A 3D-1D Young measure theory of an elastic string

Lorenzo Freddi
Dipartimento di Matematica e Informatica, Università di Udine,
via delle Scienze 206, 33100 Udine, Italy
freddi@dimi.uniud.it

and

Roberto Paroni
Dipartimento di Architettura e Pianificazione, Università di Sassari,
Palazzo del Pou Salit, Piazza Duomo, 07041 Alghero, Italy
paroni@uniss.it

Abstract

A variational limit defined on the space of one-dimensional Young measures is obtained from three-dimensional elasticity via dimension reduction. The physical requirement that the energy becomes infinite when the volume locally vanishes is taken into account. The rate at which it blows up characterizes the domain of the limit energy. The obtained limit problem uniquely determines the energy density of the elastic string.

1 Introduction

The aim of this paper is to deduce the energy density of an elastic string from non-linear three-dimensional elasticity. This result is achieved by setting the dimension reduction problem within a framework similar to one developed in a previous paper [13], where the energy density of a thin film is derived from a variational limit defined on the space of two-dimensional gradient Young measures.

A first variational derivation of the total energy of an elastic string is due to Acerbi, Buttazzo and Percivale [1]. They obtain a bulk energy with an integrand f_0^{**} which is the convex envelope of $f_0(z) := \min\{f(\bar{F}|z) \ : \ \bar{F} \in M^{3\times 2}\}$, where $z \in \mathbb{R}^3$ represents the derivative of the deformation and f is the Helmholtz free energy of the three-dimensional body. Thus, under appropriate growth conditions their convex limit problem has a solution. On the other end, the approximating problems may have integrands which are not quasi-convex, as for instance in martensitic materials. Generally, in these kind of problems the infimum of the total energy is not attained. This fact naturally leads to the conclusion that f_0^{**} is not the energy density. Hence the problem: what is the energy density of the elastic string?

It must be noticed that from the result obtained in [1] we can not even deduce that f_0 is the free energy, since there are an infinite number of functions φ such that $\varphi^{**} = f_0^{**}$. In this paper we deduce that f_0 is, in fact, the energy density of the elastic string.

The asymptotic methodology initiated with the work of Acerbi, Buttazzo and Percivale is the following: a sequence of bodies given in a cylindrical configuration with cross section of diameter ε is considered. For each of these bodies the total energy is known. Under quite general assumptions on f there are different topologies which ensure compactness to the family of minimizers (or quasi-minimizers) of these energies. Once chosen one of these topologies, the string model is obtained by passing to the limit as $\varepsilon \to 0$ in an appropriate variational sense (Γ-convergence). Roughly speaking, this variational limit ensures the convergence, in such topology, of minimizers of the energy at level ε to the minimizers of the string problem. The obtained limit problem depends on the chosen topology. As said before, typically the infimum of the total free energy of a martensitic material is not attained. The minimizing sequences shall, in general, develop fine scale oscillations, which, according to the interpretation due to Ball and James [5, 6], model the microstructure experimentally observed in specimens of phase transforming materials. Thus, in phase transforming problems the “main properties” of the minimizing sequences are to be determined. This suggests that when we pass to the limit as $\varepsilon \to 0$ we should try to use a topology which, loosely speaking, ensures the convergence of the “main properties” of the minimizing sequences at level ε to the “main properties” of the minimizing sequences of the string problem. We achieve this by embedding the problem into a suitable space of Young measures, see L.C. Young [20], with an extension by $+\infty$ of the energy functionals. In this way, we can use a suitable topology which allows us to obtain a limit problem that better describes the microstructure, that is the oscillations of the minimizing sequences.

This methodology introduces several difficulties which are completely missing in the work of Acerbi, Buttazzo and Percivale. Like them, we perform the computation of the Γ-limit by taking into account the physical requirement that the energy becomes infinite when the volume locally vanishes, that is

$$\lim_{\det F \to 0^+} f(F) = +\infty.$$

The same requirement is also met in a recent paper of Ben Belgacem [7]. In contrast with [1, 7], we need to specify the rate at which the energy blows up when the volume decreases. In fact, in the cited papers, the obtained Γ-limit involves the convexification of the energy density, which completely disregards the behaviour of the energy near vanishing-volume deformations. On the other hand, in the Young measure setting, where no convexification appears, the growth near vanishing-volume deformations is as much important as the growth for large deformations. This reflects on the fact that the domain of the limit functional strongly depends on the prescribed growths. More precisely we consider an energy which has p-growth for large deformations and q growth for vanishing-volume deformations (see assumptions (3.1) and (3.2)). In particular, in Section 7 we take $q = p/2$ which corresponds to the highest growth assumption in compression near vanishing-volume deformations, while in Section 8 the exponent q is appropriately chosen in order to take into account Ogden materials. According with the variational properties of Γ-convergence, the main theorems proved in these two sections may be interpreted as follows: if ν_ε is a Young measure minimizer of the energy of the three-dimensional body I_ε^∞ then ν_ε converges in an appropriate sense to a Young measure ν which is a minimizer of the limit energy I. In other words, the microstructure of the three-dimensional body at level ε can be approximated by the microstructure determined by the limiting problem.

In our analysis we consider an energy density which may depend also on position and deformation besides on the deformation gradient. Even the possibility that the diameter of the cross section of the body may vary with the position is taken into account.

The paper is organized as follows. In Section 2 are summarized well known definitions and properties of Young measures. In Section 3 we consider the three-dimensional problems and set the framework in which the Γ-limit will be taken. In the fourth we define and characterize a class of Young measures with a prescribed behaviour at infinity and near by zero. Section 5 is devoted to prove coerciveness. Well known properties of Γ-convergence are recalled in Section 6. In Section 7 and 8 two different refinements of the growth assumptions postulated in [1] are considered and the limit problem is completely characterized. In the former we deal with materials whose energy density blows up extremely fast under compression, while the latter contains also a large class of compressible Ogden materials and the whole class of those introduced by Antman in [2]. Finally, in Section 10, we study the relationship between our limit functional and the one obtained by Acerbi, Buttazzo and Percivale. Moreover we show that the obtained Γ-limit uniquely determines the energy density of the elastic string.

2 Young measures: notation and properties

The setting of the problem will be done immediately after this short section, where we introduce some notation and well known properties of Young measures.

Throughout the whole section Ω is an open bounded subset of $\mathbb{R}^k$. We denote by $M^{h\times k}$ the set of $h\times k$ real matrices which will be often identified with the Euclidean space $\mathbb{R}^{hk}$. By $\mathcal{M}(M^{h\times k})$ we mean the space of $\mathbb{R}$-valued Borel measures on $M^{h\times k}$ which can be viewed as the dual of the separable Banach space $C_0(M^{h\times k})$ under the duality

$$\langle \mu, \varphi \rangle = \int_{M^{h\times k}} \varphi \, d\mu.$$

We recall that a mapping $\mu : \Omega \to \mathcal{M}(M^{h\times k})$ is said to be weakly* measurable whenever the function $x \mapsto \langle \mu(x), \varphi\rangle$ is measurable for every $\varphi \in C_0(M^{h\times k})$. The space $L^\infty_w(\Omega; \mathcal{M}(M^{h\times k}))$ consists of all weakly* measurable, essentially bounded, mappings $\mu : \Omega \to \mathcal{M}(M^{h\times k})$. This space will be endowed with the weak* topology induced by the duality with $L^1(\Omega; C_0(M^{h\times k}))$. Therefore, a sequence μ^n is weakly* converging to a limit μ if and only if

$$\int_\Omega \langle \mu^n(x), \varphi\rangle g(x)\, dx \to \int_\Omega \langle \mu(x), \varphi\rangle g(x)\, dx \quad \forall \varphi \in C_0(M^{3\times 3}),\ \forall g \in L^1(\Omega).$$

A Young measure on Ω with target $M^{h\times k}$ is an element ν of $L^\infty_w(\Omega; \mathcal{M}(M^{h\times k}))$ such that $\nu_x := \nu(x)$ is a probability measure for almost every $x \in \Omega$. The space of Young measures will be denoted by $\mathcal{Y}(\Omega; M^{h\times k})$. A $(\nu_x) \in \mathcal{Y}(\Omega; M^{h\times k})$ is said to be homogeneous if it essentially does not depend on x. A Young measure $\mu \in L^\infty_w(\Omega; \mathcal{M}(M^{h\times k}))$ is said to be generated by the sequence of measurable functions $u^n : \Omega \to M^{h\times k}$ if

$$\delta_{u^n(\cdot)} \to \mu \ \text{ weakly* in } L^\infty_w(\Omega; \mathcal{M}(M^{h\times k})).$$

Every Young measure is generated by some sequence of measurable functions. The center of mass of a Young measure $\mu \in \mathcal{Y}(\Omega; M^{h\times k})$ is the function

$$x \mapsto \langle \mu_x, \mathrm{id}\rangle = \int_{M^{h\times k}} \lambda\, d\mu_x(\lambda),$$

where id denotes the identity mapping.
A large use will be done of the following theorem (see for instance Valadier [19]) which states the fundamental property of Young measures.

Theorem 2.1 *Suppose that the sequence of maps $u^n : \Omega \to \mathbb{R}^h$ generates the Young measure ν. Let $f : \Omega\times\mathbb{R}^h \to (-\infty, +\infty]$ be a Carathéodory function and assume that the negative part $f^-(x, u^n(x))$ is weakly relatively compact in $L^1(\Omega)$. Then*

$$\liminf_{k\to\infty} \int_\Omega f(x, u^n(x))\, dx \geq \int_\Omega \int_{\mathbb{R}^h} f(x, \lambda)\, d\nu_x(\lambda)\, dx.$$

Moreover, the sequence of functions $x \mapsto |f|(x, u^n(x))$ is weakly relatively compact in $L^1(\Omega)$ if and only if

$$f(\cdot, u^n(\cdot)) \to \overline{f} \text{ weakly in } L^1(\Omega) \text{ where } \overline{f}(x) = \int_{\mathbb{R}^h} f(x, \lambda)\, d\nu_x(\lambda).$$

The following lemma gives a criterion which is useful to establish when two sequences generate the same Young measure (see Lemma 6.3 of Pedregal [17]).

Lemma 2.2 *Assume that (u_n) and (v_n) be two bounded sequences in $L^p(\Omega; \mathbb{R}^h)$ and that (u_n) generate a Young measure ν. If*

$$\lim_{n\to\infty} |\{u_n \neq v_n\}| = 0$$

then also (v_n) generates ν.

3 Setting of the problem

As stated in the introduction, the goal of the paper is to deduce the energy density of an elastic string from the non-linear three-dimensional theory of elasticity. In this section we shall consider a three dimensional region Ω_ε, as the reference configuration of a hyperelastic homogeneous body, which reduces to a one-dimensional region as ε goes to zero. The energies of the three-dimensional bodies, which are classically defined on some Sobolev space, will be naturally extended to a suitable space of Young measures. Contrary to the usual Sobolev spaces setting, this extension will provide a limit problem, for the one-dimensional body, which carries information on microstucture and allows us to write the energy density of the string.

More precisely, for all $\varepsilon > 0$, let

$$\Omega_\varepsilon = \{x = (x_\alpha, x_3) \in \mathbb{R}^2 \times \mathbb{R} : x_\alpha \in \varepsilon h(x_3)\omega,\ x_3 \in (0, \ell)\},$$

where $h : [0, \ell] \to (h_0, +\infty)$ is a measurable bounded function, with $h_0 > 0$, and ω is an open, bounded subset of $\mathbb{R}^2$ containing the origin. Without loss of generality we assume that $|\omega| = 1$. We can think of Ω_ε as the reference configuration of a hyperelastic homogeneous body with stored energy density function $f : (0, \ell) \times \mathbb{R}^3 \times M^{3\times3} \to (-\infty, +\infty]$. This function $f = f(t, \xi, F)$ is assumed to be a Carathéodory integrand (measurable in t and continuous in (ξ, F) for almost every $t \in (0, \ell)$) and to satisfy the following growth assumptions

$$\begin{aligned} &\det F \leq 0 \Rightarrow f(\cdot, \cdot, F) \equiv +\infty, \\ &\forall\, \delta > 0\ \exists\, C_\delta > 0 \text{ s.t. } \det F \geq \delta \Rightarrow f(\cdot, \cdot, F) \leq C_\delta(|F|^p + 1), \\ &\exists\, c > 0 \text{ s.t. } c(|F|^p - 1) \leq f(\cdot, \cdot, F), \end{aligned} \tag{3.1}$$

for a suitable $p \in [1, +\infty)$.

These growth conditions have been introduced by Acerbi, Buttazzo and Percivale in [1] in order to take into account the physical requirement that the energy becomes infinite when the volume locally vanishes, that is

$$\lim_{\det F \to 0^+} f(\cdot, \cdot, F) = +\infty.$$

On the other hand, they are not precise enough to characterize the domain of the limit problem in terms of Young measures. Indeed, we shall see that two different refinements of the set of assumptions (3.1) will lead to limit problems defined on different spaces. For this reason we introduce the further assumption

$$\exists\, c > 0 \text{ s.t. } c\Big(\frac{1}{|z|^q} + |z|^p - 1\Big) \leq \inf_{\bar{F} \in \mathbb{R}^{3\times2}} f(\cdot, \cdot, (\bar{F}|z)) \tag{3.2}$$

for a suitable choice of $q \in (0, +\infty)$ which will be done subsequently. Above and in the sequel, we identify $M^{3\times3}$ with $M^{3\times2} \times \mathbb{R}^3$, and write $(\bar{F}|z)$ for the element in $M^{3\times3}$ which is identified with $(\bar{F}, z) \in M^{3\times2} \times \mathbb{R}^3$.

Prototypes of functions f which satisfies the prescribed growth conditions are the energy densities of neo-Hookean materials or, more generally, those of the form

$$f(F) = |F|^p + \beta(\det F)$$

where $\beta : \mathbb{R} \to [0, +\infty]$ is continuous, decreasing and $\beta(s) = +\infty$ when $s \leq 0$ (see also Examples 7.6 and 8.5).

The total energy I_ε of the body is given by

$$I_\varepsilon(y) = \int_{\Omega_\varepsilon} f(x_3, y(x), Dy(x))\,dx - \int_{\Omega_\varepsilon} \hat{g}^\varepsilon(x) \cdot y(x)\,dx,$$

where the body force densities $\hat{g}^\varepsilon$ are taken in $L^{p'}(\Omega_\varepsilon; \mathbb{R}^3)$, with $1/p + 1/p' = 1$. Assuming, for the sake of simplicity, the body to be clamped on $\varepsilon h(0)\omega \times \{0\}$, the equilibrium configuration will be found by minimizing the energy I_ε over the set of admissible deformations

$$\mathcal{A}_\varepsilon = \{y \in W^{1,p}(\Omega_\varepsilon; \mathbb{R}^3) \ : \ y(x_1, x_2, 0) = (x_1, x_2, 0)\}.$$

In order to study the behaviour of the minimizers y^ε of I_ε as $\varepsilon \to 0$, it is convenient to transform, after Ciarlet and Destuynder [11], the problems under consideration into problems over a fixed domain. To this end we define $\Omega := \Omega_1$, and a map $\theta_\varepsilon : L^p(\Omega_\varepsilon; \mathbb{R}^3) \to L^p(\Omega; \mathbb{R}^3)$ by

$$(\theta_\varepsilon y)(x) = y(\varepsilon x_1, \varepsilon x_2, x_3).$$

We accordingly rescale the energies by setting $I_\varepsilon^\Omega(y) = I_\varepsilon(\theta_\varepsilon^{-1} y)/\varepsilon^2$, which is

$$I_\varepsilon^\Omega(y) := \int_\Omega f\big(t, y, (\frac{D_\alpha y}{\varepsilon} | D_3 y)\big)\,dx - \int_\Omega g^\varepsilon \cdot y\,dx,$$

where we have set $g^\varepsilon := \theta_\varepsilon \hat{g}^\varepsilon$ and $D_\alpha y$ denotes the first two columns of the gradient of y. We assume that g^ε does not depend on ε and set $g := g^\varepsilon$. The admissible set over which the total energy has to be minimized becomes

$$\mathcal{A}_\varepsilon^\Omega = \{y \in W^{1,p}(\Omega; \mathbb{R}^3) \ : \ y(x_1, x_2, 0) = \varepsilon(x_1, x_2, 0)\}.$$

With the motivation already explained in the introduction we extend the functionals I_ε^Ω to the larger space $L_w^\infty(\Omega; \mathcal{M}(M^{3\times 3}))$ by setting

$$I_\varepsilon^\infty(\mu) = \begin{cases} I_\varepsilon^\Omega(y) & \text{if } \exists\, y \in \mathcal{A}_\varepsilon^\Omega \ s.t.\ \mu = \delta_{Dy(\cdot)} \\ +\infty & \text{otherwise in } L_w^\infty(\Omega; \mathcal{M}(M^{3\times 3})). \end{cases} \tag{3.3}$$

Let us remark that this functional is well defined because if $y_1, y_2 \in \mathcal{A}_\varepsilon^\Omega$ are such that $\delta_{Dy_1(x)} = \delta_{Dy_2(x)}$ then, due to the boundary condition, $y_1 = y_2$ almost everywhere.

4 Young measures with $(-q,p)$-growth

The growth constraints (3.1) and (3.2) that we have prescribed on the energy density motivate the introduction of the following space.

Let $p \geq 1$ and $q > 0$. Let the function $g^{-q,p} : [0,+\infty) \to [0,+\infty]$ be defined by $g^{-q,p}(t) = \dfrac{1}{t^q} + t^p$ and $g^{-q,p}(0) = +\infty$. Throughout the paper, to be more direct, instead of $g^{-q,p}(t)$ we will simply write $\dfrac{1}{t^q} + t^p$ assuming that this function takes the value $+\infty$ for $t = 0$.

Definition 4.1 *With $\mathcal{Y}^{-q,p}((0,\ell);\mathbb{R}^m)$, $m \in \mathbb{N}$, we denote the set of Young measures $\nu \in \mathcal{Y}((0,\ell);\mathbb{R}^m)$ such that*

$$\int_0^\ell \int_{\mathbb{R}^m} \frac{1}{|z|^q} + |z|^p \, d\nu_t(z)\, dt < +\infty.$$

The following theorem, similar to Theorem 7.7 of Pedregal [17], characterizes the space $\mathcal{Y}^{-q,p}((0,\ell);\mathbb{R}^m)$.

Theorem 4.2 *Let $\nu = (\nu_x)_{x\in(0,\ell)} \in \mathcal{Y}((0,\ell);\mathbb{R}^m)$. Then, $\nu \in \mathcal{Y}^{-q,p}((0,\ell);\mathbb{R}^m)$ if and only if there exists a sequence of functions (z_j) which generates ν and such that $\{g^{-q,p}(|z_j|)\}$ is bounded in $L^1(\Omega)$.*

The following lemma will be used in the proof.

Lemma 4.3 *Let ν be a probability measure on $\mathbb{R}^m$ such that*

$$\int_{\mathbb{R}^m} g^{-q,p}(|\lambda|)\, d\nu(\lambda) < +\infty.$$

Then there exists a sequence of functions (w_j) which generates ν as a homogeneous Young measure and such that $\{g^{-q,p}(|w_j|)\}$ is bounded in $L^1(\Omega)$.

PROOF. The proof proceeds along the same lines of that of Theorem 7.6 of Pedregal [17], with the only modification of the class $\mathcal{E}^g$ which here denotes the set of continuous functions $\varphi : \mathbb{R}^m \setminus \{0\} \to \mathbb{R}$ such that the limits

$$\lim_{|\lambda|\to\infty} \frac{\varphi(\lambda)}{1 + g^{-q,p}(|\lambda|)} \quad \text{and} \quad \lim_{\lambda\to 0} \frac{\varphi(\lambda)}{1 + g^{-q,p}(|\lambda|)}$$

do exist and are finite. We observe that $g^{-q,p}(|\cdot|) \in \mathcal{E}^g$ and $C_0(\mathbb{R}^m) \subset \mathcal{E}^g$ and, as in the cited proof of Pedregal, $\mathcal{E}^g$ under the norm

$$\|\varphi\|_{\mathcal{E}^g} := \left\| \frac{\varphi(\cdot)}{1 + g^{-q,p}(|\cdot|)} \right\|_{L^\infty(\mathbb{R}^m)},$$

is isometric to the separable Banach space $C(K)$ where $K = \mathbb{R}^m \cup \{\infty\}$ is the one-point compactification of $\mathbb{R}^m$. Moreover, the dual space $(\mathcal{E}^g)'$ strictly contains the probability measures in $\mathbb{R}^m$, μ, such that

$$\int_{\mathbb{R}^m} g^{-q,p}(|\lambda|)\, d\mu(\lambda) < +\infty. \tag{4.1}$$

We refer the reader to the cited proof for the remaining details. □

PROOF OF THEOREM 4.2. Let (v_n) be such that

$$\sup_n \int_0^\ell \frac{1}{|v_n(t)|^q} + |v_n(t)|^p \, dt < +\infty, \tag{4.2}$$

and v_n generates ν. Then by Theorem 2.1 we have

$$\int_0^\ell \int_{\mathbb{R}^3} \frac{1}{|z|^q} + |z|^p \, d\nu_t(z) \, dt < +\infty. \tag{4.3}$$

This proves the necessity. The proof of the sufficiency can be done exactly as in the proof of Theorem 7.7 of Pedregal [17] by observing moreover that equation (7-7) of that proof holds also for $\varphi = g = g^{-q,p}$, ensuring the boundedness of $\{g(|z_j|)\}$ in L^1. □

The proof of the next Lemma uses a standard truncations technique (see for instance Pedregal [17]).

Lemma 4.4 *Let $\nu \in \mathcal{Y}^{-q,p}((0,\ell);\mathbb{R}^m)$, with $p \geq 1$, $q > 0$. Then there always exists a sequence of functions $u_n \in L^\infty((0,\ell);\mathbb{R}^m)$ which generates ν, $1/|u_n| \in L^\infty(0,\ell)$ and $\dfrac{1}{|u_n(t)|^q} + |u_n(t)|^p$ is equi-integrable.*

PROOF. Let $\nu \in \mathcal{Y}^{-q,p}((0,\ell);\mathbb{R}^m)$. By Theorem 4.2 there exist functions $v_n \in W^{1,p}((0,\ell);\mathbb{R}^m)$ with the property written in (4.2). Let $\lambda > 1$ and consider the truncation map $\tau_\lambda : \mathbb{R}^m \to \mathbb{R}^m$ defined by

$$\tau_\lambda(z) = \begin{cases} \dfrac{1}{\lambda} e_1 & \text{if } |z| = 0, \\ \dfrac{1}{\lambda}\dfrac{z}{|z|} & \text{if } 0 < |z| \leq \dfrac{1}{\lambda}, \\ z & \text{if } \dfrac{1}{\lambda} < |z| < \lambda, \\ \lambda \dfrac{z}{|z|} & \text{if } |z| \geq \lambda. \end{cases}$$

Note that $z \mapsto |\tau_\lambda(z)|$ is a continuous map. Since

$$|\tau_\lambda(z)| \leq \lambda, \quad \text{and} \quad \frac{1}{|\tau_\lambda(z)|} \leq \lambda \quad \text{for every } z \in \mathbb{R}^m,$$

the sequence

$$\frac{1}{|\tau_\lambda(v_n)|^q} + |\tau_\lambda(v_n)|^p$$

is equi-integrable, and therefore by Theorem 2.1 we have, for every $\varphi \in L^\infty(0,\ell)$

$$\begin{aligned}\lim_{\lambda\to+\infty}\lim_{n\to+\infty}\int_0^\ell \varphi(t)\Big(\frac{1}{|\tau_\lambda(v_n)|^q}+|\tau_\lambda(v_n)|^p\Big)\,dt = \\ = \lim_{\lambda\to+\infty}\int_0^\ell \varphi(t)\int_{\mathbb{R}^m}\frac{1}{|\tau_\lambda(z)|^q}+|\tau_\lambda(z)|^p\,d\nu_t(z)\,dt \\ = \int_0^\ell \varphi(t)\int_{\mathbb{R}^m}\frac{1}{|z|^q}+|z|^p\,d\nu_t(z)\,dt,\end{aligned}$$

where the last equality is obtained after observing that

$$\frac{1}{|\tau_\lambda(z)|^q}+|\tau_\lambda(z)|^p \le 2+\frac{1}{|z|^q}+|z|^p$$

for every $z\in\mathbb{R}^m$, and by means of (4.3) and Lebesgue's convergence theorem. By a standard diagonalization argument we may find a sequence $\lambda_k\to\infty$ and a subsequence (not relabelled) of (v_n) such that

$$\frac{1}{|\tau_{\lambda_k}(v_k)|^q}+|\tau_{\lambda_k}(v_k)|^p \to \int_{\mathbb{R}^m}\frac{1}{|z|^q}+|z|^p\,d\nu_t(z)$$

weakly in $L^1(0,\ell)$. Thus, the sequence

$$\frac{1}{|\tau_{\lambda_k}(v_k)|^q}+|\tau_{\lambda_k}(v_k)|^p$$

is equi-integrable, moreover

$$\begin{aligned}|\{\tau_{\lambda_k}(v_k)\neq v_k\}| &= |\{|v_k|\le 1/\lambda_k\}|+|\{|v_k|\ge\lambda_k\}| \\ &\le \int_0^\ell \frac{1}{\lambda_k^q|v_k|^q}\,dt+\int_0^\ell\frac{|v_k|^p}{\lambda_k^p}\,dt \\ &\le c\,\frac{1}{\lambda_k^{\min\{p,q\}}}\Big(\Big\|\frac{1}{|v_k|}\Big\|_q^q+\|v_k\|_p^p\Big)\to 0\end{aligned}$$

thanks to (4.2). The proof is concluded by setting $u_n := \tau_{\lambda_n}(v_n)$ and using Lemma 2.2. □

5 Averaging and coercivity

Let us denote by π^3 the projection of $M^{3\times3}$ onto $\mathbb{R}^3$

$$\pi^3 : M^{3\times3}\to\mathbb{R}^3,\quad \pi^3(\bar F|z)=z,$$

and by $\pi^3_\#\mu_x$ the usual image measure by the corresponding projection map, that is $\pi^3_\#\mu_x(A):=\mu_x({\pi^3}^{-1}(A))$ for every Borel set A.

Let X be either $\mathbb{R}^3$ or $M^{3\times3}$. If $\nu\in L^\infty_w(\Omega;\mathcal{M}(X))$ we denote by

$$\mathrm{Av}^\alpha\,\nu : (0,\ell)\to\mathcal{M}(X)$$

the average of ν with respect to the variable x_α

$$\langle \mathrm{Av}^\alpha_{x_3}\nu, \varphi\rangle := \langle \mathrm{Av}^\alpha\,\nu(x_3), \varphi\rangle := \fint_{h(x_3)\omega} \langle \nu_{(x_\alpha,x_3)}, \varphi\rangle\, dx_\alpha,$$

for every $\varphi \in C_0(X)$, where (recalling that $|\omega| = 1$)

$$\fint_{h(x_3)\omega} \langle \nu_{(x_\alpha,x_3)}, \varphi\rangle\, dx_\alpha := \frac{1}{h(x_3)^2}\int_{h(x_3)\omega} \langle \nu_{(x_\alpha,x_3)}, \varphi\rangle\, dx_\alpha.$$

By Fubini's theorem the map $x_3 \mapsto \langle \mathrm{Av}^\alpha_{x_3}\nu, \varphi\rangle$ is measurable and, since it is also essentially bounded, we can think of Av^α as a mapping

$$\mathrm{Av}^\alpha : L^\infty_w(\Omega; \mathcal{M}(X)) \to L^\infty_w((0,\ell); \mathcal{M}(X)).$$

It is easy to check that this map is continuous and that Av^α maps Young measures into Young measures. The Γ-convergence result will be more expressive by using the average-projection mapping

$$\rho : L^\infty_w(\Omega; \mathcal{M}(M^{3\times 3})) \to L^\infty_w((0,\ell); \mathcal{M}(M^{3\times 2})),$$

defined by

$$\rho = \pi^3_\# \circ \mathrm{Av}^\alpha = \mathrm{Av}^\alpha \circ \pi^3_\#,$$

where the commutativity of composition follows directly from the definitions. By the continuity of the average and the projection mapping, even ρ is continuous. Moreover ρ maps Young measures into Young measures.

The limit problem will be written by using the following definition, which is essentially due to Acerbi, Buttazzo and Percivale [1]:

$$f_0(t,\xi,z) := \min\{f\big(t,\xi,(\bar F|z)\big) \;:\; \bar F \in M^{3\times 2}\} \tag{5.1}$$

for every $t \in (0,\ell)$, and $\xi, z \in \mathbb{R}^3$. We note that assumption (3.2) is a growth condition from below on f_0.

The following lemma states that the sequence of functionals I^∞_ε is equicoercive in the space $L^\infty_w((0,\ell); \mathcal{M}(M^{3\times 3}))$ with the weak* convergence and that the domain of the Γ-limit is contained in $\mathcal{Y}^{-q,p}((0,\ell); \mathbb{R}^3)$.

Lemma 5.1 *Let f satisfy* (3.1) *and* (3.2), *$\varepsilon_n \to 0$ and $\mu^n \in L^\infty_w(\Omega; \mathcal{M}(M^{3\times 3}))$ be such that*

$$\sup_n I^\infty_{\varepsilon_n}(\mu^n) < +\infty.$$

Then there exist $\mu \in L^\infty_w(\Omega; \mathcal{M}(M^{3\times 3}))$ and a subsequence (μ^{n_k}) such that

$$\mu^{n_k} \to \mu \quad \textit{weakly* in } L^\infty_w(\Omega; \mathcal{M}(M^{3\times 3})),$$

and $\nu := \rho(\mu) \in \mathcal{Y}^{-q,p}((0,\ell); \mathbb{R}^3)$. Moreover, denoted by $y_{n_k} \in W^{1,p}(\Omega; \mathbb{R}^3)$ and $y \in W^{1,p}((0,\ell); \mathbb{R}^3)$ the underlying deformations of μ^{n_k} and ν which satisfy the boundary conditions $y_{n_k}(x_1,x_2,0) = \varepsilon_{n_k}(x_1,x_2,0)$ and $y(0) = 0$, respectively, we have, denoting the extension of y to the domain Ω still by y, that

$$y_{n_k} \to y \textit{ weakly in } W^{1,p}(\Omega; \mathbb{R}^3).$$

PROOF. Since $\sup_n I^\infty_{\varepsilon_n}(\mu^n) < +\infty$, then $\mu^n_x = \delta_{Dy_n(x)}$ where $y_n \in W^{1,p}(\Omega;\mathbb{R}^3)$ and $y_n(x_1,x_2,0) = \varepsilon_n(x_1,x_2,0)$. By the growth assumption on f and Poincaré inequality, we get, for every n large enough so that $\varepsilon_n < 1$

$$\begin{aligned} I^\infty_{\varepsilon_n}(\mu^n) &= \int_\Omega f\big(x_3, y_n, (\frac{D_\alpha y_n}{\varepsilon_n}|D_3 y_n)\big)\,dx - \int_\Omega g\cdot y_n\,dx \\ &\geq c\|\big(\frac{D_\alpha y_n}{\varepsilon_n}|D_3 y_n\big)\|^p_p - \|g\|_{p'}\|y_n\|_p - c|\Omega| \\ &\geq c\|Dy_n\|^p_p - \|g\|_{p'}\|y_n\|_p - c|\Omega| \\ &\geq c_1\|y_n\|^p_{1,p} - c_2, \end{aligned}$$

where $\|\cdot\|_p$ and $\|\cdot\|_{1,p}$ denote the usual L^p and $W^{1,p}$ norms, respectively. Therefore

$$\sup_n \|y_n\|_{1,p} < +\infty \quad \text{and} \quad \sup_n \|(\frac{D_\alpha y_n}{\varepsilon_n}|D_3 y_n)\|_p < +\infty \tag{5.2}$$

thus, up to subsequences,

$$y_n \to y \text{ weakly in } W^{1,p}(\Omega;\mathbb{R}^3) \quad \text{and} \quad D_\alpha y_n \to 0 \text{ strongly in } L^p(\Omega;\mathbb{R}^3)$$

and

$$\mu^n \to \mu \text{ weakly* in } L^\infty_w(\Omega;\mathcal{M}(M^{3\times 3})).$$

It follows that $y = y(x_3)$ and $y(0) = 0$. From the continuity of ρ we have

$$\rho(\mu^n) \to \rho(\mu) \text{ weakly* in } L^\infty_w((0,\ell);\mathcal{M}(\mathbb{R}^3)).$$

Let $\nu := \rho(\mu)$. It remains to prove that $\nu \in \mathcal{Y}^{-q,p}((0,\ell);\mathbb{R}^3)$ and also that $\langle \nu, \mathrm{id}\rangle = y'$. Let us start by proving that ν is in the desired space. Since $\sup_n I^\infty_{\varepsilon_n}(\mu^n) < +\infty$, from (5.2), the definition of f_0 and (3.2) we have that

$$\begin{aligned} +\infty > C &> \int_\Omega f\big(x_3, y_n(x), (\frac{D_\alpha y_n(x)}{\varepsilon_n}|D_3 y_n(x))\big)\,dx \\ &\geq \int_\Omega f_0\big(x_3, y_n(x), D_3 y_n(x)\big)\,dx \\ &\geq c\int_\Omega \frac{1}{|D_3 y_n(x)|^q} + |D_3 y_n(x)|^p - 1\,dx \\ &= c\int_\Omega\int_{\mathbb{R}^3} \frac{1}{|z|^q} + |z|^p - 1\,d\pi^3_\#(\mu^n)(z)\,dx \end{aligned}$$

for every n. On the other hand, by the continuity of the projection mapping and the application of Theorem 2.1, passing to the liminf and using the fact that $h(x_3) \geq h_0 > 0$ for almost every $x_3 \in (0,\ell)$, we get

$$\begin{aligned} +\infty &> \liminf_{n\to+\infty} \int_0^\ell ⨍_{h(x_3)\omega} \int_{\mathbb{R}^3} \frac{1}{|z|^q} + |z|^p\,d\pi^3_\#\mu^n_{(x_\alpha,x_3)}(z)\,dx_\alpha\,dx_3 \\ &= \liminf_{n\to+\infty} \int_0^\ell \int_{\mathbb{R}^3} \frac{1}{|z|^q} + |z|^p\,d\rho(\mu^n)_{x_3}(z)\,dx_3 \\ &\geq \int_0^\ell \int_{\mathbb{R}^3} \frac{1}{|z|^q} + |z|^p\,d\nu_{x_3}(z)\,dx_3, \end{aligned}$$

and therefore $\nu \in \mathcal{Y}^{-q,p}((0,\ell);\mathbb{R}^3)$. Finally, the property $\langle \nu, \mathrm{id}\rangle = y'$ follows by taking the average in

$$\langle \pi^3_{\#}\mu_{(x_\alpha,x_3)}, \mathrm{id}\rangle = \langle \mu_{(x_\alpha,x_3)}, \mathrm{id}\circ\pi^3\rangle = D_3 y(x_3)$$

where the last equality can be obtained by passing to the limit as $n \to \infty$ in the identity $\langle \mu^n, \mathrm{id}\circ\pi^3\rangle = D_3 y^n(x_3)$. □

6 Γ-convergence and relaxation

In the next sections we take the limit as ε goes to 0 of the sequence of functionals (3.3). The tool we shall use is a kind of variational convergence which allows to treat families of functionals $F_\varepsilon : X \to \overline{\mathbb{R}} = (-\infty, +\infty]$ defined on a space which may be different from the domain of the limiting functional. It is a variant of De Giorgi's Γ-convergence, and has been introduced by Anzellotti, Baldo and Percivale in [3] in order to study dimension reduction problems in mechanics. Let $\rho : X \to Y$ be a map between a set X and a topological space (Y, τ).

Definition 6.1 *Given a sequence ε_n of positive real numbers we say that a sequence $F_{\varepsilon_n} : X \to \overline{\mathbb{R}}$ (sequentially) $\Gamma(\rho, \tau Y)$-converges to a functional $F : Y \to \overline{\mathbb{R}}$ at a point $y \in Y$, and we write*

$$\Gamma(\rho, \tau Y) \lim_{n\to\infty} F_{\varepsilon_n}(y) = F(y),$$

if the following two conditions are satisfied:

1. *for every sequence $x_n \in X$ such that $\rho(x_n) \xrightarrow{\tau} y$ one has*
$$\liminf_{n\to\infty} F_{\varepsilon_n}(x_n) \geq F(y);$$

2. *there exists a sequence $\overline{x}_n \in X$ such that $\rho(\overline{x}_n) \xrightarrow{\tau} y$ and*
$$\lim_{n\to\infty} F_{\varepsilon_n}(\overline{x}_n) = F(y).$$

The family of functionals $F_\varepsilon : X \to \overline{\mathbb{R}}$ (sequentially) $\Gamma(\rho, \tau Y)$-converges to a functional $F : Y \to \overline{\mathbb{R}}$ at a point $y \in Y$, and we write

$$\Gamma(\rho, \tau Y) \lim_{\varepsilon\to 0} F_\varepsilon(y) = F(y) \tag{6.1}$$

if for any sequence ε_n of positive reals such that $\lim_{n\to\infty} \varepsilon_n = 0$ we have

$$\Gamma(\rho, \tau Y) \lim_{n\to\infty} F_{\varepsilon_n}(y) = F(y).$$

A family of functionals $\Gamma(\rho, \tau Y)$-converges on a set if it $\Gamma(\rho, \tau Y)$-converges at every point of the set.

The Γ-limit defined above are, in fact, Γ^--limit but, here and in the sequel as well, we have dropped the minus sign for notational simplicity. If $X = Y$ and ρ is the identity map, then the Γ-limits defined above are the usual De Giorgi's sequential Γ^--limits which will be denoted by the classical notation

$$\Gamma^-(\tau X) \lim_{\varepsilon\to 0} F_\varepsilon = F.$$

When the topological space (Y, τ) is first countable, and the family F_ε is equi-coercive then the $\Gamma(\rho, \tau Y)$-convergence has a variational character, that is, it ensures lower semicontinuity to the limit which turns out to be coercive when uniform coercivity assumptions are made on the family F_ε; moreover, roughly speaking, it preserves convergence of minima and of minimizers. This is made precise in the following statements, see [3].

Definition 6.2 *The family F_ε is said to be $(\rho, \tau Y)$-equi-coercive if for every real number M there exists a τ-compact and τ-closed subset K_M of Y such that*

$$\{\rho(x) \ : \ F_\varepsilon(x) \le M\} \subseteq K_M \textit{ for every } \varepsilon > 0.$$

The definition above reduces to the classical one of τ-equi-coerciveness in the case $\rho = \mathrm{id}$.

Proposition 6.3 *Let us assume that $\Gamma(\rho, \tau Y) \lim_{\varepsilon\to 0} F_\varepsilon = F$ on Y and that the family F_ε be $(\rho, \tau Y)$-equi-coercive. Then we have:*

(i) F is τ-lower semicontinuous;

(ii) F is τ-coercive;

(iii) if $x_\varepsilon \in X$ satisfy $\liminf_{\varepsilon\to 0} F_\varepsilon(x_\varepsilon) = \liminf_{\varepsilon\to 0} \inf F_\varepsilon$ (e.g. if x_ε minimizes F_ε) then

(a) if ε_n is a sequence such that $\lim_{n\to\infty} \varepsilon_n = 0$ and if $\rho(x_{\varepsilon_n}) \xrightarrow{\tau} y$ then y is a minimizer of F on Y and $\lim_{n\to\infty} F_{\varepsilon_n}(x_{\varepsilon_n}) = F(y)$;

(b) there is a sequence ε_n such that $\lim_{n\to\infty} \varepsilon_n = 0$ and a minimizer y of F on Y such that $\rho(x_{\varepsilon_n}) \xrightarrow{\tau} y$.

The following proposition is an easy consequence of the weak* metrizability of compact subsets of Y and of the Urysohn property of Γ-convergence (see for instance Dal Maso [12], Chapter 8).

Proposition 6.4 *If Y is dual of a separable Banach space, τ is the weak* topology, and $F_n : X \to \overline{\mathbb{R}}$ is $(\rho, \tau Y)$-equi-coercive, then*

$$\Gamma(\rho, \tau Y) \lim_{n\to\infty} F_n(y) = F(y)$$

if and only if

(i) $\liminf_{n\to\infty} F_n(x_n) \geq F(y)$ *for every sequence* $x_n \in X$ *such that* $\rho(x_n) \xrightarrow{\tau} y$*;*

(ii) for every sequence (n_k) *of positive integers there is a subsequence* (n_{k_p}) *and a sequence* $x_p \in X$ *such that*

$$\rho(x_p) \xrightarrow{\tau} y \quad \text{and} \quad \lim_{p\to\infty} F_{n_{k_p}}(x_p) = F(y).$$

The Γ-limit of a constant sequence, $F_n = F : Y \to \overline{\mathbb{R}}$ for every n, is the τ-lower semicontinuous envelope of F and is denoted by $\Gamma^-(\tau Y)F$. Moreover, if the functional F is defined only on a proper subset of Y then $\Gamma^-(\tau Y)F$ will denote the τ-lower semicontinuous envelope of the extension by $+\infty$ of F to the whole of Y and will be briefly called the *relaxation* of F on the space (Y, τ).

7 The limit problem under the highest growth assumptions in compression

In this section we identify the limit problem when the integrand f satisfies condition (3.2) with $q = p/2$. By Proposition 7.1 below, this corresponds to the highest value that q can assume.

Besides f_0, it will be useful to introduce also the integrands

$$f_0^\delta(t,\xi,z) := \min\{f\big(t,\xi,(\bar F|z)\big) \;:\; \bar F \in M^{3\times 2},\ \det(\bar F|z) \geq \delta\}$$

where $\delta \geq 0$, $t \in (0,\ell)$ and $\xi, z \in \mathbb{R}^3$, with the usual convention $\min\varnothing = +\infty$. We note that from (3.1) follows that $f_0 = f_0^0$.

Proposition 7.1 *Let* f *satisfy* (3.1)*. Then, for every* $0 \leq \delta \leq 1$ *the integrands* $f_0^\delta : (0,\ell)\times\mathbb{R}^3\times\mathbb{R}^3 \to \overline{\mathbb{R}}$ *are Carathéodory and there exists a positive constant* C *such that*

$$f_0^\delta(\cdot,\cdot,z) \leq C\big(\frac{1}{|z|^{p/2}} + |z|^p + 1\big) \quad \text{for every } z \neq 0.$$

Moreover, f_0^δ *is increasing with respect to* δ *and* $\lim_{\delta\to 0^+} f_0^\delta = f_0$*.*

PROOF. The integrand f_0^δ is measurable and upper semicontinuous in the last two variables since it is the infimum of Carathéodory integrands. From the continuity in the last variables and the growth assumption from below on f it immediately follows that $f_0^\delta(t,\cdot,\cdot)$ is also lower semicontinuous and hence continuous. The growth condition from above follows by taking besides a vector $z \neq 0$, another vector $\hat z$ such that $z\cdot\hat z = 0$ and $|\hat z| = |z|^{-1/2}$. Let $\tilde z := \hat z \times z/|z|$. Note that $|\tilde z| = |\hat z| = |z|^{-1/2}$ and

$$\det(\tilde z|\hat z|z) = \tilde z\cdot\hat z\times z = \tilde z\cdot\tilde z\,|z| = 1.$$

Hence, for $\delta \leq 1$,

$$f_0^\delta(\cdot,\cdot,z) \leq f(\cdot,\cdot,(\tilde z|\hat z|z)) \leq \widetilde C(|(\tilde z|\hat z|z)|^p + 1) \leq C(\frac{1}{|z|^{p/2}} + |z|^p + 1).$$

The other properties are trivially proved. □

In view of the proposition above, we shall assume that (3.2) holds with $q = p/2$.

To deduce the limit problem we need to prove two lemmata which will be useful to handle the constraint on the determinant.

Lemma 7.2 *Let $g_1, g_2, g_3 \in L^\infty(0,\ell)$ such that $g_1, g_2 > 0$ and $g_1 g_2 g_3 \geq \delta > 0$ almost everywhere in $(0,\ell)$. Then there exist sequences $g_1^\varepsilon, g_2^\varepsilon, g_3^\varepsilon \in C^\infty([0,\ell])$ such that $\|g_i^\varepsilon\|_\infty \leq \|g_i\|_\infty$, $g_i^\varepsilon \to g_i$, in $L^p(0,\ell)$ for any $1 \leq p < +\infty$, and for $i = 1,2,3$, and $g_1^\varepsilon g_2^\varepsilon g_3^\varepsilon \geq \delta > 0$ for every $\varepsilon > 0$. Furthermore, if $\delta < 1$, then we can choose the g_i^ε so that $g_i^\varepsilon(0) = 1$ for $i = 1,2,3$ and for every $\varepsilon > 0$.*

PROOF. Before we start the proof we note that if g is any function in $L^\infty(0,\ell)$ such that $g \geq C > 0$, almost everywhere, and η_ε is a positive mollifier such that $\int \eta_\varepsilon \, dt = 1$, then $g^\varepsilon := \eta_\varepsilon * g \to g$ in $L^p(0,\ell)$ for any $1 \leq p < +\infty$ and $C \leq g^\varepsilon \leq \|g\|_\infty$ for all $\varepsilon > 0$. Since

$$g_1 \geq \min\Big\{1, \frac{\delta}{\|g_2\|_\infty \|g_3\|_\infty}\Big\} =: m_1 \quad \text{and} \quad g_2 \geq \min\Big\{1, \frac{\delta}{\|g_1\|_\infty \|g_3\|_\infty}\Big\} =: m_2,$$

there exist two sequences $g_1^\varepsilon, g_2^\varepsilon \in C^\infty([0,\ell])$ such that $g_\alpha^\varepsilon(0) = 1$, $g_\alpha^\varepsilon \to g_\alpha$ in $L^p(0,\ell)$ for $1 \leq p < +\infty$, and

$$m_\alpha \leq g_\alpha^\varepsilon \leq \|g_\alpha\|_\infty,$$

for $\alpha = 1,2$. Let $d := g_1 g_2 g_3 \geq \delta$. There exists a sequence $d^\varepsilon \in C^\infty([0,\ell])$ such that $d^\varepsilon \to d$ in $L^p(0,\ell)$ for any $1 \leq p < +\infty$, and $d^\varepsilon \geq \delta$. Moreover, if $\delta < 1$ we can choose d^ε so that $d^\varepsilon(0) = 1$. Define

$$g_3^\varepsilon := \frac{d^\varepsilon}{g_1^\varepsilon g_2^\varepsilon},$$

and note that $g_3^\varepsilon \in C^\infty([0,\ell])$, $g_3^\varepsilon(0) = 1$ if $d^\varepsilon(0) = 1$, and that

$$\|g_3^\varepsilon - g_3\|_p \leq \frac{1}{m_1^2 m_2^2} \|d^\varepsilon - g_1^\varepsilon g_2^\varepsilon g_3\|_p \to 0$$

for any $1 \leq p < +\infty$. Since $g_1^\varepsilon g_2^\varepsilon g_3^\varepsilon = d^\varepsilon \geq \delta$ the proof is concluded. □

In the next lemma we will use the following result: if $U : (0,\ell) \to M^{3\times 3}$ is a measurable map which takes values into the subset of symmetric matrices, then it admits eigenvalues λ_i and eigenvectors u_i, for $i = 1,2,3$, real and measurable. In fact, as a consequence of Weyl's theorem (see Horn and Johnson [14]) the eigenvalues λ_i of a symmetric matrix, arranged in an increasing order, are Lipschitz continuous functions of the matrix. Therefore the composition $\lambda_i(U(t))$ is measurable. The eigenvectors are then the non trivial solutions of a homogeneous system with measurable coefficients, and therefore there are measurable solutions. We recall, moreover that by the polar decomposition theorem, any non singular matrix can be written as a product of a proper orthogonal matrix and a positive symmetric matrix. In particular, if $F : (0, \ell) \to M^{3\times 3}$ is measurable, and for almost every $t \in (0, \ell)$, $F(t)$ is non singular, then $F(t) = R(t)U(t)$ where both the rotation R and the positive symmetric matrix U are measurable. Indeed, since $F^T F = U^2$, we deduce that U^2 is measurable. Thus, if $\eta_i > 0$ and u_i are respectively measurable eigenvalues and eigenvectors of U^2, then, by the spectral theorem, we have $U^2(t) = \sum_{i=1}^3 \eta_i(t)u_i(t) \otimes u_i(t)$, hence $U(t) = \sum_{i=1}^3 \sqrt{\eta_i(t)}u_i(t) \otimes u_i(t)$. From what we said above, we have that η_i and u_i are measurable maps, and thus U is measurable. Finally, since $R = FU^{-1}$ we deduce that also R is a measurable map.

Lemma 7.3 *Let $F \in L^\infty((0, \ell); M^{3\times 3})$ such that* $\det F \geq \delta > 0$. *Then there exists a sequence $F^\varepsilon \in C^\infty([0, \ell]; M^{3\times 3})$ such that $F^\varepsilon \to F$, in $L^p((0, \ell); M^{3\times 3})$ for every $1 \leq p < +\infty$, and* $\det F^\varepsilon \geq \delta$. *Moreover, if $\delta < 1$, F^ε can be chosen so that $F^\varepsilon(0) = I$.*

PROOF. For every $t \in (0, \ell)$, by the polar decomposition theorem, there exist an orthogonal matrix $R(t)$ and a symmetric positive definite matrix $U(t)$ such that $F(t) = R(t)U(t)$. Since $|R| = \sqrt{3}$ we have that $U = R^T F \in L^\infty((0, \ell); M^{3\times 3})$. By the spectral theorem, the matrix $U(t)$ can be written as

$$U(t) = \sum_{i=1}^{3} \lambda_i(t)u_i(t) \otimes u_i(t),$$

where $\lambda_i > 0$, $u_i \cdot u_j = \delta_{ij}$, for $i, j = 1, 2, 3$ and $\lambda_1\lambda_2\lambda_3 \geq \delta$ almost everywhere, since $\det U \geq \delta$. We further suppose that the eigenvectors u_i are ordered so that $u_3 = u_1 \times u_2$. It is easy to check that the eigenvalues $\lambda_i \in L^\infty(0, \ell)$. Hence by Lemma 7.2 we may find three sequences $\lambda_1^\varepsilon, \lambda_2^\varepsilon, \lambda_3^\varepsilon \in C^\infty([0, \ell])$ such that $\|\lambda_i^\varepsilon\|_\infty \leq \|\lambda_i\|_\infty$, $\lambda_i^\varepsilon \to \lambda_i$ in $L^p(0, \ell)$ for $1 \leq p < +\infty$, for $i = 1, 2, 3$, and $\lambda_1^\varepsilon\lambda_2^\varepsilon\lambda_3^\varepsilon \geq \delta > 0$. For every $t \in (0, \ell)$, let $Q(t)$ be the orthogonal matrix that maps the vectors of the fixed basis to the eigenvectors of $U(t)$, that is $u_i(t) = Q(t)e_i$, for $i = 1, 2, 3$. Let $Q^\varepsilon \in C^\infty([0, \ell]; M^{3\times 3})$ be a sequence of orthogonal matrices such that $Q^\varepsilon \to Q$ in $L^p((0, \ell); M^{3\times 3})$ for every $1 \leq p < +\infty$ (this can be achieved by writing the matrix Q in terms of Euler's angles and by smoothing them). Then, setting $u_i^\varepsilon := Q^\varepsilon e_i$ for $i = 1, 2, 3$, we have that $u_i^\varepsilon \cdot u_j^\varepsilon = \delta_{ij}$ for every $i, j = 1, 2, 3$, and $u_i^\varepsilon \to u_i$ in $L^p((0, \ell); \mathbb{R}^3)$ for every $1 \leq p < +\infty$. Define

$$U^\varepsilon(t) = \sum_{i=1}^{3} \lambda_i^\varepsilon(t)u_i^\varepsilon(t) \otimes u_i^\varepsilon(t).$$

It is easily seen that $U^\varepsilon \in C^\infty([0, \ell]; M^{3\times 3})$, that $U^\varepsilon \to U$ in $L^p((0, \ell); M^{3\times 3})$ for every $1 \leq p < +\infty$, and that $\det U^\varepsilon = \lambda_1^\varepsilon\lambda_2^\varepsilon\lambda_3^\varepsilon \geq \delta$. To conclude the first part of the proof it suffices to choose a sequence $R^\varepsilon \in C^\infty([0, \ell]; M^{3\times 3})$ of orthogonal matrices such that $R^\varepsilon \to R$ in $L^p((0, \ell); M^{3\times 3})$ for every $1 \leq p < +\infty$, and to set $F^\varepsilon = R^\varepsilon U^\varepsilon$. If $\delta < 1$, then we may choose $\lambda_i^\varepsilon(0) = 1$ for $i = 1, 2, 3$ so that $U^\varepsilon(0) = I$ and, choosing $R^\varepsilon(0) = I$, we obtain $F^\varepsilon(0) = I$. □

We are now ready to prove the main result of this section.

Theorem 7.4 *Let f satisfy* (3.1) *and assumption* (3.2) *with $q = p/2$. The functionals I_ε^∞ defined in* (3.3) *are $\big(\rho, w^* L_w^\infty((0,\ell);\mathcal{M}(\mathbb{R}^3))\big)$-equi-coercive (see Definition 6.2). Moreover*

$$\Gamma(\rho, w^* L_w^\infty((0,\ell);\mathcal{M}(\mathbb{R}^3))) \lim_{\varepsilon\to 0^+} I_\varepsilon^\infty(\nu) = I(\nu)$$

with

$$I(\nu) = \begin{cases} \displaystyle\int_0^\ell \Big(\langle \nu_t, f_0(t, y(t), \cdot)\rangle - \mathrm{Av}^\alpha g \cdot y \Big) h(t)\, dt & \text{if } \nu \in \mathcal{Y}^{-p/2,p}((0,\ell);\mathbb{R}^3) \\ +\infty & \text{otherwise in } L_w^\infty((0,\ell);\mathcal{M}(\mathbb{R}^3)) \end{cases}$$

where $\mathrm{Av}^\alpha g(x_3)$ denotes the average of g over the cross section with coordinate x_3, that is

$$\mathrm{Av}^\alpha g(x_3) = \frac{1}{h^2(x_3)} \int_{h(x_3)\omega} g(x_\alpha, x_3)\, dx_\alpha$$

and $y \in W^{1,p}((0,\ell);\mathbb{R}^3)$ is the underlying deformation of ν with boundary condition $y(0) = 0$.

PROOF. Since the space $L_w^\infty((0,\ell);\mathcal{M}(\mathbb{R}^3))$ is dual of the separable Banach space $L^1((0,\ell);C_0(\mathbb{R}^3))$, the claimed coerciveness follows by Lemma 5.1 and the continuity of the map ρ.

We start by finding a recovering sequence for a suitable subsequence $I_{\varepsilon_{n_k}}^\infty$; that is, for $\nu \in \mathcal{Y}^{-q,p}((0,\ell);\mathbb{R}^3)$ (where, throughout the proof, $q = p/2$) we find a sequence $y_k \in W^{1,p}(\Omega;\mathbb{R}^3)$ with $y_k(x_1,x_2,0) = \varepsilon_{n_k}(x_1,x_2,0)$ such that, setting $\mu^k = \delta_{Dy_k}$, we have $\rho(\mu^k) \to \nu$ weakly* in $L_w^\infty((0,\ell);\mathcal{M}(\mathbb{R}^3))$, and

$$\limsup_{k\to\infty} I_{\varepsilon_{n_k}}^\infty(\mu^k) \le I(\nu).$$

By Lemma 4.4 we find a sequence $\widetilde{v}_j \in L^\infty((0,\ell);\mathbb{R}^3)$ which generates ν such that $1/|\widetilde{v}_j| \in L^\infty((0,\ell);\mathbb{R}^3)$ and

$$\frac{1}{|\widetilde{v}_j|^q} + |\widetilde{v}_j|^p$$

is equi-integrable. Let

$$v_j(x_3) := \int_0^{x_3} \widetilde{v}_j(t)\, dt,$$

so that $v_j' = \widetilde{v}_j$. By a measurable selection argument (see for instance Buttazzo [9], Proposition 2.2.7), and by definition of f_0^δ, there exists a sequence of measurable functions $\bar{F}_j^\delta : (0,\ell) \to M^{3\times 2}$ such that

$$f_0^\delta(x_3, v_j(x_3), v_j'(x_3)) = f\big(x_3, v_j(x_3), (\bar{F}_j^\delta(x_3)|v_j'(x_3))\big) \quad \text{a.e. } x_3 \in (0,\ell),$$

with $\det(\bar F^\delta_j|v'_j)\geq\delta$ almost everywhere. By the growth conditions we have also

$$c(|\bar F^\delta_j|^p+|v'_j|^p-1)\leq f\big(x_3,v_j,(\bar F^\delta_j|v'_j)\big)=f^\delta_0(x_3,v_j,v'_j)\leq C\big(\frac{1}{|v'_j|^q}+|v'_j|^p+1\big)$$

almost everywhere, and therefore the sequence $(\bar F^\delta_j|v'_j)$ is p-equi-integrable, and for each j, $\bar F^\delta_j\in L^\infty((0,\ell);M^{3\times 2})$. By Lemma 7.3, for every δ and every j, there exists a function $(\overline{G}^{\,\delta}_j|\widetilde w^\delta_j)\in C^\infty([0,\ell];M^{3\times 3})$ such that $(\overline{G}^{\,\delta}_j|\widetilde w^\delta_j)(0)=I$, $\det(\overline{G}^{\,\delta}_j|\widetilde w^\delta_j)>\delta$ and

$$\|(\overline{G}^{\,\delta}_j|\widetilde w^\delta_j)-(\bar F^\delta_j|v'_j)\|_p<\frac{1}{j}. \tag{7.1}$$

Let

$$w^\delta_j(x_3):=\int_0^{x_3}\widetilde w^\delta_j(t)\,dt,$$

and $\varepsilon^\delta_{n_j}$ be a subsequence of ε_n such that

$$\varepsilon^\delta_{n_j}\big(\|\overline{G}^{\,\delta\prime}_j\|_p+\|\det(\overline{G}^{\,\delta}_j|\overline{G}^{\,\delta\prime}_j x_\alpha)\|_\infty\big)\to 0. \tag{7.2}$$

The function

$$y^\delta_j(x):=w^\delta_j(x_3)+\varepsilon^\delta_{n_j}\overline{G}^{\,\delta}_j(x_3)x_\alpha$$

satisfies the boundary condition at level ε_{n_j} and

$$\frac{D_\alpha y^\delta_j}{\varepsilon^\delta_{n_j}}=\overline{G}^{\,\delta}_j(x_3),\qquad D_3y^\delta_j=w^{\delta\prime}_j+\varepsilon^\delta_{n_j}\overline{G}^{\,\delta\prime}_j(x_3)x_\alpha. \tag{7.3}$$

Thus, from (7.1) and (7.2) we infer that, for every $\delta>0$

$$\big\|\big(\frac{D_\alpha y^\delta_j}{\varepsilon^\delta_{n_j}}|D_3y^\delta_j\big)-\big(\bar F^\delta_j|v'_j\big)\|_p\to 0\ \text{ as } j\to\infty.$$

Hence, for every $\delta>0$, the two sequences

$$\big(y^\delta_j,\big(\frac{D_\alpha y^\delta_j}{\varepsilon^\delta_{n_j}}|D_3y^\delta_j\big)\big),\qquad \big(v_j,(\bar F^\delta_j|v'_j)\big)$$

are p-equi-integrable and generate the same Young measure. Moreover, from (7.2) and (7.3) follows that, for j large enough

$$\det\big(\frac{D_\alpha y^\delta_j}{\varepsilon^\delta_{n_j}}|D_3y^\delta_j\big)>\frac{\delta}{4},\qquad \det\big(\bar F^\delta_j|v'_j\big)>\delta$$

which, together with the growth condition on f, imply that the sequences

$$f\big(x_3,y^\delta_j,\big(\frac{D_\alpha y^\delta_j}{\varepsilon_{n^\delta_j}}|D_3y^\delta_j\big)\big),\qquad f\big(x_3,v_j,(\bar F^\delta_j|v'_j)\big)$$

are equi-integrable. By applying Theorem 2.1 we get

$$\begin{aligned}\lim_{j\to\infty}\int_\Omega f\big(x_3, y_j^\delta, (\frac{D_\alpha y_j^\delta}{\varepsilon_{n_j^\delta}}|D_3 y_j^\delta)\big)\,dx &= \lim_{j\to\infty}\int_\Omega f\big(x_3, v_j, (\bar F_j^\delta|v_j')\big)\,dx\\ &= \lim_{j\to\infty}\int_\Omega f_0^\delta(x_3, v_j, v_j')\,dx\\ &= \int_\Omega\int_{\mathbb{R}^3} f_0^\delta(x_3, y(x_3), z)\,d\nu_{x_3}(z)\,dx\\ &= \int_0^\ell h(x_3)\int_{\mathbb{R}^3} f_0^\delta(x_3, y, z)\,d\nu_{x_3}(z)\,dx_3,\end{aligned} \tag{7.4}$$

where y is the underlying deformation of ν with $y(0)=0$, or, said differently, $v_j \to y$ weakly in $W^{1,p}((0,\ell);\mathbb{R}^3)$. From an application of Beppo-Levi monotone convergence theorem we have

$$\lim_{\delta\to 0^+}\lim_{j\to\infty}\int_\Omega f\big(x_3, y_j^\delta, (\frac{D_\alpha y_j^\delta}{\varepsilon_{n_j^\delta}}|D_3 y_j^\delta)\big)\,dx = \int_0^\ell h(x_3)\int_{\mathbb{R}^3} f_0(x_3, y, z)\,d\nu_{x_3}(z)\,dx_3.$$

By a standard diagonalization argument, there exist sequences $\delta_k \to 0$ and $j_k \to \infty$ such that $\varepsilon_{j_k}^{\delta_k}$ is a subsequence of ε_n which will be denoted by ε_{n_k} and, setting $y_k := y_{j_k}^{\delta_k}$, we have

$$\lim_{k\to\infty}\int_\Omega f\big(x_3, y_k, (\frac{D_\alpha y_k}{\varepsilon_{n_k}}|D_3 y_k)\big)\,dx = \int_0^\ell h(x_3)\int_{\mathbb{R}^3} f_0(x_3, y, z)\,d\nu_{x_3}(z)\,dx_3.$$

Moreover, it is easy to check that y_k satisfies the appropriate boundary condition. Furthermore, since $y_k \to y$ in $L^p(\Omega;\mathbb{R}^3)$ we also have

$$\lim_{k\to\infty}\int_\Omega g(x)\cdot y_k(x)\,dx = \int_\Omega g(x)\cdot y(x_3)\,dx = \int_0^\ell\int_{h(x_3)\omega} g(x_\alpha, x_3)\,dx_\alpha\cdot y(x_3)\,dx_3.$$

Thus, the proof of the limsup inequality is concluded.

The next step consists in proving the liminf inequality for the sequence $I_{\varepsilon_n}^\infty$. Let $\mu^n \in L_w^\infty(\Omega;\mathcal{M}(M^{3\times 3}))$ be a sequence such that $\rho(\mu^n) \to \nu$ weakly* in $L_w^\infty((0,\ell);\mathcal{M}(\mathbb{R}^3))$. We have to prove that

$$\liminf_{n\to\infty} I_{\varepsilon_n}^\infty(\mu^n) \ge I(\nu).$$

Without loss of generality we may suppose that the left hand side of the equation above be finite and that the liminf be indeed a limit. Then $\sup_n I_{\varepsilon_n}^\infty(\mu^n) < +\infty$, hence, by Lemma 5.1 there exist $\mu \in L_w^\infty(\Omega;\mathcal{M}(M^{3\times 3}))$ and a subsequence μ^{n_k} such that

$$\mu^{n_k} \to \mu \ \text{ weakly* in } L_w^\infty(\Omega;\mathcal{M}(M^{3\times 3})),$$

$\nu := \rho(\mu) \in \mathcal{Y}^{-q,p}((0,\ell);\mathbb{R}^3)$ and the corresponding underlying deformations y_{n_k} and y which satisfy the boundary conditions $y_{n_k}(x_1,x_2,0) = \varepsilon_{n_k}(x_1,x_2,0)$

and $y(0) = 0$ are such that

$$y_{n_k} \to y \text{ weakly in } W^{1,p}(\Omega;\mathbb{R}^3) \quad \text{and} \quad y = y(x_3). \tag{7.5}$$

Moreover,

$$\begin{aligned}
&\liminf_{n\to\infty} I^\infty_{\varepsilon_n}(\mu^n) = \lim_{n\to\infty} I^\infty_{\varepsilon_n}(\mu^n) = \lim_{k\to\infty} I^\infty_{\varepsilon_{n_k}}(\mu^{n_k}) = \\
&= \lim_{k\to\infty}\Big(\int_\Omega f\big(x_3, y_{n_k}, (\frac{D_\alpha y_{n_k}}{\varepsilon_{n_k}}|D_3 y_{n_k})\big)\,dx - \int_\Omega g\cdot y_{n_k}\,dx\Big) \\
&\geq \lim_{k\to\infty}\Big(\int_\Omega f_0(x_3, y_{n_k}, D_3 y_{n_k})\,dx - \int_\Omega g\cdot y_{n_k}\,dx\Big) \\
&\geq \int_\Omega \int_{\mathbb{R}^3\times\mathbb{R}^3} f_0(x_3,\xi,z)\,d\delta_{y(x_3)}\otimes \pi^3_{\#}\mu_x(\xi,z) - \int_\Omega g\cdot y(x_3)\,dx \\
&= \int_0^\ell \int_{h(x_3)\omega}\int_{\mathbb{R}^3} f_0(x_3,y(x_3),z)\,d\pi^3_{\#}\mu_x(z)\,dx_\alpha dx_3 - \int_\Omega g\cdot y(x_3)\,dx \\
&= \int_0^\ell h(x_3)\langle \mathrm{Av}^\alpha_{x_3}\,\pi^3_{\#}\mu, f_0(x_3,y(x_3),\cdot)\rangle\,dx_3 - \int_0^\ell \int_{h(x_3)\omega} g\,dx_\alpha\cdot y(x_3)\,dx_3
\end{aligned}$$

where in the last inequality we used (7.5), Theorem 2.1 and the fact that $(y_{n_k}, D_3 y_{n_k})$ generates $\delta_y\otimes\pi^3_{\#}\mu$, see Theorem 1 and Lemma 2 of [13]. □

By the properties of Γ-limits summarized in Proposition 6.3 it is possible to make precise the variational character of the limit energy $I(\nu)$, thus obtaining a characterization of the asymptotic behaviour of minima and minimizers of the energies at level ε (as, for instance, it has been done in Theorem 6 of [13] in the framework of 3D-2D dimension reduction).

Remark 7.5 In fact, the proof of Theorem 7.4 works also by assuming f only lower semicontinuous and requiring the continuity of f_0^δ for any $\delta \geq 0$.

Example 7.6 A simple example which falls into the setting above is

$$f(F) = |F|^p + \sum_{i=1}^3 \frac{c_i}{|F\,\mathrm{e}_i\,|^{p/2}},$$

if $\det F > 0$, with $c_i \geq 0$ for $i = 1,2$, $c_3 > 0$ and where $\{\mathrm{e}_1,\mathrm{e}_2,\mathrm{e}_2\}$ is an orthonormal basis of $\mathbb{R}^3$. Let us remark, moreover, that this function is frame indifferent, that is $f(QF) = f(F)$ for every $Q \in SO(3)$. In the simplest case $p = 2$ an easy computation shows that

$$f_0(z) = |z|^2 + \frac{c_3}{|z|} + \big(3/2\big)^{2/3}\big(c_1^{2/3} + c_2^{2/3}\big).$$

8 The limit problem for Ogden materials

Compressible neo-Hookean materials are characterized by an energy density of the form

$$f(F) = a|F|^2 + \beta(\det F)$$

where $a > 0$ and the function β satisfies the property $\lim_{\lambda\to 0^+}\beta(\lambda) = +\infty$. Generally, these kind of materials do not satisfy the growth assumptions made in the previous section. For this reason, we here consider a refinement of hypotheses (3.1) which covers, among other energies, a large class of compressible Ogden materials. This is a wider class, which contains the neo-Hookean materials, characterized by an energy density of the form

$$f(F) = \sum_{i=1}^{M} a_i \operatorname{tr} C^{\gamma_i/2} + \sum_{j=1}^{N} b_j \operatorname{tr}(\operatorname{cof} C)^{\delta_j/2} + \beta(\det F), \quad C = F^T F$$

where $a_i > 0$, $\gamma_i, \delta_i \geq 1$, $b_j \geq 0$ (see Ogden [15, 16] and Ciarlet [10]) and β is as above. Antman, in [2], considered energies of the form

$$f(F) = a(\operatorname{tr} C^{1/2})^{p/2} + b(\operatorname{tr} C)^{p/2} + c\frac{1}{|\det F|^s}$$

where $a, b, c, p, s > 0$. The following growth assumptions include the whole class of Antman materials.

$$\begin{aligned}
&\det F \leq 0 \;\Rightarrow\; f(\cdot,\cdot,F) \equiv +\infty,\\
&\det F > 0 \;\Rightarrow\; \text{there exists two constants } C \geq c > 0 \text{ such that}\\
&\qquad c\Big(\frac{1}{|\det F|^s} + |F|^p - 1\Big) \leq f(\cdot,\cdot,F) \leq C\Big(\frac{1}{|\det F|^s} + |F|^p + 1\Big),
\end{aligned} \tag{8.1}$$

for suitable $p \in [1, +\infty)$ and $s \in (0, +\infty)$.

Proposition 8.1 *Let f satisfy* (8.1)*. Then the function f_0 defined in* (5.1) *is a Carathéodory integrand and there exist two positive constants C and c such that*

$$c\Big(\frac{1}{|z|^q} + |z|^p - 1\Big) \leq f_0(\cdot,\cdot,z) \leq C\Big(\frac{1}{|z|^q} + |z|^p + 1\Big) \quad \textit{for every } z \neq 0,$$

with $q = \dfrac{ps}{p+2s}$.

PROOF. It has been already proved, in Proposition 7.1, that f_0 is a Carathéodory integrand.

Since $\det(\bar F|z) \leq |\bar F|^2|z|$, we have

$$\begin{aligned}
f_0(\cdot,\cdot,z) &\geq \tilde c \min\Big\{\frac{1}{|\bar F|^{2s}|z|^s} + |\bar F|^p + |z|^p - 1 \;:\; \bar F \in M^{3\times 2}\Big\}\\
&= \tilde c\Big(\min\Big\{\frac{1}{x^{2s}|z|^s} + x^p \;:\; x \in [0,+\infty)\Big\} + |z|^p - 1\Big)\\
&\geq c\Big(\frac{1}{|z|^{ps/(p+2s)}} + |z|^p - 1\Big).
\end{aligned}$$

The growth condition from above follows by observing that

$$\begin{aligned} f_0(\cdot,\cdot,z) &\le \min\big\{ f(\cdot,\cdot,(\frac{\hat z\times z}{|z|}|\hat z|z)) \ :\ z\cdot\hat z=0\big\} \\ &\le \widetilde C \min\big\{\frac{1}{|\hat z|^{2s}|z|^s}+|\hat z|^p \ :\ z\cdot\hat z=0\big\}+|z|^p+1 \\ &\le C\big(\frac{1}{|z|^{ps/(p+2s)}}+|z|^p+1\big), \end{aligned}$$

which concludes the proof. □

Let us observe that, in view of the proposition above, condition (3.2) holds with $q=ps/(p+2s)$, which is strictly smaller than $p/2$.

The following technical lemmata play the same role of Lemma 7.2 and Lemma 7.3.

Lemma 8.2 *Let $p\in[1,+\infty)$ and $s\in(0,+\infty)$. Let (g_1^j), (g_2^j), (g_3^j) be sequences of non negative functions in $L^\infty(0,\ell)$ such that*

1. *$1/g_i^j\in L^\infty(0,\ell)$ for $i=1,2,3$ and $j\in\mathbb N$;*
2. *$|g_i^j|^p$ are equi-integrable;*
3. *$1/|g_1^jg_2^jg_3^j|^s$ are equi-integrable.*

Then there exist sequences $(\tilde g_1^j)$, $(\tilde g_2^j)$, $(\tilde g_3^j)$ of functions in $C^\infty([0,\ell])$ with the properties 1, 2 and 3 above and

4. *$\tilde g_i^j(0)=1$ for $i=1,2,3$, and every $j\in\mathbb N$;*
5. *there exist constants $c_j>0$ such that $\tilde g_1^j\tilde g_2^j\tilde g_3^j\ge c_j$ for every $j\in\mathbb N$;*
6. *$\lim_{j\to\infty}\|\tilde g_i^j-g_i^j\|_p=0$ for $i=1,2,3$.*

PROOF. For $\alpha=1,2$, let us denote by $r_\alpha^j:=\big\|\frac{1}{g_\alpha^j}\big\|_\infty$ and observe that $\frac{1}{r_\alpha^j}\le g_\alpha^j$. By a standard smoothing procedure (see for instance the beginning of the proof of Lemma 7.2) there exist functions $\tilde g_\alpha^j\in C^\infty([0,\ell])$ such that

$$\begin{gathered} \min\{1,1/r_\alpha^j\}\le\tilde g_\alpha^j\le\|g_\alpha^j\|_\infty,\quad \tilde g_\alpha^j(0)=1 \text{ for } \alpha=1,2, \\ \|\tilde g_\alpha^j-g_\alpha^j\|_p<\frac{1}{j\max\{1,r_1^jr_2^j\|g_\beta^j\|_\infty\|g_3^j\|_\infty\}} \text{ with } \alpha,\beta\in\{1,2\},\ \alpha\ne\beta \end{gathered}\tag{8.2}$$

for every $j\in\mathbb N$. Let $d^j:=g_1^jg_2^jg_3^j$ and $\bar d^j:=\max\{d^j,c_j\}$ where we have set $c_j=\frac{1}{j\max\{1,r_1^jr_2^j\}}$. By smoothing again we find functions $\tilde d^j\in C^\infty([0,\ell])$ such that

$$\tilde d^j\ge c_j,\quad \tilde d^j(0)=1,\quad \|\tilde d^j-\bar d^j\|_r<c_j^3$$

for every $j \in \mathbb{N}$, where $r = \max\{s, p\}$. Then

$$\|\tilde{d}^j - d^j\|_p \le K(\|\tilde{d}^j - \bar{d}^j\|_r + \|\bar{d}^j - d^j\|_\infty) \le K(c_j^3 + c_j) \le 2Kc_j,$$

and

$$\|\frac{1}{\tilde{d}^j} - \frac{1}{\bar{d}^j}\|_r \le \|\frac{\tilde{d}^j - \bar{d}^j}{\tilde{d}^j \bar{d}^j}\|_r \le \frac{1}{c_j^2}\|\tilde{d}^j - \bar{d}^j\|_r < c_j.$$

Observing that, for every measurable subset B of $(0, \ell)$

$$\begin{aligned}\int_B \frac{1}{|\tilde{d}^j|^s}\, dx_3 &\le \int_B \Big|\frac{1}{\tilde{d}^j} - \frac{1}{\bar{d}^j}\Big|^s + \Big|\frac{1}{\bar{d}^j}\Big|^s dx_3 \\ &\le |B| + \int_B \Big|\frac{1}{\tilde{d}^j} - \frac{1}{\bar{d}^j}\Big|^r dx_3 + \int_B \Big|\frac{1}{d^j}\Big|^s dx_3\end{aligned}$$

we obtain that $\Big|\frac{1}{\tilde{d}^j}\Big|^s$ is equi-integrable, and property *3* for $\tilde{g}_i^j$, and *5*, follow by setting

$$\tilde{g}_3^j = \frac{\tilde{d}^j}{\tilde{g}_1^j \tilde{g}_2^j}.$$

Then, property *1* and *4* trivially holds for $i = 1, 2, 3$. Property *2* and *6* for $\alpha = 1, 2$, follow from (8.2).

It remains only to prove property *6* for $\tilde{g}_3^j$ (which implies *2*). To this end let us note that

$$\begin{aligned}\|\tilde{g}_3^j - g_3^j\|_p &\le r_1^j r_2^j \|\tilde{d}^j - \tilde{g}_1^j \tilde{g}_2^j g_3^j\|_p \\ &\le r_1^j r_2^j \Big(\|\tilde{d}^j - d^j\|_p + \|g_3^j\|_\infty \big(\|g_2^j\|_\infty \|\tilde{g}_1^j - g_1^j\|_p + \|g_1^j\|_\infty \|\tilde{g}_2^j - g_2^j\|_p\big)\Big) \\ &\le 2(K+1)/j\end{aligned}$$

and the proof is concluded. □

Lemma 8.3 *Let* $F_j \in L^\infty((0, \ell); M^{3\times 3})$ *such that* $\det F_j > 0$ *almost everywhere, for every* $j \in \mathbb{N}$*, and*

1. $(|F_j|^p)$ *is equi-integrable;*
2. $\dfrac{1}{\det F_j} \in L^\infty(0, \ell)$*;*
3. $\dfrac{1}{|\det F_j|^s}$ *is equi-integrable.*

Then there exists a sequence of functions (G_j) *in* $C^\infty([0, \ell]; M^{3\times 3})$ *with the properties 1, 2 and 3 above, and*

4. $G_j(0) = I$ *for every* $j \in \mathbb{N}$*;*
5. *there exist constants* $c_j > 0$ *such that* $\det G_j > c_j$ *and for every* $j \in \mathbb{N}$*;*

6. $\lim_{j\to\infty} \|G_j - F_j\|_p = 0.$

PROOF. For almost every $t \in (0,\ell)$, by the polar decomposition theorem, there exist orthogonal matrices $R_j(t)$ and symmetric positive definite matrices $U_j(t)$ such that $F_j(t) = R_j(t)U_j(t)$. Then $U_j \in L^\infty((0,\ell); M^{3\times 3})$. Let $\lambda_i^j > 0$, $i = 1,2,3$, be the eigenvalues of the matrix U_j. From hypothesis *1* follows that the sequences $(|\lambda_i^j|^p)_j$, $i = 1,2,3$, are equi-integrable and from assumption *3* we have that

$$\Big(\frac{1}{|\lambda_1^j\lambda_2^j\lambda_3^j|^s}\Big) \text{ is equi-integrable,}$$

while from *2* we obtain that

$$\frac{1}{\lambda_1^j\lambda_2^j\lambda_3^j} \in L^\infty(0,\ell) \quad \text{for every } j \in \mathbb{N}. \tag{8.3}$$

Since $F_j \in L^\infty((0,\ell); M^{3\times 3})$, then $\lambda_i^j \in L^\infty(0,\ell)$ and, by (8.3), we have also that $1/\lambda_i^j \in L^\infty(0,\ell)$. Therefore, we are in a position to apply Lemma 8.3 to the sequences (λ_i^j). The rest of the proof proceeds, with suitable adaptations, as that of Lemma 7.3. □

In the setting of the current section the Γ-convergence result is the following.

Theorem 8.4 *Let f satisfy assumptions* (8.1). *The functionals I_ε^∞ defined in* (3.3) *are $\big(\rho, w^*L_w^\infty((0,\ell);\mathcal{M}(\mathbb{R}^3))\big)$-equi-coercive. Moreover*

$$\Gamma(\rho, w^*L_w^\infty((0,\ell);\mathcal{M}(\mathbb{R}^3))) \lim_{\varepsilon\to 0^+} I_\varepsilon^\infty(\nu) = I(\nu)$$

with

$$I(\nu) = \begin{cases} \displaystyle\int_0^\ell \Big(\langle \nu_t, f_0(t,y(t),\cdot)\rangle - \mathrm{Av}^\alpha g\cdot y\Big)\, h(t)\,dt & \text{if } \nu \in \mathcal{Y}^{-q,p}((0,\ell);\mathbb{R}^3) \\ +\infty & \text{otherwise in } L_w^\infty((0,\ell);\mathcal{M}(\mathbb{R}^3)) \end{cases}$$

where $q = ps/(p+2s)$ and $y \in W^{1,p}((0,\ell);\mathbb{R}^3)$ is the underlying deformation of ν with boundary condition $y(0) = 0$.

PROOF. The equi-coercivity follows as in Theorem 7.4.

We start by finding a recovering sequence for a suitable subsequence $I_{\varepsilon_{n_k}}^\infty$; that is, for $\nu \in \mathcal{Y}^{-q,p}((0,\ell);\mathbb{R}^3)$ (where, throughout the proof, $q = ps/(p+2s)$) we find a sequence $y_k \in W^{1,p}(\Omega;\mathbb{R}^3)$ with $y_k(x_1,x_2,0) = \varepsilon_{n_k}(x_1,x_2,0)$ such that, setting $\mu^k = \delta_{Dy_k}$, we have $\rho(\mu^k) \to \nu$ weakly* in $L_w^\infty((0,\ell);\mathcal{M}(\mathbb{R}^3))$, and

$$\limsup_{k\to\infty} I_{\varepsilon_{n_k}}^\infty(\mu^k) \le I(\nu).$$

By Lemma 4.4 we can find a sequence $\widetilde{v}_j \in L^\infty((0,\ell);\mathbb{R}^3)$ which generates ν such that $1/|\widetilde{v}_j| \in L^\infty((0,\ell);\mathbb{R}^3)$ and

$$\frac{1}{|\widetilde{v}_j|^q} + |\widetilde{v}_j|^p$$

is equi-integrable. Let

$$v_j(x_3) := \int_0^{x_3} \widetilde{v}_j(t)\,dt,$$

so that $v_j' = \widetilde{v}_j$. By a measurable selection argument (see for instance Buttazzo [9], Proposition 2.2.7), and by definition of f_0, there exists a sequence of measurable functions $\bar{F}_j : (0,\ell) \to M^{3\times 2}$ such that

$$f_0(x_3, v_j(x_3), v_j'(x_3)) = f\big(x_3, v_j(x_3), (\bar{F}_j(x_3)|v_j'(x_3))\big) \quad \text{a.e. } x_3 \in (0,\ell),$$

with $\det(\bar{F}_j|v_j') \geq 0$, since $v_j' \neq 0$, almost everywhere. By the growth conditions we have also that

$$\begin{aligned} c\Big(\frac{1}{|\det(\bar{F}_j|v_j')|^s} + |(\bar{F}_j|v_j')|^p - 1\Big) \leq f\big(x_3, v_j, (\bar{F}_j|v_j')\big) = \\ = f_0(x_3, v_j, v_j') \leq C\Big(\frac{1}{|v_j'|^q} + |v_j'|^p + 1\Big) \end{aligned} \tag{8.4}$$

almost everywhere, and therefore the sequences

$$|(\bar{F}_j|v_j')|^p \quad \text{and} \quad \frac{1}{|\det(\bar{F}_j|v_j')|^s}$$

are equi-integrable, $(\bar{F}_j|v_j') \in L^\infty((0,\ell); M^{3\times 3})$ and $\dfrac{1}{\det(\bar{F}_j|v_j')} \in L^\infty(0,\ell)$, for every $j \in \mathbb{N}$. By Lemma 8.3, for each j, there exist a constant $c_j > 0$ and a function $(\overline{G}_j|\widetilde{w}_j) \in C^\infty([0,\ell]; M^{3\times 3})$ such that $(\overline{G}_j|\widetilde{w}_j)(0) = I$, $\det(\overline{G}_j|\widetilde{w}_j) > c_j$, and the sequences

$$|(\overline{G}_j|\widetilde{w}_j)|^p \quad \text{and} \quad \frac{1}{|\det(\overline{G}_j|\widetilde{w}_j)|^s}$$

are equi-integrable, and

$$\lim_{j\to\infty} \|(\overline{G}_j|\widetilde{w}_j) - (\bar{F}_j|v_j')\|_p = 0. \tag{8.5}$$

Let

$$w_j(x_3) := \int_0^{x_3} \widetilde{w}_j(t)\,dt,$$

and ε_{n_j} be a subsequence of ε_n such that

$$\varepsilon_{n_j}\big(\|\overline{G}_j{}'\|_p + \|\det(\overline{G}_j|\overline{G}_j{}' x_\alpha)\|_\infty\big) < C_j. \tag{8.6}$$

Setting

$$y_j(x) := w_j(x_3) + \varepsilon_{n_j}\overline{G}_j(x_3)x_\alpha$$

the proof proceeds along the same lines of that of Theorem 7.4 by taking $\delta = 0$ everywhere. □

Example 8.5 The following energy, which satisfies the assumptions of the current section, has been used by Burges and Levinson [8], Simpson and Spector [18]

$$f(t,\xi,F) = \alpha(t,\xi)|F|^2 + \frac{\beta(t,\xi)}{|\det F|^s} + \gamma(t,\xi)$$

if $\det F > 0$, where α, β and γ are bounded Carathéodory functions such that $\alpha, \beta \geq c > 0$ and $s > 0$. Easy computations show that

$$f_0(t,\xi,z) = \alpha(t,\xi)|z|^2 + \frac{b(t,\xi)}{|z|^{s/(s+1)}} + \gamma(t,\xi)$$

where

$$b(t,\xi) = \frac{s+1}{s^{s/(s+1)}}\Big(2^s\alpha(t,\xi)^s\beta(t,\xi)\Big)^{1/(s+1)}.$$

9 A relaxation result

The results of this section will be used in the next section in order to derive the energy density of an elastic string. The proof of the theorem below is quite similar, but simpler, to those of the Γ-convergence results proved in Section 7 and 8.

Let $f_0 : (0,\ell)\times\mathbb{R}^m\times\mathbb{R}^m \to \overline{\mathbb{R}}$, $m \in \mathbb{N}$, be a Carathéodory function which satisfies the following growth conditions

$$\exists C,\, c > 0 \text{ s.t. } c\Big(\frac{1}{|z|^q} + |z|^p - 1\Big) \leq f_0(\cdot,\cdot,z) \leq C\Big(\frac{1}{|z|^q} + |z|^p + 1\Big) \quad \forall z \neq 0 \tag{9.1}$$

for suitable $p \geq 1$ and $q > 0$, and consequently

$$z = 0 \Rightarrow f_0(\cdot,\cdot,z) \equiv +\infty.$$

Let us consider the functional

$$E(y) = \int_0^\ell f_0(t,y(t),y'(t))\,dt - \int_0^\ell g(t)y(t)\,dt$$

where $g \in L^{p'}((0,\ell);\mathbb{R}^m)$, with $1/p + 1/p' = 1$.

Theorem 9.1 *The (sequential) lower semicontinuous envelope of*

$$E^\infty(\nu) = \begin{cases} E(y) & \text{if } \exists\, y \in W^{1,p}((0,\ell);\mathbb{R}^m),\ y(0)=0 \text{ s.t. } \nu = \delta_{y'} \\ +\infty & \text{otherwise in } L^\infty_w((0,\ell);\mathcal{M}(\mathbb{R}^m)) \end{cases}$$

with respect to the weak topology of* $L^\infty_w((0,\ell);\mathcal{M}(\mathbb{R}^m))$ *is the functional*

$$\overline{E}(\nu) = \begin{cases} \displaystyle\int_0^\ell \langle \nu_t, f_0(t,y(t),\cdot)\rangle\,dt - \int_0^\ell g\cdot y\,dt & \text{if } \nu \in \mathcal{Y}^{-q,p}((0,\ell);\mathbb{R}^m) \\ +\infty & \text{otherwise in } L^\infty_w((0,\ell);\mathcal{M}(\mathbb{R}^m)) \end{cases}$$

where y *denotes the underlying deformation of* ν *such that* $y(0) = 0$. *In other words:* $\overline{E}(\nu) = \Gamma^-(w^*L^\infty_w)E(\nu)$.

PROOF. We shall prove that

$$\overline{E}(\nu) = \min\{\liminf_{n\to\infty} E^\infty(\nu_n) \;:\; \nu_n \to \nu \text{ weakly* in } L^\infty_w((0,\ell);\mathcal{M}(\mathbb{R}^m))\}.$$

First we show that, given any sequence $\nu_n \to \nu$ weakly* in $L^\infty_w((0,\ell);\mathcal{M}(\mathbb{R}^m))$ we have

$$\overline{E}(\nu) \le \liminf_{n\to\infty} E^\infty(\nu_n). \tag{9.2}$$

Without loss of generality we may suppose that the right hand side of the inequality above be finite and that the liminf be indeed a limit. Then there are functions $y_n \in W^{1,p}((0,\ell);\mathbb{R}^m)$, $y_n(0) = 0$, such that $\nu_n = \delta_{y_n'}$. By the growth condition from below on the integrand f_0, there is a constant M such that

$$\left\| \frac{1}{|y_n'|^q} + |y_n'|^p \right\|_1 \le M \quad \text{for every } n.$$

Thus, by Theorem 4.2, $\nu \in \mathcal{Y}^{-q,p}((0,\ell);\mathbb{R}^m)$. Moreover, an application of Theorem 2.1 implies that $y_n \to y$ weakly in $W^{1,p}((0,\ell);\mathbb{R}^m)$. It follows that $y(0) = 0$, and

$$\overline{E}(\nu) = \int_0^\ell \langle \nu_t, f_0(t, y(t), \cdot)\rangle \, dt - \int_0^\ell g \cdot y \, dt.$$

The liminf inequality (9.2) directly follows by applying Theorem 2.1 again. Now it remains to prove that for every $\nu \in L^\infty_w((0,\ell);\mathcal{M}(\mathbb{R}^m))$ there exists a sequence $\nu_n \in L^\infty_w((0,\ell);\mathcal{M}(\mathbb{R}^m))$ converging to ν and such that

$$\limsup_{n\to\infty} E^\infty(\nu_n) \le \overline{E}(\nu).$$

Avoiding the trivial case when the right hand side is $+\infty$, we can assume that $\nu \in \mathcal{Y}^{-q,p}((0,\ell);\mathbb{R}^m)$. Then, by Lemma 4.4, there always exists a sequence of functions $u_n \in L^\infty((0,\ell);\mathbb{R}^m)$ which generates ν and such that the sequence

$$\frac{1}{|u_n|^q} + |u_n|^p$$

is equi-integrable. Setting

$$y_n(t) := \int_0^t u_n(s)\, ds, \text{ and } \nu_n = \delta_{y_n'}$$

we have $y_n' = u_n$, $y_n(0) = 0$, $y_n \to y$ weakly in $W^{1,p}((0,\ell);\mathbb{R}^m)$. By the growth condition from above we find that the sequence $|f_0(t, y_n(t), y_n'(t))|$ is equi-integrable. Then, by Theorem 2.1, we deduce that

$$\begin{aligned}\lim_{n\to\infty} E^\infty(\nu_n) &= \lim_{n\to\infty} \int_0^\ell f_0(t, y_n(t), y_n'(t))\, dt - \int_0^\ell g(t) y_n(t)\, dt \\ &= \int_0^\ell \langle \nu_t, f_0(t, y(t), \cdot)\rangle \, dt - \int_0^\ell g \cdot y\, dt = \overline{E}(\nu)\end{aligned}$$

which concludes the proof. □

10 The energy density of the elastic string

In [1], Acerbi, Buttazzo and Percivale have studied the Γ-convergence of the functionals I_ε (see Section 3) under the strong convergence in $L^p((0,\ell);\mathbb{R}^3)$ of the average over the cross section (still denoted by Av^α) of the deformation. In fact, they consider the case of a body under free boundary conditions and with integrand f which depends only on the deformation gradient. Nevertheless, a slight modification of their proofs shows that the $\Gamma^-(\mathrm{Av}^\alpha, L^p)$-limit of the sequence of functionals

$$G_\varepsilon(y) = \begin{cases} \displaystyle\int_{\Omega_\varepsilon} f(Dy)\,dx - \int_{\Omega_\varepsilon} \hat{g}^\varepsilon \cdot y\,dx & \text{if } y \in \mathcal{A}_\varepsilon \\ +\infty & \text{otherwise in } L^p(\Omega_\varepsilon;\mathbb{R}^3) \end{cases}$$

is

$$G_0(v) = \begin{cases} \displaystyle\int_0^\ell f_0^{**}(v')\,dx_3 - \int_0^\ell Av^\alpha g \cdot v\,dx_3 & \text{if } v \in W^{1,p},\ v(0)=0 \\ +\infty & \text{otherwise in } L^p((0,\ell);\mathbb{R}^3). \end{cases}$$

Following the same steps of the proof of Theorem 7 of [13], and using Jensen's inequality, we obtain the following comparison result.

Theorem 10.1 *Let $v \in L^p((0,\ell);\mathbb{R}^3)$ and $I(\nu)$ be either the functional defined in Theorem 7.4 or in Theorem 8.4. Then*

$$G_0(v) = \inf\{I(\nu) \ : \ \nu \in L^\infty_w((0,\ell);\mathcal{M}(\mathbb{R}^3)),\ \langle\nu,\mathrm{id}\rangle = v',\ v(0)=0\} \qquad (10.1)$$

with the usual convention $\inf\varnothing = +\infty$. *Moreover the infimum is attained.*

On the other hand the $\Gamma(\rho, w^*L^\infty_w)$-limit of the sequence G_ε turns out to be

$$G(\nu) = \begin{cases} \displaystyle\int_0^\ell \Big(\langle\nu_{x_3}, f_0\rangle - \mathrm{Av}^\alpha g\cdot y\Big)\,h(x_3)\,dx_3 & \text{if } \nu \in \mathcal{Y}^{-q,p}((0,\ell);\mathbb{R}^3) \\ +\infty & \text{otherwise in } L^\infty_w((0,\ell);\mathcal{M}(\mathbb{R}^3)). \end{cases}$$

It is easily seen that $f_0 : \mathbb{R}^3 \to \overline{\mathbb{R}}$ is the unique continuous integrand satisfying the growth conditions (9.1) and such that the Γ-limit, G, is the relaxation of the functional

$$E(y) = \int_0^l f_0(y'(x_3))h(x_3)\,dx_3 - \int_\omega \mathrm{Av}^3 g\cdot y\,dx_3$$

with respect to the weak* topology in $L^\infty_w((0,\ell);\mathcal{M}(\mathbb{R}^3))$. Hence $E(y)$ can be considered as the energy functional of the elastic string and $f_0(z)h(x_3)$ as the energy density. Indeed, if f_1 has the appropriate growth and if G is the relaxation of the energy E with integrand f_1 instead of f_0, then, by the result of Section 9, we have

$$\int_\omega \langle\nu_{x_3}, f_0 - f_1\rangle h(x_3)\,dx_3 = 0 \text{ for every } \nu \in \mathcal{Y}^{-q,p}((0,\ell);\mathbb{R}^3).$$

It follows that $f_0 = f_1$ by simply taking the admissible measure $\nu_{x_3} = \delta_z$ for $z \in \mathbb{R}^3$. Hence, the variational problem G determines a unique energy density f_0. This selection is not provided by using the relaxation of E with respect the L^p topology; indeed, in general, there exists an infinite number of functions f_1 such that $f_1^{**} = f_0^{**}$. A further reason to take f_0 as the energy density of the elastic string follows from the relation between E and G, which from well known properties of relaxed functionals can be expressed in terms of correspondence between minimizing sequences of E and minimizers of G. In particular, minimizing sequences of E generate Young measures which are minimizers of G, and viceversa.

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