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ACTION FUNCTIONALS THAT ATTAIN REGULAR MINIMA IN PRESENCE OF ENERGY GAPS

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Abstract. We present three simple regular one-dimensional variational problems that present the Lavrentiev gap phenomenon, i.e.

$$\inf\left\{\int_a^b L(t,x,\dot{x}) : x \in \mathbf{W}_0^{1,1}(a,b)\right\} < \inf\left\{\int_a^b L(t,x,\dot{x}) : x \in \mathbf{W}_0^{1,\infty}(a,b)\right\},$$

(where $\mathbf{W}_0^{1,p}(a,b)$ denote the usual Sobolev spaces with zero boundary conditions) in which, in the first example, the two infima are actually minima, in the second example the infimum in $\mathbf{W}_0^{1,\infty}(a,b)$ is attained meanwhile the infimum in $\mathbf{W}_0^{1,1}(a,b)$ is not, and in the third example both infimum are not attained.

We discuss also how to construct energies with gap between any space and energies with multi-gaps.

1. **Introduction.** In 1926 M. Lavrentiev [7] published an example of a functional of the kind

$$\int_a^b L(t,x,\dot{x})dt,$$

whose infimum taken over the space of absolutely continuous functions is strictly lower than the infimum taken over the space of Lipschitzian functions with imposed boundary conditions. This energy gap is known as the Lavrentiev phenomenon since then. It is a manifestation of the high sensibility of the variational formulation upon the set of admissible minima (considering that $\mathbf{W}^{1,\infty}(a,b)$ is dense in $\mathbf{W}^{1,1}(a,b)$). The main drawback of this phenomenon is the impossibility of computing the minimum by a standard finite-element scheme.

One simple example exhibiting an energy gap is given by the Manià action [9]:

$$\mathcal{I}(x) := \int_0^1 (x^3 - t)^2 \dot{x}^6 dt,$$

with boundary conditions $x(0) = 0$, $x(1) = 1$, is such that

$$0 = \inf\left\{\mathcal{I} : \mathbf{W}_*^{1,1}(0,1)\right\} = \min\left\{\mathcal{I} : \mathbf{W}_*^{1,1}(0,1)\right\} < \inf\left\{\mathcal{I} : \mathbf{W}_*^{1,\infty}(0,1)\right\},$$

and the infimum on $\mathbf{W}_*^{1,\infty}(0,1)$ is not attained. Further examples verifying this phenomenon have been given by several authors: see [4], [10] and references therein.

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In this manuscript, we are mainly interested in provide explicitly for a new simple and regular variational problem in which, in presence of an energy gap, the infimum on the Lipschitz function and the infimum on the absolutely continuous functions are both attained. The motivation behind that is a better knowledge of the Lavrentiev phenomenon itself and the investigation of its potentialities in modelling any kind of singular phenomena. We also furnish variational problems where the infima are attained or not in all the possible combinations.

In section 2, we present our main result. We will prove also the occurrence of the repulsion property, i.e. the fact that the energy diverges to infinity as we approximate the absolutely continuous minimum by a Lipschitz function.

In sections 3 and 4, we modify the variational problem we have introduced to obtain an example where the Lipschitz infima is attained meanwhile the absolutely continuous one is not and viceversa (as it happens for the Manià's action [1]), and an example where both infima are not attained.

In the last section 5, we discuss how to construct energies with gap between any space and energies with multi-gaps by presenting in detail the case of 2-gaps Lavrentiev phenomenon.

2. **Attained regular and irregular infimum.** In Theorem 1, we present a variational problem where the infimum on the Lipschitz function and the infimum on the absolutely continuous functions are both attained in presence of the Lavrentiev phenomenon. In Corollary 1, we prove the same result for a regular Lagrangian. In Proposition 1, we discuss the repulsion property.

Theorem 1. *Let c be any constant in $(0, 1/\sqrt[3]{2})$, let p be any number greater or equal than $15/2$ and set*

$$\begin{aligned}
&E_{c,p} := \frac{1}{c^p}\left(\frac{p-1}{p-5}\right)^{p-1}\frac{2^8}{3^2\cdot 5\cdot 7},\\
&P(t,z) := \left[\frac{1}{4}z^2 + \frac{(2-c^3)(1-t^2)}{3}z + \frac{(1-c^3)(1-t^2)^2}{2}\right]z^2, \quad (t,z) \text{ in } [-1,1]\times\mathbb{R}.
\end{aligned}$$

The action functional $\mathcal{I}$ defined by

$$\int_{-1}^{1} P(t, x^3 - (1-t^2))\,(E_{c,p}|\dot{x}|^p + 1)\,dt, \tag{1}$$

with boundary conditions $x(-1) = 0$, $x(1) = 0$, presents the Lavrentiev phenomenon, i.e.

$$\begin{aligned}
0 = \quad & \inf\{\mathcal{I}(x) : x \in \mathbf{W}_0^{1,1}(-1,1)\}\\
& < \inf\{\mathcal{I}(x) : x \in \mathbf{W}_0^{1,\infty}(-1,1)\} = \frac{2(1-c^3)-1}{12}\frac{2^8}{3^2\cdot 5\cdot 7}.
\end{aligned}$$

Furthermore, defining $\tilde{x}(t) := \sqrt[3]{1-t^2} \in \mathbf{W}_0^{1,1}(-1,1)$ and $\bar{x}(t) := 0 \in \mathbf{W}_0^{1,\infty}(-1,1)$, we have

$$\begin{aligned}
&\inf\{\mathcal{I}(x) : x \in \mathbf{W}_0^{1,1}(-1,1)\} = \min\{\mathcal{I}(x) : x \in \mathbf{W}_0^{1,1}(-1,1)\} = \mathcal{I}(\tilde{x}),\\
&\inf\{\mathcal{I}(x) : x \in \mathbf{W}_0^{1,\infty}(-1,1)\} = \min\{\mathcal{I}(x) : x \in \mathbf{W}_0^{1,\infty}(-1,1)\} = \mathcal{I}(\bar{x}),
\end{aligned}$$

and $\tilde{x}$, $\bar{x}$ are unique.

Proof. Let L be the Lagrange function associated to (1), i.e.

$$L(t,x,\xi) = P(t, x^3 - (1-t^2))\,(E_{c,p}|\xi|^p + 1)\,.$$

The function $P(t,z)$ defined for $(t,z) \in [-1,1]\times\mathbb{R}$ has exactly the following extremal points:

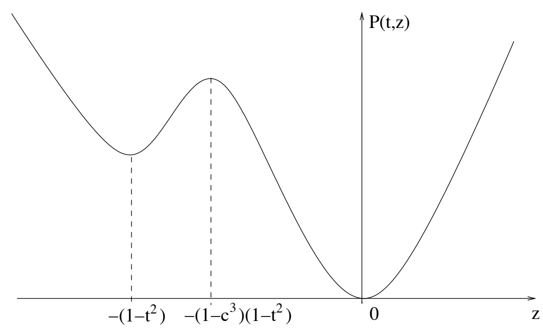

FIGURE 1. The graph of the function $P(t,\cdot)$.

I. 0 is the global minimum for $P(t,\cdot)$ and $P(t,0)=0$ (hence, P is non-negative);

II. $-(1-t^2)$ is a local minimum for $P(t,\cdot)$ and

$$P(t,-(1-t^2)) = \frac{2(1-c^3)-1}{12}(1-t^2)^4;$$

III. $-(1-c^3)(1-t^2)$ is a local maximum for $P(t,\cdot)$.

In fact, the derivative of P with respect to z is

$$P_z(t,z) = z[z+(1-c^3)(1-t^2)][z+(1-t^2)]$$

and therefore I, II and III follows directly from the assumption $c < 1/\sqrt[3]{2}$ (figure 1).

Part 1. We claim that $\tilde{x}(t) = \sqrt[3]{1-t^2}$ is the only minimum of $\mathcal{I}$ in $\mathbf{W}_0^{1,1}(-1,1)$.

The derivative of $\tilde{x}$ is equal to $-(2/3)t/\sqrt[3]{(1-t^2)^2}$; since $\tilde{x}(-1)=0$ and $\tilde{x}(1)=0$, we have that $\tilde{x}$ belongs to $\mathbf{W}_0^{1,1}(-1,1)$ (but $\tilde{x} \notin \mathbf{W}_0^{1,\infty}(-1,1)$). From I, we have $L(t,x,\xi) \geq 0$ and

$$L(t,\tilde{x},\dot{\tilde{x}}) = P(t,0)\left(E_{c,p}|\dot{\tilde{x}}|^p + 1\right) = 0.$$

Therefore, $\mathcal{I} \geq 0$ and $\mathcal{I}(\tilde{x}) = 0$, that implies that $\tilde{x}$ is a minimum.

We conclude that $\tilde{x}$ is the only minimum since, if $\tilde{x}_1$ were another minimum, we would have $\mathcal{I}(\tilde{x}_1)=0$ and, by the positivity of the Lagrangian, $L(t,\tilde{x}_1,\dot{\tilde{x}}_1)=0$. It would follow that $P(t,\tilde{x}_1^3-(1-t^2))=0$ that is possible if and only if $\tilde{x}_1^3-(1-t^2)=0$, and, hence, $\tilde{x}_1=\tilde{x}$.

Part 2. We claim that $\bar{x}(t)=0$ is the only minimum of $\mathcal{I}$ in $\mathbf{W}_0^{1,\infty}(-1,1)$.

First of all, observe that

$$\mathcal{I}(\bar{x}) = \int_{-1}^{1} P(t,-(1-t^2))dt = \frac{2(1-c^3)-1}{12}\frac{2^8}{3^2\cdot 5\cdot 7}, \tag{2}$$

since

$$\int_{-1}^{1}(1-t^2)^4 dt = \frac{2^8}{3^2\cdot 5\cdot 7}.$$

The coefficient $[2(1-c^3)-1]/12$, that we denote by ϵ_c, is strictly positive since, by assumption, $c < 1/\sqrt[3]{2}$.

Let x be any function in $\mathbf{W}_0^{1,\infty}(-1,1)\setminus\{\bar{x}\}$. We want to prove that $\mathcal{I}(x) > \mathcal{I}(\bar{x})$.

Since x has bounded derivative, $x(t)$ is smaller than the function $c\sqrt[3]{1-t^2}$ in a right neighbourhood of -1 and in a left neighbourhood of 1.

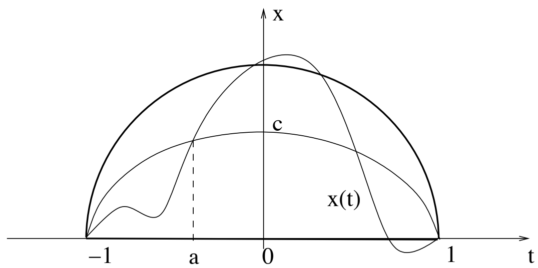

FIGURE 2. The graph of $x(t)$ intersecting $c\sqrt[3]{1-t^2}$ and $\sqrt[3]{1-t^2}$.

Whenever $x(t) < c\sqrt[3]{1-t^2}$ for any $t \in (-1,1)$, it follows that $x(t)^3 - (1-t^2) < -(1-c^3)(1-t^2)$. By II and III we have

$$L(t,x,\dot{x}) > \epsilon_c(1-t^2)^4\,(E_{c,p}|\dot{x}|^p + 1) > \epsilon_c(1-t^2)^4$$

and, hence, $\mathcal{I}(x) > \mathcal{I}(\bar{x})$.

Otherwise, suppose there exists $a \in (-1,0)$ such that (figure 2)

$$\begin{cases} x(t) < c\sqrt[3]{1-t^2}, & \mathrm{t} \in [-1,a), \\ x(a) = c\sqrt[3]{1-a^2}; \end{cases}$$

Observe that, by the symmetries of the Lagrangian, if $x(a) = c\sqrt[3]{1-a^2}$ with a in $[0,1)$, we can refraise the problem in the setting above by the change of variable $t \to -t$ in the integral (1).

By II and III, for any $t \in (-1,a)$, we have the estimate

$$L(t,x,\dot{x}) > \epsilon_c(1-t^2)^4\,(E_{c,p}|\dot{x}|^p + 1) > \epsilon_c(1-t^2)^4 E_{c,p}|\dot{x}|^p. \tag{3}$$

By the Hölder's inequality, we obtain

$$\begin{aligned} c\sqrt[3]{1-a^2} &= \int_{-1}^{a} \dot{x}dt = \int_{-1}^{a} \dot{x}(1-t^2)^{4/p}(1-t^2)^{-4/p}dt \\ &\le \left[\int_{-1}^{a} |\dot{x}|^p(1-t^2)^4 dt\right]^{1/p} \left[\int_{-1}^{a} (1-t^2)^{-4/(p-1)}dt\right]^{(p-1)/p} \\ &\le \left[\int_{-1}^{a} |\dot{x}|^p(1-t^2)^4 dt\right]^{1/p} \left[\int_{-1}^{a} (1+t)^{-4/(p-1)}dt\right]^{(p-1)/p} \\ &= \left[\int_{-1}^{a} |\dot{x}|^p(1-t^2)^4 dt\right]^{1/p} \left[(1+a)^{(p-5)/p}\right] \left[\frac{p-1}{p-5}\right]^{(p-1)/p}. \end{aligned}$$

It implies, recalling that $p \ge 15/2$,

$$\begin{aligned} \int_{-1}^{a} |\dot{x}|^p(1-t^2)^4 dt &\ge c^p(1-a^2)^{p/3}(1+a)^{-p+5}\left[\frac{p-5}{p-1}\right]^{p-1} \\ &\ge c^p\left[\frac{p-5}{p-1}\right]^{p-1}(1+a)^{-2p/3+5} \\ &\ge c^p\left[\frac{p-5}{p-1}\right]^{p-1}. \end{aligned} \tag{4}$$

We conclude that

$$\mathcal{I}(x) > \int_{-1}^{a} L(t,x,\dot{x})dt \ge \epsilon_c \frac{2^8}{3^2\cdot 5\cdot 7}.$$

Since the inequality we have obtained is strict, we infer also the uniqueness of the minimum.

The occurrence of the Lavrentiev phenomenon follows from

$$\mathcal{I}(\bar{x}) = \epsilon_c \frac{2^8}{3^2 \cdot 5 \cdot 7} > 0 = \mathcal{I}(\tilde{x}).$$

□

By the fact that $\bar{x}$ is actually in $\mathbf{C}_0^\infty(-1,1)$, we have proved that

$$\mathcal{I}(\tilde{x}) = \min\{\mathcal{I}(x) : x \in \mathbf{W}_0^{1,1}(-1,1)\} < \min\{\mathcal{I}(x) : x \in C_0^\infty(-1,1)\} = \mathcal{I}(\bar{x}).$$

We wish to point out that, whenever p is an integer number, the Lagrangian that has been presented is a polynomial of degree 8 in t, 9 in x and p in ξ, with a minimum degree equal to $p = 8 > 15/2$. Furthermore, it is convex in ξ, for any (t,x). Therefore, $L(t,x,\xi)$ is a very regular function, for instance $\mathbf{C}^\infty$.

By using a general result by P. Lowen [8] (that we state below in Theroem 2 for reader convenience), we can even provide for a coercive polynomial Lagrangian, strictly convex in ξ, for any (t,x) but $(-1,0)$ and $(1,0)$, that verifies the same statement of Theorem 1.

Corollary 1. *Let c, p, $E_{c,p}$ and $P(t,z)$ as in Theorem 1.*

The action functional $\mathcal{J}$ defined by

$$\int_{-1}^{1} P(t, x^3 - (1-t^2))\,(E_{c,p}|\dot{x}|^p + 1) + [27(1-t^2)^2\dot{x}^3 + 8t^3]^2 x^2 dt, \tag{5}$$

with boundary conditions $x(-1) = 0$, $x(1) = 0$, verifies the same statement of Theorem 1.

Theorem 2 ([8]). *Let $\mathcal{I}$ be a functional that exhibits the Lavrentiev phenomenon. Suppose that $\mathcal{I} \geq 0$ and that there exists a minimizer $\tilde{x}$ with $\mathcal{I}(\tilde{x}) = 0$.*

Then, for any action $\mathcal{P}$ with $\mathcal{P} \geq 0$ and $\mathcal{P}(\tilde{x})$ finite, there exists $\tilde{\delta} \in (0,\infty]$ such that, for any δ in $[0,\tilde{\delta})$, the functional $\mathcal{I} + \delta\mathcal{P}$ exhibits the Lavrentiev phenomenon.

Proof of Theorem 2. Let i be the infimum of $\mathcal{I}$ on the set of Lipschitz functions. Setting $\tilde{\delta} = i/[2\mathcal{P}(\tilde{x})]$, we have, for any δ in $[0,\tilde{\delta})$,

$$\begin{aligned} \mathcal{I}(\tilde{x}) + \delta\mathcal{P}(\tilde{x}) &\leq i/2, \\ \mathcal{I}(x) + \delta\mathcal{P}(x) &\geq i, \end{aligned}$$

for any Lipschitz function x. Hence, $\mathcal{I}+\delta\mathcal{P}$ exhibits the Lavrentiev phenomenon. □

Proof of Corollary 1. Let $\mathcal{P}$ be the functional given by

$$\mathcal{P}(x) = \int_{-1}^{1} [27(1-t^2)^2\dot{x}^3 + 8t^3]^2 x^2 dt.$$

It is non-negative and, recalling that $\tilde{x}(t) = \sqrt[3]{1-t^2}$ and $\dot{\tilde{x}}(t) = -(2/3)t/\sqrt[3]{(1-t^2)^2}$, $\mathcal{P}(\tilde{x}) = 0$. By Theorem 2, $\mathcal{J}$ presents the Lavrentiev phenomenon.

Since $\mathcal{J}(\tilde{x}) = \mathcal{I}(\tilde{x})$ and $\mathcal{J}(\bar{x}) = \mathcal{I}(\bar{x})$, the result follows from the fact that

$$\mathcal{J}(x) \geq \mathcal{I}(x),$$

for any function x. □

In the Proposition 1 below, we prove the occurrence of the repulsion property for the energy $\mathcal{I}$.

Proposition 1. *Let $p > 15/2$ and $\{x_n\}_n \subset \mathbf{W}_0^{1,\infty}(-1,1)$ be a sequence such that x_n converges to $\tilde{x}$ almost everywhere in $(-1,1)$ and $\mathcal{I}$ as in Theorem 1.*

Then,

$$\mathcal{I}(x_n) \to \infty,$$

as n tends to ∞.

Proof. Let $a_n \in (-1,0)$ be defined as in Theorem 1, i.e.

$$\begin{cases} x_n(t) < c\sqrt[3]{1-t^2}, \ \mathrm{t} \in [-1,a_n), \\ x_n(a_n) = c\sqrt[3]{1-a_n^2}; \end{cases}$$

By assumption, since x_n converges to $\tilde{x}$ almost everywhere in $(-1,1)$, a_n converges to -1.

Recalling the inequalities (3) and (4) in the proof of Theorem 1,

$$L(t,x_n,\dot{x}_n) > \epsilon_c(1-t^2)^4\left(E_{c,p}|\dot{x}_n|^p+1\right) > \epsilon_c(1-t^2)^4E_{c,p}|\dot{x}_n|^p$$

and

$$\begin{aligned}\int_{-1}^{a}|\dot{x}_n|^p(1-t^2)^4dt \quad &\geq c^p(1-a^2)^{p/3}(1+a)^{-p+5}\left[\frac{p-5}{p-1}\right]^{p-1} \\ &\geq c^p\left[\frac{p-5}{p-1}\right]^{p-1}(1+a_n)^{-2p/3+5},\end{aligned}$$

we conclude that

$$\mathcal{I}(x_n) > \int_{-1}^{a}L(t,x_n,\dot{x}_n)dt \geq \epsilon_c\frac{2^8}{3^2\cdot5\cdot7}\frac{1}{(1+a_n)^{2p/3-5}} \to \infty,$$

as n tends to ∞. □

3. **Attained regular infimum but not attained irregular infimum.** We provide for a coercive Lagrangian, for any (t,x), $x \neq \sqrt[3]{1-t^2}$, that presents the Lavrentiev phenomenon and attains its infimum between the Lipschitz functions but does not attain its infimum between the absolutely continuous one. More precisely:

Corollary 2. *Let c, p, $E_{c,p}$ and $P(t,z)$ be as in Theorem 1.*

The action functional $\mathcal{J}$ defined by

$$\int_{-1}^{1}P(t,x^3-(1-t^2))\left(E_{c,p}|\dot{x}|^p+1\right)+\delta[27(1-t^2)^2\dot{x}^3+16t^3]^2\dot{x}^2(1-t^2)dt, \quad (6)$$

with boundary conditions $x(-1)=0$, $x(1)=0$, presents the Lavrentiev phenomenon, i.e.

$$\begin{aligned}0 = \quad &\inf\{\mathcal{J}(x) : x \in \mathbf{W}_0^{1,1}(-1,1)\} \\ &< \inf\{\mathcal{J}(x) : x \in \mathbf{W}_0^{1,\infty}(-1,1)\} = \frac{2(1-c^3)-1}{12}\frac{2^8}{3^2\cdot5\cdot7},\end{aligned}$$

for any non-negative δ such that

$$\delta < \frac{\mathcal{I}(\bar{x})}{2\mathcal{P}(\tilde{x})},$$

where $\mathcal{P}(x) := \int_{-1}^{1}[27(1-t^2)^2\dot{x}^3+16t^3]^2\dot{x}^2(1-t^2)dt$.

Furthermore, $\bar{x}(t) = 0 \in \mathbf{W}_0^{1,\infty}(-1,1)$ is such that

$$\inf\{\mathcal{J}(x) : x \in \mathbf{W}_0^{1,\infty}(-1,1)\} = \min\{\mathcal{J}(x) : x \in \mathbf{W}_0^{1,\infty}(-1,1)\} = \mathcal{J}(\bar{x}),$$

$\bar{x}$ is unique, but $\mathcal{J}$ does not admit minimum in $\mathbf{W}_0^{1,1}(-1,1)$.

Proof. The functional $\mathcal{P}$ is non-negative and, recalling that $\tilde{x}(t) = \sqrt[3]{1-t^2}$ and $\dot{\tilde{x}}(t) = -(2/3)t/\sqrt[3]{(1-t^2)^2}$,

$$\mathcal{P}(\tilde{x}) = \int_{-1}^{1} \frac{2^8 t^8}{\sqrt[3]{1-t^2}} dt < \infty.$$

By Theorem 2, $\mathcal{J}$ presents the Lavrentiev phenomenon.

Since $\mathcal{J}(\bar{x}) = \mathcal{I}(\bar{x})$ and $\mathcal{J}(x) \geq \mathcal{I}(x)$, for any function x, we obtain that $\bar{x}$ is a minimum for $\mathcal{J}$ in $\mathbf{W}_0^{1,\infty}(-1,1)$.

If we prove that the infimum value of $\mathcal{J}$ in $\mathbf{W}_0^{1,1}(-1,1)$ is 0, then $\mathcal{J}$ cannot admit minima in $\mathbf{W}_0^{1,1}(-1,1)$. In fact, in that case if $\tilde{x}_1$ were a minimum, it would follow that $0 = \mathcal{J}(\tilde{x}_1) \geq \mathcal{I}(\tilde{x}_1) \geq 0$ and, by the uniqueness of the minimum for $\mathcal{I}$, $\tilde{x}_1 = \tilde{x}$. But $\mathcal{P}(\tilde{x}) > 0$, that contradicts $\mathcal{J}(\tilde{x}) = 0$.

Let us prove that the infimum value of $\mathcal{J}$ in $\mathbf{W}_0^{1,1}(-1,1)$ is 0.

Set $\epsilon > 0$ and let n be an integer number and $-1+\epsilon = t_0 < t_1 < \cdots < t_n = 0$ be the partition of the interval $[-1+\epsilon, 0]$ given by

$$t_k := \sqrt{1 - \left[\frac{k}{n} + \left(1 - \frac{k}{n}\right)\sqrt[3]{1-(1-\epsilon)^2}\right]^3}.$$

Define $x_{n,\epsilon}$ in $\mathbf{W}_0^{1,1}(-1,1)$ by

$$x_{n,\epsilon}(t) := \begin{cases} \sqrt[3]{1-t^2} & ,t \in [-1,-1+\epsilon), \\ \max\left\{ \sqrt[3]{1-t_k^2}, \right. & \\ \qquad \left. 2\sqrt[3]{1-t^2} - \frac{k}{n} - \left(1-\frac{k}{n}\right)\sqrt[3]{1-(1-\epsilon)^2} \right\} & \\ & ,t \in [t_k, t_{k+1}), k = 0, \cdots, n, \\ x_{n,\epsilon}(-t) & ,t \in (0,1]. \end{cases}$$

One verifies that $\|x_{n,\epsilon} - \tilde{x}\|_\infty$ converges to 0, as n converges to ∞.

Since

$$\mathcal{P}(x_{n,\epsilon}) = 2\int_{-1}^{-1+\epsilon} \frac{2^8 t^8}{\sqrt[3]{1-t^2}} dt =: j_\epsilon,$$

$|\dot{x}_n|$ is uniformly bounded with respect to n in $[-1+\epsilon, 1-\epsilon]$ and P is continuous, we have that $\mathcal{J}(x_{n,\epsilon})$ converges to j_ϵ, as n converges to ∞.

Observing that j_ϵ converges to 0, by a diagonal argument, we can find n_ϵ such that

$$\mathcal{J}(x_{n_\epsilon,\epsilon}) \to 0,$$

as ϵ tends to 0. □

4. **Not attained neither regular nor irregular infimum.** We start showing, in Corollary 3, an example of a coercive Lagrangian for any (t,x), $x \neq \sqrt[3]{1-t^2}$, that presents the Lavrentiev phenomenon that admits an absolutely continuous minimum but does not attain its Lipschitz infimum. We therefore propose, in Corollary 4, an example of a coercive Lagrangian for any (t,x) but $(-1,0)$ and $(1,0)$, that presents the Lavrentiev phenomenon that does not admit absolutely continuous nor Lipschitz minima.

Corollary 3. *Let c and $P(t,z)$ be as in Theorem 1, $p \geq 15$ and*

$$E_{c,p} := \frac{2^p}{c^{2p}}\left(\frac{p-1}{p-5}\right)^{p-1} \frac{2^8}{3^2 \cdot 5 \cdot 7}.$$

The action functional $\mathcal{I}_1$ *defined by*

$$\int_{-1}^{1} P(t, x^3 - (1-t^2)) \left(E_{c,p} \left| \dot{x}^2 - \frac{c^2}{2} \right|^p + 1 \right) dt, \tag{7}$$

with boundary conditions $x(-1) = 0$, $x(1) = 0$, *presents the Lavrentiev phenomenon, i.e.*

$$\begin{aligned} 0 = \quad & \inf\{\mathcal{I}_1(x) : x \in \mathbf{W}_0^{1,1}(-1,1)\} \\ & < \inf\{\mathcal{I}_1(x) : x \in \mathbf{W}_0^{1,\infty}(-1,1)\} = \frac{2(1-c^3)-1}{12} \frac{2^8}{3^2 \cdot 5 \cdot 7}, \end{aligned}$$

Furthermore, $\tilde{x}(t) = \sqrt[3]{1-t^2} \in \mathbf{W}_0^{1,1}(-1,1)$ *is such that*

$$\inf\{\mathcal{I}_1(x) : x \in \mathbf{W}_0^{1,1}(-1,1)\} = \min\{\mathcal{I}_1(x) : x \in \mathbf{W}_0^{1,1}(-1,1)\} = \mathcal{I}_1(\tilde{x}),$$

$\tilde{x}$ *is unique, but* $\mathcal{I}_1$ *does not admit minimum in* $\mathbf{W}_0^{1,\infty}(-1,1)$.

Proof. The proof proceeds in the same way as in the proof of Theorem 1. We outline just the main differences.

From *Part 1* of Theorem 1, it follows that $\tilde{x}(t) = \sqrt[3]{1-t^2}$ is the only minimum of $\mathcal{I}_1$ in $\mathbf{W}_0^{1,1}(-1,1)$, $\mathcal{I}_1(\tilde{x}) = 0$.

Let us prove that the infimum value of $\mathcal{I}_1$ in $\mathbf{W}_0^{1,\infty}(-1,1)$ is greater than $\mathcal{I}(\bar{x})$.

Let x be any function in $\mathbf{W}_0^{1,\infty}(-1,1)$ and a be defined as in *Part 2* of Theorem 1. We have

$$\begin{aligned} c^2 \sqrt[3]{(1+a)^2} \left(1 - \frac{1}{2}\sqrt[3]{1+a}\right) \leq \quad & c^2 \sqrt[3]{(1-a^2)^2} - \frac{c^2}{2}(1+a) \\ & = \left(\int_{-1}^{a} \dot{x} dt\right)^2 - \frac{c^2}{2}(1+a) \leq \int_{-1}^{a} \left|\dot{x}^2 - \frac{c^2}{2}\right| dt. \end{aligned}$$

Then, by proceeding as in (4) Theorem 1, we obtain

$$\int_{-1}^{a} \left|\dot{x}^2 - \frac{c^2}{2}\right|^p (1-t^2)^4 dt \geq \frac{c^{2p}}{2^p} \left[\frac{p-5}{p-1}\right]^{p-1} (1+a)^{-p/3+5} \geq \frac{c^{2p}}{2^p} \left[\frac{p-5}{p-1}\right]^{p-1}.$$

Using an analogous estimate as (3) Theorem 1, we conclude that

$$\mathcal{I}_1(x) > \int_{-1}^{a} L(t, x, \dot{x}) dt \geq \epsilon_c \frac{2^8}{3^2 \cdot 5 \cdot 7} = \mathcal{I}(\bar{x}).$$

Let use prove that the infimum value of $\mathcal{I}_1$ in $\mathbf{W}_0^{1,\infty}(-1,1)$ is exactly $\mathcal{I}(\bar{x})$.

To this purpose, let n be an integer number and $-1 = t_0 < t_1 < \cdots < t_n = 1$ be the partition of the interval $[-1,1]$ given by

$$t_k := -1 + 2\frac{k}{n}.$$

Define x_{2n} in $\mathbf{W}_0^{1,\infty}(-1,1)$ by

$$x_{2n}(t) := \begin{cases} \frac{c}{\sqrt{2}}(t - t_k) & , t \in [t_k, t_{k+1}), k \text{ even}, \\ -\frac{c}{\sqrt{2}}(t - t_{k+1}) & , t \in [t_k, t_{k+1}), k \text{ odd}. \end{cases}$$

One verifies that $\|x_{2n} - \bar{x}\|_\infty$ converges to 0.

Since $\dot{x}_{2n}^2 = c^2/2$, we conclude that

$$\mathcal{I}_1(x_{2n}) = \int_{-1}^{1} P(t, x_{2n}^3 - (1-t^2)) dt \to \mathcal{I}(\bar{x}),$$

as n tends to ∞.

We claim that $\mathcal{I}_1$ does not admit minima in $\mathbf{W}_0^{1,\infty}(-1,1)$.

In fact, If there exists a minima $\bar{x}_1$ for $\mathcal{I}_1$, we would have that $\mathcal{I}_1(\bar{x}_1)=\mathcal{I}(\bar{x})$. Furthermore, $\bar{x}_1(t)<c\sqrt[3]{1-t^2}$ for any t in $(-1,1)$, since otherwise $\mathcal{I}_1(\bar{x}_1)>\mathcal{I}(\bar{x})$, as we have shown above. Hence,

$$\begin{aligned}
&\int_{-1}^{1} P(t,-(1-t^2))\left|\dot{\bar{x}}_1^2-\frac{c^2}{2}\right|^p dt\\
&\leq \int_{-1}^{1} P(t,\bar{x}_1^3-(1-t^2))\left(E_{c,p}\left|\dot{\bar{x}}_1^2-\frac{c^2}{2}\right|^p+1\right)-P(t,-(1-t^2))dt\\
&=\mathcal{I}_1(\bar{x}_1)-\mathcal{I}(\bar{x})=0.
\end{aligned}$$

It implies $|\dot{\bar{x}}_1|=c/\sqrt{2}$ and, hence, it would follow $\mathcal{I}_1(\bar{x}_1)>\mathcal{I}(\bar{x})$, in contradiction with $\mathcal{I}_1(\bar{x}_1)=\mathcal{I}(\bar{x})$. □

In the next Corollary, we propose an example of a coercive Lagrangian for any (t,x) but $(-1,0)$ and $(1,0)$, that presents the Lavrentiev phenomenon that does not admit absolutely continuous nor Lipschitz minima.

Corollary 4. *Let c, p, $E_{c,p}$ and $P(t,z)$ be as in Corollary 3.*

The action functional $\mathcal{J}$ defined by

$$\begin{aligned}
\int_{-1}^{1}\Bigg\{&P(t,x^3-(1-t^2))\left(E_{c,p}\left|\dot{x}^2-\frac{c^2}{2}\right|^p+1\right)\\
&+\delta[27(1-t^2)^2\dot{x}^3+16t^3]^2\left(\dot{x}^2-\frac{c^2}{2}\right)^2(1-t^2)^2\Bigg\}dt,
\end{aligned}\tag{8}$$

with boundary conditions $x(-1)=0$, $x(1)=0$, presents the Lavrentiev phenomenon, i.e.

$$\begin{aligned}
0=\;&\inf\{\mathcal{J}(x):x\in\mathbf{W}_0^{1,1}(-1,1)\}\\
&<\inf\{\mathcal{J}(x):x\in\mathbf{W}_0^{1,\infty}(-1,1)\}=\frac{2(1-c^3)-1}{12}\frac{2^8}{3^2\cdot 5\cdot 7},
\end{aligned}$$

for any non-negative δ such that

$$\delta<\frac{\mathcal{I}(\bar{x})}{2\mathcal{P}(\tilde{x})},$$

where $\mathcal{P}(x):=\int_{-1}^{1}[27(1-t^2)^2\dot{x}^3+16t^3]^2(\dot{x}^2-c^2/2)^2(1-t^2)^2dt$.

Furthermore, $\mathcal{J}$ does not admit minimum in $\mathbf{W}_0^{1,1}(-1,1)$ nor in $\mathbf{W}_0^{1,\infty}(-1,1)$.

Proof. The proof is analogous to the one of Corollary 2.

The functional $\mathcal{P}$ is non-negative and, recalling that $\tilde{x}(t)=\sqrt[3]{1-t^2}$ and $\dot{\tilde{x}}(t)=-(2/3)t/\sqrt[3]{(1-t^2)^2}$,

$$\mathcal{P}(\tilde{x})=\int_{-1}^{1}\frac{2^8t^6[2/3+(c^2/2)\sqrt[3]{(1-t^2)^4}]^2}{3^2\sqrt[3]{(1-t^2)^2}}dt<\infty.$$

By Theorem 2, $\mathcal{J}$ presents the Lavrentiev phenomenon.

By definition, $\mathcal{J}(x)\geq\mathcal{I}(x)$, for any function x.

If we prove that the infimum value of $\mathcal{J}$ in $\mathbf{W}_0^{1,\infty}(-1,1)$ is $\mathcal{I}(\bar{x})$, then $\mathcal{J}$ cannot admit minima in $\mathbf{W}_0^{1,\infty}(-1,1)$. In fact, in that case if $\bar{x}_1$ were a minimum for $\mathcal{J}$, it would be a minimum also for $\mathcal{I}_1$ that contradicts Corollary 3.

Let us prove that the infimum value of $\mathcal{J}$ in $\mathbf{W}_0^{1,\infty}(-1,1)$ is $\mathcal{I}(\bar{x})$.

Let n be an integer number and $-1 = t_0 < t_1 < \cdots < t_n = 1$ be the partition of the interval $[-1, 1]$ given by

$$t_k := -1 + 2\frac{k}{n}.$$

Define x_{2n} in $\mathbf{W}_0^{1,\infty}(-1,1)$ by

$$x_{2n}(t) := \begin{cases} \dfrac{c}{\sqrt{2}}(t - t_k) & , t \in [t_k, t_{k+1}), k \text{ even}, \\ -\dfrac{c}{\sqrt{2}}(t - t_{k+1}) & , t \in [t_k, t_{k+1}), k \text{ odd}. \end{cases}$$

One verifies that $\|x_{2n} - \bar{x}\|_\infty$ converges to 0.

Since $\dot{x}_{2n}^2 = c^2/2$, we conclude that

$$\mathcal{I}_1(x_{2n}) = \int_{-1}^{1} P(t, x_{2n}^3 - (1 - t^2))dt \to \mathcal{I}(\bar{x}),$$

as n tends to ∞.

If we prove that the infimum value of $\mathcal{J}$ in $\mathbf{W}_0^{1,1}(-1,1)$ is 0, then $\mathcal{J}$ cannot admit minima in $\mathbf{W}_0^{1,1}(-1,1)$. In fact, in that case if $\tilde{x}_1$ were a minimum, it would follow that $0 = \mathcal{J}(\tilde{x}_1) \geq \mathcal{I}(\tilde{x}_1) \geq 0$ and, by the uniqueness of the minimum for $\mathcal{I}$, $\tilde{x}_1 = \tilde{x}$. But $\mathcal{P}(\tilde{x}) > 0$, that contradicts $\mathcal{J}(\tilde{x}) = 0$.

Let us prove that the infimum value of $\mathcal{J}$ in $\mathbf{W}_0^{1,1}(-1,1)$ is 0.

Set $\epsilon > 0$ and let n be an integer number and $-1 + \epsilon = t_0 < t_1 < \cdots < t_n = 0$ be the partition of the interval $[-1 + \epsilon, 0]$ given by

$$t_k := \sqrt{1 - \left[\frac{k}{n} + \left(1 - \frac{k}{n}\right)\sqrt[3]{1 - (1-\epsilon)^2}\right]^3}.$$

Define $x_{n,\epsilon}$ in $\mathbf{W}_0^{1,1}(-1,1)$ by

$$x_{n,\epsilon}(t) := \begin{cases} \sqrt[3]{1 - t^2} & , t \in [-1, -1+\epsilon), \\ \max\left\{\dfrac{c}{\sqrt{2}}(t - t_k) + \sqrt[3]{1 - t_k^2}, \right. \\ \qquad \left. 2\sqrt[3]{1 - t^2} - \dfrac{k}{n} - \left(1 - \dfrac{k}{n}\right)\sqrt[3]{1 - (1-\epsilon)^2}\right\} \\ \qquad\qquad , t \in [t_k, t_{k+1}), k = 0, \cdots, n, \\ x_{n,\epsilon}(-t) & , t \in (0, 1]. \end{cases}$$

One verifies that $\|x_{n,\epsilon} - \tilde{x}\|_\infty$ converges to 0, as n converges to ∞.

Since

$$\mathcal{P}(x_{n,\epsilon}) = 2\int_{-1}^{-1+\epsilon} \frac{2^8 t^6 [2/3 + (c^2/2)\sqrt[3]{(1-t^2)^4}]^2}{3^2\sqrt[3]{(1-t^2)^2}} dt =: j_\epsilon,$$

$|\dot{x}_{n,\epsilon}|$ is uniformly bounded with respect to n in $[-1+\epsilon, 1-\epsilon]$ and P is continuous, we have that $\mathcal{J}(x_{n,\epsilon})$ converges to j_ϵ, as n converges to ∞.

Observing that j_ϵ converges to 0, by a diagonal argument, we can find n_ϵ such that

$$\mathcal{J}(x_{n_\epsilon,\epsilon}) \to 0,$$

as ϵ tends to 0. □

5. **Energy gap between any space and multi-gaps.** In this last section we would like to point out briefly how the method we proposed in this manuscript can be extended to provide energies with gap between any couple of spaces, let us say $\mathbf{W}_0^{1,h}(-1,1)$ and $\mathbf{W}_0^{1,k}(-1,1)$, $1 < h < k < \infty$, and how to provide for energies with multi-gaps, for instance with two gaps, i.e.

$$\begin{aligned}&\min\{\mathcal{I}(x) : x \in \mathbf{W}_0^{1,h}(-1,1)\}\\ &< \min\{\mathcal{I}(x) : x \in \mathbf{W}_0^{1,k}(-1,1)\}\\ &< \min\{\mathcal{I}(x) : x \in \mathbf{W}_0^{1,l}(-1,1)\},\end{aligned}$$

with $1 < h < k < l < \infty$.

In Theorem 3, we state without proof a family of variational problems with gap in any space. In Theorem 4, we state and prove the 2-gaps Lavrentiev phenomenon for a family of variational problems.

Theorem 3. *Set $s := k(h-1) + h(k-1)$ and $r := 2(h-1)(k-1)$. Let c be any constant in $(0, 1/\sqrt[s]{2})$, let p be any number greater or equal than $(4r+1)s/(s-r)$ and set*

$$\begin{aligned}E_{c,p} &:= \frac{1}{c^p}\left(\frac{p-1}{p-4r-1}\right)^{p-1}\int_{-1}^{1}(1-t^2)^{4r},\\ P(t,z) &:= \left[\frac{1}{4}z^2 + \frac{(2-c^s)(1-t^2)^r}{3}z + \frac{(1-c^s)(1-t^2)^{2r}}{2}\right]z^2, \quad (t,z) \text{ in } [-1,1]\times\mathbb{R}.\end{aligned}$$

The action functional $\mathcal{I}$ defined by

$$\int_{-1}^{1} P(t, x^s - (1-t^2)^r)\,(E_{c,p}|\dot{x}|^p + 1)\,dt, \tag{9}$$

with boundary conditions $x(-1) = 0$, $x(1) = 0$, presents the Lavrentiev phenomenon, i.e.

$$\begin{aligned}0 = \;&\inf\{\mathcal{I}(x) : x \in \mathbf{W}_0^{1,h}(-1,1)\}\\ &< \inf\{\mathcal{I}(x) : x \in \mathbf{W}_0^{1,k}(-1,1)\} = \frac{2(1-c^s)-1}{12}\int_{-1}^{1}(1-t^2)^{4r}.\end{aligned}$$

Furthermore, defining $x_h(t) := (1-t^2)^{r/s} \in \mathbf{W}_0^{1,h}(-1,1)$ and $x_k(t) := 0 \in \mathbf{W}_0^{1,k}(-1,1)$, we have

$$\begin{aligned}\inf\{\mathcal{I}(x) : x \in \mathbf{W}_0^{1,h}(-1,1)\} &= \min\{\mathcal{I}(x) : x \in \mathbf{W}_0^{1,h}(-1,1)\} = \mathcal{I}(x_h),\\ \inf\{\mathcal{I}(x) : x \in \mathbf{W}_0^{1,k}(-1,1)\} &= \min\{\mathcal{I}(x) : x \in \mathbf{W}_0^{1,k}(-1,1)\} = \mathcal{I}(x_k),\end{aligned}$$

and x_h, x_k are unique.

Theorem 4. *Set*

$$\begin{aligned}&r := 4kl(h-1)(k-1)\\ &< w := kl[2(h-1)(k-1) + k(h-1) + h(k-1)]\\ &< s := 2kl[k(h-1) + h(k-1)].\end{aligned}$$

Let c be any constant in $(0, 1/\sqrt[s]{2})$,

$$\begin{aligned}z_1(t) &:= -2^s(1-t^2)^r + (1+c)^s(1-t^2)^r &, z_2(t) &:= -2^s(1-t^2)^r + (1-t^2)^w,\\ z_3(t) &:= -2^s(1-t^2)^r + c^s(1-t^2)^w &, z_4(t) &:= -2^s(1-t^2)^r,\end{aligned}$$

and

$$\begin{aligned} Q(t,z) := \quad & \frac{z^6}{6} - \frac{z_1+z_2+z_3+z_4}{5} z^5 + \frac{z_1 z_2 + (z_1+z_2)(z_3+z_4) + z_3 z_4}{4} z^4 + \\ & - \frac{z_1 z_2 (z_3+z_4) + z_3 z_4 (z_1+z_2)}{3} z^3 + \frac{z_1 z_2 z_3 z_4}{2} z^2. \end{aligned}$$

Let p be any number greater or equal than $(6r+2w+1)s/(s-w)$ and set

$$E_{c,p} := \max\left\{\frac{2}{c^2}, \frac{1}{(1-c)^2}\right\} \frac{2^p}{c^p} \left(\frac{p-1}{p-6r-2w-1}\right)^{p-1} \frac{1}{q_2} \int_{-1}^{1} Q(t, z_4(t)),$$

where $q_2 := \min Q(t,z_2)/(1-t^2)^{6r} > 0$.

The action functional $\mathcal{I}$ defined by

$$\int_{-1}^{1} Q(t, x^s - 2^s(1-t^2)^r) \left[E_{c,p}|\dot{x}|^p (x^s - (1-t^2)^w)^2 + 1\right] dt, \tag{10}$$

with boundary conditions $x(-1) = 0$, $x(1) = 0$, presents the 2-gaps Lavrentiev phenomenon, i.e.

$$\begin{aligned} & \inf\{\mathcal{I}(x) : x \in \mathbf{W}_0^{1,h}(-1,1)\} \\ & < \inf\{\mathcal{I}(x) : x \in \mathbf{W}_0^{1,k}(-1,1)\} \\ & < \inf\{\mathcal{I}(x) : x \in \mathbf{W}_0^{1,l}(-1,1)\}. \end{aligned}$$

Furthermore, defining $x_h(t) := 2(1-t^2)^{r/s} \in \mathbf{W}_0^{1,h}(-1,1)$, $x_k(t) := (1-t^2)^{w/s} \in \mathbf{W}_0^{1,k}(-1,1)$ and $x_l(t) := 0 \in \mathbf{W}_0^{1,l}(-1,1)$, we have

$$\begin{aligned} \inf\{\mathcal{I}(x) : x \in \mathbf{W}_0^{1,h}(-1,1)\} &= \min\{\mathcal{I}(x) : x \in \mathbf{W}_0^{1,h}(-1,1)\} = \mathcal{I}(x_h), \\ \inf\{\mathcal{I}(x) : x \in \mathbf{W}_0^{1,k}(-1,1)\} &= \min\{\mathcal{I}(x) : x \in \mathbf{W}_0^{1,k}(-1,1)\} = \mathcal{I}(x_k), \\ \inf\{\mathcal{I}(x) : x \in \mathbf{W}_0^{1,l}(-1,1)\} &= \min\{\mathcal{I}(x) : x \in \mathbf{W}_0^{1,l}(-1,1)\} = \mathcal{I}(x_l), \end{aligned}$$

and x_h, x_k, x_l are unique.

Proof. Let L be the Lagrange function associated to (10), i.e.

$$L(t,x,\xi) = Q(t, x^s - 2^s(1-t^2)^r) \left[E_{c,p}|\xi|^p (x^s - (1-t^2)^w)^2 + 1\right].$$

The function $Q(t,z)$ defined for $(t,z) \in [-1,1] \times \mathbb{R}$ has exactly the following extremal points:

I. 0 is the global minimum for $Q(t,\cdot)$ and $Q(t,0) = 0$ (hence, Q is non-negative);
II. z_2, z_4 are local minima for $Q(t,\cdot)$;
III. z_1, z_3 are local maxima for $Q(t,\cdot)$.

In fact, the derivative of Q with respect to z is

$$Q_z(t,z) = z(z-z_1)(z-z_2)(z-z_3)(z-z_4)$$

and therefore I, II and III follows directly from the assumption $c < 1/\sqrt[s]{2}$ (figure 3).

Part 1. We claim that x_h is the only minimum of $\mathcal{I}$ in $\mathbf{W}_0^{1,h}(-1,1)$.

The derivative of x_h is equal to $-4(r/s)t/(1-t^2)^{(s-r)/s}$. We have that x_h belongs to $\mathbf{W}_0^{1,h}(-1,1)$ but x_h does not belong to $\mathbf{W}_0^{1,k}(-1,1)$, by the choice of r and s.

From I, we have

$$L(t,x,\xi) \geq 0, \qquad Q(t, x_h^s - 2^s(1-t^2)^r) = Q(t,0) = 0.$$

Therefore, $\mathcal{I} \geq 0$ and $\mathcal{I}(x_h) = 0$, that implies that x_h is a minimum.

We conclude that x_h is the only minimum since, if y_h were another minimum, we would have $\mathcal{I}(y_h) = 0$ and, by the positivity of the Lagrangian, $L(t, y_h, \dot{y}_h) =$

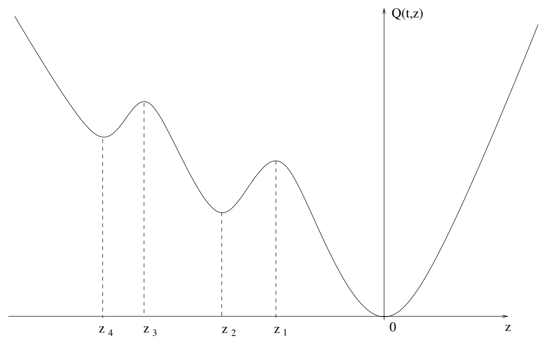

FIGURE 3. The graph of the function $Q(t, \cdot)$.

0. It would follow that $Q(t, y_h^s - 2^s(1-t^2)^r) = 0$ that is possible if and only if $y_h^s - 2^s(1-t^2)^r = 0$, and, hence, $y_h = x_h$.

Part 2. We claim that x_k is the only minimum of $\mathcal{I}$ in $\mathbf{W}_0^{1,k}(-1,1)$.

First of all, observe that

$$\mathcal{I}(x_k) = \int_1^1 Q(t, (1-t^2)^w - 2^s(1-t^2)^r)dt =: i_k. \tag{11}$$

The constant i_k is strictly positive since, by assumption, $c < 1/\sqrt[s]{2}$.

Let x be any function in $\mathbf{W}_0^{1,k}(-1,1)\backslash\{x_k\}$. We want to prove that $\mathcal{I}(x) > \mathcal{I}(x_k)$.

By the regularity of x, $x(t)$ is smaller than the function $(1+c)(1-t^2)^{r/s}$ in a right neighbourhood of -1 and in a left neighbourhood of 1.

Whenever $x(t) < (1+c)(1-t^2)^{r/s}$ for any $t \in (-1,1)$, it follows that $x(t)^s - 2^s(1-t^2)^r < z_1(t) = (1+c)^s(1-t^2)^r - 2^s(1-t^2)^r$. By II and III we have

$$L(t, x, \dot{x}) > Q(t, (1-t^2)^w - 2^s(1-t^2)^r)$$

and, hence, $\mathcal{I}(x) > \mathcal{I}(x_k)$.

Otherwise, suppose there exists $a \in (-1, 0)$ such that

$$\begin{cases} x(t) < (1+c)(1-t^2)^{r/s}, \ \mathrm{t} \in [-1, a), \\ x(a) = (1+c)(1-a^2)^{r/s}; \end{cases}$$

Let $a_0 \in [-1, a)$ be such that

$$\begin{cases} x(t) > (1+c/2)(1-t^2)^{w/s}, \ \mathrm{t} \in (a_0, a), \\ x(a_0) = (1+c/2)(1-a_0^2)^{w/s}; \end{cases}$$

Observe that, by the symmetries of the Lagrangian, if $x(a) = (1+c)(1-a^2)^{r/s}$ with a in $[0,1)$, we can refraise the problem in the setting above by the change of variable $t \to -t$ in the integral (10).

By II and III, for any $t \in (a_0, a)$, we have the estimate

$$\begin{aligned} L(t, x, \dot{x}) \ &> Q(t, (1-t^2)^w - 2^s(1-t^2)^r)E_{c,p}|\dot{x}|^p(x^s - (1-t^2)^w)^2 \\ &> q_2 \frac{c^2}{2} E_{c,p}(1-t^2)^{6r+2w}|\dot{x}|^p. \end{aligned}$$

By the Hölder's inequality, we obtain

$$\begin{aligned}
\frac{c}{2}(1-a^2)^{r/s} &\le \int_{a_0}^{a} \dot{x}dt = \int_{a_0}^{a} \dot{x}(1-t^2)^{(6r+2w)/p}(1-t^2)^{-(6r+2w)/p}dt \\
&\le \left[\int_{a_0}^{a} |\dot{x}|^p (1-t^2)^{6r+2w}dt\right]^{1/p} \left[\int_{a_0}^{a} (1-t^2)^{-(6r+2w)/(p-1)}dt\right]^{(p-1)/p} \\
&\le \left[\int_{a_0}^{a} |\dot{x}|^p (1-t^2)^{6r+2w}dt\right]^{1/p} \left[\int_{a_0}^{a} (1+t)^{-(6r+2w)/(p-1)}dt\right]^{(p-1)/p} \\
&\le \left[\int_{a_0}^{a} |\dot{x}|^p (1-t^2)^{6r+2w}dt\right]^{1/p} \left[(1+a)^{(p-6r-2w-1)/p}\right] \\
&\quad \times \left[\frac{p-1}{p-6r-2w-1}\right]^{(p-1)/p}.
\end{aligned}$$

It implies, recalling that $p \ge (6r+2w+1)(1-w/s) > (6r+2w+1)(1-r/s)$,

$$\begin{aligned}
\int_{a_0}^{a} |\dot{x}|^p (1-t^2)^{6r+2w}dt &\ge \frac{c^p}{2^p}(1-a^2)^{pr/s}(1+a)^{-p+6r+2w+1}\left[\frac{p-6r-2w-1}{p-1}\right]^{p-1} \\
&\ge \frac{c^p}{2^p}\left[\frac{p-6r-2w-1}{p-1}\right]^{p-1}(1+a)^{-p(1-r/s)+6r+2w+1} \\
&\ge \frac{c^p}{2^p}\left[\frac{p-6r-2w-1}{p-1}\right]^{p-1}.
\end{aligned} \tag{12}$$

We conclude that, by the properties of Q,

$$\mathcal{I}(x) > \int_{a_0}^{a} L(t,x,\dot{x})dt \ge \int_{-1}^{1} Q(t, z_4(t)) > i_k.$$

Since the inequality we have obtained is strict, we infer also the uniqueness of the minimum.

Part 3. We claim that x_l is the only minimum of $\mathcal{I}$ in $\mathbf{W}_0^{1,l}(-1,1)$.

First of all, observe that

$$\mathcal{I}(x_l) = \int_{1}^{1} Q(t, -2^s(1-t^2)^r)dt =: i_l. \tag{13}$$

The constant i_l is strictly positive since, by assumption, $c < 1/\sqrt[s]{2}$, and $i_l > i_k$ by definition of Q.

Let x be any function in $\mathbf{W}_0^{1,l}(-1,1)\setminus\{x_l\}$. We want to prove that $\mathcal{I}(x) > \mathcal{I}(x_l)$.

By the regularity of x, $x(t)$ is smaller than the function $c(1-t^2)^{w/s}$ in a right neighbourhood of -1 and in a left neighbourhood of 1.

Whenever $x(t) < c(1-t^2)^{w/s}$ for any $t \in (-1,1)$, it follows that $x(t)^s - 2^s(1-t^2)^r < z_1(t) = c^s(1-t^2)^w - 2^s(1-t^2)^r$. By II and III we have

$$L(t,x,\dot{x}) > Q(t, -2^s(1-t^2)^r)$$

and, hence, $\mathcal{I}(x) > \mathcal{I}(x_l)$.

Otherwise, suppose there exists $a \in (-1,0)$ such that

$$\begin{cases} x(t) < c(1-t^2)^{w/s}, \ \mathrm{t} \in [-1,a), \\ x(a) = c(1-a^2)^{w/s}; \end{cases}$$

Let $a_0 \in [-1,a)$ be such that

$$\begin{cases} x(t) > 0, \ \mathrm{t} \in (a_0,a), \\ x(a_0) = 0; \end{cases}$$

Observe that, by the symmetries of the Lagrangian, if $x(a) = c(1-a^2)^{w/s}$ with a in $[0,1)$, we can refraise the problem in the setting above by the change of variable $t \to -t$ in the integral (10).

By II and III, for any $t \in (a_0, a)$, we have the estimate

$$\begin{aligned} L(t,x,\dot{x}) &> Q(t, -2^s(1-t^2)^r)E_{c,p}|\dot{x}|^p(x^s - (1-t^2)^w)^2 \\ &> q_4(1-c)^2 E_{c,p}(1-t^2)^{6r+2w}|\dot{x}|^p, \end{aligned}$$

where $q_4 := \min Q(t, z_4)/(1-t^2)^{6r} > 0$.

By the Hölder's inequality, we obtain

$$\begin{aligned} c(1-a^2)^{w/s} &= \int_{a_0}^a \dot{x}dt = \int_{a_0}^a \dot{x}(1-t^2)^{(6r+2w)/p}(1-t^2)^{-(6r+2w)/p}dt \\ &\leq \left[\int_{a_0}^a |\dot{x}|^p(1-t^2)^{6r+2w}dt\right]^{1/p} \left[\int_{a_0}^a (1-t^2)^{-(6r+2w)/(p-1)}dt\right]^{(p-1)/p} \\ &\leq \left[\int_{a_0}^a |\dot{x}|^p(1-t^2)^{6r+2w}dt\right]^{1/p} \left[\int_{a_0}^a (1+t)^{-(6r+2w)/(p-1)}dt\right]^{(p-1)/p} \\ &\leq \left[\int_{a_0}^a |\dot{x}|^p(1-t^2)^{6r+2w}dt\right]^{1/p} \left[(1+a)^{(p-6r-2w-1)/p}\right] \\ &\quad \times \left[\frac{p-1}{p-6r-2w-1}\right]^{(p-1)/p}. \end{aligned}$$

It implies, recalling that $p \geq (6r+2w+1)(1-w/s)$,

$$\begin{aligned} \int_{a_0}^a |\dot{x}|^p(1-t^2)^{6r+2w}dt &\geq c^p(1-a^2)^{pw/s}(1+a)^{-p+6r+2w+1}\left[\frac{p-6r-2w-1}{p-1}\right]^{p-1} \\ &\geq c^p\left[\frac{p-6r-2w-1}{p-1}\right]^{p-1}(1+a)^{-p(1-w/s)+6r+2w+1} \\ &\geq c^p\left[\frac{p-6r-2w-1}{p-1}\right]^{p-1}. \end{aligned} \tag{14}$$

We conclude that

$$\mathcal{I}(x) > \int_{a_0}^a L(t,x,\dot{x})dt \geq i_l.$$

Since the inequality we have obtained is strict, we infer also the uniqueness of the minimum.

The occurrence of the 2-gaps Lavrentiev phenomenon between $\mathbf{W}_0^{1,h}(-1,1)$, $\mathbf{W}_0^{1,k}(-1,1)$ and $\mathbf{W}_0^{1,l}(-1,1)$ follows from

$$\mathcal{I}(x_l) = i_l > \mathcal{I}(x_k) = i_k > \mathcal{I}(x_h) = 0.$$

□

We would like to point out that Theorem 4 can be generalized to provide examples with three or more energy gaps by replacing $Q(t,z)$ by an analogous polynomial with as many wells as gaps (see figures 1 and 3). In that way, we will need a polynomial of degree $2 \times \{\#\text{ of gaps } +1\}$.

As last remark, we notice that it is possible to perturb the action in Theorem 4, as in Corollary 2, 3 and 4, to obtain variational problems where the infima are attained or not in all the possible combinations.

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