ &= \gamma(\psi^{-1}(t)) - \gamma(\psi^{-1}(0)) = x(t) - y. \end{aligned}$$

Hence $([0,T],x) \in X(y,u)$. Moreover

$$x(T) = \gamma(\psi^{-1}(T)) = \gamma(1) = z,$$

hence $\tau_f(y,z) \leq T$. Taking the infimum on T

$$\tau_f(y,z) \leq \mathrm{d}_{{}^t f^{-1}\, f^{-1}}(y,z).$$

Now, if $\gamma'(t) = 0$ on a set of positive measure, a simple way is to define

$$\psi(t) = \int_0^t \left|f(\gamma(s))^{-1}\gamma'(s)\right| + \varepsilon \; ds$$

and to observe that the above proof can be adapted with minor modifications, letting $\varepsilon \to 0$ at the end. ■

5.2. **Adapted Hausdorff measures.**

Lemma 5.2. *Le Ω be a nonempty open subset of $\mathbb{R}^n$, and $f : \Omega \to \mathbb{R}^n$ be a continuous map. Let M be a compact subset of Ω, and let $F = {}^t f^{-1}\, f^{-1}$. Then:*

$$\mathcal{H}^p_{\mathbf{F}}(M) = \mathcal{H}^p_f(M).$$

Proof of Lemma 5.2. Since

$$\tau_f = \mathrm{d}_{{}^t f^{-1}\, f^{-1}}$$

from Lemma 5.1, recalling the definitions of $\mathrm{diam}_{\tau_f,p}$ (2.5) and $\mathrm{diam}_{\mathbf{F},p}$ (2.4), we have

$$\mathrm{diam}_{\tau_f,p} = \mathrm{diam}_{\mathbf{F},p}.$$

Thus the proof of Lemma 5.2 is finished if we prove that $\mathcal{H}^p_f$, the adapted Hausdorff measure to the dynamic f of the set M, is also obtained by Carathéodory's construction with $\mathrm{diam}_{\tau_f,p}$ (instead of $\mathrm{diam}_{f,p}$). This is achieved with a comparison between $\mathrm{diam}_{f,p}$ and $\mathrm{diam}_{\tau_f,p}$.

Take $r > 0$ such that

$$\overline{B}(M,r) \subset \Omega$$

and $K > 1$ such that, for every $x \in \overline{B}(M,r)$,

$$\|f(x)\| \leq K, \qquad \|f(x)^{-1}\| \leq K, \qquad \frac{1}{K} \leq |\det f(x)|^{\frac{1}{p}}.$$

For $\varepsilon > 0$, using the uniform continuity of f and f^{-1} on the set $\overline{B}(M,r)$, take $\alpha > 0$ such that

$$\forall (x,y) \in \overline{B}(M,r)^2,\ |x-y| < \alpha \Rightarrow \left\{ \begin{array}{rcl} \left\|f(x)^{-1}f(y)\right\| & < & 1+\varepsilon \\ \left\|f(x)^{-1} - f(y)^{-1}\right\| & < & \varepsilon \end{array} \right.$$

Take $\delta > 0$ small enough ($\delta < \frac{r}{K^2}$, $\delta < \frac{\alpha}{K^4}$). Let $(K_i)_{i\in\mathbb{N}}$ be a countable covering of M, such that $\mathrm{diam}_{f,p}(K_i) \leq \delta$. We assume that $K_i \subset M$ (replace K_i by $K_i \cap M$). Take a set K_i, and two points y and z in K_i. Then

$$|\det f(y)|^{\frac{1}{p}} |f(y)^{-1}(y-z)| \leq \mathrm{diam}_{f,p}(K_i) \leq \delta,$$

hence

$$|f(y)^{-1}(y-z)| \leq K\delta$$

and

$$|y-z| \leq K^2\delta \leq r,$$

which implies that -since y (and z) belongs to M-

$$[y,z] \subset \overline{B}(M,r)$$

hence $[y,z]\subset\Omega$. Define
$$\gamma(t)=tz+(1-t)y.$$
In view of Lemma 5.1 and of the definition of the Riemannian distance $\mathrm{d}_{{}^tf^{-1}f^{-1}}$:
$$\begin{aligned}
\tau_f(y,z) &\leq \int_0^1 |f(\gamma(t))^{-1}\gamma'(t)|\,dt\\
&\leq \int_0^1 |f(tz+(1-t)y)^{-1}(z-y)|\\
&\leq |f(y)^{-1}(z-y)|+\sup_{t\in[0,1]}\|f(tz+(1-t)y)^{-1}-f(y)^{-1}\|\,|z-y|.
\end{aligned}$$
But $|z-y|\leq K^2\delta\leq\alpha$, and
$$\|f(tz+(1-t)y)^{-1}-f(y)^{-1}\|\leq\varepsilon \text{ for every } t\in[0,1].$$
Thus
$$\begin{aligned}
\tau_f(y,z) &\leq |f(y)^{-1}(z-y)|+\varepsilon|z-y|\\
&\leq (1+K\varepsilon)\left|f(y)^{-1}(z-y)\right|.
\end{aligned}$$
So, we obtain that
$$\mathrm{diam}_{\tau_f,p}(K_i)\leq(1+K\varepsilon)\,\mathrm{diam}_{f,p}(K_i),$$
and
$$\mathcal{H}^p_{\mathbf{F},\delta}(M)\leq(1+K\varepsilon)\,\mathcal{H}^p_{f,\delta}(M).$$
At the limit when $\delta\to 0$,
$$\mathcal{H}^p_{\mathbf{F}}(M)\leq(1+K\varepsilon)\,\mathcal{H}^p_{f}(M).$$
and, when $\varepsilon\to 0$,
$$\mathcal{H}^p_{\mathbf{F}}(M)\leq\mathcal{H}^p_{f}(M).$$

Now, to prove the other inequality, take a countable covering $(K_i)_{i\in\mathbb{N}}$ be a countable covering of M, such that $\mathrm{diam}_{\tau_f,p}(K_i)\leq\delta$, and such that $K_i\subset M$ (as noted above, possibly replacing K_i by $K_i\cap M$). Take a set K_i, and two points y and z in K_i. Then
$$|\det f(y)|^{\frac{1}{p}}\,|\tau_f(y,z)|\leq\mathrm{diam}_{\tau_f,p}(K_i)\leq\delta,$$
hence
$$|\tau_f(y,z)|\leq K\delta.$$
Take t such that $x(t;y,u)=z$. Assume that the trajectory $x(.;y,u)$ remains in th set $\overline{B}(M,r)$, i.e., $x(s;y,u)\in\overline{B}(M,r)$ for every $s\in[0,t]$. Let $x(.)=x(.;y,u)$. Then, for a.e s,
$$|\dot{x}(s)|=|f(x(s))u(s)|\leq K$$
and
$$|z-y|\leq Kt,$$
hence
$$|z-y|\leq K\tau_f(y,z)\leq K^2\delta.$$
Now, if the trajectory does not remain in $\overline{B}(M,r)$, let
$$\tau=\inf\{s\mid x(s)\notin B(M,r)\}.$$
Then $x([0,\tau])\subset\overline{B}(M,r)$ and
$$r=d(M,x(\tau))\leq|x(\tau)-y|\leq K\tau\leq Kt.$$

If t has been chosen close enough to $\tau_f(y,z)$, we obtain a contradiction with $\delta < \frac{r}{K^2}$. Thus,

$$|z-y| \le K^2\delta < \frac{r}{K^2}.$$

Define

$$\varphi(s) = f(y)^{-1}x(s).$$

Then

$$\varphi'(s) = f(y)^{-1}f(x(s))u(s).$$

If, for every s, $|x(s)-y| < \alpha$, then $\|f(y)^{-1}f(x(s))\| \le 1+\varepsilon$ and $\varphi'(s) \le 1+\varepsilon$, hence

$$\left|f(y)^{-1}(y-z)\right| = |\varphi(t)-\varphi(0)| \le (1+\varepsilon)t$$

hence

$$\left|f(y)^{-1}(y-z)\right| \le (1+\varepsilon)\tau_f(y,z).$$

If, for some s, $|x(s)-y| \ge \alpha$, take

$$\tau = \inf\{s \mid |x(s)-y| \ge \alpha\}.$$

Then $|x(\tau)-y| = \alpha$, $x([0,\tau]) \subset \overline{B}(y,\alpha)$, and

$$\left|f(y)^{-1}(x(\tau)-y)\right| \le (1+\varepsilon)\tau \le (1+\varepsilon)t.$$

Also,

$$\begin{aligned}
\left|f(y)^{-1}(y-z)\right| &\le K|y-z| \le K\frac{\alpha}{K^2} \\
&\le \frac{1}{K}|x(\tau)-y| = \frac{1}{K}\left|f(y)f(y)^{-1}(x(\tau)-y)\right| \\
&\le \left|f(y)^{-1}(x(\tau)-y)\right| \le (1+\varepsilon)t.
\end{aligned}$$

Hence

$$\left|f(y)^{-1}(y-z)\right| \le (1+\varepsilon)\tau_f(y,z).$$

We thus obtain

$$\mathrm{diam}_{f,p}(K_i) \le (1+K\varepsilon)\,\mathrm{diam}_{\tau_f,p}(K_i),$$

and

$$\mathcal{H}^p_{f,\delta}(M) \le (1+K\varepsilon)\,\mathcal{H}^p_{\mathbf{F},\delta}(M).$$

At the limit when $\delta \to 0$,

$$\mathcal{H}^p_f(M) \le (1+K\varepsilon)\,\mathcal{H}^p_{\mathbf{F}}(M).$$

and, when $\varepsilon \to 0$,

$$\mathcal{H}^p_f(M) \le \mathcal{H}^p_{\mathbf{F}}(M). \blacksquare$$

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