Entire solutions of the one-dimensional Perona-Malik equation

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Abstract

We prove that every function $u : \mathbb{R}^2 \to \mathbb{R}$ of class C^1, satisfying the Perona-Malik equation

$$u_t = \left(\frac{u_x}{1 + u_x^2} \right)_x$$

for every $(x, t) \in \mathbb{R}^2$, is a stationary affine solution, *i.e.* of the form $u(x, t) = ax + b$, where a and b are suitable real constants.

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1 Introduction

In this paper we consider the one-dimensional Perona-Malik equation

$$u_t = \left(\frac{u_x}{1+u_x^2}\right)_x = \frac{1-u_x^2}{(1+u_x^2)^2}\, u_{xx}. \tag{1.1}$$

This PDE is a *forward* parabolic equation where $|u_x| < 1$, and a *backward* parabolic equation where $|u_x| > 1$. If we set

$$\varphi(\sigma) := \frac{1}{2}\log(1+\sigma^2), \tag{1.2}$$

equation (1.1) can be written as

$$u_t = (\varphi'(u_x))_x = \varphi''(u_x)u_{xx}, \tag{1.3}$$

hence as the formal gradient flow of the integral functional

$$PM(u) = \int \varphi(u_x)\, dx.$$

The forward-backward character of (1.1) depends on the convex-concave behaviour of $\varphi(\sigma)$.

The n-dimensional analogue of (1.1) is the PDE

$$u_t = \operatorname{div}\left(\frac{\nabla u}{1+|\nabla u|^2}\right), \tag{1.4}$$

which is the formal gradient flow of the integral functional defined as above with u_x replaced by $|\nabla u|$.

Equation (1.4) with $n = 2$ was introduced by P. Perona and J. Malik [7] in the context of vision theory. They considered (1.4) on a cylinder $\{(x,t) : \ x \in \Omega, \ t \geq 0\}$, where $\Omega \subseteq \mathbb{R}^2$ is a bounded open set (for example a rectangle), with an initial datum $u(x,0) = u_0(x)$, and Neumann boundary conditions on $\partial\Omega$. Thinking the initial datum as the grey level of an image, the evolution according to (1.4) should heuristically smooth the zones where $|\nabla u_0| < 1$, and enhance the zones with $|\nabla u_0| > 1$, which may be interpreted as the edges of the image. This is what they actually observed in numerical simulations.

From the mathematical point of view, the situation seems to be not so good, due to the forward-backward character of these equations: indeed, according to [1], "no uniqueness of the solution and no stability of the process can be expected". This pessimistic position is partially supported by a result of K. Höllig [4], who gave an example of a function φ for which (1.1) admits infinitely many local solutions for large classes of initial data. However, in this example φ' is not smooth, but only piecewise affine, and also solutions are not classical, but only Lipschitz continuous.

The research on Perona-Malik equations has developed along two mail lines:

- finding a suitable notion of weak solution, consistent with numerical results;

- finding properties of classical solutions.

As far as we know, very little progress has been made in both directions.

A partial result in the first direction was provided by S. Kichenassamy [6]: he proposed a notion of generalized solutions to (1.1), which are piecewise affine and allow jumps, and he showed that their moving and merging are similar to those observed in numerical simulations.

In the second direction, B. Kawohl and N. Kutev [5] considered equation (1.1) in a bounded interval, with Neumann conditions at the endpoints, and an initial datum $u(0, x) = u_0(x)$ of class C^1. Under a technical assumption on $u_0(x)$, they showed that this problem admits at most one local solution of class C^1, which cannot exist for every $t \geq 0$ if $|u_{0x}| > 1$ at some point. However, existence of such a solution on a finite time interval is still an open problem.

In this paper we consider a different problem raised by E. De Giorgi in [3], concerning entire solutions of (1.1), *i.e.* classical solutions defined for every $(x, t) \in \mathbb{R}^2$. We prove that all such solutions are trivial. Our main result is the following.

Theorem 1.1 *Let $u : \mathbb{R}^2 \to \mathbb{R}$ be a C^1 solution of (1.1).*
Then there exist $a \in \mathbb{R}$, and $b \in \mathbb{R}$ such that

$$u(x, t) = ax + b \qquad \forall (x, t) \in \mathbb{R}^2.$$

We point out that the conclusion of Theorem 1.1 is false for weaker solutions, *e.g.* Lipschitz solutions or distributional solutions allowing jumps (see Example 6.1 and Example 6.2). It is also false for large classes of forward parabolic equations (hence also for backward parabolic equations, since it is enough to reverse the time): for example the heat equation admits a lot of entire solutions, such as $u(x, t) = e^t \sin x$.

It could be interesting better understanding which properties of the nonlinearity φ determine the nonexistence of nontrivial entire solutions of the diffusion process (1.3) (see the discussion in section 7).

Our proof of Theorem 1.1 is based on some estimates and qualitative properties of solutions of (1.1), which we think to be interesting in themselves. Just to give an example, such estimates allow to provide an alternative proof of the nonexistence result of [5] quoted above, without any technical assumption on $u_0(x)$ (see section 5).

This paper is organized as follows: in section 2 we give some preliminaries about classical solutions of (1.1); in section 3 we state the main tools used in our proof of Theorem 1.1; in section 4 we prove our results; in section 5 we present some different applications of our estimates; in section 6 we show some examples and counterexamples; in section 7 we collect some open problems.

2 Classical solutions

Throughout this paper, φ denotes the function defined by (1.2).

In this section we fix some definitions of classical solution of (1.1), and we state some properties of such solutions.

A classical solution is of course a solution which has all the derivatives written in the PDE, but in this case there are at least three possibilities.

Definition 2.1 Let $A \subseteq \mathbb{R}^2$ be an open set, and let $u : A \to \mathbb{R}$.

- We say that u is a C^1 solution of (1.1) in A if u has continuous partial derivatives u_t and u_x, and
$$u_t = (\varphi'(u_x))_x$$
for every $(x, t) \in A$. Note that this means in particular that $\varphi'(u_x)$ has a continuous partial derivative with respect to x, which coincides with u_t.

- Given $\alpha \in (0, 1]$, we say that u is a $C^{1,\alpha}$ solution of (1.1) in A if u is a C^1 solution, and, for every $t \in \mathbb{R}$, the function $x \to u_x(x, t)$ is locally α-Hölder continuous (Lipschitz continuous if $\alpha = 1$) whenever defined.

- We say that u is a C^2 solution of (1.1) in A if u is a C^1 solution, and has the partial derivative u_{xx} which is continuous in A. In this case the Perona-Malik equation is satisfied in the stronger form
$$u_t = \varphi''(u_x)u_{xx}$$
for every $(x, t) \in A$. Note that $C^{1,1}$ solutions satisfy this stronger form of the equation for almost every $(x, t) \in A$. Note also that C^2 solutions are not necessarily C^2 functions, since u_{xx} is the only second order derivative which is required to exist.

Theorem 1.1 deals with classical solutions in the weaker sense, *i.e.* C^1 solutions. This sometimes complicates proofs, but every time this happens, we begin by showing the proof for more regular solutions, and then we explain how to modify it for C^1 solutions.

In this paper we do not investigate deeply neither the relations between these three definitions, nor the further regularity of such solutions. However, for the convenience of the reader, we recall what we know at the present:

- we can have regularity problems only where $|u_x| = 1$ (see statement (1) of Proposition 2.2 below);
- any C^1 solution is actually $C^{1,1/2}$ (see statement (2) of Proposition 2.2 below);
- there exist $C^{1,1}$ solutions which are not C^2 (see Example 6.4), and so one cannot expect all C^1 solutions to be very regular;
- there exist solutions which are analytic in a neighborhood of a point where $\varphi''(u_x)$ changes its sign (see Example 6.3), and so one cannot expect all C^1 solutions to develop singularities in higher order derivatives across zeroes of $\varphi''(u_x)$.

Proposition 2.2 *Let $A \subseteq \mathbb{R}^2$ be an open set, and let $u : A \to \mathbb{R}$ be a C^1 solution of (1.1) in A. Then*

(1) $u \in C^\infty(A_0)$, where $A_0 := \{(x,t) \in A : \ |u_x(x,t)| \neq 1\}$;

(2) u is a $C^{1,1/2}$ solution in A.

The first statement follows from the classical regularity theory for parabolic equations. The second statement is a consequence of the following result, applied with $f(x) = \varphi'(x)$ and $g(x) = u_x(x,t)$ for any fixed t.

Lemma 2.3 *Let $f : \mathbb{R} \to \mathbb{R}$ be a function of class C^2 such that $|f'(x)| + |f''(x)| > 0$ for every $x \in \mathbb{R}$. Let $I \subseteq \mathbb{R}$ be an open set, and let $g : I \to \mathbb{R}$ be a continuous function.*

If the function $f(g(x))$ is locally Lipschitz continuous in I, then $g(x)$ is locally 1/2-Hölder continuous in I.

We leave the details of the proof of this Lemma to the interested reader. The main idea is the following: if $f'(g(x_0)) \neq 0$, then $g(x) = f^{-1}(f(g(x)))$ is even Lipschitz continuous in a neighborhood of x_0; if $f'(g(x_0)) = 0$, then things work as in the well known case where $f(x) = x^2$.

3 Statement of the main tools

In this section we state the estimates and qualitative properties of solutions, which we need in the proof of Theorem 1.1.

The first result is a symmetry property of solutions: roughly speaking, it says that if we take the inverse of a solution, and we reverse the time, then we get another solution.

Theorem 3.1 *Let $u : \mathbb{R}^2 \to \mathbb{R}$ be a C^1 solution of (1.1).*

Let us assume that there exists a constant ν such that

$$u_x(x,t) \geq \nu > 0 \qquad \forall (x,t) \in \mathbb{R}^2. \tag{3.1}$$

Then the function $v : \mathbb{R}^2 \to \mathbb{R}$, uniquely defined by

$$u(v(x,t), -t) = x \qquad \forall (x,t) \in \mathbb{R}^2, \tag{3.2}$$

is a C^1 solution of (1.1).

The second result is an estimate of a suitable integral of u_x. Here we state it for C^2 solutions, but in Remark 3.5 below we show how it can be used also for C^1 solutions.

Theorem 3.2 *Let $u : \mathbb{R}^2 \to \mathbb{R}$ be a C^2 solution of (1.1), let $r \in C_0^\infty(\mathbb{R})$, and let $\psi \in C^2(\mathbb{R})$ be a function such that*

(i) $\psi''(\sigma) \cdot \varphi''(\sigma) \geq 0$ for every $\sigma \in \mathbb{R}$;

(ii) there exists a nonnegative function $c : \mathbb{R} \to [0, +\infty)$ such that

$$[\psi'(\sigma)]^2 \cdot |\varphi''(\sigma)| \leq c(\sigma)\, |\psi''(\sigma)| \qquad \forall \sigma \in \mathbb{R}.$$

Then for every $t_1 \leq t_2$ we have that

$$\left[\int_{\mathbb{R}} r^2 \psi(u_x)\, dx \right]_{t_1}^{t_2} + \frac{1}{2} \int_{t_1}^{t_2} \int_{\mathbb{R}} r^2 \psi''(u_x) \varphi''(u_x) u_{xx}^2 \, dx\, dt \leq 2 \int_{t_1}^{t_2} \int_{\mathbb{R}} r_x^2\, c(u_x)\, dx\, dt, \tag{3.3}$$

where

$$\left[\int_{\mathbb{R}} r^2 \psi(u_x)\, dx \right]_{t_1}^{t_2} := \int_{\mathbb{R}} r^2(x) \psi(u_x(x,t_2))\, dx - \int_{\mathbb{R}} r^2(x) \psi(u_x(x,t_1))\, dx.$$

The next result shows that the forward region (*i.e.* the set of points where $|u_x| < 1$) is nondecreasing in time. A similar result was proved in [5] for the initial boundary value problem.

Theorem 3.3 *Let $u : \mathbb{R}^2 \to \mathbb{R}$ be a C^1 solution of (1.1), and let (x_0, t_0) be such that $|u_x(x_0, t_0)| < 1$.*

Then $|u_x(x_0, t)| < 1$ for every $t \geq t_0$.

As a consequence, the following implications are true:

$$u_x(x_0,t_0) \geq 1 \implies u_x(x_0,t) \geq 1 \qquad \forall t \leq t_0;$$
$$u_x(x_0,t_0) \leq -1 \implies u_x(x_0,t) \leq -1 \qquad \forall t \leq t_0.$$

The last result is a concentration property, which prevents nontrivial solutions from becoming affine as $t \to -\infty$.

Theorem 3.4 *Let $u : \mathbb{R}^2 \to \mathbb{R}$ be a C^1 solution of (1.1). Let us assume that u_x is not a constant (*i.e. *u is not a stationary affine solution), and that $|u_x(x_0, t_0)| < 1$ for some $(x_0, t_0) \in \mathbb{R}^2$.*

Then there exist $\gamma > 0$, $t_\star \in \mathbb{R}$, and a bounded interval $I \subseteq \mathbb{R}$ such that, setting

$$I_+(t) := \{x \in I : |u_x(x,t)| \leq 1\},$$

one has that

$$\int_{I_+(t)} \left[\varphi''(u_x) u_{xx}\right]^2 \, dx \geq \gamma \qquad \forall t \leq t_\star. \tag{3.4}$$

As it is stated here, this result seems to require that u is of class C^2. Nevertheless, for C^1 solutions the term $\varphi''(u_x)u_{xx}$ in (3.4) can be replaced by $(\varphi'(u_x))_x$ or u_t.

Remark 3.5 We conclude with a few comments on Theorem 3.2.

(1) If we know a priori that $|u_x(x,t)| \leq 1$ for every (x,t) such that $r(x) > 0$ and $t \in [t_1, t_2]$, then assumptions (i) and (ii) are to be checked only for $|\sigma| \leq 1$.

(2) If $\psi = \varphi$, then assumptions (i) and (ii) of Theorem 3.2 are satisfied, with $c(\sigma) := [\varphi'(1)]^2$. Moreover, in this case inequality (3.3) holds true also for C^1 solutions, since the term $[\varphi''(u_x)u_{xx}]^2$ is actually u_t^2. The same is true also if ψ coincides with φ only in a neighborhood of $\sigma = -1$ and $\sigma = 1$.

(3) If u is a C^1 solution, then the term $\psi''(u_x)\varphi''(u_x)u_{xx}^2$ is not well defined. However, under the same assumptions on ψ, the following weaker form of (3.3) holds true:

$$\left[\int_{\mathbb{R}} r^2 \psi(u_x) \, dx\right]_{t_1}^{t_2} \leq 2 \int_{t_1}^{t_2} \int_{\mathbb{R}} r_x^2 \, c(u_x) \, dx \, dt. \tag{3.5}$$

4 Proofs

Before we enter in the details of the proofs, we choose once for all a cut-off function $\eta \in C_0^\infty(\mathbb{R})$ such that

- $\eta(\sigma) = 1$ if $|\sigma| \leq 1$,
- $\eta(\sigma) = 0$ if $|\sigma| \geq 2$,
- $0 < \eta(\sigma) < 1$ if $1 < |\sigma| < 2$,

and we set

$$\|\eta'\|_\infty := \sup\{|\eta'(\sigma)| : \ \sigma \in \mathbb{R}\} < +\infty.$$

4.1 Proof of Theorem 3.1

Assumption (3.1) guarantees that the function $x \to u(x,-t)$ is injective and surjective for every $t \in \mathbb{R}$, hence has an inverse function $x \to v(x,t)$ uniquely defined by (3.2). By standard calculus arguments, it can be proved that v is of class C^1, and

$$v_t(x,t) = \frac{u_t(v(x,t),-t)}{u_x(v(x,t),-t)}, \qquad\qquad v_x(x,t) = \frac{1}{u_x(v(x,t),-t)},$$

hence

$$\frac{v_x(x,t)}{1+v_x^2(x,t)} = \frac{u_x(v(x,t),-t)}{1+u_x^2(v(x,t),-t)},$$

and finally by the chain rule

$$\frac{\partial}{\partial x}\left(\frac{v_x(x,t)}{1+v_x^2(x,t)}\right) = \left(\frac{\partial}{\partial x}\frac{u_x}{1+u_x^2}(v(x,t),-t)\right)\cdot v_x(x,t) = \frac{u_t(v(x,t),-t)}{u_x(v(x,t),-t)} = v_t(x,t).$$

This proves that v solves (1.1). $\square$

4.2 Proof of Theorem 3.2

To begin with, let us assume that u is more regular (C^3 should be enough).

Computing the time derivative, and using integration by parts, we have that

$$\begin{aligned}
\frac{d}{dt}\int_{\mathbb{R}} r^2\psi(u_x)\,dx &= \int_{\mathbb{R}} r^2\psi'(u_x)u_{xt}\,dx \\
&= \int_{\mathbb{R}} r^2\psi'(u_x)\left(\varphi'(u_x)\right)_{xx}\,dx \\
&= -\int_{\mathbb{R}} \left(r^2\psi'(u_x)\right)_x\left(\varphi'(u_x)\right)_x\,dx \\
&= -2\int_{\mathbb{R}} rr_x\psi'(u_x)\varphi''(u_x)u_{xx}\,dx - \int_{\mathbb{R}} r^2\psi''(u_x)\varphi''(u_x)u_{xx}^2\,dx.
\end{aligned}$$

This proves that

$$\frac{d}{dt}\int_{\mathbb{R}} r^2\psi(u_x)\,dx + \int_{\mathbb{R}} r^2\psi''(u_x)\varphi''(u_x)u_{xx}^2\,dx = -2\int_{\mathbb{R}} rr_x\psi'(u_x)\varphi''(u_x)u_{xx}\,dx. \qquad (4.1)$$

Using our assumptions on ψ, and inequality $2ab \le \varepsilon a^2 + \varepsilon^{-1}b^2$, we have that

$$-2\int_{\mathbb{R}} rr_x\psi'(u_x)\varphi''(u_x)u_{xx}\,dx \le 2\int_{\mathbb{R}} |r|\cdot|r_x|\cdot|\psi'(u_x)|\cdot|\varphi''(u_x)|\cdot|u_{xx}|\,dx$$

$$
\begin{aligned}
&\leq\ 2\int_{\mathbb{R}}|r|\cdot|r_x|\cdot\left\{|\psi'(u_x)|^2\cdot|\varphi''(u_x)|\right\}^{1/2}\cdot|\varphi''(u_x)|^{1/2}\cdot|u_{xx}|\,dx\\
&\leq\ 2\int_{\mathbb{R}}|r|\cdot|r_x|\cdot|\psi''(u_x)\varphi''(u_x)|^{1/2}\cdot|c(u_x)|^{1/2}\cdot|u_{xx}|\,dx\\
&\leq\ \left\{\varepsilon\int_{\mathbb{R}}r^2\psi''(u_x)\varphi''(u_x)u_{xx}^2\,dx+\frac{1}{\varepsilon}\int_{\mathbb{R}}r_x^2\,c(u_x)\,dx\right\}
\end{aligned}
$$

Setting $\varepsilon = 1/2$, and substituting in (4.1), we obtain that

$$
\frac{d}{dt}\int_{\mathbb{R}}r^2\psi(u_x)\,dx+\frac{1}{2}\int_{\mathbb{R}}r^2\psi''(u_x)\varphi''(u_x)u_{xx}^2\,dx\leq 2\int_{\mathbb{R}}r_x^2\,c(u_x)\,dx.
$$

Integrating in $[t_1, t_2]$ with respect to t, estimate (3.3) is proved for regular solutions.

If u is only a C^2 solution, then a standard approximation procedure is necessary. To begin with, one takes a family $\{u_\varepsilon\}$ of C^∞ approximations of u, obtained *e.g.* by convolution. It turns out that u_ε solves an approximate Perona-Malik equation

$$
u_{\varepsilon t}=\varphi''(u_{\varepsilon x})u_{\varepsilon xx}+\rho_\varepsilon,
$$

where $\rho_\varepsilon(x,t)$ tends to zero uniformly on compact subsets of $\mathbb{R}^2$. Arguing as before, one proves that u_ε satisfies an inequality similar to (3.3), with some extra terms depending on ρ_ε, which disappear as $\varepsilon \to 0^+$. We spare the reader from the details.

A similar argument proves that the weaker inequality (3.5) holds true also for C^1 solutions. $\square$

4.3 Proof of Theorem 3.3

Lemma 4.1 *There exists a family* $\{\psi_\varepsilon\}_{0<\varepsilon<1} \subseteq C^2(\mathbb{R})$ *such that*

(ψ_ε1) $\psi_\varepsilon''(\sigma)\cdot\varphi''(\sigma)\geq 0$ *for every* $\sigma\in\mathbb{R}$ *and* $\varepsilon\in(0,1)$;

(ψ_ε2) $[\psi_\varepsilon'(\sigma)]^2\cdot|\varphi''(\sigma)|\leq|\psi_\varepsilon''(\sigma)|$ *for every* $\sigma\in\mathbb{R}$ *and* $\varepsilon\in(0,1)$;

(ψ_ε3) $\psi_\varepsilon(\sigma)\geq 0$ *for every* $\sigma\in\mathbb{R}$ *and* $\varepsilon\in(0,1)$;

(ψ_ε4) for every $\delta\in(0,1)$, *there exists a constant* c_δ *such that*

$$
\psi_\varepsilon(\sigma)\leq\frac{c_\delta}{1-\sigma^2}\qquad\forall\sigma\in[-\delta,\delta],\ \forall\varepsilon\in(0,1);
$$

(ψ_ε5) one has that

$$
\lim_{\varepsilon\to 0^+}\psi_\varepsilon(\sigma)=\begin{cases}(1-\sigma^2)^{-1} & \textit{if } |\sigma|<1\\ +\infty & \textit{if } |\sigma|\geq 1.\end{cases}
$$

Proof. One can take for example

$$\psi_\varepsilon(\sigma) := \frac{1}{\varepsilon}\left(\frac{\pi}{2} + \arctan\frac{\sigma^2 + 1 - \sqrt{4-\varepsilon^2}}{\varepsilon}\right). \quad \square$$

We are now ready to prove Theorem 3.3.

Let us assume for simplicity that $t_0 = 0$, and let $\rho > 0$ and $\delta > 0$ be such that

$$u_x^2(x,0) \leq 1 - \delta \qquad \forall x \in [x_0 - 4\rho, x_0 + 4\rho]. \tag{4.2}$$

Given $t > 0$, we first claim that

$$|u_x(t,x)| < 1 \text{ for almost every } x \in [x_0 - 2\rho, x_0 + 2\rho], \tag{4.3}$$

and

$$\int_{x_0-2\rho}^{x_0+2\rho} \frac{dx}{1 - u_x^2(x,t)} \leq c_1(\delta,\rho) + t c_2(\delta,\rho), \tag{4.4}$$

where $c_1(\delta,\rho)$ and $c_2(\delta,\rho)$ are two constants depending on δ and ρ.

If the function $x \to u_x(x,t)$ is Lipschitz continuous, then this is enough to conclude that $|u_x(x,t)| < 1$ for every $x \in [x_0 - 2\rho, x_0 + 2\rho]$. However, from Proposition 2.2 we know only that this function is 1/2-Hölder continuous. We prove then that

$$\int_{x_0-\rho}^{x_0+\rho} \frac{dx}{[1 - u_x^2(x,t)]^2} < +\infty, \tag{4.5}$$

and this is enough to conclude that $|u_x(x,t)| < 1$ for every $x \in [x_0 - \rho, x_0 + \rho]$, hence in particular for $x = x_0$.

In order to prove (4.3) and (4.4), we would like to use Theorem 3.2 with $\psi(\sigma) = (1-\sigma^2)^{-1}$, but this cannot be done directly *e.g.* because this function is not defined on the whole real line.

To overcome this difficulty, we apply Theorem 3.2 to the family $\{\psi_\varepsilon\}$ given by Lemma 4.1. This is possible since $(\psi_\varepsilon 1)$ and $(\psi_\varepsilon 2)$ coincide with assumptions (i) and (ii) of Theorem 3.2 (with $c(\sigma) \equiv 1$). Choosing $r(x) = \eta((x - x_0)/(2\rho))$, and $[t_1, t_2] = [0, t]$, from inequality (3.3) we obtain that

$$\int_{\mathbb{R}} r^2(x)\psi_\varepsilon(u_x(x,t))\,dx \leq \int_{\mathbb{R}} r^2(x)\psi_\varepsilon(u_x(x,0))\,dx + 2t\int_{\mathbb{R}} r_x^2(x)\,dx,$$

and therefore, by $(\psi_\varepsilon 4)$, (4.2), and our choice of $r(x)$,

$$\begin{aligned}
\int_{x_0-2\rho}^{x_0+2\rho} \psi_\varepsilon(u_x(x,t))\,dx &\leq \int_{x_0-4\rho}^{x_0+4\rho} \psi_\varepsilon(u_x(x,0))\,dx + 2t\int_{x_0-4\rho}^{x_0+4\rho} r_x^2(x)\,dx \\
&\leq c_\delta \int_{x_0-4\rho}^{x_0+4\rho} \frac{dx}{1 - u_x^2(x,t)} + 2t\cdot\frac{1}{4\rho^2}\int_{x_0-4\rho}^{x_0+4\rho} \left|\eta'\left(\frac{x-x_0}{2\rho}\right)\right|^2 dx \\
&\leq \frac{8\rho c_\delta}{\delta} + \frac{t}{2\rho^2}\cdot 8\rho\|\eta'\|_\infty^2 =: c_1(\delta,\rho) + t c_2(\delta,\rho).
\end{aligned}$$

By $(\psi_\varepsilon 3)$ we can apply Fatou's Lemma, and obtain that

$$\int_{x_0-2\rho}^{x_0+2\rho} \liminf_{\varepsilon\to 0^+} \psi_\varepsilon(u_x(x,t))\,dx \le c_1(\delta,\rho) + tc_2(\delta,\rho),$$

which by $(\psi_\varepsilon 5)$ implies both (4.3) and (4.4).

In order to prove (4.5), we would like to use Theorem 3.2 with $\psi(\sigma) = (1-\sigma^2)^{-2}$, but once again this cannot be done directly. We therefore apply Theorem 3.2 with $r(x) = \eta((x-x_0)/\rho)$, $[t_1,t_2] = [0,t]$, and $\psi(\sigma) = \psi_\varepsilon^2(\sigma)$. Since we already know that $|u_x| \le 1$, by statement (1) of Remark 3.5 we have to verify assumptions (i) and (ii) of Theorem 3.2 only for $|\sigma| \le 1$. Assumption (i) is satisfied because

$$\left(\psi_\varepsilon^2\right)''(\sigma) = 2\left[\psi_\varepsilon'(\sigma)\right]^2 + 2\psi_\varepsilon(\sigma)\psi_\varepsilon''(\sigma) \ge 0$$

for every $|\sigma| \le 1$. Moreover, assumption (ii) is satisfied with $c(\sigma) := 2\psi_\varepsilon(\sigma)$ since

$$\begin{aligned}
\left[\left(\psi_\varepsilon^2\right)'(\sigma)\right]^2 \varphi''(\sigma) &= 4\psi_\varepsilon^2(\sigma)\left[\psi_\varepsilon'(\sigma)\right]^2 \varphi''(\sigma) \\
&\le 4\psi_\varepsilon^2(\sigma)\psi_\varepsilon''(\sigma) \\
&\le 2\psi_\varepsilon(\sigma)\left\{2\psi_\varepsilon(\sigma)\psi_\varepsilon''(\sigma) + 2\left[\psi_\varepsilon'(\sigma)\right]^2\right\} \\
&= 2\psi_\varepsilon(\sigma)\left(\psi_\varepsilon^2\right)''(\sigma)
\end{aligned}$$

for every $|\sigma| \le 1$.

From Theorem 3.2 we deduce in this case that

$$\int_{\mathbb{R}} r^2(x)\psi_\varepsilon^2(u_x(x,t))\,dx \le \int_{\mathbb{R}} r^2(x)\psi_\varepsilon^2(u_x(x,0))\,dx + 4\int_0^t \int_{\mathbb{R}} r_x^2(x)\psi_\varepsilon(u_x(x,\tau))\,dx\,d\tau,$$

hence, by $(\psi_\varepsilon 4)$, (4.2), (4.4), and our choice of $r(x)$,

$$\begin{aligned}
\int_{x_0-\rho}^{x_0+\rho} \psi_\varepsilon^2(u_x(x,t))\,dx &\le \int_{x_0-2\rho}^{x_0+2\rho} \psi_\varepsilon^2(u_x(x,0))\,dx + \frac{4\|\eta'\|_\infty^2}{\rho^2}\int_0^t \int_{x_0-2\rho}^{x_0+2\rho} \psi_\varepsilon(u_x(x,\tau))\,dx\,d\tau \\
&\le \frac{4\rho c_\delta^2}{\delta^2} + \frac{4\|\eta'\|_\infty^2}{\rho^2}\cdot t\cdot\left(c_1(\delta,\rho) + tc_2(\delta,\rho)\right) =: c_3(\delta,\rho,t).
\end{aligned}$$

Applying Fatou's Lemma we obtain that

$$\int_{x_0-\rho}^{x_0+\rho} \frac{dx}{\left[1-u_x^2(x,t)\right]^2} = \int_{x_0-\rho}^{x_0+\rho} \liminf_{\varepsilon\to 0^+} \psi_\varepsilon^2(u_x(x,t))\,dx \le c_3(\delta,\rho,t),$$

which proves (4.5) and completes the proof of Theorem 3.3. $\Box$

4.4 Proof of Theorem 3.4

The fundamental tool is the following.

Lemma 4.2 *Let $u : \mathbb{R}^2 \to \mathbb{R}$ be a C^1 solution of (1.1). Let us assume that there exist $t_\star \in \mathbb{R}$, a bounded interval $I = [A, B]$, a real constant $c > 0$, and two functions $\alpha : (-\infty, t_\star] \to \mathbb{R}$, and $\beta : (-\infty, t_\star] \to \mathbb{R}$, such that, for every $t \leq t_\star$, one has that*

(i) $A \leq \alpha(t) < \beta(t) \leq B$;

(ii) $|u_x(x, t)| \leq 1$ *for every* $x \in [\alpha(t), \beta(t)]$;

(iii) $|\varphi'(u_x(\beta(t), t)) - \varphi'(u_x(\alpha(t), t))| \geq c$.

Then

$$\int_{I_+(t)} [\varphi''(u_x)u_{xx}]^2 \, dx \geq \frac{c^2}{B - A} \tag{4.6}$$

for every $t \leq t_\star$.

Proof. By (ii) we have that $[\alpha(t), \beta(t)] \subseteq I_+(t)$ for every $t \leq t_\star$. Moreover, by (iii) and Hölder's inequality, we have that

$$\begin{aligned} c \leq |\varphi'(u_x(\beta(t), t)) - \varphi'(u_x(\alpha(t), t))| \; &\leq \; \int_{\alpha(t)}^{\beta(t)} |(\varphi'(u_x))_x| \, dx \\ &\leq \; \{\beta(t) - \alpha(t)\}^{1/2} \left\{ \int_{\alpha(t)}^{\beta(t)} [(\varphi'(u_x))_x]^2 \, dx \right\}^{1/2} \\ &\leq \; \{B - A\}^{1/2} \left\{ \int_{I_+(t)} [\varphi''(u_x)u_{xx}]^2 \, dx \right\}^{1/2}, \end{aligned}$$

which is equivalent to (4.6). □

Our strategy is now to prove Theorem 3.4 by showing that u satisfies the assumptions of Lemma 4.2. To this end, we distinguish three cases, depending on how many of the "critical values" $\sigma = -1$ and $\sigma = 1$ are assumed by u_x.

Recalling our assumption that $|u_x(x_0, t_0)| < 1$ for some $(x_0, t_0) \in \mathbb{R}^2$, the three cases are the following ones.

- *Case 1: no critical value assumed.* This means that $-1 < u_x(x, t) < 1$ for every $(x, t) \in \mathbb{R}^2$.

- *Case 2: one critical value assumed.* In this case we can assume, without loss of generality, that $u_x(x, t) < 1$ for every $(x, t) \in \mathbb{R}^2$, and there exists $(x_1, t_1) \in \mathbb{R}^2$ such that $u_x(x_1, t_1) \leq -1$.

- *Case 3: two critical values assumed.* This means that there exists $(x_1, t_1) \in \mathbb{R}^2$ and $(x_2, t_2) \in \mathbb{R}^2$ such that $u_x(x_1, t_1) \leq -1$ and $u_x(x_2, t_2) \geq 1$.

The techniques used to verify the assumptions of Lemma 4.2 in these three cases are different: we begin with case 3 (the simplest), where the main tool is Theorem 3.3; then we pass to case 1, where we need some results about linear forward parabolic equations; finally we consider case 2, where both techniques are necessary in the same time.

4.4.1 Proof in the case where two critical values are assumed

In this case we know that $u_x(x_1, t_1) \leq -1$, and $u_x(x_2, t_2) \geq 1$. We assume, without loss of generality, that $x_1 < x_2$. By Theorem 3.3 we have that $u_x(x_1, t) \leq -1$ and $u_x(x_2, t) \geq 1$ for every $t \leq t_\star := \min\{t_1, t_2\}$; so for every $t \leq t_\star$ we can define

$$\alpha(t) := \sup\{x \in [x_1, x_2] : \; u_x(x, t) \leq -1\},$$

and

$$\beta(t) := \inf\{x \in [\alpha(t), x_2] : \; u_x(x, t) \geq 1\}.$$

By the continuity of u_x, we have that $u_x(\alpha(t), t) = -1$ and $u_x(\beta(t), t) = 1$, hence the functions α and β satisfy assumptions (i), (ii), and (iii) of Lemma 4.2, with $c := 2\varphi'(1)$.

Therefore Theorem 3.4 is proved in this case. $\square$

4.4.2 Proof in the case where no critical value is assumed

In this case we need two results about linear forward parabolic equations. They look classical, but we couldn't find them in the literature.

Lemma 4.3 *Let $A < \alpha_0 < \beta_0 < B$ be real numbers, and let $a : [A, B] \times (-\infty, 0] \to \mathbb{R}$ and $v : [A, B] \times (-\infty, 0] \to \mathbb{R}$ be two functions such that*

(i) a is continuous, and $a(x, t) \geq 0$ for every $x \in [A, B]$ and $t \leq 0$;

(ii) v_t, v_x, and v_{xx} exist and are continuous functions, and

$$v_t(x, t) = a(x, t) v_{xx}(x, t) \qquad \forall (x, t) \in [A, B] \times (-\infty, 0];$$

(iii) for every $t \leq 0$, u satisfies the boundary conditions

$$v(A, t) \geq 0, \qquad v(B, t) \leq 0;$$

(iv) at $t = 0$ one has that

$$v(\alpha_0, 0) = -1, \qquad v(\beta_0, 0) = 1.$$

Then there exist two functions $\alpha : (-\infty, 0] \to [A, B]$*, and* $\beta : (-\infty, 0] \to [A, B]$ *such that*

$$A < \alpha(t) < \beta(t) < B \tag{4.7}$$

and

$$v(\alpha(t), t) \leq -1, \qquad\qquad v(\beta(t), t) \geq 1 \tag{4.8}$$

for every $t \leq 0$*.*

Proof. Roughly speaking, the function $x \to v(x, 0)$ has a minimum point, where its value is ≤ -1, followed by a maximum point, where its value is ≥ 1. We have to prove that the function $x \to v(x, t)$ has the same qualitative behaviour for every $t \leq 0$.

Now we give a rigorous proof, which is long, but completely standard (it is the same argument used in the proof of maximum principles). Given $\varepsilon > 0$, we consider the function

$$w_\varepsilon(x, t) = v(x, t)e^{-\varepsilon t}.$$

It is easy to verify that w_ε satisfies the equation

$$w_{\varepsilon t}(x, t) = a(x, t)w_{\varepsilon xx}(x, t) - \varepsilon w_\varepsilon(x, t), \tag{4.9}$$

the boundary conditions

$$w_\varepsilon(A, t) \geq 0, \qquad\qquad w_\varepsilon(B, t) \leq 0, \tag{4.10}$$

and, at $t = 0$,

$$w_\varepsilon(\alpha_0, 0) = -1, \qquad\qquad w_\varepsilon(\beta_0, 0) = 1. \tag{4.11}$$

Now we prove that there exists two functions $\alpha_\varepsilon : (-\infty, 0] \to [A, B]$, and $\beta_\varepsilon : (-\infty, 0] \to [A, B]$, such that

$$A < \alpha_\varepsilon(t) < \beta_\varepsilon(t) < B \tag{4.12}$$

and

$$w_\varepsilon(\alpha_\varepsilon(t), t) \leq -1, \qquad\qquad w_\varepsilon(\beta_\varepsilon(t), t) \geq 1 \tag{4.13}$$

for every $t \leq 0$.

In order to prove the existence of α_ε and β_ε, we take the set Ω_ε of all $\tau \leq 0$ such that we can find α_ε and β_ε satisfying (4.12) and (4.13) for every $t \in [\tau, 0]$, and we define T_ε as the infimum of Ω_ε.

Our claim is proving that $T_\varepsilon = -\infty$. To this end, we show that if $T_\varepsilon \in \mathbb{R}$, then α_ε and β_ε can be continued in a left neighborhood of T_ε, thus giving a contradiction. In order to prove that such an extension exists, the first step is to define α_ε and β_ε on the closed interval $[T_\varepsilon, 0]$. By (4.11), this is trivial if $T_\varepsilon = 0$. Otherwise, by definition of T_ε, we can assume that α_ε and β_ε are defined at least on $(T_\varepsilon, 0]$. By a compactness argument, there exists a sequence $t_n \to T_\varepsilon^+$ such that $\alpha_\varepsilon(t_n)$ and $\beta_\varepsilon(t_n)$ converge, and we denote the two limits by $\alpha_\varepsilon(T_\varepsilon)$ and $\beta_\varepsilon(T_\varepsilon)$. By the continuity of w_ε, it is clear that (4.13) is satisfied also for $t = T_\varepsilon$, while passing (4.12) to the limit we obtain only that $A \leq \alpha_\varepsilon(T_\varepsilon) \leq \beta_\varepsilon(T_\varepsilon) \leq B$. However, the three equalities are incompatible with (4.10) and (4.13), and so also (4.12) is proved for $t = T_\varepsilon$.

Now we want to extend α_ε and β_ε in a left neighborhood of T_ε. To this end, let $\xi_\varepsilon \in (\alpha_\varepsilon(T_\varepsilon), \beta_\varepsilon(T_\varepsilon))$ be such that $w_\varepsilon(\xi_\varepsilon, T_\varepsilon) = 0$, and let $\gamma_\varepsilon \in (A, \xi_\varepsilon)$ and $\delta_\varepsilon \in (\xi_\varepsilon, B)$ be such that

$$w_\varepsilon(\gamma_\varepsilon, T_\varepsilon) = \min\{w_\varepsilon(x, T_\varepsilon) : \; A \leq x \leq \xi_\varepsilon\};$$
$$w_\varepsilon(\delta_\varepsilon, T_\varepsilon) = \max\{w_\varepsilon(x, T_\varepsilon) : \; \xi_\varepsilon \leq x \leq B\}.$$

Since γ_ε is a point of local minimum for the function $x \to w_\varepsilon(x, T_\varepsilon)$, it follows that $w_{\varepsilon xx}(\gamma_\varepsilon, T_\varepsilon) \geq 0$. Moreover, since $w_\varepsilon(\gamma_\varepsilon, T_\varepsilon) \leq w_\varepsilon(\alpha_\varepsilon(T_\varepsilon), T_\varepsilon) \leq -1$, by (4.9) it follows that

$$w_{\varepsilon t}(\gamma_\varepsilon, T_\varepsilon) = a(\gamma_\varepsilon, T_\varepsilon) w_{\varepsilon xx}(\gamma_\varepsilon, T_\varepsilon) - \varepsilon w_\varepsilon(\gamma_\varepsilon, T_\varepsilon) \geq \varepsilon > 0. \tag{4.14}$$

In an analogous way, one can show that

$$w_{\varepsilon t}(\delta_\varepsilon, T_\varepsilon) \leq -\varepsilon < 0. \tag{4.15}$$

By (4.15) and (4.14) it follows that

$$w_\varepsilon(\gamma_\varepsilon, t) < -1, \qquad\qquad w_\varepsilon(\delta_\varepsilon, t) > 1,$$

for every t in a left neighborhood of T_ε, in which we can therefore extend α_ε and β_ε by setting $\alpha_\varepsilon(t) := \gamma_\varepsilon$, and $\beta_\varepsilon(t) := \delta_\varepsilon$.

This proves that α_ε and β_ε can be defined for every $t \leq 0$. It remains to show how to define $\alpha(t)$ and $\beta(t)$.

For a fixed $t \leq 0$, by a compactness argument there exists a sequence $\varepsilon_n \to 0^+$ such that $\alpha_{\varepsilon_n}(t)$ and $\beta_{\varepsilon_n}(t)$ converge to two limits, which we denote by $\alpha(t)$ and $\beta(t)$. Passing (4.13) to the limit, we obtain (4.8). Passing (4.12) to the limit, we obtain that $A \leq \alpha(t) \leq \beta(t) \leq B$.

Using the boundary conditions on v, and (4.8), it follows that the three inequalities are strict, and therefore also (4.7) is proved. $\Box$

Lemma 4.4 *Let $a : \mathbb{R} \times (-\infty, 0] \to \mathbb{R}$ and $v : \mathbb{R} \times (-\infty, 0] \to \mathbb{R}$ be two functions such that*

(i) a is continuous, and $a(x,t) \geq 0$ for every $x \in \mathbb{R}$ and $t \leq 0$;

(ii) v_t, v_x, and v_{xx} exist and are continuous functions, and

$$v_t(x,t) = a(x,t) v_{xx}(x,t) \qquad \forall (x,t) \in \mathbb{R} \times (-\infty, 0];$$

(iii) v is bounded in $\mathbb{R} \times (-\infty, 0]$.

Then for every $\alpha_0 < \beta_0$ there exists an interval $I = [A, B]$ such that

$$\max\{v(x,t) : \ x \in I\} - \min\{v(x,t) : \ x \in I\} \geq \frac{|v(\alpha_0, 0) - v(\beta_0, 0)|}{2} \tag{4.16}$$

for every $t \leq 0$.

PROOF. Since u is bounded, there exists $M \in \mathbb{R}$ such that $|v(x,t)| < M$ for every $x \in \mathbb{R}$ and every $t \leq 0$.

We assume, without loss of generality, that $v(\alpha_0, 0) < v(\beta_0, 0)$, and we set

$$m := \frac{v(\beta_0, 0) - v(\alpha_0, 0)}{2(\beta_0 - \alpha_0)}, \qquad n_1 := v(\alpha_0, 0) - m\alpha_0, \qquad n_2 := v(\beta_0, 0) - m\beta_0,$$

$$A := -\frac{2M + n_1 + n_2}{2m}, \qquad B := \frac{2M - n_1 - n_2}{2m},$$

and finally

$$w(x,t) := \frac{2v(x,t) - 2mx - n_1 - n_2}{n_2 - n_1}$$

for every $x \in \mathbb{R}$ and $t \leq 0$.

With some computations, it can be shown that

- $A < \alpha_0 < \beta_0 < B$, and $n_2 > n_1$;
- $w_t(x,t) = a(x,t)w_{xx}(x,t)$ for every $x \in \mathbb{R}$ and $t \leq 0$;
- $w(\alpha_0, 0) = -1$ and $w(\beta_0, 0) = 1$;
- $w(A,t) \geq 0$ and $w(B,t) \leq 0$ for every $t \leq 0$.

Therefore it is possible to apply Lemma 4.3 to the function w. It follows that there exist two functions $\alpha : (-\infty, 0] \to [A, B]$, and $\beta : (-\infty, 0] \to [A, B]$ such that $A < \alpha(t) < \beta(t) < B$, and

$$w(\alpha(t), t) \leq -1, \qquad w(\beta(t), t) \geq 1$$

for every $t \leq 0$. Setting $I = [A, B]$ we have therefore that

$$\begin{aligned}
\max\{v(x,t) : \ x \in I\} - \min\{v(x,t) : \ x \in I\} \geq \\
&\geq \ v(\beta(t), t)) - v(\alpha(t), t)) \\
&= \ \frac{n_2 - n_1}{2}\{w(\beta(t), t)) - w(\alpha(t), t))\} + m(\beta(t) - \alpha(t)) \\
&\geq \ n_2 - n_1 = \frac{v(\beta_0, 0) - v(\alpha_0, 0)}{2}
\end{aligned}$$

for every $t \leq 0$. This proves (4.16). $\square$

Now we are ready to prove Theorem 3.4 in the case where $|u_x(x,t)| < 1$ for every $(x,t) \in \mathbb{R}^2$.

Since u_x is nonconstant, then there exist $t_\star \in \mathbb{R}$, and $\alpha_0 < \beta_0$ such that $u_x(\alpha_0, t_\star) \neq u_x(\beta_0, t_\star)$. Indeed, if this is not the case, then u_x depends on t only, but using (1.1) it is not difficult to see that this implies that u_x is constant. Up to a time traslation, we can also assume that $t_\star = 0$.

Now we set

$$v(x,t) = \varphi'(u_x(x,t)), \qquad\qquad a(x,t) = \varphi''(u_x(x,t)),$$

and we observe that a is positive, and v is a bounded function of class C^∞ such that

$$v_t = \left(\varphi'(u_x)\right)_t = \varphi''(u_x)u_{xt} = \varphi''(u_x)\left(\varphi'(u_x)\right)_{xx} = av_{xx}.$$

Moreover, $v(\alpha_0, 0) \neq v(\beta_0, 0)$ since $\varphi'(\sigma)$ is injective for $|\sigma| \leq 1$. Applying Lemma 4.4 to the function v, we find an interval $[A, B]$ where all assumptions of Lemma 4.2 are satisfied with

$$c := \frac{|v(\alpha_0, 0) - v(\beta_0, 0)|}{2}.$$

This completes the proof of Theorem 3.4 in this case. $\square$

4.4.3 Proof in the case where one critical value is assumed

Preliminary part We show that there exist $t_\star$, α_0, β_0 such that $u_x(\alpha_0, t_\star) \leq -1$ and $u_x(\beta_0, t_\star) > -1$.

Indeed, let us assume by contradiction that this is not the case. Then for every $t \in \mathbb{R}$ we have that either

$$u_x(x,t) > -1, \qquad \forall x \in \mathbb{R}, \tag{4.17}$$

or

$$u_x(x,t) \leq -1, \qquad \forall x \in \mathbb{R}. \tag{4.18}$$

Since by assumption both behaviours are present, then by Theorem 3.3 we have that there exists T such that (4.17) holds true for every $t > T$, and (4.18) holds true for every $t \leq T$. Moreover, by a continuity argument it follows that

$$u_x(x,T) = -1, \qquad \forall x \in \mathbb{R}. \tag{4.19}$$

Now let us consider the functions

$$v(x,t) = \varphi'(u_x(x,t)), \qquad\qquad a(x,t) = \varphi''(u_x(x,t)).$$

In the halfplane $\{(x,t) \in \mathbb{R}^2 : \; t > T\}$, it turns out that a is positive, and v is a bounded solution of

$$v_t(x,t) = a(x,t)v_{xx}(x,t).$$

Moreover, v cannot be a function of t only, because in this case it would be constant, hence also u_x would be constant and greater than -1 in the halfplane, which is inconsistent with (4.19).

Therefore there exist $t_2 > T$, and x_2, x_3 such that $v(x_2,t_2) \neq v(x_3,t_2)$. Arguing as in Lemma 4.4, it follows that there exists an interval $I \subseteq \mathbb{R}$ such that

$$\max\{v(x,t) : \; x \in I\} - \min\{v(x,t) : \; x \in I\} \geq \frac{|v(x_2,t_2) - v(x_3,t_2)|}{2}$$

for every $t \in (T, t_2]$.

Since v is continuous, this would be true also for $t = T$, but this contradicts (4.19).

Main part Let $t_\star$, α_0, β_0 be as in the previous step, and let us assume, without loss of generality, that $t_\star = 0$, $\alpha_0 < \beta_0$, and $u_x(\alpha_0, 0) = -1$.

We claim that there exist a bounded interval $[A,B]$, and two functions $\alpha : (-\infty, 0] \to [A,B]$, and $\beta : (-\infty, 0] \to [A,B]$ such that

$$A \leq \alpha(t) < \beta(t) \leq B, \tag{4.20}$$

$$-1 \leq u_x(x,t) \leq 1 \qquad \forall x \in [\alpha(t), \beta(t)], \tag{4.21}$$

$$u_x(\alpha(t), t) = -1, \tag{4.22}$$

$$\varphi'(u_x(\beta(t),t)) \geq \frac{\varphi'(u_x(\beta_0,0)) + \varphi'(-1)}{2} \tag{4.23}$$

for every $t \leq 0$. Note that (4.20) and (4.21) coincide with assumptions (i) and (ii) of Lemma 4.2, while (4.22) and (4.23) imply that

$$\varphi'(u_x(\beta(t),t)) - \varphi'(u_x(\alpha(t),t)) \geq \frac{\varphi'(u_x(\beta_0,0)) - \varphi'(-1)}{2} =: C > 0,$$

which is assumption (iii) of Lemma 4.2. Therefore (4.20)–(4.23) are enough to conclude the proof of Theorem 3.4.

In order to prove them, let $A := \alpha_0$, and let m, n, and B be such that

$$m\alpha_0 + n = \varphi'(-1), \qquad m\beta_0 + n = \frac{\varphi'(u_x(\beta_0,0)) + \varphi'(-1)}{2}, \qquad mB + n = \varphi'(1).$$

Simple computations show that $m > 0$, and $B > \beta_0$.

Now for every $\varepsilon > 0$ we consider the function

$$w_\varepsilon(x,t) := e^{-\varepsilon t}\left\{\varphi'(u_x(x,t)) - mx - n\right\}$$

which turns out to be a solution of

$$w_{\varepsilon t} = \varphi''(u_x)w_{\varepsilon xx} - \varepsilon w_\varepsilon \tag{4.24}$$

at least in the open set where $u_x \neq -1$. Moreover, by definition of m, n and B, it is easy to see that

$$w_\varepsilon(B,t) \leq 0 \qquad \forall t \leq 0, \tag{4.25}$$

and

$$w_\varepsilon(\beta_0, 0) = C.$$

Now we prove that there exists two functions $\alpha_\varepsilon : (-\infty, 0] \to [A, B]$, and $\beta_\varepsilon : (-\infty, 0] \to [A, B]$, such that

$$A \leq \alpha_\varepsilon(t) < \beta_\varepsilon(t) < B, \tag{4.26}$$

$$-1 \leq u_x(x,t) \leq 1 \qquad \forall x \in [\alpha_\varepsilon(t), \beta_\varepsilon(t)], \tag{4.27}$$

$$u_x(\alpha_\varepsilon(t), t) = -1, \tag{4.28}$$

$$w_\varepsilon(\beta_\varepsilon(t), t) \geq w_\varepsilon(\beta_0, 0) = C \tag{4.29}$$

for every $t \leq 0$. Note that (4.28) implies in particular that

$$w_\varepsilon(\alpha_\varepsilon(t), t) \leq e^{-\varepsilon t}\left\{\varphi'(-1) - m\alpha_0 - n\right\} = 0. \tag{4.30}$$

The idea is that passing to the limit as $\varepsilon \to 0^+$, we get (4.20)–(4.23).

In order to prove the existence of α_ε and β_ε, we take the set Ω_ε of all $\tau \leq 0$ such that we can find α_ε and β_ε satisfying (4.26)–(4.29) for every $t \in [\tau, 0]$, and we define T_ε as the infimum of Ω_ε.

Our claim is proving that $T_\varepsilon = -\infty$. To this end, we show that if $T_\varepsilon \in \mathbb{R}$, then α_ε and β_ε can be continued in a left neighborhood of T_ε, thus giving a contradiction. In order to prove that such an extension exists, the first step is to define α_ε and β_ε on the closed interval $[T_\varepsilon, 0]$. This is trivial if $T_\varepsilon = 0$. Otherwise, by definition of T_ε, we can assume that α_ε and β_ε are defined at least on $(T_\varepsilon, 0]$.

By a compactness argument, there exists a sequence $t_n \to T_\varepsilon^+$ such that $\alpha_\varepsilon(t_n)$ and $\beta_\varepsilon(t_n)$ converge, and we denote the two limits by $\alpha_\varepsilon(T_\varepsilon)$ and $\beta_\varepsilon(T_\varepsilon)$. By the continuity of u_x and w_ε, it is clear that (4.27), (4.28), (4.29), and (4.30) are satisfied also for $t = T_\varepsilon$, while passing (4.26) to the limit we have only that $A \leq \alpha_\varepsilon(T_\varepsilon) \leq \beta_\varepsilon(T_\varepsilon) \leq B$. However, the two rightmost inequalities are strict, since equalities would be incompatible with (4.25), (4.29), and (4.30) .

Now we want to extend α_ε and β_ε in a left neighborhood of T_ε. To this end, we consider the function $g_\varepsilon(x) := w_\varepsilon(x, T_\varepsilon)$. This function is positive in $x = \beta_\varepsilon(T_\varepsilon)$, and nonpositive in $x = \alpha_\varepsilon(T_\varepsilon)$ and $x = B$. Let $(\gamma_\varepsilon, \delta_\varepsilon)$ be the larger interval, containing $\beta_\varepsilon(T_\varepsilon)$, where $g_\varepsilon(x)$ is positive. Since $u_x(\beta_\varepsilon(T_\varepsilon), T_\varepsilon) > -1$, and since w_ε is nonpositive where $u_x = -1$, it follows that $u_x(x, T_\varepsilon) > -1$ for every $x \in (\gamma_\varepsilon, \delta_\varepsilon)$. Let ξ_ε be any maximum point of $g_\varepsilon(x)$ in $(\gamma_\varepsilon, \delta_\varepsilon)$.

Then we have that

$$w_\varepsilon(\xi_\varepsilon, T_\varepsilon) \geq w_\varepsilon(\beta_\varepsilon(T_\varepsilon), T_\varepsilon) > 0,$$

and $w_{\varepsilon xx}(\xi_\varepsilon, T_\varepsilon) \leq 0$ since ξ_ε is a local maximum point. From (4.24) it follows therefore that

$$w_{\varepsilon t}(\xi_\varepsilon, T_\varepsilon) = \varphi''(u_x(\xi_\varepsilon, T_\varepsilon))w_{\varepsilon xx}(\xi_\varepsilon, T_\varepsilon) - \varepsilon w_\varepsilon(\xi_\varepsilon, T_\varepsilon) < 0,$$

hence $w_\varepsilon(\xi_\varepsilon, t) > w_\varepsilon(\beta_0, 0)$ for every t in a left neighborhood of T_ε. On the other hand, since $u_x(\alpha_\varepsilon(T_\varepsilon), T_\varepsilon) \leq -1$, by Theorem 3.3 we have that $u_x(\alpha_\varepsilon(T_\varepsilon), t) \leq -1$ for every $t \leq T_\varepsilon$.

Setting $\beta_\varepsilon(t) := \xi_\varepsilon$, and

$$\alpha_\varepsilon(t) := \sup\{x \in [A, \xi_\varepsilon] : \ u_x(x, t) \leq -1\},$$

we have thus extended α_ε and β_ε in a left neighborhood of T_ε, and this completes the proof that α_ε and β_ε can be defined for every $t \leq 0$.

It remains to show how to define $\alpha(t)$ and $\beta(t)$.

For a fixed $t \leq 0$, by a compactness argument there exists a sequence $\varepsilon_n \to 0^+$ such that $\alpha_{\varepsilon_n}(t)$ and $\beta_{\varepsilon_n}(t)$ converge to two limits, which we denote by $\alpha(t)$ and $\beta(t)$. Passing (4.27) and (4.28) to the limit, we obtain (4.21) and (4.22). Passing (4.29) to the limit, we obtain that $\varphi'(u_x(\beta(t), t) - m\beta(t) - n \geq C$, hence

$$\begin{aligned}
\varphi'(u_x(\beta(t), t) &\geq m\beta(t) + n + C \\
&\geq m\alpha_0 + n + C \\
&= \varphi'(-1) + C = \frac{\varphi'(u_x(\beta_0, 0) + \varphi'(-1)}{2}
\end{aligned}$$

which is (4.23).

Finally, passing (4.26) to the limit, we obtain that $A \leq \alpha(t) \leq \beta(t) \leq B$, but equality $\alpha(t) = \beta(t)$ is clearly incompatible with (4.22) and (4.23). Therefore also (4.20) is proved.

The proof of Theorem 3.4 is thus complete. $\square$

4.5 Proof of Theorem 1.1

Let $u : \mathbb{R}^2 \to \mathbb{R}$ be a nontrivial C^1 solution of (1.1).

The idea of the proof is to apply Theorem 3.3 with suitable choices of ψ, r, t_1, t_2, in order to get a contradiction.

Reduction of cases We can assume, without loss of generality, that $|u_x(x_0,t_0)| < 1$ at some point $(x_0,t_0) \in \mathbb{R}^2$.

Indeed, if this is not the case, there are only two possibilities.

- *Case 1*: $u_x(x,t) \geq 1$ for every $(x,t) \in \mathbb{R}^2$. In this case we replace u by the entire solution $v(x,t)$ of (1.1) given by Theorem 3.2. Since u is nontrivial, then $u_x(x,t) > 1$ at some point, hence $0 < v_x(x,t) < 1$ at some other point (we recall that $x \to v(x,t)$ is the inverse of the function $x \to u(x,-t)$).
- *Case 2*: $u_x(x,t) \leq -1$ for every $(x,t) \in \mathbb{R}^2$. In this case we reduce to case 1 by considering the function $w(x,t) = u(-x,t)$, which is still a nontrivial entire C^1 solution of (1.1).

Choice of ψ Let $g \in C^2(\mathbb{R})$ be a function such that

- $g(\sigma) = \sigma$ for every $\sigma \leq \varphi(2)$;
- $g(\sigma) = 2\varphi(2)$ for every $\sigma \geq 4\varphi(2)$;
- $0 \leq g'(\sigma) \leq 1$ for every $\sigma \in \mathbb{R}$;
- $g''(\sigma) \leq 0$ for every $\sigma \in \mathbb{R}$.

If we set $\psi(\sigma) := g(\varphi(\sigma))$, then with some computations it can be proved that ψ satisfies the following properties:

(ψ1) $\psi''(\sigma) \cdot \varphi''(\sigma) \geq 0$ for every $\sigma \in \mathbb{R}$;

(ψ2) there exists a constant c_1 such that

$$[\psi'(\sigma)]^2 \cdot |\varphi''(\sigma)| \leq c_1 \, |\psi''(\sigma)| \qquad \forall \sigma \in \mathbb{R};$$

(ψ3) there exists a constant c_2 such that

$$0 \leq \psi(\sigma) \leq c_2 \qquad \forall \sigma \in \mathbb{R};$$

(ψ4) $\psi(\sigma) = \varphi(\sigma)$ for every $\sigma \in [-2,2]$.

To be precise, one can take $c_1 := [\varphi'(1)]^2$, and $c_2 := 2\varphi(2)$.

Roughly speaking, $\psi(\sigma)$ is constructed by modifying $\varphi(\sigma)$ for $|\sigma|$ large, in order to obtain a bounded function satisfying the assumptions of Theorem 3.2.

Rescaling By (ψ1) and (ψ2) we can apply Theorem 3.3 to the function ψ defined above (with $c(\sigma) \equiv c_1$). Since ψ is non negative, we obtain that

$$\frac{1}{2}\int_{t_1}^{t_2}\int_{\mathbb{R}} r^2\psi''(u_x)\varphi''(u_x)u_{xx}^2\,dx\,dt \leq 2c_1(t_2-t_1)\int_{\mathbb{R}} r_x^2\,dx+\int_{\mathbb{R}} r^2(x)\psi(u_x(x,t_1))\,dx \quad (4.31)$$

for every $r \in C_0^\infty(\mathbb{R})$ and every $t_1 \leq t_2$. We point out that (4.31) makes sense also for C^1 solutions, due to (ψ4) (see the second statement of Remark 3.5).

Given a positive integer n, we now consider estimate (4.31) with $t_1 = -n^2$, $t_2 = 0$, and $r(x) = \eta(x/n)$.

Estimate of the right hand side Choosing ψ, r, t_1 and t_2 as above, the two summands in the right hand side of (4.31) can be estimated as follows:

$$2c_1(t_2-t_1)\int_{\mathbb{R}} r_x^2\,dx \leq 2c_1n^2\int_{-2n}^{2n}\frac{1}{n^2}\left[\eta'\left(\frac{x}{n}\right)\right]^2 dx \leq 2c_1\cdot 4n\cdot\|\eta'\|_\infty^2;$$

$$\int_{\mathbb{R}} r^2(x)\psi(u_x(x,t_1))\,dx \leq \int_{-2n}^{2n}\psi(u_x(x,-n^2))\,dx \leq 4c_2n.$$

As a consequence we have proved that

$$\text{right hand side of (4.31)} \leq c_3n. \quad (4.32)$$

Estimate of the left hand side Let ψ, r, t_1 and t_2 be as above. Now we estimate the left hand side of (4.31) using Theorem 3.4, which can be applied to the function u, since in the first step we have reduced ourselves to the case where $|u_x(x_0,t_0)| < 1$ at some point $(x_0,t_0) \in \mathbb{R}^2$.

Let I and γ be as is Theorem 3.4. If n is large enough, then we have that $I_+(t) \subseteq I \subseteq [-n,n]$ for every $t \leq 0$. Moreover, by (ψ4) we have that $\psi''(u_x) = \varphi''(u_x)$ in $I_+(t)$, hence

$$\begin{aligned}\int_{\mathbb{R}} r^2\psi''(u_x)\varphi''(u_x)u_{xx}^2\,dx &\geq \int_{-n}^{n}\psi''(u_x)\varphi''(u_x)u_{xx}^2\,dx\\ &\geq \int_{I_+(t)}\psi''(u_x)\varphi''(u_x)u_{xx}^2\,dx\\ &= \int_{I_+(t)}\left[\varphi''(u_x)u_{xx}\right]^2\,dx \geq \gamma.\end{aligned}$$

Integrating in $[-n^2,0]$ with respect to t, we obtain that

$$\text{left hand side of (4.31)} \geq c_4n^2 \quad (4.33)$$

provided that n is large enough.

Conclusion Combining (4.31), (4.32) and (4.33), we have proved that

$$c_4 n^2 \leq c_3 n$$

for n large enough, which of course gives a contradiction. □

5 Further results

In this section we show some further applications of the tools of section 3.

Let us consider the initial boundary value problem

$$u_t = (\varphi'(u_x))_x \qquad \forall (x,t) \in [-1,1] \times [0,+\infty) \tag{5.1}$$

$$u_x(-1,t) = u_x(1,t) = 0 \qquad \forall t \geq 0 \tag{5.2}$$

$$u(x,0) = u_0(x) \qquad \forall x \in [-1,1] \tag{5.3}$$

where $u_0(x)$ is of class C^1 and satisfies the compatibility condition

$$u_{0x}(-1) = u_{0x}(1) = 0. \tag{5.4}$$

If $|u_{0x}| < 1$ for every $x \in [-1,1]$, then it is well known that this problem has a unique global solution, and $|u_x(x,t)| < 1$ for every $x \in [-1,1]$ and every $t \geq 0$ (see for example [5, Theorem 6.1]).

So the most interesting case is when $|u_{0x}(x_0)| > 1$ at some $x_0 \in [-1,1]$. In this case (5.1)–(5.3) has no global solution: this was proved in [5, Theorem 3.2], under a technical assumption on the behaviour of u_0 near the points where $|u_{0x}| = 1$.

Here we present a different proof, which does not require technical assumptions.

Theorem 5.1 *Let $u_0 : [-1,1] \to \mathbb{R}$ be a C^1 function satisfying (5.4). If there exists $x_0 \in [-1,1]$ such that $|u_{0x}(x_0)| > 1$, then problem (5.1), (5.2), (5.3) has no global solution.*

Proof. Let us assume by contradiction that a global solution u exists. To begin with, let us assume that u is more regular.

Step 1. Let $\psi \in C^2(\mathbb{R})$ be a function such that $\psi'(0) = 0$. Then for every $t \geq 0$ we have that

$$\int_{-1}^{1} \psi(u_x(x,t))\,dx + \int_0^t \int_{-1}^{1} \psi''(u_x)\varphi''(u_x)u_{xx}^2\,dx\,d\tau = \int_{-1}^{1} \psi(u_x(x,0))\,dx. \tag{5.5}$$

The proof is similar to the proof of Theorem 3.2, but simpler because, due to the Neumann boundary condition, we don't need the cut-off function r and special assumptions on ψ.

Step 2. For every $t \geq 0$ there exists $x_0(t) \in [-1,1]$ such that $|u_x(x_0(t),t)| > 1$, hence also $x_1(t) \in [-1,1]$ such that $|u_x(x_1(t),t)| = 1$.

To this end, it is enough to apply equality (5.5) with

$$\psi(\sigma) := \begin{cases} 0 & \text{if } |\sigma| \leq 1 \\ (|\sigma|-1)^4 & \text{if } |\sigma| \geq 1. \end{cases} \tag{5.6}$$

In this case the right hand side of (5.5) is positive, and the second summand in the left hand side is nonpositive (since $\psi''(\sigma)\varphi''(\sigma) \leq 0$ for every $\sigma \in \mathbb{R}$): it follows that the first summand in the right hand side is necessarily positive, hence $|u_x(x,t)| > 1$ for some $x \in [-1,1]$.

Step 3. Since $|u_x(-1,t)| = 0$, and $|u_x(x_1(t),t)| = 1$, then by Hölder's inequality

$$\begin{aligned} \varphi'(1) = |\varphi'(u_x(x_1(t),t)) - \varphi'(u_x(-1,t))| &\leq \int_{-1}^{x_1(t)} |(\varphi'(u_x))_x| \, dx \\ &\leq \{x_1(t)+1\}^{1/2} \left\{ \int_{-1}^{x_1(t)} [(\varphi'(u_x))_x]^2 \, dx \right\}^{1/2} \\ &\leq \sqrt{2} \left\{ \int_{-1}^{1} [\varphi''(u_x)u_{xx}]^2 \, dx \right\}^{1/2}. \end{aligned}$$

This proves that

$$\int_{-1}^{1} [\varphi''(u_x)u_{xx}]^2 \, dx \geq \frac{[\varphi'(1)]^2}{2} \qquad \forall t \geq 0. \tag{5.7}$$

Step 4. Setting $\psi = \varphi$ in (5.5), and using (5.7), we obtain that

$$\frac{[\varphi'(1)]^2}{2} t \leq \int_0^t \int_{-1}^{1} [\varphi''(u_x)u_{xx}]^2 \, dx \, d\tau \leq \int_{-1}^{1} \varphi(u_x(x,0)) \, dx,$$

which is clearly impossible if t is large enough. This completes the proof for regular solutions.

Step 5. If u is only of class C^1, then equality (5.5) holds true for $\psi = \varphi$ (in this case $\psi''(u_x)\varphi''(u_x)u_{xx}^2 = u_t^2$), and it holds true in the weaker form

$$\int_{-1}^{1} \psi(u_x(x,t)) \, dx \geq \int_{-1}^{1} \psi(u_x(x,0)) \, dx$$

if ψ is the function defined in (5.6). Therefore we can argue exactly as in the case of regular solutions. $\square$

At this point, one could ask himself what happens if $|u_{0x}(x)| \leq 1$ for every $x \in [-1,1]$. The following result, which is an extension of Theorem 3.3, shows that such a condition is preserved by the evolution. Therefore in this case problem (5.1)–(5.3) can be reduced to a degenerate forward parabolic problem, to which the theory of maximal monotone operators can be applied to prove the existence of a unique global solution in the sense of [2, Theorem 3.2].

Theorem 5.2 *Let u be a C^1 solution of (1.1) in $(A,B) \times [t_1,t_2]$, and let $x_0 \in (A,B)$.*

If $|u_x(x_0,t_1)| < 1$, then $|u_x(x_0,t)| < 1$ for every $t \in [t_1,t_2]$.

If $|u_x(x_0,t_2)| > 1$, then $|u_x(x_0,t)| > 1$ for every $t \in [t_1,t_2]$.

PROOF. The first statement can be proved exactly as Theorem 3.3.

For the second statement, the strategy of the proof is similar, but now it is necessary first to develop a "backward analogue" of Theorem 3.2, where assumption (i) is replaced by $\psi''(\sigma)\varphi''(\sigma) \leq 0$ for every $\sigma \in \mathbb{R}$, and then to apply it to the family of functions

$$\psi_\varepsilon(\sigma) := \frac{1}{\varepsilon}\left(\frac{\pi}{2} - \arctan\frac{\sigma^2 + 1 - \sqrt{4-\varepsilon^2}}{\varepsilon}\right),$$

which converges to $(\sigma^2-1)^{-1}$ if $|\sigma| > 1$, and to $+\infty$ otherwise. We leave the details to the interested reader. $\square$

6 Examples

In the first example we show that Theorem 1.1 cannot be extended to Lipschitz solutions.

Example 6.1 Let $u : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$u(x,t) := \begin{cases} x/2 & \text{if } x \leq 0, \\ 2x & \text{if } x \geq 0. \end{cases}$$

It is easy to see that u is Lipschitz continuous, and that $\varphi'(u_x(x,t))$ is defined and constant for every $x \neq 0$ (note that $\varphi'(1/2) = \varphi'(2)$). It follows that u satisfies (1.1) almost everywhere and in the distributional sense, but it is not an affine function.

The second example shows that, if one relaxes the notion of solution in a natural way in order to allow jumps, then there exist nonstationary solutions defined on the whole real plane.

Example 6.2 Let $u : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$u(x,t) := \begin{cases} 0 & \text{if } 5x^2 + 8t \leq 0, \\ x/2 & \text{if } 5x^2 + 8t > 0. \end{cases}$$

This function is obtained by patching two different stationary solutions defined on two domains separated by the curve $5x^2+8t = 0$, along which u is discontinuous (see [6]). Since $\varphi'(\sigma)$ vanishes at infinity, it seems reasonable to neglect jumps in the computation of $\varphi'(u_x)$. One is thus led to set

$$\varphi'(u_x(x,t)) := \begin{cases} 0 & \text{if } 5x^2 + 8t < 0, \\ \varphi'(1/2) = 2/5 & \text{if } 5x^2 + 8t > 0. \end{cases}$$

With such a definition, it is possible to check that

$$\int_{\mathbb{R}^2} u(x,t)\Phi_t(x,t)\, dx\, dt = \int_{\mathbb{R}^2} \varphi'(u_x(x,t))\Phi_x(x,t)\, dx\, dt$$

for every $\Phi \in C_0^\infty(\mathbb{R}^2)$, hence u is a nonstationary distributional solution of (1.1) defined on the whole plane.

Now we give an example of an analytic solution u, defined in a connected open set, and such that u_x crosses the critical value $\sigma = 1$. We point out that, at the present, this is the only example we know with such a property (up to trivial modifications).

Example 6.3 Let us consider the Cauchy problem

$$y' = -\frac{(1+y^2)^2}{1+y}, \qquad y(0) = 0,$$

and let (σ_0, σ_1) be the maximal interval where the solution is defined. It turns out that u is analytic and decreasing in (σ_0, σ_1), and that there exists $\sigma_2 \in (\sigma_0, \sigma_1)$ such that $y(\sigma) > 1$ for every $\sigma \in (\sigma_0, \sigma_2)$, and $-1 < y(\sigma) < 1$ for every $\sigma \in (\sigma_2, \sigma_1)$.

Now let $A := \{(x,t) \in \mathbb{R}^2 : \ \sigma_0 < x + t < \sigma_1\}$, and let $u : A \to \mathbb{R}$ be defined by

$$u(x,t) := \int_0^{x+t} y(\sigma)\, d\sigma - t.$$

It is easy to check that u is an analytic solution of (1.1) in the connected open set A. Moreover $u_x(x,t) = y(x+t) > 1$ if $\sigma_0 < x + t < \sigma_2$, and $-1 < u_x(x,t) < 1$ if $\sigma_2 < x + t < \sigma_1$.

We conclude with an example of a $C^{1,1}$ solution which is not a C^2 solution.

Example 6.4 Let σ_0, σ_1, σ_2, and $y : (\sigma_0, \sigma_1) \to \mathbb{R}$ be as in Example 6.3. Let us define $\overline{y} : (-\infty, \sigma_1) \to \mathbb{R}$ as

$$\overline{y}(\sigma) := \begin{cases} 1 & \text{if } \sigma \leq \sigma_2, \\ y(\sigma) & \sigma_2 \leq \sigma < \sigma_1. \end{cases}$$

Now let $A := \{(x, t) \in \mathbb{R}^2 : \; x + t < \sigma_1\}$, and let $u : A \to \mathbb{R}$ be defined by

$$u(x, t) := \int_0^{x+t} \overline{y}(\sigma)\, d\sigma - t.$$

It is easy to check that u is a C^1 solution of (1.1) in A, and that $u_x(x, t) = \overline{y}(x + t)$ is locally Lipschitz continuous, but not differentiable where $x + t = \sigma_2$.

7 Open problems

The main open problems concerning Perona-Malik equations are probably finding a local existence result for classical solutions of the initial boundary value problem, and finding a reasonable notion of weak solution which exists for large classes of initial data and keeps into account the numerical results.

In this section we quote some other open problems, more related to this paper.

The first problem concerns the extent to which our result can be generalized to different diffusion processes.

Open problem 7.1 *Characterize all functions $\varphi \in C^2(\mathbb{R})$ for which all entire C^1 solutions of (1.3) are trivial.*

Our techniques should exclude the existence of entire solutions with $|u_x|$ bounded, or with $\varphi''(u_x)$ which vanishes at some point. Therefore, the existence of nontrivial entire C^1 solutions seems to be related to the behaviour of φ at infinity. For example, if φ is any function such that $\varphi(\sigma) = \sigma^2$ for every $\sigma \geq \sigma_0$, then $u(x, t) = \sigma_0 x + e^{x+2t}$ is a nontrivial entire solution of (1.3).

The second problem concerns regularity of classical solutions.

Open problem 7.2 *If u is a C^1 solution of (1.1) in an open set, is u necessarily a $C^{1,1}$ solution? If u is C^2, is u actually more regular?*

The last problem concerns the extension of our result to the n-dimensional case.

Open problem 7.3 *If $u : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is a classical solution of (1.4), is u of the form $u(x, t) = a \cdot x + b$, where $a \in \mathbb{R}^n$, $b \in \mathbb{R}$, and the dot denotes the scalar product?*

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