Asymptotic behaviour of a class of elliptic-parabolic operators: a unitary approach

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Abstract

We study the asymptotic behaviour of a sequence of strongly degenerate parabolic equations $\partial_t(r_h u) - \operatorname{div}(a \cdot Du)$ with $r_h(x,t) \geqslant 0$, $r_h \in L^\infty(\Omega \times (0,T))$. This leads to study, as particular cases, G-convergence for elliptic operators ($r_h \equiv 0$), G-convergence for parabolic operators ($r_h \equiv 1$), singular perturbations of an elliptic operator ($a_h \equiv a$ and $r_h \to r$, possibly $r \equiv 0$).

1 Introduction

In the papers [12], [13], [5] De Giorgi and Spagnolo introduced G-convergence for a class of elliptic operators, precisely for a class of elliptic operators in divergence form defined by an elliptic and simmetric matrix with bounded coefficients. Tartar extended this convergence to the non-simmetric (and then non-linear) case (see, for instance, [15] and [16]).
Later in [3] Colombini and Spagnolo define G-convergence for a class of parabolic operators in divergence form still defined by a simmetric matrix with bounded coefficients depending, in this case, also on time. Before introducing the aim of this paper we recall the definition of G-convergence in both cases, denoting the convergence by EG in the elliptic case and by PG in the parabolic one, as extended to non-simmetric operator by Tartar.

Consider $n \in \mathbf{N}$ fixed and denote by $\mathcal{M}_U(\lambda_0, \Lambda_0, M)$, with U open set of $\mathbf{R}^k$, $k \in \mathbf{N}$, the class of $n \times n$ matrices defined as follows:

$$\begin{aligned}
& a = [a_{ij}(y)]_{i,j=1}^n \in L^\infty(U) \qquad \text{such that} \\
& \lambda_0 |\xi|^2 \leqslant \big(a(y) \cdot \xi, \xi\big) \leqslant \Lambda_0 |\xi|^2 \\
& \big|\big(a(y) \cdot \xi, \eta\big)\big| \leqslant M \big(a(y) \cdot \xi, \xi\big)^{1/2} \big(a(y) \cdot \eta, \eta\big)^{1/2}
\end{aligned} \tag{1.1}$$

for every $\xi, \eta \in \mathbf{R}^n$, for a. e. $y \in U$.

Definition 1.1 *Let Ω be a bounded open set of $\mathbf{R}^n$ and $T > 0$. Consider a sequence $(a_h)_h \subset \mathcal{M}_\Omega(\lambda_0, \Lambda_0, M)$, $a_h = a_h(x)$ (referring to* (1.1)*, in this case $k = n$). We say that*

$$a_h \xrightarrow{EG} a \qquad \textit{in } \Omega$$

if for every $f \in H^{-1}(\Omega)$ *it results that*

$$\begin{array}{ll} u_h \to u & \textit{in } L^2(\Omega) \\ a_h \cdot Du_h \to a \cdot Du & \textit{in } L^2(\Omega)^n\textit{-weak}, \end{array}$$

where u_h *and* u *denote respectively the solutions (see Definition* (2.4))

$$\begin{cases} -\operatorname{div}(a_h \cdot Dv) = f & \textit{in } \Omega \\ v = 0 & \textit{in } \partial\Omega \end{cases} \qquad \begin{cases} -\operatorname{div}(a \cdot Dv) = f & \textit{in } \Omega \\ v = 0 & \textit{in } \partial\Omega \end{cases}$$

For a sequence $(a_h)_h \subset \mathcal{M}_{\Omega\times(0,T)}(\lambda_0, \Lambda_0, M)$, $a_h = a_h(x,t)$ *(referring to* (1.1)*, in this case* $k = n+1$*), we say that*

$$a_h \xrightarrow{PG} a \qquad \textit{in } \Omega \times (0,T)$$

if for every $f \in L^2(0,T; H^{-1}(\Omega))$ *and* $\varphi \in L^2(\Omega)$ *it results that*

$$\begin{array}{ll} u_h \to u & \textit{in } L^2(0,T,L^2(\Omega)) \\ a_h \cdot Du_h \to a \cdot Du & \textit{in } L^2(0,T,L^2(\Omega)^n)\textit{-weak}, \end{array}$$

where u_h *and* u *denote respectively the solutions (see Definition* (2.4))

$$\begin{cases} \dfrac{\partial v}{\partial t} - \operatorname{div}(a_h \cdot Dv) = f & \textit{in } \Omega \times (0,T) \\ u = 0 & \textit{in } \partial\Omega \times (0,T) \\ u = \varphi & \textit{in } \Omega \times \{0\} \end{cases} \qquad \begin{cases} \dfrac{\partial v}{\partial t} - \operatorname{div}(a \cdot Dv) = f & \textit{in } \Omega \times (0,T) \\ u = 0 & \textit{in } \partial\Omega \times (0,T) \\ u = \varphi & \textit{in } \Omega \times \{0\}\,. \end{cases}$$

In [3] the authors studied the connection between EG and PG convergence: in particular they proved that if $(a_h)_h \in \mathcal{M}_{\Omega\times(0,T)}(\lambda_0, \Lambda_0, M)$ satisfies a suitable condition on time (see (2.22))

$$a_h(\cdot,t) \xrightarrow{EG} a(\cdot,t) \quad \text{in } \Omega \text{ for a. e. } t \in (0,T) \quad \text{iff} \quad a_h \xrightarrow{PG} a \quad \text{in } \Omega \times (0,T) \tag{1.2}$$

and showed with a counterexample that this is not always true.

In this paper we consider strongly degenerate parabolic, or elliptic-parabolic, operators like

$$Pu = \frac{\partial}{\partial t}(ru) - \operatorname{div}(a \cdot Du) \qquad \text{with } r = r(x,t) \geqslant 0 \tag{1.3}$$

and study the limit behaviour of a sequence of the Cauchy-Dirichlet problems (for the existence result we refer to [7], but see also [10])

$$\begin{cases} P_h u = \dfrac{\partial}{\partial t}(r_h u) - \operatorname{div}(a_h \cdot Du) = f & \text{in } \Omega \times (0,T) \\ u = 0 & \text{in } \partial\Omega \times (0,T) \\ u = \varphi & \text{in } \Omega_{h,+}(0) \times \{0\} \end{cases} \tag{1.4}$$

where $\Omega_{h,+}(0) := \{x \in \Omega \mid r_h(x,0) > 0 \text{ for a. e. } x \in \Omega\}$, $(a_h)_h$ is a sequence in $\mathcal{M}_{\Omega\times(0,T)}(\lambda_0, \Lambda_0, M)$ satisfying moreover (2.22) (the conditions which garantees (1.2), r_h belong to a suitable class $\mathcal{F}$ defined in (2.19).
Arising from (1.2) the aim of the present paper is to give a more general definition of G-convergence, for problems (1.4) (see Definition 4.3), which is *independent* of the sequence

$(r_h)_h$ and a compactness result with respect to it (see Theorem 4.4). This in particular justyfies (1.2) and includes other phenomena, as singular perturbations (in which the result is stronger, see Proposition 5.1), but we refer to the last section for examples.
In many problems may be convenient to approximate the real problem via G-convergence or via Γ-convergence (whose definition we do not recall but refer to [4]), sometimes may be useful to perturbate an equation $Au = f$ with a term ϵBu, ϵ small parameter. With this new definition we can see these two different approaches as two different aspect of a same phenomenon.
We recall that, in the general situation, a first study in this direction was already made, in the periodic case and with $r = r(x)$, in [8].

Elliptic-parabolic operator like that in (1.3) were already studied, as regards the existence of the solution, probably first by Showalter (see, for instance, [9] for one of the first papers and [10] for a recent book) and recently in [7] for a more general class of operators (nonlinear and possibly forward, backward and stationary).
We want to stress that, since r in (1.3) may be equal to zero on some region with positive measure, a problem in this situation is the lack of the natural compactness result (see Theorem 2.6 and Theorem 2.7). Only for the sequence of the solutions (the solutions u_h to the problems (1.4)), we are able to obtain the compactness via a regularity result (see Theorem 2.8).

The interest to study such problems lies on the fact that many diffusion problems lead to differential equations like

$$\frac{\partial}{\partial t}(r(x,t)u) - \mathrm{div}(a(x,t) \cdot Du) = f$$

which may be also of mixed type (see for example [1], chap. 3, and the references therein), i. e. partially elliptic and partially parabolic. For example in some composite materials there can be a zone in which the diffusion is so quick that the problem can be considered stationary and another zone in which the problem is indeed parabolic. For some applications see also the examples in the last section.

The scheme of the paper is the following: Section 2 is dedicated to existence of the solution to an equation $Pu = f$ and to the position of the problem. In Section 3 there are some compactness result: since a "classical type" compactness result (see Theorem 2.6, with $\mathcal{R} = Id$ for the classical case) does not hold in general, we pass throught a regularity result (Theorem 2.8) to obtain it. In Section 4 we define G-convergence and prove a compactness result in two steps: Theorem 4.1 furnishes, given a sequence of operators $P_h u = \frac{\partial}{\partial t}(r_h u) - \mathrm{div}(a_h \cdot Du)$ in a suitable class, the existence, up to subsequence, of a limit operator $Pu = \frac{\partial}{\partial t}(ru) - \mathrm{div}(a_r \cdot Du)$, Theorem 4.4 states that a_r is *independent of* r. In the last section we give some examples, including also singular perturbations.

2 Elliptic-parabolic equations and position of the problem

In this section we present some results contained in [7] and then we pose the problem we want to study. Suppose to have a triplet

$$V \subset H \subset V'$$

where H is a Hilbert space, V a reflexive Banach space which continuously embeds in H and V' the dual space of V. We define the spaces

$$\mathcal{V} = L^2(0,T;V)\,, \quad \mathcal{H} = L^2(0,T;H)\,, \quad \mathcal{V}' = L^2(0,T;V')$$

endowed with the natural norms

$$\|u\|^2_{\mathcal{V}} = \int_0^T \|v(t)\|^2_V dt\,, \quad \|u\|^2_{\mathcal{H}} = \int_0^T \|v(t)\|^2_H dt\,, \quad \|u\|^2_{\mathcal{V}'} = \int_0^T \|v(t)\|^2_{V'} dt\,.$$

Suppose to have a family of non negative operators from H to H

$$R : [0,T] \longrightarrow \mathcal{L}(H)\,, \qquad \big(R(t)u, u\big)_H \geqslant 0 \quad \text{for every } u \in H\,. \tag{2.1}$$

such that for every $u, v \in V$, for almost every $t \in [0,T]$, the operators $R(t)$ satisfy

$$\begin{array}{ll} \text{(i)} & R(t) \quad \text{self-adjoint} \\ \text{(ii)} & \sup_{t\in(0,T)} \|R(t)\|_{\mathcal{L}(H)} \leqslant C_1\,, \\ \text{(iii)} & t \mapsto \big(R(t)u, v\big)_H \text{ absolutely continuos on } [0,T]\,, \\ \text{(iv)} & \big|\dfrac{d}{dt}\big(R(t)u,v\big)_H\big| \leqslant C_2\, \|u\|_V \|v\|_V \quad \text{for a. e. } t \in [0,T]\,. \end{array} \tag{2.2}$$

In this way we can define an operator $\mathcal{R}$ by

$$\mathcal{R} : \mathcal{H} \longrightarrow \mathcal{H} \qquad \text{by} \qquad \mathcal{R}u(t) := R(t)u(t)$$

which turns out to be linear and bounded by the constant C_1. Observe that we require a control on the temporal derivative of $(R(t)u, v)_H$ only by the norm of u and v in the space V and not H.
By (2.2) we can define a family of equibounded operators

$$R' : [0,T] \to \mathcal{L}(V, V') \qquad \text{by} \qquad \langle R'(t)u, v\rangle_{V'\times V} := \frac{d}{dt}\big(R(t)u, v\big)_H$$

and an operator

$$\mathcal{R}' : \mathcal{V} \to \mathcal{V}' \qquad \text{by} \qquad \langle \mathcal{R}'u, v\rangle_{\mathcal{V}'\times\mathcal{V}} := \int_0^T \langle R'(t)u(t), v(t)\rangle_{V'\times V} dt$$

which turns out to be linear and bounded by C_2.
Observe that, since $\mathcal{L}(H) \subset \mathcal{L}(V, V')$, we have that

$$R \in W^{1,\infty}(0,T;\mathcal{L}(V,V'))\,. \tag{2.3}$$

Some other notations: we will write H as sum of two subspaces for every $t \in [0,T]$

$$H = H_+(t) \oplus H_0(t)\,, \qquad R(t) = R_+(t) + R_0(t)$$

where $H_0(t)$ is the kernel of $R(t)$ and $H_+(t)$ obviously its orthogonal space and R_+ is such that $R_+(t)u = R(t)u$ for every $u \in H$ and $R_0(t)u = 0$ for every $u \in H$. We moreover define

$$\begin{array}{l} \tilde{H}_+(t) := \text{completion of } H_+(t) \\ \quad \text{with respect to the norm } \|w\|_{\tilde{H}_+(t)} = \|R(t)^{1/2}w\|_H\,. \end{array} \tag{2.4}$$

Now define the Banach space

$$\mathcal{W} = \{ u \in \mathcal{V} \mid (\mathcal{R}u)' \in \mathcal{V}' \} , \qquad \|u\|_{\mathcal{W}} := \|u\|_{\mathcal{V}} + \|(\mathcal{R}u)'\|_{\mathcal{V}'} \tag{2.5}$$

where $(\mathcal{R}u)'$ denotes the derivative in the distributional sense of $\mathcal{R}u$ with respect to the variable t.
We recall the following density result (see Proposition 2.5 in [7]).

Proposition 2.1 *If* (2.3) *holds the space* $C^1([0,T];V)$ *is dense in* $\mathcal{W}$.

Another approximation result we will need later is the following.

Lemma 2.2 *Consider* R *satisfying* (2.2) *with constants* C_1, C_2 *and let* $\mathcal{W}$ *be the space defined in* (2.5). *Then for every* $u \in \mathcal{W}$ *and* $\sigma > 0$ *there exist* $v \in C^1([0,T];V)$ *and* $S \in C^2([0,T];\mathcal{L}(V,V'))$ *satisfying* (2.2) *with the same constants* C_1, C_2 ($\mathcal{S}u(t) := S(t)u(t)$) *such that*

$$\|u - v\|_{\mathcal{V}} < \sigma , \qquad \|(\mathcal{R}u)' - (\mathcal{S}v)'\|_{\mathcal{V}'} < \sigma .$$

Proof - Fix $u \in \mathcal{W} = \mathcal{W}_{\mathcal{R}}$ and $\sigma > 0$. From Proposition 2.1 we derive the existence of $v \in C^1([0,T];V)$ such that

$$\|u - v\|_{\mathcal{W}} < \sigma/2 .$$

Consider a family of mollifiers $(\rho_\epsilon)_{\epsilon>0}$ and, after defining

$$\bar{R}(t) := \begin{cases} R(t) & \text{if } t \in [0,T] \\ 0 & \text{if } t \notin [0,T] , \end{cases}$$

consider

$$R_\epsilon(t) = \int_{\mathbf{R}} \bar{R}(\tau)\rho_\epsilon(t-\tau)d\tau$$

and the corresponding $\mathcal{R}_\epsilon u(t) := R_\epsilon(t)u(t)$. Note that $R \in W^{1,\infty}(0,T;\mathcal{L}(V,V'))$ and $R_\epsilon \to R$ in $L^\infty(0,T;\mathcal{L}(V,V')) \cap W^{1,q}(0,T;\mathcal{L}(V,V'))$ for every $q < +\infty$.
Observe that $v \in \mathcal{W}_{\mathcal{R}}$ and $v \in \mathcal{W}_{\mathcal{R}_\epsilon}$ for every $\epsilon > 0$. Then, since

$$\|(\mathcal{R}u)' - (\mathcal{R}_\epsilon v)'\|_{\mathcal{V}'} \leqslant \|(\mathcal{R}u)' - (\mathcal{R}v)'\|_{\mathcal{V}'} + \|(\mathcal{R}v)' - (\mathcal{R}_\epsilon v)'\|_{\mathcal{V}'} ,$$

to prove the result it is sufficient to estimate $\|(\mathcal{R}v)' - (\mathcal{R}_\epsilon v)'\|_{\mathcal{V}'}$. Since $R_\epsilon \to R$ in $L^\infty(0,T;\mathcal{L}(V,V'))$ we have that

$$\mathcal{R}_\epsilon v' \to \mathcal{R}v' \qquad \text{in } \mathcal{H} ,$$

and therefore it is sufficient to analyse the quantity $\mathcal{R}'_\epsilon v - \mathcal{R}'v$. We consider a function $\phi \in \mathcal{V}$ and, for any $p > 2$ we have

$$\begin{aligned}
\left| \langle \mathcal{R}'_\epsilon v - \mathcal{R}v, \phi \rangle_{\mathcal{V}' \times \mathcal{V}} \right| &= \left| \int_0^T \langle R'_\epsilon(t)v(t) - R'(t)v(t), \phi(t) \rangle_{V' \times V} dt \right| \\
&\leqslant \int_0^T \|R'_\epsilon(t) - R'(t)\|_{\mathcal{L}(V,V')} \|v(t)\|_V \|\phi(t)\|_V dt \\
&\leqslant \Big[\int_0^T \|R'_\epsilon(t) - R'(t)\|^2_{\mathcal{L}(V,V')} \|v(t)\|^2_V dt \int_0^T \|\phi(t)\|^2_V dt \Big]^{1/2} \\
&\leqslant \Big[\int_0^T \|R'_\epsilon(t) - R'(t)\|^{2p/(p-2)}_{\mathcal{L}(V,V')} dt \Big]^{(p-2)/2p} \Big[\int_0^T \|v(t)\|^p_V dt \Big]^{2/p} \|\phi\|_{\mathcal{V}}
\end{aligned}$$

since $v \in C^1([0,T];V) \subset L^p(0,T;V)$ for every $p > 2$. Taking the supremum with respect to ϕ, $\|\phi\|_{\mathcal{V}} = 1$, we obtain that

$$\|\mathcal{R}'_\epsilon v - \mathcal{R}v\|_{\mathcal{V}'} \leqslant \|R'_\epsilon - R'\|_{L^{2p/(p-2)}(0,T;\mathcal{L}(V,V'))}\|v\|_{L^p(0,T;V)}$$

for a $p > 2$ and since $R'_\epsilon \to R'$ in $L^q(0,T;\mathcal{L}(V,V'))$ for every $q < +\infty$ we conclude. $\square$

For the following result see Proposition 2.7 in [7].

Theorem 2.3 *For every $u, v \in \mathcal{W}$ the following holds:*

$$\frac{d}{dt}(\mathcal{R}u(t), v(t))_H = \langle \mathcal{R}'u(t), v(t)\rangle_{V'\times V} + \langle \mathcal{R}u'(t), v(t)\rangle_{V'\times V} + \langle \mathcal{R}v'(t), u(t)\rangle_{V'\times V}\,, \quad (2.6)$$

so in particular we have

$$\begin{aligned} &2\int_s^t \big\langle (\mathcal{R}u)'(\tau), u(\tau)\big\rangle_{V'\times V} d\tau = \\ &\qquad = (R(t)u(t), u(t))_H - (R(s)u(s), u(s))_H + \int_s^t \langle \mathcal{R}'u(\tau), u(\tau)\rangle_{V'\times V} d\tau\,. \end{aligned} \quad (2.7)$$

Moreover we have the following continuous embedding:

$$\mathcal{W} \subset C([0,T];\tilde{H}(t))$$

and there is constant $c = c(T, \|\mathcal{R}\|)$ (depending only on $T, \|\mathcal{R}\| \leqslant C_1$) such that

$$\max_{[0,T]} |(R(t)u(t), u(t))_H| \leqslant c\ \|u\|^2_{\mathcal{W}}\,. \quad (2.8)$$

Thanks to the previous proposition every $u \in \mathcal{W}$ belongs to $C([0,T];\tilde{H}(t))$ and in particular it makes sense to evaluate $u(t)$ in $\tilde{H}(t))$ for every $t \in [0,T]$. Then for every $t \in [0,T]$ we denote by $P(t)$ the orthogonal projection

$$P(t) : \tilde{H}(t) \longrightarrow \tilde{H}_+(t)\,. \quad (2.9)$$

Now we give an existence result (see Theorem 3.4 and Theorem 3.5 in [7]) for the following problem: suppose you are given $f \in \mathcal{V}'$ and $\varphi \in \tilde{H}_+(0)$. Consider the problem

$$\begin{cases} (\mathcal{R}u)' + \mathcal{A}u = f \\ P(0)u(0) = \varphi \end{cases} \quad (2.10)$$

where $\mathcal{A}$ is defined by a family of operators

$$A : [0,T] \to \mathcal{L}(V,V') \quad \text{with} \quad t \mapsto \big\langle A(t)u, v\big\rangle_{V'\times V} \quad \text{measurable on } [0,T]$$

as follows

$$\mathcal{A} : \mathcal{V} \longrightarrow \mathcal{V}'\,, \qquad \mathcal{A}u(t) = A(t)u(t) \qquad 0 \leqslant t \leqslant T\,. \quad (2.11)$$

Definition 2.4 *A function $u \in \mathcal{W}$ is a solution of* (2.10) *if*

$$\begin{aligned} &\langle (\mathcal{R}u)'(t), v\rangle_{V'\times V} + \langle \mathcal{A}u(t), v\rangle_{V'\times V} = \langle f(t), v\rangle_{V'\times V} \\ &P(0)u(0) = \varphi \qquad \text{in } \tilde{H}_+(0) \end{aligned}$$

for every $v \in V$ and for a. e. $t \in [0,T]$ and the initial datum makes sense in $\tilde{H}_+(0)$ thanks to Theorem 2.3*. If $\mathcal{R} = 0$ the initial condition has no meaning and in this case a solution is a function $u \in \mathcal{V}$ such that*

$$\langle \mathcal{A}u(t), v\rangle_{V'\times V} = \langle f(t), v\rangle_{V'\times V} \qquad \text{for every } v \in V, \text{ for a. e. } t \in [0,T]\,.$$

Theorem 2.5 *Define an operator $\mathcal{P} : \mathcal{W} \to \mathcal{V}'$ by $\mathcal{P}u = (\mathcal{R}u)' + \mathcal{A}u$. Suppose that there exist two constants $\alpha, \beta > 0$ such that*

$$\langle \mathcal{A}u + \frac{1}{2}\mathcal{R}'u, u\rangle_{\mathcal{V}'\times\mathcal{V}} \geqslant \alpha\|u\|^2_{\mathcal{V}}, \qquad \text{and} \qquad \|\mathcal{A}u + \frac{1}{2}\mathcal{R}'u\|_{\mathcal{V}'} \leqslant \beta\|u\|_{\mathcal{V}} \tag{2.12}$$

for every $u, v \in \mathcal{V}$. Then there is a constant $c = c(\alpha, \beta)$ (depending only on α and β) such that for every $u \in \mathcal{W}$

$$\|u\|_{\mathcal{W}} \leqslant c\left[\|\mathcal{P}u\|_{\mathcal{V}'} + \|R^{1/2}(0)u(0)\|_H\right]. \tag{2.13}$$

Moreover for every $f \in \mathcal{V}'$ and $\varphi \in \tilde{H}_+(0)$, problem (2.10) has a unique solution.

In this framework the natural compactness result reads as follows (see Theorem 2.12 and Theorem 2.16 in [7]).

Theorem 2.6 *Consider a family $R : [0,T] \to \mathcal{L}(H)$ of operators satisfying (2.2)-(i), (2.2)-(ii). Consider the operator $\mathcal{R}$ and suppose $\mathcal{R} \not\equiv 0$. Suppose the embedding $V \subset H$ is compact. Then the space $\mathcal{W}$ compactly embeds in $L^2(0,T;\tilde{H}_+(t))$.*

A more general result is the following (see Theorem 2.16 in [7]).

Theorem 2.7 *Consider a sequence R_h satisfying (2.2)-(i), (2.2)-(ii) and consider the operators $\mathcal{R}_h : \mathcal{H} \to \mathcal{H}$. Suppose the embedding $V \subset H$ is compact. Suppose $\mathcal{R}_h \to \mathcal{R}$ and denote $\tilde{H}(t)$ the space defined in (2.4) corresponding to the limit operator $\mathcal{R}$. Consider a sequence $(u_h)_h \subset \mathcal{V}$ such that $(\mathcal{R}_h u_h)' \in \mathcal{V}'$ and $\|u_h\|_{\mathcal{V}} + \|(\mathcal{R}_h u_h)'\|_{\mathcal{V}'} \leqslant c$ for any positive constant c. Then $(u_h)_h$ is relatively compact*

i) in $L^2(0,T;\tilde{H}(t))$ if $\mathcal{R}_h \to \mathcal{R}$ pointwise in $\mathcal{H}$,

ii) in $L^2(0,T;H)$ if $\mathcal{R}_h \to \mathcal{R}$ pointwise in $\mathcal{V}'$ and $\mathcal{R}$ is injective.

Therefore we need a regularity result to have compactness, result which we recall now before introducing the problem we want to study.
Before we give some assumptions about the operators $\mathcal{R}$ and $\mathcal{A}$. About the operator $\mathcal{A}$ we will make the following assumptions: that there are some positive constants α, β, M, N such that

$$\left|\begin{array}{ll} t \mapsto \big\langle A(t)u, v\big\rangle_{V'\times V} & \text{is absolutely continuos on } [0,T] \\ \alpha\|u\|^2_V \leqslant \big\langle A(t)u, u\big\rangle_{V'\times V} \leqslant \beta\|u\|^2_V & \\ |\big\langle A(t)u, v\big\rangle_{V'\times V}| \leqslant M\big\langle A(t)u, u\big\rangle^{1/2}_{V'\times V}\big\langle A(t)v, v\big\rangle^{1/2}_{V'\times V} & \\ |\frac{d}{dt}\big\langle A(t)u, v\big\rangle_{V'\times V}| = |\big\langle A'(t)u, v\big\rangle_{V'\times V}| \leqslant N\|u\|_V\|v\|_V & \end{array}\right. \tag{2.14}$$

with $N > 0$, for almost every $t \in (0,T)$ and for every $u, v \in V$. In this way we can define an operator $\mathcal{A}'$ as follows

$$A' : [0,T] \to \mathcal{L}(V, V') \qquad \text{with} \qquad \mathcal{A}'u(t) = A'(t)u(t)$$

bounded by N. About the operator $\mathcal{R}$ we will assume (2.2) and the same for $\mathcal{R}'$, i. e. we define $\mathcal{R}'u(t) := R'(t)u(t)$ where R' is such that

$$\left| \begin{array}{l} R' : [0,T] \to \mathcal{L}(H)\,, \quad R'(t) \ \text{self-adjoint} \\[2mm] \sup_{t\in(0,T)} \|R'(t)\|_{\mathcal{L}(H)} \leqslant K_1\,, \\[2mm] t \mapsto \big(R'(t)u, v\big)_H \ \text{absolutely continuos on } [0,T]\,, \\[2mm] \dfrac{d}{dt}\big(R'(t)u, v\big)_H \leqslant K_2 \ \|u\|_V \|v\|_V \quad \text{for a. e. } t \in [0,T]\,. \end{array} \right. \tag{2.15}$$

As done for $\mathcal{R}$ we can define

$$\langle R''(t)u, v\rangle_{V'\times V} := \frac{d}{dt}\big(R'(t)u, v\big)_H$$

and an operator $\mathcal{R}''$ by

$$\mathcal{R}'' : \mathcal{V} \to \mathcal{V}' \qquad \text{by} \qquad \langle \mathcal{R}''u, v\rangle_{\mathcal{V}'\times\mathcal{V}} := \int_0^T \langle R''(t)u(t), v(t)\rangle_{V'\times V}\, dt$$

which turns out to be linear and bounded by K_2. For the following result see Theorem 3.6 and Corollary 3.8 in [7].

Theorem 2.8 *Consider the problem* (2.10). *Assume the operators $\mathcal{R}$ and $\mathcal{A}$ satisfy* (2.2), (2.15), (2.14) *and* (2.12). *Suppose $f \in H^1(0,T;V')$ and the existence of $u_0 \in V$ such that $P(0)u_0 = \varphi$ such that $f(0) - A(0)u_0 - R'(0)u_0 \in \mathrm{Im}\ R(0)$. Then the solution u satisfies*

$$u \in H^1(0,T;V)\,.$$

Moreover there exists a positive constant c depending (only) on α, β, $\|\mathcal{A}'\|_{\mathcal{L}(\mathcal{V},\mathcal{V}')}$, $\|\mathcal{R}'\|_{\mathcal{L}(\mathcal{V},\mathcal{V}')}$, $\|\mathcal{R}''\|_{\mathcal{L}(\mathcal{V},\mathcal{V}')}$ such that

$$\begin{aligned} &\|u\|_{\mathcal{V}} + \|u'\|_{\mathcal{V}} + \|(\mathcal{R}u')'\|_{\mathcal{V}'} + \|u\|_{C([0,T];V)} \leqslant \\ &\leqslant c\, \Big[\|f\|_{\mathcal{V}'} + \|f'\|_{\mathcal{V}'} + \|R^{1/2}(0)u_0\|_H + \|R_+^{-1/2}(0)P(0)\big(f(0) - A(0)u_0 - R'(0)u_0\big)\|_H\Big]. \end{aligned}$$

Position of the problem - From now on T, λ_0, Λ_0, M will be fixed positive constants, N, C_1, C_2 non-negative constants, μ_0 a non-positive constant and Ω a bounded open set of $\mathbf{R}^n$ with Lipschitz boundary. From now on we will denote for brevity

$$\mathcal{V} := L^2(0,T;H_0^1(\Omega)), \quad \mathcal{H} := L^2(0,T;L^2(\Omega)), \quad \mathcal{V}' := L^2(0,T;H^{-1}(\Omega))\,.$$

We recall the definition of the class $\mathcal{M}_{\Omega\times(0,T)}(\lambda_0, \Lambda_0, M)$, given in (1.1), characterized by

$$\begin{array}{l} a = [a_{ij}(x,t)]_{i,j=1}^n \qquad \text{such that} \\[2mm] \lambda_0|\xi|^2 \leqslant \big(a(x,t)\cdot\xi, \xi\big) \leqslant \Lambda_0|\xi|^2 \\[2mm] \big|\big(a(x,t)\cdot\xi, \eta\big)\big| \leqslant M\big(a(x,t)\cdot\xi,\xi\big)^{1/2}\big(a(x,t)\cdot\eta,\eta\big)^{1/2} \end{array} \tag{2.16}$$

for every $\xi, \eta \in \mathbf{R}^n$, for a. e. $(x,t) \in \Omega \times (0,T)$. By $\mathcal{M}_{\Omega\times(0,T)}(\lambda_0, \Lambda_0, M, N)$ we denote the subclass of $\mathcal{M}_{\Omega\times(0,T)}(\lambda_0, \Lambda_0, M)$ satisfying the further assumption

$$\big|a(x,t) - a(x,s)\big| \leqslant N|t-s| \tag{2.17}$$

for a. e. $x \in \Omega$ and every $s,t \in [0,T]$. Observe that under this last assumption if we choose $a \in \mathcal{M}_{\Omega\times(0,T)}(\lambda_0, \Lambda_0, M, N)$ and consider the family of operators

$$A : [0,T] \to \mathcal{L}(H^1_0(\Omega), H^{-1}(\Omega)) \quad \text{defined by} \quad t \mapsto \int_\Omega \big(a(x,t)\cdot Du(x), Dv(x)\big)dx$$

we can define $A' : [0,T] \to \mathcal{L}(H^1_0(\Omega), H^{-1}(\Omega))$ as in (2.14) and, by (2.17),

$$\begin{aligned} &\big\langle A'(t)u, v\big\rangle_{H^{-1}(\Omega)\times H^1_0(\Omega)} = \int_\Omega \big(a'(x,t)\cdot Du(x), Dv(x)\big)dx \\ &\text{where } a'_{ij}(x,t) = \frac{\partial a_{ij}}{\partial t}(x,t) \end{aligned} \tag{2.18}$$

Consider $C_1, C_2 \geqslant 0$, $\mu_0 \leqslant 0$ and define the class $\mathcal{F}(C_1, C_2, \mu_0)$ by

$$\begin{aligned} & r \in L^\infty(\Omega\times(0,T)), \quad r \geqslant 0\,, \\ & \|r\|_{L^\infty(\Omega\times(0,T))} \leqslant C_1\,, \\ & t \mapsto \int_\Omega u(x)v(x)r(x,t)dx && \text{absolutely continuos on } [0,T]\,, \\ & \Big|\frac{d}{dt}\int_\Omega u(x)v(x)r(x,t)dx\Big| \leqslant C_2\ \|u\|_{H^1_0(\Omega)}\|v\|_{H^1_0(\Omega)} && \text{for a. e. } t \in [0,T]\,, \\ & \frac{d}{dt}\int_\Omega u^2(x)r(x,t)dx \geqslant \mu_0\ \|u\|^2_{H^1_0(\Omega)} && \text{for a. e. } t \in [0,T]\,. \end{aligned} \tag{2.19}$$

Remark 2.9 - The class just defined is compact, i. e. if $(r_h)_h$ is a sequence in $\mathcal{F}(C_1, C_2, \mu_0)$, there is a subsquence $(r_{h_j})_j$ and a function $r \in \mathcal{F}(C_1, C_2, \mu_0)$ such that $r_{h_j} \to r$ in $L^\infty(\Omega \times (0,T))$-weak $*$. In fact, there is a subsequence $(r_{h_j})_j$ and a function r such that $r_{h_j} \to r$ in $L^\infty(\Omega\times(0,T))$-weak $*$. Now verify that r belong to $\mathcal{F}(C_1, C_2, \mu_0)$. Observe that from this we also derive that

$$r_{h_j}(\cdot,t) \to r(\cdot,t) \qquad \text{in } L^\infty(\Omega) - \text{weak}\, * \quad \text{for a. e. } t \in [0,T]\,. \tag{2.20}$$

Clearly $r \geqslant 0$ and $\|r\|_\infty \leqslant C_1$. Moreover, if $u,v \in H^1_0(\Omega)$, the functions $F^{u,v}_h(t) = \int_\Omega u(x)v(x)r_h(x,t)dx$ are bounded in $W^{1,\infty}(0,T)$ so there if a function $F^{u,v} \in W^{1,\infty}(0,T)$ and a subsequence $F^{u,v}_{h_{j_k}}$ such that $F^{u,v}_{h_{j_k}} \to F^{u,v}$ weakly $*$ in $W^{1,\infty}$. Since $H^1_0(\Omega)$ is separable, by a diagonal process we can find a sequence of integers $(h_j)_j$ such that this is true for every $u,v \in H^1_0(\Omega)$. By (2.20) we deduce that $F^{u,v}(t) = \int_\Omega u(x)v(x)r(x,t)dx$.

Fix $f \in \mathcal{V}'$ and $\varphi \in L^2(\Omega)$ and consider two sequences

$$\begin{aligned} & (a_h)_h \subset \mathcal{M}_{\Omega\times(0,T)}(\lambda_0, \Lambda_0, M)\,, \\ & (r_h)_h \subset \mathcal{F}(C_1, C_2, \mu_0)\,, \\ & \mu_0 > -\lambda_0 \end{aligned} \tag{2.21}$$

and satisfying moreover the following additional assumption (considered by Colombini and Spagnolo in [3])

$$\lim_{\tau\to 0}\sup_h \int_I\int_\omega |a_h(x,t+\tau)-a_h(x,t)|dxdt = 0 \qquad \forall\, I\times\omega \subset\subset (0,T)\times\Omega\,. \tag{2.22}$$

Assumption $\mu_0 > -\lambda_0$ is required to have existence to problems (2.23) (see below) and also to obtain Lemma 3.1 and Remark 3.2.

Then we consider the sequence of elliptic-parabolic problems

$$\begin{cases} \dfrac{\partial}{\partial t}(r_h u) - \operatorname{div}(a_h\cdot Du) = f & \text{on } \Omega\times(0,T) \\ u = 0 & \text{on } \partial\Omega\times(0,T) \\ u = \varphi & \text{on } \Omega_{h,+}(0)\times\{0\} \end{cases} \tag{2.23}$$

where

$$\Omega_{h,+}(t) := \{x\in\Omega \mid r_h(x,t) > 0 \text{ for a. e. } x\in\Omega\}\,.$$

Considering the triple

$$V = H^1_0(\Omega) \subset H = L^2(\Omega) \subset V' = H^{-1}(\Omega)\,,$$

endow V with the usual norm $\|u\|_{H^1_0(\Omega)} := \displaystyle\int_\Omega |Du|^2 dx$ and denote by $P_{h,+}(t)$ the following orthogonal projection

$$P_{h,+}(t) : L^2\big(\Omega, r_h^{1/2}(\cdot,t)\big) \longrightarrow L^2\big(\Omega_{h,+}, r_h^{1/2}(\cdot,t)\big)$$

as done in (2.9). Then consider the spaces

$$\mathcal{W}_h = \big\{u\in L^2(0,T;H^1_0(\Omega)) \;\big|\; (r_h u)' \in L^2(0,T;H^{-1}(\Omega))\big\} \tag{2.24}$$

we define the sequences of operators ($h\in\mathbf{N}$)

$$\begin{array}{ll} \mathcal{A}_h : L^2(0,T;H^1_0(\Omega)) \to L^2(0,T;H^{-1}(\Omega)) & \mathcal{A}_h u(x,t) = -\operatorname{div}\,(a_h(x,t)Du(x,t)) \\ \mathcal{R}_h : L^2(0,T;L^2(\Omega)) \to L^2(0,T;L^2(\Omega)) & \mathcal{R}_h u(x,t) = r_h(x,t)u(x,t) \\ \mathcal{P}_h : \mathcal{W}_h \to L^2(0,T;H^{-1}(\Omega)) & \mathcal{P}_h u = (\mathcal{R}_h u)' + \mathcal{A}_h u\,. \end{array}$$

We can rewrite problems (2.23) in the following way (see Theorem 2.3)

$$\begin{cases} (\mathcal{R}_h u)' + \mathcal{A}_h u = f \\ P_{h,+}(0)u(0) = \varphi \end{cases} \tag{2.25}$$

The assumption on (2.21) ensures that (2.12) is satisfied with

$$\alpha = \lambda_0 + \frac{1}{2}\mu_0\,, \qquad \beta = \Lambda_0 + \frac{1}{2}C_2\,.$$

Then by Theorem 2.5 problem (2.23), or problem (2.25), has a unique solution

$$\begin{array}{l} u_h \in \big\{u\in\mathcal{W}_h \;\big|\; P_{h,+}(0)u(0) = \varphi\big\} \\ \|u_h\|_{\mathcal{W}_h} \leqslant c(\lambda_0 + \frac{1}{2}\mu_0, \Lambda_0, C_2)\ \Big[\|f\|_{L^2(0,T;H^{-1}(\Omega))} + \|r_h(\cdot,0)^{1/2}\varphi\|_{L^2(\Omega)}\Big]\,. \end{array}$$

We want to study the asymptotic behaviour of the solutions u_h when $h\to+\infty$ and characterize the limit problem.

3 Preliminary compactness results

In this section we will suppose, but only for the moment, more regularity than we will require to state the main theorem (see Theorem 4.4) on the sequences $(a_h)_h$ and $(r_h)_h$. Precisely in this section we require (see (2.16), (2.17) and (2.19)) not (2.21) but

$$(a_h)_h \subset \mathcal{M}_{\Omega\times(0,T)}(\lambda_0, \Lambda_0, M, N) \qquad \text{and} \qquad (r_h), \left(\frac{\partial r_h}{\partial t}\right)_h \subset \mathcal{F}(C_1, C_2, \mu_0)\,. \tag{3.1}$$

Consider the problems

$$\begin{cases} \dfrac{\partial}{\partial t}(r_h u) - \operatorname{div}(a_h \cdot Du) = g & \text{on } \Omega\times(0,T) \\ u = 0 & \text{on } \partial\Omega\times(0,T) \\ u = \psi_h & \text{on } \Omega_{h,+}(0)\times\{0\} \end{cases} \tag{3.2}$$

where $g \in H^1(0,T;H^{-1}(\Omega)) \subset C([0,T];H^{-1}(\Omega))$ and ψ_h is the solution to

$$\begin{cases} E_h w := -\operatorname{div}(a_h(x,0)\cdot Dw(x)) + \dfrac{\partial r_h}{\partial t}(x,0)w(x) = g(x,0) & \text{on } \ \Omega \\ u = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{3.3}$$

where the linear, elliptic and bounded operator $E_h : H^1_0(\Omega) \to H^{-1}(\Omega)$ can be considered because a_h and r_h are continuous in the variable t.
Before stating the main result we recall the following lemma. Denote by c the constant appearing in the Poincaré's inequality

$$\int_\Omega u^2(x)dx \leqslant c\int_\Omega |Du|^2(x)dx\,, \qquad u \in H^1_0(\Omega)\,. \tag{3.4}$$

Lemma 3.1 *Consider $a, a_1, a_2 ... \in \mathcal{M}_\Omega(\lambda_0,\Lambda_0,M)$ and suppose $a_h \xrightarrow{EG} a$ (see Definition 1.1). Consider a sequence of functions $(b_h)_h \subset L^\infty(\Omega)$, $b_h \geqslant \gamma$ for every $h \in \mathbf{N}$ where $-\lambda_0/c < \gamma \leqslant 0$ (c the Poincaré's constant) and suppose $b_h \to b$ in $L^\infty(\Omega)$-weak $*$. Then for every $f \in H^{-1}(\Omega)$ it results that*

$$\begin{array}{ll} w_h \to w & \textit{in } L^2(\Omega) \\ a_h\cdot Dw_h \to a\cdot Dw & \textit{in } L^2(\Omega)\textit{-weak}\,, \end{array}$$

where w_h and w denote respectively the solutions (see Definition (2.4))

$$\begin{cases} -\operatorname{div}(a_h\cdot Dv) + b_h v = f & \textit{in } \Omega \\ v = 0 & \textit{in } \partial\Omega \end{cases} \qquad \begin{cases} -\operatorname{div}(a\cdot Dv) + bv = f & \textit{in } \Omega \\ v = 0 & \textit{in } \partial\Omega\,. \end{cases}$$

Proof - Since $-\lambda_0/c < \gamma \leqslant 0$ we have that

$$\int_\Omega b_h v^2 dx \geqslant \int_\Omega \gamma v^2 dx \geqslant c\gamma\int_\Omega |Dv|^2 dx$$

and then the elliptic operators $v \mapsto -\operatorname{div}(a_h\cdot Dv) + b_h v$ are equicoercive. Since the solutions are compact in $L^2(\Omega)$ we have, up to subsequence, that $-b_h w_h$ converges to $-bw$ weakly in $L^2(\Omega)$ and then strongly in $H^{-1}(\Omega)$. Then we conclude observing that w_h solve the problems

$$\begin{cases} -\operatorname{div}(a_h\cdot Dv) = f_h := f - b_h w_h & \textit{in } \Omega \\ v = 0 & \textit{in } \partial\Omega\,. \end{cases}$$

$\square$

REMARK 3.2 - As a consequence we have that, if $a_h(\cdot,0) \xrightarrow{EG} a(\cdot,0)$ and $(r_h)_h, (\frac{\partial r_h}{\partial t})_h \subset \mathcal{F}(C_1, C_2, \mu_0)$ are such that $r_h \to r$ in $L^\infty(\Omega \times (0,T))$-weak $*$ (see also Remark 2.9) the equation (3.3) admit a unique solution since

$$\int_\Omega \frac{\partial r_h}{\partial t}(x,0) v^2(x) dx \geqslant \mu_0 \int_\Omega |Dv|^2 dx \,.$$

The solutions w_h of (3.3) satisfy

$$w_h \to w \qquad \text{in } L^2(\Omega)\,, \qquad a_h \cdot Dw_h \to a \cdot Dw \qquad \text{in } L^2(\Omega)\text{-weak}\,,$$

where w is the solution of

$$\begin{cases} Ew := -\mathrm{div}(a(x,0)\cdot Dw(x)) + \dfrac{\partial r}{\partial t}(x,0) w(x) = g(x,0) & \text{on } \Omega \\ u = 0 & \text{on } \partial\Omega\,. \end{cases}$$

Now we state the first compactness result.

Lemma 3.3 *Consider the problems* (3.2) *with the data* $g \in H^1(0,T; H^{-1}(\Omega))$ *and* $\psi_h = E_h^{-1} g(0)$ *and denote by* $u_h \in \mathcal{W}_h$ *the corresponding solutions. Then the sequence*

$$(u_h)_h \qquad \textit{is relatively compact in } C([0,T]; L^2(\Omega))\,.$$

As a consequence we also obtain that the sequence of functions

$$t \mapsto \int_\Omega u_h^2(x,t) r_h(x,t) dx \qquad \textit{is relatively compact in } C([0,T])\,.$$

Proof - Since $g(0) + \mathrm{div}(a_h(\cdot,0)\cdot D\psi_h) - \dfrac{\partial r_h}{\partial t}(\cdot,0)\psi_h = 0$, hypotheses of Theorem 2.8 are satisfyed. Then we have that the solutions satisfy the estimations

$$\|u_h\|_{H^1(0,T;H_0^1(\Omega))} \leqslant c\,, \qquad \|u_h\|_{C([0,T];H_0^1(\Omega))} \leqslant c\,.$$

We deduce that $(u_h)_h$ is equibounded in $C([0,T]; H_0^1(\Omega))$ and therefore the sets $\{u_h(t) \mid h \in \mathbf{N}\}$ are relatively compact in $L^2(\Omega)$ for every $t \in [0,T]$. Moreover

$$u_h(t) - u_h(s) = \int_s^t u_h'(\tau) d\tau$$

and then

$$\|u_h(t) - u_h(s)\|_{L^2(\Omega)} \leqslant \int_s^t \|u_h'(\tau)\|_{L^2(\Omega)} d\tau \leqslant |t-s|^{1/2} \|u_h'\|_{L^2(0,T;H_0^1(\Omega))}\,,$$

so $(u_h)_h$ is also equicontinuous valued in $L^2(\Omega)$. Then by Lemma 1 in [11] we obtain $(u_h)_h$ relatively compact in $C([0,T]; L^2(\Omega))$.
To prove the second statement denote for simplicity by $(u_h)_h$ a subsequence converging in $C([0,T]; L^2(\Omega))$ and call u the limit in $C([0,T]; L^2(\Omega))$. Consider the sequence $(r_h)_h$: by Remark (2.9) we have the existence of $r \in \mathcal{F}(C_1, C_2)$ such that, up to subsequence, $r_h \to r$ in $L^\infty(\Omega \times (0,T))$-weak $*$. Then, for $t, s \in [0,T]$,

$$\begin{aligned} \Big| \int_\Omega u_h^2(x,t) r_h(x,t) dx - \int_\Omega u_h^2(x,s) r_h(x,s) dx \Big| &\leqslant \\ \leqslant \Big| \int_\Omega u_h^2(x,t) r_h(x,t) dx - \int_\Omega u_h^2(x,t) r_h(x,s) dx \Big| &+ \\ + \Big| \int_\Omega u_h^2(x,t) r_h(x,s) dx - \int_\Omega u_h^2(x,s) r_h(x,s) dx \Big| &. \end{aligned}$$

By (2.19) and the continuity of $u_h : [0,T] \to L^2(\Omega)$ we conclude that $t \mapsto \int_\Omega u_h^2(x,t) r_h(x,t)dx$ is continuous. Moreover

$$\begin{aligned}
\Big| \int_\Omega u_h^2(x,t) r_h(x,t) dx - \int_\Omega u^2(x,t) r(x,t) dx \Big| \leqslant \\
\leqslant \Big| \int_\Omega u_h^2(x,t) r_h(x,t) dx - \int_\Omega u^2(x,t) r_h(x,t) dx \Big| + \\
+ \Big| \int_\Omega u^2(x,t) r_h(x,t) dx - \int_\Omega u^2(x,t) r(x,t) dx \Big| \\
\leqslant \|r_h\|_\infty \int_\Omega |u_h^2(x,t) - u^2(x,t)| dx + \Big| \int_\Omega u^2(x,t) \big(r_h(x,t) - r(x,t) \big) dx \Big|
\end{aligned}$$

from which we conlude. □

Theorem 3.4 *Consider a sequence of functions $v_h \in \mathcal{W}_h$ $(h = 1, 2, ...)$, a function $v \in \mathcal{W}$, a constant c_1 such that*

$$\|v_h\|_{\mathcal{W}_h} \leqslant c_1 \quad \text{for every } h, \quad v_h \to v \quad \text{in } C([0,T]; L^2(\Omega)), \tag{3.5}$$

and a sequence of vectorial functions $m_h, m \in L^2(0,T; (L^2(\Omega))^n)$ $(h = 1, 2, ...)$, a constant c_2 such that

$$\|m_h\|_{L^2(0,T;(L^2(\Omega))^n)} \leqslant c_2 \,, \qquad m_h \to m \quad \text{in } L^2(0,T; (L^2(\Omega))^n) - \text{weak}. \tag{3.6}$$

Moreover suppose that

$$(r_h v_h)' - \operatorname{div} (m_h) = f \in L^2(0,T; H^{-1}(\Omega)) \quad \text{in } C^1_{\mathrm{c}}(\Omega \times (0,T)). \tag{3.7}$$

Then

$$\big(m_h, Dv_h\big) \to \big(m, Dv\big) \quad \text{in } \mathcal{D}'(\Omega \times (0,T)).$$

Proof - Fix a function $\varphi \in C^\infty_{\mathrm{c}}(\Omega \times (0,T))$ and multiply the equation (3.7) by $v_h \varphi$. We obtain (see (2.6))

$$\int_0^T \int_\Omega (m_h, Dv_h) \varphi dx dt = \langle f - (r_h v_h)', v_h \varphi \rangle - \int_0^T \int_\Omega (m_h, D\varphi) v_h dx dt. \tag{3.8}$$

Clearly $\langle f, v_h \varphi \rangle \to \langle f, v \varphi \rangle$ as $h \to \infty$ and $\int_0^T \int_\Omega (m_h, D\varphi) v_h dx dt \to \int_0^T \int_\Omega (m, D\varphi) v dx dt$. By (2.6) and since φ has compact support in $\Omega \times (0,T)$ we have that

$$-2 \langle (r_h v_h)', v_h \varphi \rangle = \int_0^T \int_\Omega v_h^2(x,t) r_h(x,t) \frac{\partial \varphi}{\partial t}(x,t) dx dt - \int_0^T \int_\Omega v_h^2(x,t) \varphi \frac{\partial r_h}{\partial t}(x,t) dx dt$$

which converges to

$$\int_0^T \int_\Omega v^2(x,t) r(x,t) \frac{\partial \varphi}{\partial t}(x,t) dx dt - \int_0^T \int_\Omega v^2(x,t) \varphi \frac{\partial r}{\partial t}(x,t) dx dt = -2 \langle (rv)', v\varphi \rangle$$

Then

$$\int_0^T \int_\Omega (m_h, Dv_h) \varphi dx dt \to \langle f - (rv)', v\varphi \rangle - \int_0^T \int_\Omega (m, D\varphi) v dx dt$$

and since, multiplying (3.7) by a test function and taking the limit, we also have

$$(rv)' - \operatorname{div} (m) = f$$

we obtain the thesis. □

4 The definition of G-convergence and the compactness result

In this section we give the main result, a compactness result with respect to G-convergence defined below (see Definition 4.3) for a sequence of operators (see (2.16) and (2.19))

$$
\begin{aligned}
&\mathcal{P}_h u = \frac{\partial}{\partial t}(r_h u) - \operatorname{div}(a_h \cdot Du)\,, \\
&(a_h)_h \subset \mathcal{M}_{\Omega\times(0,T)}(\lambda_0, \Lambda_0, M)\,, \qquad (r_h)_h \subset \mathcal{F}(C_1, C_2, \mu_0)\,, \\
&\lim_{\tau\to 0}\sup_h \int_I\int_\omega |a_h(x,t+\tau) - a_h(x,t)|dxdt = 0 \qquad \forall\, I\times\omega \subset\subset (0,T)\times\Omega\,.
\end{aligned}
\tag{4.1}
$$

The statement of this result is divided in two theorems. In the first one (Theorem 4.1) we suppose the regularity required in the previous section, i. e. (3.1), and prove a partial result: the existence of a limit operator in divergence form. The second result (Theorem 4.4) is a kind of uniqueness result: with less assumptions on the coefficients, i. e. satisfying the assumptions in (4.1), we will prove that the matrix defining the limit operator is independent of the sequence (r_h).

Theorem 4.1 *Consider a sequence $(a_h)_h \subset \mathcal{M}_{\Omega\times(0,T)}(\lambda_0, \Lambda_0, M, N)$ and $(r_h)_h \subset \mathcal{F}(C_1, C_2, \mu_0)$ with $(\frac{\partial r_h}{\partial t})_h \subset \mathcal{F}(C_1, C_2, \mu_0)$. There exist a matrix $a \in \mathcal{M}_{\Omega\times(0,T)}(\lambda_0, M^2\Lambda_0, M\sqrt{\Lambda_0/\lambda_0})$ and a function $r \in \mathcal{F}(C_1, C_2, \mu_0)$ such that for every $f \in L^2(0,T;H^{-1}(\Omega))$ and $\varphi \in L^2(\Omega)$ the solutions u_h of problems (2.23), $h \in \mathbf{N}$, satisfy, up to subsequence,*

$$
u_h \to u \quad \text{in } L^2(0,T;L^2(\Omega)^n) \qquad \text{and} \qquad a_h\cdot Du_h \to a\cdot Du \quad \text{in } L^2(0,T,L^2(\Omega))\text{-weak}
$$

where, defined $\Omega_+(0) := \{x \in \Omega \mid r(x,0) > 0\}$, u is the solution of

$$
\begin{cases}
\dfrac{\partial(ru)}{\partial t} - \operatorname{div}(a\cdot Du) = f & \text{in } \Omega\times(0,T) \\
u = 0 & \text{in } \partial\Omega\times(0,T) \\
u = \varphi & \text{in } \Omega_+(0)\times\{0\}
\end{cases}
\quad .
\tag{4.2}
$$

Remark 4.2 - The result is true also if in (2.23) we consider a sequence of data $(f_h)_h \subset L^2(0,T;H^{-1}(\Omega))$, $(f_h)_h$ relatively compact in $L^2(0,T;H^{-1}(\Omega))$, and $(\varphi_h)_h \subset L^2(\Omega)$, φ_h relatively compact in $L^2(\Omega)$.

Proof - First, by Remark 2.9, from $(r_h)_h$ we can extract a subsequence, still denoted by $(r_h)_h$, such that $r_h \to r$ in $L^\infty(\Omega\times(0,T)$-weak $*$ and $r \in \mathcal{F}(C_1, C_2, \mu_0)$. Denote by $\Omega_+(0)$ the set $\{x\in\Omega \mid r(x,0) > 0\}$ and by $\mathcal{R}$ the operator defined by

$$
\mathcal{R} : L^2(0,T;L^2(\Omega)) \to L^2(0,T;L^2(\Omega))\,, \qquad (\mathcal{R}u)(x,t) = r(x,t)u(x,t)\,.
\tag{4.3}
$$

Fix X a countable and dense subset of $H^1(0,T;H^{-1}(\Omega))$ and $Y = \{y \in H^1_0(\Omega) \mid y = E^{-1}g(0)$ for $g \in X\}$ where E is the operator defined in Remark 3.2. Then consider $g \in X$ and $\psi = E^{-1}g(0)$ (and denote by $u_h(g,\psi)$ the solutions to the problems (3.2) with $\psi_h = E_h^{-1}(E\psi)$ where E_h are the operators in (3.3).
By Lemma 3.3 we have that the solutions $u_h(g,\psi)$ are compact in $C([0,T];L^2(\Omega))$, and in particular $\psi_h \to \psi$ in $L^2(\Omega)$. Denote by $\mathcal{B}_\mathcal{R}(g,\psi)$ the limit in $C([0,T];L^2(\Omega))$ of $u_h(g,\psi)$. In

this way we have defined an operator in (X, Y). $\mathcal{B}_{\mathcal{R}}$ is linear and continuous (see Theorem 2.5). For every $h \in \mathbf{N}$ we have

$$\|u_h(g,\psi)\|_{\mathcal{W}_h} \leqslant c\ \left[\|g\|_{L^2(0,T;H^{-1}(\Omega))} + \|\sqrt{r_h(\cdot,0)}\psi_h\|_{L^2(\Omega)}\right].$$

Since $(r_h u_h)' \to (ru)'$ weakly in $L^2(0,T;H^{-1}(\Omega))$ and $\sqrt{r_h(\cdot,0)}\psi_h \to \sqrt{r(\cdot,0)}\psi$ weakly in $L^2(\Omega)$, by the lower semicontinuity of the norm we obtain

$$\|\mathcal{B}_{\mathcal{R}}(g,\psi)\|_{\mathcal{W}} \leqslant c\ \left[\|g\|_{L^2(0,T;H^{-1}(\Omega))} + \|\sqrt{r(\cdot,0)}\psi\|_{L^2(\Omega)}\right]. \tag{4.4}$$

Since (X, Y) is dense in $L^2(0,T;H^{-1}(\Omega)) \times L^2\big(\Omega_+(0), (r_+(\cdot,0))^{-1/2}\big)$, where $r_+(\cdot,0)$ denotes the restriction of $r(\cdot,0)$ to $\Omega_+(0)$, we can extend $\mathcal{B}_{\mathcal{R}}$ (and denote it in the same way)

$$\mathcal{B}_{\mathcal{R}} : L^2(0,T;H^{-1}(\Omega)) \times L^2\big(\Omega_+(0), (r_+(\cdot,0))^{-1/2}\big) \to \mathcal{W}.$$

Then we define

$$\begin{aligned}
&\mathcal{K}_{\mathcal{R}} : L^2(0,T;H^{-1}(\Omega)) \times L^2\big(\Omega_+(0), (r_+(\cdot,0))^{-1/2}\big) \to L^2(0,T;H^{-1}(\Omega)),\\
&\mathcal{K}_{\mathcal{R}} f = f - \frac{d}{dt}\big(\mathcal{R}\mathcal{B}_{\mathcal{R}}(f,\varphi)\big),\\
&\|\mathcal{K}_{\mathcal{R}}(f,\varphi)\|_{L^2(0,T;H^{-1}(\Omega))} \leqslant c'\ \left[\|f\|_{L^2(0,T;H^{-1}(\Omega))} + \|\sqrt{r(\cdot,0)}\varphi\|_{L^2(\Omega)}\right].
\end{aligned}$$

So we have (see (2.25)

$$\frac{d}{dt}(\mathcal{R}_h u_h) + \mathcal{A}_h u_h = f \qquad \text{and} \qquad \frac{d}{dt}(\mathcal{R}\mathcal{B}_{\mathcal{R}}(f,\varphi)) + \mathcal{K}_{\mathcal{R}}(f,\varphi) = f \tag{4.5}$$

Multiplying the first in (4.5) by u_h we have (where $\langle\cdot,\cdot\rangle$ denotes the duality between $\mathcal{V}'$ and $\mathcal{V}$)

$$\begin{aligned}
\frac{1}{2}\int_\Omega u_h^2(x,T) r_h(x,T)dx - \frac{1}{2}\int_\Omega \varphi^2(x) r_h(x,0)dx + \int_0^T\int_\Omega \frac{\partial r_h}{\partial t} u_h^2 dxdt+&\\
+\int_0^T\int_\Omega \big(a_h\cdot Du_h, Du_h\big)dxdt = \langle f, u_h\rangle&
\end{aligned}$$

and the second by $\mathcal{B}_{\mathcal{R}}(f,\varphi)$ we have

$$\begin{aligned}
\frac{1}{2}\int_\Omega \mathcal{B}_{\mathcal{R}}(f,\varphi)^2(x,T) r(x,T)dx - \frac{1}{2}\int_\Omega \varphi^2(x) r(x,0)dx + \int_0^T\int_\Omega \frac{\partial r}{\partial t}\mathcal{B}_{\mathcal{R}}(f,\varphi)^2 dxdt+&\\
+\langle\mathcal{K}_{\mathcal{R}}(f,\varphi), \mathcal{B}_{\mathcal{R}}(f,\varphi)\rangle = \langle f, \mathcal{B}_{\mathcal{R}}(f,\varphi)\rangle.&
\end{aligned}$$

Since

$$\begin{aligned}
&\langle f, u_h\rangle \to \langle f, \mathcal{B}_{\mathcal{R}}(f,\varphi)\rangle\\
&\int_\Omega u_h^2(x,T) r_h(x,T)dx \to \int_\Omega \mathcal{B}_{\mathcal{R}}(f,\varphi)^2(x,T) r(x,T)dx\\
&\int_\Omega \varphi^2(x) r_h(x,0)dx \to \int_\Omega \varphi^2(x) r(x,0)dx\\
&\int_0^T\int_\Omega \frac{\partial r_h}{\partial t} u_h^2 dxdt \to \int_0^T\int_\Omega \frac{\partial r}{\partial t}\mathcal{B}_{\mathcal{R}}(f,\varphi)^2 dxdt
\end{aligned} \tag{4.6}$$

we deduce that

$$\int_0^T \int_\Omega \big(a_h \cdot Du_h, Du_h\big) dx dt = \langle \mathcal{A}_h u_h, u_h \rangle \to \langle \mathcal{K}_\mathcal{R}(f,\varphi), \mathcal{B}_\mathcal{R}(f,\varphi)\rangle . \tag{4.7}$$

The operator $\mathcal{B}_\mathcal{R}$ is injective: indeed if $\mathcal{B}_\mathcal{R}(f,\varphi) = 0$, $\langle \mathcal{A}_h u_h, u_h\rangle \to 0$ and then $\|u_h\|_\mathcal{V} \to 0$. Since $(\mathcal{R}_h u_h)' \to (\mathcal{R}\mathcal{B}_\mathcal{R}(f,\varphi))'$ weakly in $\mathcal{V}'$ we have that $\mathcal{A}_h u_h \to \mathcal{K}_\mathcal{R}(f,\varphi)$ weakly in $\mathcal{V}'$. Since $\mathcal{A}_h$ are equibounded and $u_h \to 0$ we conclude that $\mathcal{K}_\mathcal{R}(f,\varphi) = 0$, i. e. $f - \frac{d}{dt}(\mathcal{R}\mathcal{B}_\mathcal{R}(f,\varphi)) = 0$. Than $f = 0$. Clearly, since $\mathcal{B}_\mathcal{R}(f,\varphi) = 0$ and, by Theorem 2.3, $\mathcal{B}_\mathcal{R}(f,\varphi) = 0 \in C([0,T]; L^2(\Omega, \sqrt{r(\cdot,0)}))$ we also have that $\varphi = 0$.
Now we show that $\mathcal{B}_\mathcal{R}(\mathcal{V}' \times L^2(\Omega_+(0), r_+(\cdot,0))^{-1/2}))$ is dense in $\mathcal{V}$: fix $g \in \mathcal{V}'$ such that

$$\langle g, \mathcal{B}_\mathcal{R}(f,\varphi)\rangle = 0 \qquad \text{for every } f \in \mathcal{V}, \varphi \in L^2(\Omega_+(0), r_+(\cdot,0))^{-1/2}) .$$

In particular $\langle g, \mathcal{B}_\mathcal{R}(g,0)\rangle$ and then

$$\langle \frac{d}{dt}\big(\mathcal{R}\mathcal{B}_\mathcal{R}(g,0)\big) + \mathcal{K}_\mathcal{R}(g,0), \mathcal{B}_\mathcal{R}(g,\varphi)\rangle = 0$$

i. e. $\langle \mathcal{K}_\mathcal{R}(g,0), \mathcal{B}_\mathcal{R}(g,0)\rangle \leqslant 0$. But by (4.7) we know that $\langle \mathcal{K}_\mathcal{R}(g,0), \mathcal{B}_\mathcal{R}(g,0)\rangle \geqslant 0$, then it is zero. Therefore $u_h(g,0) \to 0$ and as above we conclude that $g = 0$.
Thus we can define the inverse of $\mathcal{B}_\mathcal{R}$: if we define

$$\begin{array}{rccc} \mathcal{A}_\mathcal{R}: & \mathcal{B}_\mathcal{R}\big(\mathcal{V}' \times L^2(\Omega_+(0), (r_+(\cdot,0))^{-1/2})\big) & \longrightarrow & \mathcal{V}' \\ & \mathcal{B}_\mathcal{R}(f,\varphi) & \mapsto & \mathcal{K}_\mathcal{R}(f,\varphi) , \end{array}$$

by density we can define an operator, still denoted by $\mathcal{A}_\mathcal{R}$,

$$\mathcal{A}_\mathcal{R} : \mathcal{V} \to \mathcal{V}' .$$

Now we need to define a last operator: for every $(g,\psi) \in (X,Y)$ the sequence $a_h \cdot Du_h(g,\psi)$ is bounded in $L^2(0,T; L^2(\Omega)^n)$ and, up to subsequences, weakly converges in $L^2(0,T; L^2(\Omega)^n)$

$$a_h \cdot Du_h(g,\psi) \to \mu(g,\psi) . \tag{4.8}$$

We have that

$$\begin{aligned} |a_h(x,t) \cdot Du_h(g,\psi)(x,t)| &= \sup_{|\eta|=1} \big(a_h(x,t) \cdot Du_h(x,t), \eta\big) \leqslant \\ &\leqslant M\Lambda_0^{1/2} \big(a_h(x,t) \cdot Du_h(x,t), Du_h(x,t)\big)^{1/2} \end{aligned}$$

By Lemma 7.8 in [2] and Theorem 3.4 we have that

$$|\mu(g,\psi)| \leqslant M\Lambda_0^{1/2} \big(\mu(g,\psi), D\mathcal{B}_\mathcal{R}(g,\psi)\big)^{1/2}$$

from which

$$|\mu(g,\psi)| \leqslant M^2 \Lambda_0 |D\mathcal{B}_\mathcal{R}(g,\psi)| . \tag{4.9}$$

Again by density, using (4.4), we can extend

$$\mu : L^2(0,T; H^{-1}(\Omega)) \times L^2\big(\Omega_+(0), (r_+(\cdot,0))^{-1/2}\big) \to L^2(0,T; L^2(\Omega)^n) .$$

We define now $\mathcal{P}_\mathcal{R} u := (\mathcal{R}u)' + \mathcal{A}_\mathcal{R} u$ and

$$\begin{array}{rcccc} M_\mathcal{R} & : & \mathcal{W} & \longrightarrow & L^2(0,T; L^2(\Omega)^n) \\ & & u & \mapsto & \mu\big(\mathcal{P}_\mathcal{R} u, P_+(0) u(0)\big) \end{array} \tag{4.10}$$

where $P_+(0)w$ is the restriction to $\Omega_+(0) := \{x \in \Omega \mid r(x,0) > 0\}$ of a function w defined in Ω. Observe that from (4.9) and by (4.5) and the definition of $\mathcal{A}_\mathcal{R}$ and $\mathcal{P}_\mathcal{R}$ one has , for every $u \in \mathcal{W}$,

$$|M_\mathcal{R} u| \leqslant M^2 \Lambda_0 |Du| \,. \tag{4.11}$$

By definition we have that for every $v \in \mathcal{V}$

$$\langle \mathcal{A}_\mathcal{R} u, v \rangle = \int_0^T \int_\Omega \big(M_\mathcal{R} u, Dv\big) dx dt \,. \tag{4.12}$$

Now we want to constract a matrix a such that for every $u, v \in \mathcal{V}$

$$\int_0^T \int_\Omega \big(M_\mathcal{R} u, Dv\big) dx dt = \int_0^T \int_\Omega \big(a \cdot Du, Dv\big) dx dt \,.$$

For this purpose fix $\omega \subset\subset \Omega$, $I \subset\subset [0,T]$ and a function $\eta \in C^1_c(\Omega \times (0,T))$ such that $\eta = 1$ on $\omega \times I$. Define $\phi_\xi(x,t) = (\xi, x)\eta(x,t)$ ((ξ, x) denotes the scalar product in $\mathbf{R}^n$). Finally, if r is the function in (4.3), define

$$a_r(x,t) \cdot \xi = M_\mathcal{R}\big(\mathcal{P}_\mathcal{R} \phi_\xi, P_+(0)\phi_\xi(x,0)\big) \qquad \text{for } (x,t) \in \omega \times I \,.$$

From (4.12) we obtain that

$$\langle \mathcal{A}_\mathcal{R} u, v \rangle = \int_0^T \int_\Omega \big(a_r Du, Dv\big) dx dt \tag{4.13}$$

and arguing as in [14], Theorem 3, we obtain

$$\lambda_0 |\xi|^2 \leqslant (a_r(x,t) \cdot \xi, \xi) \,, \qquad (a_r(x,t) \cdot \xi, \eta) \leqslant M \Lambda_0^{1/2} (a_r(x,t) \cdot \xi, \xi) |\eta|$$

for a. e. $(x,t) \in \Omega \times (0,T)$ and for every $\xi, \eta \in \mathbf{R}^n$. Then also for the operator $\mathcal{A}_\mathcal{R}$ we have

$$\lambda_0 \|u\|_\mathcal{V} \leqslant \langle \mathcal{A}_\mathcal{R} u, u \rangle \quad \text{and} \quad |\langle \mathcal{A}_\mathcal{R} u, v \rangle| \leqslant M^2 \Lambda_0 \|u\|_\mathcal{V} \|v\|_\mathcal{V} \,.$$

□

Definition 4.3 *Consider a sequence $(a_h)_h \subset \mathcal{M}_{\Omega \times (0,T)}(\lambda_0, \Lambda_0, M)$. We say that*

$$a_h \xrightarrow{G} a \qquad \text{in } \Omega \times (0,T)$$

if for every $f \in L^2(0,T; H^{-1}(\Omega))$, for every $\varphi \in L^2(\Omega)$ and for every $(r_h)_h \subset \mathcal{F}(C_1, C_2, \mu_0)$ and $r \in \mathcal{F}(C_1, C_2, \mu_0)$ with

$$r_h \to r \qquad L^\infty(\Omega \times (0,T))\text{-weak} *$$

it results that

$$\begin{array}{ll} u_h \to u & \text{in } L^2(0,T,L^2(\Omega)) \\ a_h \cdot Du_h \to a \cdot Du & \text{in } L^2(0,T,L^2(\Omega)^n)\text{-weak}, \end{array}$$

where u_h and u denote respectively the solutions of (see Definition (2.4))

$$\begin{cases} \dfrac{\partial}{\partial t}(r_h v) - \operatorname{div}(a_h \cdot Dv) = f & \text{in } \Omega \times (0,T) \\ u = 0 & \text{in } \partial\Omega \times (0,T) \\ u = \varphi & \text{in } \Omega_{h,+}(0) \times \{0\} \end{cases} \qquad \begin{cases} \dfrac{\partial}{\partial t}(r v) - \operatorname{div}(a \cdot Dv) = f & \text{in } \Omega \times (0,T) \\ u = 0 & \text{in } \partial\Omega \times (0,T) \\ u = \varphi & \text{in } \Omega_{+}(0) \times \{0\} \,. \end{cases}$$

where $\Omega_{h,+}(0) := \{x \in \Omega \mid r_h(x,0) > 0$ for a. e. $x \in \Omega\}$, $\Omega_+(0) := \{x \in \Omega \mid r(x,0) > 0$ for a. e. $x \in \Omega\}$.

Theorem 4.4 *The class $\mathcal{M}_{\Omega\times(0,T)}(\lambda_0,\Lambda_0,M)$ of matrices satisfying* (2.22) *is compact with respect to G-convergence.*

Proof - Consider first a sequence $(a_h)_h$ not in $\mathcal{M}_{\Omega\times(0,T)}(\lambda_0,\Lambda_0,M)$, but in $\mathcal{M}_{\Omega\times(0,T)}(\lambda_0,\Lambda_0, M,N)$ and a sequence and $(r_h)_h \subset \mathcal{F}(C_1,C_2,\mu_0)$ such that $(\frac{\partial r_h}{\partial t})_h \subset \mathcal{F}(C_1,C_2,\mu_0)$, as assumed in Theorem 4.1. We want to show first that the limit matrix is independent of the sequence $(r_h)_h$. Before considering a generic sequence of problems with $(r_h)_h \subset \mathcal{F}(C_1,C_2,\mu_0)$ we focalize our attention on the particular situation $r_h \equiv 0$ for every $h \in \mathbf{N}$. Consider the following problems

$$\begin{cases} -\mathrm{div}(a_h(x,t)\cdot Du(x,t)) = f_h(x,t) & \text{on } \Omega \\ u(x) = 0 & \text{on } \partial\Omega \end{cases}$$

with $f_h \to f$ in $\mathcal{V}'$, i. e. problems (2.23) in the special case $r_h = 0$ for every $h \in \mathbf{N}$ (see Definition (2.4)) and denote by w_h the corresponding solutions. By Theorem 4.1 and Remark 4.2 we have that there exists a subsequence $(a_{h_k})_k$ and a matrix

$$\bar{a} \in \mathcal{M}_{\Omega\times(0,T)}(\lambda_0, M^2\Lambda_0, M\sqrt{\Lambda_0/\lambda_0}) \tag{4.14}$$

($\bar{a}$ is the matrix associated, in the previous theorem, to $r\equiv 0$) such that the sequence $(w_{h_k})_k$ satisfies

$$w_{h_k} \to w \quad \text{in } L^2(0,T;L^2(\Omega)) \qquad \text{and} \qquad a_{h_k}\cdot Dw_{h_k} \to \bar{a}\cdot Dw \quad \text{in } L^2(0,T,L^2(\Omega))\text{-weak}$$

where w is the solution of

$$\begin{cases} -\mathrm{div}(\bar{a}(x,t)\cdot Du(x,t)) = f(x,t) & \text{on } \Omega \\ u(x) = 0 & \text{on } \partial\Omega \end{cases}$$

Now consider a sequence $(r_h)_h \subset \mathcal{F}(C_1,C_2,\mu_0)$ with $(\frac{\partial r_h}{\partial t})_h \subset \mathcal{F}(C_1,C_2,\mu_0)$. Up to subsequence

$$r_h \to r \quad \text{and} \quad \frac{\partial r_h}{\partial t} \to \frac{\partial r}{\partial t} \qquad \text{in } L^\infty(\Omega\times(0,T))\text{-weak} *$$

for a $r \in \mathcal{F}(C_1,C_2,\mu_0)$ with $\frac{\partial r}{\partial t} \in \mathcal{F}(C_1,C_2,\mu_0)$ (see Remark 2.9). Consider this function r and the matrix $\bar{a}$ in (4.14) and define the operators in $\mathcal{L}(H^1_0(\Omega),H^{-1}(\Omega))$

$$E_h(t)u := -\mathrm{div}(a_h(\cdot,t)\cdot Du) + \frac{\partial r_h}{\partial t}(\cdot,t)u\,, \qquad E(t)u := -\mathrm{div}(\bar{a}(\cdot,t)\cdot Du) + \frac{\partial r}{\partial t}(\cdot,t)u\,.$$

Now consider $f \in \mathcal{V}'$ and $\varphi \in L^2(\Omega)$, the following problems

$$\begin{cases} \dfrac{\partial}{\partial t}(r_h u) - \mathrm{div}(a_h\cdot Du) = f & \text{on } \Omega\times(0,T) \\ u = 0 & \text{on } \partial\Omega\times(0,T) \\ u = \varphi & \text{on } \Omega_{h,+}(0)\times\{0\} \end{cases}$$

and let u_h be the corresponding solutions. For every $\epsilon > 0$ we can find $\psi \in H^1_0(\Omega)$ and $g \in H^1(0,T;H^{-1}(\Omega))$ such that

$$E(0)\psi = g(0)\,, \qquad \|\varphi-\psi\|_{L^2(\Omega)} < \epsilon/2\,, \qquad \|f-g\|_{L^2(0,T;H^{-1}(\Omega))} < \epsilon/2$$

and define $\psi_h := E_h(0)^{-1}g(0)$. We have that the sequence of the solutions $(v_h)_h$ of

$$\begin{cases} \dfrac{\partial}{\partial t}(r_h u) - \operatorname{div}(a_h \cdot Du) = g & \text{on } \Omega \times (0,T) \\ u = 0 & \text{on } \partial\Omega \times (0,T) \\ u = \psi_h & \text{on } \Omega_{h,+}(0) \times \{0\}\,. \end{cases}$$

satisfies, by Theorem 2.5,

$$\begin{aligned} &\|u_h - v_h\|_{L^2(0,T;L^2(\Omega))} \leqslant c(C_1)\epsilon \\ &\|a_h \cdot Du_h - a_h \cdot Dv_h\|_{L^2(0,T;L^2(\Omega)^n)} \leqslant c(M,\Lambda_0)\epsilon \end{aligned} \tag{4.15}$$

and moreover is relatively compact in $C([0,T];L^2(\Omega))$ by Lemma 3.3. Denote by $\mathcal{R}_h$ the operator in $\mathcal{L}(L^2(0,T;L^2(\Omega)))$ defined by $\mathcal{R}_h u = r_h u$ and $\mathcal{R}$ the operator in $\mathcal{L}(L^2(0,T;L^2(\Omega)))$ defined by $\mathcal{R}u = ru$. By Theorem 2.8 we have that, in particular, $(\mathcal{R}_h v_h)'$ is relatively compact in $C([0,T];H^{-1}(\Omega))$. Then, up to subsequences,

$$(\mathcal{R}_h v_h)' \to (\mathcal{R}v)' \qquad \text{in } C([0,T];H^{-1}(\Omega))\,.$$

Observe then that v_h solves the following problem in $\mathcal{V}$

$$\begin{cases} -\operatorname{div}(a_h(x,t) \cdot Dv(x,t)) = g(x,t) - \dfrac{\partial}{\partial t}\big(r_h(x,t)v_h(x,t)\big) =: g_h(x,t) & \text{on } \Omega \\ v(x) = 0 & \text{on } \partial\Omega \end{cases}$$

with, up to subsequences, $g_h \to g - \frac{\partial(rv)}{\partial t}$ strongly in $\mathcal{V}'$ and moreover $v_h(x,0) = \psi_h(x)$ in $\Omega_{h,+}(0)$. For what said before we have that if we consider the subsequence $(a_{h_k})_k$ (or if necessary extracting from this another subsequence because of $(r_h)_h$) we have that

$$\begin{aligned} &v_{h_k} \to v \quad \text{in } C([0,T];L^2(\Omega)) \qquad \text{and} \\ &a_{h_k} \cdot Dv_{h_k} \to \bar{a} \cdot Dv \quad \text{in } L^2(0,T,L^2(\Omega))\text{-weak} \end{aligned} \tag{4.16}$$

where v is the solution of

$$\begin{cases} -\operatorname{div}(\bar{a}(x,t) \cdot Dv(x)) = f(x,t) - \dfrac{\partial}{\partial t}(r(x,t)v(x,t)) & \text{on } \Omega \\ v(x) = 0 & \text{on } \partial\Omega \end{cases}$$

and moreover, by Lemma 3.3,

$$\big(r_{h_k}(x,0)\big)^{1/2} v_{h_k}(x,0) \to \big(r(x,0)\big)^{1/2} v(x,0) \qquad \text{in } L^2(\Omega)\,.$$

Since $(r_{h_k}(x,0))^{1/2} v_{h_k}(x,0) = (r_{h_k}(x,0))^{1/2}\psi_{h_k}(x)$ and $\psi_{h_k} \to \psi$ in $L^2(\Omega)$ (by Lemma 3.1 and Remark 3.2) we conclude that v solves

$$\begin{cases} \dfrac{\partial}{\partial t}(rv) - \operatorname{div}(\bar{a} \cdot Dv) = g & \text{on } \Omega \times (0,T) \\ v = 0 & \text{on } \partial\Omega \times (0,T) \\ v = \psi & \text{on } \Omega_+(0) \times \{0\} \end{cases}$$

where $\bar{a}$ is the matrix considered in (4.14) and $\Omega_+(0)$ the set in which $r(\cdot, 0)$ is positive. Now consider the solution u to the problem

$$\begin{cases} \dfrac{\partial}{\partial t}(ru) - \operatorname{div}(\bar{a} \cdot Du) = f & \text{on } \Omega \times (0,T) \\ u = 0 & \text{on } \partial\Omega \times (0,T) \\ u = \varphi & \text{on } \Omega_+(0) \times \{0\}\,. \end{cases}$$

Then taking the limit in

$$\|u_{h_k} - u\|_{\mathcal{V}} \leqslant \|u_{h_k} - v_{h_k}\|_{\mathcal{V}} + \|v_{h_k} - v\|_{\mathcal{V}} + \|v - u\|_{\mathcal{V}}$$

we obtain thanks to (4.15) and (4.16) that $\lim_k \|u_{h_k} - u\|_{\mathcal{V}} \leqslant c\,\epsilon$. In the same way we obtain that

$$\Big| \lim_k \int_0^T \int_\Omega \big[\big(a_{h_k} \cdot Du_{h_k}, \Phi \big) - \big(\bar{a} \cdot Du, \Phi \big) \big] dxdt \Big| \leqslant c\,\epsilon$$

for every $\Phi \in C^1_c(\Omega)^n$. This concludes the proof that the matrix $\bar{a}$ does not depend on the sequence $(r_h)_h$ and on the limit r.

Now we show that the hypothesis $(\frac{\partial r_h}{\partial t})_h \subset \mathcal{F}(C_1, C_2, \mu_0)$ can be dropped. Then suppose only $(r_h)_h \subset \mathcal{F}(C_1, C_2, \mu_0)$, let r be the limit of r_h as before and denote by u_h the solutions to problems (2.25). By Lemma 2.2 for every $h \in \mathbf{N}$ we can consider a sequence of functions $v_h \in \mathcal{V}$ and a sequence of functions s_h, $(s_h)_h \subset \mathcal{F}(C_1, C_2, \mu_0)$ and $(\frac{\partial s_h}{\partial t})_h \subset \mathcal{F}(C_1, C_2, \mu_0)$, such that (we denote by $\mathcal{S}_h$ the operator defined by $\mathcal{S}_h w = s_h w$ and recall the the definition of $\mathcal{W}_h$ is given in (2.24))

$$\|u_h - v_h\|_{\mathcal{W}_h} < \frac{1}{h}\,, \qquad \|(\mathcal{R}_h u_h)' - (\mathcal{S}_h v_h)'\|_{\mathcal{V}'} < \frac{1}{h}\,.$$

By the first of these estimates and by (2.8) we obtain that

$$\int_\Omega |u_h(x,0) - v_h(x,0)|^2 r_h(x,0) dx < c\,\frac{1}{h^2}$$

and since $u_h(x,0) = \varphi(x)$ in $\Omega_{h,+}$ we have

$$\int_\Omega |\varphi(x) - v_h(x,0)|^2 r_h(x,0) dx < c\,\frac{1}{h^2}$$

Observe that the function v_h solve the following problem

$$\begin{cases} (\mathcal{S}_h w)' + \mathcal{A}_h w = f_h := f + (\mathcal{S}_h v_h)' - (\mathcal{R}_h u_h)' + \mathcal{A}_h v_h - \mathcal{A}_h u_h \\ P_{+,h}(0) w(0) = \varphi_h := \varphi + \big(v_h(0) - u_h(0) \big) \end{cases}$$

By Lemma 2.2 we have that $f_h \to f$ in $\mathcal{V}'$ and finally, by Remark 4.2 since $\mathcal{S}_h$ satisfy hypotheses of Theorem 4.1, we obtain, taking the limit, that there exist a function $v \in \mathcal{V}$

and a function $s \in \mathcal{F}(C_1, C_2, \mu_0)$ such that, up to subsequences,

$$
\begin{aligned}
& v_h \to v \quad \text{ in } L^2(0,T;L^2(\Omega)) \\
& a_h \cdot Dv_h \to \bar{a} \cdot Dv \quad \text{ in } L^2(0,T;L^2(\Omega)^n) - \text{weak} \\
& v \text{ solves the problem}
\end{aligned}
$$

$$
\begin{cases}
\dfrac{\partial}{\partial t}(sv) - \operatorname{div}(\bar{a} \cdot Dv) = f & \text{on } \Omega \times (0,T) \\
v = 0 & \text{on } \partial\Omega \times (0,T) \\
v = \varphi & \text{on } \Omega_+^s(0) \times \{0\},
\end{cases}
$$

where $\Omega_+^s(0)$ here denotes the set $\{x \in \Omega \mid s(x,0) > 0 \text{ for a. e. } x \in \Omega\}$. By estimations above we deduce that $v = u$ and that $s = r$ and that

$$
\begin{aligned}
& u_h \to u \quad \text{ in } L^2(0,T;L^2(\Omega)) \\
& a_h \cdot Du_h \to \bar{a} \cdot Du \quad \text{ in } L^2(0,T;L^2(\Omega)^n) - \text{weak}.
\end{aligned}
$$

Now the last part: we want to weaken assumptions an $(a_h)_h$, i. e. to consider simply a sequence in $\mathcal{M}_{\Omega\times(0,T)}(\lambda_0, \Lambda_0, M)$ and not in $\mathcal{M}_{\Omega\times(0,T)}(\lambda_0, \Lambda_0, M, N)$. This can be made using the result of higher sommability due to Morrey and follow the computations made by Colombini and Spagnolo in [3]. So the result is completely proved. $\square$

5 Examples

In thi section we present some particular cases of G-convergence: in subsection 1 the classical variational convergences, in 2 some possible choices in the class $\mathcal{F}$, in 3 the result in homogenization, in 4 the singular perturbations, in which the convergence of the solutions is stronger (Lemma 5.1).

1 - Variational convergences

1) If $r_h \equiv 0$ for every h we have a result for a class of elliptic operators and we have that $a_h \xrightarrow{G} a$ in $\Omega \times (0,T)$ if and only if $a_h(\cdot,t) \xrightarrow{EG} a(\cdot,t)$ in Ω for every $t \in [0,T]$.

$$
a_h \xrightarrow{G} a \quad \text{in } \Omega \times (0,T) \qquad \Longleftrightarrow \qquad a_h(\cdot,t) \xrightarrow{EG} a(\cdot,t) \quad \text{in } \Omega \quad \text{for a. e. } t \in [0,T].
$$

2) If $r_h \equiv 1$ for every h we have a result for a class of parabolic operators and we have that $a_h \xrightarrow{G} a$ in $\Omega \times (0,T)$ for every $t \in [0,T]$ if and only if $a_h \xrightarrow{PG} a$ in

$$
a_h \xrightarrow{G} a \quad \text{in } \Omega \times (0,T) \qquad \Longleftrightarrow \qquad a_h \xrightarrow{PG} a \quad \text{in } \Omega \times (0,T) \quad .
$$

3) Suppose a_h to be symmetric matrices. Then, choosing $r_h \equiv 0$ and using the classical result (see for instance [4] for the definition of Γ-convergence) we obtain that

$$
a_h \xrightarrow{G} a \quad \text{in } \Omega \times (0,T) \qquad \Longleftrightarrow \qquad a_h(\cdot,t) \xrightarrow{\Gamma} a(\cdot,t) \quad \text{in } \Omega \quad \text{for a. e. } t \in [0,T].
$$

4) $r \equiv r(x)$ - Besides $r \equiv 1$ and $r \equiv 0$, one can consider more general situations. **Every non-negative** $r \in L^\infty(\Omega)$, i. e. $r = r(x)$ depending only on x, belongs to the class $\mathcal{F}(C_1, C_2, \mu_0)$ with $C_1 = \|r\|_\infty$ and $C_2 = \mu_0 = 0$. Then also

$$r(x) = \begin{cases} 1 & \text{on a Cantor set } \Omega_+ \\ 0 & \text{outside } \Omega_+ \,, \end{cases}$$

Ω_+ with positive measure, is admitted.

5) $r = r(t)$ - Every regular function satisfying suitable bounds can be admitted. Since

$$t \mapsto \int_\Omega u(x) v(x) r(t) dx$$

is required to be absolutly continuous we will require $r \in W^{1,\infty}(0,T)$. In this way

$$\frac{d}{dt} \int_\Omega u(x) v(x) r(t) dx = r'(t) \int_\Omega u(x) v(x) dx$$

and it is sufficient to require (remember $\mu_0 \leqslant 0$)

$$0 \leqslant r \leqslant C_1 \,, \qquad \frac{\mu_0}{c} \leqslant r'(t) \leqslant C_2 \,,$$

where c is the constant appearing in (3.4), to have $r \in \mathcal{F}(C_1, C_2, \mu_0)$.
As particular case we want to observe that if $r(0) = 0$ and $r(t) > 0$ for every $t > 0$ the initial condition is non needed. For example, if $r(t) = t$ the problem

$$\begin{cases} \dfrac{\partial}{\partial t}(tv) - \operatorname{div}(a(x,t) \cdot Dv) = f & \text{in } \Omega \times (0,T) \\ v = 0 & \text{in } \partial\Omega \times (0,T) \end{cases}$$

has a unique solution (without any condition at time $t = 0$!).

6) $r = r(x,t)$ - If $r, \frac{\partial r}{\partial t} \in L^\infty(\Omega \times (0,T))$ then $r \in \mathcal{F}(C_1, C_2, \mu_0)$ with $C_1 = \|r\|_\infty$ and $C_2 = \|\frac{\partial r}{\partial t}\|_\infty$. Precisely, if c is the constant appearing in (3.4), r can be choosen such that

$$0 \leqslant r \leqslant C_1 \,, \qquad \frac{\mu_0}{c} \leqslant \frac{\partial r}{\partial t} \leqslant C_2$$

($C_2 \geqslant 0$, $c > 0$, $\mu_0 \leqslant 0$).
Also functions r for which $\frac{\partial r}{\partial t} \notin L^\infty(\Omega \times (0,T))$ can be considered. For example if

$$r(x,t) = \chi_A(x,t) \,, \qquad A \subset \Omega \times (0,T)$$

In this case, denote $\Omega_+(t) = \{x \in \Omega \mid r(\cdot,t) > 0\}$ and $\Omega_0(t) = \{x \in \Omega \mid r(\cdot,t) = 0\}$, we need

$$t \mapsto \int_{\Omega_+(t)} u(x) v(x) dx \qquad \text{differentiable}. \tag{5.1}$$

We refer to [7] for more details and to [6] (Proposition 3, section 3.4.4) for differentiability of (5.1).

3 - Homogenization

7) If $r(x)$ is a Q-periodic function in the variable x and a_{ij}, the entries of a matrix $a \in \mathcal{M}_{\Omega\times(0,T)}(\lambda_0, \Lambda_0, M)$, are Q-periodic in the variable x, Q cube of $\mathbf{R}^n$, then the solution (see [8]) of

$$\begin{cases} r(hx)\dfrac{\partial u}{\partial t} - \operatorname{div}(a(hx,t)\cdot Du) = f & \text{in } \Omega\times(0,T) \\ u = 0 & \text{in } \partial\Omega\times(0,T) \\ u = \varphi & \text{in } \Omega_{h,+}\times\{0\} \end{cases}$$

converge in $L^2(0,T;L^2(\Omega))$ to the solution of the problem

$$\begin{cases} \left[\dfrac{1}{|Q|}\displaystyle\int_Q r\right]\dfrac{\partial u}{\partial t} - \displaystyle\sum_{i,j=1}^{n} \widehat{a}_{ij}(t)\dfrac{\partial^2 v}{\partial x_i \partial x_j} = f & \text{in } \Omega\times(0,T) \\ u = 0 & \text{in } \partial\Omega\times(0,T) \\ u = \varphi & \text{in } \Omega\times\{0\} \end{cases}$$

(for the definition of $\widehat{a}$ see for example [8]). Notice that if the mean value of r $|Q|^{-1}\int_Q r > 0$ the limit problem is given by a standard parabolic equation and in this case the initial condition is obtained in all Ω even if for every h the problems above are partially elliptic and partially parabolic. This happens also if r is positive only on a Cantor set of positive measure.
If $|Q|^{-1}\int_Q r = 0$ clearly $r = 0$ and then every problem is elliptic and the initial conditions has no meaning.

4 - Singular perturbations

8) Suppose to have a fixed elliptic operator $u \mapsto -\operatorname{div}(a(x,t)\cdot Du)$, with $a \in \mathcal{M}_{\Omega\times(0,T)}(\lambda_0, \Lambda_0, M)$. Then $a \xrightarrow{G} a$ in particular means that the solutions of

$$\begin{cases} \dfrac{1}{h}\dfrac{\partial v}{\partial t} - \operatorname{div}(a\cdot Dv) = f & \text{in } \Omega\times(0,T) \\ u = 0 & \text{in } \partial\Omega\times(0,T) \\ u = \varphi & \text{in } \Omega\times\{0\} \end{cases}$$

converge in $L^2(0,T;L^2(\Omega))$ to the solution of the problem

$$\begin{cases} -\operatorname{div}(a\cdot Dv) = f & \text{in } \Omega\times(0,T) \\ u = 0 & \text{in } \partial\Omega\times(0,T) \end{cases} .$$

Indeed $r_h = 1/h$ and $r \equiv 0$ belong to the class $\mathcal{F}(C_1, C_2, \mu_0)$.
But in fact we have more. The following convergence result holds.

Proposition 5.1 *Consider* $\mathcal{A}, \mathcal{A}_h : L^2(0,T;H_0^1(\Omega)) \to L^2(0,T;H^{-1}(\Omega))$ *the operators* $\mathcal{A}u = -\operatorname{div}(a(x,t)\cdot Du)$, $\mathcal{A}_h u = -\operatorname{div}(a_h(x,t)\cdot Du)$ *with* $a, a_1, a_2, \ldots \in \mathcal{M}_{\Omega\times(0,T)}(\lambda_0, \Lambda_0, M)$. *Suppose* $\mathcal{A}_h v \to \mathcal{A}v$ *in* $L^2(0,T;H^{-1}(\Omega))$ *for every* $v \in L^2(0,T;H_0^1(\Omega))$. *Then*

$$\begin{array}{ll} u_h \to u & \text{in } L^2(0,T,H_0^1(\Omega)) \\ a_h\cdot Du_h \to a\cdot Du & \text{in } L^2(0,T,L^2(\Omega)^n)\,, \end{array}$$

where u_h *and* u *are respectively the solutions to* (2.23) *and* (4.2), *and* $a_h \xrightarrow{G} a$.

Proof - Since $\mathcal{A}_h - \mathcal{A} = \mathcal{A}_h(\mathcal{A}^{-1} - \mathcal{A}_h^{-1})\mathcal{A}$, fix $f \in L^2(0,T;H^{-1}(\Omega))$ and choose $v = \mathcal{A}^{-1} f$. Then

$$\mathcal{A}_h^{-1} f - \mathcal{A}^{-1} f = \mathcal{A}_h^{-1}(\mathcal{A}_h v - \mathcal{A} v)\,.$$

By our assumptions and (2.13) we conclude that $\|\mathcal{A}_h^{-1} f - \mathcal{A}^{-1} f\|_{L^2(0,T,H_0^1(\Omega))} \to 0$. In particular $\|\mathcal{A}_h^{-1} f - \mathcal{A}^{-1} f\|_{L^2(0,T,L^2(\Omega))} \to 0$ and, for every $\Phi \in L^2(0,T;L^2(\Omega)^n)$ by (2.16)

$$\sup_{\|\Phi\|=1} \left| \int_0^T \int_\Omega (a_h \cdot D\mathcal{A}_h^{-1} f - a \cdot D\mathcal{A}^{-1} f, \Phi) dx dt \right| \leqslant$$
$$\leqslant c(M, \Lambda_0) \langle \mathcal{A}_h u - \mathcal{A} u, u \rangle_{L^2(0,T;H^{-1}(\Omega)) \times L^2(0,T;H_0^1(\Omega))} \to 0\,.$$

By Theorem 4.4 we conlcude that $a_h \xrightarrow{G} a$.
Observe that assumptions are garanteed if $a_h \to a$ in $L^1_{loc}(\Omega \times (0,T))$, but the converse is not true (see [13]). $\square$

We conclude that under assumptions of Proposition 5.1, or if $(a_h)_h \subset \mathcal{M}_{\Omega\times(0,T)}(\lambda_0, \Lambda_0, M)$ and $a_h \to a$ in $L^1_{loc}(\Omega \times (0,T))^n$, for every $(r_h)_h \subset \mathcal{F}(C_1, C_2, \mu_0)$ converging to $r \in \mathcal{F}(C_1, C_2, \mu_0)$ in $L^\infty(\Omega \times (0,T))$-weak $*$, we have that

$$\begin{array}{ll} u_h \to u & \text{in } L^2(0,T,H_0^1(\Omega)) \\ a_h \cdot Du_h \to a \cdot Du & \text{in } L^2(0,T,L^2(\Omega)^n)\,, \end{array} \tag{5.2}$$

where u_h and u denote respectively the solutions of

$$\begin{cases} \partial_t(r_h v) - \operatorname{div}(a_h \cdot Dv) = f & \Omega \times (0,T) \\ u = 0 & \partial\Omega \times (0,T) \\ u = \varphi & \Omega_{h,+}(0) \times \{0\} \end{cases} \qquad \begin{cases} \partial_t(r v) - \operatorname{div}(a \cdot Dv) = f & \Omega \times (0,T) \\ u = 0 & \partial\Omega \times (0,T) \\ u = \varphi & \Omega_{+}(0) \times \{0\}. \end{cases}$$

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