Derivation of nonlinear bending theory for shells from three dimensional nonlinear elasticity by Gamma-convergence

Gero Friesecke
Mathematics Institute, University of Warwick
Coventry CV4 7AL, UK
gf@maths.warwick.ac.uk

Richard D. James
Department of Aerospace Engineering and Mechanics
107 Akerman Hall, University of Minnesota
Minneapolis MN 55455, USA
james@umn.edu

Maria Giovanna Mora, Stefan Müller
Max Planck Institute for Mathematics in the Sciences
Inselstr. 22-26
D-04103 Leipzig, Germany
mmora@mis.mpg.de, sm@mis.mpg.de

January 24, 2002

Abstract

We show that the nonlinear bending theory of shells arises as a Γ-limit of three dimensional nonlinear elasticity.

Résumé

Nous montrons que la théorie non linéaire des coques en flexion émerge comme Γ-limite de la théorie de l'élasticité tridimensionelle.

Version française abrégée. Dans cette note nous dérivons la théorie des coques non linéaires en flexion comme Γ-limite de la théorie d'élasticité tridimensionnelle non linéaire.

Le problème tridimensionnel. Soit M une surface orientable dans $\mathbb{R}^3$ de classe C^2. Soit μ la normale de M et soit $M_h := \{x + s\mu(x) : x \in$

$M, s \in (-h/2, h/2)\}$, pour $h > 0$ suffisamment petit. Nous supposons, pour simplifier, que M est donné par une seule application $\psi : \Omega' \subset \mathbb{R}^2 \to M$ où Ω' est un ouvert borné lipschitzien. Donc M_h est l'image de $\Omega := \Omega' \times (-1/2, 1/2)$ par l'application $\psi^{(h)}(z', z_3) = \psi(z') + hz_3\eta(z')$, où $\eta = \mu \circ \psi$.

L'énergie élastique d'une application $u : M_h \to \mathbb{R}^3$ est

$$E^{(h)}(u) = \int\limits_{M_h} W(\nabla u)\, dx.$$

Ici W est une fonction Borelienne, de classe C^2 dans un voisinage de $SO(3)$, qui satisfait (11) et (12) ci-dessous.

Le problème bidimensionnel. Considérons la classe suivante d'applications isométriques de M

$$\mathcal{A} = \{u \in W^{2,2}(M; \mathbb{R}^3) : (\nabla_{\tan} u)^T (\nabla_{\tan} u) = I \text{ p.p. dans } M\}. \tag{1}$$

Ici, la dérivée tangentielle $\nabla_{\tan} u(x)$ est une application de $T_x M$ à valeurs dans $\mathbb{R}^3$. La quantité importante est *l'application relative de Weingarten* $S_{M,u}$ qui mesure la différence entre les secondes formes fondamentales de M et de $N = u(M)$. Soit ν la normale de N et soit $Y = u \circ \psi : \Omega' \to N$. Nous définissons $S_{M,u}$ par

$$S_{M,u}(x)\nabla\psi = R^T(x)\nabla'(\nu \circ Y) - \nabla'(\mu \circ \psi), \tag{2}$$

où $\nabla' = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2})$ et $R(x) = \nabla_{\tan} u(x)$. Des caractérisations équivalentes sont données par (15) ou (16) ci-dessous.

Nous considérons les formes quadratiques $Q_3(G) := \partial^2 W / \partial F^2(I)(G, G)$, où I est l'identité, et $Q_2(x, G) := \min_{a \in \mathbb{R}^3}(G + a \otimes \mu(x))$. Évidemment, $Q_2(x, G)$ dépend seulement de la restriction de G à $T_x M$. L'énergie bidimensionnelle est donnée par

$$E(u) = \begin{cases} \frac{1}{24} \int\limits_M Q_2(x, S_{M,u}(x))\, d\mathcal{H}^2 & \text{si } u \in \mathcal{A}, \\ +\infty & \text{sinon.} \end{cases} \tag{3}$$

Le résultat principal est la Γ-convergence des fonctionnelles $h^{-3}E^{(h)}$ vers E au sens suivant.

Théorème 1.

(i) *Soit $u^{(h)} : M_h \to \mathbb{R}^3$ une suite telle que $\limsup_{h \to 0} h^{-3} E^{(h)}(u^{(h)}) < \infty$. Alors il existe $u \in \mathcal{A}$, une application $R : \Omega \to \mathbb{R}^{3\times 3}$ et des constantes $c^{(h)}$ tels que pour une sous-suite*

$$u^{(h)} \circ \psi^{(h)} - c^{(h)} \to u \circ \psi^{(0)} \quad \textit{dans } W^{1,2}(\Omega; \mathbb{R}^3), \tag{4}$$

$$\nabla u^{(h)} \circ \psi^{(h)} \to R \quad \textit{dans } L^2(\Omega; \mathbb{R}^{3\times 3}), \tag{5}$$

$$R_{,3} = 0, \quad R(z) \in SO(3) \ p.p., \quad R \in W^{1,2}(\Omega; \mathbb{R}^{3\times 3}), \tag{6}$$

$$\liminf_{h\to 0} \frac{1}{h^3} E^{(h)}(u^{(h)}) \geq E(u). \tag{7}$$

(ii) Si $u \in \mathcal{A}$ *il existe une suite* $u^{(h)} : M_h \to \mathbb{R}^3$ *telle que (4) - (6) soient vérifiées et*

$$\lim_{h\to 0} \frac{1}{h^3} E^{(h)}(u^{(h)}) = E(u).$$

On pourrait ajouter aux fonctionnelles des termes linéaires en u et des conditions au bord (voir (25), (26) ci-dessous). Par les arguments classiques de la Γ-convergence, on obtient la convergence des applications (presque) minimisantes, voir Corollary 2 ci-dessous.

Un outil essentiel de la preuve est un résultat de rigidité pour des applications proches d'un mouvement rigide (voir Theorem 3 ci-dessous ou [5], Théorème 2).

1 Introduction

The derivation of plate and shell theories is a problem having a long history with major contributions from Euler, D. Bernoulli, Kirchhoff, Love, E. and F. Cosserat, von Karman and a great many modern authors. The question which theory, if any, is predicted by three-dimensional nonlinear elasticity of thin objects has been open for a long time (formal results go back at least as far as Kirchhoff [8]; for three recent formal treatments see [3, 2, 4]). The first rigorous result for finite deformations was the derivation of nonlinear membrane theory by Le Dret and Raoult [9, 10, 11] through the use of Γ-convergence (for elastic strings a $\Gamma-$ convergence result was obtained earlier by Acerbi, Buttazzo and Percivale [1]). The more delicate case of bending theory for plates was settled recently in [5, 6], see also [13]. Here we extend this result to the nonlinear bending theory of shells. A key ingredient is an optimal oscillation estimate for deformations whose gradient is close to $SO(n)$ (see Theorem 3 below), first derived in [5] and generalizing earlier work of John [7].

2 The setting

The three dimensional problem. Let M be an oriented surface in $\mathbb{R}^3$ of class C^2. For $x \in M$ let $\mu(x)$ denote the normal at x and consider, for sufficiently small h, the thickened set M_h of thickness h

$$M_h := \left\{ x + s\mu(x) : x \in M, s \in \left(-\frac{h}{2}, \frac{h}{2}\right) \right\}.$$

In the following we will assume for convenience that M is given by a single C^2 chart

$$\psi : \Omega' \subset \mathbb{R}^2 \to M,$$

where Ω' is a bounded Lipschitz domain. Then M_h is parametrized by the map

$$\begin{aligned} \psi^{(h)} : \Omega = \Omega' \times (-\frac{1}{2}, \frac{1}{2}) \to M_h, \\ \psi^{(h)}(z', z_3) = \psi(z') + hz_3\eta(z'), \end{aligned} \tag{8}$$

where

$$\eta(z') = (\mu \circ \psi)(z'). \tag{9}$$

The case of general surfaces can be handled by standard localization arguments.

For a map $u : M_h \to \mathbb{R}^3$ its elastic energy is

$$E^{(h)}(u) = \int_{M_h} W(\nabla u)\, dx,$$

where the stored-energy density $W : \mathbb{R}^{3\times 3} \to [0, \infty]$ is Borel measurable and satisfies

$$W \in C^2 \text{ in a neighbourhood of } SO(3), \tag{10}$$

$$W \text{ is frame indifferent : } W(F) = W(RF) \text{ for all } R \in SO(3), \tag{11}$$

$$\begin{aligned} W(F) \geq C \operatorname{dist}^2(F, SO(3)), \quad C > 0, \\ W(F) = 0 \text{ if } F \in SO(3). \end{aligned} \tag{12}$$

The two dimensional problem. Let the surface M be as above and consider the following set of (infinitesimal) isometries of M

$$\mathcal{A} = \{u \in W^{2,2}(M; \mathbb{R}^3) : (\nabla_{\tan} u)^T (\nabla_{\tan} u) = I \text{ a.e. on } M\}. \tag{13}$$

Here the tangential derivative $\nabla_{\tan} u(x)$ is viewed as linear map from $T_x M$ to $\mathbb{R}^3$. Sometimes it will be convenient to extend this map to a proper rotation in $\mathbb{R}^3$. The two-dimensional energy will depend on a map $S_{M,u}(x) : T_x M \to T_x M$ which measures the difference between the second fundamental form of $N = u(M)$ and of M. Let $\psi : \Omega' \to M$ be the chart considered above, let $Y = u \circ \psi : \Omega' \to N$ and define $S_{M,u}(x)$ by

$$S_{M,u}(x) \nabla' \psi = R^T(x) \nabla'(\nu \circ Y) - \nabla'(\mu \circ \psi), \tag{14}$$

where ν is the normal of N, $\nabla' = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2})$ and $R(x) = \nabla_{\tan} u(x)$. The closely related quadratic form

$$(\nabla' \psi)^T S_{M,u} \nabla' \psi = (\nabla' Y)^T \nabla'(\nu \circ Y) - (\nabla' \psi)^T \nabla'(\mu \circ \psi) \tag{15}$$

is exactly the difference of the fundamental forms of N and M. In terms of the Weingarten maps S_M and S_N given by $S_M \nabla'\psi = \nabla'(\mu \circ \psi)$ etc. we have

$$S_{M,u}(x) = R^T(x) S_N(u(x)) R(x) - S_M(x). \tag{16}$$

We thus call $S_{M,u}$ the *relative Weingarten map.*

Consider the quadratic form

$$Q_3(G) = \frac{\partial^2 W}{\partial F^2}(I)(G,G) \tag{17}$$

related to the linearization of the energy at the identity. It follows from (11), (12) that Q_3 is positive semidefinite and its kernel consists of the skew symmetric matrices. For each $x \in M$, with normal $\mu(x)$ we define the form

$$Q_2(x,G) = \min_{a \in \mathbb{R}^3} Q_3(G + a \otimes \mu(x)). \tag{18}$$

Since Q_3 vanishes on skew-symmetric matrices we have

$$Q_2(x,G) = Q_2(x,G') = Q_2(x,G'') \tag{19}$$

where

$$G' = G(I - \mu \otimes \mu), \quad G'' = (I - \mu \otimes \mu)G'.$$

In particular Q_2 depends only on the restriction of G to $T_x M$.

The two-dimensional energy is now defined by

$$E(u) = \begin{cases} \frac{1}{24} \int\limits_M Q_2(x, S_{M,u}(x))\, d\mathcal{H}^2 & \text{if } u \in \mathcal{A} \\ +\infty & \text{else.} \end{cases} \tag{20}$$

3 Convergence results

We essentially show that the functionals $\frac{1}{h^3} E^{(h)}$ Γ-converge to E. This implies that (almost) minimizers of $\frac{1}{h^3} E^{(h)}$ converge to minimizers of E, also when force terms and boundary conditions are added.

Theorem 1.

(i) (compactness and ansatz-free lower bound)
Suppose that $u^{(h)} : M_h \to \mathbb{R}^3$ satisfy $\limsup_{h \to 0} h^{-3} E^{(h)}(u^{(h)}) < \infty$. Then there exists a $u \in \mathcal{A}$, a map $R : \Omega \to \mathbb{R}^{3\times 3}$ and constants $c^{(h)}$ such that for a subsequence

$$u^{(h)} \circ \psi^{(h)} - c^{(h)} \to u \circ \psi^{(0)} \quad \text{in } W^{1,2}(\Omega; \mathbb{R}^3), \tag{21}$$

$$\nabla u^{(h)} \circ \psi^{(h)} \to R \quad \text{in } L^2(\Omega; \mathbb{R}^{3\times 3}), \tag{22}$$

$$R_{,3} = 0, \quad R(z) \in SO(3) \text{ a.e.}, \quad R \in W^{1,2}(\Omega; \mathbb{R}^{3\times 3}), \tag{23}$$

and for the subsequence under consideration,

$$\liminf_{h\to 0} \frac{1}{h^3} E^{(h)}(u^{(h)}) \geq E(u). \tag{24}$$

(ii) *(attainment of lower bound)*
If $u \in \mathcal{A}$ then there exists a sequence $u^{(h)} : M_h \to \mathbb{R}^3$ such that (21) - (23) hold and

$$\lim_{h\to 0} \frac{1}{h^3} E^{(h)}(u^{(h)}) = E(u).$$

Remarks

1. *(body forces)* Let $f \in L^2(M)$, and let $f^{(h)}(\psi^{(h)}(x)) = f(\psi(x'))$ denote the trivial extension to M_h. Then Theorem 1 remains valid if we add the term $-h^2 \int_{M_h} f^{(h)} \cdot u\, dx$ to $E^{(h)}$ and the term $-\int_M f \cdot u\, d\mathcal{H}^2$ to E.

2. *(boundary conditions)* Let Γ be a finite union of closed intervals in the curve $\partial\Omega'$. Let

$$\hat{Y} \in C^1(\bar{\Omega}'; \mathbb{R}^3) \cap W^{2,2}(\Omega'; \mathbb{R}^3), \quad \hat{b} \in W^{1,\infty}(\Omega'; \mathbb{R}^3).$$

To describe clamped boundary conditions on $\psi(\Gamma)$ we consider the admissible sets

$$\begin{aligned} \mathcal{A}_{BC}^{(h)} &= \{u \in W^{1,2}(M_h, \mathbb{R}^3) : \\ &\qquad u^{(h)} \circ \psi^{(h)}(z', z_3) = \hat{Y}(z') + hz_3\hat{b}(z'), \forall z' \in \Gamma\}, \end{aligned} \tag{25}$$

$$\begin{aligned} \mathcal{A}_{BC} &= \{u \in \mathcal{A} : (u \circ \psi)(z') = \hat{Y}(z'), \\ &\qquad R(z')\mu(\psi(z')) = \hat{b}(z'), \forall z' \in \Gamma\}, \end{aligned} \tag{26}$$

where $R(z') \in SO(3)$ denotes the unique extension of the linear map $(\nabla_{\tan} u)(\psi(z'))$ from $T_{\psi(z')}M$ to $\mathbb{R}^3$. Then the assertions of Theorem 1 also hold if $E^{(h)}$ and E are extended by $+\infty$ outside $\mathcal{A}_{BC}^{(h)}$ and $\mathcal{A}_{BC}$, respectively.

By standard reasoning in Γ-convergence assertions (i) and (ii) of the theorem imply the convergence of (almost) minimizers.

Corollary 2. *Suppose that $\mathcal{A}_{BC} \neq \emptyset$. Let $u^{(h)} \in \mathcal{A}_{BC}^{(h)}$ be a sequence of almost minimizers of $E^{(h)}$ in $\mathcal{A}_{BC}^{(h)}$, i.e.*

$$\lim_{h\to 0} \frac{1}{h^3} \left(E^{(h)}(u^{(h)}) - \inf_{\mathcal{A}_{BC}^{(h)}} E^{(h)} \right) = 0.$$

Then a subsequence converges to $u \in \mathcal{A}_{BC}$ in the sense of (21), (22) and u minimizes E in $\mathcal{A}_{BC}$.

Remark. The proof of Theorem 1 shows that $\mathcal{A}_{BC} \neq \emptyset$ if and only if $\limsup_{h\to 0} h^{-3} \inf_{\mathcal{A}_{BC}^{(h)}} E^{(h)} < \infty$.

4 Proofs

Compactness. The key ingredient is the following quantitative rigidity estimate (see [5], [6] for a proof and a discussion of related results).

Theorem 3. *Let U be a bounded Lipschitz domain in $\mathbb{R}^n, n \geq 2$. Then there exists a constant $C(U)$ with the following property. For each $v \in W^{1,2}(U;\mathbb{R}^n)$ there exists an associated rotation $R \in SO(n)$ such that*

$$||\nabla v - R||_{L^2(U)} \leq C(U)||\text{dist }(\nabla v, SO(n))||_{L^2(U)}. \tag{27}$$

The proof of Theorem 3 shows that the constant $C(U)$ can be chosen independent of U for a family of sets that are Bilipschitz images of a cube (with uniform Lipschitz constants).

Consider a lattice of squares

$$S_{a,h} = a + (-\frac{h}{2}, \frac{h}{2})^2, \ a \in h\mathbb{Z}^2,$$

and let Ω'_h be the union of those squares with $S_{a,3h} \subset \Omega'$. For such a square we can apply Theorem 3 to the deformed cube $C_{a,h} = \psi^{(h)}(S_{a,h} \times (-\frac{1}{2}, \frac{1}{2})) \subset M_h$ since $C_{a,h}$ is a Bilipschitz image of $(-\frac{h}{2}, \frac{h}{2})^3$ under the map $\Psi^{(h)}(z) = \psi(z') + z_3\eta(z')$. We thus obtain a map $R^{(h)} : \Omega'_h \to SO(3)$ which is constant on each $S_{a,h}$ and satisfies

$$\begin{aligned}\int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{S_{a,h}} &|(\nabla u^{(h)}) \circ \psi^{(h)} - R^{(h)}|^2 \det \nabla \psi^{(h)} dz' dz_3 \\ &\leq C \int_{C_{a,h}} W(\nabla u^{(h)}) dx. \end{aligned} \tag{28}$$

Applying the same estimate to a neighbouring cell $S_{b,h}$ and to $S_{a,3h} \supset S_{b,h}$ and taking into account that $\det \nabla\psi^{(h)} \sim h$ we easily deduce (see [6]) for $|\zeta|_\infty := \max(|\zeta_1|, |\zeta_2|) \leq h$

$$\int_{\Omega'_h} |R^{(h)}(z' + \zeta) - R^{(h)}(z')|^2 \leq \frac{C}{h} \int_{M_h} W(\nabla u^{(h)}) dx.$$

If $\zeta \in \mathbb{R}^2$ is a general translation vector and $\omega' \subset \Omega'$ with dist $(\omega', \partial\Omega') \geq C|\zeta|$ then iterative application of this estimate yields

$$\int\limits_{\omega'} |R^{(h)}(z'+\zeta) - R^{(h)}(z')|^2 \leq \frac{C}{h^3}(|\zeta|+h)^2 \int\limits_{M_h} W(\nabla u^{(h)})$$
$$\leq C(|\zeta|+h)^2. \tag{29}$$

From this one deduces (21) - (23) by standard means (see [6] for the details, including convergence and regularity up to the boundary). In particular one deduces from (28) after summation over the relevant lattice points a

$$\int\limits_{\Omega'_h \times (-\frac{1}{2},\frac{1}{2})} |(\nabla u^{(h)}) \circ \psi^{(h)} - R^{(h)}|^2 dz \leq Ch^2. \tag{30}$$

Lower bound. Let $F^{(h)} := (\nabla u^{(h)}) \circ \psi^{(h)}$. In view of (30) it is natural to introduce the scaled deviation from $SO(3)$

$$G^{(h)} = \frac{(R^{(h)})^T F^{(h)} - I}{h} \chi_{\Omega'_h}.$$

Then, for a subsequence, $G^{(h)} \rightharpoonup G$ in $L^2(\Omega)$. Let $E_h = \{|G^{(h)}| > h^{-\frac{1}{2}}\}$. Then meas $E_h \to 0$ and using the positivity of W, Taylor expansion for $z \in \Omega'_h \backslash E_h$ and the fact that $h^{-1} \det \nabla\psi^{(h)}(z) \to \det((\nabla\psi)^T(\nabla\psi))^{\frac{1}{2}}(z') =: J_\psi(z')$ uniformly one deduces (see [6])

$$\liminf_{h\to 0} \frac{1}{h^3} \int\limits_{M_h} W(\nabla u^{(h)}) dx \geq \frac{1}{2} \int\limits_{\Omega' \times (-\frac{1}{2},\frac{1}{2})} Q_3(G) J_\psi dx \tag{31}$$
$$\geq \frac{1}{2} \int\limits_{\Omega' \times (-\frac{1}{2},\frac{1}{2})} Q_2(\psi(z'), G') J_\psi dz,$$

where $G'(z) = G(z)(I - \eta(z) \otimes \eta(z))$ and $\eta = \mu(\psi(z))$.

The main point is now to identify G'. To show that G' is affine in z_3 we consider the difference quotient in vertical direction

$$H^{(h)} = \Delta_3^s G^{(h)} := \frac{1}{s}(G^{(h)}(\cdot + se_3) - G^{(h)}).$$

Let $Y^{(h)} = u^{(h)} \circ \psi^{(h)}$. Then $\nabla Y^{(h)} = F^{(h)} \nabla\psi^{(h)}$ and a short calculation shows that

$$R^{(h)} H^{(h)} \nabla'\psi = \nabla' \Delta_3^s (\frac{1}{h} Y^{(h)}) - \Delta_3^s \tilde{F}^{(h)} \nabla'\eta,$$

where $\tilde{F}^{(h)}(z) = z_3 F^{(h)}(z)$. Since $H^{(h)} \rightharpoonup H$, $F^{(h)} \to R$, $R^{(h)} \to R$ and

$$\begin{aligned}\Delta_3^s \frac{1}{h} Y^{(h)} &= \frac{1}{s} \int_0^s \frac{1}{h} Y^{(h)}_{,3}(\cdot + \sigma e_3) d\sigma \\ &= \frac{1}{s} \int_0^s (F^{(h)} \eta)(\cdot + \sigma e_3) d\sigma \to R\eta\end{aligned}$$

we get

$$RH\nabla'\psi = \nabla'(R\eta) - R\nabla'\eta = RS_{M,u}\nabla'\psi$$

where $S_{M,u}$ is the relative Weingarten map introduced in (14), (16). Thus $H' = S_{M,u}$ and $G''(z', z_3) = G_0'(z') + z_3 S_{M,u}$. Expanding the quadratic form Q_2 we easily deduce (24) from (31).

Attainment of lower bound. Let $u \in \mathcal{A}_{BC}, Y = u \circ \psi : \Omega' \to \mathbb{R}^3, b = R\eta : \Omega' \to \mathbb{R}^3$ where $R(z) \in SO(3)$ is the unique extension of $(\nabla_{\tan} u)(\psi(z))$ from $T_{\psi(z)}M$ to $\mathbb{R}^3$. Then b is the unit normal, $b = Y_{,1} \wedge Y_{,2} \,/\, |Y_{,1} \wedge Y_{,2}|$. As in the case of plates [6], one can define suitable approximations $Y_\lambda \in W^{2,\infty}$ and $q_\lambda \in W^{1,\infty}$ which agree with Y and b, respectively, on a large set and satisfy the boundary conditions. Then the ansatz

$$\begin{aligned}&Y^{(h)}(z', z_3) = Y_{\lambda_h}(z') + h z_3 q_{\lambda_h}(z') + \frac{1}{2} h^2 z_3^2 d_h(z'), \\ &u^{(h)} = Y^{(h)} \circ (\psi^{(h)})^{-1},\end{aligned}$$

where $\lambda_h = c/h, d_h \in W_0^{1,\infty}(\Omega'; \mathbb{R}^3), h||d_h||_{W^{1,\infty}} \to 0, d_h \to d$ in $L^2(\Omega'; \mathbb{R}^3)$ yields the desired assertion similarly to [6]. The map d is chosen such that $Q_3(S_{M,u} + R^T d \otimes \mu) = Q_2(x, S_{M,u})$.

Boundary conditions for the two dimensional problem. The passage from $\mathcal{A}_{BC}^{(h)}$ to $\mathcal{A}_{BC}$ is achieved as in [6]. The main point is a version of the difference quotient estimate (29) which holds up to the boundary. That allows one to define a trace for (a mollification of) $R^{(h)}$ and obtain its convergence in L^2.

Acknowledgement

RDJ thanks AFOSR/MURI (F49620-98-1-0433), NSF (DMS-0074043) and ONR (ONR/UMD/Z897101) for supporting this work. GF, MGM and SM were partially supported by the TMR network FMRX–CT98–0229.

References

[1] E. Acerbi, G. Buttazzo and D. Percivale, A variational definition for the strain energy of an elastic string, J. Elas. **25** (1991), 137–148.

[2] S.S. Antman, Nonlinear problems of elasticity, Springer: New York (1995).

[3] P. G. Ciarlet, Mathematical Elasticity, Vols. II and III, Elsevier: Amsterdam (1997, 2000).

[4] D. D. Fox, A. Raoult and J. C. Simo, A justification of properly invariant plate theories, Arch. Rat. Mech. Analysis **124** (1993), 157–199.

[5] G. Friesecke, R.D. James and S. Müller, Rigorous derivation of nonlinear plate theory and geometric rigidity, C. R. Acad. Sci. Paris Ser. I 334 (2002), 173–178.

[6] G. Friesecke, R.D. James and S. Müller, A theorem on geometric rigidity and the derivation of nonlinear plate theory from three dimensional elasticity, to appear in Comm. Pure Appl. Math. (2002), preprint MPI-MIS 5/2002.

[7] F. John, Rotation and strain, Comm. Pure Appl. Math. **14** (1961), 391-413.

[8] G. Kirchhoff, Über das Gleichgewicht und die Bewegung einer elastischen Scheibe, J. Reine Angew. Math. **40** (1850), 51–88.

[9] H. LeDret and A. Raoult, Le modèle de membrane non linéaire comme limite variationelle de l'élasticité non linéaire tridimensionelle, C.R. Acad. Sci. Paris **317**, Série I (1993), 221–226.

[10] H. LeDret and A. Raoult, The nonlinear membrane model as a variational limit of nonlinear three-dimensional elasticity, J. Math. Pures Appl. **73** (1995), 549–578.

[11] H. LeDret and A. Raoult, The membrane shell model in nonlinear elasticity: a variational asymptotic derivation, J. Nonlinear Sci. Vol. **6** (1996), 59–84.

[12] A. E. H. Love, A Treatise on the Mathematical Theory of Elasticity, 4^{the} Edition, Cambridge University Press: Cambridge (1927).

[13] O. Pantz, Une justification partielle du modèle de plaque en flexion par Γ-convergence, C. R. Acad. Sci. Paris Série I **332** (2001), 587-592.