A GEOMETRICAL APPROACH TO THE SHARP HARDY INEQUALITY IN SOBOLEV–SLOBODECKIĬ SPACES

FRANCESCA BIANCHI, GIORGIO STEFANI, AND ANNA CHIARA ZAGATI

Abstract. We give a partial negative answer to a question left open in a previous work by Brasco and the first and third-named authors concerning the sharp constant in the fractional Hardy inequality on convex sets. Our approach has a geometrical flavor and equivalently reformulates the sharp constant in the limit case $p = 1$ as the Cheeger constant for the fractional perimeter and the Lebesgue measure with a suitable weight. As a by-product, we obtain new lower bounds on the sharp constant in the 1-dimensional case, even for non-convex sets, some of which optimal in the case $p = 1$.

1. Introduction

1.1. **Classical framework.** Given $N \in \mathbb{N}$ and $p \in (1, \infty)$, the *Hardy inequality* on a (non-empty) open convex set $\Omega \subsetneq \mathbb{R}^N$ states that there exists $C > 0$ such that

$$\int_{\mathbb{R}^N} |\nabla u|^p \, \mathrm{d}x \geq C \int_\Omega \frac{|u|^p}{d_\Omega^p} \, \mathrm{d}x, \quad \text{for all } u \in C_0^\infty(\Omega), \tag{1.1}$$

see [32, 35] for a detailed introduction. Here and in the following, given a (non-empty) open set $\Omega \subseteq \mathbb{R}^N$, we let $C_0^\infty(\Omega)$ be the set of all smooth functions on $\mathbb{R}^N$ with compact support contained in Ω. In addition, provided that $\Omega \neq \mathbb{R}^N$, we let $d_\Omega \colon \Omega \to [0, \infty)$,

$$d_\Omega(x) = \inf_{y \in \partial\Omega} |x - y|, \quad \text{for all } x \in \Omega, \tag{1.2}$$

be the *distance* function from $\partial\Omega$.

Date: September 22, 2025.

2020 *Mathematics Subject Classification.* Primary 49Q20. Secondary 35R11, 46E35.

Key words and phrases. Hardy inequality, fractional Sobolev spaces, fractional perimeter.

Acknowledgments. The authors thank Lorenzo Brasco for several precious comments on a preliminary version of the present work. The authors are members of the Istituto Nazionale di Alta Matematica (INdAM), Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA). F. Bianchi has received funding from INdAM under the INdAM–GNAMPA Project 2024 *Nuove frontiere nella capillarità non locale* (grant agreement No. CUP_E53C23001670001). G. Stefani has received funding from INdAM under the INdAM–GNAMPA 2023 Project *Problemi variazionali per funzionali e operatori non-locali* (grant agreement No. CUP_E53C22001930001), the INdAM–GNAMPA 2024 Project *Ottimizzazione e disuguaglianze funzionali per problemi geometrico-spettrali locali e non-locali* (grant agreement No. CUP_E53C23001670001) and the INdAM–GNAMPA Project 2025 *Metodi variazionali per problemi dipendenti da operatori frazionari isotropi e anisotropi* (grant agreement No. CUP_E53240019-50001), from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation program (grant agreement No. 945655), and from the European Union – NextGenerationEU and the University of Padua under the 2023 STARS@UNIPD Starting Grant Project *New Directions in Fractional Calculus – NewFrac* (grant agreement No. CUP_C95F21009990001). A. C. Zagati has received funding from INdAM under the INdAM–GNAMPA Project 2024 *Proprietà geometriche e regolarità in problemi variazionali locali e non locali* (grant agreement No. CUP_E53C23001670001).

The sharp constant in (1.1), defined as

$$\mathfrak{h}_{1,p}(\Omega) = \inf_{u\in C_0^\infty(\Omega)} \left\{ \int_{\mathbb{R}^N} |\nabla u|^p \,\mathrm{d}x : \int_\Omega \frac{|u|^p}{d_\Omega^p}\,\mathrm{d}x = 1 \right\} \in [0,\infty), \tag{1.3}$$

can be explicitly computed as

$$\mathfrak{h}_{1,p}(\Omega) = \mathfrak{h}_{1,p}(\mathbb{H}^N_+) = \left(\frac{p-1}{p}\right)^p, \tag{1.4}$$

see the original Hardy's works [21, 22] for $N = 1$ and [30, 31] for $N \geq 2$. As well-known, inequality (1.1) cannot hold for $p = 1$; that is,

$$\mathfrak{h}_{1,1}(\Omega) = \mathfrak{h}_{1,1}(\mathbb{H}^N_+) = 0 \tag{1.5}$$

in (1.3). Here and in the following, we set

$$\mathbb{H}^1_+ = (0,\infty) \quad \text{and} \quad \mathbb{H}^N_+ = \mathbb{R}^{N-1}\times(0,\infty) \quad \text{for } N \geq 2.$$

Noteworthy, the sharp constant (1.4) is never attained in (1.1), not even in the corresponding homogeneous Sobolev space $\mathcal{W}^{1,p}_0(\Omega)$ obtained as the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the left-hand side of (1.1).

1.2. **Fractional framework.** The fractional analog of (1.1) was achieved in [5, Th. 1.1]. Precisely, given $s \in (0,1)$ and $p \in (1,\infty)$, there exists $C > 0$ such that

$$[u]^p_{W^{s,p}(\mathbb{R}^N)} \geq C \int_\Omega \frac{|u|^p}{d_\Omega^{sp}}\,\mathrm{d}x, \quad \text{for all } u \in C_0^\infty(\Omega). \tag{1.6}$$

Here and in the rest of the paper, given $s \in (0,1)$, $p \in [1,\infty)$, an open set $A \subset \mathbb{R}^N$, and a measurable function $u\colon A \to [-\infty,\infty]$, we let

$$[u]_{W^{s,p}(A)} = \left(\int_A \int_A \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,\mathrm{d}x\,\mathrm{d}y \right)^{\frac{1}{p}} \tag{1.7}$$

be the *Sobolev–Slobodeckiĭ (s,p)-fractional seminorm* of u on A. The sharp constant in (1.6), defined analogously as in (1.3) as

$$\mathfrak{h}_{s,p}(\Omega) = \inf_{u\in C_0^\infty(\Omega)} \left\{ [u]^p_{W^{s,p}(\mathbb{R}^N)} : \int_\Omega \frac{|u|^p}{d_\Omega^{sp}}\,\mathrm{d}x = 1 \right\} \in [0,\infty), \tag{1.8}$$

was investigated by L. Brasco and the first and third-named authors in the recent work [3]. In [3, Main Th.], it was proved that, for all $p \in (1,\infty)$ and $s \in (0,1)$,

$$\mathfrak{h}_{s,p}(\mathbb{H}^N_+) = C_{N,sp}\,\Lambda_{s,p} \tag{1.9}$$

and, in analogy with the classical result (1.4), if either $p = 2$ or $sp \geq 1$, then

$$\mathfrak{h}_{s,p}(\Omega) = \mathfrak{h}_{s,p}(\mathbb{H}^N_+). \tag{1.10}$$

Moreover, as in the classical case, the sharp constant in (1.10) is never attained in (1.6), not even in the homogeneous fractional Sobolev–Slobodeckiĭ space $\mathcal{W}^{s,p}_0(\Omega)$ obtained as

the completion of $C_0^\infty(\Omega)$ with respect to the seminorm (1.7) with $A = \mathbb{R}^N$. Here and below, we set

$$C_{N,q} = \begin{cases} (N-1)\,\omega_{N-1} \displaystyle\int_0^\infty \frac{t^{N-2}}{(1+t^2)^{\frac{N+q}{2}}}\,\mathrm{d}t & \text{for } N \geq 2, \\ 1 & \text{for } N = 1, \end{cases} \tag{1.11}$$

whenever $q \in [0,\infty)$, and

$$\Lambda_{s,p} = 2\int_0^1 \frac{\left|1 - t^{\frac{sp-1}{p}}\right|^p}{(1-t)^{1+sp}}\,\mathrm{d}t + \frac{2}{sp} \tag{1.12}$$

whenever $s \in (0,1)$ and $p \in [1,\infty)$.

As mentioned in [3], concerning (1.10), the cases $p = 2$ for $\Omega = \mathbb{H}^N_+$, and $p = 2$ with $s \geq \frac{1}{2}$ for any open convex set $\Omega \subsetneq \mathbb{R}^N$, were previously established respectively in [4, Th. 1.1] and in [15, Th. 5] (see also [16]), with different techniques.

The strategy of [3] for proving (1.9) and (1.10) expands on the ideas of [4] and relies on an equivalent characterization of the constant (1.8) via the existence of positive local weak *supersolutions* of the corresponding non-local eigenvalue problem, see [3, Eq. (1.6)].

Unfortunately, this approach does not seem to work for determining the sharp constant $\mathfrak{h}_{s,p}(\Omega)$ for $\Omega \neq \mathbb{H}^N_+$ for the values of s and p not included in (1.10), see the discussion in [3, Sec. 1.3]. Nevertheless, in virtue of [3, Rem. 6.4], it holds that

$$\frac{2\,C_{N,sp}}{sp} \leq \mathfrak{h}_{s,p}(\Omega) \leq \mathfrak{h}_{s,p}(\mathbb{H}^N_+) \tag{1.13}$$

for all $p \in (1,\infty)$ and $s \in (0,1)$.

1.3. **The convex case.** Our main aim is to tackle the question left open in [3, Open Prob.] concerning the validity of (1.10) for $p \neq 2$ and $s < \frac{1}{p}$.

We first observe that a plain limiting argument yields (1.9) and (1.13), and thus (1.6), for $p = 1$—remarkably in contrast with (1.1), recall (1.5). We thus complete the picture of [3, 5] with the case $p = 1$, with the characterization of (1.8) for the half-space. Here and below, given $N \geq 1$, $s \in (0,1)$ and a non-empty open set $\Omega \subsetneq \mathbb{R}^N$, we let

$$\mathfrak{h}_{s,1}(\Omega) = \inf_{u \in C_0^\infty(\Omega)} \left\{ [u]_{W^{s,1}(\mathbb{R}^N)} : \int_\Omega \frac{|u|}{d_\Omega^s}\,\mathrm{d}x = 1 \right\} \in [0,\infty). \tag{1.14}$$

Arguing as in [3, Rem. 3.1], $C_0^\infty(\Omega)$ can be replaced with $\mathcal{W}^{s,1}_0(\Omega)$ in the infimum in (1.14).

Theorem 1.1. *For $N \geq 1$ and $s \in (0,1)$, it holds that*

$$\mathfrak{h}_{s,1}(\mathbb{H}^N_+) = \Lambda_{s,1}\,C_{N,s} \quad \textit{with} \quad \Lambda_{s,1} = \frac{4}{s}, \tag{1.15}$$

and, whenever is $\Omega \subsetneq \mathbb{R}^N$ is a (non-empty) open convex set,

$$\frac{1}{2}\,\mathfrak{h}_{s,1}(\mathbb{H}^N_+) \leq \mathfrak{h}_{s,1}(\Omega) \leq \mathfrak{h}_{s,1}(\mathbb{H}^N_+). \tag{1.16}$$

Moreover, the constant $\mathfrak{h}_{s,1}(\mathbb{H}^1_+)$ is attained in $\mathcal{W}^{s,1}_0(\mathbb{H}^1_+)$.

Having (1.6) for $p = 1$ at disposal, our first main result yields a partial negative answer to [3, Open Prob.] for $p = 1$ and $\Omega \subsetneq \mathbb{R}^N$ a (non-empty) bounded open convex set.

Theorem 1.2. *Let $N \geq 1$ and let $\Omega \subsetneq \mathbb{R}^N$ be a (non-empty) bounded open convex set. If $N = 1$, then*

$$\mathfrak{h}_{s,1}(\Omega) = 2^{-s}\,\mathfrak{h}_{s,1}(\mathbb{H}^1_+) = \frac{2^{2-s}}{s} \quad \text{for all } s \in (0,1) \tag{1.17}$$

and the constant $\mathfrak{h}_{s,1}(\Omega)$ is attained in $\mathcal{W}^{s,1}_0(\Omega)$. If $N \geq 2$, then

$$\lim_{s\to 1^-} \frac{\mathfrak{h}_{s,1}(\Omega)}{\mathfrak{h}_{s,1}(\mathbb{H}^N_+)} = \frac{1}{2}. \tag{1.18}$$

Noteworthy, for $N \geq 2$, in the special case Ω is an open ball, the limit (1.18) can be slightly refined by exploiting the main result of [20], see Proposition 4.8 below.

Theorems 1.1 and 1.2 uncover some remarkable differences between the classical inequality (1.1) and its fractional counterpart (1.6), revealing some unexpected features of the sharp constant (1.14). In particular, the first inequality in (1.16) is asymptotically optimal as $s \to 1^-$ for (non-empty) bounded open convex sets. Moreover, for $N = 1$, the sharp constant (1.14) is attained, both in the bounded and unbounded case. We do not know if this might be the case also for $N \geq 2$, but we leave this interesting question for future investigations. For further discussions, see also Remark 4.7 below.

As a consequence of Theorem 1.2, we can give the following partial negative answer to [3, Open Prob.] for $p > 1$, suggesting that the first inequality in (1.13) is asymptotically optimal as $s \to 1^-$ and $p \to 1^+$, with $sp < 1$, for (non-empty) bounded open convex sets.

Corollary 1.3. *If $N \geq 1$ and $\Omega \subsetneq \mathbb{R}^N$ is a (non-empty) bounded open convex set, then*

$$\lim_{s\to 1^-} \limsup_{p\to 1^+} \frac{\mathfrak{h}_{s,p}(\Omega)}{\mathfrak{h}_{s,p}(\mathbb{H}^N_+)} = \frac{1}{2}.$$

1.4. **Geometrical approach.** Besides refining and extending some of the results of [3] to the limit case $p = 1$, the key idea of our approach is to reinterpret the sharp constant $\mathfrak{h}_{s,1}(\Omega)$ in (1.8) in a geometrical sense. More precisely, we prove the following result (in which we do not need to assume that Ω is convex).

Theorem 1.4. *Let $N \geq 1$, $s \in (0,1)$ and let $\Omega \subsetneq \mathbb{R}^N$ be a (non-empty) open set. Letting*

$$P_s(E) = [\mathbf{1}_E]_{W^{s,1}(\mathbb{R}^N)} \quad \text{and} \quad V_{s,\Omega}(E) = \int_E \frac{1}{d^s_\Omega}\,\mathrm{d}x$$

for any measurable set $E \subseteq \Omega$, it holds that

$$\mathfrak{h}_{s,1}(\Omega) = \mathfrak{g}_s(\Omega), \tag{1.19}$$

where

$$\mathfrak{g}_s(\Omega) = \inf\left\{ \frac{P_s(E)}{V_{s,\Omega}(E)} : E \Subset \Omega,\ |E| > 0 \right\} \in [0,\infty). \tag{1.20}$$

Theorem 1.4 revisits the well-known connection between the *Cheeger constant* of a set with the *first* 1-*eigenvalue* of the underlying 1-*Laplacian* (see [17] and the references therein for a detailed account) in terms of the fractional Hardy inequality (1.6) for $p = 1$. In fact, the problem in (1.20) is the *Cheeger problem* in Ω with respect to the *fractional perimeter* P_s and the *weighted* volume $V_{s,\Omega}$, so that Theorem 1.4 can be seen as a particular case of [17, Th. 5.4] (see also [7, Th. 5.8] and [2, Th. 3.10] for similar results in

the fractional/non-local setting). The idea behind the proof of Theorem 1.4 is essentially to exploit the *fractional coarea formula* (first observed in [37])

$$[u]_{W^{s,1}(\mathbb{R}^N)} = \int_{-\infty}^{\infty} P_s(\{u > t\})\,\mathrm{d}t \tag{1.21}$$

in order to pass from a function $u \in C_0^\infty(\Omega)$ in (1.8) to its superlevel sets $\{u > t\}$ in (1.20).

With Theorem 1.4 at disposal, Theorem 1.2 basically follows from the fact that

$$\mathfrak{h}_{s,1}(\Omega) = \mathfrak{g}_s(\Omega) \leq \frac{P_s(\Omega)}{V_{s,\Omega}(\Omega)}.$$

For $N \geq 2$, the idea is first to rewrite $V_{s,\Omega}(\Omega)$ in terms of the superlevel sets of d_Ω via the usual coarea formula and then to take advantage of the well-known limit (e.g., see [9, 27])

$$\lim_{s\to 1^-} (1-s)\, P_s(\Omega) = 2\,\omega_{N-1}\, P(\Omega), \tag{1.22}$$

valid for any measurable set $\Omega \subsetneq \mathbb{R}^N$ such that $\mathbf{1}_\Omega \in BV(\mathbb{R}^N)$. The case $N = 1$ in (1.17), instead, relies on a rearrangement argument (see Lemma 4.5 below) which, unfortunately, does not work in higher dimensions (for more details, see Remark 4.6).

1.5. **The non-convex case.** The principle underlying Theorem 1.4 can be also exploited to deal with the non-convex case $\Omega = \mathbb{R}^N \setminus \{0\}$. In fact, our geometrical approach allows us to naturally recover [19, Th. 1.1] for $p = 1$ (compare also with the argument in [19, Sec. 3.4]). In more precise terms, we can (re)prove the following result.

Corollary 1.5. *Given $N \geq 1$ and $s \in (0,1)$, it holds that*

$$[u]_{W^{s,1}(\mathbb{R}^N)} \geq \mathfrak{h}_{s,1}(\mathbb{R}^N \setminus \{0\}) \int_{\mathbb{R}^N} \frac{|u(x)|}{|x|^s}\,\mathrm{d}x \tag{1.23}$$

for all $u \in \mathcal{W}_0^{s,1}(\mathbb{R}^N)$, with sharp constant given by

$$\mathfrak{h}_{s,1}(\mathbb{R}^N \setminus \{0\}) = \frac{4}{s} \frac{\pi^{\frac{N}{2}}\,\Gamma(1-s)}{\Gamma\left(\frac{N-s}{2}\right)\,\Gamma\left(1-\frac{s}{2}\right)}. \tag{1.24}$$

Equality holds in (1.23) if and only if u is proportional to a symmetric decreasing function.

Noteworthy, for $N = 1$, the constant in (1.24) equals the one in (1.17) for all $s \in (0,1)$,

$$\mathfrak{h}_{s,1}\left(\mathbb{R} \setminus \{0\}\right) = \frac{2^{2-s}}{s}, \tag{1.25}$$

by well-known properties of the Gamma function (e.g., the *Legendre duplication formula*). The equality in (1.25) is actually a particular case of the following result (including (1.17) as a special instance), which comes as a natural by-product of our geometrical approach (as customary, $\lfloor x \rfloor \in \mathbb{Z}$ denotes the largest integer smaller than $x \in \mathbb{R}$).

Theorem 1.6. *Let $N = 1$ and $\Omega \subsetneq \mathbb{R}$ be the union of $n \in \mathbb{N}$ (non-empty) disjoint bounded open intervals $\{I_k : k = 1, \dots, n\}$. For $s \in (0,1)$ and $p \in [1,\infty)$ such that $sp < 1$, it holds*

$$\mathfrak{h}_{s,p}(\Omega) \geq \frac{\mathfrak{h}_{s,p}(\mathbb{R} \setminus \{0\})}{n^{sp}}. \tag{1.26}$$

Moreover, if $p=1$ and Ω is equivalent to an interval, then (1.26) holds as an equality if and only if all the I_k's have the same measure. On the other hand, it holds that

$$\mathfrak{h}_{s,p}((-R,R)\setminus\mathbb{Z}) \le \frac{R^{1-sp}}{\lfloor R\rfloor}\,\frac{4^{1-sp}}{sp} \tag{1.27}$$

for any $R\ge 1$, with $\mathfrak{h}_{s,p}(\mathbb{R}\setminus\mathbb{Z})=0$ in the limit case $R=\infty$.

We emphasize that the intervals in the first part of the statement of Theorem 1.6 do not have to be at a positive distance apart. If, instead, one wants to keep track of the (minimal) distance between the intervals, then an alternative route is possible. This is the content of our last main result (see also Theorem 4.10 below).

Theorem 1.7. *Let $\Omega\subsetneq\mathbb{R}$ be the union of $n\in\mathbb{N}$ non-empty bounded open intervals, each with measure at most $\ell\in(0,\infty)$ and at distance at least $\delta\in(0,\infty)$ from the others. For $s\in(0,1)$ and $p\in[1,\infty)$ such that $sp\le 1$, it holds that*

$$\mathfrak{h}_{s,p}(\Omega) \ge \frac{2^{2-sp}}{sp}\left(1-\left(\frac{\ell}{\ell+\delta}\right)^{sp}\right).$$

It is worth observing that, in the case $sp<1$, due to Corollary 3.2 below, the above inequality implies that

$$\mathfrak{h}_{s,p}(\Omega) \ge \mathfrak{h}_{s,p}(\mathbb{R}\setminus\{0\})\left(1-\left(\frac{\ell}{\ell+\delta}\right)^{sp}\right),$$

which can be more easily paired with (1.26).

We point out that inequality (1.6) was addressed in the non-convex case in [8,36]. The validity of (1.6) for any open set $\Omega\subsetneq\mathbb{R}^N$ was achieved in [36, Th. 1.10] with $s\in(0,1)$ and $p\in(1,\infty)$ such that $sp>N>1$, but with an implicit constant. Recently, this result has been improved in [8, Th. 1.1], where it was shown that

$$\mathfrak{h}_{s,p}(\Omega) \ge \mathfrak{h}_{s,p}(\mathbb{R}^N\setminus\{0\}) \tag{1.28}$$

for any open set $\Omega\subsetneq\mathbb{R}^N$ with $s\in(0,1)$ and $p\in(1,\infty)$ such that $sp>N\ge 1$. To the best of our knowledge, Theorems 1.6 and 1.7 are the first results establishing inequality (1.6) on a (non-empty) open non-convex set $\Omega\subsetneq\mathbb{R}$ for $s\in(0,1)$ and $p\in[1,\infty)$ such that $sp\le N=1$. Furthermore, Theorem 1.6 shows that inequality (1.28) cannot hold in the case $p=1$ and $N=1$ for any $s\in(0,1)$.

Remark 1.8 (Regional fractional Hardy inequality)**.** For completeness, we mention that a vast literature has been devoted to the study of the *regional* version of inequality (1.6),

$$[u]^p_{W^{s,p}(\Omega)} \ge C\int_\Omega \frac{|u|^p}{d_\Omega^{sp}}\,\mathrm{d}x, \quad \text{for all } u\in C_0^\infty(\Omega). \tag{1.29}$$

Far from being complete, we refer to [10–14,36] and to the references therein for sufficient and/or necessary conditions on s, p and Ω for (1.29) to hold. The problem of determining the sharp constant in (1.29) has been addressed in [4,18,28]. In contrast to (1.6), in this case the restriction $sp>1$ is needed, since (1.29) cannot hold for $sp\le 1$ on bounded open Lipschitz sets $\Omega\subsetneq\mathbb{R}^N$, see [10, Sec. 2]. For further discussions, refer to [3, Rem. 1.1].

2. Preliminaries

In this section, we collect some useful notation and preliminary results.

2.1. **Gamma and Beta functions.** For $a, b > 0$, we let

$$\Gamma(a) = \int_0^\infty t^{a-1}\, e^{-t}\, \mathrm{d}t \quad \text{and} \quad \mathrm{B}(a,b) = \int_0^1 t^{a-1}\, (1-t)^{b-1}\, \mathrm{d}t \tag{2.1}$$

be the *Gamma* and *Beta functions*, respectively. We recall that

$$\mathrm{B}(a,b) = \frac{\Gamma(a)\,\Gamma(b)}{\Gamma(a+b)} \quad \text{for all } a, b > 0. \tag{2.2}$$

In addition, for $x \in [0,1)$ and $a > 0$ and $b \in \mathbb{R}$, we let

$$\mathrm{B}(x; a, b) = \int_0^x t^{a-1}\, (1-t)^{b-1}\, \mathrm{d}t \tag{2.3}$$

be the *incomplete Beta function*. We recall that, given $a > 0$,

$$\mathrm{B}(x; a, -a) = \frac{x^a}{a\,(1-x)^a} \quad \text{for all } x \in [0,1). \tag{2.4}$$

Formula (2.4) plainly follows from the representation

$$\mathrm{B}(x; a, b) = \frac{x^a}{a}\, F(a, 1-b; a+1; x), \quad \text{for all } x \in [0,1),$$

and the well-known properties of the *hypergeometric function* F.

For each $k \in \mathbb{N}$, we let

$$\omega_k = \frac{\pi^{\frac{k-1}{2}}}{\Gamma\left(\frac{k-1}{2}+1\right)} \tag{2.5}$$

be the k-dimensional Lebesgue measure of the unitary ball

$$\mathbb{B}^k = \{x \in \mathbb{R}^k : |x| < 1\}$$

in $\mathbb{R}^k$. Recall that $\mathscr{H}^{k-1}(\partial\mathbb{B}^k) = k\,\omega_k$, where $\mathscr{H}^k$ is the k-dimensional Hausdorff measure.

2.2. **Symmetric decreasing rearrangement.** Given a measurable set $E \subset \mathbb{R}^N$ such that $|E| \in [0,\infty)$, we let

$$E^\star = \left\{x \in \mathbb{R}^N : |x| < r_E\right\} \tag{2.6}$$

where $r_E \in [0,\infty)$ is such that $|E^\star| = |E|$, and we set $\mathbf{1}_E^\star = \mathbf{1}_{E^\star}$. The *symmetric decreasing rearrangement* of a measurable function $u\colon \mathbb{R}^N \to \mathbb{R}$ *vanishing at infinity*, i.e., such that

$$\left|\left\{x \in \mathbb{R}^N : |u| > t\right\}\right| < \infty \quad \text{for all } t > 0,$$

is defined as

$$u^\star(x) = \int_0^\infty \mathbf{1}^\star_{\{|u|>t\}}(x)\, \mathrm{d}t \quad \text{for all } x \in \mathbb{R}^N. \tag{2.7}$$

In particular, $u^\star$ is a lower semicountinuous, non-negative, radially symmetric, and non-increasing function such that $\{u^\star > t\} = \{|u| > t\}^\star$. Moreover, if $\Phi\colon [0,\infty) \to [0,\infty)$ is a non-decreasing function, then

$$(\Phi \circ |u|)^\star = \Phi \circ u^\star \tag{2.8}$$

for any measurable function $u\colon \mathbb{R}^N \to \mathbb{R}$. In particular, $(|u|^p)^\star = (u^\star)^p$ for all $p \in [1,\infty)$.

The *Hardy–Littlewood inequality* states that, if $u, v\colon \mathbb{R}^N \to \mathbb{R}$ are two measurable functions vanishing at infinity, then

$$\int_{\mathbb{R}^N} u(x)\, v(x)\, \mathrm{d}x \le \int_{\mathbb{R}^N} u^\star(x)\, v^\star(x)\, \mathrm{d}x. \tag{2.9}$$

For a detailed presentation of (2.7) and (2.9), we refer to [26, Ch. 3] for instance.

2.3. **Fractional setting.** For $s \in (0,1)$ and $p \in [1,\infty)$, we let $\mathcal{W}_0^{s,p}(\Omega)$ be the completion of $C_0^\infty(\Omega)$ with respect to the seminorm in (1.7) with $A = \mathbb{R}^N$. Thanks to [6, Th. 3.1], in the case $sp < N$ (which is the only one we actually need), we can identify

$$\mathcal{W}_0^{s,p}(\mathbb{R}^N) = \left\{ u \in L^{\frac{Np}{N-sp}}(\mathbb{R}^N) : [u]_{W^{s,p}(\mathbb{R}^N)} < \infty \right\}.$$

By [1, Th. 9.2], if $u \in \mathcal{W}_0^{s,p}(\mathbb{R}^N)$, then the symmetric decreasing rearrangement $u^\star$ of u defined in (2.7) is well defined and such that $u^\star \in \mathcal{W}_0^{s,p}(\mathbb{R}^N)$, with

$$[u^\star]_{W^{s,p}(\mathbb{R}^N)} \le [u]_{W^{s,p}(\mathbb{R}^N)}. \tag{2.10}$$

As customary, given $s \in (0,1)$ and a measurable set $E \subseteq \mathbb{R}^N$, we let

$$P_s(E) = [\mathbf{1}_E]_{W^{s,1}(\mathbb{R}^N)}$$

be the *fractional perimeter* of E. We recall that, given two measurable sets $E, F \subseteq \mathbb{R}^N$, the fractional perimeter satisfies

$$P_s(E \cap F) + P_s(E \cup F) \le P_s(E) + P_s(F), \tag{2.11}$$

see [2, Sec. 2.5] and the references therein. Thus, since $P_s(F^c) = P_s(F)$, we also have that

$$P_s(E \setminus F) = P_s(E \cap F^c) \le P_s(E) + P_s(F^c) = P_s(E) + P_s(F). \tag{2.12}$$

Moreover, we recall that, if $|E| < \infty$ and $F \subseteq \mathbb{R}^N$ is convex, then

$$P_s(E \cap F) \le P_s(E), \tag{2.13}$$

see [2, Th. 2.29] for instance.

The *fractional isoperimetric inequality* (e.g., see [20] and the references therein) states that, for all $s \in (0,1)$, if $E \subseteq \mathbb{R}^N$ is such that $|E| \in (0,\infty)$, then

$$\frac{P_s(E)}{|E|^{1-\frac{s}{N}}} \ge \frac{P_s(\mathbb{B}^N)}{|\mathbb{B}^N|^{1-\frac{s}{N}}}. \tag{2.14}$$

We recall that, thanks to [20, Prop. 1.1], we have that

$$P_s(\mathbb{B}^N) = \omega_N^{1-\frac{s}{N}}\, \frac{N\, \pi^{\frac{N+s}{2}}\, \Gamma(1-s)}{\frac{s}{2}\, \Gamma\left(\frac{N}{2}+1\right)^{\frac{s}{N}}\, \Gamma\left(1-\frac{s}{2}\right)\, \Gamma\left(\frac{N-s}{2}+1\right)} \tag{2.15}$$

for all $s \in (0,1)$. In particular, for $N = 1$, we get that

$$P_s((a,b)) = \frac{4}{s(1-s)}\, (b-a)^{1-s} \tag{2.16}$$

whenever $a, b \in \mathbb{R}$ are such that $a < b$.

2.4. **Properties of $C_{N,q}$ and $\Lambda_{s,p}$.** Concerning the constant in (1.11), the following result basically reformulates [3, Lem. B.1] and [18, Lem. 2.4] with minor refinements.

Lemma 2.1. *If $N \geq 1$ and $q \in [0,\infty)$, then the constant in (1.11) rewrites as*

$$C_{N,q} = \frac{1}{2}\int_{\mathbb{S}^{N-1}} |x_N|^q \,\mathrm{d}\mathscr{H}^{N-1}(x) = \pi^{\frac{N-1}{2}}\,\frac{\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{N+q}{2}\right)}. \tag{2.17}$$

In particular, for each $N \geq 1$, the function $q \mapsto C_{N,q}$ is continuous for $q \in [0,\infty)$.

Proof. The case $N = 1$ is trivial, so we assume $N \geq 2$.

The first equality in (2.17) has been already proved in [3, Lem. B.1]: one just needs to follow the very same argument letting $sp = q$. Note that the proof of [3, Lem. B.1] also works for $q = sp \in [0,1]$. We leave the simple check to the reader.

For the second inequality in (2.17), it is enough to observe that simple changes of variables yield the formulas (recall the definition in (2.1))

$$\mathrm{B}(a,b) = \int_0^\infty \frac{t^{a-1}}{(1+t)^{a+b}}\,\mathrm{d}t = 2\int_0^\infty \frac{t^{2a-1}}{(1+t^2)^{a+b}}\,\mathrm{d}t, \quad \text{for } a,b>0.$$

Thus, for $a = \frac{N-1}{2}$ and $b = \frac{q+1}{2}$, and owing to (2.2), we can rewrite

$$\int_0^\infty \frac{t^{N-2}}{(1+t^2)^{\frac{N+q}{2}}}\,\mathrm{d}t = \frac{1}{2}\,\mathrm{B}\left(\frac{N-1}{2},\frac{q+1}{2}\right) = \frac{\Gamma\left(\frac{N-1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\,\Gamma\left(\frac{N+q}{2}\right)}.$$

Hence, recalling (2.5), we easily get that

$$C_{N,q} = (N-1)\,\omega_{N-1}\,\frac{\Gamma\left(\frac{N-1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\,\Gamma\left(\frac{N+q}{2}\right)} = \pi^{\frac{N-1}{2}}\,\frac{\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{N+q}{2}\right)}.$$

The last part of the statement follows either by the Dominated Convergence Theorem or by the known continuity properties of the Gamma function. □

Concerning the constant in (1.12), we can prove the following result.

Lemma 2.2. *For $s \in (0,1)$ and $p \in (1,\infty)$ such that $sp < 1$, it holds that*

$$\Lambda_{s,p} < \Lambda_{sp,1} = \frac{4}{sp}. \tag{2.18}$$

As a consequence, for each $s \in (0,1)$ it holds that

$$\lim_{p\to 1^+} \Lambda_{s,p} = \Lambda_{s,1}. \tag{2.19}$$

Proof. Since $(1-t)^p < 1 - t^p$ for all $t \in (0,1)$, we can estimate

$$\int_0^1 \frac{\left|1 - t^{\frac{sp-1}{p}}\right|^p}{(1-t)^{1+sp}}\,\mathrm{d}t = \int_0^1 \frac{t^{sp-1}\left(1 - t^{\frac{1-sp}{p}}\right)^p}{(1-t)^{1+sp}}\,\mathrm{d}t < \int_0^1 \frac{t^{sp-1}\left(1 - t^{1-sp}\right)}{(1-t)^{1+sp}}\,\mathrm{d}t.$$

Letting $q = sp \in (0,1)$, we can write

$$\int_0^1 \frac{t^{q-1}\left(1-t^{1-q}\right)}{(1-t)^{1+q}}\,\mathrm{d}t = \lim_{\varepsilon\to 0^+}\left(\mathrm{B}(1-\varepsilon;q,-q) - \int_0^{1-\varepsilon}\frac{1}{(1-t)^{1+q}}\,\mathrm{d}t\right)$$
$$= \frac{1}{q} + \lim_{\varepsilon\to 0^+}\left(\mathrm{B}(1-\varepsilon;q,-q) - \frac{\varepsilon^{-q}}{q}\right),$$

where, for $\varepsilon \in (0,1)$, $\mathrm{B}(1-\varepsilon;q,-q)$ is as in (2.3). In view of (2.4), we have

$$\mathrm{B}(1-\varepsilon;q,-q) = \frac{(1-\varepsilon)^q}{q\,\varepsilon^q}$$

and thus we can easily compute

$$\lim_{\varepsilon\to 0^+}\left(\mathrm{B}(1-\varepsilon;q,-q) - \frac{\varepsilon^{-q}}{q}\right) = \lim_{\varepsilon\to 0^+}\left(\frac{(1-\varepsilon)^q}{q\,\varepsilon^q} - \frac{\varepsilon^{-q}}{q}\right) = 0.$$

We hence get that

$$\int_0^1 \frac{t^{q-1}\left(1-t^{1-q}\right)}{(1-t)^{1+q}}\,\mathrm{d}t = \frac{1}{q}$$

and the validity of (2.18) immediately follows by recalling the definition in (1.12). By Fatou's Lemma and by (2.18), we infer that

$$\Lambda_{s,1} \le \liminf_{p\to 1^+}\Lambda_{s,p} \le \limsup_{p\to 1^+}\Lambda_{s,p} \le \limsup_{p\to 1^+}\Lambda_{sp,1} = \lim_{p\to 1^+}\frac{4}{sp} = \frac{4}{s} = \Lambda_{s,1},$$

proving (2.19) and concluding the proof. □

2.5. **Rearrangements on the real line.** Given $s \in (0,1)$ and a (non-empty) open set $\Omega \subsetneq \mathbb{R}^N$, we let $\delta_{s,\Omega}\colon \mathbb{R}^N \to [0,\infty]$ be given by

$$\delta_{s,\Omega}(x) = \frac{\mathbf{1}_\Omega(x)}{d_\Omega^s(x)} \quad \text{for all } x \in \mathbb{R}^N, \tag{2.20}$$

where d_Ω is the distance function from $\partial\Omega$ defined in (1.2).

In the following result, we study the symmetric decreasing rearrangement of the function in (2.20) in the case $N = 1$ and Ω is a finite disjoint union of bounded open intervals.

Proposition 2.3. *Let $N = 1$, $s \in (0,1)$ and $\Omega \subsetneq \mathbb{R}$ be the disjoint union of $n \in \mathbb{N}$ (non-empty) bounded open intervals $\{I_k : k = 1,\dots,n\}$ and define*

$$r = \frac{1}{2}\max_{k=1,\dots,n}|I_k|.$$

Then, the function $\delta_{s,\Omega}\colon \mathbb{R} \to [0,\infty]$, defined in (2.20), satisfies

$$|\{\delta_{s,I} > t\}| \le 2nr\,\mathbf{1}_{(0,r^{-s}]}(t) + 2nt^{-\frac{1}{s}}\,\mathbf{1}_{(r^{-s},\infty)}(t), \quad \text{for all } t > 0, \tag{2.21}$$

and thus its symmetric decreasing rearrangement $\delta_{s,I}^\star\colon \mathbb{R} \to [0,\infty]$ satisfies

$$\delta_{s,\Omega}^\star(x) \le \frac{\mathbf{1}_{(-rn,rn)}(x)}{|x|^s}, \quad \text{for all } x \in \mathbb{R}. \tag{2.22}$$

Moreover, if $|I_k| = 2r$ for all $k = 1,\dots,n$, then (2.21) and (2.22) are both equalities.

In order to prove Proposition 2.3, we start with the case of a single interval.

Lemma 2.4. *Let $s \in (0,1)$, $N = 1$ and $I \subsetneq \mathbb{R}$ be a non-empty bounded open interval. Then, the function $\delta_{s,I} \colon \mathbb{R} \to [0,\infty]$, defined in (2.20), satisfies*

$$|\{\delta_{s,I} > t\}| = |I|\, \mathbf{1}_{(0,2^s|I|^{-s}]}(t) + 2\, t^{-\frac{1}{s}}\, \mathbf{1}_{(2^s|I|^{-s},\infty)}(t), \quad \text{for all } t > 0, \tag{2.23}$$

and thus its symmetric decreasing rearrangement $\delta^\star_{s,I} \colon \mathbb{R} \to [0,\infty]$ is given by

$$\delta^\star_{s,I}(x) = \frac{\mathbf{1}_{I^\star}(x)}{|x|^s}, \quad \text{for all } x \in \mathbb{R}. \tag{2.24}$$

Proof. Let $r \in (0,\infty)$ and $c \in \mathbb{R}$ be such that $I = (c-r, c+r)$, so that $d_I(x) = r - |x-c|$ for all $x \in I$. Since $\delta_{s,I}(x) > t$ if and only if $|x-c| > r - t^{-\frac{1}{s}}$ for all $x \in I$, we have

$$\{\delta_{s,I} > t\} = \begin{cases} I & 0 < t \le r^{-s}, \\ \left(c-r, c-r+t^{-\frac{1}{s}}\right) \cup \left(c+r-t^{-\frac{1}{s}}, c+r\right) & t > r^{-s}, \end{cases}$$

from which we readily deduce (2.23). We hence get that

$$\{\delta_{s,I} > t\}^\star = \begin{cases} I^\star & 0 < t \le r^{-s}, \\ \left(-t^{-\frac{1}{s}}, t^{-\frac{1}{s}}\right) & t > r^{-s}, \end{cases}$$

and thus, recalling the definition in (2.7), we can compute

$$\begin{aligned} \delta^\star_{s,I}(x) &= \int_0^\infty \mathbf{1}_{\{\delta_{s,I} > t\}^\star}(x)\, \mathrm{d}t = \int_0^{r^{-s}} \mathbf{1}_{I^\star}(x)\, \mathrm{d}t + \int_{r^{-s}}^\infty \mathbf{1}_{\left(-t^{-1/s}, t^{-1/s}\right)}(x)\, \mathrm{d}t \\ &= r^{-s}\, \mathbf{1}_{I^\star}(x) + \mathbf{1}_{I^\star}(x) \int_{r^{-s}}^\infty \mathbf{1}_{(0,|x|^{-s})}(t)\, \mathrm{d}t = |x|^{-s}\, \mathbf{1}_{I^\star}(x) \end{aligned} \tag{2.25}$$

for all $x \in \mathbb{R}$, proving (2.24) and concluding the proof. □

Remark 2.5. With the same notation of Lemma 2.4, if $I, J \subseteq \mathbb{R}$ are two non-empty bounded open intervals such that $|I| = |J|$, then $\delta^\star_{s,I} = \delta^\star_{s,J}$ for all $s \in (0,1)$.

Remark 2.6. Lemma 2.4 also holds for $N \ge 2$, but the formula corresponding to (2.24) has a completely different appearance. More precisely, taking $\Omega = \mathbb{B}^N$ for simplicity in (2.20), analogous computations yields that

$$\delta^\star_{s,\mathbb{B}^N}(x) = \mathbf{1}_{\mathbb{B}^N}(x) \left(1 - \left(1 - |x|^N\right)^{\frac{1}{N}}\right)^{-s}$$

for all $x \in \mathbb{R}^N$. We omit the detailed derivation.

Proof of Proposition 2.3. As the I_k's are disjoint, we can write

$$\delta_{s,\Omega}(x) = \sum_{k=1}^n \delta_{s,I_k}(x) \tag{2.26}$$

for all $x \in \mathbb{R}$. The decomposition in (2.26) implies that

$$\{\delta_{s,\Omega} > t\} = \bigcup_{k=1}^n \{\delta_{s,I_k} > t\}$$

for all $t > 0$ with disjoint union. By Lemma 2.4, we hence get that

$$|\{\delta_{s,\Omega} > t\}| = \sum_{k=1}^n |\{\delta_{s,I_k} > t\}| = \sum_{k=1}^n |I_k|\, \mathbf{1}_{(0,2^s|I_k|^{-s}]}(t) + 2t^{-\frac{1}{s}}\, \mathbf{1}_{(2^s|I_k|^{-s},\infty)}(t) \tag{2.27}$$

for all $t>0$. Now let us set $r_k=|I_k|/2$ for all $k=1,\dots,n$ for notational convenience. Without loss of generality, we can assume that $0<r_1\le r_2\le\cdots\le r_n<\infty$. Furthermore, let us set $r_0=0$ and $r_{n+1}=\infty$ and adopt the notation $r_0^{-s}=\infty$ and $r_{n+1}^{-s}=0$. With this conventions in force, we can write

$$\mathbf{1}_{\left(0,r_k^{-s}\right]}=\sum_{j=0}^{n-k}\mathbf{1}_{\left(r_{n+1-j}^{-s},r_{n-j}^{-s}\right]}\quad\text{and}\quad\mathbf{1}_{\left(r_k^{-s},\infty\right)}=\sum_{j=0}^{n-1}\mathbf{1}_{\left(r_{k-j}^{-s},r_{k-j-1}^{-s}\right]}$$

for all $k=1,\dots,n$. Therefore, on the one hand, we get

$$\sum_{k=1}^{n}r_k\,\mathbf{1}_{\left(0,r_k^{-s}\right]}=\sum_{j=0}^{n-k}\mathbf{1}_{\left(r_{n+1-j}^{-s},r_{n-j}^{-s}\right]}\left(\sum_{k=1}^{n-j}r_k\right)\tag{2.28}$$

for all $k=1,\dots,n$, while, on the other hand, we can compute

$$\begin{aligned}\sum_{k=1}^{n}\mathbf{1}_{\left(r_k^{-s},\infty\right)}&=\sum_{k=1}^{n}\sum_{j=0}^{n-1}\mathbf{1}_{\left(r_{k-j}^{-s},r_{k-j-1}^{-s}\right]}=\sum_{j=0}^{n-1}\sum_{k=j+1}^{n}\mathbf{1}_{\left(r_{k-j}^{-s},r_{k-j-1}^{-s}\right]}=\sum_{j=0}^{n-1}\sum_{h=1}^{n-j}\mathbf{1}_{\left(r_h^{-s},r_{h-1}^{-s}\right]}\\&=\sum_{h=1}^{n}\sum_{j=0}^{n-h}\mathbf{1}_{\left(r_h^{-s},r_{h-1}^{-s}\right]}=\sum_{h=1}^{n}(n-h+1)\,\mathbf{1}_{\left(r_h^{-s},r_{h-1}^{-s}\right]}=\sum_{j=0}^{n}j\,\mathbf{1}_{\left(r_{n+1-j}^{-s},r_{n-j}^{-s}\right]}\end{aligned}\tag{2.29}$$

for all $k=1,\dots,n$. By combining (2.28) and (2.29) with (2.27), we conclude that

$$|\{\delta_{s,\Omega}>t\}|=2\sum_{j=0}^{n}\mathbf{1}_{\left(r_{n+1-j}^{-s},r_{n-j}^{-s}\right]}(t)\left(j\,t^{-\frac{1}{s}}+\sum_{k=1}^{n-j}r_k\right)\tag{2.30}$$

for all $t>0$. Note that, for $r_{n+1-j}^{-s}<t\le r_{n-j}^{-s}$, we have $r_{n-j}\le t^{-\frac{1}{s}}<r_{n+1-j}$, and so

$$j\,t^{-\frac{1}{s}}+\sum_{k=1}^{n-j}r_k\le j\,t^{-\frac{1}{s}}+(n-j)\,r_{n-j}\le j\,t^{-\frac{1}{s}}+(n-j)\,t^{-\frac{1}{s}}=n\,t^{-\frac{1}{s}}.$$

Thus, we can estimate the measure in (2.30) (with equality if the r_k's are all equal) as

$$\begin{aligned}|\{\delta_{s,\Omega}>t\}|&\le\mathbf{1}_{\left(0,r_n^{-s}\right]}(t)\left(2\sum_{k=1}^{n}r_k\right)+2\sum_{j=1}^{n}\mathbf{1}_{\left(r_{n+1-j}^{-s},r_{n-j}^{-s}\right]}(t)\,n\,t^{-\frac{1}{s}}\\&\le 2nr_n\,\mathbf{1}_{\left(0,r_n^{-s}\right]}(t)+2nt^{-\frac{1}{s}}\,\mathbf{1}_{\left(r_n^{-s},\infty\right)}(t)\end{aligned}$$

for all $t>0$, proving (2.21). We thus obtain that (with equality if the r_k's are all equal)

$$\{\delta_{s,\Omega}>t\}^\star\subseteq\begin{cases}(-nr_n,nr_n) & 0<t\le r_n^{-s},\\ \left(-nt^{-\frac{1}{s}},nt^{-\frac{1}{s}}\right) & t>r_n^{-s},\end{cases}$$

and consequently, arguing similarly as in (2.25), we get that

$$\begin{aligned}\delta_{s,\Omega}^\star(x)&=\int_0^\infty\mathbf{1}_{\{\delta_{s,\Omega}>t\}^\star}(x)\,\mathrm{d}t\le\int_0^{r_n^{-s}}\mathbf{1}_{(-nr_n,nr_n)}(x)\,\mathrm{d}t+\int_{r_n^{-s}}^\infty\mathbf{1}_{(-nt^{-1/s},nt^{-1/s})}(x)\,\mathrm{d}t\\&=r_n^{-s}\,\mathbf{1}_{(-nr_n,nr_n)}(x)+\mathbf{1}_{(-nr_n,nr_n)}(x)\int_{r_n^{-s}}^\infty\mathbf{1}_{(0,n^s|x|^{-s})}(t)\,\mathrm{d}t=n^s\,\mathbf{1}_{(-nr_n,nr_n)}(x)\,|x|^{-s}\end{aligned}$$

for all $x\in\mathbb{R}$. Recalling that $r_n\le r$ by assumption, we obtain (2.22). □

We conclude with the following result, which is a direct consequence of Proposition 2.3.

Corollary 2.7. *Let $N = 1$, $s \in (0,1)$ and $\Omega \subsetneq \mathbb{R}$ be the disjoint union of $n \in \mathbb{N}$ (non-empty) bounded open intervals. If $u \in C_0^\infty(\Omega)$ and $p \in [1,\infty)$ is such that $sp < 1$, then*

$$\int_\Omega \frac{|u|^p}{d_\Omega^{sp}}\,\mathrm{d}x \le n^{sp} \int_{\mathbb{R}} \frac{(u^\star)^p}{|x|^{sp}}\,\mathrm{d}x. \tag{2.31}$$

Proof. Given $u \in C_0^\infty(\Omega)$, recalling (2.20) and owing to (2.9) and (2.8), we can estimate

$$\int_\Omega \frac{|u|^p}{d_\Omega^{sp}}\,\mathrm{d}x = \int_{\mathbb{R}} |u|^p\,\delta_{sp,\Omega}\,\mathrm{d}x \le \int_{\mathbb{R}} (|u|^p)^\star\,\delta_{sp,\Omega}^\star\,\mathrm{d}x = \int_{\mathbb{R}} (u^\star)^p\,\delta_{sp,\Omega}^\star\,\mathrm{d}x.$$

Therefore, since $sp < 1$, we can exploit (2.22) (with sp in place of s) to infer that

$$\int_\Omega \frac{|u|^p}{d_\Omega^{sp}}\,\mathrm{d}x \le \int_{\mathbb{R}} (u^\star)^p\,\delta_{sp,\Omega}^\star\,\mathrm{d}x \le n^{sp} \int_{\mathbb{R}} \frac{(u^\star)^p}{|x|^{sp}}\,\mathrm{d}x,$$

proving (2.31) and concluding the proof. □

3. The geometrical approach

In this section, we detail the proof of Theorem 1.4 and derive some consequences, both in the convex and non-convex case.

3.1. Proof of Theorem 1.4.

We begin with our auxiliary result concerning the geometrical interpretation of the sharp constant (1.14).

Proof of Theorem 1.4. Let us observe that $\mathfrak{g}_s(\Omega) \in [0,\infty)$ is well posed, since

$$\mathfrak{g}_s(\Omega) \le \frac{P_s(B)}{V_{s,\Omega}(B)} < \infty$$

whenever $B \Subset \Omega$ is a (non-empty) open ball. We start by showing that

$$\mathfrak{h}_{s,1}(\Omega) \ge \mathfrak{g}_s(\Omega). \tag{3.1}$$

We fix $u \in C_0^\infty(\Omega)$ and, without loss of generality, assume that $u \ge 0$. By Cavalieri's principle, we have that

$$\int_\Omega \frac{u}{d_\Omega^s}\,\mathrm{d}x = \int_0^{\|u\|_{L^\infty(\Omega)}} \int_{\{u>t\}} \frac{1}{d_\Omega^s}\,\mathrm{d}x\,\mathrm{d}t = \int_0^{\|u\|_{L^\infty(\Omega)}} V_{s,\Omega}(\{u>t\})\,\mathrm{d}t. \tag{3.2}$$

Now, for each $t \in (0, \|u\|_{L^\infty(\Omega)})$, the (non-empty, open) set $E_t = \{u > t\}$ is such that $E_t \Subset \Omega$ and $|E_t| > 0$ and thus, by the definition in (1.20), we can write

$$P_s(E_t) \ge \mathfrak{g}_s(\Omega)\,V_{s,\Omega}(E_t).$$

Hence, thanks to the fractional coarea formula in (1.21), we infer that

$$[u]_{W^{s,1}(\mathbb{R}^N)} = \int_0^{\|u\|_{L^\infty(\Omega)}} P_s(E_t)\,\mathrm{d}t \ge \int_0^{\|u\|_{L^\infty(\Omega)}} \mathfrak{g}_s(\Omega)\,V_{s,\Omega}(E_t)\,\mathrm{d}t = \mathfrak{g}_s(\Omega) \int_\Omega \frac{u}{d_\Omega^s}\,\mathrm{d}x,$$

readily implying the validity of (3.1). We now prove the converse inequality; that is,

$$\mathfrak{h}_{s,1}(\Omega) \le \mathfrak{g}_s(\Omega). \tag{3.3}$$

We let $E \Subset \Omega$ be such that $|E| > 0$, and we define

$$u_\varepsilon = \varrho_\varepsilon * \mathbf{1}_E \in C^\infty(\mathbb{R}^N),$$

where $(\varrho_\varepsilon)_{\varepsilon>0} \subseteq C_0^\infty(\mathbb{R}^N)$ is a family of standard non-negative mollifiers. Since $E \Subset \Omega$, we have that $\operatorname{supp} u_\varepsilon \Subset \Omega$ for all $\varepsilon > 0$ sufficiently small. Therefore, owing to [25, Th. 6.62] and the Dominated Convergence Theorem, we have that

$$\mathfrak{h}_{s,1}(\Omega) \leq \lim_{\varepsilon\to 0^+} \frac{[u_\varepsilon]_{W^{s,1}(\mathbb{R}^N)}}{\displaystyle\int_\Omega \frac{u_\varepsilon}{d_\Omega^s}\,\mathrm{d}x} = \frac{P_s(E)}{V_{s,\Omega}(E)},$$

from which we readily deduce (3.3) and reach the conclusion of the proof. □

Actually, in Theorem 1.4, we can restrict the infimization in (1.20) only to smooth open sets compactly contained in Ω. Precisely, we have the following result.

Corollary 3.1. *Let $N \geq 1$, $s \in (0,1)$ and let $\Omega \subsetneq \mathbb{R}^N$ be a (non-empty) open set. Then,*

$$\mathfrak{g}_s(\Omega) = \inf\left\{\frac{P_s(E)}{V_{s,\Omega}(E)} : E \in \mathcal{O}(\Omega)\right\} \in [0,\infty), \tag{3.4}$$

where $\mathcal{O}(\Omega)$ is the family of all open sets with smooth boundary compactly contained in Ω.

Proof. Letting $\widetilde{\mathfrak{g}}_s(\Omega) \in [0,\infty]$ be the right-hand side of (3.4), we have $\mathfrak{g}_s(\Omega) \leq \widetilde{\mathfrak{g}}_s(\Omega)$ by definition. On the other hand, if $u \in C_0^\infty(\Omega)$ is such that $u \geq 0$, then $\{u > t\} \in \mathcal{O}(\Omega)$ for a.e. $t \in (0, \|u\|_{L^\infty})$ (e.g., see [29, Th. 13.15]). Hence, arguing exactly as in the first part of the proof of Theorem 1.4, we see that $\mathfrak{h}_{s,1}(\Omega) \geq \widetilde{\mathfrak{g}}_s(\Omega)$, in particularly yielding that $\widetilde{\mathfrak{g}}_s(\Omega) < \infty$. Thus we have $\widetilde{\mathfrak{g}}_s(\Omega) \leq \mathfrak{g}_s(\Omega)$ by Theorem 1.4, leading to the conclusion. □

A remarkable consequence of Theorem 1.4 is the following comparison result.

Corollary 3.2. *Let $N \geq 1$, $p \in (1,\infty)$ and $s \in (0,1)$ be such that $sp < 1$. For any non-empty open set $\Omega \subsetneq \mathbb{R}^N$, it holds that*

$$\mathfrak{h}_{s,p}(\Omega) \leq \mathfrak{h}_{sp,1}(\Omega). \tag{3.5}$$

If $\Omega = \mathbb{H}^N_+$, then the inequality (3.5) is strict.

Proof. As in the second part of the proof of Theorem 1.4, given $E \Subset \Omega$ such that $|E| > 0$, we define $u_\varepsilon = \mathbf{1}_E * \varrho_\varepsilon$ for all $\varepsilon > 0$, where $(\varrho_\varepsilon)_{\varepsilon>0} \subseteq C_0^\infty(\mathbb{R}^N)$ is a family of standard non-negative mollifiers. As before, $u_\varepsilon \in C_0^\infty(\mathbb{R}^N)$ and $0 \leq u_\varepsilon \leq 1$ for all $\varepsilon > 0$, with $\operatorname{supp} u_\varepsilon \Subset \Omega$ for all $\varepsilon > 0$ sufficiently small. Therefore, by [25, Th. 6.62] and the Dominated Convergence Theorem, and since $sp < 1$, we get that

$$\mathfrak{h}_{s,p}(\Omega) \leq \lim_{\varepsilon\to 0^+} \frac{[u_\varepsilon]^p_{W^{s,p}(\mathbb{R}^N)}}{\displaystyle\int_\Omega \frac{u_\varepsilon^p}{d_\Omega^{sp}}\,\mathrm{d}x} = \frac{[\mathbf{1}_E]^p_{W^{s,p}(\mathbb{R}^N)}}{\displaystyle\int_\Omega \frac{\mathbf{1}_E^p}{d_\Omega^{sp}}\,\mathrm{d}x} = \frac{[\mathbf{1}_E]_{W^{sp,1}(\mathbb{R}^N)}}{\displaystyle\int_\Omega \frac{\mathbf{1}_E}{d_\Omega^{sp}}\,\mathrm{d}x} = \frac{P_{sp}(E)}{V_{sp,\Omega}(E)}.$$

Consequently, by Theorem 1.4, we conclude that

$$\mathfrak{h}_{s,p}(\Omega) \leq \mathfrak{g}_{sp,1}(\Omega) = \mathfrak{h}_{sp,1}(\Omega),$$

proving (3.5). For $\Omega = \mathbb{H}^N_+$, the inequality in (3.5) holds strict by (1.15) and (2.18). □

3.2. **The convex case.** If $\Omega \subsetneq \mathbb{R}^N$ is a (non-empty) convex open set, then we can refine the statement of Theorem 1.4 as follows.

Corollary 3.3. *Let $N \geq 1$, $s \in (0,1)$ and let $\Omega \subsetneq \mathbb{R}^N$ be a (non-empty) open convex set. Then, it holds that*

$$\mathfrak{h}_{s,1}(\Omega) = \mathfrak{g}_s^*(\Omega), \tag{3.6}$$

where

$$\mathfrak{g}_s^*(\Omega) = \inf\left\{ \frac{P_s(E)}{V_{s,\Omega}(E)} : E \subseteq \Omega,\ |E| \in (0,\infty) \right\} \in (0,\infty). \tag{3.7}$$

Moreover, if $u \in \mathcal{W}_0^{s,1}(\Omega) \setminus \{0\}$ is a non-negative minimizer of (1.14) then, for a.e. $t > 0$ such that $|\{u > t\}| \in (0,\infty)$, $E_t = \{u > t\}$ is a minimizer of (3.7). Vice versa, if $E \subseteq \Omega$ is a minimizer of (3.7) such that $\mathbf{1}_E \in \mathcal{W}_0^{s,1}(\Omega)$, then $u = \mathbf{1}_E / V_{s,\Omega}(E)$ is a non-negative minimizer of (1.14).

In the proof of Corollary 3.3, we are going to exploit the following simple approximation result, whose proof is omitted.

Lemma 3.4. *Let $N \geq 1$. If $\Omega \subseteq \mathbb{R}^N$ is a (non-empty) open convex set, then there exists a sequence of (non-empty) bounded open convex sets $(\Omega_k)_{k\in\mathbb{N}}$ such that*

$$\Omega_k \Subset \Omega_{k+1} \Subset \Omega \quad \text{for all } k \in \mathbb{N} \qquad \text{and} \qquad \Omega = \bigcup_{k=1}^{\infty} \Omega_k.$$

Proof of Corollary 3.3. By Theorem 1.4 and the trivial inequality $\mathfrak{g}_s^*(\Omega) \leq \mathfrak{g}_s(\Omega)$, to show (3.6) we only have to prove that $\mathfrak{g}_s(\Omega) \leq \mathfrak{g}_s^*(\Omega)$. To this aim, let $E \subseteq \Omega$ be such that $|E| \in (0,\infty)$, $(\Omega_k)_{k\in\mathbb{N}}$ be the sequence of (non-empty) bounded open convex set given by Lemma 3.4, and $E_k = E \cap \Omega_k$ for every $k \in \mathbb{N}$. Clearly, for all $k \in \mathbb{N}$ sufficiently large, we have that $E_k \Subset \Omega$ and $|E_k| \in (0,\infty)$. Moreover, by the Monotone Convergence Theorem, we also have that

$$V_{s,\Omega}(E) = \lim_{k\to\infty} V_{s,\Omega}(E_k).$$

Finally, by (2.13), we also have that $P_s(E_k) \leq P_s(E)$ for all $k \in \mathbb{N}$. Thus, we get

$$\mathfrak{g}_s(\Omega) \leq \limsup_{k\to\infty} \frac{P_s(E_k)}{V_{s,\Omega}(E_k)} \leq \lim_{k\to\infty} \frac{P_s(E)}{V_{s,\Omega}(E_k)} = \frac{P_s(E)}{V_{s,\Omega}(E)},$$

from which (3.6) follows immediately. By Theorem 1.1, this also shows that $\mathfrak{g}_s^*(\Omega) > 0$.

To prove the last part of the statement, it is enough to observe that, if $u \in \mathcal{W}_0^{s,1}(\Omega) \setminus \{0\}$ is a non-negative minimizer of (1.14) and $E_t = \{u > t\}$ for $t > 0$, then, by (3.7), the fractional coarea formula (1.21) and Cavalieri's principle (3.2), we can write

$$[u]_{W^{s,1}(\mathbb{R}^N)} - \mathfrak{h}_{s,1}(\Omega) \int_\Omega \frac{u}{d_\Omega^s}\,\mathrm{d}x = \int_0^\infty \Big(P_s(E_t) - \mathfrak{g}_s^*(\Omega)\, V_{s,\Omega}(E_t) \Big)\,\mathrm{d}t,$$

from which we deduce that E_t is a minimizer of (3.7) for a.e. $t > 0$ such that $|E_t| \in (0,\infty)$. Vice versa, if $E \subseteq \Omega$ is a minimizer of (3.7) such that $\mathbf{1}_E \in \mathcal{W}_0^{s,1}(\Omega)$, then clearly also $u = \mathbf{1}_E / V_{s,\Omega}(E) \in \mathcal{W}_0^{s,1}(\Omega) \setminus \{0\}$. Moreover, by definition and (3.6), we have that

$$[u]_{W^{s,1}(\mathbb{R}^N)} = \frac{P_s(E)}{V_{s,\Omega}(E)} = \mathfrak{g}_s^*(\Omega) = \mathfrak{h}_{s,1}(\Omega) \quad \text{and} \quad \int_\Omega \frac{u}{d_\Omega^s}\,\mathrm{d}x = 1,$$

proving that u is a minimizer of (1.14). □

As an immediate consequence of Corollary 3.3, we get the following estimate.

Corollary 3.5. *Let $N \geq 1$ and $s \in (0,1)$. If $\Omega \subsetneq \mathbb{R}^N$ is a (non-empty) bounded open convex set, then*

$$\mathfrak{h}_s(\Omega) \leq \frac{P_s(\Omega)}{V_{s,\Omega}(\Omega)}.$$

3.3. **The non-convex case.** A statement similar to Corollary 3.3 can be achieved also in the case Ω is a non-convex set, see Corollary 3.7 below. To this purpose, we need to introduce the following terminology.

Definition 3.6 (s-negligible set)**.** Let $s \in (0,1)$. A set $E \subseteq \mathbb{R}^N$ is *s-negligible* if

$$\lim_{r \to 0^+} P_s\Big(B_r(E)\Big) = 0,$$

where $B_r(E) = \bigcup_{x \in E} B_r(x)$ for every $r > 0$.

For example, by (2.11) and the fact that $P_s(B_r(x)) = P_s(\mathbb{B}^N)\, r^{N-s}$ for all $x \in \mathbb{R}^N$ and $r > 0$, any set $E \subset \mathbb{R}^N$ with finitely many points is s-negligible for every $s \in (0,1)$.

Corollary 3.7. *Let $N \geq 1$, $s \in (0,1)$, $U \subseteq \mathbb{R}^N$ be a (non-empty) open convex set and let $F \subsetneq U$ be an s-negligible closed set. If $\Omega = U \setminus F$, then*

$$\mathfrak{h}_{s,1}(\Omega) \leq \inf\left\{ \frac{P_s(E)}{V_{s,\Omega}(E)} : E \subseteq U,\ |E| \in (0,\infty) \right\}. \tag{3.8}$$

In particular, if U is bounded, then

$$\mathfrak{h}_{s,1}(\Omega) \leq \frac{P_s(U)}{V_{s,\Omega}(U)}.$$

Proof. Let $(U_k)_{k \in \mathbb{N}}$ be the sequence of (non-empty) bounded open convex sets given by Lemma 3.4 applied to U. Given $E \subseteq U$ such that $|E| \in (0,\infty)$, we let $r > 0$ and consider

$$E_k = (E \cap U_k) \quad \text{and} \quad E_{k,r} = E_k \setminus B_r(F).$$

Note that $E_{k,r} \Subset \Omega$ and $|E_{k,r}| \in (0,\infty)$ for every $k \in \mathbb{N}$ sufficiently large and every $r > 0$ sufficiently small. Now, on the one hand, by (2.12), (2.13) and (2.11), we can estimate

$$P_s(E_{k,r}) \leq P_s(E_k) + P_s\Big(B_r(F)\Big) \leq P_s(E) + P_s\Big(B_r(F)\Big).$$

On the other hand, since $E_{k,r} \subseteq E_k \subseteq E$ for $k \in \mathbb{N}$ and $r > 0$, by the Monotone Convergence Theorem, we infer that

$$\lim_{k \to \infty} \lim_{r \to 0^+} V_{s,\Omega}(E_{k,r}) = \lim_{k \to \infty} V_{s,\Omega}(E_k) = V_{s,\Omega}(E).$$

Therefore, owing to Theorem 1.4 and since F is s-negligible, we conclude that

$$\mathfrak{h}_{s,1}(\Omega) = \mathfrak{g}_s(\Omega) \leq \limsup_{k \to \infty} \limsup_{r \to 0^+} \frac{P_s(E_{k,r})}{V_{s,\Omega}(E_{k,r})} \leq \lim_{k \to \infty} \lim_{r \to 0^+} \frac{P_s(E) + P_s\Big(B_r(F)\Big)}{V_{s,\Omega}(E_{k,r})} = \frac{P_s(E)}{V_{s,\Omega}(E)},$$

from which (3.8) follows immediately. □

4. Proofs of the results

4.1. **Proof of Theorem 1.1.** In order to characterize the sharp constant in (1.8) for $\Omega = \mathbb{H}^N_+$ and $p = 1$, so to prove Theorem 1.1, we need some preliminary results.

We begin with the following stability result for $\mathfrak{h}_{s,p}(\Omega)$ as $p \to 1^+$.

Lemma 4.1. *Let $N \geq 1$, $s \in (0,1)$ and $\Omega \subsetneq \mathbb{R}^N$ be a (non-empty) open set. Then,*

$$\lim_{p\to 1^+} \int_\Omega \frac{|u|^p}{d_\Omega^{sp}}\,\mathrm{d}x = \int_\Omega \frac{|u|}{d_\Omega^{s}}\,\mathrm{d}x \tag{4.1}$$

and

$$\lim_{p\to 1^+} [u]^p_{W^{s,p}(\mathbb{R}^N)} = [u]_{W^{s,1}(\mathbb{R}^N)} \tag{4.2}$$

for every $u \in C^\infty_0(\Omega)$. As a consequence, it holds that

$$\limsup_{p\to 1^+} \mathfrak{h}_{s,p}(\Omega) \leq \mathfrak{h}_{s,1}(\Omega). \tag{4.3}$$

Proof. For $p \in (1,\infty)$, by Hölder's inequality, we can estimate

$$\int_\Omega \frac{|u|}{d_\Omega^{s}}\,\mathrm{d}x = \int_{\operatorname{supp} u} \frac{|u|}{d_\Omega^{s}}\,\mathrm{d}x \leq \left(\int_\Omega \frac{|u|^p}{d_\Omega^{sp}}\,\mathrm{d}x\right)^{\frac{1}{p}} |\operatorname{supp} u|^{\frac{p-1}{p}}, \tag{4.4}$$

and, similarly,

$$\int_\Omega \frac{|u|^p}{d_\Omega^{sp}}\,\mathrm{d}x \leq \left\|\frac{u}{d_\Omega^{s}}\right\|^{p-1}_{L^\infty(\Omega)} \int_\Omega \frac{|u|}{d_\Omega^{s}}\,\mathrm{d}x. \tag{4.5}$$

Hence the validity of (4.1) immediately follows by combining (4.4) and (4.5). Concerning (4.2), owing to Fatou's Lemma, we just need to observe that

$$[u]^p_{W^{s,p}(\mathbb{R}^N)} = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p-1}}{|x-y|^{s(p-1)}} \frac{|u(x)-u(y)|}{|x-y|^{N+s}}\,\mathrm{d}x\,\mathrm{d}y \leq [u]^{p-1}_{C^{0,s}(\mathbb{R}^N)}\,[u]_{W^{s,1}(\mathbb{R}^N)}.$$

Lastly, the validity of (4.3) readily follows by combining (1.8) with (4.1) and (4.2). □

In the case $\Omega = \mathbb{H}^N_+$, Lemma 4.1 can be complemented with the following result, yielding useful estimates on the energies of product test functions for $p = 1$.

Lemma 4.2. *If $\varphi \in C^\infty_0(\mathbb{R}^{N-1})$ and $\psi \in C^\infty_0(\mathbb{H}^1_+)$, then $u \in C^\infty_0(\mathbb{H}^N_+)$, defined as $u(x) = \varphi(x')\,\psi(x_N)$ for all $x = (x', x_N) \in \mathbb{H}^N_+$, satisfies*

$$\int_{\mathbb{H}^N_+} \frac{|u(x)|}{x_N^s}\,\mathrm{d}x = \|\varphi\|_{L^1(\mathbb{R}^{N-1})} \int_{\mathbb{H}^1_+} \frac{|\psi(t)|}{t^s}\,\mathrm{d}t \tag{4.6}$$

and

$$[u]_{W^{s,1}(\mathbb{R}^N)} \leq C_{N,s}\,\|\varphi\|_{L^1(\mathbb{R}^{N-1})}\,[\psi]_{W^{s,1}(\mathbb{R})} + 2\|\psi\|_{L^1(\mathbb{H}^1_+)}\,[\varphi]_{W^{s,1}(\mathbb{R}^{N-1})} \int_0^\infty \frac{\mathrm{d}t}{(1+t^2)^{\frac{N+s}{2}}}. \tag{4.7}$$

Proof. To prove (4.6), we simply observe that

$$\int_{\mathbb{H}^N_+} \frac{|u(x)|}{x_N^s}\,\mathrm{d}x = \int_{\mathbb{R}^{N-1}} |\varphi(x')|\,\mathrm{d}x' \int_0^\infty \frac{|\psi(x_N)|}{x_N^s}\,\mathrm{d}x_N = \|\varphi\|_{L^1(\mathbb{R}^{N-1})} \int_{\mathbb{H}^1_+} \frac{|\psi(t)|}{t^s}\,\mathrm{d}t.$$

To see (4.7), we first estimate

$$\begin{aligned}[u]_{W^{s,1}(\mathbb{R}^N)} \le &\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |\varphi(x')|\,\frac{|\psi(x_N)-\psi(y_N)|}{|x-y|^{N+s}}\,\mathrm{d}x\,\mathrm{d}y\\ &+\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |\psi(y_N)|\,\frac{|\varphi(x')-\varphi(y')|}{|x-y|^{N+s}}\,\mathrm{d}x\,\mathrm{d}y.\end{aligned} \tag{4.8}$$

The first term in the right-hand side of (4.8) rewrites as

$$\begin{aligned}&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |\varphi(x')|\,\frac{|\psi(x_N)-\psi(y_N)|}{|x-y|^{N+s}}\,\mathrm{d}x\,\mathrm{d}y\\ &\quad=\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|\psi(x_N)-\psi(y_N)|}{|x_N-y_N|^{N+s}}\int_{\mathbb{R}^{N-1}}|\varphi(x')|\int_{\mathbb{R}^{N-1}}\frac{\mathrm{d}y'}{\left(1+\frac{|x'-y'|^2}{|x_N-y_N|^2}\right)^{\frac{N+s}{2}}}\,\mathrm{d}x'\,\mathrm{d}x_N\,\mathrm{d}y_N\\ &\quad=\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|\psi(x_N)-\psi(y_N)|}{|x_N-y_N|^{1+s}}\int_{\mathbb{R}^{N-1}}|\varphi(x')|\int_{\mathbb{R}^{N-1}}\frac{\mathrm{d}z'}{(1+|z'|)^{\frac{N+s}{2}}}\,\mathrm{d}x'\,\mathrm{d}x_N\,\mathrm{d}y_N\\ &\quad=C_{N,s}\,\|\varphi\|_{L^1(\mathbb{R}^{N-1})}\,[\psi]_{W^{s,1}(\mathbb{R})},\end{aligned} \tag{4.9}$$

while the second term in (4.8) similarly corresponds to

$$\begin{aligned}&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |\psi(y_N)|\,\frac{|\varphi(x')-\varphi(y')|}{|x-y|^{N+s}}\,\mathrm{d}x\,\mathrm{d}y\\ &\quad=\int_{\mathbb{R}^{N-1}}\int_{\mathbb{R}^{N-1}} \frac{|\varphi(x')-\varphi(y')|}{|x'-y'|^{N+s}}\int_{\mathbb{R}}|\psi(y_N)|\int_{\mathbb{R}}\frac{\mathrm{d}x_N}{\left(1+\frac{|x_N-y_N|^2}{|x'-y'|^2}\right)^{\frac{N+s}{2}}}\,\mathrm{d}y_N\,\mathrm{d}x'\,\mathrm{d}y'\\ &\quad=\int_{\mathbb{R}^{N-1}}\int_{\mathbb{R}^{N-1}} \frac{|\varphi(x')-\varphi(y')|}{|x'-y'|^{N-1+s}}\int_{\mathbb{R}}|\psi(y_N)|\int_{\mathbb{R}}\frac{\mathrm{d}z_N}{(1+|z_N|^2)^{\frac{N+s}{2}}}\,\mathrm{d}y_N\,\mathrm{d}x'\,\mathrm{d}y'\\ &\quad=2\,\|\psi\|_{L^1(\mathbb{H}^1_+)}\,[\varphi]_{W^{s,1}(\mathbb{R}^{N-1})}\int_0^\infty\frac{\mathrm{d}t}{(1+t^2)^{\frac{N+s}{2}}}.\end{aligned} \tag{4.10}$$

The conclusion hence readily follows by combining (4.8), (4.9) and (4.10). □

As a consequence of the above results, we get the following partial step towards (1.15).

Corollary 4.3. *Given $N\ge 1$ and $s\in(0,1)$, it holds that*

$$C_{N,s}\,\Lambda_{s,1}\le \mathfrak{h}_{s,1}(\mathbb{H}^N_+)\le C_{N,s}\,\mathfrak{h}_{s,1}(\mathbb{H}^1_+). \tag{4.11}$$

Proof. Owing to (4.3), (1.9), Lemma 2.1 and Lemma 2.2, we plainly get that

$$\mathfrak{h}_{s,1}(\mathbb{H}^N_+)\ge \limsup_{p\to1^+}\mathfrak{h}_{s,p}(\mathbb{H}^N_+)=\lim_{p\to1^+}C_{N,sp}\,\Lambda_{s,p}=C_{N,s}\,\Lambda_{s,1},$$

proving the first inequality in (4.11).

To prove the second inequality in (4.11), instead, we argue as follows. Let $\varphi\in C^\infty_0(\mathbb{R}^{N-1})$ and $\psi\in C^\infty_0(\mathbb{H}^1_+)$ be fixed. For each $k\in\mathbb{N}$, we define $\varphi_k(x')=\varphi\left(\frac{x'}{k}\right)$ for all $x'\in\mathbb{R}^{N-1}$ and $u_k(x)=\varphi_k(x')\,\psi(x_N)$ for all $x=(x',x_N)\in\mathbb{R}^{N-1}\times(0,\infty)=\mathbb{H}^N_+$. Owing to Lemma 4.2 and observing that

$$[\varphi_k]_{W^{s,1}(\mathbb{R}^{N-1})}=k^{N-1-s}\,[\varphi]_{W^{s,1}(\mathbb{R}^{N-1})}\quad\text{and}\quad\|\varphi_k\|_{L^1(\mathbb{R}^{N-1})}=k^{N-1}\,\|\varphi\|_{L^1(\mathbb{R}^{N-1})},$$

we plainly get that

$$\begin{aligned}\mathfrak{h}_{s,1}(\mathbb{H}^N_+) &\le \frac{[u_k]_{W^{s,1}(\mathbb{R}^N)}}{\displaystyle\int_{\mathbb{H}^N_+} \frac{|u_k(x)|}{x_N^s}\,\mathrm{d}x} \\ &\le C_{N,s}\,\frac{[\psi]_{W^{s,1}(\mathbb{R})}}{\displaystyle\int_{\mathbb{H}^1_+}\frac{|\psi(t)|}{t^s}\,\mathrm{d}t} + \frac{[\varphi_k]_{W^{s,1}(\mathbb{R}^{N-1})}}{\|\varphi_k\|_{L^1(\mathbb{R}^{N-1})}}\,\frac{\|\psi\|_{L^1(\mathbb{H}^1_+)}\displaystyle\int_0^\infty \frac{\mathrm{d}t}{(1+t^2)^{\frac{N+s}{2}}}}{\displaystyle\int_{\mathbb{H}^1_+}\frac{|\psi(t)|}{t^s}\,\mathrm{d}t} \\ &= C_{N,s}\,\frac{[\psi]_{W^{s,1}(\mathbb{R})}}{\displaystyle\int_{\mathbb{H}^1_+}\frac{|\psi(t)|}{t^s}\,\mathrm{d}t} + \frac{1}{k^s}\,\frac{[\varphi]_{W^{s,1}(\mathbb{R}^{N-1})}}{\|\varphi\|_{L^1(\mathbb{R}^{N-1})}}\,\frac{\|\psi\|_{L^1(\mathbb{H}^1_+)}\displaystyle\int_0^\infty \frac{\mathrm{d}t}{(1+t^2)^{\frac{N+s}{2}}}}{\displaystyle\int_{\mathbb{H}^1_+}\frac{|\psi(t)|}{t^s}\,\mathrm{d}t}\end{aligned}$$

for all $k \in \mathbb{N}$. The second inequality in (4.11) hence follows first by passing to the limit as $k \to \infty$ and then by taking the infimum with respect to $\psi \in C_0^\infty(\mathbb{H}^1_+)$. □

At this point, the proof of Theorem 1.1 essentially reduces to a straight computation based on (2.16) and the fact that, for every $s \in (0,1)$,

$$V_{s,(a,b)}((a,b)) = \frac{2^s}{1-s}\,(b-a)^{1-s} \tag{4.12}$$

whenever $a, b \in \mathbb{R}$ are such that $a < b$.

Proof of Theorem 1.1. For $N = 1$, owing to (2.16), (4.12) and Corollary 3.3, we infer that

$$\mathfrak{h}_{s,1}(\mathbb{H}^1_+) = \mathfrak{g}^*_s(\mathbb{H}^1_+) \le \frac{P_s((0,1))}{V_{s,\mathbb{H}^1_+}((0,1))} = \frac{4}{s}$$

for all $s \in (0,1)$. The validity of (1.15) hence follows owing to (2.18) and (4.11). Consequently, we get (1.16) by simply passing to the limit as $p \to 1^+$ in (1.13). Finally, we prove that $\mathfrak{h}_{s,1}(\mathbb{H}^1_+) = \frac{4}{s}$ is attained in $\mathcal{W}^{s,1}_0(\mathbb{H}^1_+)$. Indeed, thanks to Corollary 3.3, it is enough to check that $\mathbf{1}_{(0,1)} \in \mathcal{W}^{s,1}_0(\mathbb{H}^1_+)$. This, in turn, follows by observing that

$$\lim_{t\to 0^+} [\mathbf{1}_{(t,1)} - \mathbf{1}_{(0,1)}]_{W^{s,1}(\mathbb{R})} = \lim_{t\to 0^+} P_s((0,t)) = 0$$

by (2.16), and that $1_{(t,1)} \in \mathcal{W}^{s,1}_0(\mathbb{H}^1_+)$ for every $t \in (0,1)$, since $u_{\varepsilon,t} = 1_{(t,1)} * \varrho_\varepsilon \in C_0^\infty(\mathbb{H}^1_+)$ are converging to $1_{(t,1)}$ in $W^{s,1}(\mathbb{R})$ as $\varepsilon \to 0^+$ by [25, Th. 6.65] whenever $(\varrho_\varepsilon)_{\varepsilon>0} \subseteq C_0^\infty(\mathbb{R})$ is a family of standard non-negative mollifiers. □

As an immediate consequence of Theorem 1.1 and Lemmas 2.1 and 2.2, we get the following stability result, whose simple proof is omitted.

Corollary 4.4. *Given $N \ge 1$ and $s \in (0,1)$, it holds that*

$$\lim_{p\to 1^+} \mathfrak{h}_{s,p}(\mathbb{H}^N_+) = \mathfrak{h}_{s,1}(\mathbb{H}^N_+).$$

4.2. **Proof of Theorem 1.2.** We begin with the case $N = 1$ in Theorem 1.2.

Lemma 4.5. *It holds that* $\mathfrak{h}_{s,1}(\mathbb{B}^1) = \frac{2^{2-s}}{s}$ *for all* $s \in (0,1)$. *Moreover,* $\mathfrak{h}_{s,1}(\mathbb{B}^1)$ *is attained in* $\mathcal{W}_0^{s,1}(\mathbb{B}^1)$ *for all* $s \in (0,1)$.

Proof. Owing to (2.16), (4.12) and Corollary 3.5, we can estimate

$$\mathfrak{h}_{s,1}(\mathbb{B}^1) \leq \frac{P_s(\mathbb{B}^1)}{V_{s,\mathbb{B}^1}(\mathbb{B}^1)} = \frac{2^{2-s}}{s}$$

for all $s \in (0,1)$. To prove the converse inequality, let $E \subseteq \mathbb{B}^1$ be such that $|E| > 0$ and define $E^\star = \left(-\frac{|E|}{2}, \frac{|E|}{2}\right) \subseteq \mathbb{B}^1$ as in (2.6). By (2.14), we have that

$$P_s(E) \geq P_s(E^\star) = P_s((0,1))\,|E|^{1-s} = \frac{4}{s(1-s)}\,|E|^{1-s} \tag{4.13}$$

for all $s \in (0,1)$. On the other hand, by (2.9) and Lemma 2.4, we have that

$$V_{s,\mathbb{B}^1}(E) = \int_{\mathbb{R}} \mathbf{1}_E(x)\,\delta_s(x)\,\mathrm{d}x \leq \int_{\mathbb{R}} \mathbf{1}_{E^\star}(x)\,\delta_s^\star(x)\,\mathrm{d}x = \int_{-\frac{|E|}{2}}^{\frac{|E|}{2}} \frac{\mathrm{d}x}{|x|^s} = \frac{2^s}{(1-s)}\,|E|^{1-s}. \tag{4.14}$$

By combining (4.13) and (4.14), we thus get that

$$\frac{P_s(E)}{V_{s,\mathbb{B}^1}(E)} \geq \frac{4\,|E|^{1-s}}{s(1-s)}\,\frac{(1-s)}{2^s\,|E|^{1-s}} = \frac{2^{2-s}}{s}$$

for all $s \in (0,1)$ whenever $E \subseteq \mathbb{B}^1$ is such that $|E| > 0$, readily yielding the equality $\mathfrak{h}_{s,1}(\mathbb{B}^1) = \frac{2^{2-s}}{s}$ in virtue of Corollary 3.3. To conclude, again by Corollary 3.3, it is enough to check that $\mathbf{1}_{\mathbb{B}^1} \in \mathcal{W}_0^{s,1}(\mathbb{B}^1_+)$. As in the last part of the proof of Theorem 1.1, this simply follows by observing that

$$\lim_{t\to 0^+} [\mathbf{1}_{(-1+t,1-t)} - \mathbf{1}_{\mathbb{B}^1}]_{W^{s,1}(\mathbb{R})} = \lim_{t\to 0^+} P_s((-1,-1+t)\cup(1-t,1)) = 0$$

by (2.11) and (2.16) and that $\mathbf{1}_{(-1+t,1-t)} \in \mathcal{W}_0^{s,1}(\mathbb{B}^1)$ for every $t \in (0,1)$ by a plain convolution argument. The proof is complete. □

Remark 4.6. The strategy of the proof of Lemma 4.5 is ineffectual in the higher dimensional case $N \geq 2$. Indeed, by applying Remark 2.6 below, we analogously get that

$$\frac{P_s(E)}{V_{s,\mathbb{B}^N}(E)} \geq \frac{P_s(\mathbb{B}^N)}{|\mathbb{B}^N|^{1-\frac{s}{N}}}\,\frac{|E|^{1-\frac{s}{N}}}{\displaystyle\int_{E^\star}\left(1-\left(1-|x|^N\right)^{\frac{1}{N}}\right)^{-s}\mathrm{d}x} \tag{4.15}$$

whenever $E \subseteq \mathbb{B}^N$ is such that $|E| > 0$, where $E^\star$ is as in (2.6). However, we have that

$$\inf_{R\in(0,1]} \frac{|B_R|^{1-\frac{s}{N}}}{\displaystyle\int_{B_R}\left(1-\left(1-|x|^N\right)^{\frac{1}{N}}\right)^{-s}\mathrm{d}x} = 0,$$

so that (4.15) leads to the trivial lower bound $\mathfrak{h}_{s,1}(\mathbb{B}^N) \geq 0$.

We now pass to the case $N \geq 2$ in Theorem 1.2.

Proof of Theorem 1.2. Thanks to Lemma 4.5, we can assume $N \geq 2$. Let

$$r_\Omega = \sup_{x\in\Omega} d_\Omega(x) \in (0,\infty)$$

be the *inradius* of Ω. By [23, Th. 1.2] (see also [24]), we have that

$$P(\Omega_t) \geq \left(1 - \frac{t}{r_\Omega}\right)^{N-1} P(\Omega) \quad \text{for all } t \in (0, r_\Omega), \tag{4.16}$$

where

$$\Omega_t = \{x \in \Omega : d_\Omega(x) > t\}.$$

By scale invariance of (1.8), we can assume that $r_\Omega = 1$ without loss of generality. Since d_Ω is 1-Lipschitz, $|\nabla d_\Omega| \leq 1$ a.e. in Ω and thus, by the usual coarea formula, we have that

$$V_{s,\Omega}(\Omega) = \int_\Omega \frac{1}{d_\Omega^s}\,\mathrm{d}x \geq \int_\Omega \frac{|\nabla d_\Omega|}{d_\Omega^s}\,\mathrm{d}x = \int_0^1 \int_{\{d_\Omega = t\}} \frac{1}{t^s}\,\mathrm{d}\mathscr{H}^{N-1}\,\mathrm{d}t = \int_0^1 \frac{P(\Omega_t)}{t^s}\,\mathrm{d}t,$$

since

$$\partial\Omega_t = \{x \in \Omega : d_\Omega(x) = t\} \quad \text{and} \quad P(\Omega_t) = \mathscr{H}^{N-1}(\partial\Omega_t) \quad \text{for all } t \in (0,1).$$

Owing to (4.16), we hence get that

$$V_{s,\Omega}(\Omega) \geq \int_0^1 \frac{P(\Omega_t)}{t^s}\,\mathrm{d}t \geq P(\Omega) \int_0^1 t^{-s}\,(1-t)^{N-1}\,\mathrm{d}t = P(\Omega)\ \mathrm{B}(N, 1-s).$$

By Corollary 3.5 and (2.2), we can thus estimate

$$\mathfrak{h}_{s,1}(\Omega) \leq \frac{P_s(\Omega)}{V_{s,\Omega}(\Omega)} \leq \frac{(1-s)\,P_s(\Omega)}{P(\Omega)}\,\frac{\Gamma(N+1-s)}{\Gamma(N)\,\Gamma(2-s)} \tag{4.17}$$

and so, thanks to (1.15) and (2.17), we get that

$$\frac{\mathfrak{h}_{s,1}(\Omega)}{\mathfrak{h}_{s,1}(\mathbb{H}^N_+)} \leq \frac{(1-s)\,P_s(\Omega)}{P(\Omega)}\,\frac{\Gamma(N+1-s)}{\Gamma(N)\,\Gamma(2-s)}\,\frac{s\,\Gamma\left(\frac{N+s}{2}\right)}{4\,\pi^{\frac{N-1}{2}}\,\Gamma\left(\frac{s+1}{2}\right)}$$

for all $s \in (0,1)$. Hence, by (1.22) and (2.5), we infer that

$$\begin{aligned}\limsup_{s\to 1^-} \frac{\mathfrak{h}_{s,1}(\Omega)}{\mathfrak{h}_{s,1}(\mathbb{H}^N_+)} &\leq \lim_{s\to 1^-} \frac{(1-s)\,P_s(\Omega)}{P(\Omega)}\,\frac{\Gamma(N+1-s)}{\Gamma(N)\,\Gamma(2-s)}\,\frac{s\,\Gamma\left(\frac{N+s}{2}\right)}{4\,\pi^{\frac{N-1}{2}}\,\Gamma\left(\frac{s+1}{2}\right)} \\ &= 2\,\omega_{N-1}\,\frac{\Gamma(N)}{\Gamma(N)\,\Gamma(1)}\,\frac{\Gamma\left(\frac{N+1}{2}\right)}{4\,\pi^{\frac{N-1}{2}}\,\Gamma(1)} = \frac{1}{2}\end{aligned}$$

and the conclusion immediately follows by the first inequality in (1.16). □

Remark 4.7. Given $N \geq 2$ and a (non-empty) bounded open convex set $\Omega \subsetneq \mathbb{R}^N$, from (4.17) and (1.22) we get that

$$\limsup_{s\to 1^-} \mathfrak{h}_{s,1}(\Omega) \leq \lim_{s\to 1^-} \frac{(1-s)\,P_s(\Omega)}{P(\Omega)}\,\frac{\Gamma(N+1-s)}{\Gamma(N)\,\Gamma(2-s)} = 2\,\omega_{N-1},$$

while, thanks to Theorem 1.1 and Lemma 2.1,

$$\liminf_{s\to 1^-} \mathfrak{h}_{s,1}(\Omega) \geq \frac{1}{2} \lim_{s\to 1^-} \mathfrak{h}_{s,1}(\mathbb{H}^N_+) = 2\,C_{N,1} = 2\,\omega_{N-1}.$$

This implies that (1.18) can be equivalently reformulated as

$$\lim_{s\to 1^-} \mathfrak{h}_{s,1}(\Omega) = 2\,\omega_{N-1} < 4\,\omega_{N-1} = \lim_{s\to 1^-} \mathfrak{h}_{s,1}(\mathbb{H}^N_+).$$

Noteworthy, thanks to [33, Th. 3] (see also [34]), Corollary 3.5, Lemma 2.1, and Theorem 1.1, we also have that

$$\limsup_{s\to 0^+} s\,\mathfrak{h}_{s,1}(\Omega) \le \lim_{s\to 0^+} \frac{s\,P_s(\Omega)}{V_{s,\Omega}(\Omega)} = \frac{2\,N\,\omega_N\,|\Omega|}{|\Omega|} = 2\,N\,\omega_N = 4\,C_{N,0} = \lim_{s\to 0^+} s\,\mathfrak{h}_{s,1}(\mathbb{H}^N_+).$$

Due to (1.17), in the case $N = 1$, the above improves to

$$\lim_{s\to 0^+} s\,\mathfrak{h}_{s,1}(\Omega) = \lim_{s\to 0^+} s\,\mathfrak{h}_{s,1}(\mathbb{H}^1_+),$$

but we do not know if this is also the case for $N \ge 2$.

As already mentioned in the introduction, the limit in (1.18) can be slightly improved if Ω is an open ball by exploiting the main result of [20], as follows.

Proposition 4.8. *If $N \ge 2$, then*

$$\mathfrak{h}_{s,1}(\mathbb{B}^N) \le \frac{\pi^{\frac12}}{\Gamma\left(1-\frac{s}{2}\right)\Gamma\left(\frac12+\frac{s}{2}\right)} \frac{\Gamma\left(\frac{N+s}{2}\right)}{\Gamma\left(\frac{N-s}{2}\right)} \frac{\Gamma(N-s)}{\Gamma(N)}\,\mathfrak{h}_{s,1}(\mathbb{H}^N_+) \tag{4.18}$$

for all $s \in (0,1)$.

Proof of Proposition 4.8. Since $d_{\mathbb{B}^N}(x) = 1-|x|$ for all $x \in \mathbb{B}^N$, we can compute

$$V_{s,\mathbb{B}^N}(\mathbb{B}^N) = \int_{\mathbb{B}^N} (1-|x|)^{-s}\,\mathrm{d}x = N\omega_N\,\mathrm{B}(N,1-s) = N\omega_N\,\frac{\Gamma(N)\,\Gamma(1-s)}{\Gamma(N+1-s)}$$

for all $s \in (0,1)$. Therefore, by Corollary 3.5 and recalling (2.15) and (2.5), we get that

$$\begin{aligned}
\mathfrak{h}_{s,1}(\mathbb{B}^N) &\le \frac{P_s(\mathbb{B}^N)}{V_{s,\mathbb{B}^N}(\mathbb{B}^N)} \\
&= \omega_N^{1-\frac{s}{N}}\,\frac{N\,\pi^{\frac{N+s}{2}}\,\Gamma(1-s)}{\frac{s}{2}\,\Gamma\left(\frac{N}{2}+1\right)^{\frac{s}{N}}\,\Gamma\left(1-\frac{s}{2}\right)\,\Gamma\left(\frac{N-s}{2}+1\right)}\,\frac{\Gamma(N+1-s)}{N\omega_N\,\Gamma(N)\,\Gamma(1-s)} \\
&= \omega_N^{-\frac{s}{N}}\,\frac{\pi^{\frac{N+s}{2}}}{\frac{s}{2}\,\Gamma\left(\frac{N}{2}+1\right)^{\frac{s}{N}}\,\Gamma\left(1-\frac{s}{2}\right)\,\Gamma\left(\frac{N-s}{2}+1\right)}\,\frac{\Gamma(N+1-s)}{\Gamma(N)} \\
&= \frac{2\,\pi^{\frac{N}{2}}\,\Gamma(N+1-s)}{s\,\Gamma(N)\,\Gamma\left(1-\frac{s}{2}\right)\,\Gamma\left(\frac{N-s}{2}+1\right)}
\end{aligned}$$

for all $s \in (0,1)$. We can thus estimate

$$\begin{aligned}
\frac{\mathfrak{h}_{s,1}(\mathbb{B}^N)}{\mathfrak{h}_{s,1}(\mathbb{H}^N_+)} &\le \frac{2\,\pi^{\frac{N}{2}}\,\Gamma(N+1-s)}{s\,\Gamma(N)\,\Gamma\left(1-\frac{s}{2}\right)\,\Gamma\left(\frac{N-s}{2}+1\right)}\,\frac{s\,\Gamma\left(\frac{N+s}{2}\right)}{4\,\pi^{\frac{N-1}{2}}\,\Gamma\left(\frac{s+1}{2}\right)} \\
&= \frac{\pi^{\frac12}}{\Gamma\left(1-\frac{s}{2}\right)\,\Gamma\left(\frac12+\frac{s}{2}\right)}\,\frac{\Gamma\left(\frac{N+s}{2}\right)}{\Gamma\left(\frac{N-s}{2}\right)}\,\frac{\Gamma(N-s)}{\Gamma(N)}
\end{aligned}$$

for all $s \in (0,1)$, proving (4.18) and concluding the proof. □

4.3. **Proof of Corollary 1.3.** We can now study the sharp constant in (1.8) for $s \in (0,1)$ and $p \in (1,\infty)$ such that $sp < 1$.

Proof of Corollary 1.3. By (1.13), Theorem 1.1, (4.3) and Corollary 4.4, we get that

$$\frac{1}{2} \le \limsup_{p\to 1^+} \frac{\mathfrak{h}_{s,p}(\Omega)}{\mathfrak{h}_{s,p}(\mathbb{H}^N_+)} \le \frac{\mathfrak{h}_{s,1}(\Omega)}{\mathfrak{h}_{s,1}(\mathbb{H}^N_+)}$$

for all $s \in (0,1)$. The conclusion hence plainly follows from Theorem 1.2. □

4.4. **Proof of Corollary 1.5.** We now pass to the study of the non-convex case $\Omega = \mathbb{R}^N \setminus \{0\}$, (re)proving [19, Th. 1.1] in the case $p = 1$.

Proof of Corollary 1.5. Let us set $\mathbb{U}^N = \mathbb{R}^N \setminus \{0\}$ for brevity. Since

$$V_{s,\mathbb{U}^N}(\mathbb{B}^N) = \int_{\mathbb{B}^N} \frac{\mathrm{d}x}{|x|^s} = N\,\omega_N \int_0^1 r^{N-1-s}\,\mathrm{d}r = \frac{N\,\omega_N}{N-s},$$

by Corollary 3.7 (applied with $U = \mathbb{R}^N$ and $F = \{0\}$) we can thus estimate

$$\mathfrak{g}_s(\mathbb{U}^N) \le \frac{P_s(\mathbb{B}^N)}{V_{s,\mathbb{U}^N}(\mathbb{B}^N)} = P_s(\mathbb{B}^N)\,\frac{N-s}{N\,\omega_N} \tag{4.19}$$

for all $s \in (0,1)$. On the other hand, given a measurable set $E \subseteq \mathbb{R}^N$ such that $|E| \in (0,\infty)$, for all $s \in (0,1)$, by (2.14), we can estimate

$$P_s(E) \ge \frac{P_s(\mathbb{B}^N)}{|\mathbb{B}^N|^{1-\frac{s}{N}}}\,|E|^{1-\frac{s}{N}} = \frac{P_s(\mathbb{B}^N)}{\omega_N^{1-\frac{s}{N}}}\,|E|^{1-\frac{s}{N}},$$

while, by (2.9) and owing to the fact that $x \mapsto |x|^{-s}$ is radially symmetric decreasing,

$$V_{s,\mathbb{U}^N}(E) = \int_E \frac{1}{|x|^s}\,\mathrm{d}x \le \int_{E^\star} \frac{1}{|x|^s}\,\mathrm{d}x = \frac{N\,\omega_N}{N-s}\left(\frac{|E|}{\omega_N}\right)^{1-\frac{s}{N}}$$

where E^* is as in (2.6). Thus, we get that

$$\frac{P_s(E)}{V_{s,\mathbb{U}^N}(E)} \ge \frac{P_s(\mathbb{B}^N)}{\omega_N^{1-\frac{s}{N}}}\,|E|^{1-\frac{s}{N}}\,\frac{N-s}{N\,\omega_N}\left(\frac{\omega_N}{|E|}\right)^{1-\frac{s}{N}} = P_s(\mathbb{B}^N)\,\frac{N-s}{N\,\omega_N} \tag{4.20}$$

for all $s \in (0,1)$, whenever $E \subseteq \mathbb{R}^N$ is such that $|E| \in (0,\infty)$. By combining (4.19) and (4.20), by [20, Prop. 1.1] (see (2.15)), and recalling (2.5), we conclude that

$$\mathfrak{g}_s(\mathbb{U}^N) = P_s(\mathbb{B}^N)\,\frac{N-s}{N\,\omega_N} = \frac{4}{s}\,\frac{\pi^{\frac{N}{2}}\,\Gamma(1-s)}{\Gamma\left(\frac{N-s}{2}\right)\,\Gamma\left(1-\frac{s}{2}\right)}$$

for all $s \in (0,1)$, which yields (1.24) in virtue of Theorem 1.4. The characterization of the equality cases in (1.23) follows as in [19], so we omit the details. □

4.5. **Proof of Theorem 1.6.** For the proof of Theorem 1.6, we need to establish the following simple preliminary result.

Lemma 4.9. *Let $N = 1$ and $\Omega \subsetneq \mathbb{R}$ be the union of $n \in \mathbb{N}$ (non-empty) disjoint bounded open intervals $\{I_k : k = 1, \dots, n\}$. If Ω is equivalent to an interval, then*

$$\mathfrak{h}_{s,1}(\Omega) \leq \frac{\left(\sum_{k=1}^n |I_k|\right)^{1-s}}{\sum_{k=1}^n |I_k|^{1-s}}\, \mathfrak{h}_{s,1}(\mathbb{R} \setminus \{0\}) \tag{4.21}$$

for every $s \in (0,1)$. If, in addition,

$$\mathfrak{h}_{s,1}(\Omega) \geq \frac{\mathfrak{h}_{s,1}(\mathbb{R} \setminus \{0\})}{n^s} \tag{4.22}$$

then all the I_k's have the same measure.

Proof. Since Ω is equivalent to an interval, by (2.16), we have

$$P_s(\Omega) = \frac{4}{s(1-s)}\, |\Omega|^{1-s} = \frac{4}{s(1-s)} \left(\sum_{k=1}^n |I_k|\right)^{1-s}$$

while, since the I_k's are pairwise disjoint, by (4.12) we can write

$$V_{s,\Omega}(\Omega) = \sum_{k=1}^n V_{s,\Omega}(I_k) = \sum_{k=1}^n V_{s,I_k}(I_k) = \frac{2^s}{1-s} \sum_{k=1}^n |I_k|^{1-s}$$

for all $s \in (0,1)$. Therefore, inequality (4.21) follows from Corollary 3.3 and (1.25). Finally, if (4.22) additionally holds, then from (4.21) we deduce that

$$\frac{1}{n} \sum_{k=1}^n |I_k|^{1-s} \leq \left(\frac{1}{n} \sum_{k=1}^n |I_k|\right)^{1-s}$$

which implies that the $|I_k|$'s are all equal, since $t \mapsto t^{1-s}$ is strictly concave on $(0, \infty)$. □

Proof of Theorem 1.6. We split the proof into two parts.

Proof of (1.26). By combining (2.10) with (2.31) in Corollary 2.7, we get that

$$\frac{[u]^p_{W^{s,p}(\mathbb{R})}}{\displaystyle\int_\Omega \frac{|u|^p}{d_\Omega^{sp}}\, \mathrm{d}x} \geq \frac{[u^\star]^p_{W^{s,p}(\mathbb{R})}}{n^{sp} \displaystyle\int_{\mathbb{R}} \frac{(u^\star)^p}{|x|^{sp}}\, \mathrm{d}x} \geq \frac{\mathfrak{h}_{s,p}\left(\mathbb{R}^N \setminus \{0\}\right)}{n^{sp}}$$

owing to Corollary 1.5, from which we deduce (1.26). Moreover, if $p = 1$ and Ω is equivalent to an interval, then Lemma 4.9 proves that (1.26) holds as an equality if and only if all the I_k's have the same measure.

Proof of (1.27). By Corollaries 3.2 and 3.3, for any $R \geq 1$, we can estimate

$$\mathfrak{h}_{s,p}((-R,R) \setminus \mathbb{Z}) \leq \mathfrak{h}_{sp,1}((-R,R) \setminus \mathbb{Z}) = \mathfrak{g}_{sp}((-R,R) \setminus \mathbb{Z}). \tag{4.23}$$

Now, since $|\mathbb{Z}| = 0$, we have that

$$P_{sp}((-R,R) \setminus \mathbb{Z}) = P_{sp}((-R,R)) = (2R)^{1-sp}\, \frac{4}{sp\,(1-sp)}, \tag{4.24}$$

thanks to (2.16) (with $sp < 1$ in place of s), and also

$$\begin{aligned} V_{sp,(-R,R)\setminus\mathbb{Z}}((-R,R)\setminus\mathbb{Z}) &\geq V_{sp,(-R,R)\setminus\mathbb{Z}}((-\lfloor R\rfloor,\lfloor R\rfloor)) \\ &= V_{sp,(-\lfloor R\rfloor,\lfloor R\rfloor)\setminus\mathbb{Z}}((-\lfloor R\rfloor,\lfloor R\rfloor)) \\ &= V_{sp,(-\lfloor R\rfloor,\lfloor R\rfloor)}((-\lfloor R\rfloor,\lfloor R\rfloor)) = \lfloor R\rfloor\,\frac{2^{1+sp}}{1-sp} \end{aligned} \tag{4.25}$$

thanks to (4.12). Hence, by combining (4.23), (4.24) and (4.25), we infer that

$$\mathfrak{h}_{s,p}((-R,R)\setminus\mathbb{Z}) \leq \mathfrak{g}_{sp}((-R,R)\setminus\mathbb{Z}) \leq \frac{P_{sp}((-R,R)\setminus\mathbb{Z})}{V_{sp,(-R,R)\setminus\mathbb{Z}}((-R,R)\setminus\mathbb{Z})} = \frac{R^{1-sp}}{\lfloor R\rfloor}\,\frac{4^{1-sp}}{sp}$$

proving (1.27) for any $R \geq 1$. For the limit case $R = \infty$, we can analogously bound

$$\mathfrak{h}_{s,p}(\mathbb{R}\setminus\mathbb{Z}) \leq \mathfrak{h}_{sp,1}(\mathbb{R}\setminus\mathbb{Z}) = \mathfrak{g}_{sp}(\mathbb{R}\setminus\mathbb{Z}) \leq \mathfrak{g}_{sp}((-m,m)\setminus\mathbb{Z}) \leq \frac{1}{m^{sp}}\,\frac{4^{1-sp}}{sp}$$

for all $m \in \mathbb{N}$, because $V_{s,\mathbb{R}\setminus\mathbb{Z}}(E) = V_{s,(-m,m)\setminus\mathbb{Z}}(E)$ for any $E \subset (-m,m)\setminus\mathbb{Z}$. We hence get that $\mathfrak{h}_{s,p}(\mathbb{R}\setminus\mathbb{Z}) = 0$, concluding the proof. □

4.6. **Proof of Theorem 1.7.** We now turn to the proof of Theorem 1.7—actually, of the following more precise result. Here and below, given an non-empty set $A \subseteq \mathbb{R}$ and $\delta \in (0,\infty)$, we let $A^\delta = \{x \in \mathbb{R} : \operatorname{dist}(x,A) < \delta\}$ be the *open δ-neighborhood* of A.

Theorem 4.10. *Let $\Omega \subsetneq \mathbb{R}$, $n \in \mathbb{N}$ and $\ell, \delta \in (0,\infty)$ be as in Theorem 1.7. For $s \in (0,1)$ and $p \in [1,\infty)$ such that $sp \leq 1$, it holds that*

$$[u]^p_{W^{s,p}(\Omega^\delta)} \geq \frac{2}{sp}\left(1 - \left(\frac{\ell}{\ell+\delta}\right)^{sp}\right)\int_\Omega \frac{|u|^p}{d_\Omega^{sp}}\,\mathrm{d}x$$

for all $u \in C_0^\infty(\Omega)$.

To prove Theorem 4.10, we need the following preliminary result; that is, Theorem 4.10 in the case of a single interval.

Proposition 4.11. *Let $I \subsetneq \mathbb{R}$ be a non-empty bounded open interval with $|I| = \ell \in (0,\infty)$. For $s \in (0,1)$ and $p \in [1,\infty)$ such that $sp \leq 1$, it holds that*

$$[u]^p_{W^{s,p}(I^\delta)} \geq \frac{2^{2-sp}}{sp}\left(1 - \left(\frac{\ell}{\ell+\delta}\right)^{sp}\right)\int_I \frac{|u|^p}{d_I^{sp}}\,\mathrm{d}x$$

for any $\delta \in (0,\infty)$ and $u \in C_0^\infty(I)$.

Proof. For convenience, let $a, b \in \mathbb{R}$, $a < b$, be such that $I = (a,b)$. We let $\varphi \in \operatorname{Lip}(\mathbb{R})$ be a Lipschitz function such that $\operatorname{supp}\varphi \subseteq I$ and $\varphi \geq 0$. For any $\varepsilon > 0$, we define

$$\triangle_\varepsilon = \{(x,y) \in \mathbb{R}^2 : |x-y| \leq \varepsilon\}, \tag{4.26}$$

and we let $J_p \colon \mathbb{R} \to \mathbb{R}$ be given by

$$J_p(t) = \begin{cases} |t|^{p-2}\,t & \text{for } p \in (1,\infty), \\ \operatorname{sgn}(t) & \text{for } p = 1, \end{cases}$$

for all $t \neq 0$, and $J_p(0) = 0$. With this notation in force, we observe that

$$\begin{aligned}\limsup_{\varepsilon\to 0^+} &\left| \iint_{(I^\delta\times I^\delta)\cap\triangle_\varepsilon} \frac{J_p(\mathbf{1}_I(x)-\mathbf{1}_I(y))}{|x-y|^{1+sp}}\,(\varphi(x)-\varphi(y))\,\mathrm{d}x\,\mathrm{d}y\right| \\ &\leq \limsup_{\varepsilon\to 0^+} \iint_{(I^\delta\times I^\delta)\cap\triangle_\varepsilon} \frac{|\mathbf{1}_I(x)-\mathbf{1}_I(y)|^{p-1}}{|x-y|^{1+sp}}\,|\varphi(x)-\varphi(y)|\,\mathrm{d}x\,\mathrm{d}y \\ &\leq \limsup_{\varepsilon\to 0^+} \iint_{\triangle_\varepsilon} \frac{|\varphi(x)-\varphi(y)|}{|x-y|^{1+sp}}\,\mathrm{d}x\,\mathrm{d}y = 0\end{aligned}$$

by the Dominated Convergence Theorem, since $\varphi \in W^{sp,1}(\mathbb{R})$. Therefore, we can write

$$\begin{aligned}&\iint_{I^\delta\times I^\delta} \frac{J_p(\mathbf{1}_I(x)-\mathbf{1}_I(y))}{|x-y|^{1+sp}}\,(\varphi(x)-\varphi(y))\,\mathrm{d}x\,\mathrm{d}y \\ &\qquad = \lim_{\varepsilon\to 0^+} \iint_{(I^\delta\times I^\delta)\setminus\triangle_\varepsilon} \frac{J_p(\mathbf{1}_I(x)-\mathbf{1}_I(y))}{|x-y|^{1+sp}}\,(\varphi(x)-\varphi(y))\,\mathrm{d}x\,\mathrm{d}y.\end{aligned} \tag{4.27}$$

Now, given $\varepsilon > 0$, we can rewrite

$$\begin{aligned}&\iint_{(I^\delta\times I^\delta)\setminus\triangle_\varepsilon} \frac{J_p(\mathbf{1}_I(x)-\mathbf{1}_I(y))}{|x-y|^{1+sp}}\,(\varphi(x)-\varphi(y))\,\mathrm{d}x\,\mathrm{d}y \\ &\qquad = 2\iint_{(I^\delta\times I^\delta)\setminus\triangle_\varepsilon} \frac{J_p(\mathbf{1}_I(x)-\mathbf{1}_I(y))}{|x-y|^{1+sp}}\,\varphi(x)\,\mathrm{d}x\,\mathrm{d}y \\ &\qquad = 2\int_I \varphi(x) \int_{\{y\in I^\delta\setminus I\,:\,|y-x|>\varepsilon\}} \frac{\mathrm{d}y}{|y-x|^{1+sp}}\,\mathrm{d}x.\end{aligned} \tag{4.28}$$

Observing that

$$\frac{1}{(x-a+\delta)^{sp}} \leq \frac{C_\delta^{sp}}{(x-a)^{sp}} \quad \text{and} \quad \frac{1}{(b+\delta-x)^{sp}} \leq \frac{C_\delta^{sp}}{(b-x)^{sp}}$$

for all $x \in I$, where

$$C_\delta = \frac{b-a}{b-a+\delta} \in (0,1),$$

we can estimate

$$\begin{aligned}&\lim_{\varepsilon\to 0^+} \int_{\{y\in I^\delta\setminus I:|y-x|>\varepsilon\}} \frac{\mathrm{d}y}{|y-x|^{1+sp}} \\ &\qquad = \frac{1}{sp}\left(\frac{1}{(x-a)^{sp}} - \frac{1}{(x-a+\delta)^{sp}} + \frac{1}{(b-x)^{sp}} - \frac{1}{(b+\delta-x)^{sp}}\right) \\ &\qquad \geq \frac{(1-C_\delta^{sp})}{sp}\left(\frac{1}{(x-a)^{sp}} + \frac{1}{(b-x)^{sp}}\right)\end{aligned} \tag{4.29}$$

for all $x \in I$. Since $sp \leq 1$, for each $x \in I$, we further have that

$$\frac{1}{(x-a)^{sp}} + \frac{1}{(b-x)^{sp}} = \frac{(x-a)^{sp}+(b-x)^{sp}}{(x-a)^{sp}\,(b-x)^{sp}} \geq \frac{2^{1-sp}\,(b-a)^{sp}}{(x-a)^{sp}\,(b-x)^{sp}} \geq \frac{2^{1-sp}}{d_I(x)^{sp}} \tag{4.30}$$

and thus, owing to (4.27), (4.28), Fatou's Lemma (recall that $\varphi \geq 0$), (4.29) and (4.30),

$$\begin{aligned}
&\iint_{I^\delta \times I^\delta} \frac{J_p(\mathbf{1}_I(x) - \mathbf{1}_I(y))}{|x-y|^{1+sp}} \left(\varphi(x) - \varphi(y)\right) \mathrm{d}x \, \mathrm{d}y \\
&\quad = \lim_{\varepsilon \to 0^+} \iint_{(I^\delta \times I^\delta) \setminus \triangle_\varepsilon} \frac{J_p(\mathbf{1}_I(x) - \mathbf{1}_I(y))}{|x-y|^{1+sp}} \left(\varphi(x) - \varphi(y)\right) \mathrm{d}x \, \mathrm{d}y \\
&\quad = \lim_{\varepsilon \to 0^+} 2 \int_I \varphi(x) \int_{\{y \in I^\delta \setminus I \,:\, |y-x| > \varepsilon\}} \frac{\mathrm{d}y}{|y-x|^{1+sp}} \, \mathrm{d}x \geq \frac{2^{2-sp}\left(1 - C_\delta^{sp}\right)}{sp} \int_I \frac{\varphi}{d_I^{sp}} \, \mathrm{d}x.
\end{aligned} \tag{4.31}$$

We now choose the Lipschitz function

$$\varphi = \frac{|u|^p}{(\varepsilon + \mathbf{1}_I)^{p-1}},$$

where $u \in C_0^\infty(I)$. Noticing that, by [3, Lem. 2.4],

$$J_p(\mathbf{1}_I(x) - \mathbf{1}_I(y)) \left(\frac{|u(x)|^p}{(\varepsilon + \mathbf{1}_I(x))^{p-1}} - \frac{|u(y)|^p}{(\varepsilon + \mathbf{1}_I(y))^{p-1}} \right) \leq |u(x) - u(y)|^p$$

for all $x, y \in I^\delta$ and $\varepsilon > 0$, owing to (4.31), we conclude that

$$\begin{aligned}
[u]^p_{W^{s,p}(I^\delta)} &\geq \liminf_{\varepsilon \to 0^+} \iint_{I^\delta \times I^\delta} \frac{J_p(\mathbf{1}_I(x) - \mathbf{1}_I(y))}{|x-y|^{1+sp}} \left(\frac{|u(x)|^p}{(\varepsilon + \mathbf{1}_I(x))^{p-1}} - \frac{|u(y)|^p}{(\varepsilon + \mathbf{1}_I(y))^{p-1}} \right) \mathrm{d}x \, \mathrm{d}y \\
&\geq \liminf_{\varepsilon \to 0^+} \frac{2^{2-sp}\left(1 - C_\delta^{sp}\right)}{sp} \int_I \frac{|u|^p}{d_I^{sp}(\varepsilon + \mathbf{1}_I)^{p-1}} \, \mathrm{d}x \geq \frac{2^{2-sp}\left(1 - C_\delta^{sp}\right)}{sp} \int_I \frac{|u|^p}{d_I^{sp}} \, \mathrm{d}x,
\end{aligned}$$

and the proof is complete. □

We can now prove Theorem 4.10, and thus Theorem 1.7, by exploiting Proposition 4.11.

Proof of Theorem 4.10. Let I_i, $i = 1, \dots, n$, be the n intervals composing Ω; that is, $\Omega = \bigcup_{i=1}^n I_i$ with disjoint union. By assumption, the δ-enlarged intervals I_i^δ, $i = 1, \dots, n$, are pairwise disjoint. Thus, thanks to Proposition 4.11, we can estimate

$$\begin{aligned}
[u]^p_{W^{s,p}(\Omega^\delta)} &\geq \sum_{i=1}^n [u]^p_{W^{s,p}(I_i^\delta)} \geq \sum_{i=1}^n \frac{2^{2-sp}}{sp} \left(1 - \left(\frac{|I_i|}{|I_i| + \delta}\right)^{sp}\right) \int_{I_i} \frac{|u|^p}{d_{I_i}^{sp}} \, \mathrm{d}x \\
&\geq \frac{2^{2-sp}}{sp} \left(1 - \left(\frac{\ell}{\ell + \delta}\right)^{sp}\right) \sum_{i=1}^n \int_{I_i} \frac{|u|^p}{d_{I_i}^{sp}} \, \mathrm{d}x = \frac{2^{2-sp}}{sp} \left(1 - \left(\frac{\ell}{\ell + \delta}\right)^{sp}\right) \int_\Omega \frac{|u|^p}{d_\Omega^{sp}} \, \mathrm{d}x
\end{aligned}$$

whenever $u \in C_0^\infty(\Omega)$, concluding the proof. □

Remark 4.12. Concerning the constant appearing in Theorem 4.10 and Proposition 4.11, we observe that

$$\lim_{\delta \to 0^+} \frac{2^{2-sp}}{sp} \left(1 - \left(\frac{\ell}{\ell + \delta}\right)^{sp}\right) = 0,$$

in accordance with the fact that the *regional* fractional Hardy inequality (1.29) cannot hold for $sp \leq 1$ on bounded intervals, recall Remark 1.8.

References

[1] F. J. Almgren Jr. and E. H. Lieb, *Symmetric decreasing rearrangement is sometimes continuous*, J. Amer. Math. Soc. **2** (1989), no. 4, 683–773, DOI 10.2307/1990893. MR1002633

[2] K. Bessas and G. Stefani, *Non-local BV functions and a denoising model with L^1 fidelity*, Adv. Calc. Var. **18** (2025), no. 1, 189–217, DOI 10.1515/acv-2023-0082. MR4845992

[3] F. Bianchi, L. Brasco, and A. C. Zagati, *On the sharp Hardy inequality in Sobolev-Slobodeckiĭ spaces*, Math. Ann. **390** (2024), no. 1, 493–555, DOI 10.1007/s00208-023-02770-z. MR4800921

[4] K. Bogdan and B. Dyda, *The best constant in a fractional Hardy inequality*, Math. Nachr. **284** (2011), no. 5-6, 629–638, DOI 10.1002/mana.200810109. MR2663757

[5] L. Brasco and E. Cinti, *On fractional Hardy inequalities in convex sets*, Discrete Contin. Dyn. Syst. **38** (2018), no. 8, 4019–4040, DOI 10.3934/dcds.2018175. MR3814363

[6] L. Brasco, D. Gómez-Castro, and J. L. Vázquez, *Characterisation of homogeneous fractional Sobolev spaces*, Calc. Var. Partial Differential Equations **60** (2021), no. 2, Paper No. 60, 40, DOI 10.1007/s00526-021-01934-6. MR4225499

[7] L. Brasco, E. Lindgren, and E. Parini, *The fractional Cheeger problem*, Interfaces Free Bound. **16** (2014), no. 3, 419–458, DOI 10.4171/IFB/325. MR3264796

[8] E. Cinti and F. Prinari, *On fractional Hardy-type inequalities in general open sets*, ESAIM Control Optim. Calc. Var. **30** (2024), Paper No. 77, 26, DOI 10.1051/cocv/2024066. MR4805757

[9] J. Dávila, *On an open question about functions of bounded variation*, Calc. Var. Partial Differential Equations **15** (2002), no. 4, 519–527, DOI 10.1007/s005260100135. MR1942130

[10] B. Dyda, *A fractional order Hardy inequality*, Illinois J. Math. **48** (2004), no. 2, 575–588. MR2085428

[11] B. Dyda and M. Kijaczko, *On density of compactly supported smooth functions in fractional Sobolev spaces*, Ann. Mat. Pura Appl. (4) **201** (2022), no. 4, 1855–1867, DOI 10.1007/s10231-021-01181-8. MR4454384

[12] B. Dyda and A. V. Vähäkangas, *A framework for fractional Hardy inequalities*, Ann. Acad. Sci. Fenn. Math. **39** (2014), no. 2, 675–689, DOI 10.5186/aasfm.2014.3943. MR3237044

[13] ———, *Characterizations for fractional Hardy inequality*, Adv. Calc. Var. **8** (2015), no. 2, 173–182, DOI 10.1515/acv-2013-0019. MR3331699

[14] D. E. Edmunds, R. Hurri-Syrjänen, and A. V. Vähäkangas, *Fractional Hardy-type inequalities in domains with uniformly fat complement*, Proc. Amer. Math. Soc. **142** (2014), no. 3, 897–907, DOI 10.1090/S0002-9939-2013-11818-6. MR3148524

[15] S. Filippas, L. Moschini, and A. Tertikas, *Sharp trace Hardy-Sobolev-Maz'ya inequalities and the fractional Laplacian*, Arch. Ration. Mech. Anal. **208** (2013), no. 1, 109–161, DOI 10.1007/s00205-012-0594-4. MR3021545

[16] ———, *Correction to: Sharp trace Hardy-Sobolev-Maz'ya inequalities and the fractional Laplacian [MR3021545]*, Arch. Ration. Mech. Anal. **229** (2018), no. 3, 1281–1286, DOI 10.1007/s00205-018-1264-y. MR3814603

[17] V. Franceschi, A. Pinamonti, G. Saracco, and G. Stefani, *The Cheeger problem in abstract measure spaces*, J. Lond. Math. Soc. (2) **109** (2024), no. 1, Paper No. e12840, 55, DOI 10.1112/jlms.12840. MR4754430

[18] R. L. Frank and R. Seiringer, *Sharp fractional Hardy inequalities in half-spaces*, Around the research of Vladimir Maz'ya. I, Int. Math. Ser. (N. Y.), vol. 11, Springer, New York, 2010, pp. 161–167, DOI 10.1007/978-1-4419-1341-8_6. MR2723817

[19] ———, *Non-linear ground state representations and sharp Hardy inequalities*, J. Funct. Anal. **255** (2008), no. 12, 3407–3430, DOI 10.1016/j.jfa.2008.05.015. MR2469027

[20] N. Garofalo, *On the best constant in the nonlocal isoperimetric inequality of Almgren and Lieb*, Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl. **31** (2020), no. 2, 465–470, DOI 10.4171/rlm/900. MR4120279

[21] G. H. Hardy, *Note on a theorem of Hilbert*, Math. Z. **6** (1920), no. 3-4, 314–317, DOI 10.1007/BF01199965. MR1544414

[22] ———, *An inequality between integrals*, Messenger of Math. **54** (1925), 150–156.

[23] S. Larson, *A bound for the perimeter of inner parallel bodies*, J. Funct. Anal. **271** (2016), no. 3, 610–619, DOI 10.1016/j.jfa.2016.02.022. MR3506959
[24] ———, *Corrigendum to "A bound for the perimeter of inner parallel bodies" [J. Funct. Anal. 271 (3) (2016) 610–619]*, J. Funct. Anal. **279** (2020), no. 5, 108574, 2, DOI 10.1016/j.jfa.2020.108574. MR4097280
[25] G. Leoni, *A first course in fractional Sobolev spaces*, Graduate Studies in Mathematics, vol. 229, American Mathematical Society, Providence, RI, 2023. MR4567945
[26] E. H. Lieb and M. Loss, *Analysis*, 2nd ed., Graduate Studies in Mathematics, vol. 14, American Mathematical Society, Providence, RI, 2001. MR1817225
[27] L. Lombardini, *Fractional perimeters from a fractal perspective*, Adv. Nonlinear Stud. **19** (2019), no. 1, 165–196, DOI 10.1515/ans-2018-2016. MR3912427
[28] M. Loss and C. Sloane, *Hardy inequalities for fractional integrals on general domains*, J. Funct. Anal. **259** (2010), no. 6, 1369–1379, DOI 10.1016/j.jfa.2010.05.001. MR2659764
[29] F. Maggi, *Sets of finite perimeter and geometric variational problems*, Cambridge Studies in Advanced Mathematics, vol. 135, Cambridge University Press, Cambridge, 2012. MR2976521
[30] M. Marcus, V. J. Mizel, and Y. Pinchover, *On the best constant for Hardy's inequality in* $\mathbf{R}^n$, Trans. Amer. Math. Soc. **350** (1998), no. 8, 3237–3255, DOI 10.1090/S0002-9947-98-02122-9. MR1458330
[31] T. Matskewich and P. E. Sobolevskii, *The best possible constant in generalized Hardy's inequality for convex domain in* $\mathbf{R}^n$, Nonlinear Anal. **28** (1997), no. 9, 1601–1610, DOI 10.1016/S0362-546X(96)00004-1. MR1431208
[32] V. Maz'ya, *Sobolev spaces with applications to elliptic partial differential equations*, augmented edition, Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 342, Springer, Heidelberg, 2011. MR2777530
[33] V. Maz'ya and T. Shaposhnikova, *On the Bourgain, Brezis, and Mironescu theorem concerning limiting embeddings of fractional Sobolev spaces*, J. Funct. Anal. **195** (2002), no. 2, 230–238, DOI 10.1006/jfan.2002.3955. MR1940355
[34] ———, *Erratum to: "On the Bourgain, Brezis and Mironescu theorem concerning limiting embeddings of fractional Sobolev spaces" [J. Funct. Anal.* **195** *(2002), no. 2, 230–238; MR1940355 (2003j:46051)]*, J. Funct. Anal. **201** (2003), no. 1, 298–300, DOI 10.1016/S0022-1236(03)00002-8. MR1986163
[35] B. Opic and A. Kufner, *Hardy-type inequalities*, Pitman Research Notes in Mathematics Series, vol. 219, Longman Scientific & Technical, Harlow, 1990. MR1069756
[36] F. Sk, *Characterization of fractional Sobolev-Poincaré and (localized) Hardy inequalities*, J. Geom. Anal. **33** (2023), no. 7, Paper No. 223, 20, DOI 10.1007/s12220-023-01293-y. MR4581748
[37] A. Visintin, *Generalized coarea formula and fractal sets*, Japan J. Indust. Appl. Math. **8** (1991), no. 2, 175–201, DOI 10.1007/BF03167679. MR1111612

(F. Bianchi) Dipartimento di Scienze Matematiche, Fisiche e Informatiche, Università degli Studi di Parma, Parco Area delle Scienze 53/A (Campus Scienze e Tecnologie) 43124 Parma (PR), Italy
Email address: francesca.bianchi@unipr.it

(G. Stefani) Dipartimento di Matematica "Tullio Levi-Civita", Università degli Studi di Padova, Via Trieste 63, 35121 Padova (PD), Italy
Email address: giorgio.stefani@unipd.it

(A. C. Zagati) Dipartimento di Scienze Matematiche, Fisiche e Informatiche, Università degli Studi di Parma, Parco Area delle Scienze 53/A (Campus Scienze e Tecnologie) 43124 Parma (PR), Italy
Email address: annachiara.zagati@unipr.it