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Regularity results for solutions to some classes of strongly degenerate elliptic and parabolic problems

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Introduction

The aim of this thesis is to describe the regularity results that I obtained during my PhD journey for the gradient of solutions to some classes of strongly degenerate elliptic and parabolic problems.

Regarding the elliptic problem in question, this arises as the Euler-Lagrange equation of an integral functional of the Calculus of Variations. The energy density of this functional satisfies standard p-growth and p-ellipticity conditions with respect to the gradient variable, but only outside a ball with radius $\lambda > 0$ centered at the origin.

More precisely, after introducing some notation and preliminary results in Chapter 1, in Chapter 2 we consider the following pair of variational problems in duality:

$$\inf_{\sigma \in L^{p'}(\Omega,\mathbb{R}^n)} \left\{ \int_\Omega \mathcal{H}(\sigma(x))\,dx : -\operatorname{div}\sigma = f,\ \langle \sigma, \nu\rangle = 0 \text{ on } \partial\Omega \right\} \tag{P1}$$

and

$$\sup_{u \in W^{1,p}(\Omega)} \left\{ \int_\Omega u(x) f(x)\,dx - \int_\Omega \mathcal{H}^*(Du(x))\,dx \right\}, \tag{P2}$$

where Ω is a bounded connected open subset of $\mathbb{R}^n$ ($n \geq 2$) with Lipschitz boundary, $p \in (1, +\infty)$, $p' := p/(p-1)$, f is a given function defined over $\overline{\Omega}$ with zero mean (i.e. $\int_\Omega f\,dx = 0$), ν denotes the outer normal versor of $\partial\Omega$ and the function $\mathcal{H}$ is defined by

$$\mathcal{H}(\sigma) := \frac{1}{p'}\,|\sigma|^{p'} + \lambda\,|\sigma|\,, \qquad \sigma \in \mathbb{R}^n.$$

With such a choice, we get

$$D\mathcal{H}^*(z) = (|z| - \lambda)_+^{p-1}\,\frac{z}{|z|}\,, \qquad z \in \mathbb{R}^n,$$

where $\mathcal{H}^*$ is the Legendre transform of $\mathcal{H}$ and $(\,\cdot\,)_+$ stands for the positive part (see [22]). It is well known that the Euler-Lagrange equation of the functional in (P2) is given by the PDE

$$\begin{cases} -\operatorname{div}\left(D\mathcal{H}^*(Du)\right) = f & \text{in } \Omega, \\ \quad \langle D\mathcal{H}^*(Du), \nu\rangle = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

which has to be meant in the distributional sense. If we assume that $f \in L^{p'}(\Omega)$ and that the infimum in (P1) is finite, then problem (P1) consists in minimizing a strictly convex and coercive functional on $L^{p'}$ subject to a convex and closed constraint: therefore, a solution σ_0 to (P1) exists and must be unique. Moreover, we recall that by standard convex duality (see [43] for example), the values $\inf$ (P1) and $\sup$ (P2) coincide and the primal-dual optimality

condition characterizes the minimizer σ_0 of (P1) through the equality

$$\sigma_0(x) = D\mathcal{H}^*(Du_0(x)) \quad \text{for a.e. } x \in \Omega, \tag{2}$$

where u_0 is a solution of (P2). This is equivalent to the requirement that u_0 is a weak solution of the Euler-Lagrange equation (1), in the sense that

$$\int_\Omega \langle D\mathcal{H}^*(Du_0(x)), D\varphi(x)\rangle \, dx = \int_\Omega f(x)\varphi(x)\, dx, \quad \text{for every } \varphi \in W^{1,p}(\Omega).$$

Furthermore, since f has zero mean, using the direct methods of the Calculus of Variations it is not difficult to show that the dual problem (P2) admits at least one solution u_0 belonging to

$$W^{1,p}_\diamond(\Omega) := \left\{ u \in W^{1,p}(\Omega) : \int_\Omega u(x)\, dx = 0 \right\} \tag{3}$$

and such that (2) holds, so that u_0 is a distributional solution of the strongly degenerate or singular elliptic equation

$$-\operatorname{div}\left((|Du| - \lambda)_+^{p-1} \frac{Du}{|Du|} \right) = f \quad \text{in } \Omega, \tag{4}$$

under homogeneous Neumann boundary conditions. We also note that, in general, if one looks at the solutions u of the above equation, no more than $C^{0,1}$ regularity should be expected for them: indeed, every λ-Lipschitz function u is a solution of equation (4) with $f = 0$. Moreover, when $p \geq 2$ we have

$$\frac{(|Du| - \lambda)_+^{p-1}}{|Du|} |\xi|^2 \leq \langle D^2\mathcal{H}^*(Du)\, \xi, \xi\rangle \leq (p-1)(|Du| - \lambda)_+^{p-2} |\xi|^2 \quad \text{for every } \xi \in \mathbb{R}^n,$$

that is, the Hessian matrix $D^2\mathcal{H}^*(Du)$ has eigenvalues becoming degenerate in the region $\{|Du| \leq \lambda\}$.

The regularity properties of solutions to PDEs as the one in (4) have been widely investigated. As already said, equation (4) is the Euler-Lagrange equation of the functional appearing in (P2), i.e.

$$\mathcal{F}(u, \Omega) = \int_\Omega \frac{1}{p} (|Du| - \lambda)_+^p \, dx - \int_\Omega uf \, dx,$$

which fits into the broader context of functionals having a *p-Laplacian type structure at infinity*. The local Lipschitz regularity for the minimizers of asymptotically convex problems has been proved when $f = 0$ under quadratic growth assumptions in [25], and later in [54] for superquadratic growth conditions and in [63] in the case of subquadratic growth (see also [19] and [47]). In addition, similar results have been obtained for functionals whose integrands also depend on lower-order terms (see [44, 45, 76, 80] for example).

A further topic concerning the regularity of solutions to equations as the one in (4) is the investigation of their higher differentiability of both integer and fractional order, and several results are available so far (see, for example, [19, 20, 22, 29, 36, 82]). A common aspect of nonlinear elliptic problems with growth rate $p \geq 2$ is that the higher differentiability is proved for a nonlinear function of the gradient of the weak solutions that takes into account the growth of the structure function of the equation. Indeed, already for the p-Poisson equation (which is

obtained from (4) by setting $\lambda = 0$), the higher differentiability is established for the function

$$\mathbb{V}_p(Du) := |Du|^{\frac{p-2}{2}} Du\,,$$

as can be seen in many papers, starting from the pioneering one by Uhlenbeck [93]. In the case of equation (4), this phenomenon persists and higher differentiability results hold true for the function

$$H_{\frac{p}{2}}(Du) := (|Du| - \lambda)_+^{\frac{p}{2}} \frac{Du}{|Du|}\,.$$

In particular, Brasco, Carlier and Santambrogio [22] proved that $H_{\frac{p}{2}}(Du) \in W^{1,2}_{loc}(\Omega, \mathbb{R}^n)$ if $p \geq 2$ and $f \in W^{1,p'}(\Omega)$. The main novelty in Chapter 2 is that we address the subquadratic (or *singular*) case $1 < p < 2$, which had been neglected in the literature, since extra technical difficulties arise concerning elliptic regularity whenever we are in this case. This aspect, which also occurs in the classic p-harmonic setting, has been very well explained in [13, Section 2.6].

More precisely, in the singular case we present four main results obtained in the paper [4]. First, we establish a Besov regularity result for the vector field $H_{\frac{p}{2}}(Du)$, assuming that the datum f belongs to a suitable Sobolev or Besov space. Then, we infer the Besov regularity of the solution σ_0 to (P1), under the same assumptions on f. Finally, we deduce the local higher summability of σ_0 and Du.

The main tool in the proofs of our results is the difference quotient technique, which, due to the singular character of the case under consideration, requires new ideas with respect to previous papers on the subject ([1] and [88]), where the case $1 < p < 2$ has been treated, but in the absence of singularity. For the sake of clarity, we now try to explain the difficulties associated with the singular setting. When dealing with the higher differentiability of solutions to subquadratic problems, the difference quotient method usually yields a Caccioppoli-type inequality based on the possibility of reabsorbing the terms in the right-hand side that involve the increment of the gradient of a solution by the left-hand side. In the singular setting, this possibility is actually excluded by the fact that we cannot control the difference quotient of the gradient of a solution u in terms of $H_{\frac{p}{2}}(Du)$, whose difference quotient appears in the left-hand side, and this is true regardless of the type of regularity of the datum f (for more details, see Chapter 2).

In Section 2.2 we prove that results similar to those obtained in the singular case also hold for $p \geq 2$, assuming that f belongs to the local Besov space $B^{\alpha}_{p',\infty,loc}(\Omega)$, for some $\alpha \in (0,1)$.

In Chapter 3, we present three main results contained in the paper [9]. More precisely, we establish the local $W^{1,2}$-regularity of a *novel* nonlinear function of the gradient Du of local weak solutions to equation (4), by assuming that

- $f \in B^{\frac{p-2}{p}}_{p',1,loc}(\Omega)$ if $2 < p < \infty$ (see Theorem 3.1);
- $f \in L^{\frac{np}{n(p-1)+2-p}}_{loc}(\Omega)$ if $1 < p \leq 2$ (see Theorem 3.4).

These results, in turn, imply the local higher integrability of Du under the same hypotheses on the function f (cf. Corollary 3.5).

Let us recall again that, for $\lambda = 0$, equation (4) turns into the p-Poisson equation. For the weak solutions $u \in W^{1,p}(\Omega)$ of this last equation, Irving and Koch [58] proved that

$$\mathbb{V}_p(Du) \in W^{1,2}_{loc}(\Omega, \mathbb{R}^n) \tag{5}$$

if the datum f belongs to the Besov space $B^{\frac{p-2}{p}}_{p',1}(\Omega)$, with $p > 2$. Their assumption on f is essentially sharp, in the sense that the above result is false if

$$f \in B^{s}_{p',1}(\Omega) \quad \text{with} \quad s < (p-2)/p.$$

Indeed, Brasco and Santambrogio [23, Section 5] showed with an explicit example that condition (5) may not hold if f belongs to a fractional Sobolev space $W^{\sigma,p'}_{loc}(\mathbb{R}^n)$ with $0 < \sigma < (p-2)/p$, which is continuously embedded into $B^{s}_{p',1,loc}(\mathbb{R}^n)$ whenever $s \in (0,\sigma)$ (see Lemma 1.39).

The main results reported in Chapter 3 are in the spirit of the ones mentioned above. Indeed, our primary objective is to find the assumptions to impose on the datum f *in the scale of local Besov or Lebesgue spaces* to obtain the $W^{1,2}$-regularity of a nonlinear function of the gradient of weak solutions to the widely degenerate or singular equation (4). We emphasize that, for $\lambda = 0$, our results give back those contained in [28] and [58, Remark 1.4].

A key tool in the proof of the above-mentioned Theorem 3.1 is the duality of Besov spaces. In fact, our approach has been inspired by [23], where Brasco and Santambrogio used a duality-based inequality in the setting of fractional Sobolev spaces, but limiting themselves to the p-Poisson equation.

In the case $1 < p \leq 2$, it is well known that, already for the p-Poisson equation, the higher differentiability of the weak solutions can be achieved without assuming any differentiability on f (neither of integer nor of fractional order), but only a suitable degree of integrability. The sharp assumption on f in the scale of Lebesgue spaces has been recently found in [28]. The aforementioned Theorem 3.4 simply tells us that a result analogous to [28, Theorem 1.1] holds when dealing with solutions of widely singular equations. For more details about our proof strategy, we refer the reader to Chapter 3.

In Chapter 4, we move on to the parabolic version of equation (4), that is, we consider the strongly degenerate parabolic PDE

$$u_t - \operatorname{div}\left((|Du| - \lambda)^{p-1}_{+} \frac{Du}{|Du|} \right) = f \quad \text{in } \Omega_T = \Omega \times (0,T), \tag{6}$$

where $p \geq 2$, Ω is a bounded domain in $\mathbb{R}^n$ ($n \geq 2$), $T > 0$ and λ is a positive constant. The main feature of the above equation is that the structure function

$$H_{p-1}(\xi) := \begin{cases} (|\xi| - \lambda)^{p-1}_{+} \frac{\xi}{|\xi|} & \text{if } \xi \in \mathbb{R}^n \setminus \{0\}, \\ 0 & \text{if } \xi = 0, \end{cases}$$

satisfies standard growth and ellipticity conditions for a growth rate $p \geq 2$, but only outside the ball $\{|\xi| < \lambda\}$.

Although the elliptic equation (4) has been widely studied over the years (see, e.g. [4, 17, 19, 22, 75, 82]), only a few regularity results were available for weak solutions to equation (6). In [10] and [5] we succeeded in proving higher differentiability results of integer and fractional order, respectively. These results will be exactly the subject of Chapter 4. For further results available in the literature, we refer the reader to [16, 52] and the references therein.

A motivation for studying equations of the type (6) can be found in *gas filtration problems taking into account the initial pressure gradient* (see Section 4.2 for a brief explanation).

In Chapter 4, we shall first prove the Sobolev spatial regularity of a nonlinear function of the spatial gradient Du of the weak solutions to (6) (see Theorem 4.1), which in turn implies

the Sobolev time regularity of the solutions (cf. Theorem 4.2), by assuming that the datum f belongs to a suitable Lebesgue-Sobolev parabolic space. These results are obtained by adapting the techniques for the evolutionary p-Laplacian to our more degenerate context. In fact, for less degenerate parabolic problems, these issues have been widely investigated, as one can see, for example, in [41, 53] (where $f = 0$) and in [84]. Moreover, establishing the Sobolev regularity of the solutions with respect to time, once the higher differentiability in space has been obtained, is a quite usual fact in these problems: see, for instance, [66, 67, 68].

As already mentioned, the distinguishing feature of equation (6) is that its principal part behaves like a *p-Laplace operator only at infinity*. Let us briefly summarize a few previous results on this topic: the regularity of solutions to parabolic problems with asymptotic structure of p-Laplacian type has been studied in [59], where a BMO regularity has been proved for solutions to asymptotically parabolic systems in the case $p = 2$ and $f = 0$ (see also [62], where the local Lipschitz continuity of weak solutions with respect to the spatial variable is established). In addition, we want to point out the results contained in [24], where nonhomogeneous parabolic problems involving a discontinuous nonlinearity and an asymptotic regularity in divergence form of p-Laplacian type are considered. There, Byun, Oh and Wang establish a global Calderón-Zygmund estimate by converting a given asymptotically regular problem to a suitable regular problem.

Returning to the main results of Chapter 4, one of the major novelties of our paper [10] is an observation of interpolative nature, which allows us to suitably weaken the assumptions on the datum f. This comes from an idea that has already been exploited in [28], in the elliptic setting. In fact, our assumptions on f are weaker than those considered in the works mentioned above.

With the aim of pursuing our investigation on the regularity properties of weak solutions to equation (6), in [5] we further weakened the hypotheses on f, by assuming that

$$f \in L^{p'}_{loc}\left(0, T; B^{\alpha}_{p',\infty,loc}(\Omega)\right),$$

with $2 \leq p < \infty$ and $\alpha \in (0, 1)$. This assumption led us to obtain a local higher differentiability result for the vector field $H_{\frac{p}{2}}(Du)$ in the scale of fractional Sobolev spaces (see Theorem 4.4).

In Chapter 5, we present three main results contained in the paper [6]. More precisely, we prove the spatial $W^{1,2}$-regularity of a *novel* nonlinear function of the spatial gradient of local weak solutions to equation (6), by assuming that

- $f \in L^{p'}_{loc}\left(0, T; B^{\frac{p-2}{p}}_{p',1,loc}(\Omega)\right)$ if $2 < p < \infty$ (see Theorem 5.1);
- $f \in L^2_{loc}(\Omega_T)$ if $p = 2$ (see Theorem 5.3).

These results, in turn, imply the Sobolev time regularity of the weak solutions to the evolutionary p-Poisson equation, under the same hypotheses on the function f (cf. Theorem 5.5, where we only address the case $p > 2$, since the Sobolev time regularity is well known for the heat equation with source term in $L^2_{loc}(\Omega_T)$).

Actually, in Chapter 5 we extend the elliptic results contained in Chapter 3 to the parabolic setting, in the case $p \geq 2$. Indeed, the above-mentioned Theorem 5.1 is nothing but the parabolic counterpart of Theorem 3.1, while Theorem 5.3 can be considered as the parabolic analogue of Theorem 3.4 in the case $p = 2$. In addition, Theorem 5.1 extends some known results for less degenerate parabolic equations (see e.g. [41]) to the case of data in a suitable

Lebesgue-Besov parabolic space, which turns out to be optimal, as can be seen by appropriately adapting the aforementioned example in [23, Section 5] to the parabolic context.

Since, for $\lambda > 0$, the weak solutions of (6) may not be twice weakly differentiable with respect to the spatial variable, in general we cannot differentiate the equation to estimate the second spatial derivatives of the solutions. In Chapter 5, we overcome this difficulty by introducing a suitable family of approximating problems, whose solutions are regular enough by standard parabolic regularity. The major effort in proving Theorems 5.1 and 5.3 is to establish suitable estimates for the solutions u_ε of the approximating problems that are uniform with respect to the approximation parameter ε. Next, we take advantage of these uniform estimates in the use of a comparison argument aimed to transfer the higher differentiability in space of a certain function $\mathcal{V}_\lambda(Du_\varepsilon)$ to the map $\mathcal{V}_\lambda(Du)$ (see Chapter 3 for the definition of $\mathcal{V}_\lambda$).

In the case $p > 2$, one of the main differences with respect to the arguments used in Chapter 4 lies in the derivation of the *a priori* estimates for the terms involving the datum f, which now only has a Besov spatial regularity. Indeed, if f had a suitable Sobolev spatial regularity as in Chapter 4, the terms of the estimate coming from the inhomogeneity of the equation could be controlled using the information on the integrability of the spatial gradient of the solution. In Chapter 5, instead, in order to control the terms coming from the inhomogeneity, we use the duality of Besov spaces in essentially the same way as in the elliptic setting (cf. Chapter 3).

Now we turn our attention to the case $p = 2$ and $f \in L^2_{loc}(\Omega_T)$. As far as we know, for $\lambda > 0$ the result of Theorem 5.3 appears to be completely new. Moreover, when $\lambda = 0$, Theorem 5.3 gives back a well-established result for the non-homogeneous heat equation, namely that

$$f \in L^2_{loc}(\Omega_T) \implies Du \in L^2_{loc}(0,T; W^{1,2}_{loc}(\Omega, \mathbb{R}^n))\,.$$

In Chapter 6, we consider a parabolic system that is a generalization of equation (6) with $\lambda = 1$. More precisely, we deal with the regularity of weak solutions $u : \Omega_T \to \mathbb{R}^N$ of strongly singular or degenerate parabolic systems of the type

$$\partial_t u^i - \operatorname{div} A^i(x,t,Du) = f^i \qquad \text{for } i \in \{1,\dots,N\} \qquad \text{in } \Omega_T = \Omega \times (0,T), \tag{7}$$

where Ω is a bounded and open subset of $\mathbb{R}^n$ ($n \geq 2$), $T > 0$, $N \geq 1$ and $f = (f^1,\dots,f^N)$ belongs to a suitable Orlicz space that is slightly smaller than $L^n_{loc}(\Omega_T, \mathbb{R}^N)$. The main novelty here is that the partial map $\xi \mapsto A(x,t,\xi)$ satisfies standard p-growth and p-ellipticity conditions for $p > 1$ only outside the unit ball $\{|\xi| < 1\}$.

We present two main results obtained in the paper [7]. For $p > \frac{2n}{n+2}$ we prove that any weak solution of the above system admits a locally bounded spatial gradient. Moreover, assuming that the solutions are essentially bounded, we recover the same result in the case $1 < p \leq \frac{2n}{n+2}$ and $f = 0$. We emphasize that our results include both the degenerate case $p \geq 2$ and the singular case $1 < p < 2$.

The chosen terminology can be illustrated by the model system of equations. In fact, in the prototypical system we have in mind, the vector field A is given by

$$A(x,t,\xi) = \frac{(|\xi|-1)^{p-1}_+}{|\xi|}\,\xi\,, \qquad (x,t) \in \Omega_T,\ \xi \in \mathbb{R}^{Nn}, \tag{8}$$

for some $p > 1$. We note that any time-independent 1-Lipschitz continuous function u solves the prototypical system with $f = 0$, and even more, it is a solution of the associated stationary system. Accordingly, well established methods from regularity theory, using the second weak spatial derivatives of u, cannot be utilized. We circumvent this difficulty by resorting to a

suitable approximation of system (7).

Before describing our proof strategy, we briefly discuss some results already available in the literature. So far, most progress has been made for the associated elliptic, i.e. time-independent, problem. Brasco [19] proved the Lipschitz continuity of weak solutions. Clop, Giova, Hatami and Passarelli di Napoli [29] generalized this result for systems ($N \geq 2$). The two previous works arise from the study of *strongly degenerate* functionals. As was said above, these can be regarded as *asymptotically convex* functionals, i.e. functionals having a p-Laplacian type structure only at infinity. Such class of functionals has been widely investigated in the works already mentioned at the beginning of this introduction. Extensions to various other contexts can be found in the non-exhaustive list [35, 37, 48, 49, 50, 85]. The first result concerning gradient continuity was achieved by Santambrogio and Vespri [83]. They restricted themselves to equations and the dimension $n = 2$ and proved that a gradient-dependent function of the form $g(Du)$ is continuous whenever g is continuous and vanishes on the set of degeneracy, i.e. on the unit ball in the case of the prototypical equation. Subsequently, Colombo and Figalli [31, 32] extended this result to arbitrary dimensions $n \geq 2$. A similar result in the vectorial setting was recently shown by Bögelein, Duzaar, Giova and Passarelli di Napoli [17]. Mons [75] extended this result to systems satisfying more general structural conditions. On the contrary, for the time-dependent problem, little is known so far. As already mentioned, in [6, 10] and [5] we succeeded in obtaining higher differentiability results of integer and fractional order, respectively. Another higher differentiability result has been proved by Gentile and Passarelli di Napoli in [52], where the right-hand side of equation (6) is assumed to be in $L^2_{loc}(\Omega_T)$. Very recently, again Bögelein, Duzaar, Giova and Passarelli di Napoli [16] have established the parabolic counterpart of the result obtained in [17] for the stationary case.

Now we come to the proof of the main results of Chapter 6. This is achieved by means of a parabolic Moser iteration technique, whose implementation is however quite subtle, due to the degeneracy of the differential operator. Since the weak solutions of (7) may not be twice weakly differentiable with respect to the x-variable, we consider a regularization of the original system (7). The regularized problem $(\mathcal{P}_\varepsilon)$, $\varepsilon > 0$, satisfies standard *non-degenerate* p-growth conditions (see Subsection 6.2.3 and Section 6.3). One of the advantages of introducing the regularized problem stems from the fact that its (unique) weak solution u_ε admits second weak spatial derivatives in L^2_{loc}. However, proceeding at this point with a standard Moser iteration, the constants in our estimates would blow up as $\varepsilon \downarrow 0$. We overcome this problem by choosing an appropriate test function in the derivation of the Caccioppoli-type inequality on which the Moser iteration is based. At this stage, it is helpful to realize that, for $\delta > 0$ large enough and $|Du_\varepsilon| > 1 + \delta$, the regularized systems fulfill p-growth conditions that are independent of ε. Therefore, we choose a test function vanishing on the set $\{|Du_\varepsilon| \leq 1 + \delta\}$. We note that the choice of such a test function requires the existence of second weak spatial derivatives, which is why we need to resort to the approximating solutions u_ε. The said test function allows us to obtain a Caccioppoli-type inequality, which in turn is employed to set up a kind of Moser iteration scheme.

Similarly to the treatment of the parabolic p-Laplacian, in the Moser iteration we have to distinguish between three different regimes of the parameter p. The first one is the superquadratic case $p \geq 2$, the second one is the subquadratic supercritical case $\frac{2n}{n+2} < p < 2$ and finally we have the subcritical case $1 < p \leq \frac{2n}{n+2}$. In the last case, we need to ensure that also the approximating solutions u_ε are uniformly bounded with respect to ε. This is achieved by a maximum principle, assuming that $f = 0$. In all three previous cases, we establish the local boundedness of the spatial gradient Du_ε together with quantitative

estimates. Here it is worth noting that the constants appearing in our quantitative estimates are independent of ε. This allows us to obtain the desired local L^∞-bounds of Du after passing to the limit as $\varepsilon \downarrow 0$ and using a strong convergence result in L^p (see Lemma 6.12).

Chapter 7 concludes the thesis with the discussion of a physics-informed deep learning approach for solving a certain type of strongly degenerate parabolic problems. More precisely, we will show how to exploit a novel Artificial Intelligence (AI) methodology, known as *Physics-Informed Neural Networks* (PINNs), to predict solutions to Cauchy-Dirichlet problems of the form

$$\begin{cases} \partial_t u - \operatorname{div}\left((|Du|-1)_+ \frac{Du}{|Du|}\right) = f & \text{in } \Omega_T = \Omega \times (0,T), \\ u = w & \text{on } \partial_{\text{par}}\Omega_T, \end{cases} \tag{9}$$

where Ω is a bounded connected open subset of $\mathbb{R}^n$ $(2 \leq n \leq 3)$ with Lipschitz boundary, f and w are given real-valued functions defined over $\overline{\Omega} \times [0,T]$ and $\partial_{\text{par}}\Omega_T$ respectively, while Du denotes the spatial gradient of an unknown solution $u : \overline{\Omega} \times [0,T) \to \mathbb{R}$.

As already mentioned, problem (9) may arise in gas filtration problems (see Section 4.2 again) and the regularity properties of its weak solutions have been recently studied in [5, 6, 10, 52] (see also [7] and [16] for generalizations to systems of equations).

As for the methodology presented in Chapter 7, the PINNs are a Scientific Machine Learning (SciML) technique based on Artificial Neural Networks (ANNs) with the feature of adding constraints to make the predicted results more in line with the physical laws of the addressed problem. The concept of PINNs was introduced in [61, 77, 78, 79] to solve PDE-based problems. The PINNs predict the solution to a PDE under prescribed initial-boundary conditions by training a neural network to minimize a cost function, called *loss function*, which penalizes some suitable terms on a set of admissible functions u. For more information, we refer the interested reader to [34] (see also Section 7.2).

The kind of approach we propose here can offer effective solutions to real problems such as (9) and can be applied in many other different fields: for example, in production and advanced engineering [95], for transportation problems [51], and for virtual thermal sensors using real-time simulations [56]. Additionally, it is employed to solve groundwater flow equations [33] and address petroleum and gas contamination [86].

The PINN-based approach that we present in this thesis was introduced in [8]. There, we were the first to evaluate the effectiveness of the PINN framework in solving strongly degenerate parabolic problems of the type (9). In particular, we estimated an appropriate approximation error for some test problems whose analytical solutions are known. The numerical experiments we conducted include two and three-dimensional spatial domains and have highlighted the efficacy of our approach in predicting accurate solutions.

I wish to conclude this introduction by expressing my immense gratitude to my supervisor, Professor Antonia Passarelli di Napoli, for her caring and invaluable guidance over the past three years. She has been my generous mentor since my master's studies, making me discover challenging problems to solve and strategies to tackle them. I am deeply indebted to her for her trust and precious advice.

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Chapter 1

Notations and preliminary tools

This chapter is devoted to some preliminary results that will be needed later on. Let us start with a list of classic notations. We shall denote by C or c a general positive constant that may vary on different occasions. Relevant dependencies on parameters and special constants will be suitably emphasized using parentheses or subscripts. The norm we use on $\mathbb{R}^k$, $k \in \mathbb{N}$, will be the standard Euclidean one and it will be denoted by $|\cdot|$. In particular, for the vectors $\xi, \eta \in \mathbb{R}^k$, we write $\langle \xi, \eta \rangle$ for the usual inner product and $|\xi| := \langle \xi, \xi \rangle^{\frac{1}{2}}$ for the corresponding Euclidean norm. If $a \in \mathbb{R}^n$ and $b \in \mathbb{R}^N$, their tensor product $a \otimes b$ is defined as the $n \times N$-matrix that has the element $a_i b_j$ in its i-th row and j-th column, i.e. $a \otimes b = a\, b^T$.

For points in space-time, we will frequently use abbreviations like $z = (x, t)$ or $z_0 = (x_0, t_0)$, for spatial variables $x, x_0 \in \mathbb{R}^n$ and times $t, t_0 \in \mathbb{R}$. We also denote by $B_\rho(x_0) = B(x_0, \rho) = \{x \in \mathbb{R}^n : |x - x_0| < \rho\}$ the n-dimensional open ball with radius $\rho > 0$ and center $x_0 \in \mathbb{R}^n$; when not important, or clear from the context, we shall omit to denote the center as follows: $B_\rho \equiv B_\rho(x_0)$. Unless otherwise stated, different balls in the same context will have the same center. Moreover, we use the notation

$$Q_\rho(z_0) := B_\rho(x_0) \times (t_0 - \rho^2, t_0), \quad z_0 = (x_0, t_0) \in \mathbb{R}^n \times \mathbb{R},\ \rho > 0,$$

for the backward parabolic cylinder with vertex (x_0, t_0) and width ρ. We shall sometimes omit the dependence on the vertex when all cylinders occurring in a proof share the same vertex. For a general cylinder $Q = B \times (t_0, t_1)$, where $B \subset \mathbb{R}^n$ and $t_0 < t_1$, we denote by

$$\partial_{\mathrm{par}} Q := (\overline{B} \times \{t_0\}) \cup (\partial B \times (t_0, t_1))$$

the usual *parabolic boundary* of Q, which is nothing but its standard topological boundary without the upper cap $\overline{B} \times \{t_1\}$.

If $E \subseteq \mathbb{R}^k$ is a Lebesgue-measurable set, then we will denote by $|E|$ its k-dimensional Lebesgue measure. When $0 < |E| < \infty$, the mean value of a function $v \in L^1(E)$ is defined by

$$(v)_E := \fint_E v(x)\, dx = \frac{1}{|E|} \int_E v(x)\, dx\,.$$

For every $g \in L^1_{loc}(Q, \mathbb{R}^k)$, where $Q \subset \mathbb{R}^{n+1}$ and $k \in \mathbb{N}$, we define the mollified function g_ϱ as follows:

$$g_\varrho(z) := \int_{\mathbb{R}^{n+1}} g(\tilde{z})\, \eta_\varrho(z - \tilde{z})\, d\tilde{z}\,, \tag{1.1}$$

where

$$\eta_\varrho(z) := \frac{1}{\varrho^{n+1}}\,\eta_1\left(\frac{z}{\varrho}\right), \qquad \varrho > 0,$$

with $\eta_1 \in C_0^\infty(\mathcal{B}_1(0))$[1] denoting the standard, non-negative, radially symmetric mollifier in $\mathbb{R}^{n+1}$. Obviously, here the function g is meant to be extended by the k-dimensional null vector outside Q.

We now gather some tools that will be useful to prove our results. For the auxiliary function

$$V_\gamma(\xi) := \left(\mu^2 + |\xi|^2\right)^{\gamma/2}\xi\,, \qquad \xi \in \mathbb{R}^n, \tag{1.2}$$

where $\mu \geq 0$ and $\gamma > 0$ are parameters, we record the following estimates (see [54, Formula (2.4)]).

Lemma 1.1. *Let $\gamma > 0$ and $\mu \geq 0$. Then, there exist two positive constants $c_1(\gamma)$ and $c_2(\gamma)$ such that*

$$c_1(\gamma)\,|\xi-\eta| \leq \frac{|V_\gamma(\xi) - V_\gamma(\eta)|}{\left(\mu^2 + |\xi|^2 + |\eta|^2\right)^{\gamma/2}} \leq c_2(\gamma)\,|\xi-\eta|$$

for any $\xi, \eta \in \mathbb{R}^n$.

The next result is an extension of the previous lemma to the case $\gamma \in (-1, 0)$ (see the proof of [1, Lemma 2.2]).

Lemma 1.2. *For every $\gamma \in \left(-\frac{1}{2}, 0\right)$ and $\mu \geq 0$ we have*

$$(2\gamma+1)\,|\xi-\eta| \leq \frac{\left|\left(\mu^2 + |\xi|^2\right)^\gamma \xi - \left(\mu^2 + |\eta|^2\right)^\gamma \eta\right|}{\left(\mu^2 + |\xi|^2 + |\eta|^2\right)^\gamma} \leq \frac{c(n)}{2\gamma+1}\,|\xi-\eta| \tag{1.3}$$

for any $\xi, \eta \in \mathbb{R}^n$. Moreover, setting

$$F(\zeta) := \frac{1}{2(\gamma+1)}\left(\mu^2 + |\zeta|^2\right)^{\gamma+1}, \qquad \zeta \in \mathbb{R}^n,$$

we obtain

$$DF(\zeta) = \left(\mu^2 + |\zeta|^2\right)^\gamma \zeta, \qquad \zeta \in \mathbb{R}^n, \tag{1.4}$$

and

$$\langle DF(\xi) - DF(\eta), \xi - \eta\rangle \geq (2\gamma+1)\left(\mu^2 + |\xi|^2 + |\eta|^2\right)^\gamma |\xi-\eta|^2 \tag{1.5}$$

for any $\xi, \eta \in \mathbb{R}^n$.

As an easy consequence of the two preceding lemmas, we obtain the following result:

Lemma 1.3. *Let $1 < p < \infty$. There exists a constant $c \equiv c(p,n) > 0$ such that*

$$c^{-1}(|\xi|^2 + |\eta|^2)^{\frac{p-2}{2}} \leq \frac{\left||\xi|^{\frac{p-2}{2}}\xi - |\eta|^{\frac{p-2}{2}}\eta\right|^2}{|\xi-\eta|^2} \leq c\,(|\xi|^2 + |\eta|^2)^{\frac{p-2}{2}}$$

for every $\xi, \eta \in \mathbb{R}^n$ with $\xi \neq \eta$.

[1] Here $\mathcal{B}_1(0)$ denotes the $(n+1)$-dimensional open unit ball centered at the origin.

Let $p \in [2, \infty)$. Taking $\mu = 0$ and $\gamma = \frac{p-2}{2}$ into (1.2), we obtain the function

$$\mathbb{V}_p(\xi) := |\xi|^{\frac{p-2}{2}} \xi\,, \qquad \xi \in \mathbb{R}^n.$$

For the above function, we recall the following inequalities (see [65, Chapter 12] for the proof).

Lemma 1.4. *If $2 \le p < \infty$, then for every $\xi, \eta \in \mathbb{R}^n$ we get*

$$|\mathbb{V}_p(\xi) - \mathbb{V}_p(\eta)|^2 \le \frac{p^2}{4} \langle |\xi|^{p-2}\xi - |\eta|^{p-2}\eta, \xi - \eta \rangle,$$

$$\left| |\xi|^{p-2}\xi - |\eta|^{p-2}\eta \right| \le (p-1)\left(|\xi|^{\frac{p-2}{2}} + |\eta|^{\frac{p-2}{2}} \right) |\mathbb{V}_p(\xi) - \mathbb{V}_p(\eta)|\,.$$

For further needs, we now define the auxiliary function $H_\gamma : \mathbb{R}^n \to \mathbb{R}^n$ by

$$H_\gamma(\xi) := \begin{cases} (|\xi| - \lambda)_+^\gamma \, \frac{\xi}{|\xi|} & \text{if } \xi \neq 0, \\ 0 & \text{if } \xi = 0, \end{cases} \tag{1.6}$$

where $\lambda > 0$ and $\gamma > 0$ are parameters, while $(\,\cdot\,)_+$ stands for the positive part. For the function H_γ, we establish the following estimates:

Lemma 1.5. *Let $\lambda > 0$. For every $(\alpha, \varepsilon) \in \mathbb{R}^+ \times \mathbb{R}^+$ with $\alpha < \varepsilon$, there exist two positive constants $C_1(\alpha, \varepsilon)$ and $C_2(\alpha, \varepsilon, n)$ such that*

$$C_1\, |H_\varepsilon(\xi) - H_\varepsilon(\eta)| \le \frac{|H_\alpha(\xi) - H_\alpha(\eta)|}{\left((|\xi| - \lambda)_+^\varepsilon + (|\eta| - \lambda)_+^\varepsilon \right)^{\frac{\alpha-\varepsilon}{\varepsilon}}} \le C_2\, |H_\varepsilon(\xi) - H_\varepsilon(\eta)|$$

for any $\xi, \eta \in \mathbb{R}^n$.

Proof. By Lemma 1.2, we have that for every $\gamma \in (-1, 0)$ there exist two positive constants $c_1(\gamma)$ and $c_2(\gamma, n)$ such that

$$c_1\, |z - w|^2 \le \frac{\left| |z|^\gamma z - |w|^\gamma w \right|^2}{\left(|z|^2 + |w|^2 \right)^\gamma} \le c_2\, |z - w|^2 \tag{1.7}$$

for any $z, w \in \mathbb{R}^n$. Now we recall that for every $\gamma \in (-1, 0)$ and every $z, w \in \mathbb{R}^n$ we get

$$\frac{1}{\left(|z|^2 + |w|^2 \right)^\gamma} \le \frac{1}{(|z| + |w|)^{2\gamma}} \quad \text{and} \quad \frac{1}{\left(|z|^2 + |w|^2 \right)^\gamma} \ge \frac{2^\gamma}{(|z| + |w|)^{2\gamma}}\,.$$

From this observation and from (1.7), it follows that for every $\gamma \in (-1, 0)$ there exist two positive constants $c_3(\gamma)$ and $c_4(\gamma, n)$ such that

$$c_3\, |z - w| \le \frac{\left| |z|^\gamma z - |w|^\gamma w \right|}{(|z| + |w|)^\gamma} \le c_4\, |z - w| \tag{1.8}$$

for any $z, w \in \mathbb{R}^n$. Then, for any fixed $(\alpha, \varepsilon) \in \mathbb{R}^+ \times \mathbb{R}^+$ such that $\alpha < \varepsilon$, by taking

$$z = H_\varepsilon(\xi), \quad w = H_\varepsilon(\eta) \quad \text{and} \quad \gamma = \frac{\alpha - \varepsilon}{\varepsilon}$$

into (1.8), we obtain the desired conclusion. □

For the function H_{p-1} defined by (1.6) with $\gamma = p-1$, we record the following estimates, which can be obtained by suitably modifying the proof of [22, Lemma 4.1].

Lemma 1.6. *If $2 \le p < \infty$, then for every $\xi, \eta \in \mathbb{R}^n$ we get*

$$\langle H_{p-1}(\xi) - H_{p-1}(\eta), \xi - \eta \rangle \ \ge \ \frac{4}{p^2} \left| H_{\frac{p}{2}}(\xi) - H_{\frac{p}{2}}(\eta) \right|^2, \tag{1.9}$$

$$\left| H_{p-1}(\xi) - H_{p-1}(\eta) \right| \ \le \ (p-1) \left(|H_{\frac{p}{2}}(\xi)|^{\frac{p-2}{p}} + |H_{\frac{p}{2}}(\eta)|^{\frac{p-2}{p}} \right) \left| H_{\frac{p}{2}}(\xi) - H_{\frac{p}{2}}(\eta) \right|.$$

We shall now prove that the first estimate of the previous lemma holds true also in the case $1 < p < 2$. More precisely, we have the following result:

Lemma 1.7. *If $1 < p < 2$, then there exists a constant $c \equiv c(p,n) > 0$ such that*

$$\langle H_{p-1}(\xi) - H_{p-1}(\eta), \xi - \eta \rangle \ \ge \ c \left| H_{\frac{p}{2}}(\xi) - H_{\frac{p}{2}}(\eta) \right|^2 \tag{1.10}$$

for every $\xi, \eta \in \mathbb{R}^n$.

Proof. We first note that the above inequality is trivially satisfied when $|\xi|, |\eta| \le \lambda$. Taking $\gamma = (p-2)/2$ and $\mu = 0$, equality (1.4) turns into

$$DF(\zeta) = |\zeta|^{p-2} \zeta, \qquad \zeta \in \mathbb{R}^n,$$

and estimates (1.5) and (1.3) imply

$$\begin{aligned} \langle |z|^{p-2} z - |w|^{p-2} w, z - w \rangle \ &\ge \ (p-1) \left(|z|^2 + |w|^2 \right)^{\frac{p-2}{2}} |z-w|^2 \\ &\ge \ c(p,n) \left| |z|^{\frac{p-2}{2}} z - |w|^{\frac{p-2}{2}} w \right|^2 \end{aligned} \tag{1.11}$$

for any $z, w \in \mathbb{R}^n$. Taking $z = (|\xi| - \lambda)_+ \, \xi / |\xi|$ and $w = (|\eta| - \lambda)_+ \, \eta / |\eta|$ into (1.11), we obtain

$$\left\langle H_{p-1}(\xi) - H_{p-1}(\eta), (|\xi| - \lambda)_+ \frac{\xi}{|\xi|} - (|\eta| - \lambda)_+ \frac{\eta}{|\eta|} \right\rangle \ \ge \ c(p,n) \left| H_{\frac{p}{2}}(\xi) - H_{\frac{p}{2}}(\eta) \right|^2. \tag{1.12}$$

Now let $\xi, \eta \in \mathbb{R}^n$ be such that $|\xi|, |\eta| > \lambda$ and write the left-hand side of the previous inequality as the difference of two terms $A - B$, where

$$A := \langle H_{p-1}(\xi) - H_{p-1}(\eta), \xi - \eta \rangle$$

and

$$B := \left\langle H_{p-1}(\xi) - H_{p-1}(\eta), \, \xi - (|\xi| - \lambda)_+ \frac{\xi}{|\xi|} - \eta + (|\eta| - \lambda)_+ \frac{\eta}{|\eta|} \right\rangle.$$

Using the fact that $|\xi|, |\eta| > \lambda$ and the Cauchy-Schwarz inequality, we get

$$B = \ \lambda \left\langle H_{p-1}(\xi) - H_{p-1}(\eta), \frac{\xi}{|\xi|} - \frac{\eta}{|\eta|} \right\rangle$$

$$= \lambda \left[(|\xi| - \lambda)^{p-1} + (|\eta| - \lambda)^{p-1} - \frac{\langle \xi, \eta \rangle}{|\xi|\,|\eta|} \left((|\xi| - \lambda)^{p-1} + (|\eta| - \lambda)^{p-1} \right) \right]$$
$$\geq \lambda \left[(|\xi| - \lambda)^{p-1} + (|\eta| - \lambda)^{p-1} - \frac{|\xi|\,|\eta|}{|\xi|\,|\eta|} \left((|\xi| - \lambda)^{p-1} + (|\eta| - \lambda)^{p-1} \right) \right] = 0,$$

from which the assertion directly follows.

Finally, let $|\xi| > \lambda$ and $|\eta| \leq \lambda$. In this case, we have $H_{p-1}(\eta) = H_{\frac{p}{2}}(\eta) = 0$ and we can write the left-hand side of (1.12) as the difference of two terms $D - E$, where

$$D := \langle H_{p-1}(\xi), \xi - \eta \rangle \quad \text{and} \quad E := \left\langle H_{p-1}(\xi), \frac{\lambda\,\xi}{|\xi|} - \eta \right\rangle.$$

Using the fact that $|\xi| > \lambda, |\eta| \leq \lambda$ and the Cauchy-Schwarz inequality, we find that

$$E = \lambda\,(|\xi| - \lambda)^{p-1} - \frac{\langle \xi, \eta \rangle}{|\xi|}\,(|\xi| - \lambda)^{p-1} \geq \lambda\,(|\xi| - \lambda)^{p-1} - \frac{|\xi|\,|\eta|}{|\xi|}\,(|\xi| - \lambda)^{p-1} \geq 0,$$

which concludes the proof. □

The next result can be obtained by suitably modifying the proof of [17, Lemma 2.8].

Lemma 1.8. *Let $p \in (1, \infty)$ and $\xi, \eta \in \mathbb{R}^n$ with $|\eta| > \lambda > 0$. Then, we have*

$$\langle H_{p-1}(\xi) - H_{p-1}(\eta), \xi - \eta \rangle \geq \frac{\min\{1, p-1\}}{2^{p+1}} \frac{(|\eta| - \lambda)^p}{|\eta|\,(|\xi| + |\eta|)}\,|\xi - \eta|^2.$$

In Chapters 4 and 5, we shall deal with weak solutions to the strongly degenerate parabolic equation

$$\partial_t u - \operatorname{div}\left((|Du| - \lambda)_+^{p-1} \frac{Du}{|Du|} \right) = f \qquad \text{in } \Omega_T := \Omega \times (0, T), \tag{1.13}$$

where Ω is a bounded domain in $\mathbb{R}^n$ $(n \geq 2)$, $T > 0$, $p \geq 2$, λ is a positive constant and f is a given function defined over Ω_T. Here we define a weak solution to (1.13) as follows:

Definition 1.9. Let $f \in L^1_{loc}(\Omega_T)$. A function

$$u \in C^0\left((0, T); L^2(\Omega)\right) \cap L^p\left(0, T; W^{1,p}(\Omega)\right)$$

is a *weak solution* of equation (1.13) if and only if the condition

$$\int_{\Omega_T} \left(u \cdot \partial_t \varphi - \langle H_{p-1}(Du), D\varphi \rangle \right) dz = -\int_{\Omega_T} f \varphi \, dz \tag{1.14}$$

is satisfied for all $\varphi \in C_0^\infty(\Omega_T)$.

Now we state a well-known iteration lemma (see [55, Lemma 6.1]).

Lemma 1.10. *Let $0 \leq \rho_0 < \rho_1 < \infty$ and assume that $\Psi : [\rho_0, \rho_1] \to [0, \infty)$ is a bounded function satisfying*

$$\Psi(\rho) \leq \theta\,\Psi(r) + \frac{A}{(r - \rho)^\alpha} + \frac{B}{(r - \rho)^\beta} + C$$

for all $\rho_0 \le \rho < r \le \rho_1$, *for some* $\theta \in (0,1)$ *and fixed non-negative constants* A, B, C, $\alpha > \beta > 0$. *Then, there exists a constant* $\kappa = \kappa(\alpha,\theta) > 0$ *such that*

$$\Psi(\rho_0) \le \kappa \left(\frac{A}{(\rho_1-\rho_0)^\alpha} + \frac{B}{(\rho_1-\rho_0)^\beta} + C \right).$$

For further needs, we also record the following interpolation inequality, whose proof can be found in [38, Proposition 3.1].

Lemma 1.11. *Assume that the function*

$$v : Q_r(z_0) \cup \partial_{\mathrm{par}} Q_r(z_0) \to \mathbb{R}$$

satisfies

$$v \in L^\infty\left(t_0 - r^2, t_0; L^q(B_r(x_0))\right) \cap L^p\left(t_0 - r^2, t_0; W_0^{1,p}(B_r(x_0))\right)$$

for some exponents $1 \le p, q < \infty$. *Then the estimate*

$$\int\limits_{Q_r(z_0)} |v|^{p+\frac{pq}{n}}\, dz \le c \left(\sup_{s \in (t_0-r^2,t_0)} \int\limits_{B_r(x_0)} |v(x,s)|^q\, dx \right)^{\frac{p}{n}} \int\limits_{Q_r(z_0)} |Dv|^p\, dz$$

holds true for a positive constant c *depending at most on* n, p *and* q.

1.1 Difference quotients

Here we recall the definition and some well-known properties of the difference quotients that can be found, for example, in [55].

Definition 1.12. For every vector-valued function $F : \mathbb{R}^n \to \mathbb{R}^k$ the *finite difference operator* in the direction x_j is defined by

$$\tau_{j,h} F(x) = F(x + he_j) - F(x)\,,$$

where $h \in \mathbb{R}$, e_j is the unit vector in the direction x_j and $j \in \{1, \dots, n\}$.

The *difference quotient* of F with respect to x_j is defined for $h \in \mathbb{R} \setminus \{0\}$ by

$$\Delta_{j,h} F(x) = \frac{\tau_{j,h} F(x)}{h}\,.$$

When no confusion arises, we shall omit the index j and simply write τ_h or Δ_h instead of $\tau_{j,h}$ or $\Delta_{j,h}$, respectively.

Proposition 1.13. *Let* $\Omega \subset \mathbb{R}^n$ *be an open set and let* $F \in W^{1,q}(\Omega)$, *with* $q \ge 1$. *Moreover, let* $G : \Omega \to \mathbb{R}$ *be a measurable function and consider the set*

$$\Omega_{|h|} := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > |h|\}\,.$$

Then:

(i) $\Delta_h F \in W^{1,q}(\Omega_{|h|})$ *and* $D_{x_i}(\Delta_h F) = \Delta_h(D_{x_i}F)$ *for every* $i \in \{1,\dots,n\}$.

(ii) *If at least one of the functions* F *or* G *has support contained in* $\Omega_{|h|}$, *then*

$$\int_\Omega F\,\Delta_h G\,dx \;=\; -\int_\Omega G\,\Delta_{-h}F\,dx.$$

(iii) *We have*

$$\Delta_h(FG)(x) = F(x+he_j)\Delta_h G(x) \;+\; G(x)\Delta_h F(x).$$

The next result about the finite difference operator is a kind of integral version of the Lagrange theorem and can be obtained by combining [55, Lemma 8.1] with [71, theorem on page 3].

Lemma 1.14. *If* $0 < \rho < R$, $|h| < \frac{R-\rho}{2}$, $1 < q < +\infty$ *and* $F \in L^1_{loc}(B_R,\mathbb{R}^k)$ *is such that* $DF \in L^q(B_R,\mathbb{R}^{k\times n})$, *then*

$$\int_{B_\rho} |\tau_h F(x)|^q\,dx \;\le\; c^q(n)\,|h|^q \int_{B_R} |DF(x)|^q\,dx.$$

Moreover, if $F \in L^q(B_R,\mathbb{R}^k)$, *then we have*

$$\int_{B_\rho} |F(x+he_j)|^q\,dx \;\le\; \int_{B_R} |F(x)|^q\,dx.$$

Finally, we recall the following fundamental result, whose proof can be found in [55, Lemma 8.2].

Lemma 1.15. *Let* $F:\mathbb{R}^n \to \mathbb{R}^k$, $F \in L^q(B_R,\mathbb{R}^k)$ *with* $1 < q < +\infty$. *Suppose that there exist* $\rho \in (0,R)$ *and a constant* $M > 0$ *such that*

$$\sum_{j=1}^n \int_{B_\rho} |\tau_{j,h}F(x)|^q\,dx \;\le\; M^q\,|h|^q$$

for every $h \in \mathbb{R}$ *with* $|h| < \frac{R-\rho}{2}$. *Then* $F \in W^{1,q}(B_\rho,\mathbb{R}^k)$. *Moreover,*

$$\|DF\|_{L^q(B_\rho)} \le M$$

and

$$\Delta_{j,h}F \to D_{x_j}F \quad \text{in } L^q_{loc}(B_R,\mathbb{R}^k) \;\text{ as } h \to 0\,,$$

for each $j \in \{1,\dots,n\}$.

1.2 Function spaces

In this section, we list the definitions and basic properties of some function spaces that we will use throughout our dissertation. We start with Besov spaces, then move on to fractional

Sobolev spaces (also known as *Sobolev-Slobodeckij spaces*). Finally, we turn our attention to Orlicz spaces.

1.2.1 Besov spaces

Here we recall the definition and some properties of Besov spaces (see, for example, [90, 91, 92]). We denote by $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$ the Schwartz space and the space of tempered distributions on $\mathbb{R}^n$, respectively. If $v \in \mathcal{S}(\mathbb{R}^n)$, then

$$\hat{v}(\xi) = (\mathscr{F}v)(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i\,\langle x,\xi\rangle}\, v(x)\, dx, \qquad \xi \in \mathbb{R}^n, \tag{1.15}$$

denotes the Fourier transform of v. As usual, $\mathscr{F}^{-1}v$ and $v^\vee$ stand for the inverse Fourier transform, given by the right-hand side of (1.15) with i in place of $-i$. Both $\mathscr{F}$ and $\mathscr{F}^{-1}$ are extended to $\mathcal{S}'(\mathbb{R}^n)$ in the standard way.
Now, let $\Gamma(\mathbb{R}^n)$ be the collection of all sequences $\varphi = \{\varphi_j\}_{j=0}^\infty \subset \mathcal{S}(\mathbb{R}^n)$ such that

$$\begin{cases} \operatorname{supp}\varphi_0 \subset \{x \in \mathbb{R}^n : |x| \le 2\} \\ \operatorname{supp}\varphi_j \subset \{x \in \mathbb{R}^n : 2^{j-1} \le |x| \le 2^{j+1}\} \quad \text{if } j \in \mathbb{N}, \end{cases}$$

for every multi-index β there exists a positive number c_β such that

$$2^{j|\beta|}\, |D^\beta \varphi_j(x)| \le c_\beta\,, \qquad \forall\, j \in \mathbb{N}_0,\ \forall\, x \in \mathbb{R}^n$$

and

$$\sum_{j=0}^{\infty} \varphi_j(x) = 1\,, \qquad \forall\, x \in \mathbb{R}^n.$$

Then, it is well known that $\Gamma(\mathbb{R}^n)$ is not empty (see [90, Section 2.3.1, Remark 1]). Moreover, if $\{\varphi_j\}_{j=0}^\infty \in \Gamma(\mathbb{R}^n)$, the entire analytic functions $(\varphi_j\,\hat{v})^\vee(x)$ make sense pointwise in $\mathbb{R}^n$ for any $v \in \mathcal{S}'(\mathbb{R}^n)$. Therefore, the following definition makes sense:

Definition 1.16. Let $s \in \mathbb{R}$, $1 \le p, q \le \infty$ and $\varphi = \{\varphi_j\}_{j=0}^\infty \in \Gamma(\mathbb{R}^n)$. We define the *Besov space* $B^s_{p,q}(\mathbb{R}^n)$ as the set of all $v \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|v\|_{B^s_{p,q}(\mathbb{R}^n)} := \left(\sum_{j=0}^{\infty} 2^{jsq}\, \|(\varphi_j\,\hat{v})^\vee\|^q_{L^p(\mathbb{R}^n)} \right)^{\frac{1}{q}} < +\infty \qquad \text{if } q < \infty, \tag{1.16}$$

and

$$\|v\|_{B^s_{p,q}(\mathbb{R}^n)} := \sup_{j \in \mathbb{N}_0} 2^{js}\, \|(\varphi_j\,\hat{v})^\vee\|_{L^p(\mathbb{R}^n)} < +\infty \qquad \text{if } q = \infty. \tag{1.17}$$

Remark 1.17. The space $B^s_{p,q}(\mathbb{R}^n)$ defined above is a Banach space with respect to the norm $\|\cdot\|_{B^s_{p,q}(\mathbb{R}^n)}$. Obviously, this norm depends on the chosen sequence $\varphi \in \Gamma(\mathbb{R}^n)$, but this is not the case for the spaces $B^s_{p,q}(\mathbb{R}^n)$ themselves, in the sense that two different choices for the sequence φ give rise to equivalent norms (see [90, Sections 2.3.2 and 2.3.3]). This justifies our omission of the dependence on φ in the left-hand side of (1.16)−(1.17) and in the sequel.

The norms of the *classical Besov spaces* $B^s_{p,q}(\mathbb{R}^n)$ with $s \in (0,1)$, $1 \le p < \infty$ and $1 \le q \le \infty$ can be characterized via differences of the functions involved (cf. [90, Section 2.5.12, Theorem 1]). More precisely, for $h \in \mathbb{R}^n$ and a measurable function $v : \mathbb{R}^n \to \mathbb{R}$, let us define

$$\delta_h v(x) := v(x+h) - v(x).$$

Then we have the equivalence

$$\|v\|_{B^s_{p,q}(\mathbb{R}^n)} \approx \|v\|_{L^p(\mathbb{R}^n)} + [v]_{B^s_{p,q}(\mathbb{R}^n)}, \tag{1.18}$$

where

$$[v]_{B^s_{p,q}(\mathbb{R}^n)} := \left(\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \frac{|\delta_h v(x)|^p}{|h|^{sp}}\, dx \right)^{\frac{q}{p}} \frac{dh}{|h|^n} \right)^{\frac{1}{q}}, \qquad \text{if } 1 \le q < \infty, \tag{1.19}$$

and

$$[v]_{B^s_{p,\infty}(\mathbb{R}^n)} := \sup_{h \in \mathbb{R}^n} \left(\int_{\mathbb{R}^n} \frac{|\delta_h v(x)|^p}{|h|^{sp}}\, dx \right)^{\frac{1}{p}}. \tag{1.20}$$

In (1.19), if one simply integrates for $|h| < r$ for a fixed $r > 0$, then an equivalent norm is obtained, since

$$\left(\int_{\{|h| \ge r\}} \left(\int_{\mathbb{R}^n} \frac{|\delta_h v(x)|^p}{|h|^{sp}}\, dx \right)^{\frac{q}{p}} \frac{dh}{|h|^n} \right)^{\frac{1}{q}} \le c(n,s,p,q,r)\, \|v\|_{L^p(\mathbb{R}^n)}.$$

Similarly, in (1.20) one can simply take the supremum over $|h| \le r$ and obtain an equivalent norm. By construction, $B^s_{p,q}(\mathbb{R}^n) \subset L^p(\mathbb{R}^n)$.

Remark 1.18. For $s \in (0,1)$ and $1 \le p, q < \infty$, we can simply say that $v \in B^s_{p,q}(\mathbb{R}^n)$ if and only if $v \in L^p(\mathbb{R}^n)$ and $\frac{\delta_h v}{|h|^s} \in L^q\left(\frac{dh}{|h|^n}; L^p(\mathbb{R}^n)\right)$.

Let Ω be an arbitrary open set in $\mathbb{R}^n$. As usual, $\mathcal{D}(\Omega) = C_0^\infty(\Omega)$ stands for the space of all infinitely differentiable functions in $\mathbb{R}^n$ with compact support in Ω. Let $\mathcal{D}'(\Omega)$ be the dual space of all distributions in Ω and let $g \in \mathcal{S}'(\mathbb{R}^n)$. Then we denote by $g|_\Omega$ its restriction to Ω, i.e.

$$g|_\Omega \in \mathcal{D}'(\Omega): \quad (g|_\Omega)(\phi) = g(\phi) \quad \text{for } \phi \in \mathcal{D}(\Omega).$$

Definition 1.19. Let Ω be an arbitrary domain in $\mathbb{R}^n$ with $\Omega \ne \mathbb{R}^n$ and let $s \in \mathbb{R}$, $1 \le p \le \infty$ and $1 \le q \le \infty$. Then

$$B^s_{p,q}(\Omega) := \left\{ v \in \mathcal{D}'(\Omega) : v = g|_\Omega \text{ for some } g \in B^s_{p,q}(\mathbb{R}^n) \right\}$$

and

$$\|v\|_{B^s_{p,q}(\Omega)} := \inf \|g\|_{B^s_{p,q}(\mathbb{R}^n)},$$

where the infimum is taken over all $g \in B^s_{p,q}(\mathbb{R}^n)$ such that $g|_\Omega = v$.

For a bounded C^∞-domain $\Omega \subset \mathbb{R}^n$, the classical Besov spaces $B^s_{p,q}(\Omega)$ with $s \in (0,1)$, $1 \le p < \infty$ and $1 \le q \le \infty$ can be characterized via differences of the functions involved. More precisely, we have the following result (see [91, Section 5.2.2]).

Theorem 1.20. *Let Ω be a bounded C^∞-domain in $\mathbb{R}^n$. Let $s \in (0,1)$, $1 \le p < \infty$ and $1 \le q < \infty$. Then, a function $v : \Omega \to \mathbb{R}$ belongs to the Besov space $B^s_{p,q}(\Omega)$ if and only if*

$v \in L^p(\Omega)$ *and*

$$[v]_{B^s_{p,q}(\Omega)} := \left(\int_{\mathbb{R}^n} \left(\int_\Omega \frac{|\delta_h v(x)|^p}{|h|^{sp}} \cdot \mathbb{1}_\Omega(x+h)\, dx \right)^{\frac{q}{p}} \frac{dh}{|h|^n} \right)^{\frac{1}{q}} < +\infty. \tag{1.21}$$

Moreover, the Besov space $B^s_{p,\infty}(\Omega)$ *consists of all functions* $v \in L^p(\Omega)$ *such that*

$$[v]_{B^s_{p,\infty}(\Omega)} := \sup_{h \in \mathbb{R}^n} \left(\int_\Omega \frac{|\delta_h v(x)|^p}{|h|^{sp}} \cdot \mathbb{1}_\Omega(x+h)\, dx \right)^{\frac{1}{p}} < +\infty. \tag{1.22}$$

For $s \in (0,1)$, $1 \le p < \infty$ and $1 \le q \le \infty$, we also have the equivalence

$$\|v\|_{B^s_{p,q}(\Omega)} \approx \|v\|_{L^p(\Omega)} + [v]_{B^s_{p,q}(\Omega)}.$$

If one replaces $\mathbb{R}^n$ in (1.21) by a ball $B(0,r)$ for a fixed $r > 0$, then an equivalent norm is obtained. Similarly, in (1.22) one can simply take the supremum over $|h| \le r$ and obtain an equivalent norm.

If Ω is a bounded C^∞-domain in $\mathbb{R}^n$, then the *restriction operator*

$$\mathrm{re}_\Omega : \mathcal{S}'(\mathbb{R}^n) \hookrightarrow \mathcal{D}'(\Omega), \quad \mathrm{re}_\Omega(v) = v|_\Omega$$

generates a linear and bounded map from $B^s_{p,q}(\mathbb{R}^n)$ onto $B^s_{p,q}(\Omega)$. Furthermore, the spaces $B^s_{p,q}(\Omega)$ satisfy the so-called *extension property*, as ensured by the next theorem.

Theorem 1.21. *Let* $s \in \mathbb{R}$, *let* $1 \le p, q \le \infty$ *and let* Ω *be a bounded* C^∞*-domain in* $\mathbb{R}^n$. *Then, there exists a linear and bounded extension operator* $\mathrm{ext}_\Omega : B^s_{p,q}(\Omega) \hookrightarrow B^s_{p,q}(\mathbb{R}^n)$ *such that* $\mathrm{re}_\Omega \circ \mathrm{ext}_\Omega = \mathrm{id}$, *where* id *is the identity in* $B^s_{p,q}(\Omega)$.

We refer to [92, Theorem 2.82] for a proof of the above theorem (see also [90, Theorem 3.3.4]). The next result is an immediate consequence of [90, Theorem 3.3.5]:

Theorem 1.22. *Let* $s \in \mathbb{R}$ *and* $1 \le p, q \le \infty$. *Moreover, assume that* Ω *is a bounded* C^∞*-domain in* $\mathbb{R}^n$. *Then, for every* $v \in B^s_{p,q}(\Omega)$ *and every* $j \in \{1, \dots, n\}$ *we have*

$$\|\partial_j v\|_{B^{s-1}_{p,q}(\Omega)} \le c\, \|v\|_{B^s_{p,q}(\Omega)}$$

for a positive constant c *which is independent of* v.

If $s \in \mathbb{R}$, $1 \le p < \infty$ and $1 \le q < \infty$, then $\mathcal{S}(\mathbb{R}^n)$ is a dense subset of $B^s_{p,q}(\mathbb{R}^n)$ (cf. [90, Theorem 2.3.3]). Consequently, in that case, a continuous linear functional on $B^s_{p,q}(\mathbb{R}^n)$ can be interpreted in the usual way as an element of $\mathcal{S}'(\mathbb{R}^n)$. More precisely, $g \in \mathcal{S}'(\mathbb{R}^n)$ belongs to the dual space $(B^s_{p,q}(\mathbb{R}^n))'$ of the space $B^s_{p,q}(\mathbb{R}^n)$ if and only if there exists a positive number c such that

$$|g(\phi)| \le c\, \|\phi\|_{B^s_{p,q}(\mathbb{R}^n)} \qquad \text{for all } \phi \in \mathcal{S}(\mathbb{R}^n).$$

We endow $(B^s_{p,q}(\mathbb{R}^n))'$ with the natural dual norm, defined by

$$\|g\|_{(B^s_{p,q}(\mathbb{R}^n))'} = \sup \left\{ |g(\phi)| : \phi \in \mathcal{S}(\mathbb{R}^n) \text{ and } \|\phi\|_{B^s_{p,q}(\mathbb{R}^n)} \le 1 \right\}, \qquad g \in (B^s_{p,q}(\mathbb{R}^n))'.$$

Now we recall the following duality formula, which has to be meant as an isomorphism of normed spaces (see [90, Theorem 2.11.2]).

Theorem 1.23. *Let $s \in \mathbb{R}$, $1 \leq p < \infty$ and $1 \leq q < \infty$. Then*

$$(B^s_{p,q}(\mathbb{R}^n))' = B^{-s}_{p',q'}(\mathbb{R}^n)\,,$$

where $p' = \infty$ if $p = 1$ (similarly for q').

Remark 1.24. The restrictions $p < \infty$ and $q < \infty$ in Theorem 1.23 are natural, since, if either $p = \infty$ or $q = \infty$, then $\mathcal{S}(\mathbb{R}^n)$ is not dense in $B^s_{p,q}(\mathbb{R}^n)$, and the density of $\mathcal{S}(\mathbb{R}^n)$ in $B^s_{p,q}(\mathbb{R}^n)$ is the basis of our interpretation of the dual space $(B^s_{p,q}(\mathbb{R}^n))'$.

For further needs, we now recall the following definition.

Definition 1.25. For $s \in \mathbb{R}$, $1 \leq p \leq \infty$ and $1 \leq q \leq \infty$, we define $\mathring{B}^s_{p,q}(\mathbb{R}^n)$ as the completion of $\mathcal{S}(\mathbb{R}^n)$ in $B^s_{p,q}(\mathbb{R}^n)$ with respect to the norm

$$v \mapsto \|v\|_{B^s_{p,q}(\mathbb{R}^n)}\,.$$

Of course, only the limit cases $\max\{p, q\} = \infty$ are of interest. We shall denote by $(\mathring{B}^s_{p,q}(\mathbb{R}^n))'$ the topological dual of $\mathring{B}^s_{p,q}(\mathbb{R}^n)$, which is endowed with the natural dual norm

$$\|g\|_{(\mathring{B}^s_{p,q}(\mathbb{R}^n))'} = \sup\left\{|g(\phi)| : \phi \in \mathcal{S}(\mathbb{R}^n) \text{ and } \|\phi\|_{B^s_{p,q}(\mathbb{R}^n)} \leq 1\right\}, \qquad g \in (\mathring{B}^s_{p,q}(\mathbb{R}^n))'.$$

The following duality result can be found in [90, Section 2.11.2, Remark 2] (see also [89, pages 121 and 122]).

Theorem 1.26. *Let $s \in \mathbb{R}$, $1 \leq p \leq \infty$ and $1 \leq q \leq \infty$. Then*

$$(\mathring{B}^s_{p,q}(\mathbb{R}^n))' = B^{-s}_{p',q'}(\mathbb{R}^n)\,,$$

where $p' = 1$ if $p = \infty$ (similarly for q').

One also has the following version of Sobolev embeddings (a proof can be found in [57, Proposition 7.12]).

Lemma 1.27. *Suppose that $0 < \alpha < 1$.*
(a) If $1 < p < \frac{n}{\alpha}$ and $1 \leq q \leq p^*_\alpha := \frac{np}{n-\alpha p}$, then there exists a continuous embedding $B^\alpha_{p,q}(\mathbb{R}^n) \hookrightarrow L^{p^*_\alpha}(\mathbb{R}^n)$.
(b) If $p = \frac{n}{\alpha}$ and $1 \leq q \leq \infty$, then there exists a continuous embedding $B^\alpha_{p,q}(\mathbb{R}^n) \hookrightarrow \mathrm{BMO}(\mathbb{R}^n)$, where BMO *denotes the space of functions with bounded mean oscillations* [55, Chapter 2].

We now recall the following inclusions between Besov spaces (see [57, Proposition 7.10 and Formula (7.35)]).

Lemma 1.28. *Suppose that $0 < \beta < \alpha < 1$.*
(a) If $1 < p \leq \infty$ and $1 \leq q \leq r \leq \infty$, then $B^\alpha_{p,q}(\mathbb{R}^n) \subset B^\alpha_{p,r}(\mathbb{R}^n)$.
(b) If $1 < p \leq \infty$ and $1 \leq q, r \leq \infty$, then $B^\alpha_{p,q}(\mathbb{R}^n) \subset B^\beta_{p,r}(\mathbb{R}^n)$.
(c) If $1 \leq q \leq \infty$, then $B^\alpha_{\frac{n}{\alpha},q}(\mathbb{R}^n) \subset B^\beta_{\frac{n}{\beta},q}(\mathbb{R}^n)$.

Combining Lemmas 1.27 and 1.28, we obtain the following Sobolev-type embedding theorem for Besov spaces $B^\alpha_{p,\infty}(\mathbb{R}^n)$ that are excluded from assumptions (a) in Lemma 1.27.

Lemma 1.29. *Suppose that $0 < \beta < \alpha < 1$. If $1 < p < \frac{n}{\alpha}$, then there exists a continuous embedding $B^{\alpha}_{p,\infty}(\mathbb{R}^n) \hookrightarrow L^{p^*_\beta}(\mathbb{R}^n)$. Moreover, for every $v \in B^{\alpha}_{p,\infty}(\mathbb{R}^n)$, the following local estimate*

$$\|v\|_{L^{\frac{np}{n-\beta p}}(B_\rho)} \leq c\left(\|v\|_{L^p(B_R)} + [v]_{B^{\alpha}_{p,\infty}(B_R)}\right)$$

holds for every ball $B_\rho \subset B_R$, with $c = c(n, R, \rho, \alpha, \beta)$.

We can also define local Besov spaces as follows. Given a domain $\Omega \subset \mathbb{R}^n$, we say that a function $v : \Omega \to \mathbb{R}$ belongs to the *local Besov space* $B^s_{p,q,loc}(\Omega)$ if $\phi\, v \in B^s_{p,q}(\mathbb{R}^n)$ whenever $\phi \in C^\infty_0(\Omega)$. It is worth noticing that one can prove suitable local versions of Lemmas 1.27 and 1.28, by using local Besov spaces.

The following lemma is a consequence of Remark 1.18 and its proof can be found in [12, Lemma 7].

Lemma 1.30. *Let $s \in (0,1)$ and $1 \leq p, q < \infty$. A function $v \in L^p_{loc}(\Omega)$ belongs to the local Besov space $B^s_{p,q,loc}(\Omega)$ if and only if*

$$\left\| \frac{\delta_h v}{|h|^s} \right\|_{L^q\left(\frac{dh}{|h|^n}\,;\, L^p(B)\right)} < +\infty$$

for any ball $B \subset B_{2r_B} \Subset \Omega$ with radius r_B. Here, the balls B and B_{2r_B} are assumed to be concentric and the measure $\frac{dh}{|h|^n}$ is restricted to the ball $B(0, r_B)$ on the h-space.

The next result represents the local counterpart of Lemma 1.29:

Lemma 1.31. *On any domain $\Omega \subset \mathbb{R}^n$ we have the continuous embedding $B^{\alpha}_{p,\infty,loc}(\Omega) \hookrightarrow L^r_{loc}(\Omega)$ for all $r \in \left[1, \frac{np}{n-\alpha p}\right)$, provided $\alpha \in (0,1)$ and $1 < p < \frac{n}{\alpha}$.*

We refer to [87, Sections 30-32] for a proof of this lemma. In fact, the above statement follows by localizing the corresponding result proved for functions defined on $\mathbb{R}^n$ in [87], by simply using a smooth cut-off function.

For the treatment of parabolic equations, we now give the following definition.

Definition 1.32. Let Ω be a bounded C^∞-domain in $\mathbb{R}^n$. Let $s \in (0,1)$, $1 \leq p < \infty$ and $1 \leq q \leq \infty$. Then, we say that a map $g \in L^p(\Omega \times (t_0, t_1))$ belongs to the space $L^p\left(t_0, t_1; B^s_{p,q}(\Omega)\right)$ if and only if

$$\int_{t_0}^{t_1} [g(\cdot, t)]^p_{B^s_{p,q}(\Omega)}\, dt \leq \infty\,,$$

where $[\,\cdot\,]_{B^s_{p,q}(\Omega)}$ is defined by (1.21) or (1.22).

In the sequel, we shall use the corresponding local version of the space defined above, which will be denoted by the subscript "*loc*". More precisely, given a bounded domain $\Omega \subset \mathbb{R}^n$, we write $g \in L^p_{loc}\left(0, T; B^s_{p,q,loc}(\Omega)\right)$ if and only if $g \in L^p\left(t_0, t_1; B^s_{p,q}(\Omega')\right)$ for all domains $\Omega' \times (t_0, t_1) \Subset \Omega_T := \Omega \times (0, T)$, with Ω' being a (bounded) C^∞-domain. Furthermore, we shall also use the following notation, which is typical of Bochner spaces:

$$\|g\|_{L^p(t_0,\,t_1\,;\,B^s_{p,q}(\Omega'))} := \left(\int_{t_0}^{t_1} \|g(\cdot, t)\|^p_{B^s_{p,q}(\Omega')}\, dt\right)^{\frac{1}{p}}.$$

1.2.2 Fractional Sobolev spaces

Here we recall the definition and some properties of the fractional Sobolev spaces that will be needed later on (for more details, we refer to [39]).

Let Ω be a general, possibly nonsmooth, open set in $\mathbb{R}^n$. For any $s \in (0,1)$ and for any $q \in [1,+\infty)$, we define the fractional Sobolev space $W^{s,q}(\Omega,\mathbb{R}^k)$ as follows:

$$W^{s,q}(\Omega,\mathbb{R}^k) := \left\{ v \in L^q(\Omega,\mathbb{R}^k) : \frac{|v(x)-v(y)|}{|x-y|^{\frac{n}{q}+s}} \in L^q(\Omega\times\Omega) \right\},$$

i.e. an intermediate Banach space between $L^q(\Omega,\mathbb{R}^k)$ and $W^{1,q}(\Omega,\mathbb{R}^k)$, endowed with the norm

$$\|v\|_{W^{s,q}(\Omega)} := \left(\int_\Omega |v|^q\,dx + \int_\Omega\int_\Omega \frac{|v(x)-v(y)|^q}{|x-y|^{n+sq}}\,dx\,dy \right)^{\frac{1}{q}},$$

where the term

$$[v]_{W^{s,q}(\Omega)} := \left(\int_\Omega\int_\Omega \frac{|v(x)-v(y)|^q}{|x-y|^{n+sq}}\,dx\,dy \right)^{\frac{1}{q}} \tag{1.23}$$

is the so-called *Gagliardo seminorm* of v.

Remark 1.33. For every $s \in (0,1)$ and every $q \in [1,\infty)$, we have $B^s_{q,q}(\mathbb{R}^n) = W^{s,q}(\mathbb{R}^n)$. In fact, using the change of variable $y = x + h$ in (1.23) with $\Omega = \mathbb{R}^n$, one gets the seminorm (1.19) with $p = q$.

As in the classic case with s being an integer, the space $W^{\sigma,q}(\Omega)$ is continuously embedded in $W^{s,q}(\Omega)$ when $s \le \sigma$, as shown by the next result (see [39, Proposition 2.1]).

Proposition 1.34. *Let Ω be an open set in $\mathbb{R}^n$, let $q \in [1,+\infty)$ and let $0 < s \le \sigma < 1$. Then there exists a constant $C = C(n,s,q) \ge 1$ such that, for any $v \in W^{\sigma,q}(\Omega)$, we have*

$$\|v\|_{W^{s,q}(\Omega)} \le C\,\|v\|_{W^{\sigma,q}(\Omega)}.$$

In particular, $W^{\sigma,q}(\Omega) \subseteq W^{s,q}(\Omega)$.

As is well known when $s \in \mathbb{N}$, under certain regularity assumptions on the open set $\Omega \subset \mathbb{R}^n$, any function in $W^{s,q}(\Omega)$ can be extended to a function in $W^{s,q}(\mathbb{R}^n)$. Extension results are needed to improve some embedding theorems, in the classic case as well as in the fractional one. In this regard, we now give the following definition.

Definition 1.35. For any $s \in (0,1)$ and any $q \in [1,\infty)$, we say that an open set $\Omega \subseteq \mathbb{R}^n$ is an *extension domain* for $W^{s,q}$ if there exists a positive constant $C = C(n,q,s,\Omega)$ such that for every function $v \in W^{s,q}(\Omega)$ there exists $\widetilde{v} \in W^{s,q}(\mathbb{R}^n)$ with $\widetilde{v} = v$ on Ω and

$$\|\widetilde{v}\|_{W^{s,q}(\mathbb{R}^n)} \le C\,\|v\|_{W^{s,q}(\Omega)}.$$

In general, an arbitrary open set may not be an extension domain for $W^{s,q}$. However, the following result ensures that every open Lipschitz set Ω with bounded boundary is an extension domain for $W^{s,q}$ (a proof can be found in [39, Theorem 5.4]).

Theorem 1.36. *Let $q \in [1, +\infty)$, let $s \in (0,1)$ and let $\Omega \subseteq \mathbb{R}^n$ be an open set of class $C^{0,1}$ with bounded boundary. Then $W^{s,q}(\Omega)$ is continuously embedded in $W^{s,q}(\mathbb{R}^n)$, namely for any $v \in W^{s,q}(\Omega)$ there exists $\widetilde{v} \in W^{s,q}(\mathbb{R}^n)$ such that $\widetilde{v} = v$ on Ω and*

$$\|\widetilde{v}\|_{W^{s,q}(\mathbb{R}^n)} \leq C \|v\|_{W^{s,q}(\Omega)}$$

for some positive constant $C = C(n, q, s, \Omega)$.

For more information on the problem of characterizing the class of sets that are extension domains for $W^{s,q}$, we refer the interested reader to [94], where an answer to this question has been given (see also [60] and [64, Chapters 11 and 12]).

For further needs, we recall the following Sobolev-type embedding theorem, whose proof can be found in [39, Theorem 6.7].

Theorem 1.37. *Let $s \in (0,1)$ and $q \in [1, +\infty)$ be such that $sq < n$. Let $\Omega \subseteq \mathbb{R}^n$ be an extension domain for $W^{s,q}$. Then there exists a positive constant $C = C(n, q, s, \Omega)$ such that, for any $v \in W^{s,q}(\Omega)$, we have*

$$\|v\|_{L^r(\Omega)} \leq C \|v\|_{W^{s,q}(\Omega)}$$

for any $r \in [q, q^*]$; that is, the space $W^{s,q}(\Omega)$ is continuously embedded in $L^r(\Omega)$ for any $r \in [q, q^*]$, where $q^* := nq/(n - sq)$ is the so-called "fractional critical exponent".

Moreover, if Ω is bounded, then the space $W^{s,q}(\Omega)$ is continuously embedded in $L^r(\Omega)$ for any $r \in [1, q^*]$.

Remark 1.38. In the critical case $r = q^*$, the constant C in Theorem 1.37 does not depend on Ω (see [39, Remark 6.8]).

The next embedding result can be obtained by combining [90, Section 2.2.2, Remark 3] with [90, Section 2.3.2, Proposition 2(ii)].

Lemma 1.39. *Let $s \in (0,1)$ and $q \geq 1$. Then, for every $\sigma \in (0, 1-s)$ we have the continuous embedding $W^{s+\sigma,q}_{loc}(\mathbb{R}^n) \hookrightarrow B^s_{q,1,loc}(\mathbb{R}^n)$.*

For the treatment of parabolic equations, we now give the following definition.

Definition 1.40. Let $q \in [1, +\infty)$ and let $s \in (0,1)$. We say that a map $g \in L^q(\Omega \times (t_0, t_1), \mathbb{R}^k)$ belongs to the space $L^q\left(t_0, t_1; W^{s,q}(\Omega, \mathbb{R}^k)\right)$ if and only if

$$\int_{t_0}^{t_1} \int_\Omega \int_\Omega \frac{|g(x,t) - g(y,t)|^q}{|x-y|^{n+sq}}\, dx\, dy\, dt \; < \infty.$$

In the sequel, we will use the corresponding local version of the space defined above, which will be denoted by the subscript "*loc*". More precisely, we write $g \in L^q_{loc}\left(0, T; W^{s,q}_{loc}(\Omega, \mathbb{R}^k)\right)$ if and only if $g \in L^q\left(t_0, t_1; W^{s,q}(\Omega', \mathbb{R}^k)\right)$ for all domains $\Omega' \times (t_0, t_1) \Subset \Omega_T := \Omega \times (0, T)$.

We conclude this subsection with the parabolic version of the well-known relation between Nikolskii spaces and fractional Sobolev spaces (see [2, 7.73]). This version is contained in [84, Lemma 2.4] (see also [41, Proposition 2.19]), and its proof can be obtained by a straightforward adaptation of the standard elliptic results contained in [40, 46, 72, 73, 74].

Proposition 1.41. *Let $g \in L^q(\Omega_T, \mathbb{R}^k)$, with $1 \le q < \infty$. Assume that for $\theta \in (0,1)$, $M \in [0,\infty)$ and a cylinder $Q_\rho(z_0) = B_\rho(x_0) \times (t_0 - \rho^2, t_0) \Subset \Omega_T$, the inequality*

$$|h|^{-q\theta} \int_{Q_\rho(z_0)} |g(x + he_i, t) - g(x,t)|^q \, dx \, dt \le M^q$$

holds whenever $0 < |h| \le \min\{\operatorname{dist}(B_\rho(x_0), \partial\Omega), A\}$ and $i \in \{1, \dots, n\}$, where $A > 0$ and $\{e_i\}_{1 \le i \le n}$ is the standard basis of $\mathbb{R}^n$. Then

$$g \in L^q_{loc}\left(t_0 - \rho^2, t_0; W^{s,q}_{loc}(B_\rho(x_0), \mathbb{R}^k)\right) \quad \text{for all } s \in (0, \theta).$$

1.2.3 Orlicz spaces

Here we recall some basic properties of Orlicz spaces that will be needed later on (for more details, we refer to [2]).

Let $\Psi : [0,\infty) \to [0,\infty)$ be a Young function, i.e. $\Psi(0) = 0$, Ψ is increasing and convex. If Ω is an open subset of $\mathbb{R}^n$, we define the Orlicz space $L^\Psi(\Omega)$ generated by the Young function Ψ as the set of the measurable functions $v : \Omega \to \mathbb{R}$ such that

$$\int_\Omega \Psi\left(\frac{|v|}{\lambda}\right) dx < \infty$$

for some $\lambda > 0$. When equipped with the Luxemburg norm

$$\|v\|_{L^\Psi(\Omega)} := \inf\left\{\lambda > 0 : \int_\Omega \Psi\left(\frac{|v|}{\lambda}\right) dx \le 1\right\},$$

$L^\Psi(\Omega)$ is a Banach space.

The Zygmund space $L^q \log^\alpha L(\Omega)$, for $1 \le q < \infty$, $\alpha \in \mathbb{R}$ ($\alpha \ge 0$ for $q = 1$), is defined as the Orlicz space $L^\Psi(\Omega)$ generated by the Young function $\Psi(s) \simeq s^q \log^\alpha(e+s)$ for every $s \ge s_0 \ge 0$. Therefore, a measurable function v on Ω belongs to $L^q \log^\alpha L(\Omega)$ if

$$\int_\Omega |v|^q \log^\alpha(e + |v|) \, dx < \infty.$$

Moreover, we record that for the function

$$\Psi(s) = s^q \log^\alpha(e+s), \qquad q > 1, \, \alpha \in \mathbb{R},$$

the following inequality

$$\int_\Omega |v|^q \log^\alpha(e + |v|) \, dx \le \|v\|^\theta_{L^\Psi(\Omega)}$$

holds for every $v \in L^q \log^\alpha L(\Omega)$ and some $\theta = \theta(q) > 0$ (see [29]).

Chapter 2

Besov regularity for a class of singular or degenerate elliptic equations

In this chapter, we are concerned with the regularity properties of weak solutions to the strongly singular or degenerate elliptic equation

$$-\operatorname{div}\left((|Du|-\lambda)_+^{p-1}\frac{Du}{|Du|}\right)=f \quad \text{in } \Omega, \tag{2.1}$$

where $1<p<\infty$, Ω is a bounded domain in $\mathbb{R}^n$ ($n\geq 2$), $\lambda>0$ is a fixed parameter and $(\,\cdot\,)_+$ stands for the positive part. The results that we report here are contained in the paper [4].

As already anticipated in the introduction to this thesis, we are interested in the higher differentiability of the vector field

$$H_{\frac{p}{2}}(Du):=(|Du|-\lambda)_+^{\frac{p}{2}}\,\frac{Du}{|Du|}\,,$$

where $u\in W^{1,p}_{loc}(\Omega)$ is a local weak solution of (2.1). In [22], Brasco, Carlier and Santambrogio established the local $W^{1,2}$-regularity of $H_{p/2}(Du)$ in the case $p\geq 2$ and $f\in W^{1,p'}(\Omega)$. The main novelty of our results is that we address the subquadratic (or *singular*) case $1<p<2$, which had been neglected in the literature, since extra technical difficulties arise concerning elliptic regularity whenever we are in this case. These difficulties prevent us from achieving the same kind of regularity results as those obtained in [22] for the solutions of equation (2.1) and problem (P1) (see the Introduction).

However, despite the difficulties we have mentioned, we are able to prove a Besov regularity result for the function $H_{p/2}(Du)$, when the datum f belongs either to the Sobolev class $W^{1,p'}_{loc}(\Omega)$ or to the local Besov space $B^{\alpha}_{p',\infty,loc}(\Omega)$, for some $\alpha\in(0,1)$. Next, we infer the Besov regularity of the solution σ_0 to (P1), under the same assumptions on the right-hand side f.

The first main result of this chapter is the following theorem. We refer to Chapter 1 for notation and definitions.

Theorem 2.1. *Let $n\geq 2$, $p\in(1,2)$ and $\alpha\in(0,1)$. Moreover, let $u\in W^{1,p}_{loc}(\Omega)$ be a local weak solution of equation (2.1). Then, the following implications hold:*

$$f\in W^{1,p'}_{loc}(\Omega) \;\;\Rightarrow\;\; H_{\frac{p}{2}}(Du)\in B^{\frac{1}{3-p}}_{2,\infty} \;\; \textit{locally in } \Omega,$$

$$f \in B^{\alpha}_{p',\infty,loc}(\Omega) \;\Rightarrow\; H_{\frac{p}{2}}(Du) \in B^{\min\left\{\frac{\alpha+1}{2},\frac{1}{3-p}\right\}}_{2,\infty} \quad \text{locally in } \Omega.$$

Furthermore:

(*a*) *if* $f \in W^{1,p'}_{loc}(\Omega)$, *then for any ball* $B_R \Subset \Omega$ *the following estimate*

$$\int\limits_{B_{R/2}} \left|\tau_{s,h} H_{\frac{p}{2}}(Du)\right|^2 dx \le C_1 \left(\|Df\|_{L^{p'}(B')} \|Du\|_{L^p(B')} |h|^2 + \|Du\|^p_{L^p(B')} |h|^{\frac{2}{3-p}} \right)$$

holds true for every $s \in \{1,\dots,n\}$, *for every* $h \in \mathbb{R}$ *such that* $|h| \le r_0 < \frac{1}{2}\operatorname{dist}(B_R, \partial\Omega)$, *for* $B' = B_R + B(0,r_0)$ *and a positive constant* $C_1 = C_1(n,p,R)$.

(*b*) *If, on the other hand,* $f \in B^{\alpha}_{p',\infty,loc}(\Omega)$, *then for any ball* $B_R \Subset \Omega$ *the following estimate*

$$\int\limits_{B_{R/2}} \left|\tau_{s,h} H_{\frac{p}{2}}(Du)\right|^2 dx \le C_2 \left(\|f\|_{B^{\alpha}_{p',\infty}(B')} \|Du\|_{L^p(B')} |h|^{\alpha+1} + \|Du\|^p_{L^p(B')} |h|^{\frac{2}{3-p}} \right)$$

holds true for every $s \in \{1,\dots,n\}$, *for every* $h \in \mathbb{R}$ *such that* $|h| \le r_0 < \frac{1}{2}\operatorname{dist}(B_R, \partial\Omega)$, *for* $B' = B_R + B(0,r_0)$ *and a positive constant* $C_2 = C_2(n,p,R)$.

The main tool in the proof of Theorem 2.1 is the difference quotient technique, which, due to the singular character of the case under consideration, requires new ideas with respect to previous papers on the subject ([1] and [88]), where the case $1 < p < 2$ has been treated, but in the absence of singularity. For the sake of clarity, let us now try to explain the difficulties associated with the singular setting. When dealing with the higher differentiability of solutions to subquadratic problems, the difference quotient method usually yields a Caccioppoli-type inequality based on the possibility of reabsorbing the terms in the right-hand side that involve the increment of the gradient of a solution by the left-hand side. In the singular setting, this possibility is actually excluded by the fact that we cannot control the difference quotient of the gradient of a solution u in terms of $H_{\frac{p}{2}}(Du)$, whose difference quotient appears in the left-hand side, and this is true regardless of the type of regularity of the datum f. Actually, even in the case $f = 0$, the integral appearing in the right-hand side of formula (2.3) below cannot be estimated in a standard way, since the difference quotient $\Delta_h u$ cannot be controlled in terms of $\Delta_h H_{p-1}(Du)$, unless we perform the estimate far away from the set $\{|Du| \le \lambda\}$. On the other hand, as already observed in the Introduction, every λ-Lipschitz function u is a solution of the homogeneous equation

$$-\operatorname{div}\left((|Du| - \lambda)^{p-1}_{+} \frac{Du}{|Du|} \right) = 0\,,$$

so that a more regular f may not be sufficient to ensure a higher differentiability property of the solutions.

We now go back to the variational problems (P1) and (P2) presented in the Introduction. Thanks to Theorem 2.1, we can obtain the following higher differentiability result of fractional order.

Theorem 2.2. *Let* $n \ge 2$, $p \in (1,2)$ *and* $\alpha \in (0,1)$. *Moreover, let* $\sigma_0 \in L^{p'}(\Omega,\mathbb{R}^n)$ *be the solution of* (P1). *Then, the following implications hold:*

$$f \in W^{1,p'}_{loc}(\Omega) \;\Rightarrow\; \sigma_0 \in B^{\frac{2}{p'(3-p)}}_{p',\infty} \quad \text{locally in } \Omega,$$

$$f \in B^{\alpha}_{p',\infty,loc}(\Omega) \;\Rightarrow\; \sigma_0 \in B^{\min\left\{\frac{\alpha+1}{p'},\,\frac{2}{p'(3-p)}\right\}}_{p',\infty} \quad \textit{locally in } \Omega.$$

Furthermore:

(a) if $f \in W^{1,p'}_{loc}(\Omega)$*, then for every solution* u *of problem* (P2) *and every ball* $B_R \Subset \Omega$*, the following estimate*

$$\int\limits_{B_{R/2}} |\tau_{s,h}\sigma_0|^{p'}\,dx \le C_1\left(\|Df\|_{L^{p'}(B')}\,\|Du\|_{L^p(B')}\,|h|^2 + \|Du\|^p_{L^p(B')}\,|h|^{\frac{2}{3-p}}\right)$$

holds true for every $s \in \{1,\dots,n\}$*, for every* $h \in \mathbb{R}$ *such that* $|h| \le r_0 < \frac{1}{2}\operatorname{dist}(B_R,\partial\Omega)$*, for* $B' = B_R + B(0,r_0)$ *and a positive constant* $C_1 = C_1(n,p,R)$*.*

(b) If, on the other hand, $f \in B^{\alpha}_{p',\infty,loc}(\Omega)$*, then for every solution* u *of problem* (P2) *and every ball* $B_R \Subset \Omega$*, the following estimate*

$$\int\limits_{B_{R/2}} |\tau_{s,h}\sigma_0|^{p'}\,dx \le C_2\left(\|f\|_{B^{\alpha}_{p',\infty}(B')}\,\|Du\|_{L^p(B')}\,|h|^{\alpha+1} + \|Du\|^p_{L^p(B')}\,|h|^{\frac{2}{3-p}}\right)$$

holds true for every $s \in \{1,\dots,n\}$*, for every* $h \in \mathbb{R}$ *such that* $|h| \le r_0 < \frac{1}{2}\operatorname{dist}(B_R,\partial\Omega)$*, for* $B' = B_R + B(0,r_0)$ *and a positive constant* $C_2 = C_2(n,p,R)$*.*

Remark 2.3. If we take $p = 2$ and $f \in W^{1,2}_{loc}(\Omega)$ in the statements (a) of Theorems 2.1 and 2.2, then we get back the full Sobolev regularity results of [22, Theorem 4.2 and Corollary 4.3], i.e. there is no discrepancy between our results and the ones contained in [22].

As a consequence of Theorem 2.2 and Lemma 1.31, we obtain the following higher integrability result for σ_0.

Corollary 2.4. *Under the assumptions of Theorem 2.2, we obtain the following implications:*

$$f \in W^{1,p'}_{loc}(\Omega) \;\Rightarrow\; \sigma_0 \in L^r_{loc}(\Omega,\mathbb{R}^n) \quad \textit{for all } r \in \left[1,\frac{n(3-p)\,p'}{n(3-p)-2}\right),$$

$$f \in B^{\alpha}_{p',\infty,loc}(\Omega) \;\Rightarrow\; \sigma_0 \in L^s_{loc}(\Omega,\mathbb{R}^n),$$

where

$$s = \begin{cases} \textit{any value in } \left[1,\frac{np'}{n-\alpha-1}\right), & \textit{if } 0 < \alpha < \frac{p-1}{3-p}, \\ \textit{any value in } \left[1,\frac{n(3-p)\,p'}{n(3-p)-2}\right), & \textit{if } \frac{p-1}{3-p} \le \alpha < 1. \end{cases}$$

As an easy consequence of the previous results, we also get a gain of integrability for the gradient of the local weak solutions of (2.1). Indeed, we have the following result.

Corollary 2.5. *Under the assumptions of Theorem 2.1, we obtain the following implications:*

$$f \in W^{1,p'}_{loc}(\Omega) \;\Rightarrow\; Du \in L^r_{loc}(\Omega,\mathbb{R}^n) \quad \textit{for all } r \in \left[1,\frac{np\,(3-p)}{n(3-p)-2}\right),$$

$$f \in B^{\alpha}_{p',\infty,loc}(\Omega) \;\Rightarrow\; Du \in L^{s}_{loc}(\Omega,\mathbb{R}^n)\,,$$

where

$$s = \begin{cases} \text{any value in } \left[1, \frac{np}{n-\alpha-1}\right), & \text{if } 0<\alpha<\frac{p-1}{3-p}\,, \\ \text{any value in } \left[1, \frac{np\,(3-p)}{n(3-p)-2}\right), & \text{if } \frac{p-1}{3-p}\le\alpha<1. \end{cases}$$

In this chapter, we shall prove that similar results also hold in the case $p \geq 2$, by assuming that $f \in B^{\alpha}_{p',\infty,loc}(\Omega)$ for some $\alpha \in (0,1)$. Actually, on the one hand, Theorem 2.7 below extends the results proved in [30] to a widely degenerate setting; on the other hand, it extends the aforementioned results contained in [22] to the case of data in a local Besov space.

It is worth mentioning that if $f \in W^{1,p'}_{\diamond}(\Omega)$, then for suitable values of $n \geq 2$ and $p \in (1,2)$ there are some cases in which it is possible to prove that every solution $u \in W^{1,p}_{\diamond}(\Omega)$[1] of the Neumann boundary problem (1) belongs to $W^{1,\infty}(\Omega)$ (see [19, Theorem 2.1]). However, in these cases, the boundedness of Du gives neither a weak differentiability of integer order for $H_{p/2}(Du)$ nor a fractional differentiability of higher order than that obtained in Theorem 2.1.

Before describing the structure of this chapter, we wish to briefly motivate what follows by pointing out that problems such as (P1) and (P2) and degenerate equations of the form (2.1) may arise in *optimal transport problems with congestion effects* on very dense networks. These problems lead, at the limit, to minimization problems posed on measures on curves, as shown in [11]. In [20] and [22], Brasco, Carlier and Santambrogio have gone one step further, by showing that these latter problems can be reformulated in terms of the minimization of an integral functional over a set of vector fields with prescribed divergence, exactly as in problem (P1), which was introduced by Beckmann in the 1950s (see [14]). For further details, we refer the interested reader to [19, Sections 1-2] and [21, Section 1].

This chapter is organized as follows. In Section 2.1, we proceed with the analysis of the singular case $1 < p < 2$, by proving Theorems 2.1 and 2.2 and the related Corollaries 2.4 and 2.5. Finally, in Section 2.2 we give similar results in the degenerate case $p \geq 2$, under the assumption that $f \in B^{\alpha}_{p',\infty,loc}(\Omega)$ for some $\alpha \in (0,1)$.

2.1 The singular case $1 < p < 2$

In this section, we prove Theorems 2.1 and 2.2 as well as Corollaries 2.4 and 2.5. Let us start with the

Proof of Theorem 2.1. Let e_s be the unit vector in the direction x_s of $\mathbb{R}^n$; from now on, by a slight abuse of notation, we shall write $\tau_h g(x)$ to denote $[g(x+he_s)-g(x)]$, where $h \in \mathbb{R}\setminus\{0\}$.

Now, let $\varphi \in W^{1,p}(\Omega)$ be compactly supported in Ω (i.e. a test function) and let $h \in \mathbb{R}\setminus\{0\}$ be such that $|h| < \operatorname{dist}(\operatorname{spt}(\varphi), \mathbb{R}^n \setminus \Omega)$. Since u is a local weak solution of equation (2.1), we have

$$\int_{\Omega} \langle \Delta_h H_{p-1}(Du), D\varphi\rangle \, dx \;=\; \int_{\Omega} (\Delta_h f)\,\varphi \, dx\,. \tag{2.2}$$

Let us fix two concentric balls $B_{R/2}$ and B_R with $B_R \Subset \Omega$, and consider a cut-off function $\xi \in C^{\infty}_c(B_R)$ such that $0 \leq \xi \leq 1$, $\xi \equiv 1$ on $\overline{B}_{R/2}$ and $\|D\xi\|_{\infty} \leq C/R$. Let $h \in \mathbb{R}\setminus\{0\}$ be such

[1] The space $W^{1,p}_{\diamond}(\Omega)$ is defined in (3).

that $|h| \le r_0 < \frac{1}{2}\operatorname{dist}(B_R, \mathbb{R}^n \setminus \Omega)$. In what follows, we will denote by c_k and c some positive constants which do not depend on h, but they may vary on different occasions.

Let us first assume that $f \in B^{\alpha}_{p',\infty,loc}(\Omega)$. We then choose $\varphi = \xi^2 \Delta_h u$ as a test function into (2.2). Using the fact that $u \in W^{1,p}_{loc}(\Omega)$, $f \in B^{\alpha}_{p',\infty,loc}(\Omega)$, Hölder's inequality and the properties of the difference quotients, and defining $B' := B_R + B(0, r_0)$, we get

$$\begin{aligned}
&\int_\Omega \langle \Delta_h H_{p-1}(Du), \xi^2 \Delta_h Du + 2\,\xi\, D\xi\, \Delta_h u\rangle\, dx \;\le \int_{B_R} |\Delta_h f|\, \xi^2\, |\Delta_h u|\; dx \\
&\quad \le \left(\int_{B_R} |\Delta_h f|^{p'}\, dx\right)^{\frac{1}{p'}} \left(\int_{B_R} \xi^{2p}\, |\Delta_h u|^p\, dx\right)^{\frac{1}{p}} \le \left(\int_{B_R} \frac{|\Delta_h f|^{p'}}{|h|^{\alpha p'}}\, dx\right)^{\frac{1}{p'}} |h|^{\alpha}\; \|\Delta_h u\|_{L^p(B_R)} \\
&\quad \le\; c_1(n) \left(\int_{B_R} \frac{|\tau_h f|^{p'}}{|h|^{\alpha p'}}\, dx\right)^{\frac{1}{p'}} |h|^{\alpha-1}\, \|Du\|_{L^p(B')} \;\le\; c_1(n)\, \|f\|_{B^{\alpha}_{p',\infty}(B')}\, \|Du\|_{L^p(B')}\, |h|^{\alpha-1}\,.
\end{aligned}$$

From the previous estimate, we then obtain

$$\begin{aligned}
&\int_\Omega \xi^2 \langle \Delta_h H_{p-1}(Du), \Delta_h Du\rangle\, dx \\
&\quad \le\; c_1(n)\, \|f\|_{B^{\alpha}_{p',\infty}(B')}\, \|Du\|_{L^p(B')}\, |h|^{\alpha-1} - 2\int_\Omega \langle \Delta_h H_{p-1}(Du), D\xi\rangle\, \xi\, \Delta_h u\, dx\,. \qquad (2.3)
\end{aligned}$$

Now, by Lemma 1.7 we have

$$c(p,n) \int_{B_R} \xi^2 \left|\Delta_h H_{\frac{p}{2}}(Du)\right|^2 dx \;\le \int_\Omega \xi^2 \langle \Delta_h H_{p-1}(Du), \Delta_h Du\rangle\, dx\,, \qquad (2.4)$$

where $c(p,n)$ is a positive constant. Combining estimates (2.3) and (2.4) and applying the Cauchy-Schwarz inequality as well as the properties of ξ, we get

$$\begin{aligned}
&c(p,n) \int_{B_R} \xi^2 \left|\Delta_h H_{\frac{p}{2}}(Du)\right|^2 dx \\
&\quad \le c_1(n)\, \|f\|_{B^{\alpha}_{p',\infty}(B')}\, \|Du\|_{L^p(B')}\, |h|^{\alpha-1} - 2\int_\Omega \langle \Delta_h H_{p-1}(Du), D\xi\rangle\, \xi\, \Delta_h u\, dx \\
&\quad \le c_1(n)\, \|f\|_{B^{\alpha}_{p',\infty}(B')}\, \|Du\|_{L^p(B')}\, |h|^{\alpha-1} +\; c_2(R) \int_\Omega |\Delta_h H_{p-1}(Du)|\, \xi\, |\Delta_h u|\; dx,
\end{aligned}$$

and dividing by $c(p,n)$, we obtain

$$\begin{aligned}
&\int_{B_R} \xi^2 \left|\Delta_h H_{\frac{p}{2}}(Du)\right|^2 dx \\
&\quad \le c_3(n,p)\, \|f\|_{B^{\alpha}_{p',\infty}(B')}\, \|Du\|_{L^p(B')}\, |h|^{\alpha-1} +\; c_4(n,p,R) \int_\Omega |\Delta_h H_{p-1}(Du)|\, \xi\, |\Delta_h u|\; dx\,, \qquad (2.5)
\end{aligned}$$

Now we set

$$I_1 := \int_\Omega |\Delta_h H_{p-1}(Du)|\, \xi\, |\Delta_h u|\; dx$$

and apply Hölder's inequality and the properties of the difference quotients to estimate I_1 as

follows:

$$I_1 \le c_1(n)\,\|Du\|_{L^p(B')}\left(\int_{B_R}\xi^{p'}\,|\Delta_h H_{p-1}(Du)|^{p'}\,dx\right)^{\frac{1}{p'}}. \tag{2.6}$$

Similarly, we set

$$I_2 := \int_{B_R}\xi^{p'}\,|\Delta_h H_{p-1}(Du)|^{p'}\,dx$$

and use Lemma 1.5, Hölder's inequality with exponents $\left(\frac{2}{p},\frac{2}{2-p}\right)$ and the properties of the difference quotients to control I_2 as follows:

$$\begin{aligned}
I_2 &\le c_5(n,p)\int_{B_R}\xi^{p'}\,|\Delta_h H_1(Du)|^{p'}\left[(|Du(x+he_s)|-\lambda)_+ + (|Du|-\lambda)_+\right]^{p'(p-2)}dx\\
&= c_5(n,p)\int_{B_R}\xi^{p'}\,|\Delta_h H_1(Du)|^{p}\,|\Delta_h H_1(Du)|^{p'-p}\left[(|Du(x+he_s)|-\lambda)_+ + (|Du|-\lambda)_+\right]^{p'(p-2)}dx\\
&\le \frac{c_5(n,p)}{|h|^{p'-p}}\int_{B_R}\xi^{p'}\,|\Delta_h H_1(Du)|^{p}\left[(|Du(x+he_s)|-\lambda)_+ + (|Du|-\lambda)_+\right]^{pp'-(p+p')}dx\\
&= \frac{c_5(n,p)}{|h|^{p'-p}}\int_{B_R}\xi^{p'}\,|\Delta_h H_1(Du)|^{p}\left[(|Du(x+he_s)|-\lambda)_+ + (|Du|-\lambda)_+\right]^{\frac{p}{2}(p-2)+\frac{p}{2}(2-p)}dx\\
&\le \frac{c_6(n,p)}{|h|^{p'-p}}\left(\int_{B_R}\xi^{\frac{2}{p-1}}\,|\Delta_h H_1(Du)|^{2}\left[(|Du(x+he_s)|-\lambda)_+ + (|Du|-\lambda)_+\right]^{p-2}dx\right)^{\frac{p}{2}}\|Du\|_{L^p(B')}^{\frac{p(2-p)}{2}}\\
&\le c_7(n,p)\,\|Du\|_{L^p(B')}^{\frac{p(2-p)}{2}}\,|h|^{p-p'}\left(\int_{B_R}\xi^2\left|\Delta_h H_{\frac{p}{2}}(Du)\right|^2dx\right)^{\frac{p}{2}},
\end{aligned} \tag{2.7}$$

where, in the last line, we have used Lemma 1.5 again, as well as the properties of ξ. Collecting estimates (2.5)−(2.7) and applying Young's inequality with $\theta>0$ and exponents $\left(\frac{2}{p-1},\frac{2}{3-p}\right)$, we obtain

$$\begin{aligned}
&\int_{B_R}\xi^2\left|\Delta_h H_{\frac{p}{2}}(Du)\right|^2dx\\
&\quad\le c_3\,\|f\|_{B^{\alpha}_{p',\infty}(B')}\,\|Du\|_{L^p(B')}\,|h|^{\alpha-1} + c_8(n,p,R)\,\|Du\|_{L^p(B')}\,I_2^{1/p'}\\
&\quad\le c_3\,\|f\|_{B^{\alpha}_{p',\infty}(B')}\,\|Du\|_{L^p(B')}\,|h|^{\alpha-1}\\
&\qquad + c_9(n,p,R)\,\|Du\|_{L^p(B')}^{\frac{p(3-p)}{2}}\,|h|^{p-2}\left(\int_{B_R}\xi^2\left|\Delta_h H_{\frac{p}{2}}(Du)\right|^2dx\right)^{\frac{p-1}{2}}\\
&\quad\le c_3\,\|f\|_{B^{\alpha}_{p',\infty}(B')}\,\|Du\|_{L^p(B')}\,|h|^{\alpha-1}\\
&\qquad + \frac{p-1}{2}\,\theta^{\frac{1}{p-1}}\int_{B_R}\xi^2\left|\Delta_h H_{\frac{p}{2}}(Du)\right|^2dx + \frac{3-p}{2}\,\theta^{\frac{1}{p-3}}\,c_{10}(n,p,R)\,\|Du\|_{L^p(B')}^{p}\,|h|^{\frac{2(p-2)}{3-p}}.
\end{aligned} \tag{2.8}$$

Choosing $\theta=\left(\frac{1}{p-1}\right)^{p-1}$ and reabsorbing the integral in the right-hand side of (2.8) by the left-hand side, we get

$$\int_{B_R}\xi^2\left|\Delta_h H_{\frac{p}{2}}(Du)\right|^2dx \le c\,\|f\|_{B^{\alpha}_{p',\infty}(B')}\,\|Du\|_{L^p(B')}\,|h|^{\alpha-1} + c\,\|Du\|_{L^p(B')}^{p}\,|h|^{\frac{2(p-2)}{3-p}},$$

from which we can infer

$$\int_{B_{R/2}} \left|\Delta_h H_{\frac{p}{2}}(Du)\right|^2 dx \,\le\, c\,\|f\|_{B^{\alpha}_{p',\infty}(B')}\,\|Du\|_{L^p(B')}\,|h|^{\alpha-1} + c\,\|Du\|^p_{L^p(B')}\,|h|^{\frac{2(p-2)}{3-p}}\,, \tag{2.9}$$

where $c = c(n,p,R) > 0$. Now, let $\lambda = \min\left\{\frac{\alpha+1}{2}, \frac{1}{3-p}\right\}$. Dividing both sides of (2.9) by $|h|^{2\lambda-2}$, we have

$$\int_{B_{R/2}} \left|\frac{\tau_h H_{p/2}(Du)}{|h|^{\lambda}}\right|^2 dx \,\le\, c\left(\|f\|_{B^{\alpha}_{p',\infty}(B')}\,\|Du\|_{L^p(B')}\,|h|^{\alpha+1-2\lambda} \,+\, \|Du\|^p_{L^p(B')}\,|h|^{\frac{2}{3-p}-2\lambda}\right). \tag{2.10}$$

Since the above estimate holds for every $h \in \mathbb{R}\setminus\{0\}$ such that $|h| \le r_0$, we can take the supremum over $|h| < \delta$ for some $\delta \in (0, r_0)$ and obtain

$$\sup_{|h|<\delta} \int_{B_{R/2}} \left|\frac{\tau_h H_{p/2}(Du)}{|h|^{\lambda}}\right|^2 dx \,\le\, c\left(\|f\|_{B^{\alpha}_{p',\infty}(B')}\,\|Du\|_{L^p(B')}\,r_0^{\alpha+1-2\lambda} \,+\, \|Du\|^p_{L^p(B')}\,r_0^{\frac{2}{3-p}-2\lambda}\right).$$

In particular, this means that $H_{\frac{p}{2}}(Du) \in B^{\min\left\{\frac{\alpha+1}{2}, \frac{1}{3-p}\right\}}_{2,\infty}$ locally in Ω.

Finally, let us consider the case $f \in W^{1,p'}_{loc}(\Omega)$. Arguing as above, but this time using Lemma 1.15 to estimate the $L^{p'}$-norm of the difference quotient of f, we obtain

$$\int_{B_{R/2}} \left|\frac{\tau_h H_{p/2}(Du)}{|h|^{1/(3-p)}}\right|^2 dx \,\le\, c\left(\|Df\|_{L^{p'}(B')}\,\|Du\|_{L^p(B')}\,|h|^{\frac{2(2-p)}{3-p}} \,+\, \|Du\|^p_{L^p(B')}\right), \tag{2.11}$$

which holds for every $h \in \mathbb{R}\setminus\{0\}$ such that $|h| \le r_0$. Arguing exactly as before, we may then conclude that $H_{\frac{p}{2}}(Du) \in B^{\frac{1}{3-p}}_{2,\infty}$ locally in Ω. □

We are now in a position to give the

Proof of Theorem 2.2. We shall keep the notation and the balls used for the proof of Theorem 2.1. Let us begin with the case $f \in W^{1,p'}_{loc}(\Omega)$. By duality, we know that σ_0 is related to any solution u of problem (P2) through the optimality condition

$$\sigma_0(x) = D\mathcal{H}^*(Du(x)) = H_{p-1}(Du(x)), \quad \text{for a.e. } x \in \Omega. \tag{2.12}$$

Now observe that for every $p \in (1,2)$ we have

$$0 < p - 1 < \frac{p}{2}\,.$$

Therefore, applying Lemma 1.5, we get

$$\begin{aligned}
&\int_{B_{R/2}} \left|\frac{\tau_h \sigma_0}{|h|^{\frac{2}{(3-p)p'}}}\right|^{p'} dx \\
&\quad\le\, c\,|h|^{\frac{2}{p-3}} \int_{B_{R/2}} \left|\tau_h H_{\frac{p}{2}}(Du)\right|^{p'} \left[(|Du(x+he_s)| - \lambda)_+^{\frac{p}{2}} + (|Du| - \lambda)_+^{\frac{p}{2}}\right]^{\frac{p-2}{p-1}} dx
\end{aligned}$$

$$= c \int_{B_{R/2}} \left| \frac{\tau_h H_{p/2}(Du)}{|h|^{1/(3-p)}} \right|^2 \left| \tau_h H_{\frac{p}{2}}(Du) \right|^{p'-2} \left[(|Du(x+he_s)| - \lambda)_+^{\frac{p}{2}} + (|Du| - \lambda)_+^{\frac{p}{2}} \right]^{\frac{p-2}{p-1}} dx$$

$$\leq c \int_{B_{R/2}} \left| \frac{\tau_h H_{p/2}(Du)}{|h|^{1/(3-p)}} \right|^2 dx \,, \tag{2.13}$$

where $c = c(n,p) > 0$. Combining estimates (2.13) and (2.11) and arguing as in the last part of the proof of Theorem 2.1, we obtain the desired conclusion.

Finally, when $f \in B^{\alpha}_{p',\infty,loc}(\Omega)$, a similar argument applies, but taking into account estimate (2.10) instead of (2.11). □

Now we carry on with the analysis of the singular case, by providing the

Proof of Corollary 2.4. We first note that for every $p \in (1,2)$ and every $\alpha \in (0,1)$ we have

$$\frac{2}{3-p} < 2 \leq n \quad \text{and} \quad \alpha + 1 < n.$$

If $f \in B^{\alpha}_{p',\infty,loc}(\Omega)$ and $\frac{\alpha+1}{p'} \geq \frac{2}{(3-p)p'}$ (that is, $\frac{p-1}{3-p} \leq \alpha < 1$), then from Theorem 2.2 and Lemma 1.31 it follows that $\sigma_0 \in L^s_{loc}(\Omega, \mathbb{R}^n)$ for all $s \in \left[1, \frac{n(3-p)p'}{n(3-p)-2}\right)$.

When $f \in W^{1,p'}_{loc}(\Omega)$, thanks to Theorem 2.2 we have that $\sigma_0 \in B^{\frac{2}{(3-p)p'}}_{p',\infty}$ locally in Ω. We therefore reach the same conclusion as in the previous case.

Finally, when $f \in B^{\alpha}_{p',\infty,loc}(\Omega)$ and $\frac{\alpha+1}{p'} < \frac{2}{(3-p)p'}$ (that is, $0 < \alpha < \frac{p-1}{3-p}$), Theorem 2.2 and Lemma 1.31 imply that $\sigma_0 \in L^s_{loc}(\Omega, \mathbb{R}^n)$ for all $s \in \left[1, \frac{np'}{n-\alpha-1}\right)$. □

Remark 2.6. By virtue of the optimality condition (2.12), Corollary 2.4 establishes the local higher integrability of $H_{p-1}(Du)$. This result, in turn, implies a gain of integrability for Du, as emerges from the following proof.

Proof of Corollary 2.5. In the case $f \in W^{1,p'}_{loc}(\Omega)$, we get $H_{p-1}(Du) \in L^s_{loc}(\Omega, \mathbb{R}^n)$ for every $s \in \left[1, \frac{n(3-p)p'}{n(3-p)-2}\right)$ and then

$$\int_K (|Du| - \lambda)_+^{(p-1)s} \, dx = \int_K |H_{p-1}(Du)|^s \, dx < +\infty$$

for all $s \in \left[1, \frac{n(3-p)p'}{n(3-p)-2}\right)$ and all compact subsets K of Ω, which ensures that $Du \in L^r_{loc}(\Omega, \mathbb{R}^n)$ for all $r \in \left[1, \frac{np\,(3-p)}{n(3-p)-2}\right)$. In the other two cases, a similar argument applies, thus proving the assertion. □

2.2 The degenerate case $p \geq 2$

Here we proceed with the analysis of the non-singular case, by proving that, unlike what may occur in the singular one, the Besov regularity of the datum f translates into a Besov regularity

for the function $H_{\frac{p}{2}}(Du)$ with no loss in the order of differentiation. More precisely, we have the following result.

Theorem 2.7. *Let $n \geq 2$, $p \in [2, \infty)$, $\alpha \in (0,1)$ and $f \in B^{\alpha}_{p',\infty,loc}(\Omega)$. Moreover, let $u \in W^{1,p}_{loc}(\Omega)$ be a local weak solution of equation (2.1). Then*

$$H_{\frac{p}{2}}(Du) \in B^{\frac{\alpha+1}{2}}_{2,\infty,loc}(\Omega, \mathbb{R}^n)\,.$$

Furthermore, for any ball $B_R \Subset \Omega$, the following estimate

$$\int\limits_{B_{R/2}} \left|\tau_{s,h} H_{\frac{p}{2}}(Du)\right|^2 dx \leq c\left(\|f\|_{B^{\alpha}_{p',\infty}(B')}\,\|Du\|_{L^p(B')}\,|h|^{\alpha+1} + \|Du\|^p_{L^p(B')}\,|h|^2\right)$$

holds true for every $s \in \{1,\dots,n\}$, for every $h \in \mathbb{R}$ such that $|h| \leq r_0 < \frac{1}{2}\operatorname{dist}(B_R, \partial\Omega)$, for $B' = B_R + B(0, r_0)$ and a positive constant $c = c(n,p,R)$.

Proof. In what follows, we shall keep the notation, definitions and choices used in the proof of Theorem 2.1.

Let us first assume that $p > 2$. Arguing exactly as in the first part of the proof of Theorem 2.1, we arrive at the following inequality

$$\begin{aligned}&\int_{\Omega} \xi^2 \langle \Delta_h H_{p-1}(Du), \Delta_h Du\rangle\, dx\\ &\quad\leq c_1(n)\,\|f\|_{B^{\alpha}_{p',\infty}(B')}\,\|Du\|_{L^p(B')}\,|h|^{\alpha-1} - 2\int_{\Omega}\langle \Delta_h H_{p-1}(Du), D\xi\rangle\,\xi\,\Delta_h u\,dx\,.\end{aligned}$$

Now, by Lemma 1.6 we have

$$\frac{4}{p^2}\int_{B_R}\xi^2\left|\Delta_h H_{\frac{p}{2}}(Du)\right|^2 dx \leq \int_{\Omega}\xi^2\langle \Delta_h H_{p-1}(Du), \Delta_h Du\rangle\,dx\,.$$

Combining the two previous estimates and applying the Cauchy-Schwarz inequality as well as the properties of ξ, we get

$$\begin{aligned}&\int_{B_R}\xi^2\left|\Delta_h H_{\frac{p}{2}}(Du)\right|^2 dx\\ &\quad\leq c_2(n,p)\,\|f\|_{B^{\alpha}_{p',\infty}(B')}\,\|Du\|_{L^p(B')}\,|h|^{\alpha-1} + \frac{p^2}{2}\,\|D\xi\|_{L^\infty}\int_{\Omega}|\Delta_h H_{p-1}(Du)|\,\xi\,|\Delta_h u|\;dx\\ &\quad\leq c_2(n,p)\,\|f\|_{B^{\alpha}_{p',\infty}(B')}\,\|Du\|_{L^p(B')}\,|h|^{\alpha-1} + c_3(p,R)\int_{\Omega}|\Delta_h H_{p-1}(Du)|\,\xi\,|\Delta_h u|\;dx\,. \qquad (2.14)\end{aligned}$$

Now, our aim is to estimate

$$I_1 := \int_{\Omega}|\Delta_h H_{p-1}(Du)|\,\xi\,|\Delta_h u|\;dx\,.$$

Using Lemma 1.6, Hölder's inequality with exponents $\left(p, 2, \frac{2p}{p-2}\right)$ and the properties of the

difference quotients, we obtain

$$\begin{aligned} I_1 &\le (p-1)\int_{B_R}\xi\,|\Delta_h u|\left(\left|H_{\frac{p}{2}}(Du(x+he_s))\right|^{\frac{p-2}{p}}+\left|H_{\frac{p}{2}}(Du)\right|^{\frac{p-2}{p}}\right)\left|\Delta_h H_{\frac{p}{2}}(Du)\right|dx\\ &\le (p-1)\,\|\Delta_h u\|_{L^p(B_R)}\left(\int_{B_R}\xi^2\left|\Delta_h H_{\frac{p}{2}}(Du)\right|^2dx\right)^{\frac12}\\ &\quad\cdot\left(\int_{B_R}\left(\left|H_{\frac{p}{2}}(Du(x+he_s))\right|^{\frac{p-2}{p}}+\left|H_{\frac{p}{2}}(Du)\right|^{\frac{p-2}{p}}\right)^{\frac{2p}{p-2}}dx\right)^{\frac{p-2}{2p}}\\ &\le c_4(n,p)\,\|Du\|_{L^p(B')}\left(\int_{B_R}\xi^2\left|\Delta_h H_{\frac{p}{2}}(Du)\right|^2dx\right)^{\frac12}\left\|H_{\frac{p}{2}}(Du)\right\|_{L^2(B')}^{(p-2)/p}\\ &\le c_5(n,p)\,\|Du\|_{L^p(B')}^{p/2}\left(\int_{B_R}\xi^2\left|\Delta_h H_{\frac{p}{2}}(Du)\right|^2dx\right)^{\frac12}. \end{aligned}\tag{2.15}$$

Joining (2.14) and (2.15) and applying Young's inequality with exponents $(2,2)$, we find

$$\begin{aligned} &\int_{B_R}\xi^2\left|\Delta_h H_{\frac{p}{2}}(Du)\right|^2dx\\ &\quad\le c_2(n,p)\,\|f\|_{B^\alpha_{p',\infty}(B')}\,\|Du\|_{L^p(B')}\,|h|^{\alpha-1}+c_3(p,R)\,I_1\\ &\quad\le c_2\,\|f\|_{B^\alpha_{p',\infty}(B')}\,\|Du\|_{L^p(B')}\,|h|^{\alpha-1}+c_6(n,p,R)\,\|Du\|_{L^p(B')}^{p/2}\left(\int_{B_R}\xi^2\left|\Delta_h H_{\frac{p}{2}}(Du)\right|^2dx\right)^{\frac12}\\ &\quad\le c\,\|f\|_{B^\alpha_{p',\infty}(B')}\,\|Du\|_{L^p(B')}\,|h|^{\alpha-1}+c\,\|Du\|_{L^p(B')}^{p}+\frac12\int_{B_R}\xi^2\left|\Delta_h H_{\frac{p}{2}}(Du)\right|^2dx, \end{aligned}\tag{2.16}$$

where $c=c(n,p,R)>0$. Reabsorbing the integral in the right-hand side of (2.16) by the left-hand side, we get

$$\int_{B_R}\xi^2\left|\Delta_h H_{\frac{p}{2}}(Du)\right|^2dx\le c\left(\|f\|_{B^\alpha_{p',\infty}(B')}\,\|Du\|_{L^p(B')}\,|h|^{\alpha-1}+\|Du\|_{L^p(B')}^{p}\right),$$

from which we infer

$$\int_{B_{R/2}}\left|\Delta_h H_{\frac{p}{2}}(Du)\right|^2dx\le c\left(\|f\|_{B^\alpha_{p',\infty}(B')}\,\|Du\|_{L^p(B')}\,|h|^{\alpha-1}+\|Du\|_{L^p(B')}^{p}\right),$$

since $\xi\equiv1$ on $\overline{B}_{R/2}$. Multiplying both sides of the previous inequality by $|h|^{1-\alpha}$, we then have

$$\int_{B_{R/2}}\left|\frac{\tau_h H_{p/2}(Du)}{|h|^{(\alpha+1)/2}}\right|^2dx\le c\left(\|f\|_{B^\alpha_{p',\infty}(B')}\,\|Du\|_{L^p(B')}+\|Du\|_{L^p(B')}^{p}\,|h|^{1-\alpha}\right).$$

Since the above estimate holds for every $h\in\mathbb{R}\setminus\{0\}$ such that $|h|\le r_0<\frac12\operatorname{dist}(B_R,\mathbb{R}^n\setminus\Omega)$, we can take the supremum over $|h|<\delta$ for some $\delta\in(0,r_0)$ and obtain

$$\sup_{|h|<\delta}\int_{B_{R/2}}\left|\frac{\tau_h H_{p/2}(Du)}{|h|^{(\alpha+1)/2}}\right|^2dx\le c\left(\|f\|_{B^\alpha_{p',\infty}(B')}\,\|Du\|_{L^p(B')}+\|Du\|_{L^p(B')}^{p}\,r_0^{1-\alpha}\right),$$

which gives the desired conclusion for $p > 2$.

Let us now assume that $p = 2$. Then $H_{p-1}(Du) = H_{p/2}(Du) = H_1(Du)$ and, applying Young's inequality, from (2.14) we get

$$\begin{aligned}
&\int_{B_R} \xi^2 \left|\Delta_h H_1(Du)\right|^2 dx \\
&\quad \leq c_2(n) \left\|f\right\|_{B^\alpha_{2,\infty}(B')} \left\|Du\right\|_{L^2(B')} |h|^{\alpha-1} + c_3(R) \int_{B_R} \left|\Delta_h H_1(Du)\right| \xi \left|\Delta_h u\right| dx \\
&\quad \leq c_2(n) \left\|f\right\|_{B^\alpha_{2,\infty}(B')} \left\|Du\right\|_{L^2(B')} |h|^{\alpha-1} + c_4(R) \left\|\Delta_h u\right\|^2_{L^2(B_R)} + \frac{1}{2} \int_{B_R} \xi^2 \left|\Delta_h H_1(Du)\right|^2 dx \\
&\quad \leq c_2(n) \left\|f\right\|_{B^\alpha_{2,\infty}(B')} \left\|Du\right\|_{L^2(B')} |h|^{\alpha-1} + c_5(n,R) \left\|Du\right\|^2_{L^2(B')} + \frac{1}{2} \int_{B_R} \xi^2 \left|\Delta_h H_1(Du)\right|^2 dx \,,
\end{aligned}$$

where, in the last line, we have used the first statement in Lemma 1.14. From here, arguing exactly as above, we arrive at the desired result in the case $p = 2$. This completes the proof. □

The Besov regularity of $H_{\frac{p}{2}}(Du)$ established by Theorem 2.7 allows us to get a gain of integrability for Du. More precisely, we have the following result.

Corollary 2.8. *Under the assumptions of Theorem 2.7, we get*

$$H_{\frac{p}{2}}(Du) \in L^r_{loc}(\Omega, \mathbb{R}^n) \quad \textit{for all } r \in \left[1, \frac{2n}{n-\alpha-1}\right)$$

and

$$Du \in L^s_{loc}(\Omega, \mathbb{R}^n) \quad \textit{for all } s \in \left[1, \frac{np}{n-\alpha-1}\right).$$

Proof. By observing that $\alpha + 1 < n$, from Theorem 2.7 and Lemma 1.31 we immediately obtain the first conclusion. Then

$$\int_K \left(|Du| - \lambda\right)_+^{\frac{pr}{2}} dx = \int_K \left|H_{\frac{p}{2}}(Du)\right|^r dx < +\infty$$

for all $r \in \left[1, \frac{2n}{n-\alpha-1}\right)$ and all compact subsets K of Ω, which ensures that $Du \in L^s_{loc}(\Omega, \mathbb{R}^n)$ for all $s \in \left[1, \frac{np}{n-\alpha-1}\right)$. □

Remark 2.9. When $n = 2$ (the case which is relevant for applications to network congestion), Corollary 2.8 implies that $Du \in L^s_{loc}(\Omega, \mathbb{R}^n)$ for all $s \in \left[1, \frac{2p}{1-\alpha}\right)$. This means that for every $r > 2p$ there exists a $\delta = \delta(r,p) \in (0,1)$ such that, if $\alpha \in (\delta, 1)$ and $f \in B^\alpha_{p',\infty,loc}(\Omega)$, then $Du \in L^r_{loc}(\Omega, \mathbb{R}^n)$. In fact, it is enough to take $\delta = 1 - 2p/r$. In other words, given any $r > 2p$, we can choose α close enough to 1 in order to have $Du \in L^r_{loc}(\Omega, \mathbb{R}^n)$ whenever $f \in B^\alpha_{p',\infty,loc}(\Omega)$.

Again, if we come back to the variational problem (P1), we obtain the following higher integrability result for its solution:

Corollary 2.10. *Let $n \geq 2$, $p \in [2, \infty)$, $\alpha \in (0,1)$ and $f \in B^\alpha_{p',\infty,loc}(\Omega)$. Moreover, let $\sigma_0 \in L^{p'}(\Omega, \mathbb{R}^n)$ be the solution of* (P1). *Then $\sigma_0 \in L^r_{loc}(\Omega, \mathbb{R}^n)$ for all $r \in \left[1, \frac{np'}{n-\alpha-1}\right)$.*

Proof. By duality, we know that σ_0 is related to any solution $u \in W^{1,p}(\Omega)$ of problem (P2) through the optimality condition (2.12). Since u is a weak solution of the Euler-Lagrange equation (1) and

$$|\sigma_0| = \left| H_{\frac{p}{2}}(Du) \right|^{\frac{2}{p'}} \quad \text{almost everywhere in } \Omega,$$

the assertion immediately follows from Corollary 2.8. □

Remark 2.11. When $n = 2$, Corollary 2.10 implies that $\sigma_0 \in L^r_{loc}(\Omega, \mathbb{R}^n)$ for all $r \in \left[1, \frac{2p'}{1-\alpha}\right)$. This means that for every $s > 2p'$ there exists a $\delta = \delta(s,p) \in (0,1)$ such that, if $\alpha \in (\delta, 1)$ and $f \in B^{\alpha}_{p',\infty,loc}(\Omega)$, then $\sigma_0 \in L^s_{loc}(\Omega, \mathbb{R}^n)$. In fact, it is enough to take $\delta = 1 - \frac{2p}{s(p-1)}$. In other words, given any $s > 2p'$, we can choose α close enough to 1 in order to have $\sigma_0 \in L^s_{loc}(\Omega, \mathbb{R}^n)$ whenever $f \in B^{\alpha}_{p',\infty,loc}(\Omega)$.

Chapter 3

Sharp second-order regularity for widely singular or degenerate elliptic PDEs

In this chapter, we are concerned with local $W^{1,p}$ solutions to the equation

$$-\operatorname{div}\left((|Du|-\lambda)_+^{p-1}\frac{Du}{|Du|}\right)=f \quad \text{in } \Omega, \tag{3.1}$$

where $1<p<\infty$, Ω is an open subset of $\mathbb{R}^n$ ($n\geq 2$) and $\lambda>0$. The results that we report here have been obtained in [9]. As our main achievement, we will establish the local $W^{1,2}$-regularity of a nonlinear function of the gradient Du of local weak solutions to equation (3.1), by assuming that

- $f\in B^{\frac{p-2}{p}}_{p',1,loc}(\Omega)$ if $2<p<\infty$ (see Theorem 3.1);
- $f\in L^{\frac{np}{n(p-1)+2-p}}_{loc}(\Omega)$ if $1<p\leq 2$ (see Theorem 3.4).

The above-mentioned theorems, in turn, imply the local higher integrability of Du under the same hypotheses on the function f (cf. Corollary 3.5).

The main results of this chapter are in the spirit of those contained in [23, 28, 58], which we have already discussed in the Introduction. Indeed, our primary objective here is to identify the sharp assumptions on the datum f *in the scale of local Besov or Lebesgue spaces* to obtain a second-order regularity result for weak solutions of the widely degenerate or singular equation (3.1). In order to state our main results, we introduce the function

$$\mathcal{G}_{\alpha,\lambda}(t):=\int_0^t\frac{\omega^{\frac{p-1+2\alpha}{2}}}{(\omega+\lambda)^{\frac{1+2\alpha}{2}}}\,d\omega \qquad \text{for } t\geq 0, \tag{3.2}$$

where $\alpha\geq 0$. Moreover, for $\xi\in\mathbb{R}^n$ we define the following vector-valued function:

$$\mathcal{V}_{\alpha,\lambda}(\xi)\equiv\mathcal{V}_\lambda(\xi):=\begin{cases}\mathcal{G}_{\alpha,\lambda}((|\xi|-\lambda)_+)\,\frac{\xi}{|\xi|} & \text{if } |\xi|>\lambda,\\ 0 & \text{if } |\xi|\leq\lambda.\end{cases} \tag{3.3}$$

At this point, our first result reads as follows.

Theorem 3.1. *Let $n \geq 2$, $p > 2$, $\lambda \geq 0$ and $f \in B^{\frac{p-2}{p}}_{p',1,loc}(\Omega)$. Moreover, let*

$$\alpha = \begin{cases} 0 & \text{if } \lambda = 0, \\ \text{any value in } \left[\frac{p+1}{2(p-1)}, \infty\right) & \text{if } \lambda > 0, \end{cases} \tag{3.4}$$

and assume that $u \in W^{1,p}_{loc}(\Omega)$ is a local weak solution of equation (3.1). Then

$$\mathcal{V}_{\alpha,\lambda}(Du) \in W^{1,2}_{loc}(\Omega, \mathbb{R}^n)\,.$$

Furthermore, for every pair of concentric balls $B_r \subset B_R \Subset \Omega$ we have

$$\begin{aligned} &\int_{B_{r/4}} |D\mathcal{V}_{\alpha,\lambda}(Du)|^2\,dx \\ &\quad \leq \left(C + \frac{C}{r^2}\right)\left[1 + \lambda^p + \|Du\|^p_{L^p(B_R)} + \|f\|^{p'}_{L^{p'}(B_R)}\right] + C\,\|f\|^{p'}_{B^{\frac{p-2}{p}}_{p',1}(B_R)} \end{aligned} \tag{3.5}$$

for a positive constant C depending only on n, p and R in the case $\lambda = 0$, and additionally on α if $\lambda > 0$.

Remark 3.2. Notice that, for every $\alpha \geq 0$, we have

$$\mathcal{V}_{\alpha,0}(\xi) = \frac{2}{p}\,\mathbb{V}_p(\xi) := \frac{2}{p}\,|\xi|^{\frac{p-2}{2}}\xi\,, \tag{3.6}$$

i.e. $\mathcal{V}_0$ is actually independent of the parameter α, which explains the choice $(3.4)_1$ in the statement of the above theorem. The condition $(3.4)_2$ will instead be decisive to carry out the proof of Proposition 3.14 below. Moreover, looking at (3.6), one can easily understand that Theorem 3.1 extends the result contained in [58, Remark 1.4] to a class of widely degenerate elliptic equations with standard growth, under a sharp assumption on the order of differentiation of f.

Remark 3.3. We do not know whether Theorem 3.1 is still true when we relax the regularity of f to $B^{1-2/p}_{p',q,loc}(\Omega)$ for some $q > 1$, also in the case $\lambda = 0$. In this regard we point out the paper [42], where the authors apparently prove that, for the p-Poisson equation, assertion (5) holds true under the weaker assumption $f \in W^{1-2/p,p'}_{loc}(\Omega)$. However, we believe that they made a mistake at the beginning of [42, page 373] in applying [42, Formula (4)].

As already mentioned in the Introduction, for $\lambda > 0$ the weak solutions of (3.1) are not twice differentiable. Therefore, in general we cannot differentiate the equation to estimate the second derivatives of the solutions. Here, we overcome this difficulty by considering a regularization of equation (3.1). The regularized problem satisfies standard *non-degenerate* p-growth conditions (see Section 3.2). One of the advantages of introducing this problem stems from the fact that its (unique) weak solution admits second derivatives in L^2_{loc}. The major effort in proving Theorem 3.1 is to obtain a suitable estimate for the solutions u_ε of the regularized problems that is uniform with respect to the approximation parameter ε. In establishing this estimate, we first need to identify a suitable function of the gradient that vanishes in the degeneracy set, for which a second-order *a priori* estimate holds. Next, we need to control the right-hand side of this last estimate in terms of the derivatives of such function, without assuming any Sobolev

regularity for the datum f. This is done by virtue of the following implication

$$|g|^{\frac{p-2}{2}} g \in W^{1,2}_{loc}(\Omega) \;\Rightarrow\; g \in B^{\frac{2}{p}}_{p,\infty,loc}(\Omega), \tag{3.7}$$

which allows us to use the duality of Besov spaces, provided that one imposes a suitable Besov regularity on the right-hand side f. This approach has been inspired by [23], where Brasco and Santambrogio used a duality-based inequality in the setting of fractional Sobolev spaces, but limiting themselves to the p-Poisson equation. Finally, we use a comparison argument to transfer the higher differentiability of $\mathcal{V}_{\alpha,\lambda}(Du_\varepsilon)$ to the function $\mathcal{V}_{\alpha,\lambda}(Du)$.

We would like to mention that in [26, 27] Cianchi and Maz'ya proved that the assumption $f \in L^2$ is sufficient to prove the $W^{1,2}$-regularity of $|Du|^{p-2}Du$, which is of course a different function of the gradient. At the moment, we do not know whether the analogous result can be obtained for the solutions of widely degenerate equations.

Now we turn our attention to the subquadratic case, i.e. when $1 < p \leq 2$. It is well known that, already for the p-Poisson equation, the higher differentiability of the weak solutions can be achieved without assuming any differentiability on the datum f. This different behaviour can be easily explained by observing that, if $1 < p \leq 2$,

$$|g|^{\frac{p-2}{2}} g \in W^{1,2}_{loc}(\Omega) \;\Rightarrow\; g \in W^{1,p}_{loc}(\Omega),$$

which of course does not hold true for $p > 2$ (compare with (3.7)). Therefore, the right-hand side of the above-mentioned *a priori* estimate can be controlled without assuming any differentiability for f (neither of integer nor of fractional order), but only a suitable degree of integrability. The sharp assumption on f in the scale of Lebesgue spaces has been recently found in [28].

In this chapter, we prove that a result analogous to [28, Theorem 1.1] holds when dealing with solutions of widely singular equations. More precisely, we establish the following result.

Theorem 3.4. *Let $n \geq 2$, $1 < p \leq 2$, $\lambda \geq 0$ and $f \in L^{\frac{np}{n(p-1)+2-p}}_{loc}(\Omega)$. Moreover, let α be defined according to* (3.4) *and assume that $u \in W^{1,p}_{loc}(\Omega)$ is a local weak solution of equation* (3.1)*. Then*

$$\mathcal{V}_{\alpha,\lambda}(Du) \in W^{1,2}_{loc}(\Omega, \mathbb{R}^n)\,.$$

Furthermore, for every pair of concentric balls $B_r \subset B_R \Subset \Omega$ we have

$$\begin{aligned}&\int_{B_{r/4}} |D\mathcal{V}_{\alpha,\lambda}(Du)|^2\,dx\\ &\quad\leq \frac{C}{r^2}\left[1 + \lambda^p + \|Du\|^p_{L^p(B_R)} + \|f\|^{p'}_{L^{\frac{np}{n(p-1)+2-p}}(B_R)}\right] + C\,\|f\|^{p'}_{L^{\frac{np}{n(p-1)+2-p}}(B_R)}\end{aligned} \tag{3.8}$$

for a positive constant C depending only on n, p and R in the case $\lambda = 0$, and additionally on α if $\lambda > 0$.

As an easy consequence of the higher differentiability results in Theorems 3.1 and 3.4, since the gradient of the solution is bounded in the degeneracy set and $\mathcal{G}_{\alpha,\lambda}(t) \approx t^{p/2}$ for large values of t (see Lemma 3.10 below), we are able to establish the following higher integrability result for the gradient of local weak solutions of (3.1):

Corollary 3.5. *Under the assumptions of Theorem 3.1 or Theorem 3.4, we have*

$$Du \in L^q_{loc}(\Omega, \mathbb{R}^n),$$

where

$$q = \begin{cases} \text{any value in } [1, \infty) & \text{if } n = 2, \\ \frac{np}{n-2} & \text{if } n \geq 3. \end{cases}$$

This chapter is organized as follows. Section 3.1 is devoted to some preliminary inequalities. In Section 3.2, we consider a regularization of equation (3.1) and establish some *a priori* estimates for its weak solution. These estimates will be needed to demonstrate Theorems 3.1 and 3.4, whose proofs are achieved in Sections 3.3 and 3.4, respectively.

3.1 Algebraic inequalities

In this section, we gather some relevant algebraic inequalities that will be needed later on. The first result follows from an elementary computation.

Lemma 3.6. *For $\xi, \eta \in \mathbb{R}^n \setminus \{0\}$, we have*

$$\left| \frac{\xi}{|\xi|} - \frac{\eta}{|\eta|} \right| \leq \frac{2}{|\eta|} |\xi - \eta|.$$

We now recall the following estimate, whose proof can be found in [65, Chapter 12].

Lemma 3.7. *Let $p \in (2, \infty)$ and $k \in \mathbb{N}$. Then, for every $\xi, \eta \in \mathbb{R}^k$ we get*

$$|\xi - \eta|^p \leq C \left| |\xi|^{\frac{p-2}{2}} \xi - |\eta|^{\frac{p-2}{2}} \eta \right|^2$$

for a constant $C \equiv C(p) > 0$.

The next result concerns the function $\mathcal{G}_{\alpha,\lambda}$ defined by (3.2).

Lemma 3.8. *Let $p \in (1, \infty)$, $\lambda \geq 0$ and $\alpha \geq 0$. Then*

$$\mathcal{G}_{\alpha,\lambda}(t) \leq \frac{2}{p} t^{\frac{p}{2}} \left(\frac{t}{t+\lambda} \right)^{\frac{1+2\alpha}{2}}$$

for every $t > 0$.

Proof. Since the function

$$K(\omega) := \left(\frac{\omega}{\omega + \lambda} \right)^{\frac{1+2\alpha}{2}}, \qquad \omega > 0,$$

is non-decreasing, for every $t > 0$ we have

$$\mathcal{G}_{\alpha,\lambda}(t) = \int_0^t K(\omega)\,\omega^{\frac{p}{2}-1}\,d\omega \,\le\, \left(\frac{t}{t+\lambda}\right)^{\frac{1+2\alpha}{2}} \int_0^t \omega^{\frac{p}{2}-1}\,d\omega \,=\, \frac{2}{p}\,t^{\frac{p}{2}}\left(\frac{t}{t+\lambda}\right)^{\frac{1+2\alpha}{2}}. \qquad \square$$

The next lemma relates the function $\mathcal{V}_{\alpha,\lambda}(\xi)$ with $H_{p-1}(\xi)$.

Lemma 3.9. *Let $p \in (1,\infty)$, $\lambda \ge 0$ and $\alpha \ge 0$. Then, there exists a constant $C \equiv C(n,p) > 0$ such that*

$$|\mathcal{V}_{\alpha,\lambda}(\xi) - \mathcal{V}_{\alpha,\lambda}(\eta)|^2 \,\le\, C\,\langle H_{p-1}(\xi) - H_{p-1}(\eta), \xi - \eta\rangle \tag{3.9}$$

for every $\xi, \eta \in \mathbb{R}^n$.

Proof. For $\lambda = 0$, estimate (3.9) boils down to (1.9) or (1.10). Therefore, from now on we shall assume that $\lambda > 0$. We first note that inequality (3.9) is trivially satisfied when $|\xi|, |\eta| \le \lambda$. If $|\eta| \le \lambda < |\xi|$, using the definitions (3.3), (3.2) and (1.6) along with Lemmas 1.6 and 1.7, we obtain

$$\begin{aligned}
|\mathcal{V}_{\alpha,\lambda}(\xi) - \mathcal{V}_{\alpha,\lambda}(\eta)|^2 \,&=\, [\mathcal{G}_{\alpha,\lambda}(|\xi|-\lambda)]^2 \,\le\, \left(\int_0^{|\xi|-\lambda} \omega^{\frac{p}{2}-1}\,d\omega\right)^2 \,=\, \frac{4}{p^2}\,(|\xi|-\lambda)^p \\
&=\, \frac{4}{p^2}\,|H_{\frac{p}{2}}(\xi) - H_{\frac{p}{2}}(\eta)|^2 \,\le\, c(n,p)\,\langle H_{p-1}(\xi) - H_{p-1}(\eta), \xi - \eta\rangle.
\end{aligned}$$

Now let $|\xi|, |\eta| > \lambda$. Without loss of generality, we may assume that $|\eta| \ge |\xi| > \lambda$. This implies

$$|\eta|^2 = \frac{|\eta|\,(|\eta| + |\eta|)}{2} \,\ge\, \frac{|\eta|\,(|\xi| + |\eta|)}{2}. \tag{3.10}$$

Moreover, we have

$$\begin{aligned}
|\mathcal{V}_{\alpha,\lambda}(\xi) - \mathcal{V}_{\alpha,\lambda}(\eta)| \,&=\, \left|\mathcal{V}_{\alpha,\lambda}(\xi) - \mathcal{G}_{\alpha,\lambda}(|\eta|-\lambda)\,\frac{\xi}{|\xi|} \,+\, \mathcal{G}_{\alpha,\lambda}(|\eta|-\lambda)\,\frac{\xi}{|\xi|} \,-\, \mathcal{V}_{\alpha,\lambda}(\eta)\right| \\
&\le\, |\mathcal{G}_{\alpha,\lambda}(|\xi|-\lambda) - \mathcal{G}_{\alpha,\lambda}(|\eta|-\lambda)| \,+\, \mathcal{G}_{\alpha,\lambda}(|\eta|-\lambda)\left|\frac{\xi}{|\xi|} - \frac{\eta}{|\eta|}\right| \\
&\le\, \int_{|\xi|-\lambda}^{|\eta|-\lambda} \frac{\omega^{\frac{p-1+2\alpha}{2}}}{(\omega+\lambda)^{\frac{1+2\alpha}{2}}}\,d\omega \,+\, \frac{2}{|\eta|}\,|\xi-\eta| \int_0^{|\eta|-\lambda} \frac{\omega^{\frac{p-1+2\alpha}{2}}}{(\omega+\lambda)^{\frac{1+2\alpha}{2}}}\,d\omega \\
&\le\, \int_{|\xi|-\lambda}^{|\eta|-\lambda} \omega^{\frac{p}{2}-1}\,d\omega \,+\, \frac{2}{|\eta|}\,|\xi-\eta| \int_0^{|\eta|-\lambda} \omega^{\frac{p}{2}-1}\,d\omega \\
&=\, \frac{2}{p}\,\Big||H_{\frac{p}{2}}(\eta)| - |H_{\frac{p}{2}}(\xi)|\Big| \,+\, 4\,\frac{(|\eta|-\lambda)^{\frac{p}{2}}}{p\,|\eta|}\,|\xi-\eta| \\
&\le\, \frac{2}{p}\,|H_{\frac{p}{2}}(\xi) - H_{\frac{p}{2}}(\eta)| \,+\, 4\,\frac{(|\eta|-\lambda)^{\frac{p}{2}}}{p\,|\eta|}\,|\xi-\eta|,
\end{aligned}$$

where, in the third line, we have used Lemma 3.6 and the fact that $\mathcal{G}_{\alpha,\lambda}$ is an increasing function. Now, applying Young's inequality, estimate (3.10) and Lemmas 1.6, 1.7 and 1.8, we obtain

$$|\mathcal{V}_{\alpha,\lambda}(\xi) - \mathcal{V}_{\alpha,\lambda}(\eta)|^2 \,\le\, \frac{8}{p^2}\,|H_{\frac{p}{2}}(\xi) - H_{\frac{p}{2}}(\eta)|^2 \,+\, 32\,\frac{(|\eta|-\lambda)^p}{p^2\,|\eta|^2}\,|\xi-\eta|^2$$

$$\leq \frac{8}{p^2}\,|H_{\frac{p}{2}}(\xi)-H_{\frac{p}{2}}(\eta)|^2 \,+\, \frac{64}{p^2}\,\frac{(|\eta|-\lambda)^p}{|\eta|\,(|\xi|+|\eta|)}\,|\xi-\eta|^2$$
$$\leq C(n,p)\,\langle H_{p-1}(\xi)-H_{p-1}(\eta),\xi-\eta\rangle.$$

This completes the proof. □

We conclude this section with the proof of the following lemma, on which the conclusion of Corollary 3.5 is based.

Lemma 3.10. *Let $p\in(1,\infty)$, $\lambda>0$ and $\alpha>\frac{1}{2}$. Then, there exist two positive constants $c\equiv c(p,\alpha)$ and $\tilde c\equiv\tilde c(p,\alpha)$ such that*

$$c\,(\lambda+t)^{p/2}-\tilde c\,\lambda^{p/2}\leq\ \mathcal{G}_{\alpha,\lambda}(t)\ \leq\frac{2}{p}\,t^{p/2}$$

for all $t\geq 0$.

Proof. From the very definition of the function $\mathcal{G}_{\alpha,\lambda}$, we easily get the upper bound

$$\mathcal{G}_{\alpha,\lambda}(t)\leq\int_0^t\omega^{\frac{p-2}{2}}d\omega=\frac{2}{p}\,t^{p/2}\qquad\text{for all } t\geq 0.$$

For the derivation of the lower bound, we calculate the integral in the definition of $\mathcal{G}_{\alpha,\lambda}(t)$. By the change of variable $r=\omega+\lambda$, we have

$$\int_0^t\frac{\omega^{\frac{p-1+2\alpha}{2}}}{(\omega+\lambda)^{\frac{1+2\alpha}{2}}}\,d\omega=\int_\lambda^{\lambda+t}\frac{(r-\lambda)^{\frac{p-1+2\alpha}{2}}}{r^{\frac{1+2\alpha}{2}}}\,dr.$$

Now we recall that for every $\gamma>0$ it holds

$$2^{-\gamma}a^\gamma-b^\gamma\leq(a-b)^\gamma,\quad\forall\ a\geq b\geq 0. \tag{3.11}$$

Using inequality (3.11) with $\gamma=\frac{p-1+2\alpha}{2}$, $a=r$ and $b=\lambda$, we find that

$$\begin{aligned}\int_\lambda^{\lambda+t}\frac{(r-\lambda)^{\frac{p-1+2\alpha}{2}}}{r^{\frac{1+2\alpha}{2}}}\,dr&\geq\ c(p,\alpha)\int_\lambda^{\lambda+t}\frac{r^{\frac{p-1+2\alpha}{2}}}{r^{\frac{1+2\alpha}{2}}}\,dr-\int_\lambda^{\lambda+t}\frac{\lambda^{\frac{p-1+2\alpha}{2}}}{r^{\frac{1+2\alpha}{2}}}\,dr\\&=\ c(p,\alpha)\int_\lambda^{\lambda+t}r^{\frac{p-2}{2}}\,dr-\lambda^{\frac{p-1+2\alpha}{2}}\left[\frac{2\,r^{\frac{1-2\alpha}{2}}}{1-2\alpha}\right]_\lambda^{\lambda+t}\\&=\ c(p,\alpha)\left[\frac{2\,r^{\frac{p}{2}}}{p}\right]_\lambda^{\lambda+t}+\frac{2\,\lambda^{\frac{p-1+2\alpha}{2}}}{(2\alpha-1)(\lambda+t)^{\frac{2\alpha-1}{2}}}-\frac{2\,\lambda^{\frac{p-1+2\alpha}{2}}}{(2\alpha-1)\lambda^{\frac{2\alpha-1}{2}}}\\&\geq\ c(p,\alpha)(\lambda+t)^{p/2}-c(p,\alpha)\lambda^{p/2}-\frac{2\,\lambda^{p/2}}{2\alpha-1}\,,\end{aligned}$$

where, in the last line, we have used that $\alpha>\frac{1}{2}$. Finally, setting $\tilde c:=c(p,\alpha)+\frac{2}{2\alpha-1}$, we obtain the asserted lower bound. □

3.2 Estimates for a regularized problem

The aim of this section is to establish some uniform estimates for the gradient of the weak solutions of a family of suitable approximating problems. More precisely, let $\lambda \geq 0$ and let $u \in W^{1,p}_{loc}(\Omega)$ be a local weak solution of (3.1), for some $p > 1$. Fix an open ball $B_R \Subset \Omega$ and assume without loss of generality that $R \leq 1$. For $\varepsilon \in (0,1]$, we consider the problem

$$\begin{cases} -\operatorname{div}\left(DG_\varepsilon(Du_\varepsilon)\right) = f_\varepsilon & \text{in } B_R, \\ u_\varepsilon = u & \text{on } \partial B_R, \end{cases} \tag{3.12}$$

where:

- $G_\varepsilon(z) := \frac{1}{p}\left(|z| - \lambda\right)_+^p + \frac{\varepsilon}{p}\left(1 + |z|^2\right)^{\frac{p}{2}}$, for every $z \in \mathbb{R}^n$;
- $f_\varepsilon := f * \phi_\varepsilon$ and $\{\phi_\varepsilon\}_{\varepsilon > 0}$ is a family of standard compactly supported C^∞ mollifiers.

Observe that

$$D_z G_\varepsilon(z) = H_{p-1}(z) + \varepsilon\left(1 + |z|^2\right)^{\frac{p-2}{2}} z. \tag{3.13}$$

Now we set, for $s > \lambda$,

$$\boldsymbol{\lambda}(s) := \begin{cases} \dfrac{(s-\lambda)^{p-1}}{s} & \text{if } p > 2, \\ (p-1)\,\dfrac{(s-\lambda)^{p-1}}{s} & \text{if } 1 < p \leq 2, \end{cases}$$

and

$$\boldsymbol{\Lambda}(s) := \begin{cases} (p-1)\,(s-\lambda)^{p-2} & \text{if } p > 2, \\ (s-\lambda)^{p-2} & \text{if } 1 < p \leq 2, \end{cases}$$

and $\boldsymbol{\lambda}(s) = 0 = \boldsymbol{\Lambda}(s)$ for $0 \leq s \leq \lambda$. These definitions prove to be useful in the formulation of the next lemma, whose proof follows from [17, Lemma 2.7] (see also [16]) together with standard estimates for the p-Laplace operator.

Lemma 3.11. *Let $\varepsilon \in [0,1]$ and $z \in \mathbb{R}^n \setminus \{0\}$. Then, for every $\zeta \in \mathbb{R}^n$ we have*

$$\left[\varepsilon\, c_0\left(1 + |z|^2\right)^{\frac{p-2}{2}} + \boldsymbol{\lambda}(|z|)\right]|\zeta|^2 \leq \langle D^2 G_\varepsilon(z)\,\zeta, \zeta\rangle \leq \left[\varepsilon\, c_1\left(1 + |z|^2\right)^{\frac{p-2}{2}} + \boldsymbol{\Lambda}(|z|)\right]|\zeta|^2,$$

where $c_0 = \min\{1, p-1\}$ and $c_1 = \max\{1, p-1\}$.

Proof. Actually, setting for $t > 0$

$$h(t) = \frac{(t-\lambda)_+^{p-1}}{t} \qquad \text{and} \qquad a(t) = \left(1 + t^2\right)^{\frac{p-2}{2}},$$

we can write

$$D_z G_\varepsilon(z) = \Big[h(|z|) + \varepsilon a(|z|)\Big]\, z$$

and we can easily calculate

$$D^2 G_\varepsilon(z) = \Big[h(|z|) + \varepsilon a(|z|)\Big]\mathbb{I} + \Big[h'(|z|) + \varepsilon a'(|z|)\Big]\frac{z \otimes z}{|z|}\,.$$

Thus we get

$$\langle D^2 G_\varepsilon(z)\,\eta,\zeta\rangle = \Big[h(|z|)+\varepsilon a(|z|)\Big]\langle \eta,\zeta\rangle + \Big[h'(|z|)+\varepsilon a'(|z|)\Big]\sum_{i,j=1}^n \frac{z_i\,\eta_i\,z_j\,\zeta_j}{|z|}\,,$$

for any $\eta,\zeta\in\mathbb{R}^n$. By the Cauchy-Schwarz inequality, we have

$$0\le \sum_{i,j=1}^n \frac{z_i\,\zeta_i\,z_j\,\zeta_j}{|z|} \le |z|\,|\zeta|^2. \tag{3.14}$$

At this point, if $h'(|z|)+\varepsilon a'(|z|)\ge 0$ (which occurs when $p\ge 2$), from the lower bound in (3.14) we immediately obtain

$$\langle D^2 G_\varepsilon(z)\,\zeta,\zeta\rangle \ge \big[h(|z|)+\varepsilon a(|z|)\big]|\zeta|^2 = \frac{(|z|-\lambda)_+^{p-1}}{|z|}\,|\zeta|^2 + \varepsilon(1+|z|^2)^{\frac{p-2}{2}}|\zeta|^2.$$

On the other hand, using the upper bound in (3.14), for $p\ge 2$ we deduce

$$\langle D^2 G_\varepsilon(z)\,\zeta,\zeta\rangle \le (p-1)(|z|-\lambda)_+^{p-2}\,|\zeta|^2 + \varepsilon(p-1)(1+|z|^2)^{\frac{p-2}{2}}|\zeta|^2,$$

where we have also used that

$$h(t)+th'(t) = (p-1)(t-\lambda)_+^{p-2} \tag{3.15}$$

and

$$a(t)+ta'(t)\le (p-1)(1+t^2)^{\frac{p-2}{2}} \qquad \text{when } p\ge 2.$$

Otherwise, if $h'(|z|)+\varepsilon a'(|z|)<0$ (which may happen only when $1<p<2$), we easily get

$$\langle D^2 G_\varepsilon(z)\,\zeta,\zeta\rangle \le \big[h(|z|)+\varepsilon a(|z|)\big]\,|\zeta|^2.$$

For the derivation of the lower bound, we use the right inequality in (3.14) to deduce that

$$\begin{aligned}
\langle D^2 G_\varepsilon(z)\zeta,\zeta\rangle &\ge \big[h(|z|)+\varepsilon a(|z|)\big]|\zeta|^2 + \big[h'(|z|)+\varepsilon a'(|z|)\big]\,|z|\,|\zeta|^2\\
&= \big[h(|z|)+h'(|z|)|z|\big]|\zeta|^2 + \big[a(|z|)+\varepsilon a'(|z|)|z|\big]\,|\zeta|^2\\
&\ge (p-1)(|z|-\lambda)_+^{p-2}|\zeta|^2 + \varepsilon(p-1)(1+|z|^2)^{\frac{p-2}{2}}|\zeta|^2,
\end{aligned}$$

where, in the last line, we have used (3.15) and the fact that

$$a(t)+ta'(t)\ge (p-1)(1+t^2)^{\frac{p-2}{2}} \qquad \text{when } 1<p<2.$$

This proves the claim. □

In what follows, $u_\varepsilon\in u+W_0^{1,p}(B_R)$ will be the unique weak solution to (3.12). By standard elliptic regularity [55, Chapter 8], we know that $(1+|Du_\varepsilon|^2)^{\frac{p-2}{4}}Du_\varepsilon\in W^{1,2}_{loc}(B_R)$ and therefore $u_\varepsilon\in W^{2,2}_{loc}(B_R)$. Moreover, as usual, we shall denote by p^* the Sobolev conjugate exponent of

p, defined as

$$p^* := \begin{cases} \frac{np}{n-p} & \text{if } p < n, \\ \text{any value in } (p, \infty) & \text{if } p \geq n, \end{cases}$$

and denote by $(p^*)'$ its Hölder conjugate exponent.

The proofs of Theorems 3.1 and 3.4 are crucially based on the following results.

Proposition 3.12 (**Uniform energy estimate**). *With the notation and under the assumptions above, if $f \in L^{(p^*)'}(B_R)$, there exist two positive constants $\varepsilon_0 \leq 1$ and $C \equiv C(n,p)$ such that*

$$\int_{B_R} |Du_\varepsilon|^p \, dx \, \leq \, C \left[1 + \lambda^p \, + \, \|Du\|^p_{L^p(B_R)} \, + \|f\|^{p'}_{L^{(p^*)'}(B_R)} \right] \tag{3.16}$$

for all $\varepsilon \in (0, \varepsilon_0]$.

Proof. We insert in the weak formulation of (3.12)

$$\int_{B_R} \langle DG_\varepsilon(Du_\varepsilon), D\varphi \rangle \, dx \, = \, \int_{B_R} f_\varepsilon \, \varphi \, dx \qquad \text{for every } \varphi \in W^{1,p}_0(B_R),$$

the test function $\varphi = u_\varepsilon - u$. Recalling (3.13), this gives

$$\begin{aligned} &\int_{B_R} \langle H_{p-1}(Du_\varepsilon) + \varepsilon \, (1 + |Du_\varepsilon|^2)^{\frac{p-2}{2}} \, Du_\varepsilon, Du_\varepsilon \rangle \, dx \\ &\quad = \int_{B_R} \langle H_{p-1}(Du_\varepsilon) + \varepsilon \, (1 + |Du_\varepsilon|^2)^{\frac{p-2}{2}} \, Du_\varepsilon, Du \rangle \, dx \, + \int_{B_R} f_\varepsilon \, (u_\varepsilon - u) \, dx. \end{aligned} \tag{3.17}$$

Since

$$\langle H_{p-1}(z) + \varepsilon \, (1 + |z|^2)^{\frac{p-2}{2}} z, z \rangle \, \geq \, (|z| - \lambda)^p_+ \, + \, \varepsilon \, (1 + |z|^2)^{\frac{p-2}{2}} |z|^2 \, \geq \, (|z| - \lambda)^p_+$$

for every $z \in \mathbb{R}^n$, we can estimate the integrals in (3.17), thus obtaining

$$\begin{aligned} &\int_{B_R} (|Du_\varepsilon| - \lambda)^p_+ \, dx \\ &\quad \leq \int_{B_R} |Du_\varepsilon|^{p-1} \, |Du| \, dx \, + \, \varepsilon \int_{B_R} (1 + |Du_\varepsilon|^2)^{\frac{p-1}{2}} \, |Du| \, dx \, + \, \|f_\varepsilon\|_{L^{(p^*)'}(B_R)} \, \|u_\varepsilon - u\|_{L^{p^*}(B_R)} \\ &\quad \leq \left(1 + 2^{\frac{p-1}{2}} \right) \int_{B_R} |Du_\varepsilon|^{p-1} \, |Du| \, dx \, + \, 2^{\frac{p-1}{2}} \int_{B_R} |Du| \, dx \\ &\qquad + \, c(n,p) \, \|f_\varepsilon\|_{L^{(p^*)'}(B_R)} \, \|Du_\varepsilon - Du\|_{L^p(B_R)}, \end{aligned}$$

where we have used Hölder's and Sobolev inequalities and the fact that $\varepsilon, R \leq 1$. Now, applying Young's inequality with $\sigma > 0$, we arrive at

$$\begin{aligned} &\int_{B_R} (|Du_\varepsilon| - \lambda)^p_+ \, dx \\ &\quad \leq \, \sigma \int_{B_R} |Du_\varepsilon|^p \, dx \, + \, c(n,p,\sigma) \int_{B_R} |Du|^p \, dx \, + \, c(n,p,\sigma) \left[1 + \, \|f_\varepsilon\|^{p'}_{L^{(p^*)'}(B_R)} \right], \end{aligned}$$

where we have used again that $R \le 1$. Since

$$f_\varepsilon \to f \quad \text{ strongly in } L^{(p^*)'}(B_R) \quad \text{as } \varepsilon \to 0^+, \tag{3.18}$$

there exists a positive number $\varepsilon_0 \le 1$ such that

$$\|f_\varepsilon\|_{L^{(p^*)'}(B_R)} \le 1 + \|f\|_{L^{(p^*)'}(B_R)} \quad \text{ for all } \varepsilon \in (0, \varepsilon_0].$$

Then, for $\varepsilon \in (0, \varepsilon_0]$, we have

$$\begin{aligned}
&\int_{B_R} (|Du_\varepsilon| - \lambda)_+^p \, dx \\
&\quad \le \sigma \int_{B_R} |Du_\varepsilon|^p \, dx + c \, \|Du\|^p_{L^p(B_R)} + c \left[1 + \|f\|^{p'}_{L^{(p^*)'}(B_R)}\right] \\
&\quad \le \sigma \int_{B_R} [\lambda + (|Du_\varepsilon| - \lambda)_+]^p \, dx + c \, \|Du\|^p_{L^p(B_R)} + c \left[1 + \|f\|^{p'}_{L^{(p^*)'}(B_R)}\right] \\
&\quad \le 2^{p-1}\sigma \int_{B_R} (|Du_\varepsilon| - \lambda)_+^p \, dx + c \, \lambda^p + c \, \|Du\|^p_{L^p(B_R)} + c \left[1 + \|f\|^{p'}_{L^{(p^*)'}(B_R)}\right],
\end{aligned} \tag{3.19}$$

where $c \equiv c(n, p, \sigma) > 0$. Choosing $\sigma = \frac{1}{2^p}$ and absorbing the first term on the right-hand side of (3.19) into the left-hand side, we obtain

$$\int_{B_R} (|Du_\varepsilon| - \lambda)_+^p \, dx \le C \left[1 + \lambda^p + \|Du\|^p_{L^p(B_R)} + \|f\|^{p'}_{L^{(p^*)'}(B_R)}\right]$$

for some finite positive constant C depending on n and p, but not on ε. This estimate is sufficient to ensure the validity of the assertion. □

Proposition 3.13 (**Comparison estimate**). *With the notation and under the assumptions above, if $f \in L^{(p^*)'}(B_R)$, there exists a positive constant C depending only on n and p such that the estimate*

$$\begin{aligned}
&\int_{B_R} |\mathcal{V}_{\alpha,\lambda}(Du_\varepsilon) - \mathcal{V}_{\alpha,\lambda}(Du)|^2 \, dx \\
&\quad \le C \, \|f_\varepsilon - f\|_{L^{(p^*)'}(B_R)} \left[1 + \lambda + \|Du\|_{L^p(B_R)} + \|f\|^{\frac{1}{p-1}}_{L^{(p^*)'}(B_R)}\right] \\
&\qquad + C \, \varepsilon \left[1 + \lambda^p + \|Du\|^p_{L^p(B_R)} + \|f\|^{p'}_{L^{(p^*)'}(B_R)}\right]
\end{aligned} \tag{3.20}$$

holds for every $\varepsilon \in (0, \varepsilon_0]$, where ε_0 is the constant from Proposition 3.12.

Proof. We proceed by testing equations (3.1) and $(3.12)_1$ with the map $\varphi = u_\varepsilon - u$. Thus we find

$$\begin{aligned}
&\int_{B_R} \langle H_{p-1}(Du_\varepsilon) - H_{p-1}(Du), Du_\varepsilon - Du \rangle \, dx + \varepsilon \int_{B_R} \langle (1 + |Du_\varepsilon|^2)^{\frac{p-2}{2}} \, Du_\varepsilon, Du_\varepsilon - Du \rangle \, dx \\
&\quad = \int_{B_R} (f_\varepsilon - f)(u_\varepsilon - u) \, dx.
\end{aligned} \tag{3.21}$$

Using Lemma 3.9, the Cauchy-Schwarz inequality as well as Hölder's and Young's inequalities,

from (3.21) we obtain

$$
\begin{aligned}
&C\int_{B_R}|\mathcal{V}_{\alpha,\lambda}(Du_\varepsilon)-\mathcal{V}_{\alpha,\lambda}(Du)|^2\,dx+\varepsilon\int_{B_R}(1+|Du_\varepsilon|^2)^{\frac{p-2}{2}}|Du_\varepsilon|^2\,dx\\
&\quad\le \|f_\varepsilon-f\|_{L^{(p^*)'}(B_R)}\|u_\varepsilon-u\|_{L^{p^*}(B_R)}+\varepsilon\int_{B_R}(1+|Du_\varepsilon|^2)^{\frac{p-2}{2}}|Du_\varepsilon|\,|Du|\,dx\\
&\quad\le \|f_\varepsilon-f\|_{L^{(p^*)'}(B_R)}\|u_\varepsilon-u\|_{L^{p^*}(B_R)}+\varepsilon\int_{B_R}(1+|Du_\varepsilon|^2)^{\frac{p-1}{2}}|Du|\,dx\\
&\quad\le \|f_\varepsilon-f\|_{L^{(p^*)'}(B_R)}\|u_\varepsilon-u\|_{L^{p^*}(B_R)}+\frac{\varepsilon}{p'}\int_{B_R}(1+|Du_\varepsilon|^2)^{\frac{p}{2}}\,dx+\frac{\varepsilon}{p}\|Du\|^p_{L^p(B_R)},
\end{aligned}
$$

where C is a positive constant depending only on n and p. Now, let us consider the same $\varepsilon_0\in(0,1]$ as in Proposition 3.12 and let $\varepsilon\in(0,\varepsilon_0]$. Then, applying Sobolev's and Minkowski's inequalities, we get

$$
\begin{aligned}
&\int_{B_R}|\mathcal{V}_{\alpha,\lambda}(Du_\varepsilon)-\mathcal{V}_{\alpha,\lambda}(Du)|^2\,dx\\
&\quad\le C\|f_\varepsilon-f\|_{L^{(p^*)'}(B_R)}\|Du_\varepsilon-Du\|_{L^p(B_R)}+\frac{C\,\varepsilon}{p'}\int_{B_R}(1+|Du_\varepsilon|^2)^{\frac{p}{2}}\,dx+\frac{C\,\varepsilon}{p}\|Du\|^p_{L^p(B_R)}\\
&\quad\le C\|f_\varepsilon-f\|_{L^{(p^*)'}(B_R)}\left(\|Du_\varepsilon\|_{L^p(B_R)}+\|Du\|_{L^p(B_R)}\right)\\
&\qquad+\frac{C\,\varepsilon}{p'}\int_{B_R}(1+|Du_\varepsilon|^2)^{\frac{p}{2}}\,dx+\frac{C\,\varepsilon}{p}\|Du\|^p_{L^p(B_R)}\\
&\quad\le C\|f_\varepsilon-f\|_{L^{(p^*)'}(B_R)}\left[1+\lambda+\|Du\|_{L^p(B_R)}+\|f\|^{\frac{1}{p-1}}_{L^{(p^*)'}(B_R)}\right]\\
&\qquad+C\,\varepsilon\left[1+\lambda^p+\|Du\|^p_{L^p(B_R)}+\|f\|^{p'}_{L^{(p^*)'}(B_R)}\right],
\end{aligned}
$$

where, in the last two lines, we have used inequality (3.16). This concludes the proof. □

3.3 Proof of Theorem 3.1

In this section we prove Theorem 3.1, by dividing the proof into two steps. First, we shall derive a suitable uniform *a priori* estimate for the weak solutions u_ε of the regularized problems (3.12). Then, we conclude with a standard comparison argument (see e.g. [41, 52]) which, combined with the estimates from Propositions 3.12, 3.13 and 3.14, yields the local Sobolev regularity of the function $\mathcal{V}_{\alpha,\lambda}(Du)$. We begin with the following result.

Proposition 3.14 (**Uniform Sobolev estimate**). *Under the assumptions of Theorem 3.1 and with the notation above, there exists a positive number $\varepsilon_1\le 1$ such that, for every $\varepsilon\in(0,\varepsilon_1]$ and every pair of concentric balls $B_{r/2}\subset B_r\subset B_R$, we have*

$$
\begin{aligned}
&\int_{B_{r/2}}|D\mathcal{V}_{\alpha,\lambda}(Du_\varepsilon)|^2\,dx\\
&\quad\le\left(C+\frac{C}{r^2}\right)\left[1+\lambda^p+\|Du\|^p_{L^p(B_R)}+\|f\|^{p'}_{L^{p'}(B_R)}\right]+C\,\|f\|^{p'}_{B^{\frac{p-2}{p}}_{p',1}(B_R)}
\end{aligned}
\tag{3.22}
$$

for a positive constant C depending only on n and p in the case $\lambda = 0$, and additionally on α if $\lambda > 0$.

Proof. Let us first assume that $\lambda > 0$. Differentiating the equation in (3.12) with respect to x_j for some $j \in \{1, \dots, n\}$ and then integrating by parts, we obtain

$$\int_{B_R} \langle D^2 G_\varepsilon(Du_\varepsilon)\, D(\partial_j u_\varepsilon), D\varphi \rangle \, dx \,=\, \int_{B_R} (\partial_j f_\varepsilon)\, \varphi \, dx, \qquad \forall\, \varphi \in W^{1,p}_0(B_R). \tag{3.23}$$

Let $\eta \in C^\infty_0(B_r)$ be a standard cut-off function such that

$$0 \le \eta \le 1, \quad \eta \equiv 1 \ \text{ on } \overline{B}_{r/2}\,, \quad \|D\eta\|_\infty \le \frac{\tilde{c}}{r} \tag{3.24}$$

and choose

$$\varphi \,=\, \eta^2\, (\partial_j u_\varepsilon)\, \Phi\big((|Du_\varepsilon| - \lambda)_+\big),$$

where $\Phi : [0, \infty) \to [0, \infty)$ is an increasing, locally Lipschitz continuous function, such that Φ and Φ' are bounded on $[0, \infty)$, $\Phi(0) = 0$ and

$$\Phi'(t)\, t \,\le\, c_\Phi\, \Phi(t) \tag{3.25}$$

for a suitable constant $c_\Phi > 0$. Using the above choice of φ as a test function in (3.23), we get

$$\begin{aligned}
&\int_{B_r} \langle D^2 G_\varepsilon(Du_\varepsilon)\, D(\partial_j u_\varepsilon), D(\partial_j u_\varepsilon) \rangle\, \eta^2\, \Phi\left((|Du_\varepsilon| - \lambda)_+\right) \, dx \\
&+ \int_{B_r} \langle D^2 G_\varepsilon(Du_\varepsilon)\, D(\partial_j u_\varepsilon), D[(|Du_\varepsilon| - \lambda)_+] \rangle\, \eta^2\, (\partial_j u_\varepsilon)\, \Phi'\left((|Du_\varepsilon| - \lambda)_+\right) \, dx \\
&\quad = \,-\, 2 \int_{B_r} \langle D^2 G_\varepsilon(Du_\varepsilon)\, D(\partial_j u_\varepsilon), D\eta \rangle\, \eta\, (\partial_j u_\varepsilon)\, \Phi\left((|Du_\varepsilon| - \lambda)_+\right) \, dx \\
&\qquad + \int_{B_r} (\partial_j f_\varepsilon)(\partial_j u_\varepsilon)\, \eta^2\, \Phi\left((|Du_\varepsilon| - \lambda)_+\right) \, dx.
\end{aligned} \tag{3.26}$$

As for the first term on the right-hand side of (3.26), we have

$$\begin{aligned}
&-\, 2 \int_{B_r} \langle D^2 G_\varepsilon(Du_\varepsilon)\, D(\partial_j u_\varepsilon), D\eta \rangle\, \eta\, (\partial_j u_\varepsilon)\, \Phi\left((|Du_\varepsilon| - \lambda)_+\right) \, dx \\
&\le 2 \int_{B_r} |\langle D^2 G_\varepsilon(Du_\varepsilon)\, D(\partial_j u_\varepsilon), D\eta \rangle|\, \eta\, |\partial_j u_\varepsilon|\, \Phi\left((|Du_\varepsilon| - \lambda)_+\right) \, dx \\
&\le \, 2 \int_{B_r} \sqrt{\langle D^2 G_\varepsilon(Du_\varepsilon)\, D(\partial_j u_\varepsilon), D(\partial_j u_\varepsilon) \rangle}\, \sqrt{\langle D^2 G_\varepsilon(Du_\varepsilon)\, D\eta, D\eta \rangle}\, \eta\, |\partial_j u_\varepsilon|\, \Phi\left((|Du_\varepsilon| - \lambda)_+\right) \, dx \\
&\le \, \frac{1}{2} \int_{B_r} \langle D^2 G_\varepsilon(Du_\varepsilon)\, D(\partial_j u_\varepsilon), D(\partial_j u_\varepsilon) \rangle\, \eta^2\, \Phi\left((|Du_\varepsilon| - \lambda)_+\right) \, dx \\
&\qquad + \, 2 \int_{B_r} \langle D^2 G_\varepsilon(Du_\varepsilon)\, D\eta, D\eta \rangle\, |\partial_j u_\varepsilon|^2\, \Phi\left((|Du_\varepsilon| - \lambda)_+\right) \, dx,
\end{aligned} \tag{3.27}$$

where we have used Cauchy-Schwarz and Young's inequalities. Joining (3.26) and (3.27), we

get

$$\begin{aligned}
&\int_{B_r} \langle D^2 G_\varepsilon(Du_\varepsilon)\, D(\partial_j u_\varepsilon), D(\partial_j u_\varepsilon)\rangle\, \eta^2\, \Phi\left((|Du_\varepsilon|-\lambda)_+\right)\, dx \\
&+ \int_{B_r} \langle D^2 G_\varepsilon(Du_\varepsilon)\, D(\partial_j u_\varepsilon), D[(|Du_\varepsilon|-\lambda)_+]\rangle\, \eta^2\, (\partial_j u_\varepsilon)\, \Phi'\left((|Du_\varepsilon|-\lambda)_+\right)\, dx \\
&\quad \le \frac{1}{2}\int_{B_r} \langle D^2 G_\varepsilon(Du_\varepsilon)\, D(\partial_j u_\varepsilon), D(\partial_j u_\varepsilon)\rangle\, \eta^2\, \Phi\left((|Du_\varepsilon|-\lambda)_+\right)\, dx \\
&\qquad + 2\int_{B_r} \langle D^2 G_\varepsilon(Du_\varepsilon)\, D\eta, D\eta\rangle\, |\partial_j u_\varepsilon|^2\, \Phi\left((|Du_\varepsilon|-\lambda)_+\right)\, dx \\
&\qquad + \int_{B_r} (\partial_j f_\varepsilon)(\partial_j u_\varepsilon)\, \eta^2\, \Phi\left((|Du_\varepsilon|-\lambda)_+\right)\, dx. \qquad (3.28)
\end{aligned}$$

Reabsorbing the first integral in the right-hand side of (3.28) by the left-hand side and summing the resulting inequalities with respect to j from 1 to n, we obtain

$$I_1 + I_2 \;\le\; I_3 + I_4\,, \qquad (3.29)$$

where

$$\begin{aligned}
I_1 &:= \int_{B_r} \sum_{j=1}^n \langle D^2 G_\varepsilon(Du_\varepsilon)\, D(\partial_j u_\varepsilon), D(\partial_j u_\varepsilon)\rangle\, \eta^2\, \Phi\left((|Du_\varepsilon|-\lambda)_+\right)\, dx, \\
I_2 &:= 2\int_{B_r} \sum_{j=1}^n \langle D^2 G_\varepsilon(Du_\varepsilon)\, D(\partial_j u_\varepsilon), D[(|Du_\varepsilon|-\lambda)_+]\rangle\, \eta^2\, (\partial_j u_\varepsilon)\, \Phi'\left((|Du_\varepsilon|-\lambda)_+\right)\, dx, \\
I_3 &:= 4\int_{B_r} \sum_{j=1}^n \langle D^2 G_\varepsilon(Du_\varepsilon)\, D\eta, D\eta\rangle\, |\partial_j u_\varepsilon|^2\, \Phi\left((|Du_\varepsilon|-\lambda)_+\right)\, dx, \\
I_4 &:= 2\int_{B_r} \sum_{j=1}^n (\partial_j f_\varepsilon)(\partial_j u_\varepsilon)\, \eta^2\, \Phi\left((|Du_\varepsilon|-\lambda)_+\right)\, dx.
\end{aligned}$$

We now prove that I_2 is non-negative, so that it can be dropped in the following. First of all, for $|Du_\varepsilon| > \lambda$ we have

$$\begin{aligned}
&\sum_{j=1}^n \langle D^2 G_\varepsilon(Du_\varepsilon)\, D(\partial_j u_\varepsilon), D[(|Du_\varepsilon|-\lambda)_+]\rangle\, (\partial_j u_\varepsilon) \\
&\quad = \left[(p-1)\,\frac{(|Du_\varepsilon|-\lambda)_+^{p-2}}{|Du_\varepsilon|^2} - \frac{(|Du_\varepsilon|-\lambda)_+^{p-1}}{|Du_\varepsilon|^3} + \varepsilon\,(p-2)\,(1+|Du_\varepsilon|^2)^{\frac{p-4}{2}}\right] \\
&\qquad \cdot \sum_{i,j,k=1}^n (\partial_j u_\varepsilon)(\partial_i u_\varepsilon)(\partial_k u_\varepsilon)(\partial^2_{ij} u_\varepsilon)\, \partial_k[(|Du_\varepsilon|-\lambda)_+] \\
&\qquad + \left[\frac{(|Du_\varepsilon|-\lambda)_+^{p-1}}{|Du_\varepsilon|} + \varepsilon\,(1+|Du_\varepsilon|^2)^{\frac{p-2}{2}}\right] \sum_{i,j=1}^n (\partial_j u_\varepsilon)(\partial^2_{ij} u_\varepsilon)\, \partial_i[(|Du_\varepsilon|-\lambda)_+] \\
&\quad = \left[(p-1)\,\frac{(|Du_\varepsilon|-\lambda)_+^{p-2}}{|Du_\varepsilon|} - \frac{(|Du_\varepsilon|-\lambda)_+^{p-1}}{|Du_\varepsilon|^2} + \varepsilon\,(p-2)\,(1+|Du_\varepsilon|^2)^{\frac{p-4}{2}}\,|Du_\varepsilon|\right]
\end{aligned}$$

$$\cdot \left[\sum_{k=1}^{n} (\partial_k u_\varepsilon)\, \partial_k \left[(|Du_\varepsilon| - \lambda)_+\right]\right]^2$$
$$+ \left[(|Du_\varepsilon| - \lambda)_+^{p-1} + \varepsilon\,(1 + |Du_\varepsilon|^2)^{\frac{p-2}{2}}\, |Du_\varepsilon|\right] \left|D[(|Du_\varepsilon| - \lambda)_+]\right|^2, \tag{3.30}$$

where we have used the fact that

$$\partial_k[(|Du_\varepsilon| - \lambda)_+] \,=\, \partial_k(|Du_\varepsilon|) \,=\, \frac{1}{|Du_\varepsilon|} \sum_{j=1}^{n} (\partial_j u_\varepsilon)(\partial^2_{kj} u_\varepsilon) \qquad \text{when } |Du_\varepsilon| > \lambda.$$

Thus, coming back to the estimate of I_2, from (3.30) we deduce

$$I_2 \,\geq\, 2 \int_{B_r} \eta^2\, \Phi'\left((|Du_\varepsilon| - \lambda)_+\right) \left\{ \left[(p-1)\, \frac{(|Du_\varepsilon| - \lambda)_+^{p-2}}{|Du_\varepsilon|} \,-\, \frac{(|Du_\varepsilon| - \lambda)_+^{p-1}}{|Du_\varepsilon|^2}\right] \right.$$
$$\left. \cdot \left[\sum_{k=1}^{n} (\partial_k u_\varepsilon)\, \partial_k \left[(|Du_\varepsilon| - \lambda)_+\right]\right]^2 \,+\, (|Du_\varepsilon| - \lambda)_+^{p-1} \left|D[(|Du_\varepsilon| - \lambda)_+]\right|^2 \right\} dx.$$

Now, arguing as in the proof of [70, Lemma 4.1], for $|Du_\varepsilon| > \lambda$ we have

$$\left[\sum_{k=1}^{n} (\partial_k u_\varepsilon)\, \partial_k \left[(|Du_\varepsilon| - \lambda)_+\right]\right]^2 \,\leq\, |Du_\varepsilon|^2 \left|D(|Du_\varepsilon|)\right|^2 \,=\, |Du_\varepsilon|^2 \left|D[(|Du_\varepsilon| - \lambda)_+]\right|^2. \tag{3.31}$$

This implies

$$I_2 \,\geq\, 2\,(p-1) \int_{B_r} \eta^2\, \Phi'\left((|Du_\varepsilon| - \lambda)_+\right)\, \frac{(|Du_\varepsilon| - \lambda)_+^{p-2}}{|Du_\varepsilon|} \left[\sum_{k=1}^{n} (\partial_k u_\varepsilon)\, \partial_k \left[(|Du_\varepsilon| - \lambda)_+\right]\right]^2 dx \,\geq\, 0,$$

where we have used the fact that $\Phi'\left((|Du_\varepsilon| - \lambda)_+\right) \geq 0$. From estimate (3.29) we thus obtain

$$\int_{B_r} \sum_{j=1}^{n} \langle D^2 G_\varepsilon(Du_\varepsilon)\, D(\partial_j u_\varepsilon), D(\partial_j u_\varepsilon)\rangle\, \eta^2\, \Phi\left((|Du_\varepsilon| - \lambda)_+\right)\, dx$$
$$\leq\, 4 \sum_{j=1}^{n} \int_{B_r} \langle D^2 G_\varepsilon(Du_\varepsilon)\, D\eta, D\eta\rangle\, |\partial_j u_\varepsilon|^2\, \Phi\left((|Du_\varepsilon| - \lambda)_+\right)\, dx \tag{3.32}$$
$$+\, 2 \sum_{j=1}^{n} \int_{B_r} (\partial_j f_\varepsilon)(\partial_j u_\varepsilon)\, \eta^2\, \Phi\left((|Du_\varepsilon| - \lambda)_+\right)\, dx.$$

Now we choose

$$\Phi(t) \,:=\, \frac{t^{2\alpha}}{(t^2 + \lambda^2)^\alpha} \qquad \text{for } t \geq 0, \tag{3.33}$$

and therefore

$$\Phi'(t) \,=\, \frac{2\alpha\lambda^2\, t^{2\alpha-1}}{(t^2 + \lambda^2)^{\alpha+1}}\,.$$

This function satisfies (3.25) with $c_\Phi = 2\alpha$. Using (3.24), (3.33) and Lemma 3.11, estimate

(3.32) turns into

$$\begin{aligned}&\int_{B_r} \frac{(|Du_\varepsilon|-\lambda)_+^{p-1+2\alpha}}{|Du_\varepsilon|\,[\lambda^2+(|Du_\varepsilon|-\lambda)_+^2]^\alpha}\,|D^2u_\varepsilon|^2\,\eta^2\,dx\\ &\quad\le \frac{c(p)}{r^2}\int_{B_r}(1+|Du_\varepsilon|^2)^{\frac{p}{2}}\,dx \;+\; c(p)\sum_{j=1}^n\int_{B_r}(\partial_j f_\varepsilon)(\partial_j u_\varepsilon)\,\eta^2\,\Phi((|Du_\varepsilon|-\lambda)_+)\,dx.\end{aligned}\tag{3.34}$$

By Theorem 1.21, there exists a linear and bounded extension operator

$$\mathrm{ext}_{B_r}: B^{-2/p}_{p',1}(B_r)\hookrightarrow B^{-2/p}_{p',1}(\mathbb{R}^n)$$

such that $\mathrm{re}_{B_r}\circ\mathrm{ext}_{B_r}=\mathrm{id}$, where re_{B_r} is the restriction operator defined in Subsection 1.2.1 and the symbol id denotes the identity in $B^{-2/p}_{p',1}(B_r)$. Since $\partial_j f_\varepsilon=\mathrm{ext}_{B_r}(\partial_j f_\varepsilon)$ almost everywhere in B_r, we have

$$\int_{B_r}(\partial_j f_\varepsilon)(\partial_j u_\varepsilon)\,\eta^2\,\Phi((|Du_\varepsilon|-\lambda)_+)\,dx \;=\int_{B_r}\mathrm{ext}_{B_r}(\partial_j f_\varepsilon)\cdot(\partial_j u_\varepsilon)\,\eta^2\,\Phi((|Du_\varepsilon|-\lambda)_+)\,dx.$$

At this point, we need to estimate the integral containing $\mathrm{ext}_{B_r}(\partial_j f_\varepsilon)$. To this aim, we argue as in [23, Proposition 3.2]. By definition of dual norm, we get

$$\begin{aligned}&\left|\int_{B_r}\mathrm{ext}_{B_r}(\partial_j f_\varepsilon)\cdot(\partial_j u_\varepsilon)\,\eta^2\,\Phi((|Du_\varepsilon|-\lambda)_+)\,dx\right|\\ &\quad\le \|\mathrm{ext}_{B_r}(\partial_j f_\varepsilon)\|_{(\mathring{B}^{2/p}_{p,\infty}(\mathbb{R}^n))'}\,\|(\partial_j u_\varepsilon)\,\eta^2\,\Phi((|Du_\varepsilon|-\lambda)_+)\|_{B^{2/p}_{p,\infty}(\mathbb{R}^n)}\\ &\quad= \|\mathrm{ext}_{B_r}(\partial_j f_\varepsilon)\|_{(\mathring{B}^{2/p}_{p,\infty}(\mathbb{R}^n))'}\\ &\qquad\cdot\left(\|(\partial_j u_\varepsilon)\,\eta^2\,\Phi((|Du_\varepsilon|-\lambda)_+)\|_{L^p(\mathbb{R}^n)}+\left[(\partial_j u_\varepsilon)\,\eta^2\,\Phi((|Du_\varepsilon|-\lambda)_+)\right]_{B^{2/p}_{p,\infty}(\mathbb{R}^n)}\right),\end{aligned}\tag{3.35}$$

where, in the last line, we have used the equivalence (1.18). By the properties of η and the fact that $\Phi\le 1$, we have

$$\|(\partial_j u_\varepsilon)\,\eta^2\,\Phi((|Du_\varepsilon|-\lambda)_+)\|_{L^p(\mathbb{R}^n)}\;\le\;\|Du_\varepsilon\|_{L^p(B_r)}\,.\tag{3.36}$$

Moreover, using Theorem 1.26, we obtain

$$\|\mathrm{ext}_{B_r}(\partial_j f_\varepsilon)\|_{(\mathring{B}^{2/p}_{p,\infty}(\mathbb{R}^n))'}\;\le\;c\,\|\mathrm{ext}_{B_r}(\partial_j f_\varepsilon)\|_{B^{-2/p}_{p',1}(\mathbb{R}^n)}\,,$$

for some positive constant $c\equiv c(n,p)$. Combining the above inequality and the boundedness of the operator ext_{B_r} yields

$$\|\mathrm{ext}_{B_r}(\partial_j f_\varepsilon)\|_{(\mathring{B}^{2/p}_{p,\infty}(\mathbb{R}^n))'}\;\le\;c\,\|\partial_j f_\varepsilon\|_{B^{-2/p}_{p',1}(B_r)}\,.$$

Furthermore, applying Theorem 1.22, we find that

$$\|\partial_j f_\varepsilon\|_{B^{-2/p}_{p',1}(B_r)}\;\le\;c\,\|f_\varepsilon\|_{B^{\frac{p-2}{p}}_{p',1}(B_r)}.$$

Combining the preceding inequalities, we infer

$$\|\mathrm{ext}_{B_r}(\partial_j f_\varepsilon)\|_{(\mathring{B}^{2/p}_{p,\infty}(\mathbb{R}^n))'} \le c\,\|f_\varepsilon\|_{B^{\frac{p-2}{p}}_{p',1}(B_r)}, \tag{3.37}$$

for a positive constant c depending only on n and p. Now, recalling that

$$\left[(\partial_j u_\varepsilon)\,\eta^2\,\Phi((|Du_\varepsilon|-\lambda)_+)\right]^p_{B^{2/p}_{p,\infty}(\mathbb{R}^n)} = \sup_{|h|>0}\int_{\mathbb{R}^n}\frac{|\delta_h\left((\partial_j u_\varepsilon)\,\eta^2\,\Phi((|Du_\varepsilon|-\lambda)_+)\right)|^p}{|h|^2}\,dx$$

and using Lemma 3.7, we deduce

$$\begin{aligned}
&\int_{\mathbb{R}^n}\frac{|\delta_h\left((\partial_j u_\varepsilon)\,\eta^2\,\Phi((|Du_\varepsilon|-\lambda)_+)\right)|^p}{|h|^2}\,dx\\
&\quad\le \frac{c}{|h|^2}\int_{\mathbb{R}^n}\left|\delta_h\left(|\partial_j u_\varepsilon|^{\frac{p-2}{2}}(\partial_j u_\varepsilon)\,\eta^p\,[\Phi((|Du_\varepsilon|-\lambda)_+)]^{\frac{p}{2}}\right)\right|^2 dx\\
&\quad\le c(n,p)\int_{\mathbb{R}^n}\left|D\left(|\partial_j u_\varepsilon|^{\frac{p-2}{2}}(\partial_j u_\varepsilon)\,\eta^p\,[\Phi((|Du_\varepsilon|-\lambda)_+)]^{\frac{p}{2}}\right)\right|^2 dx,
\end{aligned}$$

where, in the last line, we have used the first statement in Lemma 1.14. By the properties of η at (3.24) and the boundedness of Φ, one can easily obtain

$$\begin{aligned}
\left[(\partial_j u_\varepsilon)\,\eta^2\,\Phi((|Du_\varepsilon|-\lambda)_+)\right]^p_{B^{2/p}_{p,\infty}(\mathbb{R}^n)} \le\ & c\int_{B_r}\left|D\left(|\partial_j u_\varepsilon|^{\frac{p-2}{2}}(\partial_j u_\varepsilon)\,[\Phi((|Du_\varepsilon|-\lambda)_+)]^{\frac{p}{2}}\right)\right|^2\eta^2\,dx\\
&+\ \frac{c}{r^2}\int_{B_r}|Du_\varepsilon|^p\,dx,
\end{aligned}\tag{3.38}$$

where $c\equiv c(n,p)>0$. Now, a straightforward computation reveals that, for every $k\in\{1,\dots,n\}$, we have

$$\begin{aligned}
&\partial_k\left[|\partial_j u_\varepsilon|^{\frac{p-2}{2}}(\partial_j u_\varepsilon)\,[\Phi((|Du_\varepsilon|-\lambda)_+)]^{\frac{p}{2}}\right]\\
&\quad=\frac{p}{2}\,|\partial_j u_\varepsilon|^{\frac{p-2}{2}}(\partial^2_{kj}u_\varepsilon)\,[\Phi((|Du_\varepsilon|-\lambda)_+)]^{\frac{p}{2}}\\
&\qquad+\ \frac{p}{2}\,|\partial_j u_\varepsilon|^{\frac{p-2}{2}}(\partial_j u_\varepsilon)\,[\Phi((|Du_\varepsilon|-\lambda)_+)]^{\frac{p-2}{2}}\,\Phi'((|Du_\varepsilon|-\lambda)_+)\,\frac{\langle Du_\varepsilon,\partial_k Du_\varepsilon\rangle}{|Du_\varepsilon|}.
\end{aligned}$$

This yields

$$\left|D\left(|\partial_j u_\varepsilon|^{\frac{p-2}{2}}(\partial_j u_\varepsilon)\,[\Phi((|Du_\varepsilon|-\lambda)_+)]^{\frac{p}{2}}\right)\right|^2 \le c(n,p)\,(\mathbf{A}_1+\mathbf{A}_2),\tag{3.39}$$

where we set

$$\mathbf{A}_1 := |Du_\varepsilon|^{p-2}\,|D^2u_\varepsilon|^2\,[\Phi((|Du_\varepsilon|-\lambda)_+)]^p$$

and

$$\mathbf{A}_2 := |Du_\varepsilon|^{p}\,|D^2u_\varepsilon|^2\,[\Phi((|Du_\varepsilon|-\lambda)_+)]^{p-2}\,[\Phi'((|Du_\varepsilon|-\lambda)_+)]^2.$$

We now estimate $\mathbf{A}_1$ and $\mathbf{A}_2$ separately in the set where $|Du_\varepsilon|>\lambda$, since both $\mathbf{A}_1$ and $\mathbf{A}_2$ vanish in the set $\{|Du_\varepsilon|\le\lambda\}$. Recalling the definition of Φ in (3.33), we can write $\mathbf{A}_1$ as the product of two terms:

$$\mathbf{A}_1 = \frac{|Du_\varepsilon|^{p-1}\,(|Du_\varepsilon|-\lambda)_+^{(2\alpha-1)(p-1)}}{[\lambda^2+(|Du_\varepsilon|-\lambda)_+^2]^{\alpha(p-1)}}\cdot\frac{|D^2u_\varepsilon|^2\,(|Du_\varepsilon|-\lambda)_+^{p-1+2\alpha}}{|Du_\varepsilon|\,[\lambda^2+(|Du_\varepsilon|-\lambda)_+^2]^{\alpha}}\,.$$

Then we have

$$\frac{|Du_\varepsilon|^{p-1}\,(|Du_\varepsilon|-\lambda)_+^{(2\alpha-1)(p-1)}}{[\lambda^2+(|Du_\varepsilon|-\lambda)_+^2]^{\alpha(p-1)}} \;\le\; 2^{\alpha(p-1)}\,\frac{|Du_\varepsilon|^{2\alpha(p-1)}}{|Du_\varepsilon|^{2\alpha(p-1)}} \;=\; 2^{\alpha(p-1)},$$

where we have used that $2\alpha-1>0$, since by assumption $\alpha \ge \frac{p+1}{2(p-1)} > \frac{1}{2}$. This implies that

$$\mathbf{A}_1 \;\le\; 2^{\alpha(p-1)}\,\frac{|D^2u_\varepsilon|^2\,(|Du_\varepsilon|-\lambda)_+^{p-1+2\alpha}}{|Du_\varepsilon|\,[\lambda^2+(|Du_\varepsilon|-\lambda)_+^2]^{\alpha}}\,. \tag{3.40}$$

Now we deal with $\mathbf{A}_2$. First, we observe that

$$[\Phi(t)]^{p-2}\,[\Phi'(t)]^2 = \frac{4\,\alpha^2\,\lambda^4\,t^{2(\alpha p-1)}}{(t^2+\lambda^2)^{\alpha p+2}}\,.$$

Therefore, $\mathbf{A}_2$ can be written as follows:

$$\mathbf{A}_2 = \,4\,\alpha^2\,\lambda^4\,\frac{|Du_\varepsilon|^{p+1}\,(|Du_\varepsilon|-\lambda)_+^{2\alpha(p-1)-p-1}}{[\lambda^2+(|Du_\varepsilon|-\lambda)_+^2]^{\alpha(p-1)+2}}\;\cdot\;\frac{|D^2u_\varepsilon|^2\,(|Du_\varepsilon|-\lambda)_+^{p-1+2\alpha}}{|Du_\varepsilon|\,[\lambda^2+(|Du_\varepsilon|-\lambda)_+^2]^{\alpha}}\,.$$

Again, the assumption $\alpha \ge \frac{p+1}{2(p-1)}$ implies $(|Du_\varepsilon|-\lambda)_+^{2\alpha(p-1)-p-1} \le |Du_\varepsilon|^{2\alpha(p-1)-p-1}$, and so

$$\lambda^4\,\frac{|Du_\varepsilon|^{p+1}\,(|Du_\varepsilon|-\lambda)_+^{2\alpha(p-1)-p-1}}{[\lambda^2+(|Du_\varepsilon|-\lambda)_+^2]^{\alpha(p-1)+2}} \;\le\; 2^{\alpha(p-1)+2}\lambda^4\,\frac{|Du_\varepsilon|^{2\alpha(p-1)}}{|Du_\varepsilon|^4\,|Du_\varepsilon|^{2\alpha(p-1)}} \;<\; 2^{\alpha(p-1)+2},$$

where we have used that $|Du_\varepsilon|>\lambda$. Thus we have

$$\mathbf{A}_2 \;\le\; 2^{\alpha(p-1)+4}\,\alpha^2\,\frac{|D^2u_\varepsilon|^2\,(|Du_\varepsilon|-\lambda)_+^{p-1+2\alpha}}{|Du_\varepsilon|\,[\lambda^2+(|Du_\varepsilon|-\lambda)_+^2]^{\alpha}}\,. \tag{3.41}$$

Joining estimates (3.38)−(3.41), we find

$$\begin{aligned}\big[(\partial_j u_\varepsilon)\,\eta^2\,\Phi((|Du_\varepsilon|-\lambda)_+)\big]^p_{B^{2/p}_{p,\infty}(\mathbb{R}^n)} \le\; & c\int_{B_r}\frac{(|Du_\varepsilon|-\lambda)_+^{p-1+2\alpha}\,|D^2u_\varepsilon|^2}{|Du_\varepsilon|\,[\lambda^2+(|Du_\varepsilon|-\lambda)_+^2]^{\alpha}}\,\eta^2\,dx\\ &+\;\frac{c}{r^2}\int_{B_r}|Du_\varepsilon|^p\,dx,\end{aligned}$$

where $c\equiv c(n,p,\alpha)>0$. Combining the previous inequality, (3.36) and (3.37) with (3.35), we get

$$\begin{aligned}&\left|\int_{B_r}(\partial_j f_\varepsilon)(\partial_j u_\varepsilon)\,\eta^2\,\Phi((|Du_\varepsilon|-\lambda)_+)\,dx\right|\\ &\quad\le\; c\,\|f_\varepsilon\|_{B^{\frac{p-2}{p}}_{p',1}(B_r)}\left[\left(\int_{B_r}\frac{(|Du_\varepsilon|-\lambda)_+^{p-1+2\alpha}\,|D^2u_\varepsilon|^2}{|Du_\varepsilon|\,[\lambda^2+(|Du_\varepsilon|-\lambda)_+^2]^{\alpha}}\,\eta^2\,dx\right)^{\frac{1}{p}}+\left(1+\frac{1}{r^{2/p}}\right)\|Du_\varepsilon\|_{L^p(B_r)}\right],\end{aligned}$$

for a constant $c \equiv c(n,p,\alpha) > 0$. Going back to (3.34) and using the above estimate gives

$$\begin{aligned}
&\int_{B_r} \frac{(|Du_\varepsilon|-\lambda)_+^{p-1+2\alpha}\,|D^2u_\varepsilon|^2}{|Du_\varepsilon|\,[\lambda^2+(|Du_\varepsilon|-\lambda)_+^2]^\alpha}\,\eta^2\,dx\\
&\quad\le c\left(\int_{B_r} \frac{(|Du_\varepsilon|-\lambda)_+^{p-1+2\alpha}\,|D^2u_\varepsilon|^2}{|Du_\varepsilon|\,[\lambda^2+(|Du_\varepsilon|-\lambda)_+^2]^\alpha}\,\eta^2\,dx\right)^{\frac{1}{p}} \|f_\varepsilon\|_{B^{\frac{p-2}{p}}_{p',1}(B_r)}\\
&\qquad+\left(c+\frac{c}{r^{2/p}}\right)\|Du_\varepsilon\|_{L^p(B_r)}\,\|f_\varepsilon\|_{B^{\frac{p-2}{p}}_{p',1}(B_r)} \;+\; \frac{c}{r^2}\int_{B_r}(1+|Du_\varepsilon|^2)^{\frac{p}{2}}\,dx,
\end{aligned}$$

where $c \equiv c(n,p,\alpha) > 0$. Now we apply Young's inequality to the first two terms on the right-hand side of the previous estimate. This yields

$$\begin{aligned}
&\int_{B_r} \frac{(|Du_\varepsilon|-\lambda)_+^{p-1+2\alpha}\,|D^2u_\varepsilon|^2}{|Du_\varepsilon|\,[\lambda^2+(|Du_\varepsilon|-\lambda)_+^2]^\alpha}\,\eta^2\,dx\\
&\qquad\le\left(C+\frac{C}{r^2}\right)\int_{B_r}(1+|Du_\varepsilon|^2)^{\frac{p}{2}}\,dx \;+\; C\,\|f_\varepsilon\|^{p'}_{B^{\frac{p-2}{p}}_{p',1}(B_r)},
\end{aligned}\tag{3.42}$$

for some constant $C \equiv C(n,p,\alpha) > 0$. At this point, recalling the definition of $\mathcal{V}_{\alpha,\lambda}$ in (3.2)−(3.3), a straightforward computation reveals that, for every $j \in \{1,\dots,n\}$, we have

$$\begin{aligned}
\partial_j \mathcal{V}_{\alpha,\lambda}(Du_\varepsilon) = \;&\frac{(|Du_\varepsilon|-\lambda)_+^{\frac{p-1+2\alpha}{2}}}{|Du_\varepsilon|^2\,[\lambda+(|Du_\varepsilon|-\lambda)_+]^{\frac{1+2\alpha}{2}}}\,\langle Du_\varepsilon,\partial_j Du_\varepsilon\rangle\,Du_\varepsilon\\
&+\mathcal{G}_{\alpha,\lambda}((|Du_\varepsilon|-\lambda)_+)\left[\frac{\partial_j Du_\varepsilon}{|Du_\varepsilon|}\;-\;\frac{\langle Du_\varepsilon,\partial_j Du_\varepsilon\rangle}{|Du_\varepsilon|^3}\,Du_\varepsilon\right]
\end{aligned}$$

if $|Du_\varepsilon| > \lambda$, and $\partial_j \mathcal{V}_{\alpha,\lambda}(Du_\varepsilon) = 0$ otherwise. In the set $\{|Du_\varepsilon| > 0\}$, this yields

$$|D\mathcal{V}_{\alpha,\lambda}(Du_\varepsilon)|^2 \le \mathbf{B}_1 + \mathbf{B}_2\,,\tag{3.43}$$

where we define

$$\mathbf{B}_1 := 2\,\frac{(|Du_\varepsilon|-\lambda)_+^{p-1+2\alpha}\,|D^2u_\varepsilon|^2}{[\lambda+(|Du_\varepsilon|-\lambda)_+]^{1+2\alpha}}$$

and

$$\mathbf{B}_2 := 8\,\frac{[\mathcal{G}_{\alpha,\lambda}((|Du_\varepsilon|-\lambda)_+)]^2\,|D^2u_\varepsilon|^2}{|Du_\varepsilon|^2}\,.$$

We now estimate $\mathbf{B}_1$ and $\mathbf{B}_2$ separately in the set where $|Du_\varepsilon| > \lambda$, since both $\mathbf{B}_1$ and $\mathbf{B}_2$ vanish for $0 < |Du_\varepsilon| \le \lambda$. We immediately have

$$\mathbf{B}_1 \le 2\,\frac{(|Du_\varepsilon|-\lambda)_+^{p-1+2\alpha}\,|D^2u_\varepsilon|^2}{|Du_\varepsilon|\,[\lambda^2+(|Du_\varepsilon|-\lambda)_+^2]^\alpha}\,.\tag{3.44}$$

As for $\mathbf{B}_2$, by Lemma 3.8 we obtain

$$\begin{aligned}
\mathbf{B}_2 &\le \frac{32}{p^2}\,\frac{(|Du_\varepsilon|-\lambda)_+^{p+1+2\alpha}\,|D^2u_\varepsilon|^2}{|Du_\varepsilon|^2\,[\lambda+(|Du_\varepsilon|-\lambda)_+]^{1+2\alpha}}\\
&\le \frac{32}{p^2}\,\frac{(|Du_\varepsilon|-\lambda)_+^{p-1+2\alpha}\,(|Du_\varepsilon|-\lambda)_+^2\,|D^2u_\varepsilon|^2}{|Du_\varepsilon|^3\,[\lambda^2+(|Du_\varepsilon|-\lambda)_+^2]^\alpha}
\end{aligned}\tag{3.45}$$

$$\leq \frac{32}{p^2}\,\frac{(|Du_\varepsilon|-\lambda)_+^{p-1+2\alpha}\,|D^2u_\varepsilon|^2}{|Du_\varepsilon|\,[\lambda^2+(|Du_\varepsilon|-\lambda)_+^2]^\alpha}\,. \tag{3.46}$$

Joining estimates (3.43)−(3.45), we then find

$$\int_{B_r}|D\mathcal{V}_{\alpha,\lambda}(Du_\varepsilon)|^2\,\eta^2\,dx \;\leq\; c(p)\int_{B_r}\frac{(|Du_\varepsilon|-\lambda)_+^{p-1+2\alpha}\,|D^2u_\varepsilon|^2}{|Du_\varepsilon|\,[\lambda^2+(|Du_\varepsilon|-\lambda)_+^2]^\alpha}\,\eta^2\,dx, \tag{3.47}$$

which combined with (3.42), gives

$$\int_{B_r}|D\mathcal{V}_{\alpha,\lambda}(Du_\varepsilon)|^2\,\eta^2\,dx \;\leq\; \left(C+\frac{C}{r^2}\right)\int_{B_r}(1+|Du_\varepsilon|^2)^{\frac{p}{2}}\,dx \;+\; C\,\|f_\varepsilon\|^{p'}_{B^{\frac{p-2}{p}}_{p',1}(B_r)}.$$

Let us now consider the same $\varepsilon_0\in(0,1]$ as in Proposition 3.12 and let $\varepsilon\in(0,\varepsilon_0]$. Then, recalling that $\eta\equiv 1$ on $\overline{B}_{r/2}$ and applying estimate (3.16), we obtain

$$\int_{B_{r/2}}|D\mathcal{V}_{\alpha,\lambda}(Du_\varepsilon)|^2\,dx \;\leq\; \left(C+\frac{C}{r^2}\right)\Big[1+\lambda^p+\;\|Du\|^p_{L^p(B_R)}\;+\|f\|^{p'}_{L^{(p^*)'}(B_R)}\Big]+\;C\,\|f_\varepsilon\|^{p'}_{B^{\frac{p-2}{p}}_{p',1}(B_r)},$$

where we have used the fact that $r<R\leq 1$. Since $(p^*)'<p'$, using Hölder's inequality, from the above estimate we get

$$\int_{B_{r/2}}|D\mathcal{V}_{\alpha,\lambda}(Du_\varepsilon)|^2\,dx \;\leq\; \left(C+\frac{C}{r^2}\right)\Big[1+\lambda^p+\;\|Du\|^p_{L^p(B_R)}\;+\|f\|^{p'}_{L^{p'}(B_R)}\Big]+\;C\,\|f_\varepsilon\|^{p'}_{B^{\frac{p-2}{p}}_{p',1}(B_r)}.$$

Furthermore, there exists a positive number $\varepsilon_1\leq\varepsilon_0$ such that

$$\|f_\varepsilon\|_{B^{\frac{p-2}{p}}_{p',1}(B_r)}\;\leq\;\|f\|_{B^{\frac{p-2}{p}}_{p',1}(B_R)}<+\infty\qquad\text{for every }\varepsilon\in(0,\varepsilon_1].$$

Combining the last two estimates for $\varepsilon\in(0,\varepsilon_1]$, we conclude the proof in the case $\lambda>0$.

Finally, when $\lambda=0$ the above proof can be greatly simplified, since the parameter α plays no role, $\Phi\equiv 1$ and

$$\mathcal{V}_0(Du_\varepsilon)=\frac{2}{p}\,H_{\frac{p}{2}}(Du_\varepsilon)=\frac{2}{p}\,|Du_\varepsilon|^{\frac{p-2}{2}}Du_\varepsilon\,.$$

In this regard, we leave the details to the reader. □

Combining Lemma 1.14 with estimate (3.22), we obtain the following corollary.

Corollary 3.15. *Let $\varepsilon_1\in(0,1]$ be the constant from Proposition 3.14. Under the assumptions of Theorem 3.1 and with the notation above, for every pair of concentric balls $B_{r/4}\subset B_r\subset B_R$ we have*

$$\begin{aligned}&\int_{B_{r/4}}|\tau_{j,h}\mathcal{V}_{\alpha,\lambda}(Du_\varepsilon)|^2\,dx\\&\quad\leq\;\frac{C\,|h|^2\,(1+r^2)}{r^2}\Big[1+\lambda^p+\;\|Du\|^p_{L^p(B_R)}\;+\|f\|^{p'}_{L^{p'}(B_R)}\Big]+\;C\,|h|^2\,\|f\|^{p'}_{B^{\frac{p-2}{p}}_{p',1}(B_R)},\end{aligned}\tag{3.48}$$

for every $j\in\{1,\dots,n\}$, for every $h\in\mathbb{R}$ such that $|h|<\frac{r}{8}$, for every $\varepsilon\in(0,\varepsilon_1]$ and a positive constant C depending only on n and p in the case $\lambda=0$, and additionally on α if $\lambda>0$.

We are now in a position to prove Theorem 3.1.

Proof of Theorem 3.1. Consider the same $\varepsilon_1 \in (0,1]$ as in Proposition 3.14 and let $\varepsilon \in (0,\varepsilon_1]$. Moreover, let u_ε be the unique energy solution of the Dirichlet problem (3.12). Now we fix three concentric balls $B_{r/4}$, $B_{r/2}$ and B_r, with $B_r \subset B_R \Subset \Omega$, $R \le 1$, and use the finite difference operator $\tau_{j,h}$ defined in Section 1.1, for increments $h \in \mathbb{R}\setminus\{0\}$ such that $|h| < \frac{r}{8}$. In what follows, we will denote by C a positive constant which neither depends on ε nor on h. In order to obtain an estimate for the finite difference $\tau_{j,h}\mathcal{V}_{\alpha,\lambda}(Du)$, we use the following comparison argument:

$$\begin{aligned}
&\int_{B_{r/4}} |\tau_{j,h}\mathcal{V}_{\alpha,\lambda}(Du)|^2\,dx\\
&\quad\le 4\int_{B_{r/4}} |\tau_{j,h}\mathcal{V}_{\alpha,\lambda}(Du_\varepsilon)|^2\,dx + 4\int_{B_{r/4}} |\mathcal{V}_{\alpha,\lambda}(Du_\varepsilon)-\mathcal{V}_{\alpha,\lambda}(Du)|^2\,dx\\
&\qquad + 4\int_{B_{r/4}} |\mathcal{V}_{\alpha,\lambda}(Du_\varepsilon(x+he_j))-\mathcal{V}_{\alpha,\lambda}(Du(x+he_j))|^2\,dx\\
&\quad\le 4\int_{B_{r/4}} |\tau_{j,h}\mathcal{V}_{\alpha,\lambda}(Du_\varepsilon)|^2\,dx + 8\int_{B_R} |\mathcal{V}_{\alpha,\lambda}(Du_\varepsilon)-\mathcal{V}_{\alpha,\lambda}(Du)|^2\,dx,
\end{aligned}$$

where we have used the second statement in Lemma 1.14. Combining the previous estimate with (3.48) and (3.20), for every $j \in \{1,\dots,n\}$ we get

$$\begin{aligned}
&\int_{B_{r/4}} |\tau_{j,h}\mathcal{V}_{\alpha,\lambda}(Du)|^2\,dx\\
&\quad\le \frac{C\,|h|^2\,(1+r^2)}{r^2}\left[1+\lambda^p+\|Du\|^p_{L^p(B_R)}+\|f\|^{p'}_{L^{p'}(B_R)}\right] + C\,|h|^2\,\|f\|^{p'}_{B^{\frac{p-2}{p}}_{p',1}(B_R)}\\
&\qquad + C\,\|f_\varepsilon - f\|_{L^{(p^*)'}(B_R)}\left[1+\lambda+\|Du\|_{L^p(B_R)}+\|f\|^{\frac{1}{p-1}}_{L^{(p^*)'}(B_R)}\right]\\
&\qquad + C\,\varepsilon\left[1+\lambda^p+\|Du\|^p_{L^p(B_R)}+\|f\|^{p'}_{L^{(p^*)'}(B_R)}\right],
\end{aligned}\tag{3.49}$$

which holds for every sufficiently small $h \in \mathbb{R}\setminus\{0\}$ and a constant C depending only on n and p in the case $\lambda = 0$, and additionally on α if $\lambda > 0$. Therefore, recalling (3.18) and letting $\varepsilon \searrow 0$ in (3.49), we obtain

$$\begin{aligned}
&\int_{B_{r/4}} |\Delta_{j,h}\mathcal{V}_{\alpha,\lambda}(Du)|^2\,dx\\
&\quad\le \left(C+\frac{C}{r^2}\right)\left[1+\lambda^p+\|Du\|^p_{L^p(B_R)}+\|f\|^{p'}_{L^{p'}(B_R)}\right] + C\,\|f\|^{p'}_{B^{\frac{p-2}{p}}_{p',1}(B_R)}.
\end{aligned}$$

Since the above inequality holds for every $j \in \{1,\dots,n\}$ and every sufficiently small $h \neq 0$, by Lemma 1.15 we may conclude that $\mathcal{V}_{\alpha,\lambda}(Du) \in W^{1,2}_{loc}(\Omega,\mathbb{R}^n)$. Moreover, letting $h \to 0$ in the previous inequality, we obtain estimate (3.5) for every ball $B_R \Subset \Omega$ with $R \le 1$. The validity of (3.5) for arbitrary balls follows from a standard covering argument. □

3.4 Proof of Theorem 3.4

This section is devoted to the proof of Theorem 3.4. Actually, here we limit ourselves to deriving the *a priori* estimates for $1 < p \le 2$, since inequality (3.8) can be obtained using the same arguments presented in Section 3.3. In what follows, we shall keep the notations used for the proof of Proposition 3.14.

Proof of Theorem 3.4. Let us first assume that $\lambda > 0$. Arguing as in the proof of Proposition 3.14, we define the integrals $I_1 - I_4$ exactly as in (3.29). We need to treat differently only the integrals I_2 and I_4, in which the new assumptions $1 < p \le 2$ and $f \in L^{\frac{np}{n(p-1)+2-p}}_{loc}(\Omega)$ are involved. It comes out that I_2 is non-negative, as in the super-quadratic case. Indeed, estimates (3.30) and (3.31) lead us to

$$
\begin{aligned}
I_2 \;\ge\; & 2\int_{B_r} \eta^2\,\Phi'\left((|Du_\varepsilon|-\lambda)_+\right) \\
& \cdot\Bigg\{\left[(p-1)\,\frac{(|Du_\varepsilon|-\lambda)_+^{p-2}}{|Du_\varepsilon|} \;-\; \frac{(|Du_\varepsilon|-\lambda)_+^{p-1}}{|Du_\varepsilon|^2} + \varepsilon(p-2)(1+|Du_\varepsilon|^2)^{\frac{p-4}{2}}|Du_\varepsilon|\right] \\
& \quad\cdot\left[\sum_{k=1}^n (\partial_k u_\varepsilon)\,\partial_k\left[(|Du_\varepsilon|-\lambda)_+\right]\right]^2 \;+\; (|Du_\varepsilon|-\lambda)_+^{p-1}\left|D[(|Du_\varepsilon|-\lambda)_+]\right|^2 \\
& \quad+\;\varepsilon\,(1+|Du_\varepsilon|^2)^{\frac{p-4}{2}}\,|Du_\varepsilon|^3\left|D[(|Du_\varepsilon|-\lambda)_+]\right|^2\Bigg\}\,dx \\
\ge\; & 2\int_{B_r} \eta^2\,\Phi'\left((|Du_\varepsilon|-\lambda)_+\right)\left[(p-1)\,\frac{(|Du_\varepsilon|-\lambda)_+^{p-2}}{|Du_\varepsilon|} \;+\; \varepsilon(p-1)(1+|Du_\varepsilon|^2)^{\frac{p-4}{2}}|Du_\varepsilon|\right] \\
& \cdot\left[\sum_{k=1}^n (\partial_k u_\varepsilon)\,\partial_k\left[(|Du_\varepsilon|-\lambda)_+\right]\right]^2 dx \ge 0.
\end{aligned}
$$

Therefore, using (3.24), (3.33) and Lemma 3.11, from (3.29) we now obtain

$$
\begin{aligned}
& \int_{B_r} \frac{(|Du_\varepsilon|-\lambda)_+^{p-1+2\alpha}}{|Du_\varepsilon|\,[\lambda^2+(|Du_\varepsilon|-\lambda)_+^2]^\alpha}\,|D^2u_\varepsilon|^2\,\eta^2\,dx \\
& \quad\le\; \frac{c(p)}{r^2}\int_{B_r}(1+|Du_\varepsilon|^2)^{\frac{p}{2}}\,dx \;+\; \frac{c(p)}{r^2}\int_{B_r}\frac{(|Du_\varepsilon|-\lambda)_+^{p-2+2\alpha}}{[\lambda^2+(|Du_\varepsilon|-\lambda)_+^2]^\alpha}\,|Du_\varepsilon|^2\,dx \\
& \qquad+\; c(p)\sum_{j=1}^n\int_{B_r}(\partial_j f_\varepsilon)(\partial_j u_\varepsilon)\,\eta^2\,\Phi((|Du_\varepsilon|-\lambda)_+)\,dx \\
& \quad\le\; \frac{c(p,\alpha)}{r^2}\int_{B_r}(1+|Du_\varepsilon|^2)^{\frac{p}{2}}\,dx \;+ c(p)\sum_{j=1}^n\int_{B_r}(\partial_j f_\varepsilon)(\partial_j u_\varepsilon)\,\eta^2\,\Phi((|Du_\varepsilon|-\lambda)_+)\,dx, \qquad (3.50)
\end{aligned}
$$

where the function Φ is defined in (3.33). At this point, we integrate by parts and then apply

Hölder's inequality in the second integral on right-hand side of (3.50). This gives

$$\begin{aligned}&\left|\int_{B_r}(\partial_j f_\varepsilon)(\partial_j u_\varepsilon)\,\eta^2\,\Phi((|Du_\varepsilon|-\lambda)_+)\,dx\right|\\&\quad\le\ \|f_\varepsilon\|_{L^{\frac{np}{n(p-1)+2-p}}(B_r)}\,\|\partial_j[(\partial_j u_\varepsilon)\,\eta^2\,\Phi((|Du_\varepsilon|-\lambda)_+)]\|_{L^{\frac{np}{n-2+p}}(\mathbb{R}^n)}.\end{aligned}\tag{3.51}$$

From now on, we will only deal with the case $n\ge 3$ and $1<p<2$, since the remaining cases imply that

$$\frac{np}{n(p-1)+2-p}\ =\ \frac{np}{n-2+p}\ =\ 2$$

and can be addressed by suitably modifying the arguments used in Section 3.3. Note that in the case $p=n=2$, we have $(p^*)'<p'=2$, and therefore we can continue to argue as in the proof of Proposition 3.14.
For ease of notation, we now set

$$Z(x):=\ \eta^2(x)\cdot(\partial_j u_\varepsilon(x))\cdot\Phi((|Du_\varepsilon(x)|-\lambda)_+)\,.$$

Since $\frac{np}{n-2+p}<2$ for $n\ge 3$ and $1<p<2$, an application of Hölder's inequality and Lemma 1.3 yield

$$\begin{aligned}&\int_{\mathbb{R}^n}|\tau_{j,h}Z(x)|^{\frac{np}{n-2+p}}\,dx\\&=\ \int_{\mathbb{R}^n}|\tau_{j,h}Z(x)|^{\frac{np}{n-2+p}}\left(|Z(x+he_j)|^2+|Z(x)|^2\right)^{\frac{np\,(p-2)}{4(n-2+p)}}\left(|Z(x+he_j)|^2+|Z(x)|^2\right)^{\frac{np\,(2-p)}{4(n-2+p)}}dx\\&\le\ c(n,p)\left(\int_{\mathbb{R}^n}|\tau_{j,h}Z(x)|^2\left(|Z(x+he_j)|^2+|Z(x)|^2\right)^{\frac{p-2}{2}}dx\right)^{\frac{np}{2(n-2+p)}}\left(\int_{\mathbb{R}^n}|Z(x)|^{\frac{np}{n-2}}dx\right)^{\frac{(n-2)(2-p)}{2(n-2+p)}}\\&\le\ c(n,p)\left(\int_{\mathbb{R}^n}\left|\tau_{j,h}\left(|Z(x)|^{\frac{p-2}{2}}Z(x)\right)\right|^2dx\right)^{\frac{np}{2(n-2+p)}}\left(\int_{\mathbb{R}^n}|Z(x)|^{\frac{np}{n-2}}dx\right)^{\frac{(n-2)(2-p)}{2(n-2+p)}}\end{aligned}$$

for every $h\in\mathbb{R}\setminus\{0\}$. Dividing both sides by $|h|^{\frac{np}{n-2+p}}$ and letting $h\to 0$, by virtue of Lemma 1.15 we obtain

$$\begin{aligned}&\int_{\mathbb{R}^n}|\partial_j Z(x)|^{\frac{np}{n-2+p}}\,dx\\&\le c(n,p)\left(\int_{\mathbb{R}^n}\left|\partial_j\left(|Z(x)|^{\frac{p-2}{2}}Z(x)\right)\right|^2dx\right)^{\frac{np}{2(n-2+p)}}\left(\int_{\mathbb{R}^n}|Z(x)|^{\frac{np}{n-2}}dx\right)^{\frac{(n-2)(2-p)}{2(n-2+p)}}\\&\le c(n,p)\left(\int_{\mathbb{R}^n}\left|D\left(|Z(x)|^{\frac{p-2}{2}}Z(x)\right)\right|^2dx\right)^{\frac{np}{2(n-2+p)}}\left(\int_{\mathbb{R}^n}|Z(x)|^{\frac{np}{n-2}}dx\right)^{\frac{(n-2)(2-p)}{2(n-2+p)}}\\&=c(n,p)\left(\int_{\mathbb{R}^n}\left|D\left(|Z(x)|^{\frac{p-2}{2}}Z(x)\right)\right|^2dx\right)^{\frac{np}{2(n-2+p)}}\left(\int_{\mathbb{R}^n}\left||Z(x)|^{\frac{p-2}{2}}Z(x)\right|^{\frac{2n}{n-2}}dx\right)^{\frac{(n-2)(2-p)}{2(n-2+p)}}\\&\le c(n,p)\left(\int_{\mathbb{R}^n}\left|D\left(|Z(x)|^{\frac{p-2}{2}}Z(x)\right)\right|^2dx\right)^{\frac{np}{2(n-2+p)}}\left(\int_{\mathbb{R}^n}\left|D\left(|Z(x)|^{\frac{p-2}{2}}Z(x)\right)\right|^2dx\right)^{\frac{n(2-p)}{2(n-2+p)}}\\&=c(n,p)\left(\int_{\mathbb{R}^n}\left|D\left(|Z(x)|^{\frac{p-2}{2}}Z(x)\right)\right|^2dx\right)^{\frac{n}{n-2+p}}.\end{aligned}\tag{3.52}$$

Recalling the definition of Z, calculating the gradient in the right-hand side of (3.52), using the properties of η and recalling that $\Phi \le 1$, we get

$$\begin{aligned}
&\| \partial_j[\eta^2(\partial_j u_\varepsilon)\,\Phi((|Du_\varepsilon|-\lambda)_+)]\|_{L^{\frac{np}{n-2+p}}(\mathbb{R}^n)} \\
&\quad\le c\left(\int_{\mathbb{R}^n}\left|D\left(\eta^p|\partial_j u_\varepsilon|^{\frac{p-2}{2}}(\partial_j u_\varepsilon)\,[\Phi((|Du_\varepsilon|-\lambda)_+)]^{\frac{p}{2}}\right)\right|^2 dx\right)^{\frac{1}{p}} \\
&\quad\le c\left(\int_{B_r}\eta^{2p}\left|D\left(|\partial_j u_\varepsilon|^{\frac{p-2}{2}}(\partial_j u_\varepsilon)\,[\Phi((|Du_\varepsilon|-\lambda)_+)]^{\frac{p}{2}}\right)\right|^2 dx\right)^{\frac{1}{p}} \\
&\qquad + c\left(\int_{B_r}\eta^{2p-2}\,|D\eta|^2\,|Du_\varepsilon|^p\,dx\right)^{\frac{1}{p}}. \qquad (3.53)
\end{aligned}$$

Inserting (3.53) into (3.51) and using the properties of η, we obtain

$$\begin{aligned}
&\left|\int_{B_r}(\partial_j f_\varepsilon)(\partial_j u_\varepsilon)\,\eta^2\,\Phi((|Du_\varepsilon|-\lambda)_+)\,dx\right| \\
&\quad\le c\,\|f_\varepsilon\|_{L^{\frac{np}{n(p-1)+2-p}}(B_r)}\left(\int_{B_r}\eta^2\left|D\left(|\partial_j u_\varepsilon|^{\frac{p-2}{2}}(\partial_j u_\varepsilon)\,[\Phi((|Du_\varepsilon|-\lambda)_+)]^{\frac{p}{2}}\right)\right|^2 dx\right)^{\frac{1}{p}} \\
&\qquad + \frac{c}{r^{2/p}}\,\|f_\varepsilon\|_{L^{\frac{np}{n(p-1)+2-p}}(B_r)}\left(\int_{B_r}|Du_\varepsilon|^p\,dx\right)^{\frac{1}{p}}. \qquad (3.54)
\end{aligned}$$

Now, combining (3.54) with (3.50), we have

$$\begin{aligned}
&\int_{B_r}\frac{(|Du_\varepsilon|-\lambda)_+^{p-1+2\alpha}}{|Du_\varepsilon|\,[\lambda^2+(|Du_\varepsilon|-\lambda)_+^2]^\alpha}\,|D^2u_\varepsilon|^2\,\eta^2\,dx \\
&\quad\le \frac{c}{r^2}\int_{B_r}(1+|Du_\varepsilon|^2)^{\frac{p}{2}}\,dx + \frac{c}{r^{2/p}}\,\|f_\varepsilon\|_{L^{\frac{np}{n(p-1)+2-p}}(B_r)}\,\|Du_\varepsilon\|_{L^p(B_r)} \\
&\qquad + c\,\|f_\varepsilon\|_{L^{\frac{np}{n(p-1)+2-p}}(B_r)}\sum_{j=1}^n\left(\int_{B_r}\eta^2\left|D\left(|\partial_j u_\varepsilon|^{\frac{p-2}{2}}(\partial_j u_\varepsilon)\,[\Phi((|Du_\varepsilon|-\lambda)_+)]^{\frac{p}{2}}\right)\right|^2 dx\right)^{\frac{1}{p}},
\end{aligned}$$

where now $c\equiv c(n,p,\alpha)>0$. The last integral can be estimated using (3.39), (3.40) and (3.41). Thus we infer

$$\begin{aligned}
&\int_{B_r}\frac{(|Du_\varepsilon|-\lambda)_+^{p-1+2\alpha}}{|Du_\varepsilon|\,[\lambda^2+(|Du_\varepsilon|-\lambda)_+^2]^\alpha}\,|D^2u_\varepsilon|^2\,\eta^2\,dx \\
&\quad\le \frac{c}{r^2}\int_{B_r}(1+|Du_\varepsilon|^2)^{\frac{p}{2}}\,dx + \frac{c}{r^{2/p}}\,\|f_\varepsilon\|_{L^{\frac{np}{n(p-1)+2-p}}(B_r)}\,\|Du_\varepsilon\|_{L^p(B_r)} \\
&\qquad + c\,\|f_\varepsilon\|_{L^{\frac{np}{n(p-1)+2-p}}(B_r)}\left(\int_{B_r}\frac{(|Du_\varepsilon|-\lambda)_+^{p-1+2\alpha}}{|Du_\varepsilon|\,[\lambda^2+(|Du_\varepsilon|-\lambda)_+^2]^\alpha}\,|D^2u_\varepsilon|^2\,\eta^2\,dx\right)^{\frac{1}{p}}.
\end{aligned}$$

Applying Young's inequality to reabsorb the last integral by the left-hand side, and then using inequality (3.47), we derive

$$\int_{B_r}|D\mathcal{V}_{\alpha,\lambda}(Du_\varepsilon)|^2\,\eta^2\,dx \le \frac{C}{r^2}\int_{B_r}(1+|Du_\varepsilon|^2)^{\frac{p}{2}}\,dx + C\,\|f_\varepsilon\|^{p'}_{L^{\frac{np}{n(p-1)+2-p}}(B_r)},$$

for a positive constant C depending only on n, p and α. The desired conclusion follows by arguing as in the proofs of Proposition 3.14 and Theorem 3.1, observing that

$$(p^*)' = \frac{np}{np-n+p} < \frac{np}{n(p-1)+2-p} \qquad \text{for every } p>1.$$

Finally, when $\lambda = 0$ the above proof can be greatly simplied, since the parameter α plays no role, $\Phi \equiv 1$ and

$$\mathcal{V}_0(Du_\varepsilon) = \frac{2}{p}\,|Du_\varepsilon|^{\frac{p-2}{2}} Du_\varepsilon\,.$$

We leave the details to the reader. □

Chapter 4

Regularity results for a class of widely degenerate parabolic equations

In this chapter, we are interested in the regularity properties of weak solutions $u : \Omega_T \to \mathbb{R}$ to the strongly degenerate parabolic equation

$$u_t - \operatorname{div}\left((|Du| - \lambda)_+^{p-1} \frac{Du}{|Du|} \right) = f \quad \text{in } \Omega_T = \Omega \times (0,T), \tag{4.1}$$

where $2 \le p < \infty$, Ω is a bounded domain in $\mathbb{R}^n$ ($n \ge 2$), $T > 0$ and λ is a positive constant. The results reported here have been obtained in [10] and [5].

Notice that the above PDE is nothing but the evolutionary version of the elliptic equation (2.1). A motivation for studying equations of the type (4.1) can be found in *gas filtration problems taking into account the initial pressure gradient* (see Section 4.2 for a brief explanation).

In what follows, we shall first prove the Sobolev spatial regularity of a nonlinear function of the spatial gradient Du of the weak solutions to (4.1), which in turn implies the Sobolev time regularity of the solutions, by assuming that the datum f belongs to a suitable Lebesgue-Sobolev parabolic space.

More precisely, the first main result of this chapter is the following theorem, which can be considered as the parabolic counterpart of [22, Theorem 4.2]. We refer to Chapter 1 for notation and definitions.

Theorem 4.1. *Let $n \ge 2$, $p \in [2,\infty)$, $\frac{np+4}{np+4-n} \le \vartheta < \infty$ and $f \in L^{\vartheta}\left(0,T; W^{1,\vartheta}(\Omega)\right)$. Moreover, assume that*

$$u \in C^0\left((0,T); L^2(\Omega)\right) \cap L^p\left(0,T; W^{1,p}(\Omega)\right)$$

is a weak solution of equation (4.1). Then

$$H_{\frac{p}{2}}(Du) \in L^2_{loc}\left(0,T; W^{1,2}_{loc}(\Omega,\mathbb{R}^n)\right).$$

Furthermore, the estimate

$$\int\limits_{Q_{\varrho/2}(z_0)} |DH_{\frac{p}{2}}(Du)|^2 \, dz \le c\left(\lambda \|Df\|_{L^{\vartheta}(Q_{R_0})} + \|Df\|^{\frac{np+4}{np+2-n}}_{L^{\vartheta}(Q_{R_0})} \right)$$

$$+\frac{c}{R^2}\left(\|Du\|^p_{L^p(Q_{R_0})}+\|Du\|^2_{L^p(Q_{R_0})}+\lambda^p+\lambda^2\right) \tag{4.2}$$

holds true for any parabolic cylinder $Q_\varrho(z_0)\subset Q_R(z_0)\subset Q_{R_0}(z_0)\Subset\Omega_T$ *and a positive constant* c *depending at most on* n, p, ϑ *and* R_0.

One of the main tools in the proof of Theorem 4.1 is the difference quotients technique used in the spatial directions. In this chapter, we will argue as in [41, Lemma 5.1] and [53, Theorem 4.1], but we need to take into account the strong degeneracy of equation (4.1). This is why we obtain the Sobolev spatial regularity not for the usual function

$$\mathbb{V}_p(Du):=|Du|^{\frac{p-2}{2}}\,Du\,,$$

but for the vector field $H_{\frac{p}{2}}(Du)$, which vanishes in the set where equation (4.1) becomes degenerate.

The proof of Theorem 4.1 is also based on a comparison argument with the solutions of a family of much less degenerate parabolic problems having a smooth inhomogeneity. To the solutions u_ε of these problems we can apply the *a priori* estimates that we establish in Section 4.3, whose constants are independent of the approximation parameter ε. Next, we show that the L^p-norms of the spatial gradients of such solutions are uniformly bounded with respect to ε, and this allows us to transfer the higher differentiability in space of $H_{\frac{p}{2}}(Du_\varepsilon)$ to the function $H_{\frac{p}{2}}(Du)$ (see Section 4.4 below).

As we anticipated earlier, from Theorem 4.1 we can easily deduce that any weak solution u of (4.1) admits a weak time derivative u_t, which belongs to the local Lebesgue space $L^{\min\{\vartheta,p'\}}_{loc}(\Omega_T)$. The idea is roughly as follows. Consider equation (4.1); since the above theorem tells us that in a certain pointwise sense the second spatial derivatives of u exist, we may develop the expression under the divergence symbol. This will give us an expression that equals u_t, from which we get the desired summability of the time derivative. Such an argument must be made more rigorous. Furthermore, we also need to make explicit *a priori* local estimates. These are provided in the following theorem.

Theorem 4.2. *Under the assumptions of Theorem 4.1, the time derivative of the solution exists in the weak sense and satisfies*

$$\partial_t u\in L^{\min\{\vartheta,p'\}}_{loc}(\Omega_T).$$

Furthermore, the estimate

$$\begin{aligned}
&\left(\int_{Q_{\varrho/2}(z_0)}|\partial_t u|^{\min\{\vartheta,p'\}}\,dz\right)^{\frac{1}{\min\{\vartheta,p'\}}}\\
&\quad\le c\,\|f\|_{L^\vartheta(Q_{R_0})}+c\,\|Du\|^{\frac{p-2}{2}}_{L^p(Q_{R_0})}\left(\lambda\,\|Df\|_{L^\vartheta(Q_{R_0})}+\|Df\|^{\frac{np+4}{np+2-n}}_{L^\vartheta(Q_{R_0})}\right)^{\frac12}\\
&\qquad+\frac{c}{R}\left(\|Du\|^{2p-2}_{L^p(Q_{R_0})}+\|Du\|^p_{L^p(Q_{R_0})}+(\lambda^p+\lambda^2)\,\|Du\|^{p-2}_{L^p(Q_{R_0})}\right)^{\frac12}
\end{aligned} \tag{4.3}$$

holds true for any parabolic cylinder $Q_\varrho(z_0)\subset Q_R(z_0)\subset Q_{R_0}(z_0)\Subset\Omega_T$ *and a positive constant* c *depending on* n, p, ϑ *and* R_0.

Remark 4.3. It is worth pointing out that, starting from the weaker assumption

$$f \in L^{\frac{np+4}{np+4-n}}\left(0,T;W^{1,\frac{np+4}{np+4-n}}(\Omega)\right), \tag{4.4}$$

Sobolev regularity results such as those of Theorems 4.1 and 4.2 seemed not to have been established yet for weak solutions to parabolic PDEs that are far less degenerate than equation (4.1). In particular, the results of Theorems 4.1 and 4.2 can be easily extended to the case $\lambda = 0$, i.e. to the evolutionary p-Poisson equation

$$u_t - \operatorname{div}\left(|Du|^{p-2}Du\right) = f \qquad \text{in } \Omega_T, \tag{4.5}$$

under the assumption (4.4). Therefore, our results permit to improve the existing literature, already for equations of the form (4.5), which exhibit a milder degeneracy.

With the aim of pursuing our investigation on the regularity properties of weak solutions to equation (4.1), in [5] we weakened the hypotheses of Theorem 4.1, by assuming that

$$f \in L^{p'}_{loc}\left(0,T;B^{\alpha}_{p',\infty,loc}(\Omega)\right),$$

with $2 \le p < \infty$ and $\alpha \in (0,1)$ (cf. Definition 1.32 and the lines following it). This assumption led us to obtain a local higher differentiability result for the vector field $H_{\frac{p}{2}}(Du)$ in the scale of *fractional* Sobolev spaces. More precisely, our result reads as follows.

Theorem 4.4. *Let $n \ge 2$, $p \in [2,\infty)$, $\alpha \in (0,1)$ and $f \in L^{p'}_{loc}\left(0,T;B^{\alpha}_{p',\infty,loc}(\Omega)\right)$. Moreover, assume that*

$$u \in C^0\left((0,T);L^2(\Omega)\right) \cap L^p\left(0,T;W^{1,p}(\Omega)\right)$$

is a weak solution of equation (4.1). *Then*

$$H_{\frac{p}{2}}(Du) \in L^2_{loc}\left(0,T;W^{s,2}_{loc}(\Omega,\mathbb{R}^n)\right) \qquad \text{for all } s \in \left(0,\frac{\alpha+1}{2}\right).$$

Furthermore, the local estimate

$$\begin{aligned}
&\sup_{t\in(t_0-(\varrho/2)^2,t_0)} \int_{B_{\varrho/2}(x_0)} |\tau_{i,h}u(x,t)|^2\,dx + \int_{Q_{\varrho/2}(z_0)} |\tau_{i,h}H_{\frac{p}{2}}(Du)|^2\,dz \\
&\le c\left[\frac{|h|^2}{\varrho^2}\int_{Q_{2\varrho}(z_0)} (|Du|^p+1)\,dz + |h|^{\alpha+1}\,\|Du\|_{L^p(Q_{2\varrho})}\,\|f\|_{L^{p'}\left(t_0-4\varrho^2,t_0\,;B^{\alpha}_{p',\infty}(B_{2\varrho})\right)}\right]
\end{aligned}$$

holds true for any $i \in \{1,\dots,n\}$, for any parabolic cylinder $Q_\varrho(z_0) \subset Q_{2\varrho}(z_0) \Subset \Omega_T$, for any $h \in \mathbb{R}$ such that $|h| < \varrho/4$ and a positive constant c depending only on n and p.

The proof of the above theorem is equally achieved using the well-known difference quotients technique in the spatial directions (see Section 4.6 below).

We wish to emphasize that Theorem 4.4 can be considered as the parabolic counterpart of Theorem 2.7, where however we obtain a higher fractional differentiability result in the scale of *Besov* spaces, starting from a datum in the local Besov space $B^{\alpha}_{p',\infty,loc}(\Omega)$ for some $\alpha \in (0,1)$. Furthermore, Theorem 4.4 can also be viewed as the fractional version of Theorem 4.1.

As a consequence of Theorems 4.4, 1.36 and 1.37, we get a gain of summability for Du with respect to the spatial variable. More precisely, we have the following result.

Corollary 4.5. *Under the assumptions of Theorem 4.4, we obtain that*

$$H_{\frac{p}{2}}(Du) \in L^2_{loc}\left(0,T;L^r_{loc}(\Omega,\mathbb{R}^n)\right) \quad \text{for all } r \in \left[1, \frac{2n}{n-\alpha-1}\right)$$

and

$$Du \in L^p_{loc}\left(0,T;L^q_{loc}(\Omega,\mathbb{R}^n)\right) \quad \text{for all } q \in \left[1, \frac{np}{n-\alpha-1}\right).$$

Remark 4.6. In the above corollary, the fact that our assumption on f is now even weaker prevents us from obtaining a (local) higher summability for Du over Ω_T (i.e. with respect to the space-time variable $z=(x,t)$), contrary to what can be easily proven under the assumption $f \in L^{p'}(0,T;W^{1,p'}(\Omega))$, by arguing exactly as in the proof of Proposition 4.13 below. For the same reason, we are led to deal with fractional Sobolev spaces, rather than the traditional Sobolev spaces of integer order (compare Theorem 4.4 with Theorem 4.1).

Remark 4.7. It is worth emphasizing that, starting from the assumption

$$f \in L^{p'}_{loc}\left(0,T;B^{\alpha}_{p',\infty,loc}(\Omega)\right),$$

higher fractional differentiability results such as that of Theorem 4.4 seemed not to have been established yet for the spatial gradient of weak solutions to the strongly degenerate equation (4.1). However, we would like to mention the result contained in [24, Theorem 2.10], where Byun et al. consider a weak solution

$$u \in C^0\left([0,T];L^2(\Omega)\right) \cap L^p\left(0,T;W^{1,p}_0(\Omega)\right)$$

to the following parabolic problem in divergence form:

$$\begin{cases} u_t - \operatorname{div}\mathbf{a}(x,t,Du) = \operatorname{div}\left(|F|^{p-2}F\right) & \text{in } \Omega_T, \\ u = 0 & \text{on } \partial_{\text{par}}\Omega_T, \end{cases}$$

where $\frac{2n}{n+2} < p < \infty$, $\mathbf{a}$ is a discontinuous nonlinearity with an asymptotic regularity, Ω is a bounded domain whose boundary $\partial\Omega$ is nonsmooth, and

$$F = F(x,t) = (f_1(x,t),\dots,f_n(x,t)) \in L^p(\Omega_T,\mathbb{R}^n)$$

is a given vector-valued function. In fact, the just mentioned result gives an affirmative answer as to what are both the weakest regularity requirement on $\mathbf{a}$ and the lowest level of geometric assumption on $\partial\Omega$ under which the implication

$$|F|^p \in L^q(\Omega_T) \implies |Du|^p \in L^q(\Omega_T)$$

holds true for every $q \in [1,\infty)$.

This chapter is organized as follows. Section 4.1 contains some considerations on the interplay between the regularity of the right-hand side of (4.1) and that of the function $H_{p/2}(Du)$. In

Section 4.2, we give a motivation for studying equations of the type (4.1). In Section 4.3, we consider a regularization of equation (4.1) and establish some *a priori* estimates for its weak solution. These estimates will be needed to demonstrate Theorem 4.1, whose proof is contained in Section 4.4. Section 4.5 is devoted to the proof of Theorem 4.2. Finally, in Section 4.6 we prove Theorem 4.4 and the related Corollary 4.5.

4.1 Comparison with a less degenerate elliptic equation

In this section, we make some considerations about the interplay between the regularity of the right-hand side of (4.1) and that of the vector field $H_{p/2}(Du)$, starting from the comparison with a known result for an elliptic equation which is less degenerate than (4.1).

In [23, Theorem 1.1], Brasco and Santambrogio established the sharp assumptions on the function g in order to obtain that the $W^{1,p}$ solutions to the Poisson equation for the p-Laplace operator

$$-\Delta_p v := -\operatorname{div}\left(|Dv|^{p-2}Dv\right) = g \quad \text{in } \Omega, \tag{4.6}$$

still satisfy the following Uhlenbeck's result (see [93, Lemma 3.1]):

$$|Dv|^{\frac{p-2}{2}}Dv \in W^{1,2}_{loc}(\Omega,\mathbb{R}^n) \tag{4.7}$$

in the superquadratic case $p > 2$, where $\Omega \subset \mathbb{R}^n$ is an open set. For the sake of completeness, we recall their result here below, by noting that g must belong at least to a suitable (local) fractional Sobolev space for condition (4.7) to be true.

Theorem 4.8 ([23, Theorem 1.1]). *Let $p > 2$ and let $U \in W^{1,p}_{loc}(\Omega)$ be a local weak solution of equation (4.6). If*

$$g \in W^{s,p'}_{loc}(\Omega) \qquad \text{with} \quad \frac{p-2}{p} < s \leq 1, \tag{4.8}$$

then

$$\mathbb{V}_p(DU) := |DU|^{\frac{p-2}{2}}DU \in W^{1,2}_{loc}(\Omega,\mathbb{R}^n)$$

and

$$DU \in W^{\sigma,p}_{loc}(\Omega,\mathbb{R}^n), \qquad \text{for} \quad 0 < \sigma < \frac{2}{p}.$$

As for the sharpness of assumption (4.8), we refer the interested reader to [23, Sections 1.2 and 5]. Under the hypotheses of Theorem 4.8, we also find that DU belongs to some fractional Sobolev space (locally in Ω).

The interplay between the regularity of the right-hand side g and that of the vector field $\mathbb{V}_p(DU)$ has been considered in detail also in [73], where the point of view is slightly different: the main concern there is to obtain (fractional) differentiability of the vector field $\mathbb{V}_p(DU)$ when g is not regular. In particular, in [73] the datum g may not belong to the relevant dual Sobolev space and the concept of solution to (4.6) has to be carefully defined. We refer to [23, Remark 1.3] and the references therein for comparison with other results.

As already mentioned, when considering the strongly degenerate parabolic equation (4.1), the vector-valued function $\mathbb{V}_p(DU)$ is replaced by the function $H_{p/2}(Du)$, which belongs to $L^2_{loc}\left(0,T;W^{s,2}_{loc}(\Omega,\mathbb{R}^n)\right)$ for all $s \in \left(0,\frac{\alpha+1}{2}\right)$ under the assumptions of Theorem 4.4. One might

then ask what are the optimal (weakest) assumptions to be imposed on the datum f so that the weak solutions u of equation (4.1) still satisfy the condition

$$H_{\frac{p}{2}}(Du) \in L^2_{loc}\left(0,T;W^{1,2}_{loc}(\Omega,\mathbb{R}^n)\right) \tag{4.9}$$

for $p \geq 2$, since we cannot achieve the result (4.9) under the hypothesis

$$f \in L^{p'}_{loc}\left(0,T;B^{\alpha}_{p',\infty,loc}(\Omega)\right) \qquad \text{with } \alpha \in (0,1).$$

However, it is worth emphasizing that equation (4.1) exhibits a more severe degeneracy than equation (4.6) and that the crucial point in [23] that

$$|DU|^{\frac{p-2}{2}}\, DU \in W^{1,2}_{loc} \quad \text{implies} \quad DU \in W^{\sigma,p}_{loc}$$

cannot be retrieved under the assumptions of Theorem 4.4.

4.2 Motivation

In this section, we wish to motivate our study by pointing out that, in the physical cases $n = 2$ and $n = 3$, degenerate equations of the form (4.1) may arise in *gas filtration problems taking into account the initial pressure gradient.*

The existence of significant deviations from the linear Darcy filtration law has been established for many systems consisting of a fluid and a porous medium (e.g., the filtration of a gas in argillous rocks). One of the manifestations of this nonlinearity is the existence of a limiting (initial) pressure gradient, i.e. the minimum value of the pressure gradient for which fluid motion occurs. In general, fluid motion still takes place for subcritical values of the pressure gradient, but very slowly; on reaching the limiting value of the pressure gradient, there is a marked acceleration of the filtration. Therefore, the limiting-gradient concept provides a good approximation for velocities which are not too low.

In accordance with some experimental results (see [3]), under certain physical conditions one can take the gas filtration law in the very simple form

$$\begin{cases} \mathbf{v} = -\frac{k}{\mu}\, D\mathrm{P}\left[1 - \frac{G}{|D\mathrm{P}^2|}\right] & \text{if } |D\mathrm{P}^2| \geq G, \\ \mathbf{v} = \mathbf{0} & \text{if } |D\mathrm{P}^2| < G, \end{cases}$$

where $\mathbf{v} = \mathbf{v}(x,t)$ is the filtration velocity, k is the rock permeability, μ is the gas viscosity, $\mathrm{P} = \mathrm{P}(x,t)$ is the pressure and G is a positive constant. Under this assumption, we obtain a particularly simple expression for the gas mass velocity (flux) $\boldsymbol{j}$, which contains only the gradient of the pressure squared, just as in the usual gas filtration problems:

$$\begin{cases} \boldsymbol{j} = \varrho\mathbf{v} = -\frac{k}{2\mu C}\left[D\mathrm{P}^2 - G\,\frac{D\mathrm{P}^2}{|D\mathrm{P}^2|}\right] & \text{if } |D\mathrm{P}^2| > G, \\ \boldsymbol{j} = \mathbf{0} & \text{if } |D\mathrm{P}^2| \leq G, \end{cases} \tag{4.10}$$

where ϱ is the gas density and C is a positive constant. Substituting expression (4.10) into the gas mass-conservation equation, we obtain the basic equation for the pressure:

$$\begin{cases} \frac{\partial \mathrm{P}}{\partial t} = \frac{k}{2m\mu} \operatorname{div}\left[D\mathrm{P}^2 - G\,\frac{D\mathrm{P}^2}{|D\mathrm{P}^2|}\right] & \text{if } |D\mathrm{P}^2| > G, \\ \frac{\partial \mathrm{P}}{\partial t} = 0 & \text{if } |D\mathrm{P}^2| \le G, \end{cases} \tag{4.11}$$

where m is a positive constant. Equation (4.11) implies, first of all, that the steady gas motion is described by the same relations as in the steady motion of an incompressible fluid if we replace the pressure of the incompressible fluid with the square of the gas pressure. Moreover, if the gas pressure differs very little from some constant pressure P_0, or if the gas pressure differs considerably from a constant value only in regions where the gas motion is nearly steady, then the equation for the gas filtration in the region of motion can be "linearized" following Leibenson, and thus obtaining (see [3] again)

$$\begin{cases} \frac{\partial \mathrm{P}^2}{\partial t} = \frac{k\,\mathrm{P}_0}{m\mu} \operatorname{div}\left[D\mathrm{P}^2 - G\,\frac{D\mathrm{P}^2}{|D\mathrm{P}^2|}\right] & \text{if } |D\mathrm{P}^2| > G, \\ \frac{\partial \mathrm{P}^2}{\partial t} = 0 & \text{if } |D\mathrm{P}^2| \le G. \end{cases} \tag{4.12}$$

Now, setting $u = \mathrm{P}^2$ and performing an appropriate scaling, equation (4.12) turns into

$$\frac{\partial u}{\partial t} - \operatorname{div}\left[(|Du| - 1)_+ \,\frac{Du}{|Du|}\right] = 0,$$

which is nothing but equation (4.1) in the case $p = 2$, $\lambda = 1$ and $f = 0$. This is why (4.1) is sometimes called the *Leibenson equation* in the literature.

4.3 The regularization

In this section, we consider a regularization of equation (4.1) and establish some *a priori* estimates for its weak solution. These estimates will play a key role in the proof of Theorem 4.1 (see Section 4.4).

Let $n \in \mathbb{N}$ with $n \ge 2$, $p \in [2, \infty)$, $\frac{np+4}{np+4-n} \le \vartheta < \infty$ and $f \in L^{\vartheta}\left(0, T; W^{1,\vartheta}(\Omega)\right)$. For $\varepsilon \in [0, 1]$ and a couple of standard, non-negative, radially symmetric mollifiers $\phi_1 \in C_0^\infty(B_1(0))$ and $\phi_2 \in C_0^\infty((-1, 1))$, we write

$$f_\varepsilon(x, t) := \int_{-1}^{1} \int_{B_1(0)} f(x - \varepsilon y, t - \varepsilon s)\, \phi_1(y)\, \phi_2(s)\, dy\, ds, \tag{4.13}$$

where f is meant to be extended by zero outside Ω_T. Observe that $f_0 = f$ and $f_\varepsilon \in C^\infty(\Omega_T)$ for every $\varepsilon \in (0, 1]$.

Now we consider a domain in space-time denoted by $\Omega'_{1,2} := \Omega' \times (T_1, T_2)$, where $\Omega' \subseteq \Omega$ is a bounded domain with smooth boundary and $(T_1, T_2) \Subset (0, T)$. For our purposes, in the following we will need the definitions below.

Definition 4.9. Let $\varepsilon \in (0, 1]$. A function

$$u_\varepsilon \in C^0\left((T_1, T_2); L^2(\Omega')\right) \cap L^p\left(T_1, T_2; W^{1,p}(\Omega')\right)$$

is a *weak solution* of the equation

$$\partial_t u_\varepsilon - \operatorname{div}\left(H_{p-1}(Du_\varepsilon) + \varepsilon\,|Du_\varepsilon|^{p-2}Du_\varepsilon\right) = f_\varepsilon \quad \text{in } \Omega'_{1,2} \tag{4.14}$$

if and only if, for any test function $\varphi \in C_0^\infty(\Omega'_{1,2})$, the following integral identity holds:

$$\int_{\Omega'_{1,2}} \left(u_\varepsilon \cdot \partial_t\varphi - \langle H_{p-1}(Du_\varepsilon) + \varepsilon\,|Du_\varepsilon|^{p-2}Du_\varepsilon, D\varphi\rangle\right)\,dz \;=\; -\int_{\Omega'_{1,2}} f_\varepsilon\,\varphi\,dz. \tag{4.15}$$

Definition 4.10. Let $\varepsilon \in (0,1]$ and

$$g \;\in\; C^0\left([T_1,T_2];L^2(\Omega')\right) \cap L^p\left(T_1,T_2;W^{1,p}(\Omega')\right).$$

In this framework, we identify a function

$$u_\varepsilon \;\in\; C^0\left([T_1,T_2];L^2(\Omega')\right) \cap L^p\left(T_1,T_2;W^{1,p}(\Omega')\right)$$

as a *weak solution of the Cauchy-Dirichlet problem*

$$\begin{cases} \partial_t u_\varepsilon - \operatorname{div}\left(H_{p-1}(Du_\varepsilon) + \varepsilon\,|Du_\varepsilon|^{p-2}Du_\varepsilon\right) = f_\varepsilon & \text{in } \Omega'_{1,2}, \\ u_\varepsilon = g & \text{on } \partial_{\mathrm{par}}\Omega'_{1,2}, \end{cases} \tag{4.16}$$

if and only if (4.15) holds and, moreover,

$$u_\varepsilon \;\in\; g + L^p\left(T_1,T_2;W_0^{1,p}(\Omega')\right)$$

and $u_\varepsilon(\cdot,T_1) = g(\cdot,T_1)$ in the L^2-sense, that is,

$$\lim_{t\searrow T_1} \|u_\varepsilon(\cdot,t) - g(\cdot,T_1)\|_{L^2(\Omega')} \;=\; 0. \tag{4.17}$$

Therefore, the initial condition $u_\varepsilon = g$ on $\Omega' \times \{T_1\}$ has to be understood in the usual L^2-sense (4.17), while the condition $u_\varepsilon = g$ on the lateral boundary $\partial\Omega' \times (T_1,T_2)$ has to be meant in the sense of traces, i.e.

$$(u_\varepsilon - g)\,(\cdot,t) \in W_0^{1,p}(\Omega')$$

for almost every $t \in (T_1,T_2)$.

Remark 4.11. The regularized parabolic equation (4.14) fulfills standard p-growth conditions. The advantage of considering the associated Cauchy-Dirichlet problem (4.16) arises from the fact that the existence of a unique solution

$$u_\varepsilon \;\in\; C^0\left([T_1,T_2];L^2(\Omega')\right) \cap L^p\left(T_1,T_2;W^{1,p}(\Omega')\right)$$

satisfying the requirements of Definition 4.10 can be ensured by the classic existence theory for parabolic equations (see [69, Chapter 2, Theorem 1.2 and Remark 1.2]).

Now we shall use the well-known difference quotients method in the spatial directions together with the properties of the functions H_{p-1} and $\mathbb{V}_p$ to establish the following result.

Proposition 4.12. *Let $\varepsilon \in (0,1]$, $n \geq 2$, $p \geq 2$, $\frac{np+4}{np+4-n} \leq \vartheta < \infty$ and $f \in L^\vartheta\left(0,T;W^{1,\vartheta}(\Omega)\right)$.*

Moreover, assume that

$$u_\varepsilon \in C^0\left((T_1,T_2);L^2(\Omega')\right)\cap L^p\left(T_1,T_2;W^{1,p}(\Omega')\right)$$

is a weak solution of equation (4.14)*. Then the solution satisfies*

$$H_{\frac{p}{2}}(Du_\varepsilon)\in L^2_{loc}\left(T_1,T_2;W^{1,2}_{loc}(\Omega',\mathbb{R}^n)\right)\qquad\text{and}\qquad Du_\varepsilon\in L^\infty_{loc}\left(T_1,T_2;L^2_{loc}(\Omega',\mathbb{R}^n)\right). \tag{4.18}$$

Furthermore, the estimate

$$\begin{aligned}\sup_{t\in(t_0-(\rho/2)^2,t_0)}\|Du_\varepsilon(\cdot,t)\|^2_{L^2(B_{\rho/2}(x_0))}&+\int\limits_{Q_{\rho/2}(z_0)}|DH_{\frac{p}{2}}(Du_\varepsilon)|^2\,dz\\ &\le c\left[\|Df_\varepsilon\|_{L^{p'}(Q_{R_0})}\|Du_\varepsilon\|_{L^p(Q_{R_0})}+\frac{1}{\rho^2}\int\limits_{Q_{R_0}}(|Du_\varepsilon|^p+1)\,dz\right]\end{aligned} \tag{4.19}$$

holds true for any parabolic cylinder $Q_\rho(z_0)\subset Q_{R_0}(z_0)\Subset\Omega'_{1,2}$ and a positive constant c depending only on n and p.

Proof. By a slight abuse of notation, for $w\in L^1_{loc}(\Omega'_{1,2},\mathbb{R}^N)$, $s\in\{1,\dots,n\}$ and $h\neq 0$, we set (when $x+he_s\in\Omega'$)

$$\tau_h w(x,t)\equiv\tau_{s,h}w(x,t):=w(x+he_s,t)-w(x,t)\,,$$

$$\Delta_h w(x,t)\equiv\Delta_{s,h}w(x,t):=\frac{w(x+he_s,t)-w(x,t)}{h}\,,$$

where e_s is the unit vector in the direction x_s.

Since u_ε is a weak solution of equation (4.14), we have

$$\int_{\Omega'_{1,2}}\left(u_\varepsilon\cdot\partial_t\varphi-\langle H_{p-1}(Du_\varepsilon)+\varepsilon\,|Du_\varepsilon|^{p-2}Du_\varepsilon,D\varphi\rangle\right)dz=-\int_{\Omega'_{1,2}}f_\varepsilon\,\varphi\,dz$$

for every $\varphi\in C^\infty_0(\Omega'_{1,2})$. Replacing φ with $\tau_{-h}\varphi$, where $0<|h|<\operatorname{dist}(\operatorname{spt}\varphi,\partial\Omega'_{1,2})$, by virtue of the properties of the finite difference operator, we get

$$\int_{\Omega'_{1,2}}\left(\tau_h u_\varepsilon\cdot\partial_t\varphi-\langle\tau_h H_{p-1}(Du_\varepsilon)+\varepsilon\,\tau_h\left[|Du_\varepsilon|^{p-2}Du_\varepsilon\right],D\varphi\rangle\right)dz=-\int_{\Omega'_{1,2}}\tau_h f_\varepsilon\cdot\varphi\,dz.$$

We now replace φ by $\varphi_\sigma\equiv\phi_\sigma*\varphi$ in the previous equation, where $\{\phi_\sigma\}$, $\sigma>0$, denotes the family of standard, non-negative, radially symmetric mollifiers in $\mathbb{R}^{n+1}$. This yields, for $0<\sigma\ll 1$,

$$\int\limits_{\Omega'_{1,2}}\left((\tau_h u_\varepsilon)_\sigma\cdot\partial_t\varphi-\langle(\tau_h H_{p-1}(Du_\varepsilon))_\sigma+\varepsilon\left(\tau_h\left[|Du_\varepsilon|^{p-2}Du_\varepsilon\right]\right)_\sigma,D\varphi\rangle\right)dz=-\int\limits_{\Omega'_{1,2}}(\tau_h f_\varepsilon)_\sigma\cdot\varphi\,dz\,.$$

Now, in the last equation we choose the test function $\varphi\equiv\Phi(\tau_h u_\varepsilon)_\sigma$, where $\Phi\in C^\infty_0(\Omega'_{1,2})$ is a cut-off function which will be specified later. After an integration by parts and then letting

$\sigma \to 0$, we obtain

$$
\begin{aligned}
&-\frac{1}{2}\int_{\Omega'_{1,2}} |\tau_h u_\varepsilon|^2\, \partial_t\Phi \; dz + \int_{\Omega'_{1,2}} \Phi\, \langle \tau_h H_{p-1}(Du_\varepsilon) + \varepsilon\, \tau_h \left[|Du_\varepsilon|^{p-2}Du_\varepsilon\right], D\tau_h u_\varepsilon\rangle \, dz \\
&\quad = -\int_{\Omega'_{1,2}} \langle \tau_h H_{p-1}(Du_\varepsilon), D\Phi\rangle \tau_h u_\varepsilon \; dz - \varepsilon \int_{\Omega'_{1,2}} \langle \tau_h \left[|Du_\varepsilon|^{p-2}Du_\varepsilon\right], D\Phi\rangle \tau_h u_\varepsilon \; dz \\
&\qquad + \int_{\Omega'_{1,2}} \tau_h f_\varepsilon \cdot \Phi \cdot \tau_h u_\varepsilon \; dz
\end{aligned} \tag{4.20}
$$

for any $\Phi \in C_0^\infty(\Omega'_{1,2})$, provided that $|h|$ is small enough. Note that an approximation argument yields the same identity for any $\Phi \in W^{1,\infty}(\Omega'_{1,2})$ with compact support in $\Omega'_{1,2}$ and any sufficiently small $h \in \mathbb{R} \setminus \{0\}$. In what follows, we will denote by c_k and c some positive constants which neither depend on h nor on ε.

Now, let us consider a parabolic cylinder

$$
Q_\rho(z_0) \subset Q_{R_0}(z_0) \Subset \Omega'_{1,2}\,.
$$

For a fixed time $t_1 \in (t_0 - \rho^2, t_0)$ and $\delta \in (0, t_0 - t_1)$, we choose

$$
\Phi(x,t) \;=\; \widetilde{\chi}(t)\, \chi(t)\, \eta^2(x)
$$

with $\chi \in W^{1,\infty}((T_1,T_2),[0,1])$, $\chi \equiv 0$ on $(T_1, t_0-\rho^2)$ and $\partial_t\chi \geq 0$, $\eta \in C_0^\infty(B_\rho(x_0),[0,1])$, and with the Lipschitz continuous function $\widetilde{\chi} : (T_1,T_2) \to \mathbb{R}$ defined by

$$
\widetilde{\chi}(t) = \begin{cases} 1 & \text{if } t \leq t_1, \\ \text{affine} & \text{if } t_1 < t < t_1 + \delta, \\ 0 & \text{if } t \geq t_1 + \delta. \end{cases}
$$

With such a choice of Φ, equation (4.20) becomes

$$
\begin{aligned}
&-\frac{1}{2}\int_{\Omega'_{1,2}} |\tau_h u_\varepsilon|^2\, \eta^2(x)\chi(t)\partial_t\widetilde{\chi}(t)\, dz \;-\; \frac{1}{2}\int_{\Omega'_{1,2}} |\tau_h u_\varepsilon|^2\, \eta^2(x)\widetilde{\chi}(t)\partial_t\chi(t)\, dz \\
&\quad + \int_{\Omega'_{1,2}} \widetilde{\chi}(t)\chi(t)\eta^2(x)\langle \tau_h H_{p-1}(Du_\varepsilon), D\tau_h u_\varepsilon\rangle \, dz \\
&\quad + \varepsilon \int_{\Omega'_{1,2}} \widetilde{\chi}(t)\chi(t)\eta^2(x)\langle \tau_h[|Du_\varepsilon|^{p-2}Du_\varepsilon], D\tau_h u_\varepsilon\rangle \, dz \\
&\qquad = -2\int_{\Omega'_{1,2}} \widetilde{\chi}(t)\chi(t)\eta(x)\langle \tau_h H_{p-1}(Du_\varepsilon), D\eta\rangle \tau_h u_\varepsilon \, dz \\
&\qquad\quad - 2\,\varepsilon \int_{\Omega'_{1,2}} \widetilde{\chi}(t)\chi(t)\eta(x)\langle \tau_h[|Du_\varepsilon|^{p-2}Du_\varepsilon], D\eta\rangle \tau_h u_\varepsilon \, dz \\
&\qquad\quad + \int_{\Omega'_{1,2}} (\tau_h f_\varepsilon)(\tau_h u_\varepsilon)\widetilde{\chi}(t)\chi(t)\eta^2(x)\, dz.
\end{aligned}
$$

Setting

$$
Q^{t_1} \;:=\; B_\rho(x_0) \times (t_0 - \rho^2, t_1)
$$

and letting $\delta \to 0$ in the previous equality, for every $t_1 \in (t_0 - \rho^2, t_0)$ we get

$$\begin{aligned}
&\frac{1}{2}\int_{B_\rho(x_0)} \chi(t_1)\eta^2(x)\,|\tau_h u_\varepsilon(x,t_1)|^2\,dx \,+\, \int_{Q^{t_1}} \chi(t)\eta^2(x)\langle \tau_h H_{p-1}(Du_\varepsilon), D\tau_h u_\varepsilon\rangle\,dz \\
&\quad+\,\varepsilon\int_{Q^{t_1}} \chi(t)\eta^2(x)\langle \tau_h[|Du_\varepsilon|^{p-2}Du_\varepsilon], D\tau_h u_\varepsilon\rangle\,dz \\
&\qquad=\,-2\int_{Q^{t_1}} \chi(t)\eta(x)\langle \tau_h H_{p-1}(Du_\varepsilon), D\eta\rangle \tau_h u_\varepsilon\,dz \,+\, \int_{Q^{t_1}} (\tau_h f_\varepsilon)(\tau_h u_\varepsilon)\chi(t)\eta^2(x)\,dz \\
&\qquad\quad-\,2\,\varepsilon\int_{Q^{t_1}} \chi(t)\eta(x)\langle \tau_h[|Du_\varepsilon|^{p-2}Du_\varepsilon], D\eta\rangle \tau_h u_\varepsilon\,dz \,+\, \frac{1}{2}\int_{Q^{t_1}} (\partial_t\chi)\eta^2(x)\,|\tau_h u_\varepsilon|^2\,dz \\
&\qquad=:\, I_1 + I_2 + I_3 + I_4\,,
\end{aligned} \tag{4.21}$$

where we have used that $\partial_t\widetilde{\chi}$ converges to a Dirac delta distribution as $\delta \to 0$, together with the $L^2(\Omega')$-valued continuity of u_ε. In the following, we estimate I_1, I_2 and I_3 separately. Let us first consider I_1. Using Lemma 1.6 together with Young's inequality, we obtain

$$\begin{aligned}
|I_1| \,\le\, &\frac{2(p-1)^2}{\beta}\int_{Q^{t_1}} \chi(t)\,|D\eta|^2\left(|H_{\frac{p}{2}}(Du_\varepsilon(x+he_s,t))|^{\frac{p-2}{p}} \,+\, |H_{\frac{p}{2}}(Du_\varepsilon)|^{\frac{p-2}{p}}\right)^2 |\tau_h u_\varepsilon|^2\,dz \\
&+\,\frac{\beta}{2}\int_{Q^{t_1}} \chi(t)\eta^2(x)\,|\tau_h H_{\frac{p}{2}}(Du_\varepsilon)|^2\,dz,
\end{aligned} \tag{4.22}$$

where $\beta > 0$ will be chosen later. As for I_3, by Lemma 1.4, we similarly have

$$\begin{aligned}
|I_3| \,\le\, &\frac{2\varepsilon\,(p-1)^2}{\beta}\int_{Q^{t_1}} \chi(t)\,|D\eta|^2\left(|Du_\varepsilon(x+he_s,t)|^{\frac{p-2}{2}} + |Du_\varepsilon|^{\frac{p-2}{2}}\right)^2 |\tau_h u_\varepsilon|^2\,dz \\
&+\,\frac{\varepsilon\beta}{2}\int_{Q^{t_1}} \chi(t)\eta^2(x)\,|\tau_h \mathbb{V}_p(Du_\varepsilon)|^2\,dz.
\end{aligned} \tag{4.23}$$

We now turn our attention to I_2. Taking advantage of the properties of f_ε, u_ε, χ and η, and using Hölder's inequality with exponents p and p' together with Lemma 1.14, we obtain

$$|I_2| \,\le\, c_1(n)\,|h|^2\,\|Df_\varepsilon\|_{L^{p'}(Q_{R_0})}\,\|Du_\varepsilon\|_{L^p(Q_{R_0})}\,, \tag{4.24}$$

provided that $|h|$ is suitably small. Now, by virtue of Lemma 1.6, we also have

$$\frac{4}{p^2}\int_{Q^{t_1}} \chi(t)\eta^2(x)\,|\tau_h H_{\frac{p}{2}}(Du_\varepsilon)|^2\,dz \,\le\, \int_{Q^{t_1}} \chi(t)\eta^2(x)\langle \tau_h H_{p-1}(Du_\varepsilon), D\tau_h u_\varepsilon\rangle\,dz. \tag{4.25}$$

Similarly, by Lemma 1.4, we obtain

$$\frac{4\,\varepsilon}{p^2}\int_{Q^{t_1}} \chi(t)\eta^2(x)\,|\tau_h \mathbb{V}_p(Du_\varepsilon)|^2\,dz \,\le\, \varepsilon\int_{Q^{t_1}} \chi(t)\eta^2(x)\langle \tau_h[|Du_\varepsilon|^{p-2}Du_\varepsilon], D\tau_h u_\varepsilon\rangle\,dz. \tag{4.26}$$

Joining estimates (4.21)−(4.26) and choosing $\beta = \frac{4}{p^2}$, we arrive at

$$\begin{aligned}
&\int_{B_\rho(x_0)} \chi(t_1)\eta^2(x)\,|\tau_h u_\varepsilon(x,t_1)|^2\,dx \,+\, \int_{Q^{t_1}} \chi(t)\eta^2(x)\,|\tau_h H_{\frac{p}{2}}(Du_\varepsilon)|^2\,dz \\
&+\,\varepsilon\int_{Q^{t_1}} \chi(t)\eta^2(x)\,|\tau_h \mathbb{V}_p(Du_\varepsilon)|^2\,dz
\end{aligned}$$

$$
\begin{aligned}
&\leq c_2(n,p)\int_{Q^{t_1}}\left[\chi\,|D\eta|^2\left(|H_{\frac{p}{2}}(Du_\varepsilon(x+he_s,t))|^{\frac{p-2}{p}}+|H_{\frac{p}{2}}(Du_\varepsilon)|^{\frac{p-2}{p}}\right)^2+(\partial_t\chi)\eta^2\right]|\tau_h u_\varepsilon|^2\,dz\\
&\quad+c_2(n,p)\int_{Q^{t_1}}\left[\varepsilon\,\chi\,|D\eta|^2\left(|Du_\varepsilon(x+he_s,t)|^{\frac{p-2}{2}}+|Du_\varepsilon|^{\frac{p-2}{2}}\right)^2+(\partial_t\chi)\eta^2\right]|\tau_h u_\varepsilon|^2\,dz\\
&\quad+c_2(n,p)\,|h|^2\,\|Df_\varepsilon\|_{L^{p'}(Q_{R_0})}\,\|Du_\varepsilon\|_{L^p(Q_{R_0})}\,,
\end{aligned}
$$

which holds for every $t_1\in(t_0-\rho^2,t_0)$ and every sufficiently small $h\in\mathbb{R}\setminus\{0\}$. Recalling that $0<\varepsilon\leq 1$ and using the definition of the function $H_{\frac{p}{2}}$, from the above estimate we get

$$
\begin{aligned}
&\int_{B_\rho(x_0)}\chi(t_1)\eta^2(x)\,|\tau_h u_\varepsilon(x,t_1)|^2\,dx+\int_{Q^{t_1}}\chi(t)\eta^2(x)\,|\tau_h H_{\frac{p}{2}}(Du_\varepsilon)|^2\,dz\\
&\quad\leq c_3(n,p)\int_{Q^{t_1}}\left[\chi\,|D\eta|^2\left(|Du_\varepsilon(x+he_s,t)|^{\frac{p-2}{2}}+|Du_\varepsilon|^{\frac{p-2}{2}}\right)^2+(\partial_t\chi)\eta^2\right]|\tau_h u_\varepsilon|^2\,dz\\
&\qquad+c_3(n,p)\,|h|^2\,\|Df_\varepsilon\|_{L^{p'}(Q_{R_0})}\,\|Du_\varepsilon\|_{L^p(Q_{R_0})}\,,
\end{aligned}
$$

which holds for every $t_1\in(t_0-\rho^2,t_0)$ and every suitably small $h\neq 0$.

We now choose a cut-off function $\eta\in C_0^\infty(B_\rho(x_0))$ with $\eta\equiv 1$ on $B_{\rho/2}(x_0)$ such that $0\leq\eta\leq 1$ and $|D\eta|\leq\frac{C}{\rho}$. For the cut-off function in time, we choose the piecewise affine function $\chi:(T_1,T_2)\to[0,1]$ with

$$
\begin{aligned}
\chi&\equiv 0 \qquad \text{on } (T_1,t_0-\rho^2)\,,\\
\chi&\equiv 1 \qquad \text{on } \left(t_0-\left(\tfrac{\rho}{2}\right)^2,T_2\right),\\
\partial_t\chi&\equiv\tfrac{4}{3\rho^2} \qquad \text{on } \left(t_0-\rho^2,t_0-\left(\tfrac{\rho}{2}\right)^2\right).
\end{aligned}
$$

Dividing both sides of the previous estimate by $|h|^2$ and using the properties of χ and η, we obtain

$$
\begin{aligned}
&\sup_{t\in(t_0-(\rho/2)^2,t_0)}\int\limits_{B_{\rho/2}(x_0)}|\Delta_h u_\varepsilon(x,t)|^2\,dx+\int\limits_{Q_{\rho/2}(z_0)}|\Delta_h H_{\frac{p}{2}}(Du_\varepsilon)|^2\,dz\\
&\quad\leq c_4(n,p)\,\rho^{-2}\int\limits_{Q_\rho(z_0)}\left[\left(|Du_\varepsilon(x+he_s,t)|^{\frac{p-2}{2}}+|Du_\varepsilon|^{\frac{p-2}{2}}\right)^2+1\right]|\Delta_h u_\varepsilon|^2\,dz\\
&\qquad+c_4(n,p)\,\|Df_\varepsilon\|_{L^{p'}(Q_{R_0})}\,\|Du_\varepsilon\|_{L^p(Q_{R_0})}\,.
\end{aligned}
\tag{4.27}
$$

Now we set

$$
I_5:=\int\limits_{Q_\rho(z_0)}\left[\left(|Du_\varepsilon(x+he_s,t)|^{\frac{p-2}{2}}+|Du_\varepsilon|^{\frac{p-2}{2}}\right)^2+1\right]|\Delta_h u_\varepsilon|^2\,dz.
$$

If $p>2$, using Hölder's inequality with exponents $\left(\frac{p}{2},\frac{p}{p-2}\right)$ and the properties of the difference quotients, we can control I_5 as follows

$$
I_5\;\leq\;c_5(n)\left(\int_{Q_{R_0}}|Du_\varepsilon|^p\,dz\right)^{\frac{2}{p}}\left(\int_{Q_{R_0}}\left[|Du_\varepsilon|^{p-2}+1\right]^{\frac{p}{p-2}}dz\right)^{\frac{p-2}{p}}
$$

$$\leq c_6(n,p)\left(\int_{Q_{R_0}}|Du_\varepsilon|^p\,dz\right)^{\frac{2}{p}}\left(\int_{Q_{R_0}}\left[|Du_\varepsilon|^p+1\right]\,dz\right)^{\frac{p-2}{p}}$$
$$\leq c_6(n,p)\int_{Q_{R_0}}(|Du_\varepsilon|^p+1)\,dz, \tag{4.28}$$

provided that $|h|$ is sufficiently small. Joining estimates (4.27) and (4.28), for $p>2$ we then have

$$\sup_{t\in(t_0-(\rho/2)^2,t_0)}\int_{B_{\rho/2}(x_0)}|\Delta_{s,h}u_\varepsilon(x,t)|^2\,dx+\int_{Q_{\rho/2}(z_0)}|\Delta_{s,h}H_{\frac{p}{2}}(Du_\varepsilon)|^2\,dz$$
$$\leq c\left[\|Df_\varepsilon\|_{L^{p'}(Q_{R_0})}\|Du_\varepsilon\|_{L^p(Q_{R_0})}+\frac{1}{\rho^2}\int_{Q_{R_0}}(|Du_\varepsilon|^p+1)\,dz\right],$$

with $c=c(n,p)>0$. Since the above estimate holds for every $s\in\{1,\dots,n\}$ and every sufficiently small $h\in\mathbb{R}\setminus\{0\}$, for $p>2$, by Lemma 1.15, we may conclude that

$$H_{\frac{p}{2}}(Du_\varepsilon)\in L^2_{loc}\left(T_1,T_2;W^{1,2}_{loc}(\Omega',\mathbb{R}^n)\right).$$

Finally, if $p=2$, arguing in a similar fashion, we reach the same conclusions. Moreover, letting $h\to 0$ in the above estimate, we obtain the Caccioppoli-type inequality (4.19), which in turn implies the validity of (4.18). □

As a consequence of Proposition 4.12, we obtain the following higher integrability result for the spatial gradient Du_ε, whose proof is based on the Sobolev embedding theorem in space applied slicewise.

Proposition 4.13. *Under the assumptions of Proposition 4.12, we obtain that*

$$Du_\varepsilon\in L^{p+\frac{4}{n}}_{loc}\left(\Omega'_{1,2},\mathbb{R}^n\right).$$

Moreover, there exists a positive constant $C_1=C_1(n,p)$ such that, for any cylinder

$$Q_\gamma(z_0)\subset Q_\rho(z_0)\Subset\Omega'_{1,2},$$

there holds

$$\int_{Q_\gamma(z_0)}(|Du_\varepsilon|-\lambda)_+^{p+\frac{4}{n}}\,dz$$
$$\leq C_1\left[\sup_{t\in(t_0-\rho^2,t_0)}\int_{B_\rho(x_0)}|Du_\varepsilon(x,t)|^2\,dx\right]^{\frac{2}{n}}\int_{Q_\rho(z_0)}\left(|DH_{\frac{p}{2}}(Du_\varepsilon)|^2+\frac{1}{(\rho-\gamma)^2}|Du_\varepsilon|^p\right)dz.$$

In addition, the estimate

$$\int_{Q_{\rho/2}(z_0)}(|Du_\varepsilon|-\lambda)_+^{p+\frac{4}{n}}\,dz$$

$$\leq C_2 \left[\|Df_\varepsilon\|_{L^{p'}(Q_{R_0})} \|Du_\varepsilon\|_{L^p(Q_{R_0})} + \frac{1}{\rho^2} \fint_{Q_{R_0}} (|Du_\varepsilon|^p + 1)\, dz \right]^{1+\frac{2}{n}} \tag{4.29}$$

holds true for any parabolic cylinder $Q_\rho(z_0) \subset Q_{2\rho}(z_0) \subset Q_{R_0}(z_0) \Subset \Omega'_{1,2}$ *and a positive constant* $C_2 = C_2(n,p)$.

Proof. Let us introduce the following function:

$$G_\varepsilon := \frac{2n}{np+4} |H_{\frac{p}{2}}(Du_\varepsilon)|^{\frac{4}{np}+1}.$$

In what follows, we will denote by c_k some positive constants which do not depend on ε. From the definition of G_ε, we obtain

$$|DG_\varepsilon| = \frac{2}{p} |H_{\frac{p}{2}}(Du_\varepsilon)|^{\frac{4}{np}} |D|H_{\frac{p}{2}}(Du_\varepsilon)|| \leq c_1 |H_{\frac{p}{2}}(Du_\varepsilon)|^{\frac{4}{np}} |DH_{\frac{p}{2}}(Du_\varepsilon)|, \tag{4.30}$$

where $c_1 = c_1(n,p) > 0$. Now, let us fix a ball $B_\rho(x_0) \Subset \Omega'$ and a time $t_0 \in (T_1, T_2)$, and let

$$\varphi \in C_0^\infty(B_\rho(x_0)) \quad \text{and} \quad \chi \in W^{1,\infty}((T_1,T_2))$$

be two non-negative cut-off functions with $\chi(T_1) = 0$ and $\partial_t \chi \geq 0$. Applying the Sobolev embedding theorem on the time slices $\Sigma_t := B_\rho(x_0) \times \{t\}$, for almost every $t \in (T_1, t_0)$ we have

$$\begin{aligned}
\int_{\Sigma_t} \varphi^2 G_\varepsilon^2\, dx &\leq c_2(n) \left(\int_{\Sigma_t} |D(\varphi\, G_\varepsilon)|^{\frac{2n}{n+2}}\, dx \right)^{\frac{n+2}{n}} \\
&= c_2(n) \left(\int_{\Sigma_t} |\varphi\, DG_\varepsilon + G_\varepsilon\, D\varphi|^{\frac{2n}{n+2}}\, dx \right)^{\frac{n+2}{n}} \\
&\leq c_3(n) \left(\int_{\Sigma_t} |\varphi\, DG_\varepsilon|^{\frac{2n}{n+2}}\, dx \right)^{\frac{n+2}{n}} + c_3(n) \left(\int_{\Sigma_t} |G_\varepsilon\, D\varphi|^{\frac{2n}{n+2}}\, dx \right)^{\frac{n+2}{n}} \\
&=: c_3\, J_1(t) + c_3\, J_2(t),
\end{aligned}$$

where, in the second to last line, we have applied Minkowski's and Young's inequalities one after the other. In the following, we estimate $J_1(t)$ and $J_2(t)$ separately. Let us first consider $J_1(t)$. Using (4.30) and Hölder's inequality, we deduce

$$\begin{aligned}
J_1(t) &\leq c_4 \left(\int_{\Sigma_t} \varphi^{\frac{2n}{n+2}} \left[(|Du_\varepsilon| - \lambda)_+^{\frac{2}{n}} |DH_{\frac{p}{2}}(Du_\varepsilon)| \right]^{\frac{2n}{n+2}} dx \right)^{\frac{n+2}{n}} \\
&\leq c_4 \int_{\Sigma_t} \varphi^2 |DH_{\frac{p}{2}}(Du_\varepsilon)|^2\, dx \left(\int_{\mathrm{spt}(\varphi)} (|Du_\varepsilon| - \lambda)_+^2\, dx \right)^{\frac{2}{n}} \\
&\leq c_4 \int_{\Sigma_t} \varphi^2 |DH_{\frac{p}{2}}(Du_\varepsilon)|^2\, dx \left(\int_{\mathrm{spt}(\varphi)} |Du_\varepsilon|^2\, dx \right)^{\frac{2}{n}},
\end{aligned}$$

where $c_4 = c_4(n,p) > 0$. We now turn our attention to $J_2(t)$. Using the definition of G_ε and

Hölder's inequality, we can conclude

$$\begin{aligned} J_2(t) \;&=\; \frac{4n^2}{(np+4)^2}\left(\int_{\Sigma_t}\left(|Du_\varepsilon|-\lambda\right)_+^{\frac{np+4}{n+2}}|D\varphi|^{\frac{2n}{n+2}}\,dx\right)^{\frac{n+2}{n}}\\ &\le\; \frac{4n^2}{(np+4)^2}\left(\int_{\Sigma_t}\left[|D\varphi|^2\,|Du_\varepsilon|^p\right]^{\frac{n}{n+2}}|Du_\varepsilon|^{\frac{4}{n+2}}\,dx\right)^{\frac{n+2}{n}}\\ &\le\; \frac{4n^2}{(np+4)^2}\int_{\Sigma_t}|D\varphi|^2\,|Du_\varepsilon|^p\,dx\,\left(\int_{\mathrm{spt}(\varphi)}|Du_\varepsilon|^2\,dx\right)^{\frac{2}{n}}. \end{aligned}$$

Setting

$$Q_{T_1,t_0}:=\,B_\rho(x_0)\times(T_1,t_0)\,,$$

putting together the last three estimates and integrating with respect to time, we obtain

$$\begin{aligned} &\int_{Q_{T_1,t_0}}\chi\varphi^2\left(|Du_\varepsilon|-\lambda\right)_+^{p+\frac{4}{n}}\,dz\\ &\quad\le\; c_5\int_{T_1}^{t_0}\chi\left[\int_{\mathrm{spt}(\varphi)}|Du_\varepsilon(x,t)|^2\,dx\right]^{\frac{2}{n}}\left[\int_{\Sigma_t}\left(\varphi^2\,|DH_{\frac{p}{2}}(Du_\varepsilon)|^2\,+\,|D\varphi|^2\,|Du_\varepsilon|^p\right)dx\right]dt\\ &\quad\le\; c_5\left[\sup_{T_1<t<t_0,\,\chi(t)\neq0}\int_{\mathrm{spt}(\varphi)}|Du_\varepsilon(x,t)|^2\,dx\right]^{\frac{2}{n}}\int_{Q_{T_1,t_0}}\chi\left(\varphi^2\,|DH_{\frac{p}{2}}(Du_\varepsilon)|^2\,+\,|D\varphi|^2\,|Du_\varepsilon|^p\right)dz, \end{aligned}\tag{4.31}$$

where $c_5=c_5(n,p)>0$. Now we perform a particular choice of the cut-off functions χ and φ involved above. For a parabolic cylinder $Q_\rho(z_0)\Subset\Omega'_{1,2}$, we choose $\chi\in W^{1,\infty}((T_1,T_2))$ such that

$$\chi\equiv0\quad\text{on }(T_1,t_0-\rho^2]\,,\qquad\chi\equiv1\quad\text{on }[t_0-\gamma^2,T_2)\,,\qquad\partial_t\chi\ge0\,,$$

with $0<\gamma<\rho$. As for $\varphi\in C_0^\infty(B_\rho(x_0))$, we assume that $\varphi\equiv1$ on $B_\gamma(x_0)$, $0\le\varphi\le1$ and $|D\varphi|\le\frac{C}{\rho-\gamma}$. With these choices, estimate (4.31) turns into

$$\begin{aligned} &\fint_{Q_\gamma(z_0)}\left(|Du_\varepsilon|-\lambda\right)_+^{p+\frac{4}{n}}\,dz\\ &\quad\le\; c_6(n,p)\left[\sup_{t\,\in\,(t_0-\rho^2,t_0)}\fint_{B_\rho(x_0)}|Du_\varepsilon(x,t)|^2\,dx\right]^{\frac{2}{n}}\fint_{Q_\rho(z_0)}\left(|DH_{\frac{p}{2}}(Du_\varepsilon)|^2\,+\,\frac{1}{(\rho-\gamma)^2}\,|Du_\varepsilon|^p\right)dz. \end{aligned}\tag{4.32}$$

We now choose $\gamma=\frac{\rho}{2}$ and use (4.19) with $Q_\rho(z_0)$ replaced by $Q_{2\rho}(z_0)$, in order to estimate the first and second integrals on the right-hand side of (4.32). After changing notation about the cylinders involved, we finally obtain inequality (4.29), which ensures that

$$Du_\varepsilon\,\in\,L^{p+\frac{4}{n}}_{loc}\left(\Omega'_{1,2},\mathbb{R}^n\right).$$

□

4.4 Proof of Theorem 4.1

We now prove Theorem 4.1, by dividing the proof into three steps. The first step consists in constructing a family of Cauchy-Dirichlet problems, for which we are allowed to use the *a priori* estimates from Propositions 4.12 and 4.13. At this stage, Lemma 1.10 will play a key role in deriving an *a priori* estimate for the solutions to these problems (i.e. the comparison maps): we specifically refer to estimate (4.44) below.

In the second step, we will show that the L^p-norms of the spatial gradients of the comparison maps are actually uniformly bounded (see estimate (4.54)), and this is where Lemma 1.11 will come into play.

Finally, in the third step, we shall use a standard comparison argument, as well as the results obtained in the previous steps, to reach the desired conclusion.

Proof of Theorem 4.1. We conduct this proof in three steps.

Step 1: A priori estimate for the comparison maps. We shall keep the notation introduced for the proof of Proposition 4.12, starting from the case $p > 2$. For $\varepsilon \in (0,1]$ and a couple of standard, non-negative, radially symmetric mollifiers $\phi_1 \in C_0^\infty(B_1(0))$ and $\phi_2 \in C_0^\infty((-1,1))$, we define the function f_ε as in (4.13), where f is meant to be extended by zero outside Ω_T. Therefore, we have that $f_\varepsilon \in C^\infty(\Omega_T)$.

Now, for any fixed $\varepsilon \in (0,1]$, let us define the comparison map

$$u_\varepsilon \in C^0\left([t_0 - R_0^2, t_0]; L^2(B_{R_0}(x_0))\right) \cap L^p\left(t_0 - R_0^2, t_0; W^{1,p}(B_{R_0}(x_0))\right)$$

as the unique energy solution of the Cauchy-Dirichlet problem

$$\begin{cases} \partial_t u_\varepsilon - \operatorname{div}\left(H_{p-1}(Du_\varepsilon) + \varepsilon\,|Du_\varepsilon|^{p-2}Du_\varepsilon\right) = f_\varepsilon & \text{in } Q_{R_0}(z_0), \\ u_\varepsilon = u & \text{on } \partial_{\mathrm{par}} Q_{R_0}(z_0), \end{cases} \tag{4.33}$$

where $Q_{R_0}(z_0) \Subset \Omega_T$ and the initial-boundary condition is meant in the sense of Definition 4.10 (see [69, Chapter 2] or [38, Chapter 9] for the existence). Moreover, let us fix a positive number $R < R_0$ and arbitrary radii

$$\frac{R}{2} \le r < \ell < \rho < \gamma < \alpha r < R,$$

with $1 < \alpha < 2$. In what follows, we will denote by c_k some positive constants which neither depend on h nor on ε.

For a fixed time $t_1 \in (t_0 - \rho^2, t_0)$ and $\delta \in (0, t_0 - t_1)$, we choose

$$\Phi(x,t) = \widetilde{\chi}(t)\,\chi(t)\,\eta^2(x)$$

with $\chi \in W^{1,\infty}((t_0 - R_0^2, t_0), [0,1])$, $\chi \equiv 0$ on $(t_0 - R_0^2, t_0 - \rho^2)$ and $\partial_t \chi \ge 0$, $\eta \in C_0^\infty(B_\rho(x_0), [0,1])$, and with the Lipschitz continuous function $\widetilde{\chi} : (t_0 - R_0^2, t_0) \to \mathbb{R}$ defined by

$$\widetilde{\chi}(t) = \begin{cases} 1 & \text{if } t \le t_1, \\ \text{affine} & \text{if } t_1 < t < t_1 + \delta, \\ 0 & \text{if } t \ge t_1 + \delta. \end{cases}$$

Setting

$$Q^{t_1} := B_\rho(x_0) \times (t_0 - \rho^2, t_1)$$

and arguing as in the first part of the proof of Proposition 4.12, from $(4.33)_1$ we get

$$\begin{aligned}
&\frac{1}{2}\int_{B_\rho(x_0)} \chi(t_1)\eta^2(x)\,|\tau_h u_\varepsilon(x,t_1)|^2\,dx \,+\, \int_{Q^{t_1}} \chi(t)\eta^2(x)\langle \tau_h H_{p-1}(Du_\varepsilon), D\tau_h u_\varepsilon\rangle\,dz \\
&\quad+\,\varepsilon\int_{Q^{t_1}} \chi(t)\eta^2(x)\langle \tau_h\left[|Du_\varepsilon|^{p-2}Du_\varepsilon\right], D\tau_h u_\varepsilon\rangle\,dz \\
&\qquad=\,-2\int_{Q^{t_1}} \chi(t)\eta(x)\langle \tau_h H_{p-1}(Du_\varepsilon), D\eta\rangle \tau_h u_\varepsilon\;dz \,+\, \int_{Q^{t_1}} (\tau_h f_\varepsilon)(\tau_h u_\varepsilon)\chi(t)\eta^2(x)\,dz \\
&\qquad\quad-\,2\,\varepsilon\int_{Q^{t_1}} \chi(t)\eta(x)\langle \tau_h\left[|Du_\varepsilon|^{p-2}Du_\varepsilon\right], D\eta\rangle \tau_h u_\varepsilon\;dz \,+\, \frac{1}{2}\int_{Q^{t_1}} (\partial_t\chi)\eta^2(x)\,|\tau_h u_\varepsilon|^2\,dz \\
&\qquad=:\, A_1 + A_2 + A_3 + A_4
\end{aligned} \tag{4.34}$$

for every sufficiently small $h \in \mathbb{R}\setminus\{0\}$ and every $t_1 \in (t_0-\rho^2, t_0)$. In the following, we estimate A_1, A_2 and A_3 separately. Let us first consider A_1. Using Lemma 1.6 together with Young's inequality with $\sigma > 0$ and exponents $(2,2)$, we obtain

$$\begin{aligned}
|A_1| \,\le\, &\frac{2(p-1)^2}{\sigma}\int_{Q^{t_1}} \chi(t)\,|D\eta|^2\left(|H_{\frac{p}{2}}(Du_\varepsilon(x+he_s,t))|^{\frac{p-2}{p}} \,+\, |H_{\frac{p}{2}}(Du_\varepsilon)|^{\frac{p-2}{p}}\right)^2 |\tau_h u_\varepsilon|^2\,dz \\
&+\,\frac{\sigma}{2}\int_{Q^{t_1}} \chi(t)\eta^2(x)\,|\tau_h H_{\frac{p}{2}}(Du_\varepsilon)|^2\,dz.
\end{aligned} \tag{4.35}$$

As for A_3, by Lemma 1.4, we similarly have

$$\begin{aligned}
|A_3| \,\le\, &\frac{2\varepsilon\,(p-1)^2}{\sigma}\int_{Q^{t_1}} \chi(t)\,|D\eta|^2\left(|Du_\varepsilon(x+he_s,t)|^{\frac{p-2}{2}} + |Du_\varepsilon|^{\frac{p-2}{2}}\right)^2 |\tau_h u_\varepsilon|^2\,dz \\
&+\,\frac{\varepsilon\sigma}{2}\int_{Q^{t_1}} \chi(t)\eta^2(x)\,|\tau_h \mathbb{V}_p(Du_\varepsilon)|^2\,dz.
\end{aligned} \tag{4.36}$$

We now turn our attention to A_2. Thanks to Propositions 4.12 and 4.13, we know that

$$H_{\frac{p}{2}}(Du_\varepsilon) \,\in\, L^2_{loc}\left(t_0-R_0^2, t_0; W^{1,2}_{loc}(B_{R_0}(x_0),\mathbb{R}^n)\right),$$

$$Du_\varepsilon \,\in\, L^\infty_{loc}\left(t_0-R_0^2, t_0; L^2_{loc}(B_{R_0}(x_0),\mathbb{R}^n)\right) \cap L^{p+\frac{4}{n}}_{loc}\left(Q_{R_0}(z_0),\mathbb{R}^n\right)$$

and

$$\begin{aligned}
\int_{Q_\gamma(z_0)} (|Du_\varepsilon|-\lambda)_+^{p+\frac{4}{n}}\;dz \,\le\, c_1(n,p)&\left[\sup_{t_0-(\alpha r)^2<t<t_0}\int_{B_{\alpha r}(x_0)} |Du_\varepsilon(x,t)|^2\,dx\right]^{\frac{2}{n}} \\
&\cdot\int_{Q_{\alpha r}(z_0)}\left(|DH_{\frac{p}{2}}(Du_\varepsilon)|^2 \,+\, \frac{1}{(\alpha r-\gamma)^2}\,|Du_\varepsilon|^p\right)dz.
\end{aligned} \tag{4.37}$$

Therefore, taking advantage of the properties of f_ε, u_ε, χ and η, and using Hölder's inequality

with exponents $\left(\frac{np+4}{np+4-n}, p+\frac{4}{n}\right)$ together with Lemma 1.14, we obtain

$$A_2 \leq c_2(n,p)\,|h|^2 \left(\int_{Q_\gamma(z_0)} |Df_\varepsilon|^{\frac{np+4}{np+4-n}}\,dz\right)^{\frac{np+4-n}{np+4}} \left(\int_{Q_\gamma(z_0)} |Du_\varepsilon|^{p+\frac{4}{n}}\,dz\right)^{\frac{n}{np+4}}, \tag{4.38}$$

provided that $|h|$ is suitably small. Now, by virtue of Lemma 1.6, we have

$$\frac{4}{p^2}\int_{Q^{t_1}} \chi(t)\eta^2(x)\,|\tau_h H_{\frac{p}{2}}(Du_\varepsilon)|^2\,dz \leq \int_{Q^{t_1}} \chi(t)\eta^2(x)\langle \tau_h H_{p-1}(Du_\varepsilon), D\tau_h u_\varepsilon\rangle\,dz. \tag{4.39}$$

Similarly, by Lemma 1.4, we obtain

$$\frac{4\varepsilon}{p^2}\int_{Q^{t_1}} \chi(t)\eta^2(x)\,|\tau_h \mathbb{V}_p(Du_\varepsilon)|^2\,dz \leq \varepsilon\int_{Q^{t_1}} \chi(t)\eta^2(x)\langle \tau_h\left[|Du_\varepsilon|^{p-2}Du_\varepsilon\right], D\tau_h u_\varepsilon\rangle\,dz. \tag{4.40}$$

Joining estimates (4.34)−(4.36) and (4.38)−(4.40), we arrive at

$$\begin{aligned}
&\int_{B_\rho(x_0)} \chi(t_1)\eta^2(x)\,|\tau_h u_\varepsilon(x,t_1)|^2\,dx + \int_{Q^{t_1}} \chi(t)\eta^2(x)\,|\tau_h H_{\frac{p}{2}}(Du_\varepsilon)|^2\,dz\\
&+ \varepsilon\int_{Q^{t_1}} \chi(t)\eta^2(x)\,|\tau_h \mathbb{V}_p(Du_\varepsilon)|^2\,dz\\
&\quad\leq \sigma\,c_3(p)\int_{Q^{t_1}} \chi(t)\eta^2(x)\,|\tau_h H_{\frac{p}{2}}(Du_\varepsilon)|^2\,dz + \varepsilon\,\sigma\,c_3(p)\int_{Q^{t_1}} \chi(t)\eta^2(x)\,|\tau_h \mathbb{V}_p(Du_\varepsilon)|^2\,dz\\
&\qquad+ \frac{c_3(p)}{\sigma}\int_{Q^{t_1}} \chi(t)\,|D\eta|^2\left(|H_{\frac{p}{2}}(Du_\varepsilon(x+he_s,t))|^{\frac{p-2}{p}} + |H_{\frac{p}{2}}(Du_\varepsilon)|^{\frac{p-2}{p}}\right)^2|\tau_h u_\varepsilon|^2\,dz\\
&\qquad+ \frac{\varepsilon\,c_3(p)}{\sigma}\int_{Q^{t_1}} \chi(t)\,|D\eta|^2\left(|Du_\varepsilon(x+he_s,t)|^{\frac{p-2}{2}} + |Du_\varepsilon|^{\frac{p-2}{2}}\right)^2|\tau_h u_\varepsilon|^2\,dz\\
&\qquad+ c_4(n,p)\,|h|^2\left(\int_{Q_\gamma(z_0)} |Df_\varepsilon|^{\frac{np+4}{np+4-n}}\,dz\right)^{\frac{np+4-n}{np+4}}\left(\int_{Q_\gamma(z_0)} |Du_\varepsilon|^{p+\frac{4}{n}}\,dz\right)^{\frac{n}{np+4}}\\
&\qquad+ c_3(p)\int_{Q^{t_1}} (\partial_t\chi)\eta^2(x)\,|\tau_h u_\varepsilon|^2\,dz.
\end{aligned} \tag{4.41}$$

Choosing $\sigma = (2\,c_3(p))^{-1}$ and reabsorbing the first two integrals in the right-hand side of (4.41) by the left-hand side, we get

$$\begin{aligned}
&\int_{B_\rho(x_0)} \chi(t_1)\eta^2(x)\,|\tau_h u_\varepsilon(x,t_1)|^2\,dx + \int_{Q^{t_1}} \chi(t)\eta^2(x)\,|\tau_h H_{\frac{p}{2}}(Du_\varepsilon)|^2\,dz\\
&+ \varepsilon\int_{Q^{t_1}} \chi(t)\eta^2(x)\,|\tau_h \mathbb{V}_p(Du_\varepsilon)|^2\,dz\\
&\quad\leq c_5(p)\int_{Q^{t_1}} \chi(t)\,|D\eta|^2\left(|H_{\frac{p}{2}}(Du_\varepsilon(x+he_s,t))|^{\frac{p-2}{p}} + |H_{\frac{p}{2}}(Du_\varepsilon)|^{\frac{p-2}{p}}\right)^2|\tau_h u_\varepsilon|^2\,dz\\
&\qquad+ \varepsilon\,c_5(p)\int_{Q^{t_1}} \chi(t)\,|D\eta|^2\left(|Du_\varepsilon(x+he_s,t)|^{\frac{p-2}{2}} + |Du_\varepsilon|^{\frac{p-2}{2}}\right)^2|\tau_h u_\varepsilon|^2\,dz\\
&\qquad+ c_6(n,p)\,|h|^2\left(\int_{Q_\gamma(z_0)} |Df_\varepsilon|^{\frac{np+4}{np+4-n}}\,dz\right)^{\frac{np+4-n}{np+4}}\left(\int_{Q_\gamma(z_0)} |Du_\varepsilon|^{p+\frac{4}{n}}\,dz\right)^{\frac{n}{np+4}}
\end{aligned}$$

$$+\ c_5(p)\int_{Q^{t_1}}(\partial_t\chi)\eta^2(x)\,|\tau_h u_\varepsilon|^2\,dz,$$

which holds for every $t_1 \in (t_0-\rho^2, t_0)$ and every sufficiently small $h \in \mathbb{R}\setminus\{0\}$.

We now choose a cut-off function $\eta \in C_0^\infty(B_\rho(x_0))$ with $\eta \equiv 1$ on $B_\ell(x_0)$ such that

$$0 \le \eta \le 1 \qquad \text{and} \qquad |D\eta| \le \frac{C}{\rho-\ell}\,.$$

For the cut-off function in time, we choose $\chi \in W^{1,\infty}((t_0-R_0^2, t_0),[0,1])$ such that

$$\begin{aligned} \chi &\equiv 0 && \text{on } (t_0-R_0^2, t_0-\rho^2]\,,\\ \chi &\equiv 1 && \text{on } [t_0-\ell^2, t_0)\,,\\ \partial_t\chi &\le \tfrac{C}{(\rho-\ell)^2} && \text{on } (t_0-\rho^2, t_0-\ell^2)\,. \end{aligned}$$

Dividing both sides of the previous estimate by $|h|^2$, using the properties of χ, η and u_ε, and applying Young's inequality with exponents $\left(\frac{p}{p-2}, \frac{p}{2}\right)$, we get

$$\begin{aligned}
&\sup_{t\in(t_0-\ell^2,t_0)}\int_{B_\ell(x_0)}|\Delta_h u_\varepsilon(x,t)|^2\,dx \;+\; \int_{Q_\ell(z_0)}|\Delta_h H_{\frac{p}{2}}(Du_\varepsilon)|^2\,dz \;+\; \varepsilon\int_{Q_\ell(z_0)}|\Delta_h \mathbb{V}_p(Du_\varepsilon)|^2\,dz\\
&\le \frac{c_7(p)}{(\rho-\ell)^2}\int_{Q_\rho(z_0)}\left[\left(|H_{\frac{p}{2}}(Du_\varepsilon(x+he_s,t))|^{\frac{p-2}{p}} + |H_{\frac{p}{2}}(Du_\varepsilon)|^{\frac{p-2}{p}}\right)^2+1\right]|\Delta_h u_\varepsilon|^2\,dz\\
&\quad+\ \frac{\varepsilon\,c_7(p)}{(\rho-\ell)^2}\int_{Q_\rho(z_0)}\left(|Du_\varepsilon(x+he_s,t)|^{\frac{p-2}{2}}+|Du_\varepsilon|^{\frac{p-2}{2}}\right)^2|\Delta_h u_\varepsilon|^2\,dz\\
&\quad+\ c_6(n,p)\left(\int_{Q_\gamma(z_0)}|Df_\varepsilon|^{\frac{np+4}{np+4-n}}\,dz\right)^{\frac{np+4-n}{np+4}}\left(\int_{Q_\gamma(z_0)}|Du_\varepsilon|^{p+\frac{4}{n}}\,dz\right)^{\frac{n}{np+4}}\\
&\le \frac{c_8(p)}{(\rho-\ell)^2}\int_{Q_\gamma(z_0)}\left(|Du_\varepsilon|^{p-2}+1\right)|\Delta_h u_\varepsilon|^2\,dz\\
&\quad+\ c_6(n,p)\left(\int_{Q_\gamma(z_0)}|Df_\varepsilon|^{\frac{np+4}{np+4-n}}\,dz\right)^{\frac{np+4-n}{np+4}}\left(\int_{Q_\gamma(z_0)}|Du_\varepsilon|^{p+\frac{4}{n}}\,dz\right)^{\frac{n}{np+4}}\\
&= \frac{c_8(p)}{(\rho-\ell)^2}\int_{Q_\gamma(z_0)}|Du_\varepsilon|^{p-2}|\Delta_h u_\varepsilon|^2\,dz \;+\; \frac{c_8(p)}{(\rho-\ell)^2}\int_{Q_\gamma(z_0)}|\Delta_h u_\varepsilon|^2\,dz\\
&\quad+\ c_6(n,p)\left(\int_{Q_\gamma(z_0)}|Df_\varepsilon|^{\frac{np+4}{np+4-n}}\,dz\right)^{\frac{np+4-n}{np+4}}\left(\int_{Q_\gamma(z_0)}|Du_\varepsilon|^{p+\frac{4}{n}}\,dz\right)^{\frac{n}{np+4}}\\
&\le \frac{c_9(p)}{(\rho-\ell)^2}\int_{Q_\gamma}|Du_\varepsilon|^p\,dz \;+\; \frac{c_{10}(p)}{(\rho-\ell)^2}\int_{Q_\gamma}|\Delta_h u_\varepsilon|^p\,dz \;+\; \frac{c_{11}(n,p)}{(\rho-\ell)^2}\int_{Q_R}|Du_\varepsilon|^2\,dz\\
&\quad+\ c_6(n,p)\left(\int_{Q_\gamma(z_0)}|Df_\varepsilon|^{\frac{np+4}{np+4-n}}\,dz\right)^{\frac{np+4-n}{np+4}}\left(\int_{Q_\gamma(z_0)}|Du_\varepsilon|^{p+\frac{4}{n}}\,dz\right)^{\frac{n}{np+4}}\\
&\le \frac{c_{12}(n,p)}{(\rho-\ell)^2}\left(\int_{Q_R(z_0)}|Du_\varepsilon|^p\,dz+\int_{Q_R(z_0)}|Du_\varepsilon|^2\,dz\right)
\end{aligned}$$

$$+\ c_{12}(n,p)\left(\int_{Q_R(z_0)}|Df_\varepsilon|^{\frac{np+4}{np+4-n}}\,dz\right)^{\frac{np+4-n}{np+4}}\left(\int_{Q_\gamma(z_0)}|Du_\varepsilon|^{p+\frac{4}{n}}\,dz\right)^{\frac{n}{np+4}}, \tag{4.42}$$

where we have used Lemma 1.14. Therefore, letting $h\to 0$ in (4.42), we obtain

$$\begin{aligned}
&\sup_{t\in(t_0-\ell^2,t_0)}\|Du_\varepsilon(\cdot,t)\|^2_{L^2(B_\ell(x_0))}+\int_{Q_\ell(z_0)}|DH_{\frac{p}{2}}(Du_\varepsilon)|^2\,dz+\varepsilon\int_{Q_\ell(z_0)}|D\mathbb{V}_p(Du_\varepsilon)|^2\,dz\\
&\quad\le\frac{c_{12}(n,p)}{(\rho-\ell)^2}\left(\int_{Q_R(z_0)}|Du_\varepsilon|^p\,dz+\int_{Q_R(z_0)}|Du_\varepsilon|^2\,dz\right)\\
&\qquad+c_{12}(n,p)\left(\int_{Q_R(z_0)}|Df_\varepsilon|^{\frac{np+4}{np+4-n}}\,dz\right)^{\frac{np+4-n}{np+4}}\left(\int_{Q_\gamma(z_0)}|Du_\varepsilon|^{p+\frac{4}{n}}\,dz\right)^{\frac{n}{np+4}}.
\end{aligned}\tag{4.43}$$

Now we use (4.37) in order to estimate the last integral as follows:

$$\begin{aligned}
&\int_{Q_\gamma(z_0)}|Du_\varepsilon|^{p+\frac{4}{n}}\,dz\\
&\quad=\int_{Q_\gamma\cap\{|Du_\varepsilon|\ge\lambda\}}(|Du_\varepsilon|-\lambda+\lambda)^{p+\frac{4}{n}}\,dz+\int_{Q_\gamma\cap\{|Du_\varepsilon|<\lambda\}}|Du_\varepsilon|^{p+\frac{4}{n}}\,dz\\
&\quad\le c_{13}(n,p)\int_{Q_\gamma}(|Du_\varepsilon|-\lambda)_+^{p+\frac{4}{n}}\,dz+c_{13}(n,p)\,\lambda^{p+\frac{4}{n}}\,|Q_{R_0}|\\
&\quad\le c_{14}(n,p)\left[\sup_{t\in(t_0-(\alpha r)^2,t_0)}\int_{B_{\alpha r}(x_0)}|Du_\varepsilon(x,t)|^2\,dx\right]^{\frac{2}{n}}\int_{Q_{\alpha r}(z_0)}\left(|DH_{\frac{p}{2}}(Du_\varepsilon)|^2+\frac{|Du_\varepsilon|^p}{(\alpha r-\gamma)^2}\right)dz\\
&\qquad+c_{14}(n,p)\,\lambda^{p+\frac{4}{n}}\,|Q_{R_0}|\,,
\end{aligned}$$

where $|Q_{R_0}|$ denotes the $(n+1)$-dimensional Lebesgue measure of the parabolic cylinder $Q_{R_0}(z_0)$. Plugging the above estimate into (4.43) and applying Young's inequality with $\theta\in(0,1)$ and exponents $\left(\frac{np+4}{np+2-n},\frac{np+4}{2},\frac{np+4}{n}\right)$, we get

$$\begin{aligned}
&\sup_{t\in(t_0-\ell^2,t_0)}\|Du_\varepsilon(\cdot,t)\|^2_{L^2(B_\ell(x_0))}+\int_{Q_\ell(z_0)}|DH_{\frac{p}{2}}(Du_\varepsilon)|^2\,dz\\
&\quad\le\frac{c_{12}(n,p)}{(\rho-\ell)^2}\left(\|Du_\varepsilon\|^p_{L^p(Q_R)}+\|Du_\varepsilon\|^2_{L^2(Q_R)}\right)+c_{15}(n,p)\,\lambda\,\|Df_\varepsilon\|_{L^{\frac{np+4}{np+4-n}}(Q_R)}\,|Q_{R_0}|^{\frac{n}{np+4}}\\
&\qquad+c_{15}(n,p)\,\|Df_\varepsilon\|_{L^{\frac{np+4}{np+4-n}}(Q_R)}\left[\sup_{t\in(t_0-(\alpha r)^2,t_0)}\|Du_\varepsilon(\cdot,t)\|^2_{L^2(B_{\alpha r}(x_0))}\right]^{\frac{2}{np+4}}\\
&\qquad\cdot\left[\int_{Q_{\alpha r}(z_0)}\left(|DH_{\frac{p}{2}}(Du_\varepsilon)|^2+\frac{|Du_\varepsilon|^p}{(\alpha r-\gamma)^2}\right)dz\right]^{\frac{n}{np+4}}\\
&\quad\le\frac{c_{12}(n,p)}{(\rho-\ell)^2}\left(\|Du_\varepsilon\|^p_{L^p(Q_R)}+\|Du_\varepsilon\|^2_{L^2(Q_R)}\right)+c_{16}(n,p,R_0)\,\lambda\,\|Df_\varepsilon\|_{L^{\frac{np+4}{np+4-n}}(Q_R)}\\
&\qquad+c_{17}(n,p)\,\theta^{\frac{n+2}{n-np-2}}\,\|Df_\varepsilon\|^{\frac{np+4}{np+2-n}}_{L^{\frac{np+4}{np+4-n}}(Q_R)}+\theta\left[\sup_{t\in(t_0-(\alpha r)^2,t_0)}\|Du_\varepsilon(\cdot,t)\|^2_{L^2(B_{\alpha r}(x_0))}\right]\\
&\qquad+\theta\int_{Q_{\alpha r}(z_0)}|DH_{\frac{p}{2}}(Du_\varepsilon)|^2\,dz+\frac{\theta}{(\alpha r-\gamma)^2}\,\|Du_\varepsilon\|^p_{L^p(Q_R)}\,.
\end{aligned}$$

Now, if we choose $\theta = \frac{1}{2}$ and set

$$\Psi(\xi) := \sup_{t\,\in\,(t_0-\xi^2,t_0)} \|Du_\varepsilon(\cdot,t)\|^2_{L^2(B_\xi(x_0))} + \int_{Q_\xi(z_0)} |DH_{\frac{p}{2}}(Du_\varepsilon)|^2\,dz\,,$$

then the previous estimate gives

$$\begin{aligned}\Psi(r) \,\le\, \Psi(\ell) \,\le\, \frac{1}{2}\,\Psi(\alpha r) \,+\, \frac{c_{12}(n,p)}{(\rho-\ell)^2}\,\Big(\|Du_\varepsilon\|^p_{L^p(Q_R)} \,+\, \|Du_\varepsilon\|^2_{L^2(Q_R)}\Big)\\ +\, \frac{1}{2\,(\alpha r-\gamma)^2}\,\|Du_\varepsilon\|^p_{L^p(Q_R)}\\ +\, c_{18}(n,p,R_0)\left(\lambda\,\|Df_\varepsilon\|_{L^{\frac{np+4}{np+4-n}}(Q_R)} \,+\, \|Df_\varepsilon\|^{\frac{np+4}{np+2-n}}_{L^{\frac{np+4}{np+4-n}}(Q_R)}\right).\end{aligned}$$

Since the above estimate holds for all

$$\frac{R}{2} \le r < \ell < \rho < \gamma < \alpha r < R \qquad \text{with } 1<\alpha<2,$$

we can now choose

$$\ell \,=\, r+\frac{1}{4}(\alpha r - r), \qquad \rho \,=\, r+\frac{1}{2}(\alpha r - r), \qquad \gamma \,=\, r+\frac{3}{4}(\alpha r - r).$$

Thus, we obtain

$$\begin{aligned}\Psi(r) \,\le\, \frac{1}{2}\,\Psi(\alpha r) \,+\, \frac{c_{19}(n,p)}{r^2(\alpha-1)^2}\,\Big(\|Du_\varepsilon\|^p_{L^p(Q_R)} \,+\, \|Du_\varepsilon\|^2_{L^2(Q_R)}\Big)\\ +\, c_{18}(n,p,R_0)\left(\lambda\,\|Df_\varepsilon\|_{L^{\frac{np+4}{np+4-n}}(Q_R)} \,+\, \|Df_\varepsilon\|^{\frac{np+4}{np+2-n}}_{L^{\frac{np+4}{np+4-n}}(Q_R)}\right),\end{aligned}$$

which holds for all

$$\frac{R}{2} \le r < \alpha r < R \qquad \text{with } 1<\alpha<2.$$

Therefore, the use of Lemma 1.10 with $\rho_0 = \frac{R}{2}$ and $\rho_1 = R$ yields

$$\begin{aligned}\Psi\left(\frac{R}{2}\right) \le\, \frac{c_{20}(n,p)}{R^2}\,\Big(\|Du_\varepsilon\|^p_{L^p(Q_R)} \,+\, \|Du_\varepsilon\|^2_{L^2(Q_R)}\Big)\\ +\, c_{21}(n,p,R_0)\left(\lambda\,\|Df_\varepsilon\|_{L^{\frac{np+4}{np+4-n}}(Q_R)} \,+\, \|Df_\varepsilon\|^{\frac{np+4}{np+2-n}}_{L^{\frac{np+4}{np+4-n}}(Q_R)}\right),\end{aligned}$$

that is,

$$\begin{aligned}\sup_{t\,\in\,(t_0-R^2/4,t_0)} \|Du_\varepsilon(\cdot,t)\|^2_{L^2(B_{R/2}(x_0))} + \int_{Q_{R/2}(z_0)} |DH_{\frac{p}{2}}(Du_\varepsilon)|^2\,dz\\ \le\, \frac{c_{20}(n,p)}{R^2}\,\Big(\|Du_\varepsilon\|^p_{L^p(Q_R)} \,+\, \|Du_\varepsilon\|^2_{L^2(Q_R)}\Big)\\ +\, c_{21}(n,p,R_0)\left(\lambda\,\|Df_\varepsilon\|_{L^{\frac{np+4}{np+4-n}}(Q_R)} \,+\, \|Df_\varepsilon\|^{\frac{np+4}{np+2-n}}_{L^{\frac{np+4}{np+4-n}}(Q_R)}\right),\end{aligned} \tag{4.44}$$

which is the *a priori* estimate we were looking for.

Our next aim is to prove that the norms $\|Du_\varepsilon\|_{L^p(Q_{R_0})}$ are all bounded by a constant independent of ε, provided that ε is sufficiently small (and therefore the same will be true for the norms $\|Du_\varepsilon\|_{L^2(Q_{R_0})}$, since we are assuming that $p > 2$).

Step 2: Uniform boundedness of the norms $\|Du_\varepsilon\|_{L^p(Q_{R_0})}$. In order to obtain an (uniform in ε) energy estimate, we now proceed by testing equations (4.1) and $(4.33)_1$, for u and u_ε, respectively, with the map $\varphi = g(t)(u_\varepsilon - u)$, where $g \in W^{1,\infty}(\mathbb{R})$ is chosen such that

$$g(t) = \begin{cases} 1 & \text{if } t \le t_2, \\ -\frac{1}{\omega}(t - t_2 - \omega) & \text{if } t_2 < t < t_2 + \omega, \\ 0 & \text{if } t \ge t_2 + \omega, \end{cases}$$

with

$$t_0 - R_0^2 \,<\, t_2 \,<\, t_2 \,+\, \omega \,<\, t_0\,,$$

and then letting $\omega \to 0$. We observe that, at this stage, it is important that u_ε and u agree on the parabolic boundary $\partial_{\rm par} Q_{R_0}(z_0)$. We also note that the following computations are somewhat formal concerning the use of the time derivative, but they can easily be made rigorous, for example by the use of Steklov averages. We skip this, since it is a standard procedure. With the previous choice of φ, for almost every $t_2 \in (t_0 - R_0^2, t_0)$ we find

$$\begin{aligned} &\frac{1}{2}\int_{B_{R_0}(x_0)} |u_\varepsilon(x,t_2) - u(x,t_2)|^2\,dx \,+ \int_{Q_{R_0,t_2}} \langle H_{p-1}(Du_\varepsilon) - H_{p-1}(Du), Du_\varepsilon - Du\rangle\,dz \\ &\quad + \varepsilon \int_{Q_{R_0,t_2}} \langle |Du_\varepsilon|^{p-2} Du_\varepsilon, Du_\varepsilon - Du\rangle\,dz \\ &\qquad = \int_{Q_{R_0,t_2}} (f - f_\varepsilon)(u_\varepsilon - u)\,dz, \end{aligned} \tag{4.45}$$

where we have used the abbreviation

$$Q_{R_0,t_2} := B_{R_0}(x_0) \times (t_0 - R_0^2, t_2)\,.$$

Using Lemma 1.6, the Cauchy-Schwarz inequality as well as Young's inequality with $\mu > 0$ and exponents (p, p'), from (4.45) we infer

$$\begin{aligned} &\frac{1}{2}\sup_{t\,\in\,(t_0 - R_0^2, t_0)} \|u_\varepsilon(\cdot,t) - u(\cdot,t)\|^2_{L^2(B_{R_0}(x_0))} \,+\, \frac{4}{p^2}\int_{Q_{R_0}(z_0)} |H_{\frac{p}{2}}(Du_\varepsilon) - H_{\frac{p}{2}}(Du)|^2\,dz \\ &\quad + \varepsilon \int_{Q_{R_0}(z_0)} |Du_\varepsilon|^p\,dz \\ &\qquad \le \int_{Q_{R_0}(z_0)} |f - f_\varepsilon|\,|u_\varepsilon - u|\;dz \,+\, \varepsilon \int_{Q_{R_0}(z_0)} |Du_\varepsilon|^{p-1}\,|Du|\;dz \\ &\qquad \le \int_{Q_{R_0}} |f - f_\varepsilon|\,|u_\varepsilon - u|\;dz \,+\, \frac{\varepsilon}{p\,\mu^{p/2}}\int_{Q_{R_0}} |Du|^p\,dz \,+\, \frac{\varepsilon}{p'}\,\mu^{\frac{p'}{2}} \int_{Q_{R_0}} |Du_\varepsilon|^p\,dz. \end{aligned} \tag{4.46}$$

Choosing $\mu = \left(\frac{p'}{2}\right)^{\frac{2}{p'}}$ and reabsorbing the last integral in the right-hand side of (4.46) by the

left-hand side, we arrive at

$$\sup_{t\in(t_0-R_0^2,t_0)}\|u_\varepsilon(\cdot,t)-u(\cdot,t)\|^2_{L^2(B_{R_0}(x_0))}+\int_{Q_{R_0}}|H_{\frac{p}{2}}(Du_\varepsilon)-H_{\frac{p}{2}}(Du)|^2\,dz+\varepsilon\int_{Q_{R_0}}|Du_\varepsilon|^p\,dz$$
$$\le\varepsilon\,c_1(p)\int_{Q_{R_0}}|Du|^p\,dz+c_1(p)\int_{Q_{R_0}}|f-f_\varepsilon|\,|u_\varepsilon-u|\,dz. \tag{4.47}$$

Now we set

$$A_5:=\int_{Q_{R_0}}|f-f_\varepsilon|\,|u_\varepsilon-u|\,dz$$

and observe that

$$\left(p+\frac{4}{n}\right)'=\frac{np+4}{np+4-n}\ge\left(p+\frac{2p}{n}\right)'=\frac{np+2p}{np+2p-n}\qquad\text{for every } p\ge 2. \tag{4.48}$$

Therefore, we can apply Hölder's inequality with exponents $\left(\frac{np+2p}{np+2p-n},p+\frac{2p}{n}\right)$ and Lemma 1.11 with $v=u_\varepsilon-u$, $r=R_0$ and $q=2$ to estimate A_5 as follows:

$$\begin{aligned}A_5&\le\left(\int_{Q_{R_0}}|f-f_\varepsilon|^{\frac{np+2p}{np+2p-n}}\,dz\right)^{\frac{np+2p-n}{np+2p}}\left(\int_{Q_{R_0}}|u_\varepsilon-u|^{p+\frac{2p}{n}}\,dz\right)^{\frac{n}{np+2p}}\\&\le c_2(n,p)\,\|f-f_\varepsilon\|_{L^{\frac{np+2p}{np+2p-n}}(Q_{R_0})}\left(\sup_{t\in(t_0-R_0^2,t_0)}\|u_\varepsilon(\cdot,t)-u(\cdot,t)\|^2_{L^2(B_{R_0}(x_0))}\right)^{\frac{1}{n+2}}\\&\qquad\cdot\left(\int_{Q_{R_0}}|Du_\varepsilon-Du|^p\,dz\right)^{\frac{n}{np+2p}}.\end{aligned} \tag{4.49}$$

Combining estimates (4.47) and (4.49), recalling that $0<\varepsilon\le 1$ and applying Young's inequality with $\beta>0$ and exponents

$$\left(\frac{np+2p}{np+p-n},\,n+2,\,\frac{np+2p}{n}\right),$$

we obtain

$$\begin{aligned}&\sup_{t\in(t_0-R_0^2,t_0)}\|u_\varepsilon(\cdot,t)-u(\cdot,t)\|^2_{L^2(B_{R_0}(x_0))}+\int_{Q_{R_0}}|H_{\frac{p}{2}}(Du_\varepsilon)-H_{\frac{p}{2}}(Du)|^2\,dz+\varepsilon\int_{Q_{R_0}}|Du_\varepsilon|^p\,dz\\&\le c_1(p)\int_{Q_{R_0}}|Du|^p\,dz+c_3(n,p)\,\beta^{\frac{n+p}{n-np-p}}\,\|f-f_\varepsilon\|^{\frac{np+2p}{np+p-n}}_{L^{\frac{np+2p}{np+2p-n}}(Q_{R_0})}\\&\quad+\beta\left[\sup_{t\in(t_0-R_0^2,t_0)}\|u_\varepsilon(\cdot,t)-u(\cdot,t)\|^2_{L^2(B_{R_0}(x_0))}\right]+\beta\int_{Q_{R_0}}|Du_\varepsilon-Du|^p\,dz\\&\le\left(c_1(p)+2^{p-1}\beta\right)\int_{Q_{R_0}}|Du|^p\,dz+c_3(n,p)\,\beta^{\frac{n+p}{n-np-p}}\,\|f-f_\varepsilon\|^{\frac{np+2p}{np+p-n}}_{L^{\frac{np+2p}{np+2p-n}}(Q_{R_0})}\\&\quad+\beta\left[\sup_{t\in(t_0-R_0^2,t_0)}\|u_\varepsilon(\cdot,t)-u(\cdot,t)\|^2_{L^2(B_{R_0}(x_0))}\right]+2^{p-1}\beta\int_{Q_{R_0}}|Du_\varepsilon|^p\,dz.\end{aligned} \tag{4.50}$$

Now, let us notice that

$$\begin{aligned}\int_{Q_{R_0}} |Du_\varepsilon|^p\,dz &= \int_{Q_{R_0}\cap\{|Du_\varepsilon|\geq\lambda\}} (|Du_\varepsilon|-\lambda+\lambda)^p\,dz + \int_{Q_{R_0}\cap\{|Du_\varepsilon|<\lambda\}} |Du_\varepsilon|^p\,dz\\ &\leq 2^{p-1}\int_{Q_{R_0}} \left[(|Du_\varepsilon|-\lambda)_+^p+\lambda^p\right]\,dz + \lambda^p\,|Q_{R_0}|\\ &= 2^{p-1}\int_{Q_{R_0}} \left(|H_{\frac{p}{2}}(Du_\varepsilon)-H_{\frac{p}{2}}(Du)+H_{\frac{p}{2}}(Du)|^2+\lambda^p\right)\,dz + \lambda^p\,|Q_{R_0}|\\ &\leq 2^{p}\int_{Q_{R_0}} \left(|H_{\frac{p}{2}}(Du_\varepsilon)-H_{\frac{p}{2}}(Du)|^2+|H_{\frac{p}{2}}(Du)|^2+\lambda^p\right)\,dz + \lambda^p\,|Q_{R_0}|\\ &\leq 2^{p}\int_{Q_{R_0}} |H_{\frac{p}{2}}(Du_\varepsilon)-H_{\frac{p}{2}}(Du)|^2\,dz + 2^{p+1}\int_{Q_{R_0}} (|Du|^p+\lambda^p)\,dz. \qquad (4.51)\end{aligned}$$

Multiplying all sides of (4.50) by 2^p and then adding the resulting expression and (4.51) side by side, we get

$$\begin{aligned}&(1+2^p\varepsilon)\int_{Q_{R_0}} |Du_\varepsilon|^p\,dz + 2^p \sup_{t\in(t_0-R_0^2,t_0)} \|u_\varepsilon(\cdot,t)-u(\cdot,t)\|^2_{L^2(B_{R_0}(x_0))}\\ &\quad\leq \left(c_4(p)+2^{2p-1}\beta\right)\int_{Q_{R_0}} (|Du|^p+\lambda^p)\,dz + c_5(n,p)\,\beta^{\frac{n+p}{n-np-p}}\,\|f-f_\varepsilon\|^{\frac{np+2p}{np+p-n}}_{L^{\frac{np+2p}{np+2p-n}}(Q_{R_0})}\\ &\qquad + 2^p\beta\left[\sup_{t\in(t_0-R_0^2,t_0)} \|u_\varepsilon(\cdot,t)-u(\cdot,t)\|^2_{L^2(B_{R_0}(x_0))}\right] + 2^{2p-1}\beta\int_{Q_{R_0}} |Du_\varepsilon|^p\,dz. \qquad (4.52)\end{aligned}$$

Now, choosing $\beta = 1/2^{2p}$ and reabsorbing the last two terms in the right-hand side of (4.52) by the left-hand side, we obtain

$$\begin{aligned}&\int_{Q_{R_0}} |Du_\varepsilon|^p\,dz + \sup_{t\in(t_0-R_0^2,t_0)} \|u_\varepsilon(\cdot,t)-u(\cdot,t)\|^2_{L^2(B_{R_0}(x_0))}\\ &\quad\leq c_6(p)\int_{Q_{R_0}} (|Du|^p+\lambda^p)\,dz + c_7(n,p)\,\|f-f_\varepsilon\|^{\frac{np+2p}{np+p-n}}_{L^{\frac{np+2p}{np+2p-n}}(Q_{R_0})}.\end{aligned}$$

Moreover, since

$$f\in L^{\frac{np+4}{np+4-n}}(\Omega_T)\,,$$

recalling (4.13), by virtue of (4.48) we have

$$f_\varepsilon\to f \qquad \text{strongly in } L^{\frac{np+2p}{np+2p-n}}(Q_{R_0}),\quad \text{as } \varepsilon\to 0. \qquad (4.53)$$

Hence, there exists a positive number $\varepsilon_0\leq 1$ such that

$$\|f-f_\varepsilon\|_{L^{\frac{np+2p}{np+2p-n}}(Q_{R_0})}\leq 1 \quad \text{for all } \varepsilon\in(0,\varepsilon_0]\,.$$

From this we can finally deduce

$$\begin{aligned}&\|Du_\varepsilon\|^p_{L^p(Q_{R_0})} + \sup_{t\in(t_0-R_0^2,t_0)} \|u_\varepsilon(\cdot,t)-u(\cdot,t)\|^2_{L^2(B_{R_0}(x_0))} \qquad (4.54)\\ &\quad\leq c_8(n,p,R_0)\left[\|Du\|^p_{L^p(Q_{R_0})}+\lambda^p+\|f-f_\varepsilon\|^{\frac{np+2p}{np+p-n}}_{L^{\frac{np+2p}{np+2p-n}}(Q_{R_0})}\right]\leq K \quad \text{for all } \varepsilon\in(0,\varepsilon_0],\end{aligned}$$

for some positive constant K depending only on n, p, λ, R_0 and $\|Du\|_{L^p(Q_{R_0})}$. Estimate (4.54) is exactly the uniform energy estimate we are looking for.

Moreover, joining (4.47), (4.49) and (4.54), and applying Minkowski's inequality, we get the following *comparison estimate*:

$$
\begin{aligned}
&\sup_{t\,\in\,(t_0-R_0^2,t_0)} \|u_\varepsilon(\cdot,t)-u(\cdot,t)\|^2_{L^2(B_{R_0}(x_0))} + \int_{Q_{R_0}} |H_{\frac{p}{2}}(Du_\varepsilon)-H_{\frac{p}{2}}(Du)|^2\,dz\\
&\quad\le \varepsilon\, c_1(p)\,\|Du\|^p_{L^p(Q_{R_0})} + c_9(n,p)\,\|f-f_\varepsilon\|_{L^{\frac{np+2p}{np+2p-n}}(Q_{R_0})}\\
&\qquad\cdot\left(\sup_{t\,\in\,(t_0-R_0^2,t_0)} \|u_\varepsilon(\cdot,t)-u(\cdot,t)\|^2_{L^2(B_{R_0}(x_0))}\right)^{\frac{1}{n+2}}\left(\int_{Q_{R_0}}|Du_\varepsilon-Du|^p\,dz\right)^{\frac{n}{np+2p}}\\
&\quad\le \varepsilon\, c_1(p)\,\|Du\|^p_{L^p(Q_{R_0})} + c_{10}(n,p,R_0)\,\|f-f_\varepsilon\|_{L^{\frac{np+2p}{np+2p-n}}(Q_{R_0})}\\
&\qquad\cdot\left(\|Du\|^p_{L^p(Q_{R_0})}+\lambda^p+\|f-f_\varepsilon\|^{\frac{np+2p}{np+p-n}}_{L^{\frac{np+2p}{np+2p-n}}(Q_{R_0})}\right)^{\frac{1}{n+2}}\left(\|Du_\varepsilon\|_{L^p(Q_{R_0})}+\|Du\|_{L^p(Q_{R_0})}\right)^{\frac{n}{n+2}}\\
&\quad\le \varepsilon\, c_1(p)\,\|Du\|^p_{L^p(Q_{R_0})}\\
&\qquad + c_{11}(n,p,R_0)\,\|f-f_\varepsilon\|_{L^{\frac{np+2p}{np+2p-n}}(Q_{R_0})}\left(\|Du\|^p_{L^p(Q_{R_0})}+\lambda^p+\|f-f_\varepsilon\|^{\frac{np+2p}{np+p-n}}_{L^{\frac{np+2p}{np+2p-n}}(Q_{R_0})}\right)^{\frac{1}{n+2}}\\
&\qquad\cdot\left(\|Du\|_{L^p(Q_{R_0})}+\lambda+\|f-f_\varepsilon\|^{\frac{n+2}{np+p-n}}_{L^{\frac{np+2p}{np+2p-n}}(Q_{R_0})}\right)^{\frac{n}{n+2}},
\end{aligned}
\tag{4.55}
$$

where, in the last line, we have used estimate (4.54) again.

Step 3: The conclusion. Now we shall fix $\varrho \in (0,R)$ and consider the finite difference operator $\tau_{s,h}$, defined in Section 1.1, for increments $h \in \mathbb{R}\setminus\{0\}$ such that $|h|$ is suitably small.

In order to obtain an estimate for the finite difference $\tau_{s,h}H_{\frac{p}{2}}(Du)$, we use the following comparison argument:

$$
\begin{aligned}
&\int_{Q_{\varrho/2}} |\tau_{s,h}H_{\frac{p}{2}}(Du)|^2\,dz\\
&\quad\le 4\int_{Q_{\varrho/2}} |\tau_{s,h}H_{\frac{p}{2}}(Du_\varepsilon)|^2\,dz + 4\int_{Q_{\varrho/2}} |H_{\frac{p}{2}}(Du_\varepsilon(x+he_s,t))-H_{\frac{p}{2}}(Du(x+he_s,t))|^2\,dx\,dt\\
&\qquad + 4\int_{Q_{\varrho/2}} |H_{\frac{p}{2}}(Du_\varepsilon)-H_{\frac{p}{2}}(Du)|^2\,dz.
\end{aligned}
$$

Combining the above inequality with estimates (4.44), (4.54) and (4.55), for every $s\in\{1,\dots,n\}$ we get

$$
\begin{aligned}
&\int_{Q_{\varrho/2}} |\tau_{s,h}H_{\frac{p}{2}}(Du)|^2\,dz\\
&\quad\le 4\int_{Q_{R/2}} |\tau_{s,h}H_{\frac{p}{2}}(Du_\varepsilon)|^2\,dz + 8\int_{Q_R} |H_{\frac{p}{2}}(Du_\varepsilon)-H_{\frac{p}{2}}(Du)|^2\,dz\\
&\quad\le \frac{c_1(n,p)}{R^2}\,|h|^2\left(\|Du_\varepsilon\|^p_{L^p(Q_R)} + \|Du_\varepsilon\|^2_{L^2(Q_R)}\right) + \varepsilon\, c_2(p)\,\|Du\|^p_{L^p(Q_{R_0})}
\end{aligned}
$$

$$
\begin{aligned}
&+\ c_3(n,p,R_0)\,|h|^2\left(\lambda\,\|Df_\varepsilon\|_{L^{\frac{np+4}{np+4-n}}(Q_R)}+\|Df_\varepsilon\|^{\frac{np+4}{np+2-n}}_{L^{\frac{np+4}{np+4-n}}(Q_R)}\right)\\
&+\ c_3(n,p,R_0)\|f-f_\varepsilon\|_{L^{\frac{np+2p}{np+2p-n}}(Q_{R_0})}\left(\|Du\|^p_{L^p(Q_{R_0})}+\lambda^p+\|f-f_\varepsilon\|^{\frac{np+2p}{np+p-n}}_{L^{\frac{np+2p}{np+2p-n}}(Q_{R_0})}\right)^{\frac{1}{n+2}}\\
&\cdot\left(\|Du\|_{L^p(Q_{R_0})}+\lambda+\|f-f_\varepsilon\|^{\frac{n+2}{np+p-n}}_{L^{\frac{np+2p}{np+2p-n}}(Q_{R_0})}\right)^{\frac{n}{n+2}}\\
\le\ &\frac{c_4(n,p,R_0)}{R^2}\,|h|^2\left(\|Du\|^p_{L^p(Q_{R_0})}+\|Du\|^2_{L^p(Q_{R_0})}+\lambda^p+\lambda^2\right)\\
&+\ \frac{c_4(n,p,R_0)}{R^2}\,|h|^2\left(\|f-f_\varepsilon\|^{\frac{np+2p}{np+p-n}}_{L^{\frac{np+2p}{np+2p-n}}(Q_{R_0})}+\|f-f_\varepsilon\|^{\frac{2n+4}{np+p-n}}_{L^{\frac{np+2p}{np+2p-n}}(Q_{R_0})}\right)\\
&+\ \varepsilon\,c_2(p)\,\|Du\|^p_{L^p(Q_{R_0})}+\ c_4(n,p,R_0)\,|h|^2\left(\lambda\,\|Df_\varepsilon\|_{L^{\frac{np+4}{np+4-n}}(Q_R)}+\|Df_\varepsilon\|^{\frac{np+4}{np+2-n}}_{L^{\frac{np+4}{np+4-n}}(Q_R)}\right)\\
&+\ c_4(n,p,R_0)\|f-f_\varepsilon\|_{L^{\frac{np+2p}{np+2p-n}}(Q_{R_0})}\left(\|Du\|^p_{L^p(Q_{R_0})}+\lambda^p+\|f-f_\varepsilon\|^{\frac{np+2p}{np+p-n}}_{L^{\frac{np+2p}{np+2p-n}}(Q_{R_0})}\right)^{\frac{1}{n+2}}\\
&\cdot\left(\|Du\|_{L^p(Q_{R_0})}+\lambda+\|f-f_\varepsilon\|^{\frac{n+2}{np+p-n}}_{L^{\frac{np+2p}{np+2p-n}}(Q_{R_0})}\right)^{\frac{n}{n+2}},
\end{aligned}
$$

which holds for every $\varepsilon\in(0,1]$ and every sufficiently small $h\in\mathbb{R}\setminus\{0\}$. Therefore, recalling (4.13) and letting $\varepsilon\to 0$ in the above estimate, by virtue of (4.53) we obtain

$$
\begin{aligned}
\int_{Q_{\varrho/2}}|\Delta_{s,h}H_{\frac{p}{2}}(Du)|^2\,dz\ \le\ &c_4\left(\lambda\,\|Df\|_{L^{\frac{np+4}{np+4-n}}(Q_{R_0})}+\|Df\|^{\frac{np+4}{np+2-n}}_{L^{\frac{np+4}{np+4-n}}(Q_{R_0})}\right)\\
&+\ \frac{c_4}{R^2}\left(\|Du\|^p_{L^p(Q_{R_0})}+\|Du\|^2_{L^p(Q_{R_0})}+\lambda^p+\lambda^2\right),
\end{aligned}
\tag{4.56}
$$

which holds for every $s\in\{1,\dots,n\}$ and every sufficiently small $h\in\mathbb{R}\setminus\{0\}$. This proves the desired result for $p>2$. Moreover, letting $h\to 0$ in the preceding inequality, we also obtain estimate (4.2).

Finally, when $p=2$, arguing in a similar fashion, we can reach the same conclusions. □

4.5 Proof of Theorem 4.2

This section is devoted to the study of the existence and regularity of the time derivative of the weak solutions to equation (4.1), under the assumptions of Theorem 4.1. Indeed, we are now in a position to give the proof of Theorem 4.2.

Proof of Theorem 4.2. We shall keep the notation and the parabolic cylinders used for the proof of Theorem 4.1. Let us first show that

$$
H_{p-1}(Du)\ \in\ L^{p'}_{loc}\left(0,T;W^{1,p'}_{loc}(\Omega,\mathbb{R}^n)\right).
\tag{4.57}
$$

Observe that, for $p=2$, we have $H_{p-1}(Du)=H_{p/2}(Du)$, so that assertion (4.57) immediately follows from Theorem 4.1 in this case. Therefore, from now on we will assume that $p>2$.

For every $\xi \in \mathbb{R}^n$ we have

$$H_{p-1}(\xi) \,=\, \mathcal{F}(H_{\frac{p}{2}}(\xi))\,,$$

where $\mathcal{F}:\mathbb{R}^n \to \mathbb{R}^n$ is the function defined by

$$\mathcal{F}(\eta) := |\eta|^{\frac{p-2}{p}}\eta\,,$$

which is locally Lipschitz continuous for $p > 2$. Thus, the function $H_{p-1}(Du)$ is weakly differentiable with respect to the x-variable by virtue of the chain rule in Sobolev spaces. From the definitions of $\mathcal{F}$ and $H_{p/2}$, it follows that

$$|D_\eta\, \mathcal{F}(H_{\frac{p}{2}}(Du))| \,\le\, c_1\, |H_{\frac{p}{2}}(Du)|^{\frac{p-2}{p}} \,\le\, c_1\, |Du|^{\frac{p-2}{2}} \tag{4.58}$$

for some positive constant $c_1 = c(n,p)$. Now, applying the chain rule, the Cauchy-Schwarz inequality and estimate (4.58), we obtain

$$\begin{aligned} |D_x\, H_{p-1}(Du)|^{p'} &\le\, c_2(n,p)\, |D_\eta\, \mathcal{F}(H_{\frac{p}{2}}(Du))|^{p'}\, |D_x\, H_{\frac{p}{2}}(Du)|^{p'} \\ &\le\, c_3\, |Du|^{\frac{(p-2)p'}{2}}\, |D_x\, H_{\frac{p}{2}}(Du)|^{p'}, \end{aligned} \tag{4.59}$$

where $c_3 = c_3(n,p) > 0$. Using (4.59), Hölder's inequality with exponents $\left(\frac{2(p-1)}{p-2}, \frac{2}{p'}\right)$ and estimate (4.2), we get

$$\begin{aligned} &\left(\int_{Q_{\varrho/2}} |DH_{p-1}(Du)|^{p'}\,dz\right)^{\frac{1}{p'}} \\ &\le\, c_4(n,p)\, \|Du\|^{\frac{p-2}{2}}_{L^p(Q_{\varrho/2})} \left(\int_{Q_{\varrho/2}} |DH_{\frac{p}{2}}(Du)|^2\,dz\right)^{\frac{1}{2}} \\ &\le\, c_5\, \|Du\|^{\frac{p-2}{2}}_{L^p(Q_{R_0})} \left[\lambda\, \|Df\|_{L^\vartheta(Q_{R_0})} \,+\, \|Df\|^{\frac{np+4}{np+2-n}}_{L^\vartheta(Q_{R_0})}\right]^{\frac{1}{2}} \\ &\quad +\, \frac{c_5}{R}\left[\|Du\|^{2p-2}_{L^p(Q_{R_0})} \,+\, \|Du\|^{p}_{L^p(Q_{R_0})} + (\lambda^p + \lambda^2)\, \|Du\|^{p-2}_{L^p(Q_{R_0})}\right]^{\frac{1}{2}}, \end{aligned} \tag{4.60}$$

where c_5 is a positive constant depending on n, p, ϑ and R_0. Note that the right-hand side of (4.60) is finite, and this implies (4.57).

Now, let $\kappa = \min\{\vartheta, p'\}$. Going back to the weak formulation (1.14), thanks to (4.57) we can perform a partial integration in the second term on the left-hand side with respect to the spatial variables. We thus obtain

$$\int_{Q_{\varrho/2}} u\cdot\partial_t\varphi\; dz \,=\, -\int_{Q_{\varrho/2}} \left(\sum_{i=1}^n D_{x_i}[(H_{p-1}(Du))_i] \,+ f\right)\cdot\varphi\; dz$$

for any $\varphi \in C^\infty_0(Q_{\varrho/2})$, and the desired conclusion immediately follows from (4.2) if $p = 2$, and from (4.60) if $p > 2$, since

$$f \in L^\vartheta\left(0,T;W^{1,\vartheta}(\Omega)\right) \qquad \text{with} \quad \frac{np+4}{np+4-n} \,\le\, \vartheta \,<\infty$$

and

$$\frac{np+4}{np+4-n} = \left(p+\frac{4}{n}\right)' < p' \qquad \text{for every } p \geq 2.$$

Furthermore, we can now observe that

$$\partial_t u = \sum_{i=1}^{n} D_{x_i}[(H_{p-1}(Du))_i] + f \qquad \text{in } Q_{\varrho/2}(z_0).$$

From this we can infer

$$\begin{aligned}\left(\int_{Q_{\varrho/2}} |\partial_t u|^\kappa \, dz\right)^{\frac{1}{\kappa}} &\leq n \, \|DH_{p-1}(Du)\|_{L^\kappa(Q_{\varrho/2})} + \|f\|_{L^\kappa(Q_{\varrho/2})} \\ &\leq c_6 \, \|DH_{p-1}(Du)\|_{L^{p'}(Q_{\varrho/2})} + c_6 \, \|f\|_{L^\vartheta(Q_{\varrho/2})}, \end{aligned} \tag{4.61}$$

where c_6 is a positive constant depending on n, p, ϑ and R_0. Thus, we can now deduce estimate (4.3) from (4.61) and (4.2) if $p = 2$, and by combining (4.60) and (4.61) when $p > 2$. □

4.6 Proofs of Theorem 4.4 and Corollary 4.5

We will now use the well-known difference quotients method in the spatial directions, as well as the properties of the function H_{p-1} and Proposition 1.41, to prove Theorem 4.4.

Proof of Theorem 4.4. Again, by a slight abuse of notation, for $w \in L^1_{loc}(\Omega_T, \mathbb{R}^N)$, $i \in \{1, \dots, n\}$ and $h \neq 0$, we set (when $x + he_i \in \Omega$)

$$\tau_h w(x,t) \equiv \tau_{i,h} w(x,t) := w(x+he_i,t) - w(x,t),$$

$$\Delta_h w(x,t) \equiv \Delta_{i,h} w(x,t) := \frac{w(x+he_i,t) - w(x,t)}{h},$$

where e_i is the unit vector in the direction x_i.

Since u is a weak solution of equation (4.1), we have

$$\int_{\Omega_T} (u \cdot \partial_t \varphi - \langle H_{p-1}(Du), D\varphi \rangle) \, dz = -\int_{\Omega_T} f\varphi \, dz$$

for every test function $\varphi \in C_0^\infty(\Omega_T)$. Replacing φ by $\tau_{-h}\varphi$, where $0 < |h| < \operatorname{dist}(\operatorname{spt}\varphi, \partial\Omega_T)$, by virtue of the properties of the finite difference operator, we get

$$\int_{\Omega_T} (\tau_h u \cdot \partial_t \varphi - \langle \tau_h H_{p-1}(Du), D\varphi \rangle) \, dz = -\int_{\Omega_T} \tau_h f \cdot \varphi \, dz.$$

We now replace φ by $\varphi_\varepsilon \equiv \phi_\varepsilon * \varphi$ in the previous equation, where $\{\phi_\varepsilon\}$, $\varepsilon > 0$, denotes the family of standard, non-negative, radially symmetric mollifiers in $\mathbb{R}^{n+1}$. This yields, for $0 < \varepsilon \ll 1$,

$$\int_{\Omega_T} ((\tau_h u)_\varepsilon \cdot \partial_t \varphi - \langle (\tau_h H_{p-1}(Du))_\varepsilon, D\varphi \rangle) \, dz = -\int_{\Omega_T} (\tau_h f)_\varepsilon \cdot \varphi \, dz.$$

Now, in the last equation we choose the test function $\varphi \equiv \Phi(\tau_h u)_\varepsilon$, where $\Phi \in C_0^\infty(\Omega_T)$ is a cut-off function which will be specified later. After an integration by parts and then letting $\varepsilon \searrow 0$, we obtain

$$\begin{aligned} &-\frac{1}{2}\int_{\Omega_T} |\tau_h u|^2\, \partial_t \Phi \; dz + \int_{\Omega_T} \Phi \langle \tau_h H_{p-1}(Du), D\tau_h u\rangle \, dz \\ &\quad = -\int_{\Omega_T} \langle \tau_h H_{p-1}(Du), D\Phi\rangle \tau_h u \; dz + \int_{\Omega_T} \tau_h f \cdot \Phi \cdot \tau_h u \; dz\,. \end{aligned} \tag{4.62}$$

Note that an approximation argument yields the same identity for any $\Phi \in W^{1,\infty}(\Omega_T)$ with compact support in Ω_T and any sufficiently small $h \in \mathbb{R} \setminus \{0\}$. In what follows, we will denote by c_k some positive constants which do not depend on h.

Now, let us consider a parabolic cylinder

$$Q_\varrho(z_0) \subset Q_{2\varrho}(z_0) \Subset \Omega_T.$$

For a fixed time $t_1 \in (t_0 - \varrho^2, t_0)$ and $\delta \in (0, t_0 - t_1)$, we choose

$$\Phi(x,t) \;=\; \widetilde{\chi}(t)\, \chi(t)\, \eta^2(x)$$

with $\chi \in W^{1,\infty}((0,T),[0,1])$, $\chi \equiv 0$ on $(0, t_0 - \varrho^2)$ and $\partial_t \chi \geq 0$, $\eta \in C_0^\infty(B_\varrho(x_0),[0,1])$, and with the Lipschitz continuous function $\widetilde{\chi} : (0,T) \to \mathbb{R}$ defined by

$$\widetilde{\chi}(t) = \begin{cases} 1 & \text{if } t \leq t_1, \\ \text{affine} & \text{if } t_1 < t < t_1 + \delta, \\ 0 & \text{if } t \geq t_1 + \delta. \end{cases}$$

With such a choice of Φ, equation (4.62) turns into

$$\begin{aligned} &-\frac{1}{2}\int_{\Omega_T} |\tau_h u|^2\, \eta^2(x) \chi(t) \partial_t \widetilde{\chi}(t) \; dz - \frac{1}{2}\int_{\Omega_T} |\tau_h u|^2\, \eta^2(x) \widetilde{\chi}(t) \partial_t \chi(t) \; dz \\ &\quad + \int_{\Omega_T} \widetilde{\chi}(t)\chi(t)\eta^2(x) \langle \tau_h H_{p-1}(Du), D\tau_h u\rangle \, dz \\ &\qquad = -2\int_{\Omega_T} \widetilde{\chi}(t)\chi(t)\eta(x) \langle \tau_h H_{p-1}(Du), D\eta\rangle \tau_h u \; dz + \int_{\Omega_T} (\tau_h f)(\tau_h u) \widetilde{\chi}(t)\chi(t)\eta^2(x)\, dz\,. \end{aligned}$$

Setting

$$Q^{t_1} := B_\varrho(x_0) \times (t_0 - \varrho^2, t_1)$$

and letting $\delta \to 0$ in the previous equality, we get

$$\begin{aligned} &\frac{1}{2}\int_{B_\varrho(x_0)} \chi(t_1)\eta^2(x)\, |\tau_h u(x,t_1)|^2 \, dx + \int_{Q^{t_1}} \chi(t)\eta^2(x) \langle \tau_h H_{p-1}(Du), D\tau_h u\rangle \, dz \\ &\quad = -2\int_{Q^{t_1}} \chi(t)\eta(x) \langle \tau_h H_{p-1}(Du), D\eta\rangle \tau_h u \; dz + \int_{Q^{t_1}} (\tau_h f)(\tau_h u)\chi(t)\eta^2(x)\, dz \\ &\qquad + \frac{1}{2}\int_{Q^{t_1}} (\partial_t \chi)\eta^2\, |\tau_h u|^2 \, dz \end{aligned} \tag{4.63}$$

for every $h \in \mathbb{R} \setminus \{0\}$ such that $|h| < \varrho/4$ and for almost every $t_1 \in (t_0 - \varrho^2, t_0)$.

Now, by Lemma 1.6, we have

$$\frac{4}{p^2}\int_{Q^{t_1}}\chi(t)\eta^2(x)\,|\tau_h H_{\frac{p}{2}}(Du)|^2\,dz\;\le\;\int_{Q^{t_1}}\chi(t)\eta^2(x)\langle\tau_h H_{p-1}(Du),D\tau_h u\rangle\,dz \tag{4.64}$$

and

$$\begin{aligned}&\left|2\int_{Q^{t_1}}\chi(t)\eta(x)\langle\tau_h H_{p-1}(Du),D\eta\rangle\tau_h u\;dz\right|\\&\le 2(p-1)\int_{Q^{t_1}}\chi(t)\eta(x)\left(|H_{\frac{p}{2}}(Du(x+he_i,t))|^{\frac{p-2}{p}}\;+\;|H_{\frac{p}{2}}(Du)|^{\frac{p-2}{p}}\right)|\tau_h H_{\frac{p}{2}}(Du)|\,|D\eta|\,|\tau_h u|\,dz.\end{aligned}$$

Using Young's inequality with exponents $(2,2)$ in the right-hand side of the previous estimate, we obtain

$$\begin{aligned}&\left|2\int_{Q^{t_1}}\chi(t)\eta(x)\langle\tau_h H_{p-1}(Du),D\eta\rangle\tau_h u\;dz\right|\\&\quad\le\;\frac{2(p-1)^2}{\sigma}\int_{Q^{t_1}}\chi(t)\,|D\eta|^2\left(|H_{\frac{p}{2}}(Du(x+he_i,t))|^{\frac{p-2}{p}}\;+\;|H_{\frac{p}{2}}(Du)|^{\frac{p-2}{p}}\right)^2|\tau_h u|^2\,dz\\&\qquad+\;\frac{\sigma}{2}\int_{Q^{t_1}}\chi(t)\eta^2(x)\,|\tau_h H_{\frac{p}{2}}(Du)|^2\,dz,\end{aligned}\tag{4.65}$$

where $\sigma>0$ will be chosen later. Moreover, by Hölder's inequality with exponents p and p', as well as by the properties of the difference quotients, of χ, η, u and f, we find that

$$\begin{aligned}&\left|\int_{Q^{t_1}}(\tau_h f)(\tau_h u)\chi(t)\eta^2(x)\,dz\right|\\&\quad\le\;c_1(n)\;|h|\;\|Du\|_{L^p(Q_{2\varrho})}\left(\int_{Q^{t_1}}|\tau_h f|^{p'}\,dz\right)^{\frac{1}{p'}}\\&\quad=\;c_1\;|h|^{\alpha+1}\;\|Du\|_{L^p(Q_{2\varrho})}\left(\int_{Q^{t_1}}\frac{|\tau_h f|^{p'}}{|h|^{\alpha p'}}\,dz\right)^{\frac{1}{p'}}\\&\quad\le\;c_1\;|h|^{\alpha+1}\;\|Du\|_{L^p(Q_{2\varrho})}\\&\qquad\cdot\left[\int_{t_0-4\varrho^2}^{t_0}\left(\sup_{|y|<\frac{\varrho}{4}}\int_{B_{2\varrho}(x_0)}\frac{|f(x+y,t)-f(x,t)|^{p'}}{|y|^{\alpha p'}}\cdot\mathbb{1}_{B_{2\varrho}(x_0)}(x+y)\,dx\right)dt\right]^{\frac{1}{p'}}\\&\quad\le\;c_2(n,p)\;|h|^{\alpha+1}\;\|Du\|_{L^p(Q_{2\varrho})}\left(\int_{t_0-4\varrho^2}^{t_0}\|f(\cdot,t)\|^{p'}_{B^{\alpha}_{p',\infty}(B_{2\varrho})}\,dt\right)^{\frac{1}{p'}}\\&\quad=\;c_2\;|h|^{\alpha+1}\;\|Du\|_{L^p(Q_{2\varrho})}\,\|f\|_{L^{p'}\left(t_0-4\varrho^2,t_0\,;\,B^{\alpha}_{p',\infty}(B_{2\varrho}(x_0))\right)}\end{aligned}\tag{4.66}$$

for every $h\in\mathbb{R}\setminus\{0\}$ such that $|h|<\varrho/4$. Joining estimates (4.63)−(4.66) and choosing $\sigma=4/p^2$, we arrive at

$$\begin{aligned}&\int_{B_\varrho(x_0)}\chi(t_1)\eta^2(x)\,|\tau_h u(x,t_1)|^2\,dx\;+\;\int_{Q^{t_1}}\chi(t)\eta^2(x)\,|\tau_h H_{\frac{p}{2}}(Du)|^2\,dz\\&\quad\le\;c_3(p)\int_{Q^{t_1}}\left[\chi(t)\,|D\eta|^2\left(|H_{\frac{p}{2}}(Du(x+he_i,t))|^{\frac{p-2}{p}}\;+\;|H_{\frac{p}{2}}(Du)|^{\frac{p-2}{p}}\right)^2+(\partial_t\chi)\eta^2\right]|\tau_h u|^2\,dz\end{aligned}$$

$$+\ c_4(n,p)\ |h|^{\alpha+1}\ \|Du\|_{L^p(Q_{2\varrho})}\,\|f\|_{L^{p'}\left(t_0-4\varrho^2,t_0\,;\,B^{\alpha}_{p',\infty}(B_{2\varrho}(x_0))\right)},$$

which holds for almost every $t_1 \in (t_0-\varrho^2, t_0)$.

We now choose a cut-off function $\eta \in C_0^\infty(B_\varrho(x_0))$ with $\eta \equiv 1$ on $B_{\varrho/2}(x_0)$ such that $0 \le \eta \le 1$ and $|D\eta| \le C/\varrho$. For the cut-off function in time, we choose the piecewise affine function $\chi : (0,T) \to [0,1]$ with

$$\begin{aligned} \chi &\equiv 0 \quad && \text{on } \ (0, t_0-\varrho^2)\,, \\ \chi &\equiv 1 \quad && \text{on } \ \left(t_0-\left(\tfrac{\varrho}{2}\right)^2, T\right), \\ \partial_t\chi &\equiv \tfrac{4}{3\varrho^2} \quad && \text{on } \ \left(t_0-\varrho^2, t_0-\left(\tfrac{\varrho}{2}\right)^2\right). \end{aligned}$$

Dividing both sides of the previous estimate by $|h|$ and using the properties of χ and η, we obtain

$$\begin{aligned} &\sup_{t\in(t_0-(\varrho/2)^2,t_0)}\int_{B_{\varrho/2}}\left|\frac{\tau_{i,h}u(x,t)}{|h|^{1/2}}\right|^2 dx\ +\int_{Q_{\varrho/2}}\left|\frac{\tau_{i,h}H_{p/2}(Du)}{|h|^{1/2}}\right|^2 dz \\ &\quad\le\ c_5(p)\ |h|\ \varrho^{-2}\int_{Q_\varrho}\left[\left(|H_{\frac{p}{2}}(Du(x+he_i,t))|^{\frac{p-2}{p}}\ +\ |H_{\frac{p}{2}}(Du)|^{\frac{p-2}{p}}\right)^2+1\right]|\Delta_h u|^2\,dz \\ &\qquad+\ c_4(n,p)\ |h|^{\alpha}\ \|Du\|_{L^p(Q_{2\varrho})}\,\|f\|_{L^{p'}\left(t_0-4\varrho^2,t_0\,;\,B^{\alpha}_{p',\infty}(B_{2\varrho})\right)}. \end{aligned} \tag{4.67}$$

Now we set

$$I := \int_{Q_\varrho}\left[\left(|H_{\frac{p}{2}}(Du(x+he_i,t))|^{\frac{p-2}{p}}\ +\ |H_{\frac{p}{2}}(Du)|^{\frac{p-2}{p}}\right)^2+1\right]|\Delta_h u|^2\,dz$$

and we will assume that $p>2$. By Hölder's inequality with exponents $\left(\frac{p}{2}, \frac{p}{p-2}\right)$, as well as by the properties of the difference quotients, we can control I as follows:

$$\begin{aligned} I\ &\le\ c_6(n)\left(\int_{Q_{2\varrho}}|Du|^p\,dz\right)^{\frac{2}{p}}\left(\int_{Q_{2\varrho}}\left[|H_{\frac{p}{2}}(Du)|^{\frac{2(p-2)}{p}}\ +\ 1\right]^{\frac{p}{p-2}}dz\right)^{\frac{p-2}{p}} \\ &\le\ c_7(n,p)\left(\int_{Q_{2\varrho}}|Du|^p\,dz\right)^{\frac{2}{p}}\left(\int_{Q_{2\varrho}}(|Du|^p+1)\,dz\right)^{\frac{p-2}{p}} \\ &\le\ c_7(n,p)\int_{Q_{2\varrho}}(|Du|^p+1)\ dz, \end{aligned} \tag{4.68}$$

provided that $|h| < \varrho/4$. Combining estimates (4.67) and (4.68), we then have

$$\begin{aligned} &\sup_{t\in(t_0-(\varrho/2)^2,t_0)}\int_{B_{\varrho/2}}\left|\frac{\tau_{i,h}u(x,t)}{|h|^{(\alpha+1)/2}}\right|^2 dx\ +\int_{Q_{\varrho/2}}\left|\frac{\tau_{i,h}H_{p/2}(Du)}{|h|^{(\alpha+1)/2}}\right|^2 dz \\ &\quad\le\ c\ |h|^{1-\alpha}\ \varrho^{-2}\int_{Q_{2\varrho}}(|Du|^p+1)\ dz\ +\ c\ \|Du\|_{L^p(Q_{2\varrho})}\,\|f\|_{L^{p'}\left(t_0-4\varrho^2,t_0\,;\,B^{\alpha}_{p',\infty}(B_{2\varrho})\right)}, \end{aligned} \tag{4.69}$$

with $c = c(n,p) > 0$. Since the previous estimate holds for every $h \in \mathbb{R}\setminus\{0\}$ such that

$|h| < \varrho/4$, from (4.69) we obtain

$$\int_{Q_{\varrho/2}} \left| \frac{\tau_{i,h} H_{p/2}(Du)}{|h|^{(\alpha+1)/2}} \right|^2 dz \,\le\, 4^{\alpha-1}\, c\, \varrho^{-1-\alpha} \int_{Q_{2\varrho}} (|Du|^p + 1)\, dz$$

$$+\, c\, \|Du\|_{L^p(Q_{2\varrho})}\, \|f\|_{L^{p'}\left(t_0-4\varrho^2,\, t_0\,;\, B^{\alpha}_{p',\infty}(B_{2\varrho})\right)}.$$

From this we can deduce

$$\int_{Q_{\varrho/2}} |\tau_{i,h} H_{\frac{p}{2}}(Du)|^2\, dz \,\le\, M\, |h|^{\alpha+1}$$

for some finite positive constant M depending on

$$n, \quad p, \quad \varrho, \quad \alpha, \quad \|Du\|_{L^p(\Omega_T)}\,, \quad \|f\|_{L^{p'}\left(t_0-4\varrho^2,\, t_0\,;\, B^{\alpha}_{p',\infty}(B_{2\varrho})\right)},$$

but not on h. Note that the above estimate holds for every $i \in \{1, \dots, n\}$ and every sufficiently small $h \in \mathbb{R} \setminus \{0\}$. Therefore, using Proposition 1.41 with the choices $g = H_{p/2}(Du)$, $q = 2$ and $\theta = \frac{\alpha+1}{2}$, as well as a standard covering argument, we infer that

$$H_{\frac{p}{2}}(Du) \,\in\, L^2_{loc}\left(0, T; W^{s,2}_{loc}(\Omega, \mathbb{R}^n)\right) \quad \text{ for all } s \in \left(0, \frac{\alpha+1}{2}\right).$$

Finally, when $p = 2$, arguing in a similar fashion, we reach the same conclusions. This completes the proof. □

We are now in a position to give the

Proof of Corollary 4.5. By virtue of Theorems 4.4, 1.36 and 1.37, for any fixed $s \in \left(0, \frac{\alpha+1}{2}\right)$ we have

$$H_{\frac{p}{2}}(Du) \,\in\, L^2\left(t_0, t_1; L^{\gamma}(\Omega', \mathbb{R}^n)\right) \qquad \text{for every } \gamma \in \left[1, \frac{2n}{n-2s}\right]$$

for any open Lipschitz set $\Omega' \Subset \Omega$ and any $(t_0, t_1) \subset (0, T)$. Now, let us observe that the function $g_n : \left(0, \frac{\alpha+1}{2}\right) \to \mathbb{R}$ defined by

$$g_n(s) = \frac{2n}{n-2s}$$

is continuous and strictly increasing, and $g_n(s) \nearrow \frac{2n}{n-\alpha-1}$ as $s \nearrow \frac{\alpha+1}{2}$. Hence, letting s tend to $\frac{\alpha+1}{2}$ from below, we obtain that

$$H_{\frac{p}{2}}(Du) \,\in\, L^2_{loc}\left(0, T; L^r_{loc}(\Omega, \mathbb{R}^n)\right) \qquad \text{for all } r \in \left[1, \frac{2n}{n-\alpha-1}\right),$$

since every open ball $B_\varrho \Subset \Omega$ is a set of class $C^{0,1}$ with bounded boundary. In particular, for all Lipschitz open subsets $\Omega' \Subset \Omega$ and all $(t_0, t_1) \subset (0, T)$ we have

$$\int_{t_0}^{t_1} \left(\int_{\Omega'} (|Du(x,t)| - \lambda)_+^{rp/2}\, dx \right)^{\frac{2}{r}} dt \,=\, \int_{t_0}^{t_1} \|H_{\frac{p}{2}}(Du(\cdot, t))\|^2_{L^r(\Omega')}\, dt \,< +\infty \tag{4.70}$$

for any $r \in \left[1, \frac{2n}{n-\alpha-1}\right)$. Now notice that

$$1 \le r < \frac{2n}{n-\alpha-1} \quad \text{implies} \quad 1 \le \frac{p}{2} \le \frac{rp}{2} < \frac{np}{n-\alpha-1},$$

since $p \ge 2$. Therefore, from (4.70) it follows that

$$\int_{t_0}^{t_1} \|(|Du(\cdot,t)| - \lambda)_+\|^p_{L^q(\Omega')}\, dt = \int_{t_0}^{t_1} \left(\int_{\Omega'} (|Du(x,t)| - \lambda)_+^q \, dx \right)^{\frac{p}{q}} dt < +\infty$$

for all $q \in \left[1, \frac{np}{n-\alpha-1}\right)$, for any open Lipschitz set $\Omega' \Subset \Omega$ and any $(t_0, t_1) \subset (0, T)$. This is sufficient to ensure that

$$Du \in L^p_{loc}(0,T; L^q_{loc}(\Omega, \mathbb{R}^n)) \qquad \text{for all } q \in \left[1, \frac{np}{n-\alpha-1}\right). \qquad \square$$

Chapter 5

Sharp Sobolev regularity for widely degenerate parabolic equations

In this chapter, we are interested in local weak solutions $u : \Omega_T \to \mathbb{R}$ to the strongly degenerate parabolic equation

$$\partial_t u - \operatorname{div}\left((|Du| - \lambda)_+^{p-1} \frac{Du}{|Du|} \right) = f \quad \text{in } \Omega_T = \Omega \times (0,T), \tag{5.1}$$

where $2 \le p < \infty$, Ω is a bounded domain in $\mathbb{R}^n$ ($n \ge 2$), $T > 0$ and $\lambda \ge 0$. The results that we report here have been obtained in [6]. As our main achievement, we will establish the Sobolev spatial regularity of a *novel* nonlinear function of the spatial gradient of the weak solutions to equation (5.1), by assuming that

- $f \in L^{p'}_{loc}\left(0,T; B^{\frac{p-2}{p}}_{p',1,loc}(\Omega)\right)$ if $p > 2$ (see Theorem 5.1);
- $f \in L^2_{loc}(\Omega_T)$ if $p = 2$ (see Theorem 5.3).

The above-mentioned theorems, in turn, imply the Sobolev time regularity of the weak solutions to the evolutionary p-Poisson equation, under the same hypotheses on the function f (cf. Theorem 5.5, where we only address the case $p > 2$, since the Sobolev time regularity is well known for the heat equation with source term in $L^2_{loc}(\Omega_T)$).

Actually, our aim here is to extend the elliptic results contained in Chapter 3 to the parabolic setting, in the case $p \ge 2$. The first main result of this chapter is the following theorem, which can be considered as the parabolic counterpart of Theorem 3.1. We refer to Chapter 1 for notation and definitions.

Theorem 5.1. *Let $n \ge 2$, $p > 2$, $\lambda \ge 0$ and $f \in L^{p'}_{loc}\left(0,T; B^{\frac{p-2}{p}}_{p',1,loc}(\Omega)\right)$. Moreover, let*

$$\alpha = \begin{cases} 0 & \text{if } \lambda = 0, \\ \text{any value in } \left[\frac{p+1}{2(p-1)}, \infty\right) & \text{if } \lambda > 0, \end{cases} \tag{5.2}$$

and assume that

$$u \in C^0\left(0,T; L^2(\Omega)\right) \cap L^p_{loc}\left(0,T; W^{1,p}_{loc}(\Omega)\right)$$

is a local weak solution of equation (5.1). *Then*

$$\mathcal{V}_{\alpha,\lambda}(Du) \in L^2_{loc}\left(0,T;W^{1,2}_{loc}(\Omega,\mathbb{R}^n)\right),$$

where the function $\mathcal{V}_{\alpha,\lambda}$ *is defined according to* (3.2)−(3.3). *Furthermore, for any parabolic cylinder* $Q_r(z_0) \subset Q_\rho(z_0) \subset Q_R(z_0) \Subset \Omega_T$ *we have*

$$\begin{aligned}\int_{Q_{r/2}(z_0)} |D_x \mathcal{V}_{\alpha,\lambda}(Du)|^2\,dz \le \left(C+\frac{C}{\rho^2}\right)\left[\|Du\|^p_{L^p(Q_R)}+\|Du\|^2_{L^p(Q_R)}+\lambda^p+\lambda^2+1\right]\\ + C\,\|f\|^{p'}_{L^{p'}\left(t_0-R^2,t_0;B^{\frac{p-2}{p}}_{p',1}(B_R(x_0))\right)}\end{aligned} \tag{5.3}$$

for a positive constant C *depending only on* n, p *and* R *in the case* $\lambda = 0$, *and additionally on* α *if* $\lambda > 0$. *Besides, if* $\lambda = 0$ *we get*

$$Du \in L^p_{loc}\left(0,T;W^{\sigma,p}_{loc}(\Omega,\mathbb{R}^n)\right) \qquad \text{for all } \sigma \in \left(0,\frac{2}{p}\right).$$

Remark 5.2. As already noted in Remark 3.2, for every $\alpha \ge 0$ we have

$$\mathcal{V}_{\alpha,0}(\xi) = \frac{2}{p}\,\mathbb{V}_p(\xi) := \frac{2}{p}\,|\xi|^{\frac{p-2}{2}}\xi\,, \tag{5.4}$$

i.e. $\mathcal{V}_{\alpha,0}$ is actually independent of the parameter α. This explains the choice $(5.2)_1$ in the statement of the above theorem. The condition $(5.2)_2$ will instead be decisive to carry out the proof of Proposition 5.12 below. Moreover, looking at (5.4), one can easily understand that, on the one hand, Theorem 5.1 extends the results proved in [41] for the parabolic p-Laplace equation to a widely degenerate parabolic setting. On the other hand, it extends the aforementioned results to the case of data in a suitable Lebesgue-Besov parabolic space, which turns out to be optimal, as can be seen by appropriately adapting the example in [23, Section 5] to the parabolic context (in this regard, see also the Introduction).

Now we turn our attention to the case $p = 2$. It is well known that for the non-homogeneous heat equation

$$\partial_t u - \operatorname{div} Du = f \quad \text{in } \Omega_T, \tag{5.5}$$

the spatial $W^{1,2}$-regularity of the weak spatial gradient Du can be achieved by simply assuming that $f \in L^2_{loc}(\Omega_T)$. Here, we prove that a similar result holds even when dealing with much more degenerate equations. More precisely, we establish the following result.

Theorem 5.3. *Let* $n \ge 2$, $\lambda \ge 0$ *and* $f \in L^2_{loc}(\Omega_T)$. *Moreover, let*

$$\alpha = \begin{cases} 0 & \text{if } \lambda = 0,\\ \text{any value in } \left[\frac{3}{2},\infty\right) & \text{if } \lambda > 0,\end{cases}$$

and assume that

$$u \in C^0\left(0,T;L^2(\Omega)\right) \cap L^2_{loc}\left(0,T;W^{1,2}_{loc}(\Omega)\right)$$

is a local weak solution of the equation

$$\partial_t u - \operatorname{div}\left((|Du| - \lambda)_+ \frac{Du}{|Du|} \right) = f \quad \text{in } \Omega_T.$$

Then

$$\mathcal{V}_{\alpha,\lambda}(Du) \in L^2_{loc}\left(0,T; W^{1,2}_{loc}(\Omega, \mathbb{R}^n)\right).$$

Furthermore, for any parabolic cylinder $Q_r(z_0) \subset Q_\rho(z_0) \subset Q_R(z_0) \Subset \Omega_T$ we have

$$\int_{Q_{r/2}(z_0)} |D_x \mathcal{V}_{\alpha,\lambda}(Du)|^2 \, dz \le \frac{C}{\rho^2}\left(\|Du\|^2_{L^2(Q_R)} + \lambda^2 + 1 \right) + C\left(\|f\|^2_{L^2(Q_R)} + 1 \right)$$

for a positive constant C depending only on n and R in the case $\lambda = 0$, and additionally on α if $\lambda > 0$.

Remark 5.4. In the case $p = 2$, we have $\mathcal{V}_{\alpha,0}(Du) = Du$. Then, for $\lambda = 0$, it follows from Theorem 5.3 that

$$Du \in L^2_{loc}\left(0,T; W^{1,2}_{loc}(\Omega, \mathbb{R}^n)\right).$$

As already mentioned, this is a well-established result for equation (5.5) when $f \in L^2_{loc}(\Omega_T)$.

For $p > 2$, we now consider the evolutionary p-Poisson equation

$$\partial_t u - \operatorname{div}\left(|Du|^{p-2} Du\right) = f \quad \text{in } \Omega_T, \tag{5.6}$$

which is obtained from equation (5.1) by setting $\lambda = 0$. As we anticipated earlier, from Theorem 5.1 we can easily deduce that the weak solutions of (5.6) admit a weak time derivative which belongs to the local Lebesgue space $L^{p'}_{loc}(\Omega_T)$. The idea is essentially the same as the proof of Theorem 4.2 and leads us to the following result.

Theorem 5.5. *Let $n \ge 2$, $p > 2$ and $f \in L^{p'}_{loc}\left(0,T; B^{\frac{p-2}{p}}_{p',1,loc}(\Omega)\right)$. Moreover, assume that*

$$u \in C^0\left(0,T; L^2(\Omega)\right) \cap L^p_{loc}\left(0,T; W^{1,p}_{loc}(\Omega)\right)$$

is a local weak solution of equation (5.6). Then, the time derivative of u exists in the weak sense and satisfies

$$\partial_t u \in L^{p'}_{loc}(\Omega_T).$$

Furthermore, for any parabolic cylinder $Q_r(z_0) \subset Q_\rho(z_0) \subset Q_R(z_0) \Subset \Omega_T$ we have

$$\begin{aligned}\left(\int_{Q_{r/2}(z_0)} |\partial_t u|^{p'} \, dz \right)^{\frac{1}{p'}} \le{}& \left(C + \frac{C}{\rho} \right) \left[\|Du\|^{p-1}_{L^p(Q_R)} + \|Du\|^{\frac{p}{2}}_{L^p(Q_R)} + \|Du\|^{\frac{p-2}{2}}_{L^p(Q_R)} \right] \\ &+ C \, \|Du\|^{\frac{p-2}{2}}_{L^p(Q_R)} \, \|f\|^{\frac{p'}{2}}_{L^{p'}\left(t_0 - R^2, t_0; B^{\frac{p-2}{p}}_{p',1}(B_R(x_0))\right)} + \|f\|_{L^{p'}(Q_R)}\end{aligned} \tag{5.7}$$

for a positive constant C depending only on n, p and R.

Remark 5.6. It is worth pointing out that, starting from the weaker assumption

$$f \in L^{p'}_{loc}\left(0,T; B^{\frac{p-2}{p}}_{p',1,loc}(\Omega)\right) \qquad \text{with } p>2,$$

Sobolev regularity results such as those of Theorems 5.1 and 5.5 seemed not to have been established yet for weak solutions to parabolic PDEs that are far less degenerate than equation (5.1) with $\lambda > 0$. In particular, the results of Theorems 5.1 and 5.5 permit to improve the existing literature, already for the evolutionary p-Poisson equation (5.6), which exhibits a milder degeneracy. Moreover, as far as we know, for $\lambda > 0$ the result of Theorem 5.3 is completely new.

This chapter is organized as follows. In Section 5.1, we consider a regularization of equation (5.1) and establish some *a priori* estimates for its weak solution. In particular, we prove Sobolev estimates which are independent of the regularization parameter (Propositions 5.12 and 5.13). These estimates will be needed to demonstrate Theorems 5.1 and 5.3, whose proofs are achieved in Section 5.2. Finally, in Section 5.3 we prove Theorem 5.5.

5.1 The regularization

In this section, we shall assume that λ, α, f and u are as in the statement of Theorem 5.1 if $p > 2$, and as in the statement of Theorem 5.3 if $p = 2$.
For $\varepsilon \in [0,1]$ and a couple of standard, non-negative, radially symmetric mollifiers $\varphi_1 \in C^\infty_0(B_1(0))$ and $\varphi_2 \in C^\infty_0((-1,1))$, we define

$$f_\varepsilon(x,t) := \int_{-1}^{1}\int_{B_1(0)} f(x-\varepsilon y, t-\varepsilon s)\,\varphi_1(y)\,\varphi_2(s)\,dy\,ds, \tag{5.8}$$

where f is meant to be extended by zero outside Ω_T. Let us observe that $f_0 = f$ and $f_\varepsilon \in C^\infty(\Omega_T)$ for every $\varepsilon \in (0,1]$. In addition, we define the function $A_\varepsilon : \mathbb{R}^n \to \mathbb{R}$ by

$$A_\varepsilon(\xi) := \frac{1}{p}\,(|\xi|-\lambda)^p_+ + \frac{\varepsilon}{p}\,(1+|\xi|^2)^{\frac{p}{2}}.$$

Therefore, we obtain

$$D_\xi A_\varepsilon(\xi) = H_{p-1}(\xi) + \varepsilon\,(1+|\xi|^2)^{\frac{p-2}{2}}\xi\,, \qquad \xi \in \mathbb{R}^n.$$

The next lemma provides the p-ellipticity property of the operator $D_\xi A_\varepsilon(\xi)$ and its proof follows from [17, Lemma 2.7].

Lemma 5.7. *Let $\varepsilon \in [0,1]$, $p \geq 2$ and $z \in \mathbb{R}^n \setminus \{0\}$. Then, for every $\zeta \in \mathbb{R}^n$ we have*

$$\left[\varepsilon\,(1+|z|^2)^{\frac{p-2}{2}} + \frac{(|z|-\lambda)^{p-1}_+}{|z|}\right]|\zeta|^2 \leq \langle D^2 A_\varepsilon(z)\,\zeta,\zeta\rangle \leq (p-1)\left[\varepsilon\,(1+|z|^2)^{\frac{p-2}{2}} + (|z|-\lambda)^{p-2}_+\right]|\zeta|^2.$$

Now we consider a parabolic cylinder $Q_R(z_0) := B_R(x_0)\times(t_0-R^2,t_0) \Subset \Omega_T$. For our purposes, in the sequel we will need the following definition.

Definition 5.8. Let $\varepsilon \in (0,1]$. In this framework, we identify a function

$$u_\varepsilon \in C^0\left([t_0-R^2,t_0];L^2(B_R(x_0))\right)\cap L^p\left(t_0-R^2,t_0;W^{1,p}(B_R(x_0))\right)$$

as a *weak solution of the Cauchy-Dirichlet problem*

$$\begin{cases} \partial_t u_\varepsilon - \operatorname{div}\left[DA_\varepsilon(Du_\varepsilon)\right] = f_\varepsilon & \text{in } Q_R(z_0),\\ u_\varepsilon = u & \text{on } \partial_{\mathrm{par}}Q_R(z_0), \end{cases} \tag{5.9}$$

if and only if, for any $\varphi \in C_0^\infty(Q_R(z_0))$, the following integral identity holds

$$\int_{Q_R(z_0)} \left(u_\varepsilon\cdot\partial_t\varphi - \langle DA_\varepsilon(Du_\varepsilon),D\varphi\rangle\right)dz \;=\; -\int_{Q_R(z_0)} f_\varepsilon\,\varphi\,dz \tag{5.10}$$

and, moreover,

$$u_\varepsilon \;\in\; u + L^p\left(t_0-R^2,t_0;W_0^{1,p}(B_R(x_0))\right)$$

and $u_\varepsilon(\cdot,t_0-R^2)=u(\cdot,t_0-R^2)$ in the L^2-sense, that is,

$$\lim_{t\downarrow t_0-R^2}\left\|u_\varepsilon(\cdot,t)-u(\cdot,t_0-R^2)\right\|_{L^2(B_R(x_0))} \;=\; 0. \tag{5.11}$$

Therefore, the initial condition $u_\varepsilon = u$ on $B_R(x_0)\times\{t_0-R^2\}$ has to be understood in the usual L^2-sense (5.11), while the condition $u_\varepsilon = u$ on the lateral boundary $\partial B_R(x_0)\times(t_0-R^2,t_0)$ has to be meant in the sense of traces, i.e.

$$(u_\varepsilon-u)(\cdot,t)\in W_0^{1,p}(B_R(x_0))$$

for almost every $t\in(t_0-R^2,t_0)$.

Remark 5.9. The advantage of considering the Cauchy-Dirichlet problem (5.9) stems from the fact that the existence of a unique solution

$$u_\varepsilon \;\in\; C^0\left([t_0-R^2,t_0];L^2(B_R(x_0))\right)\cap L^p\left(t_0-R^2,t_0;W^{1,p}(B_R(x_0))\right)$$

satisfying the requirements of Definition 5.8 can be ensured by the classic existence theory for parabolic equations (see, for example, [69, Chapter 2, Theorem 1.2 and Remark 1.2]). Moreover, the operator $D_\xi A_\varepsilon(\xi)$ fulfills p-growth and p-ellipticity conditions with constants depending on ε (cf. Lemma 5.7 above). Therefore, by the results in [41], we have

$$V_1(Du_\varepsilon) := (1+|Du_\varepsilon|^2)^{\frac{p-2}{4}}Du_\varepsilon \;\in\; L^2_{loc}\left(t_0-R^2,t_0;W^{1,2}_{loc}(B_R(x_0),\mathbb{R}^n)\right)$$

and

$$Du_\varepsilon \;\in\; L^{p+\frac{4}{n}}_{loc}(Q_R(z_0),\mathbb{R}^n),$$

and, by the definition of $V_1(Du_\varepsilon)$, this yields

$$DV_1(Du_\varepsilon)\approx(1+|Du_\varepsilon|^2)^{\frac{p-2}{4}}D^2u_\varepsilon \;\in\; L^2_{loc}(Q_R(z_0),\mathbb{R}^{n\times n}) \;\Longrightarrow\; |D^2u_\varepsilon| \;\in\; L^2_{loc}(Q_R(z_0)).$$

5.1.1 Uniform a priori estimates

The first step in the proofs of Theorems 5.1 and 5.3 is the following estimate for the weak solutions of the regularized problems (5.9). Similar estimates are scattered in the literature, but not in the exact form needed here.

Proposition 5.10 (**Uniform energy estimate**). *With the notation and under the assumptions above, there exist two positive constants $\varepsilon_0 \le 1$ and $C \equiv C(n,p,R)$ such that*

$$\int_{Q_R(z_0)} |Du_\varepsilon|^p \, dz \; + \sup_{t\in(t_0-R^2,t_0)} \big\|u_\varepsilon(\cdot,t)-u(\cdot,t)\big\|^2_{L^2(B_R(x_0))} \le C\left(\|Du\|^p_{L^p(Q_R(z_0))} + \lambda^p + 1\right) \tag{5.12}$$

for all $\varepsilon \in (0,\varepsilon_0]$.

Proof. In order to obtain an (uniform in ε) energy estimate for $|Du_\varepsilon|$, we proceed by testing equations (5.1) and $(5.9)_1$ with the map $\varphi = \psi(t)(u_\varepsilon - u)$, where $\psi \in W^{1,\infty}(\mathbb{R})$ is chosen such that

$$\psi(t) = \begin{cases} 1 & \text{if } t \le t_2, \\ -\frac{1}{\omega}(t-t_2-\omega) & \text{if } t_2 < t < t_2+\omega, \\ 0 & \text{if } t \ge t_2+\omega, \end{cases}$$

with

$$t_0 - R^2 < t_2 < t_2 + \omega < t_0,$$

and then letting $\omega \to 0$. We observe that, at this stage, it is important that u_ε and u agree on the parabolic boundary $\partial_{\rm par}Q_R(z_0)$. We also note that the following computations are somewhat formal concerning the use of the time derivative, but they can easily be made rigorous, for example by the use of Steklov averages. We skip this, since it is a standard procedure. With the previous choice of φ, for every $t_2 \in (t_0-R^2,t_0)$ we find

$$\begin{aligned} &\frac{1}{2}\int_{B_R(x_0)} |u_\varepsilon(x,t_2)-u(x,t_2)|^2\,dx + \int_{Q_{R,t_2}} \langle H_{p-1}(Du_\varepsilon) - H_{p-1}(Du), Du_\varepsilon - Du\rangle\,dz \\ &+ \varepsilon\int_{Q_{R,t_2}} \langle (1+|Du_\varepsilon|^2)^{\frac{p-2}{2}} Du_\varepsilon, Du_\varepsilon - Du\rangle\,dz = \int_{Q_{R,t_2}} (f-f_\varepsilon)(u_\varepsilon-u)\,dz, \end{aligned} \tag{5.13}$$

where we have used the abbreviation

$$Q_{R,t_2} = B_R(x_0)\times(t_0-R^2,t_2).$$

In what follows, we shall denote by c_k some positive constants depending only on n, p and R. Using Lemma 1.6, the Cauchy-Schwarz inequality as well as Young's inequality with $\beta > 0$, from (5.13) we infer

$$\begin{aligned} &\sup_{t\in(t_0-R^2,t_0)} \big\|u_\varepsilon(\cdot,t)-u(\cdot,t)\big\|^2_{L^2(B_R)} + \frac{8}{p^2}\int_{Q_R(z_0)} \left|H_{\frac{p}{2}}(Du_\varepsilon) - H_{\frac{p}{2}}(Du)\right|^2 dz + 2\,\varepsilon\int_{Q_R(z_0)} |Du_\varepsilon|^p\,dz \\ &\le 2\,\varepsilon\int_{Q_R(z_0)} (1+|Du_\varepsilon|^2)^{\frac{p-1}{2}}|Du|\,dz + 2\int_{Q_R(z_0)} |f-f_\varepsilon|\,|u_\varepsilon-u|\,dz \\ &\le \frac{2\,\varepsilon\beta^{p'}}{p'}\int_{Q_R} (1+|Du_\varepsilon|^2)^{\frac{p}{2}}\,dz + \frac{2\,\varepsilon}{p\beta^p}\int_{Q_R}|Du|^p\,dz + 2\int_{Q_R}|f-f_\varepsilon|\,|u_\varepsilon-u|\,dz \end{aligned}$$

$$\leq \frac{2^{\frac{p}{2}}\,\varepsilon\beta^{p'}}{p'}\int_{Q_R}|Du_\varepsilon|^p\,dz \,+\, \frac{2^{\frac{p}{2}}\,\varepsilon\beta^{p'}}{p'}|Q_R| + \frac{2\,\varepsilon}{p\beta^p}\int_{Q_R}|Du|^p\,dz \,+\, 2\int_{Q_R}|f-f_\varepsilon|\,|u_\varepsilon-u|\,dz. \tag{5.14}$$

Choosing $\beta = \left(\frac{p'}{2^{p/2}}\right)^{\frac{1}{p'}}$ and reabsorbing the first integral in the right-hand side of (5.14) by the left-hand side, we arrive at

$$\begin{aligned}&\sup_{t\in(t_0-R^2,t_0)}\big\|u_\varepsilon(\cdot,t)-u(\cdot,t)\big\|^2_{L^2(B_R)} \,+\, \int_{Q_R}\Big|H_{\frac{p}{2}}(Du_\varepsilon)-H_{\frac{p}{2}}(Du)\Big|^2\,dz \,+\, \varepsilon\int_{Q_R}|Du_\varepsilon|^p\,dz\\ &\quad\leq\, c_1\,\varepsilon + c_1\,\varepsilon\int_{Q_R}|Du|^p\,dz \,+\, c_1\int_{Q_R}|f-f_\varepsilon|\,|u_\varepsilon-u|\,dz.\end{aligned} \tag{5.15}$$

Now we apply Hölder's and Sobolev inequalities to estimate the last integral as follows

$$\begin{aligned}\int_{Q_R}|f-f_\varepsilon|\,|u_\varepsilon-u|\,dz \,&\leq\, \|f-f_\varepsilon\|_{L^{p'}(Q_R)}\,\|u_\varepsilon-u\|_{L^p(Q_R)}\\ &\leq\, c_2\,\|f-f_\varepsilon\|_{L^{p'}(Q_R)}\,\|Du_\varepsilon-Du\|_{L^p(Q_R)}.\end{aligned} \tag{5.16}$$

Joining estimates (5.15) and (5.16), recalling that $0<\varepsilon\leq 1$ and applying Young's inequality with $\gamma>0$, we obtain

$$\begin{aligned}&\sup_{t\in(t_0-R^2,t_0)}\big\|u_\varepsilon(\cdot,t)-u(\cdot,t)\big\|^2_{L^2(B_R)} \,+\, \int_{Q_R}\Big|H_{\frac{p}{2}}(Du_\varepsilon)-H_{\frac{p}{2}}(Du)\Big|^2\,dz \,+\, \varepsilon\int_{Q_R}|Du_\varepsilon|^p\,dz\\ &\quad\leq\, c_3 \,+\, c_3\int_{Q_R}|Du|^p\,dz \,+\, c_3\,\gamma^{\frac{1}{1-p}}\|f-f_\varepsilon\|^{p'}_{L^{p'}(Q_R)} + \gamma\,\|Du_\varepsilon-Du\|^p_{L^p(Q_R)}\\ &\quad\leq\, c_3 \,+\, (c_3+2^{p-1}\gamma)\int_{Q_R}|Du|^p\,dz \,+\, 2^{p-1}\gamma\int_{Q_R}|Du_\varepsilon|^p\,dz \,+\, c_3\,\gamma^{\frac{1}{1-p}}\|f-f_\varepsilon\|^{p'}_{L^{p'}(Q_R)}.\end{aligned} \tag{5.17}$$

Now, arguing as in (4.51), we have

$$\int_{Q_R}|Du_\varepsilon|^p\,dz \,\leq\, 2^p\int_{Q_R}\Big|H_{\frac{p}{2}}(Du_\varepsilon)-H_{\frac{p}{2}}(Du)\Big|^2\,dz \,+\, 2^{p+1}\int_{Q_R}(|Du|^p+\lambda^p)\,dz. \tag{5.18}$$

Multiplying all sides of (5.17) by 2^p and then adding the resulting expression and (5.18) side by side, we get

$$\begin{aligned}&(1+2^p\varepsilon)\int_{Q_R}|Du_\varepsilon|^p\,dz \,+\, 2^p\sup_{t\in(t_0-R^2,t_0)}\big\|u_\varepsilon(\cdot,t)-u(\cdot,t)\big\|^2_{L^2(B_R)}\\ &\quad\leq\, c_4 + (c_4+2^{2p-1}\gamma)\int_{Q_R}(|Du|^p+\lambda^p)\,dz \,+\, c_4\,\gamma^{\frac{1}{1-p}}\|f-f_\varepsilon\|^{p'}_{L^{p'}(Q_R)} + 2^{2p-1}\gamma\int_{Q_R}|Du_\varepsilon|^p\,dz.\end{aligned}$$

Choosing $\gamma=\frac{1}{2^{2p}}$ and reabsorbing the last term in the above estimate by the left-hand side, we find

$$\begin{aligned}&\int_{Q_R}|Du_\varepsilon|^p\,dz \,+\, \sup_{t\in(t_0-R^2,t_0)}\big\|u_\varepsilon(\cdot,t)-u(\cdot,t)\big\|^2_{L^2(B_R)}\\ &\qquad\leq\, c_5\int_{Q_R}(|Du|^p+\lambda^p)\,dz \,+\, c_5\,\|f-f_\varepsilon\|^{p'}_{L^{p'}(Q_R)} + c_5\,.\end{aligned} \tag{5.19}$$

Since, by virtue of (5.8),

$$f_\varepsilon \to f \quad \text{ strongly in } L^{p'}(Q_R) \quad \text{as } \varepsilon \to 0^+, \tag{5.20}$$

there exists a positive number $\varepsilon_0 \le 1$ such that

$$\|f - f_\varepsilon\|_{L^{p'}(Q_R)} \le 1 \qquad \text{for all } \varepsilon \in (0, \varepsilon_0]. \tag{5.21}$$

Combining this with (5.19), we obtain the desired conclusion. □

Thanks to the previous proposition, we can now obtain the following result, which will be decisive to carry out the proofs of Theorems 5.1 and 5.3.

Proposition 5.11 (**Comparison estimate**). *With the notation and under the assumptions of Proposition 5.10, there exists a positive constant C depending only on n, p and R such that the estimate*

$$\begin{aligned}
&\sup_{t\in(t_0-R^2,t_0)} \big\|u_\varepsilon(\cdot,t) - u(\cdot,t)\big\|^2_{L^2(B_R(x_0))} + \int_{Q_R(z_0)} |\mathcal{V}_{\alpha,\lambda}(Du_\varepsilon) - \mathcal{V}_{\alpha,\lambda}(Du)|^2\,dz \\
&\quad \le C\,\varepsilon \left(\|Du\|^p_{L^p(Q_R(z_0))} + 1\right) + C\,\|f - f_\varepsilon\|_{L^{p'}(Q_R(z_0))} \left(\|Du\|_{L^p(Q_R(z_0))} + \lambda + 1\right)
\end{aligned} \tag{5.22}$$

holds for every $\varepsilon \in (0, \varepsilon_0]$, where ε_0 is the constant from Proposition 5.10. In particular, this estimate implies that

$$u_\varepsilon \to u \quad \textit{strongly in } L^2(Q_R(z_0)) \quad \textit{as } \varepsilon \to 0$$

and

$$\mathcal{V}_{\alpha,\lambda}(Du_\varepsilon) \to \mathcal{V}_{\alpha,\lambda}(Du) \quad \textit{strongly in } L^2(Q_R(z_0), \mathbb{R}^n) \quad \textit{as } \varepsilon \to 0.$$

Proof. Arguing as in the proof of Proposition 5.10, but using Lemma 3.9 instead of Lemma 1.6, we arrive at

$$\begin{aligned}
&\sup_{t\in(t_0-R^2,t_0)} \big\|u_\varepsilon(\cdot,t) - u(\cdot,t)\big\|^2_{L^2(B_R)} + \int_{Q_R} |\mathcal{V}_{\alpha,\lambda}(Du_\varepsilon) - \mathcal{V}_{\alpha,\lambda}(Du)|^2\,dz \\
&\quad \le c\,\varepsilon \left(\|Du\|^p_{L^p(Q_R)} + 1\right) + c\,\|f - f_\varepsilon\|_{L^{p'}(Q_R)}\,\|Du_\varepsilon - Du\|_{L^p(Q_R)}\,,
\end{aligned}$$

where $c \equiv c(n,p,R) > 0$. Now, let us consider the same $\varepsilon_0 \in (0,1]$ as in the proof of Proposition 5.10. Then, applying Minkowski's inequality, for every $\varepsilon \in (0, \varepsilon_0]$ we get

$$\begin{aligned}
&\sup_{t\in(t_0-R^2,t_0)} \big\|u_\varepsilon(\cdot,t) - u(\cdot,t)\big\|^2_{L^2(B_R)} + \int_{Q_R} |\mathcal{V}_{\alpha,\lambda}(Du_\varepsilon) - \mathcal{V}_{\alpha,\lambda}(Du)|^2\,dz \\
&\quad \le c\,\varepsilon \left(\|Du\|^p_{L^p(Q_R)} + 1\right) + c\,\|f - f_\varepsilon\|_{L^{p'}(Q_R)} \left(\|Du_\varepsilon\|_{L^p(Q_R)} + \|Du\|_{L^p(Q_R)}\right) \\
&\quad \le C\,\varepsilon \left(\|Du\|^p_{L^p(Q_R)} + 1\right) + C\,\|f - f_\varepsilon\|_{L^{p'}(Q_R)} \left(\|Du\|_{L^p(Q_R)} + \lambda + 1\right),
\end{aligned}$$

where, in the last line, we have used inequality (5.12). Thus we obtain the comparison estimate

(5.22). Now, just observe that

$$\int_{Q_R(z_0)} |u_\varepsilon - u|^2 \, dz \;\le\; R^2 \sup_{t\in(t_0-R^2,t_0)} \big\|u_\varepsilon(\cdot,t) - u(\cdot,t)\big\|^2_{L^2(B_R(x_0))}.$$

Combining this with (5.22) and (5.20), the statement is proved. □

We are now in a position to establish uniform estimates for the derivatives of the function $\mathcal{V}_{\alpha,\lambda}(Du_\varepsilon)$. Let us start with the following result.

Proposition 5.12 (**Uniform Sobolev estimate**). *Under the assumptions of Theorem 5.1 and with the notation above, there exists a positive number $\varepsilon_1 \le 1$ such that, for every $\varepsilon \in (0,\varepsilon_1]$, for every parabolic cylinder $Q_\rho(z_0) \Subset Q_R(z_0)$ and every $j \in \{1,\dots,n\}$, we have*

$$\begin{aligned}\int_{Q_{\rho/2}(z_0)} |D_{x_j}\mathcal{V}_{\alpha,\lambda}(Du_\varepsilon)|^2 \, dz \;\le\; \left(C + \frac{C}{\rho^2}\right)\Big[\|Du\|^p_{L^p(Q_R)} + \|Du\|^2_{L^p(Q_R)} + \lambda^p + \lambda^2 + 1\Big] \\ + \, C\,\|f\|^{p'}_{L^{p'}\left(t_0-R^2,t_0;B^{\frac{p-2}{p}}_{p',1}(B_R(x_0))\right)}\end{aligned} \tag{5.23}$$

for a positive constant C depending only on n, p and R in the case $\lambda = 0$, and additionally on α if $\lambda > 0$.

Proof. Let us first assume that $\lambda > 0$. To shorten our notation, we introduce the function $P : Q_R(z_0) \to \mathbb{R}^+_0$ defined by

$$P := (|Du_\varepsilon| - \lambda)_+ \tag{5.24}$$

and its "mollified" version

$$P_\varrho \;:=\; (|[Du_\varepsilon]_\varrho| - \lambda)_+ \,, \qquad \varrho > 0, \tag{5.25}$$

using the notation (1.1) on the right-hand side of (5.25), while, on the left-hand side, an intentional abuse of the same notation has been made. We now test the weak formulation (5.10) with the map

$$\varphi = D_{x_j}\left(D_{x_j}[u_\varepsilon]_\varrho \cdot \psi \cdot \Phi(P_\varrho)\right), \qquad j \in \{1,\dots,n\},$$

where $\psi \in W^{1,\infty}_0(Q_R(z_0))$ is a non-negative cut-off function that will be specified later and $\Phi : \mathbb{R}^+_0 \to \mathbb{R}^+_0$ is an increasing, locally Lipschitz continuous function, such that Φ and Φ' are bounded on $\mathbb{R}^+_0$, $\Phi(0) = 0$ and

$$\Phi'(t)\, t \;\le\; c_\Phi\, \Phi(t) \tag{5.26}$$

for a suitable constant $c_\Phi > 0$. After summing with respect to j from 1 to n, we obtain

$$\begin{aligned}&\frac12 \int_{Q_R} \partial_t(|D[u_\varepsilon]_\varrho|^2)\,\psi\,\Phi(P_\varrho)\,dz + \int_{Q_R}\sum_{j=1}^n \left\langle D_{x_j}[DA_\varepsilon(Du_\varepsilon)]_\varrho, D\left(D_{x_j}[u_\varepsilon]_\varrho \cdot \psi\,\Phi(P_\varrho)\right)\right\rangle dz \\ &\quad = -\int_{Q_R}\sum_{j=1}^n [f_\varepsilon]_\varrho\, D_{x_j}\left(D_{x_j}[u_\varepsilon]_\varrho \cdot \psi\,\Phi(P_\varrho)\right) dz.\end{aligned} \tag{5.27}$$

Arguing as in [7, page 521], the integral involving the time derivative can be expressed as follows

$$\begin{aligned}\frac{1}{2}\int_{Q_R}\partial_t(|D[u_\varepsilon]_\varrho|^2)\,\psi\,\Phi(P_\varrho)\,dz &= \frac{1}{2}\int_{Q_R}\partial_t\left[\int_0^{|D[u_\varepsilon]_\varrho|^2}\Phi((\sqrt{w}-\lambda)_+)\,dw\right]\psi\,dz\\ &= -\frac{1}{2}\int_{Q_R}\left[\int_0^{|D[u_\varepsilon]_\varrho|^2}\Phi((\sqrt{w}-\lambda)_+)\,dw\right]\partial_t\psi\,dz.\end{aligned}$$

Therefore, integrating by parts and then letting $\varrho\to 0$ in (5.27), we get

$$\begin{aligned}&-\frac{1}{2}\int_{Q_R}\left[\int_0^{|Du_\varepsilon|^2}\Phi((\sqrt{w}-\lambda)_+)\,dw\right]\partial_t\psi\,dz\\ &+\int_{Q_R}\sum_{j=1}^n\langle D^2A_\varepsilon(Du_\varepsilon)\,D(D_{x_j}u_\varepsilon),D(D_{x_j}u_\varepsilon)\rangle\,\psi\,\Phi(P)\,dz\\ &+\int_{Q_R}\sum_{j=1}^n\langle D^2A_\varepsilon(Du_\varepsilon)\,D(D_{x_j}u_\varepsilon),D\psi\rangle(D_{x_j}u_\varepsilon)\,\Phi(P)\,dz\\ &+\int_{Q_R}\sum_{j=1}^n\langle D^2A_\varepsilon(Du_\varepsilon)\,D(D_{x_j}u_\varepsilon),DP\rangle(D_{x_j}u_\varepsilon)\,\psi\,\Phi'(P)\,dz\\ &=\int_{Q_R}\sum_{j=1}^n(D_{x_j}f_\varepsilon)\,(D_{x_j}u_\varepsilon)\,\psi\,\Phi(P)\,dz.\end{aligned}\tag{5.28}$$

In what follows, we will denote by c_k and C some positive constants that do not depend on ε. Now, let us fix an arbitrary radius $\rho\in(0,R)$. For a fixed time $t_1\in(t_0-\rho^2,t_0)$ and $\sigma\in(0,t_0-t_1)$, we choose

$$\psi(x,t)=\widetilde{\chi}(t)\,\chi(t)\,\eta^2(x)$$

with $\chi\in W^{1,\infty}((t_0-R^2,t_0),[0,1])$, $\chi\equiv 0$ on $(t_0-R^2,t_0-\rho^2)$ and $\partial_t\chi\geq 0$, $\eta\in C_0^\infty(B_\rho(x_0),[0,1])$, and with the Lipschitz continuous function $\widetilde{\chi}:(t_0-R^2,t_0)\to\mathbb{R}$ defined by

$$\widetilde{\chi}(t)=\begin{cases}1 & \text{if } t\leq t_1,\\ \text{affine} & \text{if } t_1<t<t_1+\sigma,\\ 0 & \text{if } t\geq t_1+\sigma.\end{cases}$$

With such a choice of ψ, equation (5.28) turns into

$$\begin{aligned}&-\frac{1}{2}\int_{Q_R}\left[\int_0^{|Du_\varepsilon|^2}\Phi((\sqrt{w}-\lambda)_+)\,dw\right]\eta^2(x)\,\chi(t)\,\partial_t\widetilde{\chi}(t)\,dz\\ &-\frac{1}{2}\int_{Q_R}\left[\int_0^{|Du_\varepsilon|^2}\Phi((\sqrt{w}-\lambda)_+)\,dw\right]\eta^2(x)\,\widetilde{\chi}(t)\,\partial_t\chi(t)\,dz\\ &+\int_{Q_R}\sum_{j=1}^n\langle D^2A_\varepsilon(Du_\varepsilon)\,D(D_{x_j}u_\varepsilon),D(D_{x_j}u_\varepsilon)\rangle\,\eta^2(x)\,\widetilde{\chi}(t)\,\chi(t)\,\Phi(P)\,dz\\ &+\int_{Q_R}\sum_{j=1}^n\langle D^2A_\varepsilon(Du_\varepsilon)\,D(D_{x_j}u_\varepsilon),DP\rangle(D_{x_j}u_\varepsilon)\,\eta^2(x)\,\widetilde{\chi}(t)\,\chi(t)\,\Phi'(P)\,dz\end{aligned}$$

$$= -2\int_{Q_R}\sum_{j=1}^{n}\langle D^2A_\varepsilon(Du_\varepsilon)\,D(D_{x_j}u_\varepsilon),D\eta\rangle(D_{x_j}u_\varepsilon)\,\eta(x)\,\widetilde{\chi}(t)\,\chi(t)\,\Phi(P)\,dz$$
$$+\int_{Q_R}\sum_{j=1}^{n}(D_{x_j}f_\varepsilon)\,(D_{x_j}u_\varepsilon)\,\eta^2(x)\,\widetilde{\chi}(t)\,\chi(t)\,\Phi(P)\,dz.$$

Setting

$$Q^{t_1} := B_\rho(x_0)\times(t_0-\rho^2,t_1)$$

and letting $\sigma\to 0$ in the previous equality, for every $t_1\in(t_0-\rho^2,t_0)$ we get

$$\frac{1}{2}\int_{B_\rho(x_0)}\chi(t_1)\,\eta^2(x)\left[\int_0^{|Du_\varepsilon(x,t_1)|^2}\Phi((\sqrt{w}-\lambda)_+)\,dw\right]dx$$
$$+\int_{Q^{t_1}}\sum_{j=1}^{n}\langle D^2A_\varepsilon(Du_\varepsilon)\,D(D_{x_j}u_\varepsilon),D(D_{x_j}u_\varepsilon)\rangle\,\eta^2(x)\,\chi(t)\,\Phi(P)\,dz$$
$$+\int_{Q^{t_1}}\sum_{j=1}^{n}\langle D^2A_\varepsilon(Du_\varepsilon)\,D(D_{x_j}u_\varepsilon),DP\rangle(D_{x_j}u_\varepsilon)\,\eta^2(x)\,\chi(t)\,\Phi'(P)\,dz$$
$$= -2\int_{Q^{t_1}}\sum_{j=1}^{n}\langle D^2A_\varepsilon(Du_\varepsilon)\,D(D_{x_j}u_\varepsilon),D\eta\rangle(D_{x_j}u_\varepsilon)\,\eta(x)\,\chi(t)\,\Phi(P)\,dz$$
$$+\int_{Q^{t_1}}\sum_{j=1}^{n}(D_{x_j}f_\varepsilon)\,(D_{x_j}u_\varepsilon)\,\eta^2(x)\,\chi(t)\,\Phi(P)\,dz$$
$$+\frac{1}{2}\int_{Q^{t_1}}(\partial_t\chi)\,\eta^2(x)\left[\int_0^{|Du_\varepsilon(x,t)|^2}\Phi((\sqrt{w}-\lambda)_+)\,dw\right]dz, \tag{5.29}$$

where we have used that $\partial_t\widetilde{\chi}$ converges to a Dirac delta distribution as $\sigma\to 0$, together with the $L^2(B_R(x_0))$-valued continuity of u_ε. We now observe that the second integral on the left-hand side of (5.29) is non-negative, since A_ε is convex and $\chi,\Phi\geq 0$. As for the first term on the right-hand side of (5.29), for every $t_1\in(t_0-\rho^2,t_0)$ we have

$$-2\int_{Q^{t_1}}\langle D^2A_\varepsilon(Du_\varepsilon)\,D(D_{x_j}u_\varepsilon),D\eta\rangle(D_{x_j}u_\varepsilon)\,\eta(x)\,\chi(t)\,\Phi(P)\,dz$$
$$\leq 2\int_{Q^{t_1}}|\langle D^2A_\varepsilon(Du_\varepsilon)\,D(D_{x_j}u_\varepsilon),D\eta\rangle|\,\chi(t)\,\eta(x)\,|D_{x_j}u_\varepsilon|\,\Phi(P)\,dz$$
$$\leq 2\int_{Q^{t_1}}\sqrt{\langle D^2A_\varepsilon(Du_\varepsilon)\,D(D_{x_j}u_\varepsilon),D(D_{x_j}u_\varepsilon)\rangle}\,\sqrt{\langle D^2A_\varepsilon(Du_\varepsilon)\,D\eta,D\eta\rangle}\,\chi(t)\,\eta(x)\,|D_{x_j}u_\varepsilon|\,\Phi(P)\,dz$$
$$\leq \frac{1}{2}\int_{Q^{t_1}}\langle D^2A_\varepsilon(Du_\varepsilon)\,D(D_{x_j}u_\varepsilon),D(D_{x_j}u_\varepsilon)\rangle\,\chi(t)\,\eta^2(x)\,\Phi(P)\,dz$$
$$+2\int_{Q^{t_1}}\langle D^2A_\varepsilon(Du_\varepsilon)\,D\eta,D\eta\rangle\,\chi(t)\,|D_{x_j}u_\varepsilon|^2\,\Phi(P)\,dz, \tag{5.30}$$

where we have used the Cauchy-Schwarz inequality together with Young's inequality. Joining (5.29) and (5.30), we obtain

$$I_1+I_2+I_3\leq I_4+I_5+I_6\,, \tag{5.31}$$

where

$$I_1 := \int_{B_\rho(x_0)} \chi(t_1)\,\eta^2(x) \left[\int_0^{|Du_\varepsilon(x,t_1)|^2} \Phi((\sqrt{w}-\lambda)_+)\,dw\right] dx,$$

$$I_2 := 2\int_{Q^{t_1}} \sum_{j=1}^n \langle D^2A_\varepsilon(Du_\varepsilon)\,D(D_{x_j}u_\varepsilon), DP\rangle (D_{x_j}u_\varepsilon)\,\eta^2(x)\,\chi(t)\,\Phi'(P)\,dz,$$

$$I_3 := \int_{Q^{t_1}} \sum_{j=1}^n \langle D^2A_\varepsilon(Du_\varepsilon)\,D(D_{x_j}u_\varepsilon), D(D_{x_j}u_\varepsilon)\rangle\,\chi(t)\,\eta^2(x)\,\Phi(P)\,dz,$$

$$I_4 := 4\int_{Q^{t_1}} \sum_{j=1}^n \langle D^2A_\varepsilon(Du_\varepsilon)\,D\eta, D\eta\rangle\,\chi(t)\,|D_{x_j}u_\varepsilon|^2\,\Phi(P)\,dz,$$

$$I_5 := \int_{Q^{t_1}} (\partial_t\chi)\,\eta^2(x) \left[\int_0^{|Du_\varepsilon(x,t)|^2} \Phi((\sqrt{w}-\lambda)_+)\,dw\right] dz,$$

$$I_6 := 2\int_{Q^{t_1}} \sum_{j=1}^n (D_{x_j}f_\varepsilon)\,(D_{x_j}u_\varepsilon)\,\eta^2(x)\,\chi(t)\,\Phi(P)\,dz.$$

Now we observe that $I_1 \geq 0$, thus we can drop it in the following. Furthermore, arguing as in the proof of Proposition 3.14, one can show that $I_2 \geq 0$. Therefore, inequality (5.31) boils down to

$$I_3 \leq I_4 + I_5 + I_6\,. \tag{5.32}$$

At this stage, we choose a cut-off function $\eta \in C_0^\infty(B_\rho(x_0))$ with $\eta \equiv 1$ on $B_{\rho/2}(x_0)$ such that

$$0 \leq \eta \leq 1 \qquad \text{and} \qquad |D\eta| \leq \frac{\tilde{c}}{\rho}\,. \tag{5.33}$$

For the cut-off function in time, we choose the piecewise affine function $\chi : (t_0 - R^2, t_0) \to [0,1]$ with

$$\begin{aligned} \chi &\equiv 0 \quad \text{on } (t_0 - R^2, t_0 - \rho^2),\\ \chi &\equiv 1 \quad \text{on } \left(t_0 - \left(\frac{\rho}{2}\right)^2, t_0\right),\\ \partial_t\chi &\equiv \frac{4}{3\rho^2} \quad \text{on } \left(t_0 - \rho^2, t_0 - \left(\frac{\rho}{2}\right)^2\right). \end{aligned}$$

Moreover, as in the proof of Proposition 3.14, we choose

$$\Phi(t) := \frac{t^{2\alpha}}{(t^2+\lambda^2)^\alpha} \qquad \text{for } t \geq 0, \tag{5.34}$$

and therefore

$$\Phi'(t) = \frac{2\alpha\lambda^2\,t^{2\alpha-1}}{(t^2+\lambda^2)^{\alpha+1}}\,. \tag{5.35}$$

This function satisfies (5.26) with $c_\Phi = 2\alpha$. With the above choices, we now estimate I_3, I_4 and I_5 separately. Let us first consider I_3. Recalling the definition of P in (5.24), by Lemma 5.7 we have

$$I_3 \geq \int_{Q^{t_1}} \frac{(|Du_\varepsilon|-\lambda)_+^{p-1+2\alpha}}{|Du_\varepsilon|\,[\lambda^2 + (|Du_\varepsilon|-\lambda)_+^2]^\alpha}\,|D^2u_\varepsilon|^2\,\chi(t)\,\eta^2(x)\,dz. \tag{5.36}$$

Using Lemma 5.7 again, the fact that $\varepsilon, \chi, \Phi \le 1$ and the properties (5.33) of η, we obtain

$$I_4 \le 8(p-1)\int_{Q_R}(1+|Du_\varepsilon|^2)^{\frac{p}{2}}|D\eta|^2\,dz \le \frac{c_1(p)}{\rho^2}\int_{Q_R}(1+|Du_\varepsilon|^2)^{\frac{p}{2}}\,dz. \tag{5.37}$$

Now we use again the fact that $\Phi \le 1$, the properties of χ and η, and Hölder's inequality, in order to get

$$I_5 \le \int_{Q^{t_1}}(\partial_t\chi)\,\eta^2(x)\,|Du_\varepsilon|^2\,dz \le \frac{4}{3\rho^2}\int_{Q_R}|Du_\varepsilon|^2\,dz \le \frac{c_2}{\rho^2}\,\|Du_\varepsilon\|^2_{L^p(Q_R)}, \tag{5.38}$$

where $c_2 \equiv c_2(n,p,R) > 0$. Combining estimates (5.32), (5.36), (5.37) and (5.38), we find

$$\begin{aligned}\int_{Q^{t_1}}&\frac{(|Du_\varepsilon|-\lambda)_+^{p-1+2\alpha}}{|Du_\varepsilon|\,[\lambda^2+(|Du_\varepsilon|-\lambda)_+^2]^\alpha}\,|D^2u_\varepsilon|^2\,\chi(t)\,\eta^2(x)\,dz\\ &\le \frac{c_2}{\rho^2}\left[\int_{Q_R}(1+|Du_\varepsilon|^2)^{\frac{p}{2}}\,dz + \|Du_\varepsilon\|^2_{L^p(Q_R)}\right]\\ &\quad + 2\sum_{j=1}^n\int_{Q^{t_1}}(D_{x_j}f_\varepsilon)\,(D_{x_j}u_\varepsilon)\,\eta^2(x)\,\chi(t)\,\Phi(P)\,dz,\end{aligned} \tag{5.39}$$

which holds for every $t_1 \in (t_0-\rho^2, t_0)$.

It remains to estimate the integral containing $D_{x_j}f_\varepsilon$. With this aim in mind, we now argue as in the proof of Proposition 3.14. By Theorem 1.21, there exists a linear and bounded extension operator

$$\mathrm{ext}_{B_\rho(x_0)} : B^{-2/p}_{p',1}(B_\rho(x_0)) \hookrightarrow B^{-2/p}_{p',1}(\mathbb{R}^n)$$

such that $\mathrm{re}_{B_\rho(x_0)} \circ \mathrm{ext}_{B_\rho(x_0)} = \mathrm{id}$, where $\mathrm{re}_{B_\rho(x_0)}$ is the restriction operator defined in Subsection 1.2.1 and the symbol id denotes the identity in $B^{-2/p}_{p',1}(B_\rho(x_0))$. Since, for almost every $t \in (t_0-\rho^2, t_0)$, we have $D_{x_j}f_\varepsilon(\cdot,t) = \mathrm{ext}_{B_\rho(x_0)}(D_{x_j}f_\varepsilon(\cdot,t))$ almost everywhere in $B_\rho(x_0)$, we find

$$\int_{Q^{t_1}}(D_{x_j}f_\varepsilon)(D_{x_j}u_\varepsilon)\,\eta^2(x)\,\chi(t)\,\Phi(P)\,dz = \int_{Q^{t_1}}\mathrm{ext}_{B_\rho(x_0)}(D_{x_j}f_\varepsilon(x,t))\cdot(D_{x_j}u_\varepsilon)\,\eta^2(x)\,\chi(t)\,\Phi(P)\,dz. \tag{5.40}$$

By definition of dual norm, we get

$$\begin{aligned}&\left|\int_{Q^{t_1}}\mathrm{ext}_{B_\rho(x_0)}(D_{x_j}f_\varepsilon(x,t))\cdot(D_{x_j}u_\varepsilon)\,\eta^2(x)\,\chi(t)\,\Phi(P)\,dz\right|\\ &\le \int_{t_0-\rho^2}^{t_1}\|\mathrm{ext}_{B_\rho(x_0)}(D_{x_j}f_\varepsilon(\cdot,t))\|_{(\mathring{B}^{2/p}_{p,\infty}(\mathbb{R}^n))'}\,\|(D_{x_j}u_\varepsilon(\cdot,t))\,\eta^2\,\chi(t)\,\Phi(P(\cdot,t))\|_{B^{2/p}_{p,\infty}(\mathbb{R}^n)}\,dt\\ &= \int_{t_0-\rho^2}^{t_1}\|\mathrm{ext}_{B_\rho(x_0)}(D_{x_j}f_\varepsilon(\cdot,t))\|_{(\mathring{B}^{2/p}_{p,\infty}(\mathbb{R}^n))'}\\ &\qquad\cdot\Big(\|(D_{x_j}u_\varepsilon(\cdot,t))\,\eta^2\,\chi(t)\,\Phi(P(\cdot,t))\|_{L^p(\mathbb{R}^n)} + [(D_{x_j}u_\varepsilon(\cdot,t))\,\eta^2\,\chi(t)\,\Phi(P(\cdot,t))]_{B^{2/p}_{p,\infty}(\mathbb{R}^n)}\Big)\,dt,\end{aligned} \tag{5.41}$$

where, in the last line, we have used the equivalence (1.18). By the properties of η and the fact that $0 \le \chi, \Phi \le 1$, we have

$$\|(D_{x_j}u_\varepsilon(\cdot,t))\,\eta^2\,\chi(t)\,\Phi(P(\cdot,t))\|_{L^p(\mathbb{R}^n)} \le \|Du_\varepsilon(\cdot,t)\|_{L^p(B_\rho(x_0))} \tag{5.42}$$

for almost every $t \in (t_0-\rho^2,t_0)$. Furthermore, using Theorem 1.26, for almost every $t \in (t_0-\rho^2,t_0)$ we obtain

$$\|\mathrm{ext}_{B_\rho(x_0)}(D_{x_j}f_\varepsilon(\cdot,t))\|_{(\mathring{B}^{2/p}_{p,\infty}(\mathbb{R}^n))'} \le c_3\,\|\mathrm{ext}_{B_\rho(x_0)}(D_{x_j}f_\varepsilon(\cdot,t))\|_{B^{-2/p}_{p',1}(\mathbb{R}^n)},$$

for some positive constant $c_3 \equiv c_3(n,p)$. Combining the above inequality with the boundedness of the operator $\mathrm{ext}_{B_\rho(x_0)}$ yields

$$\|\mathrm{ext}_{B_\rho(x_0)}(D_{x_j}f_\varepsilon(\cdot,t))\|_{(\mathring{B}^{2/p}_{p,\infty}(\mathbb{R}^n))'} \le c_3\,\|D_{x_j}f_\varepsilon(\cdot,t)\|_{B^{-2/p}_{p',1}(B_\rho(x_0))}$$

for almost every $t \in (t_0-\rho^2,t_0)$. Moreover, applying Theorem 1.22, we find that

$$\|D_{x_j}f_\varepsilon(\cdot,t)\|_{B^{-2/p}_{p',1}(B_\rho(x_0))} \le c_3\,\|f_\varepsilon(\cdot,t)\|_{B^{\frac{p-2}{p}}_{p',1}(B_\rho(x_0))}$$

for almost every $t \in (t_0-\rho^2,t_0)$. Combining the preceding inequalities, we infer

$$\|\mathrm{ext}_{B_\rho(x_0)}(D_{x_j}f_\varepsilon(\cdot,t))\|_{(\mathring{B}^{2/p}_{p,\infty}(\mathbb{R}^n))'} \le c_3\,\|f_\varepsilon(\cdot,t)\|_{B^{\frac{p-2}{p}}_{p',1}(B_\rho(x_0))}, \tag{5.43}$$

which holds for almost every $t \in (t_0-\rho^2,t_0)$ and a positive constant c_3 depending only on n and p. Now, recalling that

$$\begin{aligned}&[(D_{x_j}u_\varepsilon(\cdot,t))\,\eta^2\,\chi(t)\,\Phi(P(\cdot,t))]^p_{B^{2/p}_{p,\infty}(\mathbb{R}^n)}\\ &\quad=\sup_{|h|>0}\frac{1}{|h|^2}\int_{\mathbb{R}^n}\chi^p(t)\left|\delta_h\left((D_{x_j}u_\varepsilon(x,t))\,\eta^2(x)\,\Phi(P(x,t))\right)\right|^p dx\end{aligned}$$

and using Lemma 3.7, we deduce

$$\begin{aligned}&\frac{1}{|h|^2}\int_{\mathbb{R}^n}\chi^p(t)\left|\delta_h\left((D_{x_j}u_\varepsilon(x,t))\,\eta^2(x)\,\Phi(P(x,t))\right)\right|^p dx\\ &\quad\le\frac{c_4(p)}{|h|^2}\,\chi(t)\int_{\mathbb{R}^n}\left|\delta_h\left(|D_{x_j}u_\varepsilon(x,t)|^{\frac{p-2}{2}}(D_{x_j}u_\varepsilon(x,t))\,\eta^p(x)\,[\Phi(P(x,t))]^{\frac{p}{2}}\right)\right|^2 dx\\ &\quad\le c_5(n,p)\,\chi(t)\int_{\mathbb{R}^n}\left|D\left(|D_{x_j}u_\varepsilon(x,t)|^{\frac{p-2}{2}}(D_{x_j}u_\varepsilon(x,t))\,\eta^p(x)\,[\Phi(P(x,t))]^{\frac{p}{2}}\right)\right|^2 dx,\end{aligned}$$

where, in the last line, we have used the first statement in Lemma 1.14. Using the properties (5.33) of η and the boundedness of Φ, with simple manipulations we thus obtain

$$\begin{aligned}&[(D_{x_j}u_\varepsilon(\cdot,t))\,\eta^2\,\chi(t)\,\Phi(P(\cdot,t))]^p_{B^{2/p}_{p,\infty}(\mathbb{R}^n)}\\ &\quad\le c_5\,\chi(t)\int_{B_\rho(x_0)}\left|D\left(|D_{x_j}u_\varepsilon(x,t)|^{\frac{p-2}{2}}(D_{x_j}u_\varepsilon(x,t))\,[\Phi(P(x,t))]^{\frac{p}{2}}\right)\right|^2\eta^2(x)\,dx\\ &\qquad+\frac{c_5}{\rho^2}\,\chi(t)\int_{B_\rho(x_0)}|Du_\varepsilon(x,t)|^p\,dx\end{aligned}\tag{5.44}$$

for almost every $t \in (t_0-\rho^2,t_1)$. Now, from (3.39) we know that

$$\left|D\left(|D_{x_j}u_\varepsilon|^{\frac{p-2}{2}}(D_{x_j}u_\varepsilon)\,[\Phi(P)]^{\frac{p}{2}}\right)\right|^2 \le c_6(n,p)\,(\mathbf{A}_1+\mathbf{A}_2), \tag{5.45}$$

where

$$\mathbf{A}_1 := |Du_\varepsilon|^{p-2}\,|D^2u_\varepsilon|^2\,[\Phi(P)]^p$$

and

$$\mathbf{A}_2 := |Du_\varepsilon|^{p}\,|D^2u_\varepsilon|^2\,[\Phi(P)]^{p-2}\,[\Phi'(P)]^2.$$

At this point, we estimate $\mathbf{A}_1$ and $\mathbf{A}_2$ exactly as in the proof of Proposition 3.14. This gives

$$\mathbf{A}_1 \,\le\, 2^{\alpha(p-1)}\,\frac{|D^2u_\varepsilon|^2\,(|Du_\varepsilon|-\lambda)_+^{p-1+2\alpha}}{|Du_\varepsilon|\,[\lambda^2+(|Du_\varepsilon|-\lambda)_+^2]^\alpha} \tag{5.46}$$

and

$$\mathbf{A}_2 \,\le\, 2^{\alpha(p-1)+4}\,\alpha^2\,\frac{|D^2u_\varepsilon|^2\,(|Du_\varepsilon|-\lambda)_+^{p-1+2\alpha}}{|Du_\varepsilon|\,[\lambda^2+(|Du_\varepsilon|-\lambda)_+^2]^\alpha}\,. \tag{5.47}$$

Joining estimates (5.44)−(5.47), for almost every $t\in(t_0-\rho^2,t_1)$ we find

$$\begin{aligned}[(D_{x_j}u_\varepsilon(\cdot,t))\,\eta^2\,\chi(t)\,\Phi(P(\cdot,t))]^p_{B^{2/p}_{p,\infty}(\mathbb{R}^n)} \,&\le\, c_7\int_{B_\rho}\frac{(|Du_\varepsilon(x,t)|-\lambda)_+^{p-1+2\alpha}\,|D^2u_\varepsilon(x,t)|^2}{|Du_\varepsilon(x,t)|\,[\lambda^2+(|Du_\varepsilon(x,t)|-\lambda)_+^2]^\alpha}\,\chi(t)\,\eta^2\,dx\\ &\quad+\,\frac{c_7}{\rho^2}\,\chi(t)\int_{B_\rho}|Du_\varepsilon(x,t)|^p\,dx,\end{aligned}$$

where $c_7\equiv c_7(n,p,\alpha)>0$. Combining the previous inequality, (5.42) and (5.43) with (5.41), after some algebraic manipulation, from (5.40) we get

$$\begin{aligned}&\int_{Q^{t_1}}(D_{x_j}f_\varepsilon)(D_{x_j}u_\varepsilon)\,\eta^2(x)\,\chi(t)\,\Phi(P)\,dz\\ &\quad\le\, c_7\int_{t_0-\rho^2}^{t_1}\|f_\varepsilon(\cdot,t)\|_{B^{\frac{p-2}{p}}_{p',1}(B_\rho)}\Bigg[\left(\int_{B_\rho}\frac{(|Du_\varepsilon(x,t)|-\lambda)_+^{p-1+2\alpha}\,|D^2u_\varepsilon(x,t)|^2}{|Du_\varepsilon(x,t)|\,[\lambda^2+(|Du_\varepsilon(x,t)|-\lambda)_+^2]^\alpha}\,\chi(t)\,\eta^2\,dx\right)^{\frac1p}\\ &\qquad\qquad+\left(1\,+\,\frac{1}{\rho^{2/p}}\right)\|Du_\varepsilon(\cdot,t)\|_{L^p(B_\rho)}\Bigg]dt.\end{aligned}$$

Now we go back to (5.39) and use the above estimate in combination with Young's inequality. This yields

$$\begin{aligned}&\int_{Q^{t_1}}\frac{(|Du_\varepsilon|-\lambda)_+^{p-1+2\alpha}}{|Du_\varepsilon|\,[\lambda^2+(|Du_\varepsilon|-\lambda)_+^2]^\alpha}\,|D^2u_\varepsilon|^2\,\chi(t)\,\eta^2(x)\,dz\\ &\quad\le\, c_8\left(1\,+\,\frac{1}{\rho^2}\right)\left[\int_{Q_R}(1+|Du_\varepsilon|^2)^{\frac p2}\,dz\,+\,\|Du_\varepsilon\|^2_{L^p(Q_R)}\right]+\,c_8\int_{t_0-\rho^2}^{t_0}\|f_\varepsilon(\cdot,t)\|^{p'}_{B^{\frac{p-2}{p}}_{p',1}(B_\rho)}\,dt\end{aligned}$$

for every $t_1\in(t_0-\rho^2,t_0)$ and a positive constant c_8 depending only on n, p, α and R. Using the properties of χ and η, from the above inequality we deduce

$$\begin{aligned}&\int_{Q_{\rho/2}(z_0)}\frac{(|Du_\varepsilon|-\lambda)_+^{p-1+2\alpha}}{|Du_\varepsilon|\,[\lambda^2+(|Du_\varepsilon|-\lambda)_+^2]^\alpha}\,|D^2u_\varepsilon|^2\,dz\\ &\quad\le\left(c_8\,+\,\frac{c_8}{\rho^2}\right)\left[\int_{Q_R}(1+|Du_\varepsilon|^2)^{\frac p2}\,dz\,+\,\|Du_\varepsilon\|^2_{L^p(Q_R)}\right]+\,c_8\int_{t_0-\rho^2}^{t_0}\|f_\varepsilon(\cdot,t)\|^{p'}_{B^{\frac{p-2}{p}}_{p',1}(B_\rho)}\,dt.\end{aligned} \tag{5.48}$$

Continuing to argue as in the proof of Proposition 3.14, we also find

$$\int_{Q_{\rho/2}(z_0)} |D_x \mathcal{V}_{\alpha,\lambda}(Du_\varepsilon)|^2 \, dz \, \leq \, c(p) \int_{Q_{\rho/2}(z_0)} \frac{(|Du_\varepsilon| - \lambda)_+^{p-1+2\alpha} \, |D^2 u_\varepsilon|^2}{|Du_\varepsilon| \, [\lambda^2 + (|Du_\varepsilon| - \lambda)_+^2]^\alpha} \, dz, \tag{5.49}$$

which combined with (5.48), gives

$$\begin{aligned}
&\int_{Q_{\rho/2}(z_0)} |D_x \mathcal{V}_{\alpha,\lambda}(Du_\varepsilon)|^2 \, dz \\
&\quad \leq \left(c_8 \, + \, \frac{c_8}{\rho^2} \right) \left[\int_{Q_R} (1 + |Du_\varepsilon|^2)^{\frac{p}{2}} \, dz \, + \, \|Du_\varepsilon\|^2_{L^p(Q_R)} \right] + \, c_8 \int_{t_0 - \rho^2}^{t_0} \|f_\varepsilon(\cdot, t)\|^{p'}_{B^{\frac{p-2}{p}}_{p',1}(B_\rho)} \, dt.
\end{aligned}$$

Let us now consider the same $\varepsilon_0 \in (0,1]$ as in Proposition 5.10 and let $\varepsilon \in (0, \varepsilon_0]$. Then, applying estimate (5.12), we obtain

$$\begin{aligned}
&\int_{Q_{\rho/2}(z_0)} |D_x \mathcal{V}_{\alpha,\lambda}(Du_\varepsilon)|^2 \, dz \\
&\quad \leq \left(C \, + \, \frac{C}{\rho^2} \right) \left[\|Du\|^p_{L^p(Q_R)} + \|Du\|^2_{L^p(Q_R)} + \lambda^p + \lambda^2 + 1 \right] + \, C \, \|f_\varepsilon\|^{p'}_{L^{p'}\left(t_0 - \rho^2, t_0; B^{\frac{p-2}{p}}_{p',1}(B_\rho) \right)},
\end{aligned}$$

for a constant $C \equiv C(n, p, \alpha, R) > 0$. Moreover, arguing again as in the proof of Proposition 3.14, we find a positive number $\varepsilon_1 \leq \varepsilon_0$ such that

$$\begin{aligned}
\|f_\varepsilon\|^{p'}_{L^{p'}\left(t_0 - \rho^2, t_0; B^{\frac{p-2}{p}}_{p',1}(B_\rho) \right)} &= \int_{t_0 - \rho^2}^{t_0} \|f_\varepsilon(\cdot, t)\|^{p'}_{B^{\frac{p-2}{p}}_{p',1}(B_\rho)} \, dt \, \leq \, \int_{t_0 - R^2}^{t_0} \|f(\cdot, t)\|^{p'}_{B^{\frac{p-2}{p}}_{p',1}(B_R)} \, dt \\
&= \, \|f\|^{p'}_{L^{p'}\left(t_0 - R^2, t_0; B^{\frac{p-2}{p}}_{p',1}(B_R) \right)} < +\infty, \qquad \text{for every } \varepsilon \in (0, \varepsilon_1].
\end{aligned}$$

Combining the last two estimates for $\varepsilon \in (0, \varepsilon_1]$, we conclude the proof in the case $\lambda > 0$.

Finally, when $\lambda = 0$ the above proof can be greatly simplified, since the parameter α plays no role and

$$\mathcal{V}_{\alpha,0}(Du_\varepsilon) = \, \frac{2}{p} \, H_{\frac{p}{2}}(Du_\varepsilon) = \, \frac{2}{p} \, |Du_\varepsilon|^{\frac{p-2}{2}} Du_\varepsilon \, .$$

In this regard, we leave the details to the reader. □

We conclude this section with the following result, which will play a crucial role in the proof of Theorem 5.3.

Proposition 5.13 (**Uniform Sobolev estimate**). *Under the assumptions of Theorem 5.3 and with the notation above, there exists a positive number $\varepsilon_0 \leq 1$ such that, for every $\varepsilon \in (0, \varepsilon_0]$, for every parabolic cylinder $Q_\rho(z_0) \Subset Q_R(z_0)$ and every $j \in \{1, \dots, n\}$, we have*

$$\int_{Q_{\rho/2}(z_0)} |D_{x_j} \mathcal{V}_{\alpha,\lambda}(Du_\varepsilon)|^2 \, dz \, \leq \, \frac{C}{\rho^2} \left(\|Du\|^2_{L^2(Q_R)} + \lambda^2 + 1 \right) \, + \, C \left(\|f\|^2_{L^2(Q_R)} + 1 \right) \tag{5.50}$$

for a positive constant C depending only on n and R in the case $\lambda = 0$, and additionally on α if $\lambda > 0$.

Proof. In what follows, we shall keep the notation, definitions and choices used in the proof of Proposition 5.12.

Let us first assume that $\lambda > 0$. Letting $\varrho \to 0$ in (5.27) and arguing exactly as in the preceding proof, we arrive at the following inequality

$$I_3 \le I_4 + I_5 + \tilde{I}_6 + I_7 + I_8\,, \tag{5.51}$$

where

$$\begin{aligned}
I_3 &:= \int_{Q^{t_1}} \sum_{j=1}^{n} \langle D^2 A_\varepsilon(Du_\varepsilon)\, D(D_{x_j} u_\varepsilon), D(D_{x_j} u_\varepsilon)\rangle\, \chi(t)\, \eta^2(x)\, \Phi(P)\, dz,\\
I_4 &:= 4 \int_{Q^{t_1}} \sum_{j=1}^{n} \langle D^2 A_\varepsilon(Du_\varepsilon)\, D\eta, D\eta\rangle\, \chi(t)\, |D_{x_j} u_\varepsilon|^2\, \Phi(P)\, dz,\\
I_5 &:= \int_{Q^{t_1}} (\partial_t \chi)\, \eta^2(x) \left[\int_0^{|Du_\varepsilon(x,t)|^2} \Phi((\sqrt{w} - \lambda)_+)\, dw \right] dz,\\
\tilde{I}_6 &:= -4 \int_{Q^{t_1}} \sum_{j=1}^{n} f_\varepsilon\, (D_{x_j} u_\varepsilon)\, \eta(x)\, (D_{x_j}\eta)\, \chi(t)\, \Phi(P)\, dz,\\
I_7 &:= -2 \int_{Q^{t_1}} \sum_{j=1}^{n} f_\varepsilon\, (D^2_{x_j} u_\varepsilon)\, \eta^2(x)\, \chi(t)\, \Phi(P)\, dz,\\
I_8 &:= -2 \int_{Q^{t_1}} \sum_{j=1}^{n} f_\varepsilon\, (D_{x_j} u_\varepsilon)\, \eta^2(x)\, \chi(t)\, \Phi'(P)\, (D_{x_j} P)\, dz,
\end{aligned}$$

while P, η, χ and Φ are chosen as in the previous proof. Note that we have already estimated I_3, I_4 and I_5 in (5.36), (5.37) and (5.38) respectively. More precisely, for $p = 2$ these estimates yield

$$I_3 \ge \int_{Q^{t_1}} \frac{(|Du_\varepsilon| - \lambda)_+^{1+2\alpha}}{|Du_\varepsilon|\,[\lambda^2 + (|Du_\varepsilon| - \lambda)_+^2]^\alpha}\, |D^2 u_\varepsilon|^2\, \chi(t)\, \eta^2(x)\, dz, \tag{5.52}$$

$$I_4 \le \frac{8\,\tilde{c}^2}{\rho^2} \int_{Q_R} (1 + |Du_\varepsilon|^2)\, dz \tag{5.53}$$

and

$$I_5 \le \frac{4}{3\rho^2}\, \|Du_\varepsilon\|^2_{L^2(Q_R)}\,. \tag{5.54}$$

We now estimate $\tilde{I}_6$, I_7 and I_8 separately. Let us first consider $\tilde{I}_6$. Using the fact that $0 \le \chi, \Phi \le 1$, the properties (5.33) of η and Young's inequality, we obtain

$$\tilde{I}_6 \le 4n \int_{Q_R} |f_\varepsilon|\, |Du_\varepsilon|\, |D\eta|\, dz \le 2n\, \|f_\varepsilon\|^2_{L^2(Q_R)} + \frac{2n\,\tilde{c}^2}{\rho^2}\, \|Du_\varepsilon\|^2_{L^2(Q_R)}\,. \tag{5.55}$$

As for the term I_7, recalling (5.24) and (5.34), we have

$$\begin{aligned}
I_7 &\le 2n \int_{Q^{t_1} \cap \{|Du_\varepsilon| > \lambda\}} |f_\varepsilon|\, |D^2 u_\varepsilon|\, \eta^2(x)\, \chi(t)\, \frac{(|Du_\varepsilon| - \lambda)_+^{2\alpha}}{[\lambda^2 + (|Du_\varepsilon| - \lambda)_+^2]^\alpha}\, dz\\
&\le 2n \int_{Q^{t_1} \cap \{|Du_\varepsilon| > \lambda\}} |f_\varepsilon|\, |D^2 u_\varepsilon|\, \eta^2(x)\, \chi(t)\, \frac{(|Du_\varepsilon| - \lambda)_+^{2\alpha-1}\, |Du_\varepsilon|}{[\lambda^2 + (|Du_\varepsilon| - \lambda)_+^2]^\alpha}\, dz.
\end{aligned}$$

Now we turn our attention to I_8. Recalling (5.35) and applying Young's inequality, we get

$$\begin{aligned} I_8 &\le 4\alpha \int_{Q^{t_1}\cap\{|Du_\varepsilon|>\lambda\}} \sum_{j=1}^n |f_\varepsilon|\,|D_{x_j}u_\varepsilon|\,\eta^2(x)\,\chi(t)\,\frac{\lambda^2\,(|Du_\varepsilon|-\lambda)_+^{2\alpha-1}}{[\lambda^2+(|Du_\varepsilon|-\lambda)_+^2]^{\alpha+1}}\cdot\frac{|\langle Du_\varepsilon, D_{x_j}Du_\varepsilon\rangle|}{|Du_\varepsilon|}\,dz \\ &\le 8\alpha n \int_{Q^{t_1}\cap\{|Du_\varepsilon|>\lambda\}} |f_\varepsilon|\,|D^2u_\varepsilon|\,\eta^2(x)\,\chi(t)\,\frac{|Du_\varepsilon|^3\,(|Du_\varepsilon|-\lambda)_+^{2\alpha-1}}{[\lambda^2+(|Du_\varepsilon|-\lambda)_+^2]^{\alpha}\,|Du_\varepsilon|^2}\,dz \\ &= 8\alpha n \int_{Q^{t_1}\cap\{|Du_\varepsilon|>\lambda\}} |f_\varepsilon|\,|D^2u_\varepsilon|\,\eta^2(x)\,\chi(t)\,\frac{(|Du_\varepsilon|-\lambda)_+^{2\alpha-1}\,|Du_\varepsilon|}{[\lambda^2+(|Du_\varepsilon|-\lambda)_+^2]^{\alpha}}\,dz. \end{aligned}$$

Joining the last two estimates and using Young's inequality again, we find

$$\begin{aligned} &I_7+I_8 \\ &\le 2n\,(4\alpha+1)\int_{Q^{t_1}\cap\{|Du_\varepsilon|>\lambda\}} \frac{|f_\varepsilon|\,|Du_\varepsilon|^{\frac32}\,(|Du_\varepsilon|-\lambda)^{\alpha-\frac32}}{[\lambda^2+(|Du_\varepsilon|-\lambda)^2]^{\frac{\alpha}{2}}}\cdot\frac{(|Du_\varepsilon|-\lambda)_+^{\alpha+\frac12}|D^2u_\varepsilon|\,\eta^2(x)\,\chi(t)}{\sqrt{|Du_\varepsilon|}\,[\lambda^2+(|Du_\varepsilon|-\lambda)_+^2]^{\frac{\alpha}{2}}}\,dz \\ &\le 2n^2(4\alpha+1)^2\int_{Q^{t_1}\cap\{|Du_\varepsilon|>\lambda\}} \frac{|f_\varepsilon|^2\,|Du_\varepsilon|^3\,(|Du_\varepsilon|-\lambda)^{2\alpha-3}\,\eta^2(x)\,\chi(t)}{[\lambda^2+(|Du_\varepsilon|-\lambda)^2]^{\alpha}}\,dz \\ &\quad+\frac12\int_{Q^{t_1}}\frac{(|Du_\varepsilon|-\lambda)_+^{1+2\alpha}\,|D^2u_\varepsilon|^2}{|Du_\varepsilon|\,[\lambda^2+(|Du_\varepsilon|-\lambda)_+^2]^{\alpha}}\,\eta^2(x)\,\chi(t)\,dz \\ &\le 4n^2(4\alpha+1)^2\int_{Q^{t_1}\cap\{|Du_\varepsilon|>\lambda\}} \frac{|f_\varepsilon|^2\,|Du_\varepsilon|^3\,(|Du_\varepsilon|-\lambda)_+^{2\alpha-3}}{|Du_\varepsilon|^{2\alpha}}\,dz \\ &\quad+\frac12\int_{Q^{t_1}}\frac{(|Du_\varepsilon|-\lambda)_+^{1+2\alpha}\,|D^2u_\varepsilon|^2}{|Du_\varepsilon|\,[\lambda^2+(|Du_\varepsilon|-\lambda)_+^2]^{\alpha}}\,\eta^2(x)\,\chi(t)\,dz \\ &\le 4n^2(4\alpha+1)^2\,\|f_\varepsilon\|^2_{L^2(Q_R)}+\frac12\int_{Q^{t_1}}\frac{(|Du_\varepsilon|-\lambda)_+^{1+2\alpha}\,|D^2u_\varepsilon|^2}{|Du_\varepsilon|\,[\lambda^2+(|Du_\varepsilon|-\lambda)_+^2]^{\alpha}}\,\eta^2(x)\,\chi(t)\,dz, \end{aligned} \tag{5.56}$$

where, in the last line, we have used the fact that $|Du_\varepsilon|^3\,(|Du_\varepsilon|-\lambda)_+^{2\alpha-3}\le|Du_\varepsilon|^{2\alpha}$ in the set $\{|Du_\varepsilon|>\lambda\}$, since $\alpha\ge\frac32$ by assumption. Now we combine estimates (5.51)−(5.56), thus obtaining

$$\int_{Q^{t_1}}\frac{(|Du_\varepsilon|-\lambda)_+^{1+2\alpha}\,|D^2u_\varepsilon|^2}{|Du_\varepsilon|\,[\lambda^2+(|Du_\varepsilon|-\lambda)_+^2]^{\alpha}}\,\eta^2(x)\,\chi(t)\,dz\le\frac{c_1}{\rho^2}\int_{Q_R}(1+|Du_\varepsilon|^2)\,dz+c_1\,\|f_\varepsilon\|^2_{L^2(Q_R)},$$

which holds for every $t_1\in(t_0-\rho^2,t_0)$ and a positive constant c_1 depending only on n and α. Using the properties of η and χ, from the above inequality we deduce

$$\int_{Q_{\rho/2}(z_0)}\frac{(|Du_\varepsilon|-\lambda)_+^{1+2\alpha}\,|D^2u_\varepsilon|^2}{|Du_\varepsilon|\,[\lambda^2+(|Du_\varepsilon|-\lambda)_+^2]^{\alpha}}\,dz\le\frac{c_1}{\rho^2}\int_{Q_R}(1+|Du_\varepsilon|^2)\,dz+c_1\,\|f_\varepsilon\|^2_{L^2(Q_R)}. \tag{5.57}$$

Taking $p=2$ into (5.49) and combining the resulting estimate with (5.57), we then find

$$\int_{Q_{\rho/2}(z_0)}|D_x\mathcal{V}_{\alpha,\lambda}(Du_\varepsilon)|^2\,dz\le\frac{c_1}{\rho^2}\int_{Q_R}(1+|Du_\varepsilon|^2)\,dz+c_1\,\|f_\varepsilon\|^2_{L^2(Q_R)}.$$

Let us now consider the same $\varepsilon_0\in(0,1]$ as in Proposition 5.10 and let $\varepsilon\in(0,\varepsilon_0]$. Then,

applying estimate (5.12) with $p = 2$, we get

$$\int_{Q_{\rho/2}(z_0)} |D_x \mathcal{V}_{\alpha,\lambda}(Du_\varepsilon)|^2 \, dz \, \leq \, \frac{C}{\rho^2} \left(\|Du\|^2_{L^2(Q_R)} + \lambda^2 + 1 \right) \, + \, C \, \|f_\varepsilon\|^2_{L^2(Q_R)}$$

for a constant $C \equiv C(n, \alpha, R) > 0$. Moreover, for $p = 2$, from (5.21) it follows that

$$\|f_\varepsilon\|_{L^2(Q_R)} \, \leq \, \|f\|_{L^2(Q_R)} + 1.$$

Combining the last two inequalities, we conclude the proof in the case $\lambda > 0$.

Finally, when $\lambda = 0$ the above proof can be greatly simplified, since the parameter α plays no role and

$$\mathcal{V}_{\alpha,0}(Du_\varepsilon) \, = \, Du_\varepsilon \, .$$

We leave the details to the reader. □

5.2 Proofs of Theorems 5.1 and 5.3

In this section, we prove Theorems 5.1 and 5.3 by combining a standard comparison argument (see e.g. [41, 52]) with the estimates from Propositions 5.10, 5.11, 5.12 and 5.13.

Proof of Theorem 5.1. In what follows, we will refer to the notation introduced in Section 5.1. For any fixed $\varepsilon \in (0, 1]$, we let

$$u_\varepsilon \in C^0 \left([t_0 - R^2, t_0]; L^2(B_R(x_0)) \right) \cap L^p \left(t_0 - R^2, t_0; W^{1,p}(B_R(x_0)) \right)$$

be the unique energy solution of the Cauchy-Dirichlet problem (5.9). Moreover, by a slight abuse of notation, for $w \in L^1_{loc}(Q_R(z_0), \mathbb{R}^k)$, $j \in \{1, \ldots, n\}$ and $h \neq 0$, we set (when $x + he_j \in B_R(x_0)$)

$$\tau_{j,h} w(x,t) := \, w(x + he_j, t) - w(x,t),$$

$$\Delta_{j,h} w(x,t) := \, \frac{w(x + he_j, t) - w(x,t)}{h},$$

where e_j is the unit vector in the direction x_j. Now we fix arbitrary radii

$$0 < r < \rho < R$$

and use the finite difference operator $\tau_{j,h}$ defined above, for increments $h \in \mathbb{R} \setminus \{0\}$ such that $|h| < \frac{\rho - r}{4}$.

Let us first assume that $\lambda > 0$. Furthermore, consider the same $\varepsilon_1 \in (0, 1]$ as in Proposition 5.12 and let $\varepsilon \in (0, \varepsilon_1]$. In the following, we will denote by C a positive constant which neither depends on h nor on ε. In order to obtain an estimate for the finite difference $\tau_{j,h} \mathcal{V}_{\alpha,\lambda}(Du)$, we use the following comparison argument:

$$\begin{aligned} &\int_{Q_{r/2}(z_0)} |\tau_{j,h} \mathcal{V}_{\alpha,\lambda}(Du)|^2 \, dz \\ &\quad \leq \, 4 \int_{Q_{r/2}(z_0)} |\tau_{j,h} \mathcal{V}_{\alpha,\lambda}(Du_\varepsilon)|^2 \, dx \, dt \, + \, 4 \int_{Q_{r/2}(z_0)} |\mathcal{V}_{\alpha,\lambda}(Du_\varepsilon) - \mathcal{V}_{\alpha,\lambda}(Du)|^2 \, dx \, dt \end{aligned}$$

$$
\begin{aligned}
&+ 4\int_{Q_{r/2}(z_0)} |\mathcal{V}_{\alpha,\lambda}(Du_\varepsilon(x+he_j,t)) - \mathcal{V}_{\alpha,\lambda}(Du(x+he_j,t))|^2\,dx\,dt \\
&\le 4\int_{Q_{r/2}(z_0)} |\tau_{j,h}\mathcal{V}_{\alpha,\lambda}(Du_\varepsilon)|^2\,dz + 8\int_{Q_R(z_0)} |\mathcal{V}_{\alpha,\lambda}(Du_\varepsilon) - \mathcal{V}_{\alpha,\lambda}(Du)|^2\,dz.
\end{aligned}
$$

Combining the previous estimate with (5.22) and (5.23), for every $j \in \{1,\dots,n\}$ we get

$$
\begin{aligned}
&\int_{Q_{r/2}} |\tau_{j,h}\mathcal{V}_{\alpha,\lambda}(Du)|^2\,dz \\
&\quad\le \frac{C(\rho^2+1)}{\rho^2}\,|h|^2\left(\|Du\|^p_{L^p(Q_R)} + \|Du\|^2_{L^p(Q_R)} + \lambda^p + \lambda^2 + 1\right) \\
&\qquad + C\,|h|^2\,\|f\|^{p'}_{L^{p'}\left(t_0-R^2,t_0;B^{\frac{p-2}{p}}_{p',1}(B_R(x_0))\right)} + C\,\varepsilon\,\left(\|Du\|^p_{L^p(Q_R)} + 1\right) \\
&\qquad + C\,\|f-f_\varepsilon\|_{L^{p'}(Q_R)}\,\left(\|Du\|_{L^p(Q_R)} + \lambda + 1\right),
\end{aligned}
\tag{5.58}
$$

which holds for every sufficiently small $h \in \mathbb{R}\setminus\{0\}$ and a constant $C \equiv C(n,p,\alpha,R) > 0$. Therefore, recalling (5.20) and letting $\varepsilon \searrow 0$ in (5.58), we obtain

$$
\begin{aligned}
\int_{Q_{r/2}} |\Delta_{j,h}\mathcal{V}_{\alpha,\lambda}(Du)|^2\,dz \le \left(C + \frac{C}{\rho^2}\right)\left[\|Du\|^p_{L^p(Q_R)} + \|Du\|^2_{L^p(Q_R)} + \lambda^p + \lambda^2 + 1\right] \\
+ C\,\|f\|^{p'}_{L^{p'}\left(t_0-R^2,t_0;B^{\frac{p-2}{p}}_{p',1}(B_R(x_0))\right)}.
\end{aligned}
\tag{5.59}
$$

Since the above estimate holds for every $j \in \{1,\dots,n\}$ and every suitably small $h \neq 0$, by Lemma 1.15 we may conclude that

$$
\mathcal{V}_{\alpha,\lambda}(Du) \in L^2_{loc}\left(0,T;W^{1,2}_{loc}(\Omega,\mathbb{R}^n)\right).
$$

Moreover, letting $h \to 0$ in the previous inequality, we also obtain estimate (5.3).

Let us now assume that $\lambda = 0$. In this case, we have already observed that the constant C is independent of α (see Proposition 5.12) and

$$
\mathcal{V}_{\alpha,0}(Du) = \frac{2}{p}\,|Du|^{\frac{p-2}{2}}Du\,.
$$

Combining this fact with Lemma 3.7 and estimate (5.59), for every sufficiently small $h \neq 0$ we find

$$
\int_{Q_{r/2}} \frac{|\tau_{j,h}Du(x,t)|^p}{|h|^2}\,dx\,dt \le C_1(p)\int_{Q_{r/2}} |\Delta_{j,h}\mathcal{V}_{\alpha,0}(Du)|^2\,dz \le M
$$

for some finite positive constant M depending on

$$
n, \quad p, \quad R, \quad \rho, \quad \|Du\|_{L^p(Q_R)}, \quad \|f\|_{L^{p'}\left(t_0-R^2,t_0;B^{\frac{p-2}{p}}_{p',1}(B_R)\right)},
$$

but not on h. Note that the above estimate also holds for every $j \in \{1,\dots,n\}$. Therefore, using Proposition 1.41 with the choices $g = Du$, $q = p$ and $\theta = \frac{2}{p}$, as well as a standard covering

argument, we infer that

$$Du \in L^p_{loc}\left(0,T;W^{\sigma,p}_{loc}(\Omega,\mathbb{R}^n)\right) \qquad \text{for all } \sigma \in \left(0,\frac{2}{p}\right).$$

This completes the proof. □

We now conclude this section by giving the

Proof of Theorem 5.3. Actually, it is sufficient to proceed as in the proof of Theorem 5.1, but using estimate (5.50) in place of (5.23). In particular, since $\mathcal{V}_{\alpha,0}(Du) = Du$, in the case $\lambda = 0$ we retrieve

$$u \in L^2_{loc}(0,T;W^{2,2}_{loc}(\Omega))\,,$$

which is a well-established result for the non-homogeneous heat equation (5.5) when $f \in L^2_{loc}(\Omega_T)$. □

5.3 Proof of Theorem 5.5

This section is devoted to the study of the existence and regularity of the time derivative of the weak solutions to equation (5.6), under the assumptions of Theorem 5.5. Indeed, we are now in a position to prove the aforementioned theorem.

Proof of Theorem 5.5. We shall keep the notation and the parabolic cylinders used for the proof of Theorem 5.1. Let us notice that

$$|Du|^{p-2}Du = \mathcal{F}(\mathcal{V}_{\alpha,0}(Du)),$$

where $\mathcal{F}:\mathbb{R}^n \to \mathbb{R}^n$ is the function defined by

$$\mathcal{F}(\xi) := \left(\frac{p}{2}\right)^{\frac{2}{p'}} |\xi|^{\frac{p-2}{p}}\xi\,,$$

which is locally Lipschitz continuous for $p > 2$. Thus, the vector field $|Du|^{p-2}Du$ is weakly differentiable with respect to the x-variable by virtue of the chain rule in Sobolev spaces. From the definitions of $\mathcal{F}$ and $\mathcal{V}_{\alpha,0}$, it follows that

$$|D_\xi\,\mathcal{F}(\mathcal{V}_{\alpha,0}(Du))| \,\le\, c_1\,|Du|^{\frac{p-2}{2}} \tag{5.60}$$

for some positive constant $c_1 \equiv c(n,p)$. Now, applying the chain rule, the Cauchy-Schwarz inequality and estimate (5.60), we obtain

$$\begin{aligned} |D_x\,\mathcal{F}(\mathcal{V}_{\alpha,0}(Du))|^{p'} &\le c_2(n,p)\,|D_\xi\,\mathcal{F}(\mathcal{V}_{\alpha,0}(Du))|^{p'}\,|D_x\mathcal{V}_{\alpha,0}(Du)|^{p'} \\ &\le c_3\,|Du|^{\frac{(p-2)p'}{2}}\,|D_x\mathcal{V}_{\alpha,0}(Du)|^{p'}, \end{aligned} \tag{5.61}$$

where $c_3 \equiv c_3(n,p) > 0$. Using (5.61), Hölder's inequality with exponents $\left(\frac{2(p-1)}{p-2}, \frac{2}{p'}\right)$ and estimate (5.3) with $\lambda = 0$, we get

$$\begin{aligned}\left(\int_{Q_{r/2}} |D_x \mathcal{F}(\mathcal{V}_{\alpha,0}(Du))|^{p'}\,dz\right)^{\frac{1}{p'}} &\leq c_4(n,p)\,\|Du\|^{\frac{p-2}{2}}_{L^p(Q_{r/2})}\left(\int_{Q_{r/2}} |D_x \mathcal{V}_{\alpha,0}(Du)|^2\,dz\right)^{\frac{1}{2}}\\ &\leq \left(C+\frac{C}{\rho}\right)\left[\|Du\|^{p-1}_{L^p(Q_R)} + \|Du\|^{\frac{p}{2}}_{L^p(Q_R)} + \|Du\|^{\frac{p-2}{2}}_{L^p(Q_R)}\right]\\ &\quad + C\,\|Du\|^{\frac{p-2}{2}}_{L^p(Q_R)}\,\|f\|^{\frac{p'}{2}}_{L^{p'}\left(t_0-R^2,t_0;B^{\frac{p-2}{p}}_{p',1}(B_R)\right)} \end{aligned} \tag{5.62}$$

for a constant $C \equiv C(n,p,R) > 0$. Note that the right-hand side of (5.62) is finite, and this implies that

$$|Du|^{p-2}Du \in L^{p'}_{loc}(0,T;W^{1,p'}_{loc}(\Omega,\mathbb{R}^n)). \tag{5.63}$$

Therefore, going back to the weak formulation (1.14), we can now perform a partial integration in the second term on the left-hand side with respect to the spatial variables. We thus obtain

$$\int_{Q_{r/2}(z_0)} u\cdot\partial_t\varphi\,dz = -\int_{Q_{r/2}(z_0)}\left(\sum_{j=1}^n D_{x_j}[|Du|^{p-2}D_{x_j}u] + f\right)\cdot\varphi\,dz$$

for any $\varphi \in C^\infty_0(Q_{r/2}(z_0))$, and the desired conclusion immediately follows from (5.63), since

$$f \in L^{p'}_{loc}\left(0,T;B^{\frac{p-2}{p}}_{p',1,loc}(\Omega)\right).$$

Furthermore, we can now observe that

$$\partial_t u = \sum_{j=1}^n D_{x_j}[|Du|^{p-2}D_{x_j}u] + f \quad \text{in } Q_{r/2}(z_0).$$

From this we can infer

$$\left(\int_{Q_{r/2}(z_0)} |\partial_t u|^{p'}\,dz\right)^{\frac{1}{p'}} \leq n\,\|D_x\mathcal{F}(\mathcal{V}_{\alpha,0}(Du))\|_{L^{p'}(Q_{r/2})} + \|f\|_{L^{p'}(Q_{r/2})},$$

and combining this inequality with (5.62), we finally obtain estimate (5.7). □

Chapter 6

Gradient bounds for strongly singular or degenerate parabolic systems

In this chapter, we are interested in the regularity of weak solutions $u : \Omega_T \to \mathbb{R}^N$ of strongly singular or degenerate parabolic systems of the type

$$\partial_t u^i - \operatorname{div} A^i(x,t,Du) = f^i \qquad \text{for } i \in \{1,\dots,N\} \qquad \text{in } \Omega_T = \Omega \times (0,T). \tag{6.1}$$

Here and in the following, Ω is a bounded and open subset of $\mathbb{R}^n$ ($n \geq 2$), $T > 0$ and $N \geq 1$. The exact assumptions on the vector field $A = (A^1,\dots,A^N)\colon \Omega_T \times \mathbb{R}^{Nn} \to \mathbb{R}^{Nn}$ and the vector-valued inhomogeneity $f = (f^1,\dots,f^N)$, $N \geq 1$, will be presented in a while. The results we report here have been obtained in [7]. As our main achievement, we will show that weak solutions of (6.1) are locally Lipschitz-continuous in the spatial directions, i.e. that the spatial gradient Du is locally bounded (see Theorem 6.1 below).

To specify the structure of the vector field $A : \Omega_T \times \mathbb{R}^{Nn} \to \mathbb{R}^{Nn}$, we consider a function

$$F\colon \Omega_T \times [0,\infty) \to [0,\infty), \qquad (x,t,s) \mapsto F(x,t,s),$$

which is convex with respect to the s-variable and vanishes for all $s \in [0,1]$. Furthermore, we shall assume that the partial map $s \mapsto F(x,t,s)$ is in $C^1(\mathbb{R}^+) \cap C^2(\mathbb{R}^+ \setminus \{1\})$ for almost every $(x,t) \in \Omega_T$, while for every $(t,s) \in (0,T) \times [0,\infty)$ the map $x \mapsto F(x,t,s)$ is differentiable almost everywhere. We additionally suppose that there exist an exponent $p > 1$ and some positive constants L, $C_1 > 1$ and K such that, for all $s > 1$ and for almost every $x, y \in \Omega$ and $t \in (0,T)$, the function F satisfies the following growth assumptions:

$$\begin{cases} \frac{1}{L}(s-1)^p \leq F(x,t,s) \leq Ls^p & (\mathrm{H}_1) \\ \frac{1}{C_1}(s-1)^{p-1} \leq \partial_s F(x,t,s) \leq C_1(s-1)^{p-1} & (\mathrm{H}_2) \\ \frac{1}{C_1}(s-1)^{p-2} \leq \partial_{ss} F(x,t,s) \leq C_1(s-1)^{p-2} & (\mathrm{H}_3) \\ |\partial_s F(x,t,s) - \partial_s F(y,t,s)| \leq K\,|x-y|\,s^{p-1}. & (\mathrm{H}_4) \end{cases}$$

Then, the vector field $A : \Omega_T \times \mathbb{R}^{Nn} \to \mathbb{R}^{Nn}$ is defined by

$$A(x,t,\xi) := \begin{cases} \frac{\partial_s F(x,t,|\xi|)}{|\xi|}\,\xi & \text{if } \xi \in \mathbb{R}^{Nn} \setminus \{0\}, \\ 0 & \text{if } \xi = 0. \end{cases} \tag{6.2}$$

Note that for the prototypical case

$$F(x,t,s) = \frac{a(x,t)}{p}\,(s-1)_+^p\,,$$

where $a : \Omega_T \to \mathbb{R}^+$ and $x \mapsto a(x,t)$ is a Lipschitz continuous function that is bounded from below by a positive constant, one can easily deduce that the growth conditions (H_1)–(H_4) are fulfilled. If furthermore $a(x,t) \equiv 1$, then we recover the vector field in (8).

6.1 Statement of the main result

In this chapter, we prove the local boundedness of the spatial gradient of weak solutions to (6.1). In particular, this can be seen as a parabolic analogue of Brasco's Lipschitz-continuity result established in [19]. Let us note that system (6.1) can be written in the vector form

$$u_t - \operatorname{div} A(x,t,Du) = f \qquad \text{in } \Omega_T\,.$$

A function $u \in C^0\left((0,T);L^2\left(\Omega,\mathbb{R}^N\right)\right) \cap L^p\left(0,T;W^{1,p}\left(\Omega,\mathbb{R}^N\right)\right)$ is a *weak solution* of (6.1) if and only if

$$\int_{\Omega_T} \left(u\cdot\partial_t\phi - \langle A(x,t,Du), D\phi\rangle\right) dz \;=\; -\int_{\Omega_T} f\cdot\phi\, dz$$

holds true for any test function $\phi \in C_0^\infty(\Omega_T,\mathbb{R}^N)$.

The main result of this chapter is the following theorem. For notation and definitions we refer to Chapter 1.

Theorem 6.1. *Let $n \geq 2$, $N \geq 1$, $1 < p < \infty$ and $f \in L^{\hat n+2}\log^\alpha L_{loc}(\Omega_T,\mathbb{R}^N)$, where $\hat n$ is defined according to (6.3) – (6.4) below and $\alpha > 2\hat n + 3$. Moreover, assume that*

$$u \in C^0\left((0,T);L^2\left(\Omega,\mathbb{R}^N\right)\right) \cap L^p\left(0,T;W^{1,p}\left(\Omega,\mathbb{R}^N\right)\right)$$

is a weak solution to (6.1), where A is defined by (6.2) and the structure conditions $(\mathrm{H}_1)-(\mathrm{H}_4)$ above are in force. Then, for any parabolic cylinder $Q_r(z_1) \Subset Q_R(z_0) \Subset \Omega_T$ with $r \in (0,1)$ and any $s \in (0,1)$, we have that:

(a) if $p > \frac{2n}{n+2}$, the estimate

$$\operatorname*{ess\,sup}_{Q_{sr}(z_1)} |Du| \;\leq\; \frac{c}{(1-s)^{\frac{\hat n+2}{\kappa}}}\cdot\exp\left(c\,\frac{\|f\|^\Theta_{L^{\hat n+2}\log^\alpha L(Q_r(z_1))}}{r^{\frac{n-\hat n}{\alpha}}}\right)\cdot\left[\fint_{Q_r(z_1)}(|Du|+1)^p\,dz\right]^{\frac{1}{\kappa}}$$

holds true for some positive constants $c \equiv c(N,n,\hat n,\alpha,p,C_1,K)$ and $\Theta \equiv \Theta(\hat n,\alpha)$, and for

$$\kappa = \begin{cases} \frac{p(\hat n+2)-2\hat n}{2} & \text{if } \frac{2n}{n+2} < p < 2,\\ 2 & \text{if } p \geq 2\,; \end{cases}$$

(*b*) *if* $1 < p \le \frac{2n}{n+2}$, $f = 0$ *and* $u \in L^\infty(Q_R(z_0), \mathbb{R}^N)$, *the estimate*

$$\operatorname*{ess\,sup}_{Q_{sr}(z_1)} |Du| \;\le\; c\,\frac{\left(1 + \|u\|^2_{L^\infty(Q_R(z_0))}\right)^{\frac{n+2}{2p}}}{\left[(1-s)^2\, r\right]^{\frac{n+2}{p}}} \left[\fint_{Q_r(z_1)} (|Du|+1)^p dz\right]^{\frac{1}{p}}$$

holds true for some positive constant $c \equiv c(N, n, \alpha, p, C_1, K)$.

In the subcritical case $1 < p \le \frac{2n}{n+2}$, the weak solutions to (6.1) may not be bounded (see [38, Sub-chapter 5.4] and note that the p-Laplacian satisfies our growth assumptions). Therefore, the extra assumption $u \in L^\infty(Q_R(z_0), \mathbb{R}^N)$ in Theorem 6.1 (*b*) is natural.

Note that some widely degenerate systems can be interpreted, similarly to functionals, as asymptotically regular systems. Therefore, in the special case that $A(x,t,Du) \equiv A(Du) \approx |Du|^{p-2}Du$ for $|Du| \gg 1$ and $p > \frac{2n}{n+2}$, we recover the results obtained in [15] and [62].

Finally, we explain the choice of the parameter $\hat{n}$, which is defined by

$$\hat{n} := \begin{cases} n & \text{if } n > 2 \\ 2+\beta & \text{if } n = 2 \end{cases} \tag{6.3}$$

for some $\beta > 0$. If $n = 2$ and $1 < p < 2$, we choose β in such a way that

$$0 < \beta < \frac{4(p-1)}{2-p}\,. \tag{6.4}$$

This choice ensures that in the case $n = 2$ we have $p > \frac{2\hat{n}}{\hat{n}+2}$ for any $p > 1$. Indeed, this condition will be decisive to carry out the proof in Subsection 6.6.3.

6.1.1 Strategy of the proof

Our approach is inspired by [29, 35, 44]. The proof of Theorem 6.1 will be achieved by a parabolic Moser iteration technique, whose implementation is however quite subtle, due to the degeneracy of the differential operator. Since the weak solutions of (6.1) may not be twice weakly differentiable with respect to the x-variable, we consider a regularization of the original system (6.1). The regularized problem $(\mathcal{P}_\varepsilon)$, $\varepsilon > 0$, satisfies standard *non-degenerate* p-growth conditions (see Subsection 6.2.3 and Section 6.3). One of the advantages of introducing the regularized problem stems from the fact that its (unique) weak solution u_ε admits second weak spatial derivatives in L^2_{loc} (cf. Remark 6.10 and Theorem 6.11). However, proceeding at this point with a standard Moser iteration, the constants in our estimates would blow up as $\varepsilon \downarrow 0$. This problem will be overcome by choosing an appropriate test function in the derivation of the Caccioppoli-type inequality on which the Moser iteration is based. At this stage, it is helpful to realize that, for $\delta > 0$ large enough and $|Du_\varepsilon| > 1 + \delta$, the regularized systems satisfy p-growth conditions that are independent of ε. Therefore, we choose a test function vanishing on the set $\{|Du_\varepsilon| \le 1 + \delta\}$. We notice that the choice of such a test function requires the existence of second weak spatial derivatives, which is why we need to resort to the approximating solutions u_ε. The said test function allows us to obtain a Caccioppoli-type inequality, which will in turn be used to set up a kind of Moser iteration scheme.

Similarly to the treatment of the parabolic p-Laplacian, in the Moser iteration we have

to distinguish between three different regimes of the parameter p. The first one is the superquadratic case $p \geq 2$, the second one is the subquadratic supercritical case $\frac{2n}{n+2} < p < 2$ and finally we have the subcritical case $1 < p \leq \frac{2n}{n+2}$. In the last case, we need to ensure that also the approximating solutions u_ε are uniformly bounded with respect to ε. This is achieved by a maximum principle, assuming that $f = 0$. In all three previous cases, we establish the local boundedness of the spatial gradient Du_ε together with quantitative estimates. Here it is worth noting that the constants appearing in our quantitative estimates are independent of ε. This will allow us to obtain the conclusions of Theorem 6.1 after passing to the limit as $\varepsilon \downarrow 0$ and using the strong convergence result in Lemma 6.12 below.

This chapter is organized as follows. In Subsection 6.2.1, we recall some basic facts on the regularization in time via Steklov averages. In the following Subsections 6.2.2 and 6.2.3, we define the approximation of the vector field A and derive its growth properties. Thereby, we distinguish the subquadratic and superquadratic cases. Subsection 6.2.4 is devoted to some algebraic inequalities needed for our purposes. In Section 6.3, we introduce the regularized problems $(\mathcal{P}_\varepsilon)$ and prove a strong convergence result in L^p. As a by-product, we obtain the uniqueness of weak solutions to a Cauchy-Dirichlet problem associated with system (6.1). In Section 6.4, we establish a maximum principle for the solutions of the regularized problems in the case $1 < p \leq \frac{2n}{n+2}$ and $f = 0$. This allows us to show that the approximating solutions are essentially bounded, provided that the solution of (6.1) is itself essentially bounded. In Section 6.5 we lay the groundwork for Section 6.6, where we prove local L^∞-bounds for the spatial gradient of the approximating solutions. The proof is divided into several subsections. In Subsection 6.6.1 we establish a Caccioppoli-type inequality, from which we derive a reverse Hölder-type inequality in Subsection 6.6.2. In these subsections, we need to address separately the three regimes of the parameter p that we referred to above. For this reason, the Moser iteration procedure is split into two steps: in Subsection 6.6.3 we deal with the supercritical case $p > \frac{2n}{n+2}$, while in Subsection 6.6.4 we consider the subcritical one. Finally, in Section 6.7 we complete the proof of Theorem 6.1.

6.2 Preliminary results

6.2.1 Steklov averages

Here we recall the definition and some elementary properties of Steklov averages. Let us denote a domain in space-time by $Q' := \Omega' \times I$, where $\Omega' \subseteq \Omega$ is a bounded domain and $I := (t_1, t_2) \subseteq (0, T)$. For every $h \in (0, t_2 - t_1)$ and $v \in L^1(\Omega' \times I, \mathbb{R}^k)$, where $k \in \mathbb{N}$, the *Steklov average* $[v]_h(\cdot, t)$ is defined by

$$[v]_h(x,t) := \begin{cases} \frac{1}{h}\int_t^{t+h} v(x,s)\,ds & \text{for } t \in (t_1, t_2 - h), \\ 0 & \text{for } t \in [t_2 - h, t_2), \end{cases}$$

for $x \in \Omega'$. This definition implies, for almost every $(x,t) \in \Omega' \times (t_1, t_2 - h)$,

$$\frac{\partial [v]_h}{\partial t}(x,t) = \frac{v(x,t+h) - v(x,t)}{h}\,.$$

The proof of the following result is straightforward from the theory of Lebesgue spaces (see [38, Lemma 3.2]).

Lemma 6.2. *Let $q, r \geq 1$ and $v \in L^r(t_1, t_2; L^q(\Omega'))$. Then, as $h \to 0$, $[v]_h$ converges to v in $L^r(t_1, t_2 - \varepsilon; L^q(\Omega'))$ for every $\varepsilon \in (0, t_2 - t_1)$. If $v \in C^0((t_1, t_2); L^q(\Omega'))$, then as $h \to 0$, $[v]_h(\cdot, t)$ converges to $v(\cdot, t)$ in $L^q(\Omega')$ for every $t \in (t_1, t_2 - \varepsilon)$, $\forall\, \varepsilon \in (0, t_2 - t_1)$.*

For further needs, we now recall the well-known Steklov average formulation of (6.1) in $Q' = \Omega' \times (t_1, t_2)$. Assume that $u \in C^0\left((0,T); L^2(\Omega, \mathbb{R}^N)\right) \cap L^p\left(0, T; W^{1,p}(\Omega, \mathbb{R}^N)\right)$ is a weak solution to (6.1) in Ω_T and let $0 < h < t_2 - t_1$. Then, the Steklov average $[u]_h(\cdot, t)$ satisfies, for all times $t \in (t_1, t_2 - h)$,

$$\int_{\Omega' \times \{t\}} \left(\partial_t [u]_h \cdot \phi + \langle [A(x,t,Du)]_h, D\phi \rangle\right) dx = \int_{\Omega' \times \{t\}} [f]_h \cdot \phi \, dx$$

for all $\phi \in C_0^\infty(\Omega', \mathbb{R}^N)$.

6.2.2 Approximation of the function F for $1 < p \leq 2$

As already mentioned, we assume that for almost every $(x,t) \in \Omega_T$ the map $s \mapsto F(x,t,s)$ is in $C^1(\mathbb{R}^+) \cap C^2(\mathbb{R}^+ \setminus \{1\})$. Moreover, we notice that in the case $1 < p < 2$ both bounds for $\partial_{ss} F(x,t,\cdot)$ in (H_3) blow up as $s \to 1^+$. This very singular behavior of F must be avoided, since we need to use the second derivative $\partial_{ss} F$ to establish a local bound for the spatial gradient of the weak solutions to (6.1). Therefore, for $1 < p < 2$ and for almost every $(x,t) \in \Omega_T$ we approximate the partial map $s \mapsto F(x,t,s)$ by smoothing it around the singularity of $\partial_{ss} F(x,t,\cdot)$, in such a way that the resulting approximation $F_\varepsilon(x,t,\cdot)$ coincides with $F(x,t,\cdot)$ outside a small neighborhood of the singularity $s = 1$.

Thus, in this section we will assume that $1 < p \leq 2$, unless otherwise stated. For $\varepsilon \in \left(0, \frac{1}{2}\right)$ we define the function $v_\varepsilon : \mathbb{R}_0^+ \to \mathbb{R}^+$ by

$$v_\varepsilon(s) := \frac{1}{\varepsilon} \int_{\mathbb{R}} \eta_1\Big(\frac{w - s}{\varepsilon}\Big) \max\{\varepsilon, w - 1\} \, dw, \tag{6.5}$$

where $\eta_1 \in C_0^\infty((-1,1))$ denotes the standard, non-negative, radially symmetric mollifier in $\mathbb{R}$. Keeping this definition in mind, in what follows we will show that an approximation of $F(x,t,\cdot)$ is given by

$$\tilde{F}_\varepsilon(x,t,s) := F(x,t,v_\varepsilon(s) + 1), \qquad \varepsilon \in \left(0, \tfrac{1}{2}\right).$$

Firstly, we need to prove that $\tilde{F}_\varepsilon$ satisfies non-degenerate growth conditions for $0 < \varepsilon < \frac{1}{2}$. Therefore, we begin our analysis by studying the growth of this function.

By assumption, we know that for almost every $(x,t) \in \Omega_T$ the map $s \mapsto \tilde{F}_\varepsilon(x,t,s)$ belongs to $C^2(\mathbb{R}^+)$. For later purposes, we now note that one can easily check that

$$v_\varepsilon(s) = \begin{cases} \varepsilon & \text{if } 0 \leq s \leq 1 \\ s - 1 & \text{if } s \geq 1 + 2\varepsilon \end{cases} \tag{6.6}$$

and

$$0 \leq v_\varepsilon'(s) \leq \mathbb{1}_{\{\cdot > 1\}}(s), \qquad 0 \leq v_\varepsilon''(s) \leq \tfrac{C}{\varepsilon} \mathbb{1}_{\{1 < \cdot < 1 + 2\varepsilon\}}(s) \tag{6.7}$$

for all $s \geq 0$ and for some universal constant $C > 0$. Furthermore, from the growth assumption (H_1) and from definition (6.5) we can deduce that

$$\frac{1}{cL}(s^p - 1) \leq \tilde{F}_\varepsilon(x,t,s) \leq cL(s^p + 1)$$

for all $s \geq 1 + 2\varepsilon$ and for almost every $(x,t) \in \Omega_T$. As for the derivatives of $\tilde{F}_\varepsilon$ with respect to the s-variable, for almost every $(x,t) \in \Omega_T$ we have

$$\partial_s \tilde{F}_\varepsilon(x,t,s) \,=\, \partial_s F(x,t,v_\varepsilon(s)+1)\, v_\varepsilon'(s) \tag{6.8}$$

and

$$\partial_{ss} \tilde{F}_\varepsilon(x,t,s) \,=\, \partial_{ss} F(x,t,v_\varepsilon(s)+1)(v_\varepsilon'(s))^2 \,+\, \partial_s F(x,t,v_\varepsilon(s)+1)\, v_\varepsilon''(s), \tag{6.9}$$

and from assumptions (H_2) and (H_3) it follows that $\partial_s \tilde{F}_\varepsilon,\ \partial_{ss} \tilde{F}_\varepsilon \geq 0$. Moreover, combining (6.7), (6.8), (H_2) and the fact that $v_\varepsilon(s) \leq \max\{2\varepsilon, s-1\} \leq s$ for every $s > 1$ and every $\varepsilon \in \left(0, \frac{1}{2}\right)$, for almost every $(x,t) \in \Omega_T$ we obtain

$$\partial_s \tilde{F}_\varepsilon(x,t,s) \,\leq\, C_1\,(v_\varepsilon(s))^{p-1}\,\mathbb{1}_{\{\cdot\,>1\}}(s) \,\leq\, C_1\, s^{p-1}\,\mathbb{1}_{\{\cdot\,>1\}}(s) \tag{6.10}$$

for any $s > 0$ and any $\varepsilon \in \left(0, \frac{1}{2}\right)$. As for the second derivative $\partial_{ss} \tilde{F}_\varepsilon$, combining (6.7), (6.9), (H_2) and (H_3), for every $s > 0$ and for almost every $(x,t) \in \Omega_T$ we find

$$\partial_{ss} \tilde{F}_\varepsilon(x,t,s) \,\leq\, C_1\,(v_\varepsilon(s))^{p-2}\,\mathbb{1}_{\{\cdot\,>1\}}(s) \,+\, \frac{C_1 C}{\varepsilon}(v_\varepsilon(s))^{p-1}\,\mathbb{1}_{\{1<\cdot\,<1+2\varepsilon\}}(s). \tag{6.11}$$

Now, using the fact $v_\varepsilon(s) \leq 2\varepsilon$ for $s < 1+2\varepsilon$, we can estimate the second term on the right-hand side of (6.11) as follows:

$$\begin{aligned}\frac{(v_\varepsilon(s))^{p-1}}{\varepsilon}\,\mathbb{1}_{\{1<\cdot\,<1+2\varepsilon\}}(s) &\leq\, 2^{p-1}\,\varepsilon^{p-2}\,\frac{(1+s^2)^{\frac{p-2}{2}}}{(1+s^2)^{\frac{p-2}{2}}}\,\mathbb{1}_{\{1<\cdot\,<1+2\varepsilon\}}(s)\\ &\leq\, 2^{p-1}\,5^{\frac{2-p}{2}}\,\varepsilon^{p-2}\,(1+s^2)^{\frac{p-2}{2}}.\end{aligned} \tag{6.12}$$

In order to deal with the first term on the right-hand side of (6.11), we distinguish between the cases $1 < s < 1 + 2\varepsilon$ and $s \geq 1 + 2\varepsilon$. In the first case, we have $v_\varepsilon(s) \geq \varepsilon$ and therefore we get

$$(v_\varepsilon(s))^{p-2}\,\mathbb{1}_{\{\cdot\,>1\}}(s) \,\leq\, \varepsilon^{p-2} \,\leq\, 5^{\frac{2-p}{2}}\,\varepsilon^{p-2}\,(1+s^2)^{\frac{p-2}{2}}. \tag{6.13}$$

If, on the other hand, $s \geq 1 + 2\varepsilon$, then we have $v_\varepsilon(s) = s - 1$ due to (6.6). In this case, using that $(s-1)^2 \geq \frac{\varepsilon^2}{4}(1+s^2)$ we obtain

$$(v_\varepsilon(s))^{p-2}\,\mathbb{1}_{\{\cdot\,>1\}}(s) \,=\, (s-1)^{p-2} \,\leq\, 2^{2-p}\,\varepsilon^{p-2}\,(1+s^2)^{\frac{p-2}{2}} \,\leq\, 5^{\frac{2-p}{2}}\,\varepsilon^{p-2}\,(1+s^2)^{\frac{p-2}{2}}. \tag{6.14}$$

Joining estimates (6.11)$-$(6.14), for every $s > 0$ and for almost every $(x,t) \in \Omega_T$ we then have

$$\partial_{ss} \tilde{F}_\varepsilon(x,t,s) \leq c\,\varepsilon^{p-2}(1+s^2)^{\frac{p-2}{2}}, \tag{6.15}$$

where $c \equiv c(p, C_1) > 0$. This concludes the necessary growth estimates of $\tilde{F}_\varepsilon$.

Now, in order to prove that the function $\tilde{F}_\varepsilon$ is indeed a good approximation of F, it remains

to analyze the behavior of $\tilde{F}_\varepsilon$ as $\varepsilon \searrow 0$. To this end, we notice that (6.6) immediately implies

$$\partial_s \tilde{F}_\varepsilon(x,t,s) \equiv \partial_s F(x,t,s) \qquad \text{for } s \notin (1, 1+2\varepsilon). \tag{6.16}$$

Furthermore, for $s \in [1, 1+2\varepsilon]$ we can estimate the difference of these two derivatives as follows:

$$\begin{aligned} |\partial_s \tilde{F}_\varepsilon(x,t,s) - \partial_s F(x,t,s)| &\le |\partial_s \tilde{F}_\varepsilon(x,t,s)| + |\partial_s F(x,t,s)| \\ &\le 2^p\, C_1\, \varepsilon^{p-1} \le\ 2^p\, C_1\, \varepsilon^{p-1} s^{p-1}. \end{aligned} \tag{6.17}$$

Let us explicitly observe that the last term tends to zero as $\varepsilon \searrow 0$. To ensure the convergence result of Lemma 6.12 below, we need to accelerate the rate of convergence. For this reason, for any $1 < p < 2$ we now define the new approximation

$$F_\varepsilon(x,t,s) := \tilde{F}_{\varepsilon^{\frac{1}{p-1}}}(x,t,s), \qquad s \ge 0. \tag{6.18}$$

Collecting the above conclusions, we note that F_ε has the following properties:

Lemma 6.3. *For every $\varepsilon \in (0, 2^{1-p})$, for almost every $(x,t) \in \Omega_T$ and every $s \in \mathbb{R}_0^+$, we have*

$$\begin{cases} \frac{1}{cL}(s^p - 1) \le F_\varepsilon(x,t,s) \le cL(s^p+1), \\ 0 \le \partial_s F_\varepsilon(x,t,s) \le C_1\, s^{p-1}\, \mathbb{1}_{\{\cdot > 1\}}(s), \\ 0 \le \partial_{ss} F_\varepsilon(x,t,s) \le c\, \varepsilon^{\frac{p-2}{p-1}} (1+s^2)^{\frac{p-2}{2}}, \\ |\partial_s F_\varepsilon(x,t,s) - \partial_s F(x,t,s)| \le 2^p\, C_1\, \varepsilon\, s^{p-1}, \\ \partial_s F_\varepsilon(x,t,s) \equiv \partial_s F(x,t,s) \qquad \text{if } s \notin \left(1, 1+2\varepsilon^{\frac{1}{p-1}}\right). \end{cases}$$

However, notice that $2^{1-p} \ge \frac{1}{2}$ whenever $1 < p \le 2$, which implies that the previous lemma holds for any $\varepsilon \in \left(0, \frac{1}{2}\right)$.

6.2.3 Approximation of the vector field A

With the approximation (6.18) of F and Lemma 6.3 in mind, we can define for every $\varepsilon \in (0, 1/2)$ the vector field $A_\varepsilon : \Omega_T \times \mathbb{R}^{Nn} \to \mathbb{R}^{Nn}$ by

$$A_\varepsilon(x,t,\xi) := h_\varepsilon(x,t,|\xi|)\, \xi, \tag{6.19}$$

where

$$h_\varepsilon(x,t,s) := \begin{cases} \frac{\partial_s F_\varepsilon(x,t,s)}{s} + \varepsilon(1+s^2)^{\frac{p-2}{2}} & \text{if } 1 < p \le 2, \\ \frac{\partial_s F(x,t,s)}{s} + \varepsilon(1+s^2)^{\frac{p-2}{2}} & \text{if } p > 2. \end{cases} \tag{6.20}$$

We thus approximate the structure function A by means of the vector fields A_ε, in order to be allowed to apply some results from the theory of singular or degenerate parabolic systems to the weak solutions of problem $(\mathcal{P}_\varepsilon)$, introduced in Section 6.3. Therefore, we now need to check whether A_ε fulfills non-degenerate growth conditions. This is what we will do hereafter.

From the growth conditions of $\partial_s F_\varepsilon$ and from the structure of the approximation, for any $p > 1$ and for any $\varepsilon \in (0, \frac{1}{2})$ we immediately obtain

$$\varepsilon(1+|\xi|^2)^{\frac{p-2}{2}}|\xi|^2 \le \langle A_\varepsilon(x,t,\xi), \xi\rangle \le (C_1+\varepsilon)(1+|\xi|^2)^{\frac{p}{2}} \tag{6.21}$$

for every $\xi \in \mathbb{R}^{Nn}$ and for almost every $(x,t) \in \Omega_T$. As for the spatial gradient of A_ε, by the assumption (H_4) we have

$$|D_x A_\varepsilon(x,t,\xi)| \le 2^{p-1} K (1+|\xi|)^{p-1} \tag{6.22}$$

for every $\xi \in \mathbb{R}^{Nn}$, for every $\varepsilon \in (0,\frac{1}{2})$ and for almost every $(x,t) \in \Omega_T$.

Now we want to examine the structure of $D_\xi A_\varepsilon(x,t,\xi)$. To this end, for any $\xi \in \mathbb{R}^{Nn} \setminus \{0\}$ and for any $\varepsilon \in (0,\frac{1}{2})$ we define the bilinear form $\mathcal{A}_\varepsilon(x,t,\xi) : \mathbb{R}^{Nn^2} \times \mathbb{R}^{Nn^2} \to \mathbb{R}$ by

$$\mathcal{A}_\varepsilon(x,t,\xi)(\lambda,\zeta) := h_\varepsilon(x,t,|\xi|)\,\lambda\cdot\zeta + \partial_s h_\varepsilon(x,t,|\xi|) \sum_{i,j=1}^{N} \sum_{k,\ell,m=1}^{n} \frac{\xi_k^i\,\lambda_{km}^i\,\xi_\ell^j\,\zeta_{\ell m}^j}{|\xi|} \qquad \text{for } \lambda,\zeta \in \mathbb{R}^{Nn^2},$$

and observe that

$$\sum_{i,j=1}^{N} \sum_{k,\ell,m=1}^{n} D_{\xi_k^i}(A_\varepsilon)_\ell^j(x,t,\xi)\,\lambda_{km}^i\,\zeta_{\ell m}^j = \mathcal{A}_\varepsilon(x,t,\xi)(\lambda,\zeta). \tag{6.23}$$

The next lemma provides the relevant ellipticity and boundedness properties of the bilinear form $\mathcal{A}_\varepsilon(x,t,\xi)$.

Lemma 6.4. *Let $1 < p < \infty$ and $\varepsilon \in (0,\frac{1}{2})$. Then, there exists a positive constant $C \equiv C(p,C_1,\varepsilon)$ such that, for any $\xi \in \mathbb{R}^{Nn} \setminus \{0\}$ and any $\lambda \in \mathbb{R}^{Nn^2}$, we have*

$$\varepsilon\,\min\{1,p-1\}\,(1+|\xi|^2)^{\frac{p-4}{2}}|\xi|^2\,|\lambda|^2 \le \mathcal{A}_\varepsilon(x,t,\xi)(\lambda,\lambda) \le C\,(1+|\xi|^2)^{\frac{p-2}{2}}\,|\lambda|^2\,. \tag{6.24}$$

Moreover, for $\delta > 4\varepsilon^{\frac{1}{p-1}}$ and $|\xi| \ge 1+\frac{\delta}{2}$ we get

$$\frac{1}{c}\,|\xi|^{p-2}\,|\lambda|^2 \le \mathcal{A}_\varepsilon(x,t,\xi)(\lambda,\lambda) \le c\,|\xi|^{p-2}\,|\lambda|^2\,, \tag{6.25}$$

where $c \equiv c(p,C_1,\delta) > 1$.

Proof. From (6.20) it follows that

$$\partial_s h_\varepsilon(x,t,s) = \begin{cases} \frac{\partial_{ss}F_\varepsilon(x,t,s)}{s} - \frac{\partial_s F_\varepsilon(x,t,s)}{s^2} + (p-2)\varepsilon s(1+s^2)^{\frac{p-4}{2}} & \text{if } 1<p\le 2 \\ \frac{\partial_{ss}F(x,t,s)}{s} - \frac{\partial_s F(x,t,s)}{s^2} + (p-2)\varepsilon s(1+s^2)^{\frac{p-4}{2}} & \text{if } p>2. \end{cases}$$

In order to prove the assertion, we distinguish between two cases.

When $\partial_s h_\varepsilon(x,t,|\xi|) < 0$, from Lemma 6.3 and the growth assumption (H_2) we obtain

$$\begin{aligned} \mathcal{A}_\varepsilon(x,t,\xi)(\lambda,\lambda) &\le h_\varepsilon(x,t,|\xi|)\,|\lambda|^2 \\ &\le \begin{cases} C_1\,|\xi|^{p-2}\,\mathbb{1}_{\{|\xi|\ge 1\}}\,|\lambda|^2 + \varepsilon(1+|\xi|^2)^{\frac{p-2}{2}}\,|\lambda|^2 & \text{for } 1<p\le 2 \\ (C_1+\varepsilon)(1+|\xi|^2)^{\frac{p-2}{2}}\,|\lambda|^2 & \text{for } p>2 \end{cases} \\ &\le \begin{cases} \left(2^{\frac{2-p}{2}}C_1+1\right)(1+|\xi|^2)^{\frac{p-2}{2}}\,|\lambda|^2 & \text{for } 1<p\le 2 \\ (C_1+1)(1+|\xi|^2)^{\frac{p-2}{2}}\,|\lambda|^2 & \text{for } p>2, \end{cases} \end{aligned}$$

where we have applied the inequality $|\xi|^2\,\mathbb{1}_{\{|\xi|\ge 1\}} \ge \frac{1}{2}(1+|\xi|^2)\,\mathbb{1}_{\{|\xi|\ge 1\}}$ in the case $1<p\le 2$ and

$$\frac{(s-1)_+^{p-1}}{s} \le s^{p-2} \le (1+s^2)^{\frac{p-2}{2}} \qquad \text{for } s>0 \text{ and } p>2.$$

This proves the upper bound in (6.24), and the one in (6.25) is an immediate consequence. Moreover, using the Cauchy-Schwarz inequality, we get

$$\mathcal{A}_\varepsilon(x,t,\xi)(\lambda,\lambda) \geq \begin{cases} \partial_{ss}F_\varepsilon(x,t,|\xi|)\,|\lambda|^2 + (p-1)\varepsilon(1+|\xi|^2)^{\frac{p-4}{2}}\,|\xi|^2\,|\lambda|^2 & \text{for } 1<p\leq 2,\\ \partial_{ss}F(x,t,|\xi|)\,|\lambda|^2 + (p-1)\varepsilon(1+|\xi|^2)^{\frac{p-4}{2}}\,|\xi|^2\,|\lambda|^2 & \text{for } p>2. \end{cases}$$

Since $\partial_{ss}F_\varepsilon(x,t,|\xi|)$, $\partial_{ss}F(x,t,|\xi|) \geq 0$, we find that

$$\mathcal{A}_\varepsilon(x,t,\xi)(\lambda,\lambda) \geq (p-1)\varepsilon(1+|\xi|^2)^{\frac{p-4}{2}}|\xi|^2\,|\lambda|^2 \qquad \text{for every } p>1,$$

thus proving the lower bound in (6.24). To obtain the lower bound in (6.25), we observe that $F_\varepsilon(x,t,|\xi|) = F(x,t,|\xi|)$ for $|\xi| \geq 1+\frac{\delta}{2}$, so that in view of (H_3) we have

$$\mathcal{A}_\varepsilon(x,t,\xi)(\lambda,\lambda) \;\geq\; \partial_{ss}F(x,t,|\xi|)\,|\lambda|^2 \;\geq\; \frac{1}{C_1}(|\xi|-1)^{p-2}\,|\lambda|^2 \;\geq\; \frac{1}{c}\,|\xi|^{p-2}\,|\lambda|^2$$

for some positive constant $c \equiv c(p, C_1, \delta)$.

In the case $\partial_s h_\varepsilon(x,t,|\xi|) \geq 0$, applying the Cauchy-Schwarz inequality again, from Lemma 6.3 and the growth condition (H_3) we get

$$\begin{aligned}\mathcal{A}_\varepsilon(x,t,\xi)(\lambda,\lambda) &\leq \begin{cases} \partial_{ss}F_\varepsilon(x,t,|\xi|)\,|\lambda|^2 + (p-1)\varepsilon(1+|\xi|^2)^{\frac{p-4}{2}}\,|\xi|^2\,|\lambda|^2 & \text{for } 1<p\leq 2\\ \partial_{ss}F(x,t,|\xi|)\,|\lambda|^2 + (p-1)\varepsilon(1+|\xi|^2)^{\frac{p-4}{2}}\,|\xi|^2\,|\lambda|^2 & \text{for } p>2 \end{cases}\\ &\leq \begin{cases} \left(p-1+C\varepsilon^{\frac{p-2}{p-1}}\right)(1+|\xi|^2)^{\frac{p-2}{2}}\,|\lambda|^2 & \text{for } 1<p\leq 2\\ (p-1+C_1)(1+|\xi|^2)^{\frac{p-2}{2}}\,|\lambda|^2 & \text{for } p>2, \end{cases}\end{aligned}$$

where we have used the inequality $(s-1)_+^{p-2} \leq (s^2+1)^{\frac{p-2}{2}}$, which holds for every $s \geq 0$ and every $p>2$. To obtain the lower bound in (6.24), we neglect the term $\partial_s h_\varepsilon(x,t,|\xi|)$ and use the fact that $\partial_s F_\varepsilon(x,t,|\xi|)$, $\partial_s F(x,t,|\xi|) \geq 0$. Thus we have

$$\mathcal{A}_\varepsilon(x,t,\xi)(\lambda,\lambda) \;\geq\; h_\varepsilon(x,t,|\xi|)\,|\lambda|^2 \;\geq\; \varepsilon(1+|\xi|^2)^{\frac{p-4}{2}}\,|\xi|^2\,|\lambda|^2 \qquad \text{for every } p>1.$$

Finally, to establish the bounds in (6.25), one can argue as above. This time, for $|\xi| \geq 1+\frac{\delta}{2}$ we are allowed to use the growth assumptions (H_2) and (H_3) also in the case $1<p\leq 2$, since $F_\varepsilon(x,t,|\xi|) = F(x,t,|\xi|)$. Furthermore, we can estimate the term $\varepsilon(1+|\xi|^2)^{\frac{p-4}{2}}|\xi|^2$ by zero from below and make use of $\varepsilon(1+|\xi|^2)^{\frac{p-4}{2}}|\xi|^2 \leq \varepsilon(1+|\xi|^2)^{\frac{p-2}{2}} \leq 2^{\frac{(p-2)_+}{2}}|\xi|^{p-2}$. After doing this, we get the desired estimates by means of the following inequalities, which hold for $|\xi| \geq 1+\frac{\delta}{2}$:

$$\left(\frac{\delta}{2+\delta}\right)^{p-1}|\xi|^{p-2} \leq \frac{(|\xi|-1)_+^{p-1}}{|\xi|} \leq |\xi|^{p-2} \tag{6.26}$$

and

$$\min\left\{1, \left(\frac{\delta}{2+\delta}\right)^{p-2}\right\}|\xi|^{p-2} \leq (|\xi|-1)_+^{p-2} \leq \max\left\{1, \left(\frac{\delta}{2+\delta}\right)^{p-2}\right\}|\xi|^{p-2}. \tag{6.27}$$

□

From the previous lemma, it follows that the bilinear form $(\lambda,\zeta) \mapsto \mathcal{A}_\varepsilon(x,t,\xi)(\lambda,\zeta)$ defines a scalar product on the Euclidean space $\mathbb{R}^{Nn^2}$. As for the modulus of $\mathcal{A}_\varepsilon$, we get the following

result.

Lemma 6.5. *Let $1 < p < \infty$ and $\varepsilon \in (0, \frac{1}{2})$. Then, there exists a positive constant $C \equiv C(p, C_1, \varepsilon)$ such that, for any $\xi \in \mathbb{R}^{Nn} \setminus \{0\}$ and any $\lambda, \zeta \in \mathbb{R}^{Nn^2}$, we have*

$$|\mathcal{A}_\varepsilon(x,t,\xi)(\lambda,\zeta)| \le C(1+|\xi|^2)^{\frac{p-2}{2}} |\lambda|\,|\zeta|\,.$$

Moreover, for $\delta > 4\varepsilon^{\frac{1}{p-1}}$ and $|\xi| \ge 1 + \frac{\delta}{2}$ we get

$$|\mathcal{A}_\varepsilon(x,t,\xi)(\lambda,\zeta)| \le C_2\,|\xi|^{p-2}\,|\lambda|\,|\zeta|\,,$$

where $C_2 \equiv C_2(\delta, p, C_1) > 0$.

Proof. For every $\xi \in \mathbb{R}^{Nn} \setminus \{0\}$ and every $\lambda, \zeta \in \mathbb{R}^{Nn^2}$, we obtain

$$|\mathcal{A}_\varepsilon(x,t,\xi)(\lambda,\zeta)| \le \Big[\,|h_\varepsilon(x,t,|\xi|)| + |\partial_s h_\varepsilon(x,t,|\xi|)|\,|\xi|\,\Big]\,|\lambda|\,|\zeta|\,.$$

Now, by (H_2) and (H_3), in the case $p > 2$ we get

$$\begin{aligned}
|\mathcal{A}_\varepsilon(x,t,\xi)(\lambda,\zeta)| &\le \left(2\,\frac{\partial_s F(x,t,|\xi|)}{|\xi|} + \partial_{ss}F(x,t,|\xi|) + \varepsilon(p-1)(1+|\xi|^2)^{\frac{p-2}{2}}\right)|\lambda|\,|\zeta| \\
&\le \left(2C_1\,\frac{(|\xi|-1)_+^{p-1}}{|\xi|} + C_1(|\xi|-1)_+^{p-2} + \varepsilon(p-1)(1+|\xi|^2)^{\frac{p-2}{2}}\right)|\lambda|\,|\zeta| \\
&\le (3C_1+p-1)(1+|\xi|^2)^{\frac{p-2}{2}}\,|\lambda|\,|\zeta|\,. \qquad (6.28)
\end{aligned}$$

In the case $1 < p \le 2$, we use the estimates from Lemma 6.3 to find that

$$\begin{aligned}
|\mathcal{A}_\varepsilon(x,t,\xi)(\lambda,\zeta)| &\le \left(2\,\frac{\partial_s F_\varepsilon(x,t,|\xi|)}{|\xi|} + \partial_{ss}F_\varepsilon(x,t,|\xi|) + \varepsilon(p-1)(1+|\xi|^2)^{\frac{p-2}{2}}\right)|\lambda|\,|\zeta| \\
&\le \left(2C_1\,|\xi|^{p-2}\,\mathbb{1}_{\{|\xi|>1\}}(|\xi|) + \left(c\,\varepsilon^{\frac{p-2}{p-1}} + (p-1)\varepsilon\right)(1+|\xi|^2)^{\frac{p-2}{2}}\right)|\lambda|\,|\zeta| \\
&\le \left(2^{\frac{4-p}{2}}C_1 + c\,\varepsilon^{\frac{p-2}{p-1}} + (p-1)\varepsilon\right)(1+|\xi|^2)^{\frac{p-2}{2}}\,|\lambda|\,|\zeta|\,.
\end{aligned}$$

We thus obtain the first conclusion of this lemma.

Finally, due to equality (6.16), if $|\xi| \ge 1 + \frac{\delta}{2}$ we only need to use the assumptions (H_2) and (H_3) together with the inequalities (6.26) and (6.27) to obtain from (6.28)

$$\begin{aligned}
|\mathcal{A}_\varepsilon(x,t,\xi)(\lambda,\zeta)| &\le \left(2C_1\,\frac{(|\xi|-1)_+^{p-1}}{|\xi|} + C_1(|\xi|-1)_+^{p-2} + \varepsilon(p-1)(1+|\xi|^2)^{\frac{p-2}{2}}\right)|\lambda|\,|\zeta| \\
&\le \begin{cases} \left(2C_1 + C_1\left(\frac{\delta}{2+\delta}\right)^{p-2} + p - 1\right)|\xi|^{p-2}\,|\lambda|\,|\zeta| & \text{for } 1 < p \le 2 \\ \left(3C_1 + (p-1)2^{\frac{p-2}{2}}\right)|\xi|^{p-2}\,|\lambda|\,|\zeta| & \text{for } p > 2. \end{cases}
\end{aligned}$$

These inequalities conclude the proof. □

6.2.4 Algebraic inequalities

In this section, we gather some relevant algebraic inequalities that will be needed later on. We start with an elementary assertion, which will be used in the Moser iteration.

Lemma 6.6. *Let $M > 1$, $\kappa > 1$, $0 \le \alpha < 1$, $C, c > 0$ and $i \in \mathbb{N}$. Furthermore, let $\{\beta_j\}_{j\in\mathbb{N}_0}$ satisfy*

$$\beta_j \ge C\kappa^j$$

for each $j \in \mathbb{N}_0$. Then we have

$$\prod_{j=0}^{i} M^{\frac{\kappa^{i-(1-\alpha)j}}{\beta_{i+1}}} \le M^{\frac{1}{C\kappa(1-\kappa^{\alpha-1})}} \tag{6.29}$$

and

$$\prod_{j=0}^{i} M^{jc\frac{\kappa^{i+1-j}}{\beta_{i+1}}} \le M^{\frac{c\kappa}{C(1-\kappa)^2}}. \tag{6.30}$$

Proof. The proof of the second inequality is similar to the one of Lemma 2.3. in [18]. Thus we only prove the first inequality:

$$\prod_{j=0}^{i} M^{\frac{\kappa^{i-(1-\alpha)j}}{\beta_{i+1}}} \le M^{\frac{1}{C\kappa}\sum_{j=0}^{i}\kappa^{(\alpha-1)j}} \le M^{\frac{1}{C\kappa}\sum_{j=0}^{\infty}\kappa^{(\alpha-1)j}} = M^{\frac{1}{C\kappa(1-\kappa^{\alpha-1})}}. \qquad \square$$

For $\delta \in (0,1]$ we define the auxiliary function $G_\delta : \mathbb{R}^k \to \mathbb{R}^k$, $k \in \mathbb{N}$, as follows:

$$G_\delta(\xi) := \begin{cases} (|\xi| - 1 - \delta)_+ \frac{\xi}{|\xi|} & \text{if } \xi \in \mathbb{R}^k \setminus \{0\}, \\ 0 & \text{if } \xi = 0. \end{cases} \tag{6.31}$$

The following two lemmas are concerned with auxiliary estimates for the functions A_ε and G_δ, defined by (6.19) and (6.31) respectively.

Lemma 6.7. *Let $1 < p < \infty$, $\delta > 0$ and $\varepsilon \in (0, \frac{1}{2})$. Then, there exists a positive constant $C \equiv C(p, C_1, \delta)$ such that for every $\xi, \tilde{\xi} \in \mathbb{R}^{Nn}$ with $|\xi| > 1 + \frac{\delta}{2}$ we have*

$$\langle A_\varepsilon(x,t,\xi) - A_\varepsilon(x,t,\tilde{\xi}), \xi - \tilde{\xi}\rangle \ge C \frac{(|\xi| - 1 - \frac{\delta}{2})^p}{|\xi|\,(|\xi| + |\tilde{\xi}|)} |\xi - \tilde{\xi}|^2.$$

Proof. Here we argue as in [17, Lemma 2.8]. The inequality on the left-hand side of (6.25) implies that

$$\mathcal{A}_\varepsilon(x,t,\xi)(\lambda,\lambda) \ge c \frac{(|\xi| - 1 - \frac{\delta}{2})_+^{p-1}}{|\xi|} |\lambda|^2$$

holds for any $\xi \in \mathbb{R}^{Nn} \setminus \{0\}$ and any $\lambda \in \mathbb{R}^{Nn^2}$, where $c \equiv c(p, C_1, \delta) > 0$. Abbreviating $\xi_s := \xi + s(\tilde{\xi} - \xi)$, for $s \in [0,1]$, we find

$$\langle A_\varepsilon(x,t,\xi) - A_\varepsilon(x,t,\tilde{\xi}), \xi - \tilde{\xi}\rangle \ge c \int_0^1 \frac{(|\xi_s| - 1 - \frac{\delta}{2})_+^{p-1}}{|\xi_s|}\, ds\, |\xi - \tilde{\xi}|^2$$

$$\geq \frac{c}{|\xi|+|\tilde{\xi}|}\int_0^1 \big(|\xi_s|-1-\tfrac{\delta}{2}\big)_+^{p-1}\,ds\,|\xi-\tilde{\xi}|^2. \tag{6.32}$$

It remains to estimate the integral in the right-hand side of (6.32). To this end, we distinguish whether or not $|\tilde{\xi}| \leq |\xi|$. If $|\tilde{\xi}| \leq |\xi|$, then

$$|\xi_s| \geq (1-s)|\xi| - s|\tilde{\xi}| \geq (1-2s)|\xi| > 1+\frac{\delta}{2} \qquad \forall\, s \in \left[0, \frac{|\xi|-1-\frac{\delta}{2}}{2\,|\xi|}\right).$$

For $s \in \big[0, \frac{|\xi|-1-\delta/2}{4|\xi|}\big]$ this implies a bound from below in the form

$$\left(|\xi_s|-1-\frac{\delta}{2}\right)_+ \geq (1-2s)|\xi|-1-\frac{\delta}{2} \geq \left(1-\frac{|\xi|-1-\frac{\delta}{2}}{2\,|\xi|}\right)|\xi|-1-\frac{\delta}{2} \geq \frac{1}{2}\left(|\xi|-1-\tfrac{\delta}{2}\right).$$

Thus, we obtain that

$$\int_0^1 \big(|\xi_s|-1-\tfrac{\delta}{2}\big)_+^{p-1}\,ds \geq \int_0^{\frac{|\xi|-1-\frac{\delta}{2}}{4|\xi|}} \big(|\xi_s|-1-\tfrac{\delta}{2}\big)_+^{p-1}\,ds \geq \frac{\big(|\xi|-1-\frac{\delta}{2}\big)^p}{2^{p+1}|\xi|}. \tag{6.33}$$

In the case $|\tilde{\xi}| > |\xi|$, we estimate $|\xi_s|$ from below as follows

$$|\xi_s| \geq s|\tilde{\xi}| - (1-s)|\xi| \geq (2s-1)|\xi| > 1+\frac{\delta}{2} \qquad \forall\, s \in \left(\frac{|\xi|+1+\frac{\delta}{2}}{2\,|\xi|}, 1\right].$$

Therefore, for $s \in \big[\frac{3|\xi|+1+\delta/2}{4|\xi|}, 1\big]$ we get

$$\big(|\xi_s|-1-\tfrac{\delta}{2}\big)_+ \geq (2s-1)|\xi|-1-\frac{\delta}{2} \geq \left(\frac{3\,|\xi|+1+\frac{\delta}{2}}{2\,|\xi|}-1\right)|\xi|-1-\frac{\delta}{2} \geq \frac{1}{2}\left(|\xi|-1-\tfrac{\delta}{2}\right),$$

which yields

$$\int_0^1 \big(|\xi_s|-1-\tfrac{\delta}{2}\big)_+^{p-1}\,ds \geq \int_{\frac{3|\xi|+1+\frac{\delta}{2}}{4|\xi|}}^1 \big(|\xi_s|-1-\tfrac{\delta}{2}\big)_+^{p-1}\,ds \geq \frac{\big(|\xi|-1-\frac{\delta}{2}\big)^p}{2^{p+1}|\xi|}. \tag{6.34}$$

Combining estimates (6.33) and (6.34) with (6.32), we conclude the proof. □

Using the previous lemma, we can easily achieve the following result:

Lemma 6.8. *Let $p > 1$, $\delta > 0$, $0 < \varepsilon < \min\{\frac{1}{2}, (\frac{\delta}{4})^{p-1}\}$ and $\xi, \tilde{\xi} \in \mathbb{R}^{Nn}$. Then, for every $\nu > 0$ and almost every $x \in \Omega$ we have*

$$|G_\delta(\xi) - G_\delta(\tilde{\xi})|^p \leq \varepsilon^\nu(\max\{|\xi|, 1+\delta\})^p + C\varepsilon^{-\nu}\langle A_\varepsilon(x,t,\xi) - A_\varepsilon(x,t,\tilde{\xi}), \xi-\tilde{\xi}\rangle$$

for a positive constant $C \equiv C(p, C_1, \delta)$.

Proof. If $|\xi|, |\tilde{\xi}| \leq 1+\delta$, we have $G_\delta(\xi) = G_\delta(\tilde{\xi}) = 0$. Therefore, the claimed inequality immediately follows from Lemma 6.7, by the positivity of the right-hand side.

Thus, we only need to consider the case where either $|\xi| > 1+\delta$ or $|\tilde{\xi}| > 1+\delta$. In order to do this, we first assume that $|\xi| \geq |\tilde{\xi}|$. Note that this implies $|\xi| > 1+\delta$. From [17, Lemma

2.3] we know that

$$|G_\delta(\xi) - G_\delta(\tilde{\xi})| \le 3\,|\xi - \tilde{\xi}|.$$

For $p \ge 2$ this yields

$$|G_\delta(\xi) - G_\delta(\tilde{\xi})|^p \;\le\; 3^p|\xi - \tilde{\xi}|^p \;\le\; 3^p(|\xi| + |\tilde{\xi}|)^{p-2}\,|\xi - \tilde{\xi}|^2 \;\le\; 3^p\,2^{p-2}\,|\xi|^{p-2}\,|\xi - \tilde{\xi}|^2,$$

while in the case $1 < p < 2$ we use Young's inequality with exponents $(\frac{2}{p}, \frac{2}{2-p})$ to obtain

$$\begin{aligned}
|G_\delta(\xi) - G_\delta(\tilde{\xi})|^p \;&=\; |\xi|^{\frac{(2-p)p}{2}}|\xi|^{\frac{(p-2)p}{2}}\varepsilon^{\frac{(2-p)\nu}{2}}\varepsilon^{\frac{(p-2)\nu}{2}}|G_\delta(\xi) - G_\delta(\tilde{\xi})|^p \\
&\le\; \frac{(2-p)\varepsilon^\nu}{4}\,|\xi|^p + 2^{\frac{2(1-p)}{p}}\,p\,\varepsilon^{\frac{(p-2)\nu}{p}}|\xi|^{p-2}\,|G_\delta(\xi) - G_\delta(\tilde{\xi})|^2 \\
&\le\; \frac{\varepsilon^\nu}{4}\,|\xi|^p + 9p\,\varepsilon^{-\nu}|\xi|^{p-2}\,|\xi - \tilde{\xi}|^2,
\end{aligned}$$

where we have used the fact that $0 < \varepsilon < \frac{1}{2}$ and $0 < \frac{2-p}{p} < 1$ whenever $p \in (1,2)$. Combining both cases, taking into account the inequalities $\frac{1}{|\xi|} \le \frac{2}{|\xi|+|\tilde{\xi}|}$ and $|\xi| \le \big(2 + \frac{2}{\delta}\big)\,(|\xi| - 1 - \frac{\delta}{2})$, and finally applying Lemma 6.7, we get

$$\begin{aligned}
|G_\delta(\xi) - G_\delta(\tilde{\xi})|^p \;&\le\; \varepsilon^\nu|\xi|^p + \varepsilon^{-\nu}c(\delta,p)\,\frac{\big(|\xi| - 1 - \frac{\delta}{2}\big)^p}{|\xi|(|\xi| + |\tilde{\xi}|)}\,|\xi - \tilde{\xi}|^2 \\
&\le\; \varepsilon^\nu|\xi|^p + C(p, C_1, \delta)\,\varepsilon^{-\nu}\langle A_\varepsilon(x,t,\xi) - A_\varepsilon(x,t,\tilde{\xi}), \xi - \tilde{\xi}\rangle. \qquad (6.35)
\end{aligned}$$

This proves the claimed inequality in the case $|\xi| \ge |\tilde{\xi}|$. In the remaining case, i.e. $|\tilde{\xi}| > |\xi|$, by interchanging the roles of ξ and $\tilde{\xi}$ and replacing ε with $\frac{\varepsilon}{2^{\frac{2+(p-2)_+}{\nu}}}$, we obtain from (6.35) that

$$\begin{aligned}
|G_\delta(\xi) - G_\delta(\tilde{\xi})|^p \;&\le\; \frac{\varepsilon^\nu}{2^{2+(p-2)_+}}\,|\tilde{\xi}|^p + C\varepsilon^{-\nu}\langle A_\varepsilon(x,t,\xi) - A_\varepsilon(x,t,\tilde{\xi}), \xi - \tilde{\xi}\rangle \\
&=\; \frac{\varepsilon^\nu}{2^{2+(p-2)_+}}\,(|G_\delta(\tilde{\xi})| + 1 + \delta)^p + C\varepsilon^{-\nu}\langle A_\varepsilon(x,t,\xi) - A_\varepsilon(x,t,\tilde{\xi}), \xi - \tilde{\xi}\rangle \\
&\le\; \frac{\varepsilon^\nu}{2}\,(|G_\delta(\xi)| + 1 + \delta)^p + C\varepsilon^{-\nu}\langle A_\varepsilon(x,t,\xi) - A_\varepsilon(x,t,\tilde{\xi}), \xi - \tilde{\xi}\rangle \\
&\quad + \frac{\varepsilon^\nu}{2}\,|G_\delta(\xi) - G_\delta(\tilde{\xi})|^p \\
&\le\; \frac{\varepsilon^\nu}{2}\,(\max\{|\xi|, 1+\delta\})^p + C\varepsilon^{-\nu}\langle A_\varepsilon(x,t,\xi) - A_\varepsilon(x,t,\tilde{\xi}), \xi - \tilde{\xi}\rangle \\
&\quad + \frac{1}{2}\,|G_\delta(\xi) - G_\delta(\tilde{\xi})|^p.
\end{aligned}$$

Absorbing the last term on the right-hand side into the left-hand side, we obtain the desired result. □

6.3 A family of regularized parabolic problems

In this section, we let u be a weak solution of (6.1) and introduce an approximating problem $(\mathcal{P}_\varepsilon)$, where $\varepsilon \in (0, \frac{1}{2})$ is the approximation parameter. Denoting by u_ε the unique weak solution

of this problem, we will prove the strong convergence $G_\delta(Du_\varepsilon) \to G_\delta(Du)$ in L^p as $\varepsilon \to 0$. As an easy consequence of this convergence, we then establish the uniqueness of weak solutions to an initial-boundary value problem associated with (6.1).

To set up the approximating problem, for $\varepsilon \in (0, \frac{1}{2})$ we define the truncated vector-valued function f_ε by

$$f_\varepsilon^i := \max\left\{-\tfrac{1}{\varepsilon}, \min\left\{f^i, \tfrac{1}{\varepsilon}\right\}\right\}, \qquad i \in \{1, \dots, N\}. \tag{6.36}$$

Moreover, we consider a space-time cylinder $Q' := \Omega' \times I$, where $\Omega' \subseteq \Omega$ is a bounded domain and $I := (t_1, t_2) \subseteq (0, T)$. In what follows, it will suffice to assume that $f \in L^2(Q', \mathbb{R}^N)$.

Definition 6.9. Let $\varepsilon \in (0, \frac{1}{2})$ and $u \in C^0\left([t_1, t_2]; L^2\left(\Omega', \mathbb{R}^N\right)\right) \cap L^p\left(t_1, t_2; W^{1,p}\left(\Omega', \mathbb{R}^N\right)\right)$. In this framework, we identify a function

$$u_\varepsilon \in C^0\left([t_1, t_2]; L^2\left(\Omega', \mathbb{R}^N\right)\right) \cap L^p\left(t_1, t_2; W^{1,p}\left(\Omega', \mathbb{R}^N\right)\right)$$

as a *weak solution of the Cauchy-Dirichlet problem*

$$\begin{cases} \partial_t u_\varepsilon - \operatorname{div}[A_\varepsilon(x, t, Du_\varepsilon)] = f_\varepsilon & \text{in } Q', \\ u_\varepsilon = u & \text{on } \partial_{\mathrm{par}} Q', \end{cases} \tag{$\mathcal{P}_\varepsilon$}$$

if and only if u_ε is a weak solution of $(\mathcal{P}_\varepsilon)_1$ and, moreover,

$$u_\varepsilon \in u + L^p\left(t_1, t_2; W_0^{1,p}\left(\Omega', \mathbb{R}^N\right)\right)$$

and $u_\varepsilon(\cdot, t_1) = u(\cdot, t_1)$ in the L^2-sense, that is

$$\lim_{t \searrow t_1} \|u_\varepsilon(\cdot, t) - u(\cdot, t_1)\|_{L^2(\Omega')} = 0. \tag{6.37}$$

Remark 6.10. We know that the regularized parabolic system $(\mathcal{P}_\varepsilon)_1$ fulfills standard p-growth conditions by virtue of (6.23) and Lemma 6.4. The existence of a unique weak solution to $(\mathcal{P}_\varepsilon)$ can be inferred from the classic existence theory, cf. [69, Chapter 2, Theorem 1.2 and Remark 1.2]. By a difference quotient method, one can show that Du_ε is locally bounded and that u_ε admits second weak spatial derivatives in L^2_{loc} (see [38, Chapter 8]). For this to hold in the subcritical case $1 < p \leq \frac{2n}{n+2}$, we additionally have to require that u_ε belongs to $L^r_{loc}(Q', \mathbb{R}^N)$ for some $r \geq 2$ such that $n(p-2) + rp > 0$ (see again [38, Chapter 8]). Since, in the subcritical case, we will always assume that $u_\varepsilon \in L^\infty_{loc}(Q', \mathbb{R}^N)$, the latter requirement is trivially fulfilled. By virtue of what has just been said, we have the following result.

Theorem 6.11. *Let $p > 1$ and let u_ε be the weak solution of $(\mathcal{P}_\varepsilon)$ with $Q' = Q_R(z_0) \Subset \Omega_T$. Moreover, assume that u_ε satisfies the requirements of Remark 6.10 in the case $1 < p \leq \frac{2n}{n+2}$. Then*

$$Du_\varepsilon \in L^2_{loc}\left(t_0 - R^2, t_0; W^{1,2}_{loc}\left(B_R(x_0), \mathbb{R}^{Nn}\right)\right) \cap L^\infty_{loc}\left(Q_R(z_0), \mathbb{R}^{Nn}\right).$$

We now prove the following strong convergence result:

Lemma 6.12. *Let $p > 1$, $\delta > 0$, $0 < \varepsilon < \min\{\frac{1}{2}, (\frac{\delta}{4})^{p-1}\}$ and $f \in L^2(\Omega' \times I, \mathbb{R}^N)$. Moreover, let*

$$u \in C^0\left([t_1, t_2]; L^2\left(\Omega', \mathbb{R}^N\right)\right) \cap L^p\left(t_1, t_2; W^{1,p}\left(\Omega', \mathbb{R}^N\right)\right)$$

be a weak solution of (6.1) *and assume that*

$$u_\varepsilon \in C^0\left([t_1,t_2];L^2\left(\Omega',\mathbb{R}^N\right)\right)\cap L^p\left(t_1,t_2;W^{1,p}\left(\Omega',\mathbb{R}^N\right)\right)$$

is the unique energy solution of problem $(\mathcal{P}_\varepsilon)$*. Then, there exists a constant* $\widetilde{C}\equiv\widetilde{C}(p,\hat{n},\delta)\in(0,\min\{\frac{1}{2},(\frac{\delta}{4})^{p-1}\})$ *such that for every* $\varepsilon\in(0,\widetilde{C})$*, the estimate*

$$\begin{aligned}\sup_{\tau\in I}\int_{\Omega'}|u_\varepsilon-u|^2(x,\tau)\,dx\,+\int_{\Omega'\times I}|G_\delta(Du_\varepsilon)-G_\delta(Du)|^p\,dz\\ \le C\varepsilon^{\frac{\hat{n}}{2(\hat{n}+2)}}\int_{\Omega'\times I}(\max\{|Du|,1+\delta\})^p\,dz\,+\,C|I|\int_{(\Omega'\times I)\cap\{|f|>\frac{1}{\varepsilon}\}}|f|^2\,dz\end{aligned}$$

holds for some positive constant $C\equiv C(p,C_1,\delta)$*. In particular, this estimate implies that*

$$G_\delta(Du_\varepsilon)\to G_\delta(Du)\quad\textit{strongly in}\quad L^p(\Omega'\times I,\mathbb{R}^{Nn}),\quad\textit{as}\quad\varepsilon\to0^+.$$

Proof. We observe that $(u_\varepsilon-u)\in L^p(I;W_0^{1,p}(\Omega',\mathbb{R}^N))$ by the lateral boundary condition. Unfortunately, we cannot test systems (6.1) and $(\mathcal{P}_\varepsilon)_1$ with the function $u_\varepsilon-u$, since weak time derivatives might not exist. Therefore, we resort to the equivalent Steklov average formulations of (6.1) and $(\mathcal{P}_\varepsilon)_1$, thus obtaining

$$\int\limits_{\Omega'\times\{t\}}\big(\partial_t[u_\varepsilon-u]_h\cdot\phi+\langle[A_\varepsilon(x,t,Du_\varepsilon)-A(x,t,Du)]_h,D\phi\rangle\big)dx\,=\int\limits_{\Omega'\times\{t\}}[f_\varepsilon-f]_h\cdot\phi\,dx\quad(6.38)$$

for every $t\in I=(t_1,t_2)$, for every $h\in(0,t_2-t_1)$ and every $\phi\in W_0^{1,p}(\Omega',\mathbb{R}^N)\cap L^2(\Omega',\mathbb{R}^N)$. Then, for a fixed time slice $\Omega'\times\{t\}$ we can choose the function $\phi(\cdot,t)$ defined by

$$\phi^i(x,t):=[u_\varepsilon^i-u^i]_h(x,t),\qquad i\in\{1,\dots,N\},\qquad(6.39)$$

as a test function in (6.38). For any fixed $\tau\in I$, the term involving the time derivatives yields

$$\begin{aligned}\int_{t_1}^{\tau}\int_{\Omega'}\partial_t[u_\varepsilon-u]_h\cdot\phi\,dx\,dt\,&=\,\frac12\int_{t_1}^{\tau}\int_{\Omega'}\partial_t|[u_\varepsilon-u]_h|^2\,dx\,dt\\&=\,\frac12\int_{\Omega'}|[u_\varepsilon-u]_h(x,\tau)|^2\,dx-\frac12\int_{\Omega'}|[u_\varepsilon-u]_h(x,t_1)|^2\,dx\qquad(6.40)\end{aligned}$$

for every $h\in(0,\tau-t_1)$. Now we pass to the limit as $h\to0$. Using Lemma 6.2 and taking into account the growth conditions of A_ε and the fact that $u_\varepsilon=u$ on $\overline{\Omega'}\times\{t_1\}$ in the L^2-sense, from (6.38) and (6.40) we obtain the following inequality

$$\begin{aligned}&\frac12\int_{\Omega'}|u_\varepsilon-u|^2(x,\tau)\,dx+\int_{\Omega'\times(t_1,\tau)}\langle A_\varepsilon(x,t,Du_\varepsilon)-A_\varepsilon(x,t,Du),Du_\varepsilon-Du\rangle\,dx\,dt\\&\quad\le\int_{\Omega'\times(t_1,\tau)}|A(x,t,Du)-A_\varepsilon(x,t,Du)||Du_\varepsilon-Du|\,dx\,dt+\int_{\Omega'\times(t_1,\tau)\cap\{|f|>\frac{1}{\varepsilon}\}}|f||u_\varepsilon-u|\,dx\,dt\end{aligned}$$

for every $\tau\in I$. Taking the supremum over $\tau\in I$, we thus obtain

$$\frac12\,\sup_{\tau\in I}\int_{\Omega'}|u_\varepsilon-u|^2(x,\tau)\,dx\,+\int_{\Omega'\times I}\langle A_\varepsilon(x,t,Du_\varepsilon)-A_\varepsilon(x,t,Du),Du_\varepsilon-Du\rangle\,dx\,dt$$

$$\leq \int_{\Omega'\times I} |A(x,t,Du) - A_\varepsilon(x,t,Du)||Du_\varepsilon - Du|\,dx\,dt + \int_{(\Omega'\times I)\cap\{|f|>\frac{1}{\varepsilon}\}} |f||u_\varepsilon - u|\,dx\,dt.$$

We now apply Young's inequality with exponents $(2,2)$ to control the last integral as follows:

$$\begin{aligned}\int_{(\Omega'\times I)\cap\{|f|>\frac{1}{\varepsilon}\}} |f||u_\varepsilon - u|\,dx\,dt \;&\leq\; |I| \int_{(\Omega'\times I)\cap\{|f|>\frac{1}{\varepsilon}\}} |f|^2\,dx\,dt \;+\; \frac{1}{4|I|}\int_{\Omega'\times I} |u_\varepsilon - u|^2\,dx\,dt\\ &\leq\; |I| \int_{(\Omega'\times I)\cap\{|f|>\frac{1}{\varepsilon}\}} |f|^2\,dx\,dt \;+\; \frac{1}{4}\sup_{\tau\in I}\int_{\Omega'} |u_\varepsilon - u|^2(x,\tau)\,dx.\end{aligned}$$

Joining the two previous estimates, we get

$$\begin{aligned}&\frac{1}{4}\sup_{\tau\in I}\int_{\Omega'} |u_\varepsilon - u|^2(x,\tau)\,dx \;+\; \int_{\Omega'\times I} \langle A_\varepsilon(x,t,Du_\varepsilon) - A_\varepsilon(x,t,Du), Du_\varepsilon - Du\rangle\,dx\,dt\\ &\quad\leq \int_{\Omega'\times I} |A(x,t,Du) - A_\varepsilon(x,t,Du)||Du_\varepsilon - Du|\,dx\,dt \;+\; |I| \int_{(\Omega'\times I)\cap\{|f|>\frac{1}{\varepsilon}\}} |f|^2\,dx\,dt.\end{aligned} \tag{6.41}$$

In order to estimate the first integral on the right-hand side of (6.41), we now distinguish whether or not $p \geq 2$. If $p \geq 2$, then we have

$$A_\varepsilon(x,t,Du) = A(x,t,Du) + \varepsilon(1+|Du|^2)^{\frac{p-2}{2}}Du,$$

which implies

$$\begin{aligned}&\frac{1}{4}\sup_{\tau\in I}\int_{\Omega'} |u_\varepsilon - u|^2(x,\tau)\,dx \;+\; \int_{\Omega'\times I} \langle A_\varepsilon(x,t,Du_\varepsilon) - A_\varepsilon(x,t,Du), Du_\varepsilon - Du\rangle\,dx\,dt\\ &\quad\leq \varepsilon\int_{\Omega'\times I} (1+|Du|^2)^{\frac{p-2}{2}}|Du||Du_\varepsilon - Du|\,dx\,dt \;+\; |I| \int_{(\Omega'\times I)\cap\{|f|>\frac{1}{\varepsilon}\}} |f|^2\,dx\,dt.\end{aligned}$$

In what follows, we will denote by C a general positive constant that only depends on p, C_1 and δ. Using the previous estimate, Lemma 6.8 with $\nu := \frac{\hat{n}}{2(\hat{n}+2)} < \frac{1}{2}$ and Young's inequality with exponents $(p, \frac{p}{p-1})$, we have

$$\begin{aligned}&\frac{1}{4}\sup_{\tau\in I}\int_{\Omega'} |u_\varepsilon - u|^2(x,\tau)\,dx \;+\; \int_{\Omega'\times I} |G_\delta(Du_\varepsilon) - G_\delta(Du)|^p\,dx\,dt\\ &\quad\leq\; C\varepsilon^\nu \int_{\Omega'\times I} (\max\{|Du|, 1+\delta\})^p\,dx\,dt \;+\; \varepsilon^\nu \int_{\Omega'\times I} |Du_\varepsilon - Du|^p\,dx\,dt\\ &\qquad+\; C|I| \int_{(\Omega'\times I)\cap\{|f|>\frac{1}{\varepsilon}\}} |f|^2\,dx\,dt,\end{aligned} \tag{6.42}$$

where we have also exploited the facts that $\varepsilon < \varepsilon^{1-\nu}$ and $\varepsilon^{1-\nu} < \varepsilon^\nu$, since $0 < \varepsilon < \frac{1}{2}$.

In the case $1 < p < 2$, we need to use (6.17) and (6.18), which imply

$$\begin{aligned}|A_\varepsilon(x,t,\xi) - A(x,t,\xi)| \;&\leq\; \frac{|\partial_s F_\varepsilon(x,t,|\xi|) - \partial_s F(x,t,|\xi|)|}{|\xi|} \;+\; \varepsilon(1+|\xi|^2)^{\frac{p-2}{2}}|\xi|\\ &\leq\; \varepsilon(2^p\,C_1 + 1)|\xi|^{p-1}\end{aligned}$$

for every $\xi \in \mathbb{R}^{Nn} \setminus \{0\}$ and for almost every $(x,t) \in \Omega' \times I$. Using the above estimate with $\xi = Du$ and arguing as in the case $p > 2$, we now obtain from (6.41) that

$$\begin{aligned}\frac{1}{4}\sup_{\tau\in I}\int_{\Omega'}|u_\varepsilon-u|^2(x,\tau)\,dx &+\int_{\Omega'\times I}|G_\delta(Du_\varepsilon)-G_\delta(Du)|^p\,dz\\ &\le C\varepsilon^\nu\int_{\Omega'\times I}(\max\{|Du|,1+\delta\})^p\,dz+\varepsilon^\nu\int_{\Omega'\times I}|Du_\varepsilon-Du|^p\,dz\\ &\quad+C|I|\int_{(\Omega'\times I)\cap\{|f|>\frac{1}{\varepsilon}\}}|f|^2\,dz\end{aligned}\tag{6.43}$$

for any $0<\varepsilon<\min\{\frac12,(\frac{\delta}{4})^{p-1}\}$. Now, from the proof of [17, Lemma 2.3] we have that

$$|Du_\varepsilon-Du|^p\le\left(\frac{\delta+2}{\delta}\right)^p|G_\delta(Du_\varepsilon)-G_\delta(Du)|^p$$

if either $|Du_\varepsilon|>1+\delta$ or $|Du|>1+\delta$. Using this information to estimate the right-hand side of both (6.42) and (6.43), for every $p>1$ we get

$$\begin{aligned}&\frac{1}{4}\sup_{\tau\in I}\int_{\Omega'}|u_\varepsilon-u|^2(x,\tau)\,dx+\int_{\Omega'\times I}|G_\delta(Du_\varepsilon)-G_\delta(Du)|^p\,dz\\ &\quad\le C\varepsilon^\nu\int_{\Omega'\times I}(\max\{|Du|,1+\delta\})^p\,dz+\varepsilon^\nu\,2^p\,|\Omega'\times I|(1+\delta)^p\\ &\qquad+\varepsilon^\nu\left(\frac{\delta+2}{\delta}\right)^p\int_{\Omega'\times I}|G_\delta(Du_\varepsilon)-G_\delta(Du)|^p\,dz+C|I|\int_{(\Omega'\times I)\cap\{|f|>\frac{1}{\varepsilon}\}}|f|^2\,dz\\ &\quad\le(2^p+C)\varepsilon^\nu\int_{\Omega'\times I}(\max\{|Du|,1+\delta\})^p\,dz+\varepsilon^\nu\left(\frac{\delta+2}{\delta}\right)^p\int_{\Omega'\times I}|G_\delta(Du_\varepsilon)-G_\delta(Du)|^p\,dz\\ &\qquad+C|I|\int_{(\Omega'\times I)\cap\{|f|>\frac{1}{\varepsilon}\}}|f|^2\,dz.\end{aligned}$$

At this point, notice that

$$\varepsilon^\nu\left(\frac{\delta+2}{\delta}\right)^p\searrow 0\qquad\text{as}\qquad\varepsilon\searrow 0,\tag{6.44}$$

since $\nu:=\frac{\hat n}{2(\hat n+2)}>0$. Also, recall that ε is small enough to have $0<\varepsilon<\min\{\frac12,(\frac{\delta}{4})^{p-1}\}$. From this and from (6.44) it follows that there exists a constant $\widetilde C\equiv\widetilde C(p,\hat n,\delta)\in(0,\min\{\frac12,(\frac{\delta}{4})^{p-1}\})$ such that, for every $\varepsilon\in(0,\widetilde C)$, we have

$$\varepsilon^\nu\left(\frac{\delta+2}{\delta}\right)^p\le\frac12\,.$$

This allows to absorb the second term on the right-hand side of the last estimate into the left-hand side, so that we finally obtain the desired conclusion. □

We are now in a position to prove the uniqueness of weak solutions to the Cauchy-Dirichlet problem

$$\begin{cases}\partial_t u-\operatorname{div}A(x,t,Du)=f & \text{in }\ \Omega_T,\\ u=g & \text{on }\ \partial_{\rm par}\Omega_T,\end{cases}\tag{6.45}$$

where Ω is a bounded domain in $\mathbb{R}^n$, $g \in C^0([0,T];L^2(\Omega,\mathbb{R}^N)) \cap L^p(0,T;W^{1,p}(\Omega,\mathbb{R}^N))$ and $f \in L^2(\Omega_T,\mathbb{R}^N)$.

Theorem 6.13. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and let $f \in L^2(\Omega_T,\mathbb{R}^N)$. Then, the Cauchy-Dirichlet problem* (6.45) *admits at most one weak solution.*

Proof. Let $u \in C^0([0,T];L^2(\Omega,\mathbb{R}^N)) \cap L^p(0,T;W^{1,p}(\Omega,\mathbb{R}^N))$ be a weak solution of (6.45) and let $u_\varepsilon \in C^0([0,T];L^2(\Omega,\mathbb{R}^N)) \cap L^p(0,T;W^{1,p}(\Omega,\mathbb{R}^N))$ be the unique weak solution to $(\mathcal{P}_\varepsilon)$ with $Q' := \Omega' \times I = \Omega_T$. Then, by Lemma 6.12, for every $\delta > 0$ there exists a constant $\widetilde{C} \in (0, \min\{\frac{1}{2}, (\frac{\delta}{4})^{p-1}\})$ such that for any $\varepsilon \in (0,\widetilde{C})$ we have

$$\sup_{\tau\in(0,T)} \int_\Omega |u_\varepsilon - u|^2(x,\tau)\,dx \;\le\; C\varepsilon^{\frac{\hat{n}}{2(\hat{n}+2)}} \int_{\Omega_T} (\max\{|Du|, 1+\delta\})^p\,dz + CT \int_{\Omega_T\cap\{|f|>\frac{1}{\varepsilon}\}} |f|^2\,dz \tag{6.46}$$

for some positive constant C that is independent of ε. Notice that the right-hand side of (6.46) converges to zero as $\varepsilon \to 0^+$. Moreover, we have

$$\int_{\Omega_T} |u_\varepsilon - u|^2\,dz \;\le\; T \sup_{\tau\in(0,T)} \int_\Omega |u_\varepsilon - u|^2(x,\tau)\,dx. \tag{6.47}$$

Combining estimates (6.46) and (6.47), we infer that $u_\varepsilon \to u$ strongly in $L^2(\Omega_T,\mathbb{R}^N)$ as $\varepsilon \to 0^+$. This implies the uniqueness of the weak solutions to problem (6.45), by virtue of the uniqueness of limits in Lebesgue spaces and the uniqueness of the energy solutions to problem $(\mathcal{P}_\varepsilon)$. □

6.4 Maximum principle for the homogeneous system

In this section, we establish a maximum principle for the weak solution of problem $(\mathcal{P}_\varepsilon)$ in the case $1 < p \le \frac{2n}{n+2}$ and $f = 0$. Let us note right away that the boundedness of u_ε assumed in Remark 6.10 is an immediate consequence of the maximum principle. Therefore, this boundedness assumption is not restrictive whenever $f = 0$. Also observe that, for $f = 0$, we have $f_\varepsilon = 0$ for every $\varepsilon \in (0, \frac{1}{2})$ (by virtue of (6.36)). Keeping in mind the notation introduced in Section 6.3, we now let

$$k := \|u\|_{L^\infty(\Omega'\times I)} < +\infty$$

and denote by w the N-dimensional vector whose components are all equal to k. Notice that w is a trivial solution of system (6.1) with $f = 0$. Moreover, one can easily check that, for every $i \in \{1,\dots,N\}$ and for almost every $t \in I := (t_1,t_2)$, we have

$$\big(u_\varepsilon^i(\cdot,t) - k\big)_+, \big(u_\varepsilon^i(\cdot,t) + k\big)_- \in W_0^{1,p}(\Omega').$$

Now, for $h \in (0, t_2 - t_1)$ we define

$$\phi^i(x,t) := [(u_\varepsilon^i - k)_+]_h(x,t), \qquad i \in \{1,\dots,N\}. \tag{6.48}$$

Using the Steklov average formulations of $(\mathcal{P}_\varepsilon)_1$ and (6.1) with w instead of u, and arguing as in the proof of Lemma 6.12 with (6.48) in place of (6.39), one can easily obtain

$$\frac{1}{2}\int_{\Omega'} |(u_\varepsilon(x,\tau)-w)_+|^2\,dx + \int_{\Omega'\times(t_1,\tau)} \langle A_\varepsilon(x,t,Du_\varepsilon) - A_\varepsilon(x,t,Dw), D[(u_\varepsilon-w)_+]\rangle\,dx\,dt \;=\; 0$$

for every $\tau \in I$. Since $A_\varepsilon(x,t,\cdot)$ is a monotone vector field and $Dw = A_\varepsilon(x,t,Dw) = 0$, we can omit the latter integral, thus obtaining

$$\int_{\Omega'} |(u_\varepsilon(x,\tau)-w)_+|^2\,dx \;=\; 0 \qquad \text{for every } \tau \in I.$$

Therefore, for every $i \in \{1,\dots,N\}$ and every $\varepsilon \in (0,\frac{1}{2})$, we have

$$u^i_\varepsilon(x,\tau) \le k \qquad \text{for almost every } (x,\tau) \in Q' := \Omega' \times I.$$

Similarly, but using the function $(u^i_\varepsilon + k)_-$ instead of $(u^i_\varepsilon - k)_+$ in (6.48), we get

$$u^i_\varepsilon(x,\tau) \ge -k$$

for any $i \in \{1,\dots,N\}$, for any $\varepsilon \in (0,\frac{1}{2})$ and for almost every $(x,\tau) \in Q'$. Thus, we can conclude that

$$\|u_\varepsilon\|_{L^\infty(\Omega'\times I)} \;\le\; \sqrt{N}k \;=\; \sqrt{N}\,\|u\|_{L^\infty(\Omega'\times I)} \qquad \text{for all } \varepsilon \in \left(0,\tfrac{1}{2}\right).$$

6.5 Weak differentiability

Here we derive some higher differentiability results that will be useful in the sequel. These results involve the spatial gradient of the weak solution to problem $(\mathcal{P}_\varepsilon)$ with $Q' = Q_R(z_0) \Subset \Omega_T$.

To begin with, for each $\gamma \ge 0$ and $a > 0$ we consider the function

$$\Phi_{\gamma,a}(w) := \; w^2(a+w)^{\gamma-2}, \qquad w \ge 0. \tag{6.49}$$

For this function, one can easily check that

$$\Phi'_{\gamma,a}(w) \le 2(\gamma+1)w(a+w)^{\gamma-2}, \tag{6.50}$$

from which we can immediately deduce

$$w\,\Phi'_{\gamma,a}(w) \le 2(\gamma+1)\Phi_{\gamma,a}(w) \tag{6.51}$$

and

$$\Phi'_{\gamma,a}(w) \le (2a)^{\gamma+1}(\gamma+1) \qquad \text{for every } w \in (0,a).$$

Using these results, we obtain the following lemma.

Lemma 6.14. *Let $p > 1$, $\varepsilon \in (0,\frac{1}{2})$, $\gamma \ge 0$ and $a > 0$. Moreover, assume that*

$$u_\varepsilon \in C^0\left([t_0-R^2,t_0];L^2\left(B_R(x_0),\mathbb{R}^N\right)\right) \cap L^p\left(t_0-R^2,t_0;W^{1,p}\left(B_R(x_0),\mathbb{R}^N\right)\right)$$

is the unique energy solution of problem $(\mathcal{P}_\varepsilon)$ *with* $Q' = Q_R(z_0) := B_R(x_0) \times (t_0 - R^2, t_0) \Subset \Omega_T$. *Then*

$$\Phi_{\gamma,a}((|Du_\varepsilon| - a)_+) \in L^2_{loc}\left(t_0 - R^2, t_0; W^{1,2}_{loc}\left(B_R(x_0)\right)\right).$$

In the case $1 < p \leq \frac{2n}{n+2}$ *we additionally assume that* $u_\varepsilon \in L^\infty_{loc}\left(Q_R(z_0), \mathbb{R}^N\right)$.

Proof. Notice that for $m \in \mathbb{R}^+$ the function

$$\Phi_{\gamma,a,m}(w) := \min\{\Phi_{\gamma,a}(w), m\}, \qquad w > 0,$$

is Lipschitz continuous. By Theorem 6.11 (see also Remark 6.10 for the case $1 < p \leq \frac{2n}{n+2}$), we have that

$$Du_\varepsilon \in L^2_{loc}(t_0 - R^2, t_0; W^{1,2}_{loc}(B_R(x_0), \mathbb{R}^{Nn})).$$

Then, by virtue of the chain rule in Sobolev spaces, we obtain

$$\Phi_{\gamma,a,m}((|Du_\varepsilon| - a)_+) \in L^2_{loc}(t_0 - R^2, t_0; W^{1,2}_{loc}(B_R(x_0)))$$

along with

$$D\big(\Phi_{\gamma,a,m}((|Du_\varepsilon| - a)_+)\big) \,=\, \Phi'_{\gamma,a,m}((|Du_\varepsilon| - a)_+)\,\frac{D^2u_\varepsilon \cdot Du_\varepsilon}{|Du_\varepsilon|}\,\mathbb{1}_{\{a < |Du_\varepsilon| < m\}}\,.$$

Theorem 6.11 also tells us that $Du_\varepsilon \in L^\infty_{loc}(Q_R(z_0), \mathbb{R}^{Nn})$. This implies that for every compact subset $\mathcal{K}$ of $Q_R(z_0)$ there exists a positive number $m = m_{\mathcal{K}}$ such that

$$\Phi_{\gamma,a,m}((|Du_\varepsilon| - a)_+) \,=\, \Phi_{\gamma,a}((|Du_\varepsilon| - a)_+) \qquad \text{in } \mathcal{K}.$$

This is sufficient to prove the assertion. Moreover, we have

$$D(\Phi_{\gamma,a}((|Du_\varepsilon| - a)_+)) \,=\, \Phi'_{\gamma,a}((|Du_\varepsilon| - a)_+)\,\frac{D^2u_\varepsilon \cdot Du_\varepsilon}{|Du_\varepsilon|}\,\mathbb{1}_{\{|Du_\varepsilon| > a\}}\,. \qquad \square$$

Now we focus on the map

$$(x,t) \in \Omega_T \longmapsto A_\varepsilon(x,t,Du_\varepsilon(x,t)) \in \mathbb{R}^{Nn}.$$

Using the notation (1.1), we prove the following result:

Lemma 6.15. *Let* $p > 1$, $\delta > 0$ *and* $0 < \varepsilon < \min\{\frac{1}{2}, (\frac{\delta}{4})^{p-1}\}$. *Moreover, assume that*

$$u_\varepsilon \in C^0\left([t_0 - R^2, t_0]; L^2\left(B_R(x_0), \mathbb{R}^N\right)\right) \cap L^p\left(t_0 - R^2, t_0; W^{1,p}\left(B_R(x_0), \mathbb{R}^N\right)\right)$$

is the unique energy solution of problem $(\mathcal{P}_\varepsilon)$ *with* $Q' = Q_R(z_0) := B_R(x_0) \times (t_0 - R^2, t_0) \Subset \Omega_T$. *Then*

$$A_\varepsilon(\cdot, Du_\varepsilon) \in L^2_{loc}\left(t_0 - R^2, t_0; W^{1,2}_{loc}(B_R(x_0), \mathbb{R}^{Nn})\right).$$

In the case $1 < p \leq \frac{2n}{n+2}$ *we additionally assume that* $u_\varepsilon \in L^\infty_{loc}\left(Q_R(z_0), \mathbb{R}^N\right)$.

Proof. Integrating by parts, for $0 < \varrho \ll 1$, for every $\phi \in C^\infty_0(Q_R(z_0))$ and every $i \in \{1, \dots, n\}$, we obtain

$$\int_{Q_R(z_0)} A_\varepsilon(x,t,(Du_\varepsilon)_\varrho) D_i\phi \, dz \,=\, -\int_{Q_R(z_0)} (D_{x_i}A_\varepsilon)(x,t,(Du_\varepsilon)_\varrho)\,\phi \, dz$$

$$-\int_{Q_R(z_0)} (D_\xi A_\varepsilon)(x,t,(Du_\varepsilon)_\varrho)\cdot D_i(Du_\varepsilon)_\varrho\,\phi\,dz, \tag{6.52}$$

where, in the last two lines, we have used the notation (1.1). In order to pass to the limit as $\varrho \searrow 0$ under the integral signs, we need to estimate the above integrands. From (6.21) it follows that

$$|A_\varepsilon(x,t,(Du_\varepsilon)_\varrho)|\,|D_i\phi| \le c(1+|(Du_\varepsilon)_\varrho|^2)^{\frac{p-1}{2}}|D_i\phi| \tag{6.53}$$

for some positive constant $c \equiv c(p, C_1, \varepsilon)$. As for the first integrand on the right-hand side of (6.52), thanks to (6.22) we get

$$|(D_{x_i}A_\varepsilon)(x,t,(Du_\varepsilon)_\varrho)|\,|\phi| \le 2^{p-1}K\,(1+|(Du_\varepsilon)_\varrho|)^{p-1}|\phi|. \tag{6.54}$$

Moreover, by Lemma 6.5, we obtain

$$|(D_\xi A_\varepsilon)(x,t,(Du_\varepsilon)_\varrho)\cdot D_i(Du_\varepsilon)_\varrho|\,|\phi| \le C\,(1+|(Du_\varepsilon)_\varrho|^2)^{\frac{p-2}{2}}\,|D^2(u_\varepsilon)_\varrho|\,|\phi|, \tag{6.55}$$

where $C \equiv C(p, C_1, \varepsilon) > 0$. Now we use a well-known result on the convergence of mollified functions to deduce that

$$(Du_\varepsilon)_\varrho \to Du_\varepsilon \quad \text{in} \quad L^2_{loc}(t_0-R^2,t_0;W^{1,2}_{loc}(B_R(x_0),\mathbb{R}^{Nn}))\cap L^p(Q_R(z_0),\mathbb{R}^{Nn}), \quad \text{as} \quad \varrho\to 0^+.$$

Combining this with estimates (6.53)−(6.55) and applying the Generalized Lebesgue's Dominated Convergence Theorem (see [81, page 92, Theorem 17]) to both sides of (6.52), we find that

$$\begin{aligned}\int_{Q_R(z_0)} A_\varepsilon(x,t,Du_\varepsilon)D_i\phi\,dz \;=\; &-\int_{Q_R(z_0)}(D_{x_i}A_\varepsilon)(x,t,Du_\varepsilon)\,\phi\,dz\\ &-\int_{Q_R(z_0)}(D_\xi A_\varepsilon)(x,t,Du_\varepsilon)\cdot D_iDu_\varepsilon\,\phi\,dz\end{aligned} \tag{6.56}$$

for every $\phi \in C_0^\infty(Q_R(z_0))$ and every $i \in \{1,\dots,n\}$. This implies the assertion. In addition, from (6.56) we obtain that

$$D_iA_\varepsilon(x,t,Du_\varepsilon) = (D_{x_i}A_\varepsilon)(x,t,Du_\varepsilon)+(D_\xi A_\varepsilon)(x,t,Du_\varepsilon)\cdot D_iDu_\varepsilon \quad \text{for every } i\in\{1,\dots,n\}. \ \square$$

For further needs, we now introduce the auxiliary function $\tilde{\mathsf{H}}_\lambda : \mathbb{R}^{Nn} \to \mathbb{R}^+$ defined by

$$\tilde{\mathsf{H}}_\lambda(\xi) := \max\{1+\lambda, |\xi|\},$$

where $\lambda > 0$ is a parameter. For this function we record the following result, whose proof is omitted, since it is similar to that of Lemma 6.14.

Lemma 6.16. *Let $p > 1$, $\varepsilon \in (0, \frac{1}{2})$ and $\lambda > 0$. Moreover, assume that*

$$u_\varepsilon \in C^0\left([t_0-R^2,t_0];L^2\left(B_R(x_0),\mathbb{R}^N\right)\right)\cap L^p\left(t_0-R^2,t_0;W^{1,p}\left(B_R(x_0),\mathbb{R}^N\right)\right)$$

is the unique energy solution of problem $(\mathcal{P}_\varepsilon)$ with $Q' = Q_R(z_0) := B_R(x_0)\times(t_0-R^2,t_0) \Subset \Omega_T$. Then

$$\tilde{\mathsf{H}}_\lambda(Du_\varepsilon) \in L^2_{loc}\left(t_0-R^2,t_0;W^{1,2}_{loc}\left(B_R(x_0)\right)\right).$$

In the case $1 < p \leq \frac{2n}{n+2}$ *we additionally assume that* $u_\varepsilon \in L^\infty_{loc}\left(Q_R(z_0), \mathbb{R}^N\right)$.

6.6 Local boundedness of Du_ε: quantitative estimates

As before, we denote by u a weak solution of (6.1) and we let u_ε be the unique weak solution of the regularized problem $(\mathcal{P}_\varepsilon)$ with $Q' = Q_R(z_0) := B_R(x_0) \times (t_0 - R^2, t_0) \Subset \Omega_T$. Our aim in this section is to establish local L^∞ estimates for Du_ε with constants independent of ε. We will achieve this result by using the Moser iteration technique, which is based on a Caccioppoli-type inequality. This inequality will be derived in Subsection 6.6.1, by testing the weak formulation of system $(\mathcal{P}_\varepsilon)_1$ with a suitable admissible function. The groundwork for doing this has been laid in Section 6.5.

To move forward, we now fix $\delta > 0$ and $\gamma \geq 0$. Moreover, to shorten our notation we set

$$a := 1 + \frac{\delta}{2} \qquad , \qquad b := 1 + \delta,$$

and we drop the subscript ε for the weak solution u_ε and the subscripts γ, a for the function $\Phi_{\gamma,a}$ defined in (6.49). Therefore, from now on we will simply write u and Φ in place of u_ε and $\Phi_{\gamma,a}$ respectively, unless otherwise stated. To simplify our notation even more, we additionally introduce the function $P : Q_R(z_0) \to \mathbb{R}^+_0$ defined by

$$P := (|Du| - a)_+ ,$$

and its "mollified" version

$$P_\varrho := (|Du_\varrho| - a)_+ , \qquad \varrho > 0, \tag{6.57}$$

using the notation (1.1) on the right-hand side of (6.57), while, on the left-hand side, an intentional abuse of the same notation has been made. By the elementary properties of Sobolev functions, the weak spatial gradient of P is given by

$$DP \;=\; \frac{D^2u \cdot Du}{|Du|}\, \mathbb{1}_{\{|Du| > a\}} . \tag{6.58}$$

6.6.1 Step 1: Caccioppoli-type inequality

In order to establish the Caccioppoli-type inequality that we need, we now test the weak formulation of $(\mathcal{P}_\varepsilon)_1$ with the function $D_i\tilde{\phi}_\varrho$, where

$$\tilde{\phi}_\varrho \equiv \tilde{\phi}_{i,\varrho} := D_i u_\varrho \cdot \Phi(P_\varrho)\, \psi, \qquad i \in \{1, \dots, n\},$$

and $\psi \in W^{1,\infty}_0(Q_R(z_0))$ is a non-negative cut-off function that will be specified later. We thus obtain

$$\int_{Q_R(z_0)} \sum_{i=1}^{n} \left(\partial_t(D_i u_\varrho) \cdot \tilde{\phi}_{i,\varrho} + \langle D_i[A_\varepsilon(x,t,Du)]_\varrho, D\tilde{\phi}_{i,\varrho} \rangle + (f_\varepsilon)_\varrho \cdot D_i\tilde{\phi}_{i,\varrho} \right) dz = 0. \tag{6.59}$$

At this stage, we fix $z_1 = (x_1, t_1) \in Q_R(z_0)$ and $r \in (0,1)$ such that $Q_r(z_1) \Subset Q_R(z_0)$. Moreover, we choose $\psi(x,t) = \chi(t)\omega^2(t)\eta^2(x)$ with $\chi, \omega \in W^{1,\infty}(\mathbb{R}, [0,1])$, $\chi(t_1) = 0$, $\partial_t\chi \le 0$, $\omega(t_1 - r^2) = 0$, $\partial_t\omega \ge 0$ and $\eta \in C_0^\infty(B_r(x_1), [0,1])$. With such a choice of ψ and integrating by parts, for the term involving the time derivative we get

$$
\begin{aligned}
J_{0,\varrho} &:= \int_{Q_R(z_0)} \sum_{i=1}^n \partial_t(D_i u_\varrho)\cdot D_i u_\varrho\, \Phi(P_\varrho)\psi\, dx\, dt \;=\; \frac{1}{2}\int_{Q_R(z_0)} \partial_t |Du_\varrho|^2\, \Phi(P_\varrho)\psi\, dx\, dt \\
&= \frac{1}{2}\int_{Q_R(z_0)} \partial_t \left[\int_0^{|Du_\varrho|^2} \Phi((\sqrt{w}-a)_+)\, dw\right]\psi\, dx\, dt \\
&= -\frac{1}{2}\int_{Q_R(z_0)}\int_0^{|Du_\varrho|^2} \Phi((\sqrt{w}-a)_+)\, dw\, \partial_t\chi(t)\omega^2(t)\eta^2(x)\, dx\, dt \\
&\quad - \int_{Q_R(z_0)}\int_0^{|Du_\varrho|^2} \Phi((\sqrt{w}-a)_+)\, dw\, \chi(t)\omega(t)\partial_t\omega(t)\eta^2(x)\, dx\, dt. \qquad (6.60)
\end{aligned}
$$

Now, for $\gamma > 0$ we estimate the inner integral from above and below as follows:

$$
\begin{aligned}
\int_0^{|Du_\varrho|^2} \Phi((\sqrt{w}-a)_+)\, dw &= \int_{a^2}^{|Du_\varrho|^2} (\sqrt{w}-a)_+^2\, w^{\frac{\gamma-2}{2}} dw = 2\int_a^{|Du_\varrho|} (w-a)_+^2\, w^{\gamma-1} dw \\
&\le 2\int_a^{|Du_\varrho|} w^{\gamma+1} dw \cdot \mathbb{1}_{\{|Du_\varrho|>a\}} = \frac{2}{\gamma+2}\,|Du_\varrho|^{\gamma+2}\,\mathbb{1}_{\{|Du_\varrho|>a\}} \le \frac{2}{\gamma+2}\,(\tilde{\mathsf{H}}_{\frac{\delta}{2}}(Du_\varrho))^{\gamma+2}
\end{aligned}
$$

and

$$
\begin{aligned}
\int_0^{|Du_\varrho|^2} \Phi((\sqrt{w}-a)_+)\, dw &= 2\int_a^{|Du_\varrho|} (w-a)_+^2\, w^{\gamma-1} dw \ge 2\int_b^{|Du_\varrho|} \frac{(w-a)^2}{w^2} w^{\gamma+1}\, dw \cdot \mathbb{1}_{\{|Du_\varrho|>b\}} \\
&\ge \frac{\delta^2}{2b^2(\gamma+2)}\,(|Du_\varrho|^{\gamma+2} - b^{\gamma+2})\,\mathbb{1}_{\{|Du_\varrho|>b\}} = \frac{\delta^2}{2b^2(\gamma+2)}\,(|Du_\varrho|^{\gamma+2} - b^{\gamma+2})_+ .
\end{aligned}
$$

Recalling that $\partial_t\chi \le 0$, $\partial_t\omega \ge 0$ and $\chi, \omega \ge 0$, and plugging the two previous inequalities into (6.60), we obtain

$$
\begin{aligned}
J_{0,\varrho} \ge &-\frac{\delta^2}{4b^2(\gamma+2)}\int_{Q_r(z_1)} (|Du_\varrho|^{\gamma+2} - b^{\gamma+2})_+\, \partial_t\chi(t)\omega^2(t)\eta^2(x)\, dx\, dt \\
&- \frac{2}{\gamma+2}\int_{Q_r(z_1)} (\tilde{\mathsf{H}}_{\frac{\delta}{2}}(Du_\varrho))^{\gamma+2}\, \chi(t)\omega(t)\partial_t\omega(t)\eta^2(x)\, dx\, dt.
\end{aligned}
$$

Inserting this into (6.59) and letting $\varrho \searrow 0$ yields the following inequality

$$
J_0 + J_1 + J_2 + J_3 + J_4 + J_5 + J_6 + J_7 + J_8 + J_9 \le 0, \qquad (6.61)
$$

where the terms $J_0 - J_9$ are defined by

$$
\begin{aligned}
J_0 := &-\frac{\delta^2}{4b^2(\gamma+2)}\int_{Q_r(z_1)} \big(|Du|^{\gamma+2} - b^{\gamma+2}\big)_+\, \partial_t\chi\, \omega^2\eta^2\, dz \\
&- \frac{2}{\gamma+2}\int_{Q_r(z_1)} \big(\tilde{\mathsf{H}}_{\frac{\delta}{2}}(Du)\big)^{\gamma+2}\chi\omega\, \partial_t\omega\, \eta^2\, dz,
\end{aligned}
$$

$$
\begin{aligned}
J_1 &:= \int_{Q_r(z_1)} \mathcal{A}_\varepsilon(x,t,Du)(D^2u, D^2u)\,\Phi(P)\psi\,dz,\\
J_2 &:= \int_{Q_r(z_1)} \sum_{i=1}^n \sum_{j=1}^N \sum_{k=1}^n \big\langle D_{\xi_k^j} A_\varepsilon(x,t,Du) D_i D_k u^j, D_i u\, DP\big\rangle\,\Phi'(P)\psi\,dz,\\
J_3 &:= 2\int_{Q_r(z_1)} \mathcal{A}_\varepsilon(x,t,Du)(D^2u, Du\otimes D\eta)\,\Phi(P)\chi\omega^2\eta\,dz,\\
J_4 &:= \int_{Q_r(z_1)} \sum_{i=1}^n f_\varepsilon \cdot D_i^2 u\,\Phi(P)\psi\,dz,\\
J_5 &:= \int_{Q_r(z_1)} \sum_{i=1}^n f_\varepsilon \cdot D_i u\,\Phi'(P) D_i P\,\psi\,dz,\\
J_6 &:= 2\int_{Q_r(z_1)} \sum_{i=1}^n f_\varepsilon \cdot D_i u\,\Phi(P)\chi\omega^2\eta D_i\eta\,dz,\\
J_7 &:= \int_{Q_r(z_1)} \sum_{i=1}^n \big\langle \big[D_{x_i} A_\varepsilon\big](x,t,Du), D_i Du\big\rangle\,\Phi(P)\psi\,dz,\\
J_8 &:= \int_{Q_r(z_1)} \sum_{i=1}^n \big\langle \big[D_{x_i} A_\varepsilon\big](x,t,Du), D_i u\, DP\big\rangle\,\Phi'(P)\psi\,dz,\\
J_9 &:= 2\int_{Q_r(z_1)} \sum_{i=1}^n \big\langle \big[D_{x_i} A_\varepsilon\big](x,t,Du), D\eta D_i u\big\rangle\,\Phi(P)\chi\omega^2\eta\,dz.
\end{aligned}
$$

For convenience of notation, we now use the abbreviations $Q_r = Q_r(z_1)$ and $B_r = B_r(x_1)$. In what follows, we will estimate each of the last nine terms above. We first prove that J_2 is non-negative, thus we can drop it in the following. We limit ourselves to dealing with the case $p > 2$, since for $1 < p \le 2$ the same result follows in a similar way. For $p > 2$ we have

$$
\begin{aligned}
D_{\xi_k^j}[(A_\varepsilon)_m^\ell(x,t,\xi)] &= D_{\xi_k^j}\left[\frac{\partial_s F(x,t,|\xi|)}{|\xi|}\,\xi_m^\ell + \varepsilon\big(|\xi|^2+1\big)^{\frac{p-2}{2}}\xi_m^\ell\right]\\
&= \left(\frac{\partial_s F(x,t,|\xi|)}{|\xi|} + \varepsilon\big(|\xi|^2+1\big)^{\frac{p-2}{2}}\right)\delta_{mk}\,\delta_{\ell j}\\
&\quad + \left(\frac{|\xi|\,\partial_{ss}F(x,t,|\xi|) - \partial_s F(x,t,|\xi|)}{|\xi|^3} + (p-2)\varepsilon\big(|\xi|^2+1\big)^{\frac{p-4}{2}}\right)\xi_k^j\,\xi_m^\ell\,,
\end{aligned}
$$

where δ_{mk} and $\delta_{\ell j}$ denote the Kronecker delta. Using the above equality and the expression for $D|Du|$, we then obtain

$$
\begin{aligned}
&\langle D_{\xi_k^j} A_\varepsilon(x,t,Du) D_i D_k u^j, D_i u\, DP\rangle\\
&\quad= \sum_{m=1}^n \sum_{\ell=1}^N D_{\xi_k^j}[(A_\varepsilon)_m^\ell(x,t,Du)] D_i D_k u^j\, D_i u^\ell\, D_m|Du|\\
&\quad= \sum_{m=1}^n \sum_{\ell=1}^N \left(\frac{\partial_s F(x,t,|Du|)}{|Du|} + \varepsilon\big(|Du|^2+1\big)^{\frac{p-2}{2}}\right) D_i D_m u^\ell\, D_i u^\ell\, D_m|Du|\\
&\qquad+ \sum_{m=1}^n \sum_{\ell=1}^N \left(\frac{|Du|\,\partial_{ss}F(x,t,|Du|) - \partial_s F(x,t,|Du|)}{|Du|^3} + (p-2)\varepsilon\big(|Du|^2+1\big)^{\frac{p-4}{2}}\right)\cdot
\end{aligned}
$$

$$
\begin{aligned}
&\cdot D_k u^j\, D_m u^\ell\, D_i D_k u^j\, D_i u^\ell\, D_m|Du| \\
&= \big(\partial_s F(x,t,|Du|) + \varepsilon\big(|Du|^2+1\big)^{\frac{p-2}{2}}|Du|\big)\,|D|Du||^2 \\
&\quad + \sum_{m=1}^{n}\sum_{\ell=1}^{N}\left(\frac{|Du|\,\partial_{ss}F(x,t,|Du|) - \partial_s F(x,t,|Du|)}{|Du|^2} + (p-2)\varepsilon\big(|Du|^2+1\big)^{\frac{p-4}{2}}|Du|\right)\cdot \\
&\quad \cdot D_m u^\ell\, D_i u^\ell\, D_m|Du|\, D_i|Du|
\end{aligned}
\tag{6.62}
$$

almost everywhere in the set $Q_r \cap \{|Du| > a\}$. Now, the Cauchy-Schwarz inequality implies that

$$
\sum_{m=1}^{n}\sum_{\ell=1}^{N} D_m u^\ell\, D_i u^\ell\, D_m|Du|\, D_i|Du| \;\le\; |Du|^2\,|D|Du||^2.
$$

Combining this with the fact that $\partial_s F(x,t,|Du|), \partial_{ss}F(x,t,|Du|) \ge 0$, from (6.62) we infer that

$$
\begin{aligned}
&\langle D_{\xi_k^j} A_\varepsilon(x,t,Du) D_i D_k u^j, D_i u\, DP\rangle \\
&\quad \ge\; \partial_s F(x,t,|Du|)\,|D|Du||^2 - \frac{\partial_s F(x,t,|Du|)}{|Du|^2}\sum_{m=1}^{n}\sum_{\ell=1}^{N} D_m u^\ell\, D_i u^\ell\, D_m|Du|\, D_i|Du| \ge 0
\end{aligned}
$$

almost everywhere in $Q_r \cap \{|Du| > a\}$. Furthermore, we obviously have

$$
\langle D_{\xi_k^j} A_\varepsilon(x,t,Du) D_i D_k u^j, D_i u\, DP\rangle = 0 \qquad \text{when} \quad |Du| \le a.
$$

Thus, taking into account that $\psi \ge 0$ and $\Phi'(w) = (a+w)^{\gamma-3}w(\gamma w + 2a) \ge 0$ for every $w \ge 0$, we arrive at the desired conclusion, that is $J_2 \ge 0$.

We now deal with the terms J_1 and J_3. We let $0 < \varepsilon < \min\{\frac{1}{2}, (\frac{\delta}{4})^{p-1}\}$. By Lemma 6.4, we have

$$
J_1 \;\ge\; \frac{1}{c_2}\int_{Q_r} |Du|^{p-2}|D^2u|^2\,\Phi(P)\psi\,dz. \tag{6.63}
$$

Using the Cauchy-Schwarz inequality together with Young's inequality and again Lemma 6.4, we get

$$
\begin{aligned}
|J_3| &\le 2\int_{Q_r} |\mathcal{A}_\varepsilon(x,t,Du)(D^2u, Du\otimes D\eta)|\,\Phi(P)\chi\omega^2\eta\,dz \\
&\le 2\int_{Q_r} \sqrt{\mathcal{A}_\varepsilon(x,t,Du)(D^2u,D^2u)}\sqrt{\mathcal{A}_\varepsilon(x,t,Du)(Du\otimes D\eta, Du\otimes D\eta)}\,\Phi(P)\chi\omega^2\eta\,dz \\
&\le \frac{1}{2}J_1 + 2\int_{Q_r} \mathcal{A}_\varepsilon(x,t,Du)(Du\otimes D\eta, Du\otimes D\eta)\,\Phi(P)\chi\omega^2\,dz \\
&\le \frac{1}{2}J_1 + 2c_2\int_{Q_r} |Du|^p\,\Phi(P)\chi\omega^2|D\eta|^2\,dz,
\end{aligned}
\tag{6.64}
$$

where $c_2 \equiv c_2(p, C_1, \delta) > 1$.

Now we estimate the terms containing f_ε. To this end, we will use that $|f_\varepsilon| \le |f|$. Let us first consider J_6. Applying Young's inequality with exponents $(2,2)$ and using the definitions of P, Φ and $\tilde{\mathsf{H}}_\delta$, we find that

$$
|J_6| \;\le\; 2\int_{Q_r} |f||Du|\Phi(P)\chi\omega^2\eta|D\eta|\,dz
$$

$$\leq \int_{Q_r} |f|^2 |Du| \Phi(P) \psi \, dz + \int_{Q_r} |Du| \Phi(P) \chi \omega^2 |D\eta|^2 \, dz$$
$$\leq \int_{Q_r} |f|^2 (\tilde{\mathsf{H}}_\delta(Du))^{\gamma+1} \psi \, dz + \int_{Q_r} |Du|^p \, \Phi(P) \chi \omega^2 |D\eta|^2 \, dz. \tag{6.65}$$

We now deal with the terms J_4 and J_5 at the same time. Using equality (6.58), the fact that $|Du| \leq 2P$ on the set $Q_r \cap \{|Du| \geq 2a\}$ and inequality (6.51) with $w = P$, we get

$$\begin{aligned} |J_4| + |J_5| \, &\leq \, n \int_{Q_r} |f| |D^2 u| \Phi(P) \psi \, dz + \int_{Q_r \cap \{|Du| \geq 2a\}} |f| |Du| \Phi'(P) |D^2 u| \psi \, dz \\ &\quad + \int_{Q_r \cap \{a < |Du| < 2a\}} |f| |Du| \Phi'(P) \, |D^2 u| \psi \, dz \\ &\leq \, 5n(\gamma+1) \int_{Q_r} |f| \Phi(P) |D^2 u| \psi \, dz + \int_{Q_r \cap \{a < |Du| < 2a\}} |f| |Du| \Phi'(P) |D^2 u| \psi \, dz. \end{aligned} \tag{6.66}$$

At this point, we apply Young's inequality with $\kappa > 0$ and exponents $(2, 2)$ to control the first term on the right-hand side of (6.66). In the case $p > 2$, we obtain

$$\begin{aligned} &5n(\gamma+1) \int_{Q_r} |f| \Phi(P) |D^2 u| \psi \, dz \\ &\quad \leq \kappa \int_{Q_r} \Phi(P) |D^2 u|^2 \psi \, dz \, + \, \frac{25n^2(\gamma+1)^2}{4\kappa} \int_{Q_r} |f|^2 \, \Phi(P) \psi \, dz \\ &\quad \leq \kappa \int_{Q_r} \Phi(P) |D^2 u|^2 \psi \, dz \, + \, \frac{25n^2(\gamma+1)^2}{4\kappa} \int_{Q_r} |f|^2 (\tilde{\mathsf{H}}_{\frac{\delta}{2}}(Du))^\gamma \psi \, dz \\ &\quad \leq \kappa \int_{Q_r} \Phi(P) |D^2 u|^2 |Du|^{p-2} \psi \, dz \, + \, \frac{25n^2(\gamma+1)^2}{4\kappa} \int_{Q_r} |f|^2 (\tilde{\mathsf{H}}_\delta(Du))^{\gamma+1} \psi \, dz. \end{aligned} \tag{6.67}$$

If, on the other hand, $1 < p \leq 2$, then we have

$$\begin{aligned} &5n(\gamma+1) \int_{Q_r} |f| \Phi(P) |D^2 u| \psi \, dz \\ &\quad \leq \kappa \int_{Q_r} \Phi(P) |D^2 u|^2 |Du|^{p-2} \psi \, dz \, + \, \frac{25n^2(\gamma+1)^2}{4\kappa} \int_{Q_r} |f|^2 \, \Phi(P) |Du|^{2-p} \psi \, dz \\ &\quad \leq \kappa \int_{Q_r} \Phi(P) |D^2 u|^2 |Du|^{p-2} \psi \, dz \, + \, \frac{25n^2(\gamma+1)^2}{4\kappa} \int_{Q_r} |f|^2 (\tilde{\mathsf{H}}_\delta(Du))^{\gamma+1} \psi \, dz, \end{aligned} \tag{6.68}$$

where, in the last line, we have used that $\Phi(P) = 0$ in $Q_r \cap \{|Du| \leq a\}$, while $\Phi(P)|Du|^{2-p} \leq |Du|^{\gamma+1} \leq (\tilde{\mathsf{H}}_\delta(Du))^{\gamma+1}$ in $Q_r \cap \{|Du| > a\}$. Now we estimate the second integral on the right-hand side of (6.66). Using Young's inequality with exponents $(2, 2)$ as well as inequalities (6.50) and (6.51) with $w = P$, for every $\zeta > 0$ we obtain

$$\begin{aligned} &\int_{Q_r \cap \{a < |Du| < 2a\}} |f| |Du| \Phi'(P) |D^2 u| \psi \, dz \\ &\quad = \int_{Q_r \cap \{a < |Du| < 2a\}} \left[|f|^2 |Du|^{4-p} \, \frac{\gamma+1}{P+\zeta} \right]^{\frac{1}{2}} \left[|Du|^{p-2} |D^2 u|^2 \, \frac{P+\zeta}{\gamma+1} \right]^{\frac{1}{2}} \Phi'(P) \psi \, dz \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{\kappa}{2(\gamma+1)} \int_{Q_r\cap\{a<|Du|<2a\}} |Du|^{p-2}|D^2u|^2(P+\zeta)\Phi'(P)\psi\,dz \\
&\quad + \frac{\gamma+1}{2\kappa} \int_{Q_r\cap\{a<|Du|<2a\}} |f|^2|Du|^{4-p}\,\frac{\Phi'(P)}{P+\zeta}\,\psi\,dz \\
&\leq \kappa \int_{Q_r\cap\{a<|Du|<2a\}} |Du|^{p-2}|D^2u|^2\,\Phi(P)\psi\,dz \\
&\quad + \kappa\zeta \int_{Q_r\cap\{a<|Du|<2a\}} |Du|^{p-2}|D^2u|^2|Du|^{\gamma}\psi\,dz \\
&\quad + \frac{(\gamma+1)^2}{\kappa} \int_{Q_r\cap\{a<|Du|<2a\}} |f|^2|Du|^{4-p}|Du|^{\gamma-2}\psi\,dz \\
&\leq \kappa \int_{Q_r} |Du|^{p-2}|D^2u|^2\,\Phi(P)\psi\,dz \,+\, (2a)^{\gamma}\kappa\zeta \int_{Q_r\cap\{a<|Du|<2a\}} |Du|^{p-2}|D^2u|^2\,dz \\
&\quad + \frac{(\gamma+1)^2}{\kappa} \int_{Q_r\cap\{a<|Du|<2a\}} |f|^2(\tilde{\mathsf{H}}_\delta(Du))^{\gamma+1}\psi\,dz,
\end{aligned}
$$

where the last inequality is due to the fact that $|Du|^{4-p}|Du|^{\gamma-2} = |Du|^{\gamma+1}|Du|^{1-p} \leq |Du|^{\gamma+1} \leq (\tilde{\mathsf{H}}_\delta(Du))^{\gamma+1}$ for every $p > 1$ on the set $Q_r \cap \{a < |Du| < 2a\}$. We now recall that, by virtue of Theorem 6.11, we have

$$
|D^2u|^2|Du|^{p-2} \in L^1_{loc}(Q_R(z_0)).
$$

Therefore, letting $\zeta \searrow 0$ in the preceding inequality, we get

$$
\begin{aligned}
&\int_{Q_r\cap\{a<|Du|<2a\}} |f||Du|\Phi'(P)|D^2u|\psi\,dz \\
&\quad \leq \kappa \int_{Q_r} |Du|^{p-2}|D^2u|^2\,\Phi(P)\psi\,dz \,+\, \frac{(\gamma+1)^2}{\kappa} \int_{Q_r\cap\{a<|Du|<2a\}} |f|^2(\tilde{\mathsf{H}}_\delta(Du))^{\gamma+1}\psi\,dz. \quad (6.69)
\end{aligned}
$$

At this point, joining estimates (6.66)−(6.69), we obtain

$$
\begin{aligned}
|J_4| + |J_5| \,&\leq\, \kappa(n+1) \int_{Q_r} \Phi(P)|D^2u|^2|Du|^{p-2}\psi\,dz \\
&\quad + \frac{8n^2(\gamma+1)^2}{\kappa} \int_{Q_r} |f|^2(\tilde{\mathsf{H}}_\delta(Du))^{\gamma+1}\psi\,dz. \quad (6.70)
\end{aligned}
$$

We now turn our attention to J_7 and J_9. Using (6.22) and Young's inequality, we find

$$
\begin{aligned}
|J_7| \,&\leq\, 2^{2p-2}Kn \int_{Q_r} |Du|^{p-1}|D^2u|\,\Phi(P)\psi\,dz \\
&\leq\, \kappa \int_{Q_r} |Du|^{p-2}|D^2u|^2\,\Phi(P)\psi\,dz \,+\, \frac{16^pK^2n^2}{2\kappa} \int_{Q_r} |Du|^p\,\Phi(P)\psi\,dz. \quad (6.71)
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
|J_9| \,&\leq\, 2^{2p-1}Kn \int_{Q_r} |Du|^p|D\eta|\Phi(P)\chi\omega^2\eta\,dz \\
&\leq\, 2^{2p-2}K^2n \int_{Q_r} |Du|^p\,\Phi(P)\psi\,dz \,+\, 2^{2p-2}n \int_{Q_r} |Du|^p|D\eta|^2\,\Phi(P)\chi\omega^2\,dz. \quad (6.72)
\end{aligned}
$$

Finally we estimate J_8. Using estimate (6.22) and equality (6.58), we obtain

$$\begin{aligned}
|J_8| &\le 2^{2p-2}Kn\int_{Q_r}|Du|^p|DP|\Phi'(P)\psi\,dz \le 2^{2p-2}Kn\int_{Q_r}|Du|^p|D^2u|\Phi'(P)\psi\,dz\\
&= 2^{2p-2}Kn\int_{Q_r\cap\{a<|Du|<2a\}}|Du|^p|D^2u|\Phi'(P)\psi\,dz\\
&\quad + 2^{2p-2}Kn\int_{Q_r\cap\{|Du|\ge 2a\}}|Du|^p|D^2u|\Phi'(P)\psi\,dz. \qquad (6.73)
\end{aligned}$$

Now we deal with the second integral on the right-hand side of (6.73). Using the fact that $|Du|\le 2P$ in $Q_r\cap\{|Du|\ge 2a\}$, the inequality (6.51) with $w=P$ as well as Young's inequality, we get

$$\begin{aligned}
&2^{2p-2}Kn\int_{Q_r\cap\{|Du|\ge 2a\}}|Du|^p|D^2u|\Phi'(P)\psi\,dz\\
&\le 2^{2p-1}Kn\int_{Q_r\cap\{|Du|\ge 2a\}}|Du|^{p-1}|D^2u|P\,\Phi'(P)\psi\,dz\\
&\le 2^{2p}(\gamma+1)Kn\int_{Q_r\cap\{|Du|\ge 2a\}}|Du|^{p-1}|D^2u|\Phi(P)\psi\,dz\\
&\le \kappa\int_{Q_r}|Du|^{p-2}|D^2u|^2\,\Phi(P)\psi\,dz + \frac{16^p(\gamma+1)^2K^2n^2}{4\kappa}\int_{Q_r}|Du|^p\,\Phi(P)\psi\,dz\\
&\le \kappa\int_{Q_r}|Du|^{p-2}|D^2u|^2\,\Phi(P)\psi\,dz + \frac{16^p(\gamma+1)^2K^2n^2}{4\kappa}\int_{Q_r}|Du|^{\gamma+p}\psi\,dz. \qquad (6.74)
\end{aligned}$$

We now estimate the first term on the right-hand side of (6.73) by resorting to the same procedure leading to (6.69). Thus, for every $\sigma>0$ we have

$$\begin{aligned}
&2^{2p-2}Kn\int_{Q_r\cap\{a<|Du|<2a\}}|Du|^p|D^2u|\Phi'(P)\psi\,dz\\
&= \int_{Q_r\cap\{a<|Du|<2a\}}\left[2^{4p-4}K^2n^2\,|Du|^{p+2}\,\frac{\gamma+1}{P+\sigma}\right]^{\frac12}\left[|Du|^{p-2}|D^2u|^2\,\frac{P+\sigma}{\gamma+1}\right]^{\frac12}\Phi'(P)\psi\,dz\\
&\le \frac{\kappa}{2(\gamma+1)}\int_{Q_r\cap\{a<|Du|<2a\}}|Du|^{p-2}|D^2u|^2(P+\sigma)\Phi'(P)\psi\,dz\\
&\quad + \frac{16^p(\gamma+1)K^2n^2}{32\kappa}\int_{Q_r\cap\{a<|Du|<2a\}}|Du|^{p+2}\,\frac{\Phi'(P)}{P+\sigma}\,\psi\,dz\\
&\le \kappa\int_{Q_r}|Du|^{p-2}|D^2u|^2\,\Phi(P)\psi\,dz + (2a)^\gamma\kappa\sigma\int_{Q_r}|Du|^{p-2}|D^2u|^2\,dz\\
&\quad + \frac{16^p(\gamma+1)^2K^2n^2}{4\kappa}\int_{Q_r}|Du|^{\gamma+p}\psi\,dz.
\end{aligned}$$

Letting $\sigma\searrow 0$ in the last inequality, we then obtain

$$\begin{aligned}
&2^{2p-2}Kn\int_{Q_r\cap\{a<|Du|<2a\}}|Du|^p|D^2u|\Phi'(P)\psi\,dz\\
&\le \kappa\int_{Q_r}|Du|^{p-2}|D^2u|^2\,\Phi(P)\psi\,dz + \frac{16^p(\gamma+1)^2K^2n^2}{4\kappa}\int_{Q_r}|Du|^{\gamma+p}\psi\,dz. \qquad (6.75)
\end{aligned}$$

At this point, combining estimates (6.73)−(6.75), we find

$$|J_8| \,\le\, 2\kappa \int_{Q_r} |Du|^{p-2}|D^2u|^2\,\Phi(P)\psi\,dz \,+\, \frac{16^p(\gamma+1)^2K^2n^2}{2\kappa}\int_{Q_r}|Du|^{\gamma+p}\psi\,dz. \tag{6.76}$$

Now, observe that from inequality (6.61) it follows that

$$J_0 + J_1 \le |J_3| + |J_4| + |J_5| + |J_6| + |J_7| + |J_8| + |J_9|, \tag{6.77}$$

since $J_2 \ge 0$. Plugging estimates (6.63)−(6.65), (6.70)−(6.72) and (6.76) into (6.77) and choosing $\kappa = \frac{1}{4c_2(n+4)}$, we arrive at

$$\begin{aligned}
&-\frac{\delta^2}{4b^2(\gamma+2)}\int_{Q_r}(|Du|^{\gamma+2}-b^{\gamma+2})_+\,\partial_t\chi\,\omega^2\eta^2\,dz \,+\, \frac{1}{4c_2}\int_{Q_r}|Du|^{p-2}|D^2u|^2\,\Phi(P)\psi\,dz\\
&\quad\le\, c(\gamma+1)^2\bigg[\int_{Q_r}\Big[|Du|^p\,\Phi(P)\chi\omega^2|D\eta|^2+|Du|^{\gamma+p}\psi\Big]\,dz \,+\, \int_{Q_r}|f|^2(\tilde{\mathsf{H}}_\delta(Du))^{\gamma+1}\psi\,dz\bigg]\\
&\qquad+\,\frac{2}{\gamma+2}\int_{Q_r}(\tilde{\mathsf{H}}_{\frac{\delta}{2}}(Du))^{\gamma+2}\,\chi\omega\,\partial_t\omega\,\eta^2\,dz,
\end{aligned}\tag{6.78}$$

where $c \equiv c(n,p,\delta,C_1,K) > 1$.

At this stage, we perform a particular choice of the function χ. For a fixed time $\tau \in (t_1-r^2,t_1)$ and $\vartheta \in (0,t_1-\tau)$, we define the Lipschitz continuous function χ by

$$\chi(t) := \begin{cases} 1 & \text{if } t \le \tau,\\ 1-\frac{t-\tau}{\vartheta} & \text{if } \tau < t \le \tau+\vartheta,\\ 0 & \text{if } t > \tau+\vartheta. \end{cases}$$

Therefore, we have

$$-\int_{Q_r(z_1)}(|Du|^{\gamma+2}-b^{\gamma+2})_+\,\partial_t\chi\,\omega^2\eta^2\,dz \,=\, \frac{1}{\vartheta}\int_\tau^{\tau+\vartheta}\int_{B_r(x_1)}(|Du|^{\gamma+2}-b^{\gamma+2})_+\,\omega^2\eta^2\,dx\,dt,$$

which converges for almost every $\tau \in (t_1-r^2,t_1)$ to

$$\int_{B_r(x_1)\times\{\tau\}}(|Du|^{\gamma+2}-b^{\gamma+2})_+\,\omega^2\eta^2\,dx \qquad \text{as} \qquad \vartheta \searrow 0.$$

Hence, letting $\vartheta \searrow 0$ and taking the supremum over $\tau \in (t_1-r^2,t_1)$, estimate (6.78) turns into

$$\begin{aligned}
&\frac{\delta^2}{4b^2(\gamma+2)}\sup_{\tau\in(t_1-r^2,t_1)}\int_{B_r\times\{\tau\}}(|Du|^{\gamma+2}-b^{\gamma+2})_+\,\omega^2\eta^2\,dx \,+\, \frac{1}{4c_2}\int_{Q_r}|Du|^{p-2}|D^2u|^2\,\Phi(P)\omega^2\eta^2\,dz\\
&\quad\le\, c(\gamma+1)^2\int_{Q_r}\Big[|Du|^p\,\Phi(P)\omega^2|D\eta|^2+|Du|^{\gamma+p}\,\omega^2\eta^2\Big]\,dz\\
&\qquad+\int_{Q_r}(\tilde{\mathsf{H}}_{\frac{\delta}{2}}(Du))^{\gamma+2}\,\omega\,\partial_t\omega\,\eta^2\,dz \,+\, c(\gamma+1)^2\int_{Q_r}|f|^2(\tilde{\mathsf{H}}_\delta(Du))^{\gamma+1}\omega^2\eta^2\,dz\\
&\quad\le\, c(\gamma+1)^2\int_{Q_r}(\tilde{\mathsf{H}}_\delta(Du))^{\gamma+p}\,(\omega^2|D\eta|^2+\omega^2\eta^2)\,dz
\end{aligned}$$

$$+\int_{Q_r}(\tilde{\mathsf{H}}_{\frac{\delta}{2}}(Du))^{\gamma+2}\,\omega\,\partial_t\omega\,\eta^2\,dz\;+\;c(\gamma+1)^2\int_{Q_r}|f|^2(\tilde{\mathsf{H}}_\delta(Du))^{\gamma+1}\omega^2\eta^2\,dz.$$

In summary, we have so far obtained the following result.

Proposition 6.17. *Let $p>1$, $\gamma>0$, $\delta>0$, $b=1+\delta$ and $0<\varepsilon<\min\{\frac{1}{2},(\frac{\delta}{4})^{p-1}\}$. Moreover, assume that*

$$u_\varepsilon\in C^0\left([t_0-R^2,t_0];L^2\left(B_R(x_0),\mathbb{R}^N\right)\right)\cap L^p\left(t_0-R^2,t_0;W^{1,p}\left(B_R(x_0),\mathbb{R}^N\right)\right)$$

is the unique energy solution of problem $(\mathcal{P}_\varepsilon)$ with $Q'=Q_R(z_0)\Subset\Omega_T$ and u a weak solution of (6.1), satisfying the additional assumption of Remark 6.10 if $1<p\le\frac{2n}{n+2}$. Then, for any parabolic cylinder $Q_r(z_1)\Subset Q_R(z_0)$ with $r\in(0,1)$ and any cut-off functions $\eta\in C_0^\infty(B_r(x_1),[0,1])$ and $\omega\in W^{1,\infty}(\mathbb{R},[0,1])$ satisfying $\omega(t_1-r^2)=0$ and $\partial_t\omega\ge0$, the estimate

$$\begin{aligned}
&\frac{\delta^2}{4b^2(\gamma+2)}\sup_{\tau\in(t_1-r^2,t_1)}\int_{B_r(x_1)\times\{\tau\}}(|Du_\varepsilon|^{\gamma+2}-b^{\gamma+2})_+\,\omega^2\eta^2\,dx\\
&\quad+\frac{1}{4c_2}\int_{Q_r(z_1)}|Du_\varepsilon|^{p-2}|D^2u_\varepsilon|^2\,\Phi(P)\omega^2\eta^2\,dz\\
&\le c(\gamma+1)^2\int_{Q_r(z_1)}(\tilde{\mathsf{H}}_\delta(Du_\varepsilon))^{\gamma+p}\,(\omega^2|D\eta|^2+\omega^2\eta^2)\,dz\\
&\quad+\int_{Q_r(z_1)}(\tilde{\mathsf{H}}_{\frac{\delta}{2}}(Du_\varepsilon))^{\gamma+2}\,\omega\,\partial_t\omega\,\eta^2\,dz\;+\;c(\gamma+1)^2\int_{Q_r(z_1)}|f|^2(\tilde{\mathsf{H}}_\delta(Du_\varepsilon))^{\gamma+1}\omega^2\eta^2\,dz
\end{aligned}\tag{6.79}$$

holds true for some constants $c_2\equiv c_2(p,C_1,\delta)>1$ and $c\equiv c(n,p,\delta,C_1,K)>1$.

In what follows, we will write again u and Φ in place of u_ε and $\Phi_{\gamma,a}$ respectively, unless otherwise specified. First, we estimate the second integral on the left-hand side of (6.79) from below. To this end, we note that

$$D\left[(\tilde{\mathsf{H}}_\delta(Du))^{\frac{\gamma+p}{2}}\right]=\frac{\gamma+p}{2}\,(\tilde{\mathsf{H}}_\delta(Du))^{\frac{\gamma+p-2}{2}}\sum_{i=1}^n\sum_{j=1}^N\frac{D_iu^jD(D_iu^j)}{|Du|}\,\mathbb{1}_{\{|Du|>1+\delta\}}\,.$$

From this identity, we infer

$$\begin{aligned}
\left|D\left[(\tilde{\mathsf{H}}_\delta(Du))^{\frac{\gamma+p}{2}}\right]\right|^2&\le n\,\frac{(\gamma+p)^2}{4}\,(\tilde{\mathsf{H}}_\delta(Du))^{\gamma+p-2}\,|D^2u|^2\,\mathbb{1}_{\{|Du|>1+\delta\}}\\
&\le n\,\frac{(\gamma+p)^2}{4}\,|Du|^{\gamma+p-4}\,|D^2u|^2\,|Du|^2\,\mathbb{1}_{\{|Du|>1+\delta\}}\\
&\le n\,\frac{(\gamma+p)^2(1+\delta)^2}{\delta^2}\,|Du|^{\gamma+p-4}\,P^2\,|D^2u|^2\\
&\le n\,\frac{(\gamma+p)^2(1+\delta)^2}{\delta^2}\,|Du|^{p-2}\,\Phi(P)\,|D^2u|^2,
\end{aligned}\tag{6.80}$$

where, in the second to last line, we have used the fact that

$$|Du|\,\mathbb{1}_{\{|Du|>1+\delta\}}=\left(P+1+\tfrac{\delta}{2}\right)\mathbb{1}_{\{|Du|>1+\delta\}}\le\frac{2(1+\delta)}{\delta}\,P\,\mathbb{1}_{\{|Du|>1+\delta\}}\le\frac{2(1+\delta)}{\delta}\,P.$$

Using estimate (6.80) in combination with Young's inequality, we then obtain

$$\begin{aligned}\left|D\left[\eta(\tilde{\mathsf{H}}_\delta(Du))^{\frac{\gamma+p}{2}}\right]\right|^2 &\le 2\,(\tilde{\mathsf{H}}_\delta(Du))^{\gamma+p}\,|D\eta|^2 \,+\, 2\left|D\left[(\tilde{\mathsf{H}}_\delta(Du))^{\frac{\gamma+p}{2}}\right]\right|^2\eta^2\\ &\le 2\,(\tilde{\mathsf{H}}_\delta(Du))^{\gamma+p}\,|D\eta|^2 \,+\, 2n\,\frac{(\gamma+p)^2\,b^2}{\delta^2}\,|Du|^{p-2}\,\Phi(P)\,|D^2u|^2\,\eta^2,\end{aligned}$$

from which we deduce

$$\begin{aligned}&\frac{\delta^2}{16n(\gamma+p)^2\,b^2\,c_2}\left|D\left[\eta(\tilde{\mathsf{H}}_\delta(Du))^{\frac{\gamma+p}{2}}\right]\right|^2\omega^2\\ &\quad\le\,\frac{\delta^2}{8n(\gamma+p)^2\,b^2\,c_2}\,(\tilde{\mathsf{H}}_\delta(Du))^{\gamma+p}\,|D\eta|^2\,\omega^2 \,+\, \frac{1}{8c_2}\,|Du|^{p-2}\,\Phi(P)\,|D^2u|^2\,\omega^2\eta^2. \qquad (6.81)\end{aligned}$$

Now we consider the case $1<p\le\frac{2n}{n+2}$. This implies

$$\gamma+\frac{4}{n+2}\,\le\,\gamma+2-p\,<\,\gamma+1\,<\,\gamma+p\,. \qquad (6.82)$$

Our goal now is to reduce the exponent of $\tilde{\mathsf{H}}_{\frac{\delta}{2}}(Du)$ in the second integral on the right-hand side of (6.79). To this end, we observe that

$$\begin{aligned}&\int_{Q_r(z_1)}(\tilde{\mathsf{H}}_{\frac{\delta}{2}}(Du))^{\gamma+2}\,\omega\,\partial_t\omega\,\eta^2\,dz\\ &\quad\le 2a^2\int_{Q_r(z_1)}(\tilde{\mathsf{H}}_{\frac{\delta}{2}}(Du))^{\gamma}\,\omega\,\partial_t\omega\,\eta^2\,dz \,+\, 2\int_{Q_r(z_1)}(\tilde{\mathsf{H}}_{\frac{\delta}{2}}(Du))^{\gamma}\,P^2\,\omega\,\partial_t\omega\,\eta^2\,dz\\ &\quad\le 2a^2\int_{Q_r(z_1)}(\tilde{\mathsf{H}}_{\delta}(Du))^{\gamma}\,\omega\,\partial_t\omega\,\eta^2\,dz \,+\, 2\int_{Q_r(z_1)}|Du|^{\gamma}P^2\,\omega\,\partial_t\omega\,\eta^2\,dz. \qquad (6.83)\end{aligned}$$

In addition, for $|Du|>a$ we have

$$\begin{aligned}\sum_{i=1}^n\sum_{j=1}^N D_i[u^j|Du|^{\gamma-2}D_iu^jP^2\eta^2] \,=\,& |Du|^\gamma P^2\eta^2+|Du|^{\gamma-2}P^2\eta^2\sum_{i=1}^n\sum_{j=1}^N u^j\,D_i^2u^j\\ &+(\gamma-2)|Du|^{\gamma-3}P^2\eta^2\sum_{i,k=1}^n\sum_{j,\ell=1}^N u^j\,D_iu^j\,\frac{D_ku^\ell\,D_iD_ku^\ell}{|Du|}\\ &+2\,|Du|^{\gamma-2}P\eta^2\sum_{i,k=1}^n\sum_{j,\ell=1}^N u^j\,D_iu^j\,\frac{D_ku^\ell\,D_iD_ku^\ell}{|Du|}\\ &+2\,|Du|^{\gamma-2}P^2\eta\sum_{i=1}^n\sum_{j=1}^N u^j\,D_iu^j\,D_i\eta.\end{aligned}$$

Using the above identity to estimate the last integral of (6.83) and taking into account that $\eta\in C_0^\infty(B_r(x_1))$ and ω is independent of the x-variable, we obtain

$$\int_{Q_r(z_1)}|Du|^\gamma P^2\,\omega\,\partial_t\omega\,\eta^2\,dz\,\le\,\|u\|_{L^\infty(Q_r(z_1))}\int_{Q_r(z_1)}|Du|^{\gamma-2}|D^2u|P^2\,\omega\,\partial_t\omega\,\eta^2\,dz$$

$$
\begin{aligned}
&+ |\gamma-2|\,\|u\|_{L^\infty(Q_r(z_1))}\int_{Q_r(z_1)}|Du|^{\gamma-2}|D^2u|P^2\,\omega\,\partial_t\omega\,\eta^2\,dz\\
&+ 2\,\|u\|_{L^\infty(Q_r(z_1))}\int_{Q_r(z_1)}|Du|^{\gamma-1}|D^2u|P\,\omega\,\partial_t\omega\,\eta^2\,dz\\
&+ 2\,\|u\|_{L^\infty(Q_r(z_1))}\int_{Q_r(z_1)}|Du|^{\gamma-1}P^2\,\omega\,\partial_t\omega\,\eta\,|D\eta|\,dz\\
\le\ &(\gamma+5)\|u\|_{L^\infty(Q_r(z_1))}\int_{Q_r(z_1)}|Du|^{\gamma-1}|D^2u|P\,\omega\,\partial_t\omega\,\eta^2\,dz\\
&+ 2\,\|u\|_{L^\infty(Q_r(z_1))}\int_{Q_r(z_1)}|Du|^{\gamma+1}\omega\,\partial_t\omega\,\eta\,|D\eta|\,dz. \qquad (6.84)
\end{aligned}
$$

Joining estimates (6.83) and (6.84) and applying Young's inequality, we then have

$$
\begin{aligned}
&\int_{Q_r(z_1)}(\tilde{\mathsf{H}}_{\frac{\delta}{2}}(Du))^{\gamma+2}\,\omega\,\partial_t\omega\,\eta^2\,dz\\
&\quad\le\ 2a^2\int_{Q_r(z_1)}(\tilde{\mathsf{H}}_\delta(Du))^{\gamma}\,\omega\,\partial_t\omega\,\eta^2\,dz\ +\ 4\,\|u\|_{L^\infty(Q_r(z_1))}\int_{Q_r(z_1)}|Du|^{\gamma+1}\omega\,\partial_t\omega\,\eta\,|D\eta|\,dz\\
&\qquad+10(\gamma+1)\|u\|_{L^\infty(Q_r(z_1))}\int_{Q_r(z_1)}|Du|^{\gamma-1}|D^2u|P\,\omega\,\partial_t\omega\,\eta^2\,dz\\
&\quad=\ 2a^2\int_{Q_r(z_1)}(\tilde{\mathsf{H}}_\delta(Du))^{\gamma}\,\omega\,\partial_t\omega\,\eta^2\,dz\ +\ 4\,\|u\|_{L^\infty(Q_r(z_1))}\int_{Q_r(z_1)}|Du|^{\gamma+1}\omega\,\partial_t\omega\,\eta\,|D\eta|\,dz\\
&\qquad+10(\gamma+1)\|u\|_{L^\infty(Q_r(z_1))}\int_{Q_r(z_1)}(|Du|^{\frac{\gamma+2-p}{2}}\partial_t\omega\,\eta)(|Du|^{\frac{p-2}{2}}|D^2u||Du|^{\frac{\gamma-2}{2}}P\omega\eta)\,dz\\
&\quad\le\ 2a^2\int_{Q_r(z_1)}(\tilde{\mathsf{H}}_\delta(Du))^{\gamma}\,\omega\,\partial_t\omega\,\eta^2\,dz\ +\ 4\,\|u\|_{L^\infty(Q_r(z_1))}\int_{Q_r(z_1)}|Du|^{\gamma+1}\omega\,\partial_t\omega\,\eta\,|D\eta|\,dz\\
&\qquad+c(\gamma+1)^2\,\|u\|^2_{L^\infty(Q_r(z_1))}\int_{Q_r(z_1)}|Du|^{\gamma+2-p}(\partial_t\omega)^2\,\eta^2\,dz\\
&\qquad+\ \frac{1}{8c_2}\int_{Q_r(z_1)}|Du|^{p-2}|D^2u|^2\,\Phi(P)\,\omega^2\eta^2\,dz.
\end{aligned}
$$

Now we notice that $a^2\le b(\tilde{\mathsf{H}}_\delta(Du))^p$, $|Du|^{\gamma+1}\le(\tilde{\mathsf{H}}_\delta(Du))^{\gamma+p}$ and $|Du|^{\gamma+2-p}\le(\tilde{\mathsf{H}}_\delta(Du))^{\gamma+p}$ (by virtue of (6.82)). Using this information and Young's inequality, from the previous estimate we deduce

$$
\begin{aligned}
&\int_{Q_r(z_1)}(\tilde{\mathsf{H}}_{\frac{\delta}{2}}(Du))^{\gamma+2}\,\omega\,\partial_t\omega\,\eta^2\,dz\\
&\quad\le\ 2b\int_{Q_r(z_1)}(\tilde{\mathsf{H}}_\delta(Du))^{\gamma+p}\,\omega\,\partial_t\omega\,\eta^2\,dz\ +\ 4\,\|u\|_{L^\infty(Q_r(z_1))}\int_{Q_r(z_1)}(\tilde{\mathsf{H}}_\delta(Du))^{\gamma+p}\,\omega\,\partial_t\omega\,\eta\,|D\eta|\,dz\\
&\qquad+\ c(\gamma+1)^2\,\|u\|^2_{L^\infty(Q_r(z_1))}\int_{Q_r(z_1)}(\tilde{\mathsf{H}}_\delta(Du))^{\gamma+p}(\partial_t\omega)^2\,\eta^2\,dz\\
&\qquad+\ \frac{1}{8c_2}\int_{Q_r(z_1)}|Du|^{p-2}|D^2u|^2\,\Phi(P)\,\omega^2\eta^2\,dz\\
&\quad\le\ 2b\int_{Q_r(z_1)}(\tilde{\mathsf{H}}_\delta(Du))^{\gamma+p}\,[\omega\,\partial_t\omega\,\eta^2+\omega^2\,|D\eta|^2]\,dz
\end{aligned}
$$

$$+\ c(\gamma+1)^2\,\|u\|^2_{L^\infty(Q_r(z_1))}\int_{Q_r(z_1)}(\tilde{\mathsf{H}}_\delta(Du))^{\gamma+p}(\partial_t\omega)^2\,\eta^2\,dz$$
$$+\ \frac{1}{8c_2}\int_{Q_r(z_1)}|Du|^{p-2}|D^2u|^2\,\Phi(P)\,\omega^2\eta^2\,dz. \tag{6.85}$$

At this point, in order to deal jointly with the cases where p is greater or less than the critical exponent $\frac{2n}{n+2}$, we introduce a new exponent $\mathfrak{p}$, defined as follows:

$$\mathfrak{p}:=\begin{cases} p & \text{if } 1<p\le\frac{2n}{n+2},\\ 2 & \text{if } \frac{2n}{n+2}<p<2,\\ p & \text{if } p\ge 2.\end{cases} \tag{6.86}$$

Combining (6.79) and (6.85), in the case $1<p\le\frac{2n}{n+2}$ we obtain the following inequality

$$\begin{aligned}&\frac{\delta^2}{4b^2(\gamma+2)}\sup_{\tau\in(t_1-r^2,t_1)}\int_{B_r(x_1)\times\{\tau\}}(|Du|^{\gamma+2}-b^{\gamma+2})_+\,\omega^2\eta^2\,dx\\ &+\ \frac{1}{8c_2}\int_{Q_r(z_1)}|Du|^{p-2}|D^2u|^2\,\Phi(P)\,\omega^2\eta^2\,dz\\ &\quad\le\ c(\gamma+1)^2\int_{Q_r(z_1)}(\tilde{\mathsf{H}}_\delta(Du))^{\gamma+\mathfrak{p}}\,[\omega^2|D\eta|^2+\omega\,\partial_t\omega\,\eta^2+(\|u\|^2_{L^\infty(Q_r(z_1))}(\partial_t\omega)^2+\omega^2)\,\eta^2]\,dz\\ &\qquad+\ c(\gamma+1)^2\int_{Q_r(z_1)}|f|^2(\tilde{\mathsf{H}}_\delta(Du))^{\gamma+1}\,\omega^2\eta^2\,dz.\end{aligned} \tag{6.87}$$

For convenience of notation, we now use the indicator function

$$\mathbb{1}_{(1,\frac{2n}{n+2}]}(p):=\begin{cases}1 & \text{if } 1<p\le\frac{2n}{n+2},\\ 0 & \text{otherwise,}\end{cases}$$

to indicate that the terms multiplied by it need to be taken into account only in the subcritical case $1<p\le\frac{2n}{n+2}$. Integrating (6.81) over $Q_r(z_1)$ and combining the resulting inequality with (6.87) in the case $1<p\le\frac{2n}{n+2}$, after some algebraic manipulation, for every $p>1$ we get the following estimate:

Proposition 6.18 (**Caccioppoli-type inequality**). *Let $p>1$, $\gamma>0$, $\delta>0$, $b=1+\delta$ and $0<\varepsilon<\min\{\frac12,(\frac{\delta}{4})^{p-1}\}$. Moreover, let $\mathfrak{p}$ be defined as in* (6.86) *and assume that*

$$u_\varepsilon\in C^0\left([t_0-R^2,t_0];L^2\left(B_R(x_0),\mathbb{R}^N\right)\right)\cap L^p\left(t_0-R^2,t_0;W^{1,p}\left(B_R(x_0),\mathbb{R}^N\right)\right)$$

is the unique energy solution of problem $(\mathcal{P}_\varepsilon)$ with $Q'=Q_R(z_0)\Subset\Omega_T$ and u a weak solution of (6.1)*, satisfying the additional assumption of Remark 6.10 if $1<p\le\frac{2n}{n+2}$. Then, for any parabolic cylinder $Q_r(z_1)\Subset Q_R(z_0)$ with $r\in(0,1)$ and any cut-off functions $\eta\in C_0^\infty(B_r(x_1),[0,1])$ and $\omega\in W^{1,\infty}(\mathbb{R},[0,1])$ satisfying $\omega(t_1-r^2)=0$ and $\partial_t\omega\ge0$, the estimate*

$$\begin{aligned}&\frac{\gamma+p}{2}\sup_{\tau\in(t_1-r^2,t_1)}\int_{B_r(x_1)\times\{\tau\}}(|Du_\varepsilon|^{\gamma+2}-b^{\gamma+2})_+\,\omega^2\eta^2\,dx+\int_{Q_r(z_1)}\left|D\left[\eta(\tilde{\mathsf{H}}_\delta(Du_\varepsilon))^{\frac{\gamma+p}{2}}\right]\right|^2\omega^2\,dz\\ &\qquad\le\ k(\gamma+p)^4\int_{Q_r(z_1)}(\tilde{\mathsf{H}}_\delta(Du_\varepsilon))^{\gamma+\mathfrak{p}}\left[\omega^2|D\eta|^2+\omega\,\partial_t\omega\,\eta^2+\omega^2\eta^2\right]dz\end{aligned}$$

$$+\ k(\gamma+p)^4\int_{Q_r(z_1)}|f|^2(\tilde{\mathsf{H}}_\delta(Du_\varepsilon))^{\gamma+1}\,\omega^2\eta^2\,dz$$
$$+\ \mathbb{1}_{(1,\frac{2n}{n+2}]}(p)\ k(\gamma+p)^4\,\|u_\varepsilon\|^2_{L^\infty(Q_r(z_1))}\int_{Q_r(z_1)}(\tilde{\mathsf{H}}_\delta(Du_\varepsilon))^{\gamma+\mathfrak{p}}\,(\partial_t\omega)^2\,\eta^2\,dz$$

holds true for some constant $k>1$ depending on n, p, δ, C_1 and K.

6.6.2 Step 2: reverse Hölder-type inequality

In this subsection, we shall use the estimate from Proposition 6.18 to derive a reverse Hölder-type inequality, which will in turn be employed to start the Moser iteration procedure. With this aim in mind, in what follows we keep the assumptions, notations and cylinders used for the proof of Proposition 6.18.

Let us start with the case $p>\frac{2n}{n+2}$. Applying Hölder's inequality with exponents $\big(\frac{\hat n}{\hat n-2},\frac{\hat n}{2}\big)$ and the Sobolev embedding theorem on the time slices $B^t:=B_r(x_1)\times\{t\}$ for almost every $t\in(t_1-r^2,t_1)$, we obtain

$$\begin{aligned}
&\fint_{B^t}\omega^{2+\frac{4}{\hat n}}\,\eta^{2+\frac{4}{\hat n}}\,(\tilde{\mathsf{H}}_\delta(Du))^{\gamma+p+\frac{2(\gamma+2)}{\hat n}}\,dx\\
&\quad\le\ (\omega(t))^{2+\frac{4}{\hat n}}\bigg[\fint_{B^t}\big[\eta(\tilde{\mathsf{H}}_\delta(Du))^{\frac{\gamma+p}{2}}\big]^{\frac{2\hat n}{\hat n-2}}\,dx\bigg]^{\frac{\hat n-2}{\hat n}}\bigg[\fint_{B^t}\eta^2(\tilde{\mathsf{H}}_\delta(Du))^{\gamma+2}\,dx\bigg]^{\frac{2}{\hat n}}\\
&\quad\le\ C_S\,r^2\bigg[\fint_{B^t}\big|D\big[\eta(\tilde{\mathsf{H}}_\delta(Du))^{\frac{\gamma+p}{2}}\big]\big|^2\omega^2\,dx\bigg]\bigg[\fint_{B^t}\eta^2(\tilde{\mathsf{H}}_\delta(Du))^{\gamma+2}\,\omega^2\,dx\bigg]^{\frac{2}{\hat n}},
\end{aligned}\tag{6.88}$$

where $C_S\equiv C_S(N,n,p)\ge 1$. Now, using the definitions of $\tilde{\mathsf{H}}_\delta$ and b, the properties of ω and η, and applying Proposition 6.18, we can estimate the second mean value on the right-hand side of (6.88) as follows:

$$\begin{aligned}
\fint_{B^t}\eta^2\,[\tilde{\mathsf{H}}_\delta(Du(x,t))]^{\gamma+2}\,\omega^2\,dx\ &\le\ \frac{c(n)}{r^n}\sup_{\tau\in(t_1-r^2,t_1)}\int_{B_r(x_1)\times\{\tau\}}(\tilde{\mathsf{H}}_\delta(Du))^{\gamma+2}\,\omega^2\eta^2\,dx\\
&=\ \frac{c(n)}{r^n}\sup_{\tau\in(t_1-r^2,t_1)}\int_{B_r(x_1)\times\{\tau\}}[b^{\gamma+2}+(|Du|^{\gamma+2}-b^{\gamma+2})_+]\,\omega^2\eta^2\,dx\\
&\le\ \frac{\tilde c\,(\gamma+p)^3}{r^n}\bigg[\mathbf{I}_1+\int_{Q_r(z_1)}|f|^2(\tilde{\mathsf{H}}_\delta(Du))^{\gamma+1}\,\omega^2\eta^2\,dz\bigg],
\end{aligned}$$

where $\tilde c\equiv\tilde c(n,p,\delta,C_1,K)>1$ and

$$\mathbf{I}_1:=\ b^{\gamma+2}\,|\operatorname{spt}\eta|\ +\int_{Q_r(z_1)}(\tilde{\mathsf{H}}_\delta(Du))^{\gamma+\mathfrak{p}}\,\big[\omega^2(|D\eta|^2+\eta^2)+\omega\,\partial_t\omega\,\eta^2\big]\,dz.$$

Inserting the preceding inequality into (6.88), integrating with respect to time over (t_1-r^2,t_1) and using Proposition 6.18 again, we obtain

$$\mathbf{I}_2:=\fint_{Q_r(z_1)}\omega^{2+\frac{4}{\hat n}}\,\eta^{2+\frac{4}{\hat n}}\,(\tilde{\mathsf{H}}_\delta(Du))^{\gamma+p+\frac{2(\gamma+2)}{\hat n}}\,dz$$

$$\leq \frac{C_2\,(\gamma+p)^{4+\frac{6}{\hat n}}}{r^{n(1+\frac{2}{\hat n})}}\left[\mathbf{I}_1^{1+\frac{2}{\hat n}}+\left(\int_{Q_r(z_1)}|f|^2(\tilde{\mathsf{H}}_\delta(Du))^{\gamma+1}\,\omega^2\eta^2\,dz\right)^{1+\frac{2}{\hat n}}\right], \tag{6.89}$$

where $C_2\equiv C_2(N,n,\hat n,p,\delta,C_1,K)>1$. For a fixed $\mu>0$ that will be chosen later, we now split the cylinder $Q_r(z_1)$ as follows

$$Q_r(z_1)=\left\{z\in Q_r(z_1):|f|^2\leq \exp\left(\mu^{-1}\right)-e\right\}\cup\left\{z\in Q_r(z_1):|f|^2>\exp\left(\mu^{-1}\right)-e\right\}=:E_\mu^1\cup E_\mu^2,$$

so that

$$\begin{aligned}\int_{Q_r(z_1)}|f|^2(\tilde{\mathsf{H}}_\delta(Du))^{\gamma+1}\omega^2\eta^2\,dz&=\int_{E_\mu^1}|f|^2(\tilde{\mathsf{H}}_\delta(Du))^{\gamma+1}\omega^2\eta^2\,dz+\int_{E_\mu^2}|f|^2(\tilde{\mathsf{H}}_\delta(Du))^{\gamma+1}\omega^2\eta^2\,dz\\&=:J_{10}+J_{11}.\end{aligned} \tag{6.90}$$

From the definition of E_μ^1, it immediately follows that

$$J_{10}\leq \exp\left(\frac{1}{\mu}\right)\int_{Q_r(z_1)}(\tilde{\mathsf{H}}_\delta(Du))^{\gamma+1}\,\omega^2\eta^2\,dz\leq \exp\left(\frac{1}{\mu}\right)\int_{Q_r(z_1)}(\tilde{\mathsf{H}}_\delta(Du))^{\gamma+\mathfrak{p}}\,\omega^2\eta^2\,dz. \tag{6.91}$$

In order to deal with the term J_{11}, we first observe that

$$\log^{\frac{2\alpha}{\hat n+2}}(e+|f|^2)<\log^{\frac{2\alpha}{\hat n+2}}\left((e+|f|)^2\right)=2^{\frac{2\alpha}{\hat n+2}}\log^{\frac{2\alpha}{\hat n+2}}(e+|f|) \tag{6.92}$$

and

$$(\gamma+1)\left(1+\frac{2}{\hat n}\right)<\gamma+p+\frac{2(\gamma+2)}{\hat n}\,. \tag{6.93}$$

Using (6.92) and (6.93), Hölder's inequality, the definition of E_μ^2 and the assumption $|f|\in L^{\hat n+2}\log^\alpha L_{loc}(\Omega_T)$, we obtain

$$\begin{aligned}J_{11}&=\int_{E_\mu^2}|f|^2\log^{\frac{2\alpha}{\hat n+2}}(e+|f|^2)\log^{-\frac{2\alpha}{\hat n+2}}(e+|f|^2)(\tilde{\mathsf{H}}_\delta(Du))^{\gamma+1}\,\omega^2\eta^2\,dz\\&\leq 2^{\frac{2\alpha}{\hat n+2}}\int_{E_\mu^2}|f|^2\log^{\frac{2\alpha}{\hat n+2}}(e+|f|)\log^{-\frac{2\alpha}{\hat n+2}}(e+|f|^2)(\tilde{\mathsf{H}}_\delta(Du))^{\gamma+1}\,\omega^2\eta^2\,dz\\&\leq 2^{\frac{2\alpha}{\hat n+2}}\left[\int_{E_\mu^2}|f|^{\hat n+2}\log^\alpha(e+|f|)\,dz\right]^{\frac{2}{\hat n+2}}\cdot\\&\qquad\cdot\left[\int_{E_\mu^2}\log^{-\frac{2\alpha}{\hat n}}(e+|f|^2)(\tilde{\mathsf{H}}_\delta(Du))^{(\gamma+1)\left(1+\frac{2}{\hat n}\right)}\,\omega^{2+\frac{4}{\hat n}}\,\eta^{2+\frac{4}{\hat n}}\,dz\right]^{\frac{\hat n}{\hat n+2}}\\&\leq 2^{\frac{2\alpha}{\hat n+2}}\,\|f\|^\theta_{L^{\hat n+2}\log^\alpha L(Q_r(z_1))}\,\mu^{\frac{2\alpha}{\hat n+2}}\left[\int_{E_\mu^2}(\tilde{\mathsf{H}}_\delta(Du))^{\gamma+p+\frac{2(\gamma+2)}{\hat n}}\,\omega^{2+\frac{4}{\hat n}}\,\eta^{2+\frac{4}{\hat n}}\,dz\right]^{\frac{\hat n}{\hat n+2}}\\&\leq 2^{\frac{2\alpha}{\hat n+2}}\,\mu^{\frac{2\alpha}{\hat n+2}}\,\|f\|^\theta_{L^{\hat n+2}\log^\alpha L(Q_r(z_1))}\left[\int_{Q_r(z_1)}(\tilde{\mathsf{H}}_\delta(Du))^{\gamma+p+\frac{2(\gamma+2)}{\hat n}}\,\omega^{2+\frac{4}{\hat n}}\,\eta^{2+\frac{4}{\hat n}}\,dz\right]^{\frac{\hat n}{\hat n+2}},\end{aligned} \tag{6.94}$$

where $\theta\equiv\theta(\hat n)>0$. Joining (6.89) − (6.91) and (6.94), after some algebraic manipulation we

arrive at

$$\mathbf{I}_2 \,\le\, \frac{C_3\,(\gamma+p)^{4+\frac{6}{\hat n}}}{r^{\frac{2n}{\hat n}}}\left[r^2\mu^{\frac{2\alpha}{\hat n}}\,\|f\|^{\frac{\theta(\hat n+2)}{\hat n}}_{L^{\hat n+2}\log^\alpha L(Q_r(z_1))}\,\mathbf{I}_2\,+\,\frac{\exp\left(\frac{1}{\mu}\left(1+\frac{2}{\hat n}\right)\right)}{r^n}\,\mathbf{I}_1^{1+\frac{2}{\hat n}}\right], \tag{6.95}$$

where $C_3 \equiv C_3(N,n,\hat n,\alpha,p,\delta,C_1,K) > 1$.

At this point, we consider estimate (6.89) in the case $\|f\|_{L^{\hat n+2}\log^\alpha L(Q_r(z_1))} = 0$ and the above inequality if $\|f\|_{L^{\hat n+2}\log^\alpha L(Q_r(z_1))} > 0$. In the latter case, choosing μ such that

$$\frac{C_3\,(\gamma+p)^{4+\frac{6}{\hat n}}}{r^{2(\frac{n}{\hat n}-1)}}\,\mu^{\frac{2\alpha}{\hat n}}\,\|f\|^{\frac{\theta(\hat n+2)}{\hat n}}_{L^{\hat n+2}\log^\alpha L(Q_r(z_1))} \,=\, \frac12\,,$$

from (6.95) we obtain

$$\mathbf{I}_2 \,\le\, C_4\,(\gamma+p)^{4+\frac{6}{\hat n}}\,\exp\left(C_5\,(\gamma+p)^{\frac{2\hat n+3}{\alpha}}\right)\mathbf{I}_1^{1+\frac{2}{\hat n}}, \tag{6.96}$$

where the constants on the right-hand side are of the type

$$C_4 \,=\, \frac{C}{r^{n(1+\frac{2}{\hat n})}} \qquad\text{and}\qquad C_5 \,=\, C\,\|f\|^{\frac{\theta(\hat n+2)}{2\alpha}}_{L^{\hat n+2}\log^\alpha L(Q_r(z_1))}\,r^{\frac{\hat n-n}{\alpha}}, \tag{6.97}$$

for some constant $C \equiv C(N,n,\hat n,\alpha,p,\delta,C_1,K) > 1$. Moreover, from (6.89) it immediately follows that inequality (6.96) holds true also when $\|f\|_{L^{\hat n+2}\log^\alpha L(Q_r(z_1))} = 0$.

Let us now consider the case $1 < p \le \frac{2n}{n+2}$. Arguing exactly as above, this time we arrive at

$$\begin{aligned}\mathbf{I}_2 \,\le\, & C_4\,(\gamma+p)^{4+\frac{6}{\hat n}}\,\exp\left(C_5\,(\gamma+p)^{\frac{2\hat n+3}{\alpha}}\right)\Bigg[b^{\gamma+2}\,|\operatorname{spt}\eta| \\ & +\int_{Q_r(z_1)}(\tilde{\mathsf{H}}_\delta(Du))^{\gamma+\mathfrak{p}}\big[\omega^2(|D\eta|^2+\eta^2)+\omega\,\partial_t\omega\,\eta^2+\|u\|^2_{L^\infty(Q_r(z_1))}\,(\partial_t\omega)^2\,\eta^2\big]\,dz\Bigg]^{1+\frac{2}{\hat n}},\end{aligned}$$

where the constants C_4 and C_5 are of the same type as in (6.97).

In summary, we have obtained the following result:

Proposition 6.19 (**Reverse Hölder-type inequality**). *Under the assumptions of Proposition 6.18, for any parabolic cylinder $Q_r(z_1) \Subset Q_R(z_0)$ with $r \in (0,1)$ and any cut-off functions $\eta \in C_0^\infty(B_r(x_1),[0,1])$ and $\omega \in W^{1,\infty}(\mathbb{R},[0,1])$ satisfying $\omega(t_1-r^2)=0$ as well as $\partial_t\omega \ge 0$, the estimate*

$$\begin{aligned}&\fint_{Q_r(z_1)}(\omega\eta)^{2+\frac{4}{\hat n}}\,(\tilde{\mathsf{H}}_\delta(Du_\varepsilon))^{\gamma+p+\frac{2(\gamma+2)}{\hat n}}\,dz \\ &\quad\le\, C\,(\gamma+p)^{4+\frac{6}{\hat n}}\,\exp\left[\frac{C\,\|f\|^\Theta_{L^{\hat n+2}\log^\alpha L(Q_r(z_1))}\,(\gamma+p)^{\frac{2\hat n+3}{\alpha}}}{r^{\frac{n-\hat n}{\alpha}}}\right] \\ &\qquad\cdot\Bigg[r^2\fint_{Q_r(z_1)}(\tilde{\mathsf{H}}_\delta(Du_\varepsilon))^{\gamma+\mathfrak{p}}\big[\omega^2(|D\eta|^2+\eta^2)+\omega\,\partial_t\omega\,\eta^2\big]\,dz\,+\,\frac{b^{\gamma+2}}{r^n}\,|\operatorname{spt}\eta| \\ &\qquad\quad+\mathbb{1}_{(1,\frac{2n}{n+2}]}(p)\,\|u_\varepsilon\|^2_{L^\infty(Q_r(z_1))}\,r^2\fint_{Q_r(z_1)}(\tilde{\mathsf{H}}_\delta(Du_\varepsilon))^{\gamma+\mathfrak{p}}\,(\partial_t\omega)^2\,\eta^2\,dz\Bigg]^{1+\frac{2}{\hat n}}\end{aligned} \tag{6.98}$$

holds true for some constants $C \equiv C(N,n,\hat n,\alpha,p,\delta,C_1,K) > 1$ and $\Theta \equiv \Theta(\hat n,\alpha) > 0$.

6.6.3 Step 3: the iteration for $p > \frac{2n}{n+2}$

Thanks to Proposition 6.19, we can now start the Moser iteration procedure. Once again, we shall keep both the assumptions and the notations used in Propositions 6.18 and 6.19. At this step, we assume that $p > \frac{2n}{n+2}$. By virtue of (6.3) and (6.4), this implies that $p > \frac{2\hat{n}}{\hat{n}+2}$. We define by induction a sequence $\{\gamma_k\}_{k\in\mathbb{N}_0}$ by setting $\gamma_0 = 0$ and

$$\gamma_k := \begin{cases} \left(1+\frac{2}{\hat{n}}\right)\gamma_{k-1} + \frac{4}{\hat{n}} - 2 + p & \text{if } \frac{2\hat{n}}{\hat{n}+2} < p < 2 \\ \left(1+\frac{2}{\hat{n}}\right)\gamma_{k-1} + \frac{4}{\hat{n}} & \text{if } p \geq 2 \end{cases}, \ k \in \mathbb{N}.$$

Notice that $\gamma_k > 0$ for every $k \in \mathbb{N}$, since $p > \frac{2\hat{n}}{\hat{n}+2}$. Furthermore, this sequence diverges to $+\infty$ as $k \to \infty$ and by induction we have

$$\gamma_k = \begin{cases} \left[2 - \frac{\hat{n}(2-p)}{2}\right]\left[\left(1+\frac{2}{\hat{n}}\right)^k - 1\right] & \text{if } \frac{2\hat{n}}{\hat{n}+2} < p < 2 \\ 2\left[\left(1+\frac{2}{\hat{n}}\right)^k - 1\right] & \text{if } p \geq 2 \end{cases}, \ k \in \mathbb{N}_0.$$

In addition, one can easily check that

$$\gamma_k + p + \frac{2(\gamma_k+2)}{\hat{n}} = \gamma_{k+1} + \mathfrak{p} \qquad \forall\, k \in \mathbb{N}_0, \forall\, p > \frac{2\hat{n}}{\hat{n}+2}\,. \tag{6.99}$$

Now, for $k \in \mathbb{N}_0$ and $s \in (0,1)$ we set

$$r_k := sr + \frac{(1-s)r}{2^k}, \qquad \mathcal{B}_k := B_{r_k}(x_1) \qquad \text{and} \qquad \mathcal{Q}_k := Q_{r_k}(z_1) = \mathcal{B}_k \times (t_1 - r_k^2, t_1)$$

and choose cut-off functions $\eta_k \in C_0^\infty(\mathcal{B}_k, [0,1])$ such that $\eta_k \equiv 1$ on $\mathcal{B}_{k+1}$ and $|D\eta_k| \leq \frac{2^{k+2}}{(1-s)r}$ and $\omega_k \in W^{1,\infty}((t_1 - r_k^2, t_1), [0,1])$ such that $\omega_k(t_1 - r_k^2) = 0$, $\omega_k \equiv 1$ on $(t_1 - r_{k+1}^2, t_1)$ and $0 \leq \partial_t \omega_k \leq \frac{2^{2k+2}}{(1-s)^2 r^2}$. Note that this is possible since

$$r_k^2 - r_{k+1}^2 = \frac{3(1-s)^2 r^2}{2^{2k+2}} + \frac{(1-s)sr^2}{2^k} > \frac{(1-s)^2 r^2}{2^{2k+2}}\,.$$

Using equality (6.99) and replacing γ, η and ω with γ_k, η_k and ω_k respectively, from (6.98) we obtain the following *recursive reverse Hölder-type inequality*

$$\begin{aligned} &\fint_{Q_r(z_1)} (\omega_k \eta_k)^{2+\frac{4}{\hat{n}}} \left(\tilde{\mathsf{H}}_\delta(Du_\varepsilon)\right)^{\gamma_{k+1}+\mathfrak{p}} dz \\ &\quad \leq C(\gamma_k + p)^{4+\frac{6}{\hat{n}}} \exp\left[\frac{C\,\|f\|^{\Theta}_{L^{\hat{n}+2}\log^\alpha L(Q_r(z_1))}\,(\gamma_k+p)^{\frac{2\hat{n}+3}{\alpha}}}{r^{\frac{n-\hat{n}}{\alpha}}}\right] \\ &\qquad \cdot \left[\frac{b^{\gamma_k+2}}{r^n}\,|\mathrm{spt}\,\eta_k| + r^2 \fint_{Q_r(z_1)} \left(\tilde{\mathsf{H}}_\delta(Du_\varepsilon)\right)^{\gamma_k+\mathfrak{p}} \left[\omega_k^2(|D\eta_k|^2 + \eta_k^2) + \omega_k\,\partial_t\omega_k\,\eta_k^2\right] dz\right]^{1+\frac{2}{\hat{n}}}, \end{aligned} \tag{6.100}$$

which holds for any $k \in \mathbb{N}_0$. Exploiting the properties of η_k and ω_k as well as the definitions of r_k, $\mathcal{Q}_k$, b, $\tilde{\mathsf{H}}_\delta$ and $\mathfrak{p}$, for every $k \in \mathbb{N}_0$ we obtain that

$$\begin{aligned}\frac{b^{\gamma_k+2}}{r^n}\,|\mathrm{spt}\,\eta_k| \;&\le\; \frac{1}{r^n\,r_k^2}\int_{\mathcal{Q}_k} b^{\gamma_k+2}\,dz \;\le\; \frac{4^k}{(1-s)^2\,r^{n+2}}\int_{\mathcal{Q}_k}(1+\delta)^{\gamma_k+p-p+2}\,dz\\ &\le\; \frac{4^k\,r_k^{n+2}\,c(n,p,\delta)}{(1-s)^2\,r^{n+2}}\fint_{\mathcal{Q}_k}(\tilde{\mathsf{H}}_\delta(Du_\varepsilon))^{\gamma_k+\mathfrak{p}}\,dz \qquad (6.101)\end{aligned}$$

and

$$\begin{aligned}&r^2\fint_{Q_r(z_1)}(\tilde{\mathsf{H}}_\delta(Du_\varepsilon))^{\gamma_k+\mathfrak{p}}\big[\omega_k^2(|D\eta_k|^2+\eta_k^2)+\omega_k\,\partial_t\omega_k\,\eta_k^2\big]\,dz\\ &\quad\le\;\frac{4^{k+4}\,r_k^{n+2}}{(1-s)^2\,r^{n+2}}\fint_{\mathcal{Q}_k}(\tilde{\mathsf{H}}_\delta(Du_\varepsilon))^{\gamma_k+\mathfrak{p}}\,dz, \qquad (6.102)\end{aligned}$$

where, in the last line, we have used the fact that

$$\omega_k^2(|D\eta_k|^2+\eta_k^2)\;\le\;\frac{2^{2k+4}}{(1-s)^2\,r^2}\;+\;1\;\le\;\frac{4^{k+3}}{(1-s)^2\,r^2}\,. \qquad (6.103)$$

In addition, one can easily deduce that

$$\fint_{\mathcal{Q}_{k+1}}(\tilde{\mathsf{H}}_\delta(Du_\varepsilon))^{\gamma_{k+1}+\mathfrak{p}}\,dz\;\le\;\frac{r^{n+2}}{r_{k+1}^{n+2}}\fint_{Q_r(z_1)}(\omega_k\eta_k)^{2+\frac{4}{\hat n}}\,(\tilde{\mathsf{H}}_\delta(Du_\varepsilon))^{\gamma_{k+1}+\mathfrak{p}}\,dz \qquad (6.104)$$

for any $k \in \mathbb{N}_0$. Joining estimates (6.100)−(6.102) and (6.104), and using the fact that $r_k = 2r_{k+1} - sr \le 2r_{k+1}$ and $r_k \le r$ for every $k \in \mathbb{N}_0$, we find

$$\begin{aligned}\fint_{\mathcal{Q}_{k+1}}(\tilde{\mathsf{H}}_\delta(Du_\varepsilon))^{\gamma_{k+1}+\mathfrak{p}}\,dz\;&\le C(\gamma_k+p)^{4+\frac{6}{\hat n}}\,\exp\left[\frac{C\,\|f\|^{\Theta}_{L^{\hat n+2}\log^\alpha L(Q_r(z_1))}\,(\gamma_k+p)^{\frac{2\hat n+3}{\alpha}}}{r^{\frac{n-\hat n}{\alpha}}}\right]\\ &\cdot\left[\frac{4^k}{(1-s)^2}\fint_{\mathcal{Q}_k}(\tilde{\mathsf{H}}_\delta(Du_\varepsilon))^{\gamma_k+\mathfrak{p}}\,dz\right]^{1+\frac{2}{\hat n}} \qquad (6.105)\end{aligned}$$

for any $k \in \mathbb{N}_0$, where $C \equiv C(N,n,\hat n,\alpha,p,\delta,C_1,K) > 1$ and $\Theta \equiv \Theta(\hat n,\alpha) > 0$. To shorten our notation, we now set

$$M_k := \fint_{\mathcal{Q}_k}(\tilde{\mathsf{H}}_\delta(Du_\varepsilon))^{\gamma_k+\mathfrak{p}}\,dz, \qquad k \in \mathbb{N}_0, \qquad (6.106)$$

$$c_s := \frac{C}{(1-s)^{2+\frac{4}{\hat n}}} \qquad \text{and} \qquad c_r := C\,\|f\|^{\Theta}_{L^{\hat n+2}\log^\alpha L(Q_r(z_1))}\,r^{\frac{\hat n-n}{\alpha}}, \qquad (6.107)$$

so that the last inequality turns into

$$M_{k+1}\;\le\;c_s(\gamma_k+p)^{4+\frac{6}{\hat n}}\,4^{k(1+\frac{2}{\hat n})}\exp\Big(c_r(\gamma_k+p)^{\frac{2\hat n+3}{\alpha}}\Big)\,M_k^{1+\frac{2}{\hat n}}$$

for $k \in \mathbb{N}_0$. Iterating the above estimate, we obtain

$$M_k\;\le\;\prod_{j=0}^{k-1}\Big[c_s(\gamma_j+p)^{4+\frac{6}{\hat n}}\,4^{j(1+\frac{2}{\hat n})}\exp\Big(c_r(\gamma_j+p)^{\frac{2\hat n+3}{\alpha}}\Big)\Big]^{(1+\frac{2}{\hat n})^{k-1-j}}\;M_0^{(1+\frac{2}{\hat n})^k}$$

for any $k \in \mathbb{N}$. This inequality can be rewritten as follows:

$$M_k^{\frac{1}{\gamma_k+\mathfrak{p}}} \leq \prod_{j=0}^{k-1} \Big[c_s(\gamma_j+p)^{4+\frac{6}{\hat{n}}}\, 4^{j(1+\frac{2}{\hat{n}})} \exp\Big(c_r(\gamma_j+p)^{\frac{2\hat{n}+3}{\alpha}} \Big) \Big]^{\frac{(1+\frac{2}{\hat{n}})^{k-1-j}}{\gamma_k+\mathfrak{p}}} M_0^{\frac{(1+\frac{2}{\hat{n}})^k}{\gamma_k+\mathfrak{p}}}. \tag{6.108}$$

Next, we observe that

$$\begin{aligned} \lim_{k\to\infty} \frac{\big(1+\frac{2}{\hat{n}}\big)^k}{\gamma_k+\mathfrak{p}} &= \lim_{k\to\infty} \begin{cases} \dfrac{\big(1+\frac{2}{\hat{n}}\big)^k}{2+\big[2-\frac{\hat{n}(2-p)}{2}\big]\big[\big(1+\frac{2}{\hat{n}}\big)^k-1\big]} & \text{if } \frac{2\hat{n}}{\hat{n}+2} < p < 2 \\ \dfrac{\big(1+\frac{2}{\hat{n}}\big)^k}{p+2\big[\big(1+\frac{2}{\hat{n}}\big)^k-1\big]} & \text{if } p \geq 2 \end{cases} \\ &= \begin{cases} \frac{2}{4-\hat{n}(2-p)} & \text{if } \frac{2\hat{n}}{\hat{n}+2} < p < 2 \\ \frac{1}{2} & \text{if } p \geq 2, \end{cases} \end{aligned} \tag{6.109}$$

so that

$$\lim_{k\to\infty} M_0^{\frac{(1+\frac{2}{\hat{n}})^k}{\gamma_k+\mathfrak{p}}} = \left[\fint_{Q_r(z_1)} (\tilde{\mathsf{H}}_\delta(Du_\varepsilon))^{\mathfrak{p}}\, dz \right]^{\frac{1}{\varphi}}, \tag{6.110}$$

where

$$\varphi = \begin{cases} 2-\frac{\hat{n}(2-p)}{2} & \text{if } \frac{2\hat{n}}{\hat{n}+2} < p < 2 \\ 2 & \text{if } p \geq 2. \end{cases} \tag{6.111}$$

One can easily check that $\mathfrak{p}-\varphi \geq 0$ and $\varphi \leq p$ whenever $p > \frac{2\hat{n}}{\hat{n}+2}$. Using this information and the fact that $c_s > 1$, we can apply inequality (6.29) to obtain the following estimate:

$$\prod_{j=0}^{k-1} c_s^{\frac{\big(1+\frac{2}{\hat{n}}\big)^{k-1-j}}{\gamma_k+\mathfrak{p}}} \leq c_s^{\frac{\hat{n}}{2\varphi}}. \tag{6.112}$$

Similarly, by inequality (6.30) we get

$$\prod_{j=0}^{k-1} 4^{j\frac{\big(1+\frac{2}{\hat{n}}\big)^{k-j}}{\gamma_k+\mathfrak{p}}} \leq 2^{\frac{\hat{n}(\hat{n}+2)}{2\varphi}}. \tag{6.113}$$

Now we use that $1 < \gamma_j + p \leq p(\frac{\hat{n}+2}{\hat{n}})^j$ for any $j \in \mathbb{N}_0$ to estimate the following product by means of (6.29) and (6.30):

$$\prod_{j=0}^{k-1} \Big[(\gamma_j+p)^{4+\frac{6}{\hat{n}}} \Big]^{\frac{\big(1+\frac{2}{\hat{n}}\big)^{k-1-j}}{\gamma_k+\mathfrak{p}}} \leq \prod_{j=0}^{k-1} \left[p\left(\frac{\hat{n}+2}{\hat{n}}\right)^j \right]^{4\frac{\big(1+\frac{2}{\hat{n}}\big)^{k-j}}{\gamma_k+\mathfrak{p}}} \leq p^{\frac{2(\hat{n}+2)}{\varphi}} \left(\frac{\hat{n}+2}{\hat{n}}\right)^{\frac{\hat{n}(\hat{n}+2)}{\varphi}}. \tag{6.114}$$

It remains to estimate the product involving the exponential terms with inequality (6.29). Recalling that $\frac{2\hat{n}+3}{\alpha} < 1$, we obtain

$$\prod_{j=0}^{k-1} \Big[\exp\Big(c_r(\gamma_j+p)^{\frac{2\hat{n}+3}{\alpha}} \Big) \Big]^{\frac{\big(1+\frac{2}{\hat{n}}\big)^{k-1-j}}{\gamma_k+\mathfrak{p}}} \leq \prod_{j=0}^{k-1} \left[\exp\left(c_r\, p^{\frac{2\hat{n}+3}{\alpha}} \left(\frac{\hat{n}+2}{\hat{n}}\right)^{j\frac{2\hat{n}+3}{\alpha}} \right) \right]^{\frac{\big(1+\frac{2}{\hat{n}}\big)^{k-1-j}}{\gamma_k+\mathfrak{p}}}$$

$$\leq \exp\left[\frac{c_r}{\gamma_k+\mathfrak{p}}\,p^{\frac{2\hat{n}+3}{\alpha}}\left(\frac{\hat{n}+2}{\hat{n}}\right)^{k-1}\frac{(\hat{n}+2)^{1-\frac{2\hat{n}+3}{\alpha}}}{(\hat{n}+2)^{1-\frac{2\hat{n}+3}{\alpha}}-(\hat{n})^{1-\frac{2\hat{n}+3}{\alpha}}}\right]. \tag{6.115}$$

Combining estimates (6.112)−(6.115) and recalling the definitions of c_s and c_r in (6.107), from (6.108) we obtain that

$$M_k^{\frac{1}{\gamma_k+\mathfrak{p}}} \leq \frac{c}{(1-s)^{\frac{\hat{n}+2}{\varphi}}}\cdot\exp\left[c\,\frac{\|f\|^{\Theta}_{L^{\hat{n}+2}\log^{\alpha}L(Q_r(z_1))}}{r^{\frac{n-\hat{n}}{\alpha}}}\,\cdot\,\frac{\left(1+\frac{2}{\hat{n}}\right)^k}{\gamma_k+\mathfrak{p}}\right]M_0^{\frac{\left(1+\frac{2}{\hat{n}}\right)^k}{\gamma_k+\mathfrak{p}}} \tag{6.116}$$

for any $k\in\mathbb{N}$, with $c\equiv c(N,n,\hat{n},\alpha,p,\delta,C_1,K)>0$. Since the constant c is independent of $k\in\mathbb{N}$, recalling (6.106), (6.109), (6.110) and (6.111), and passing to the limit as $k\to\infty$ in both sides of (6.116), we arrive at

$$\operatorname*{ess\,sup}_{Q_{sr}(z_1)}\tilde{\mathsf{H}}_\delta(Du_\varepsilon)\leq\frac{c}{(1-s)^{\frac{\hat{n}+2}{\varphi}}}\exp\left(\frac{c\,\|f\|^{\Theta}_{L^{\hat{n}+2}\log^{\alpha}L(Q_r(z_1))}}{r^{\frac{n-\hat{n}}{\alpha}}}\right)\left[\fint_{Q_r(z_1)}(\tilde{\mathsf{H}}_\delta(Du_\varepsilon))^{\mathfrak{p}}\,dz\right]^{\frac{1}{\varphi}}, \tag{6.117}$$

due to the fact that $r_k\searrow sr$ and $\gamma_k\nearrow\infty$ as $k\to\infty$. At this stage we need to separate the cases $\frac{2\hat{n}}{\hat{n}+2}<p<2$ and $p\geq 2$.

Let us first assume that $\frac{2\hat{n}}{\hat{n}+2}<p<2$. In this case we have

$$\mathfrak{p}=2>p,\qquad \varphi=2-\frac{\hat{n}(2-p)}{2}\qquad\text{and}\qquad\frac{\varphi}{2-p}>1.$$

From Theorem 6.11 we know that $\tilde{\mathsf{H}}_\delta(Du_\varepsilon)\in L^\infty_{loc}(Q_R(z_0))$. Therefore, from (6.117) we deduce

$$\begin{aligned}
&\operatorname*{ess\,sup}_{Q_{sr}(z_1)}\tilde{\mathsf{H}}_\delta(Du_\varepsilon)\\
&\quad\leq\frac{c}{(1-s)^{\frac{\hat{n}+2}{\varphi}}}\exp\left(\frac{c\,\|f\|^{\Theta}_{L^{\hat{n}+2}\log^{\alpha}L(Q_r(z_1))}}{r^{\frac{n-\hat{n}}{\alpha}}}\right)\operatorname*{ess\,sup}_{Q_r(z_1)}(\tilde{\mathsf{H}}_\delta(Du_\varepsilon))^{\frac{2-p}{\varphi}}\left[\fint_{Q_r(z_1)}(\tilde{\mathsf{H}}_\delta(Du_\varepsilon))^{p}\,dz\right]^{\frac{1}{\varphi}}\\
&\quad\leq\frac{1}{2}\operatorname*{ess\,sup}_{Q_r(z_1)}\tilde{\mathsf{H}}_\delta(Du_\varepsilon)\\
&\qquad+\frac{c}{(1-s)^{\frac{2\hat{n}+4}{p(\hat{n}+2)-2\hat{n}}}}\exp\left(\frac{c\,\|f\|^{\Theta}_{L^{\hat{n}+2}\log^{\alpha}L(Q_r(z_1))}}{r^{\frac{n-\hat{n}}{\alpha}}}\right)\left[\fint_{Q_r(z_1)}(\tilde{\mathsf{H}}_\delta(Du_\varepsilon))^{p}\,dz\right]^{\frac{2}{p(\hat{n}+2)-2\hat{n}}},
\end{aligned}$$

where, in the second to last line, we have applied Young's inequality with exponents $(\frac{\varphi}{2-p},\frac{\varphi}{\varphi-2+p})$. Note that the preceding inequality holds for any $s\in(0,1)$. Hence, we can absorb the essential supremum on the right-hand side using Lemma 1.10 with $\rho_0=sr$ and $\rho_1=r$. This yields

$$\begin{aligned}
&\operatorname*{ess\,sup}_{Q_{sr}(z_1)}\tilde{\mathsf{H}}_\delta(Du_\varepsilon)\\
&\qquad\leq\frac{c}{(1-s)^{\frac{2\hat{n}+4}{p(\hat{n}+2)-2\hat{n}}}}\exp\left(\frac{c\,\|f\|^{\Theta}_{L^{\hat{n}+2}\log^{\alpha}L(Q_r(z_1))}}{r^{\frac{n-\hat{n}}{\alpha}}}\right)\left[\fint_{Q_r(z_1)}(\tilde{\mathsf{H}}_\delta(Du_\varepsilon))^{p}\,dz\right]^{\frac{2}{p(\hat{n}+2)-2\hat{n}}}
\end{aligned}$$

for a positive constant c depending on $N,n,\hat{n},\alpha,p,\delta,C_1$ and K, but not on ε.

Finally, when $p \geq 2$, inequality (6.117) reads as

$$\operatorname*{ess\,sup}_{Q_{sr}(z_1)} \tilde{\mathsf{H}}_\delta(Du_\varepsilon) \leq \frac{c}{(1-s)^{\frac{\hat{n}+2}{2}}} \exp\left(\frac{c\,\|f\|^{\Theta}_{L^{\hat{n}+2}\log^\alpha L(Q_r(z_1))}}{r^{\frac{n-\hat{n}}{\alpha}}}\right)\left[\fint_{Q_r(z_1)} (\tilde{\mathsf{H}}_\delta(Du_\varepsilon))^p\,dz\right]^{\frac{1}{2}}.$$

We have thus proved the following result, which ensures the desired local L^∞-bound of Du_ε for all $p > \frac{2n}{n+2}$.

Theorem 6.20. *Let $p > \frac{2n}{n+2}$, $\delta > 0$ and $0 < \varepsilon < \min\{\frac{1}{2}, (\frac{\delta}{4})^{p-1}\}$. Moreover, assume that*

$$u_\varepsilon \in C^0\left([t_0 - R^2, t_0]; L^2\left(B_R(x_0), \mathbb{R}^N\right)\right) \cap L^p\left(t_0 - R^2, t_0; W^{1,p}\left(B_R(x_0), \mathbb{R}^N\right)\right)$$

is the unique energy solution of problem $(\mathcal{P}_\varepsilon)$ with $Q' = Q_R(z_0) \Subset \Omega_T$ and u a weak solution of (6.1). Then, for any parabolic cylinder $Q_r(z_1) \Subset Q_R(z_0)$ with $r \in (0,1)$ and any $s \in (0,1)$, we have

$$\operatorname*{ess\,sup}_{Q_{sr}(z_1)} \tilde{\mathsf{H}}_\delta(Du_\varepsilon) \leq \frac{c}{(1-s)^{\frac{\hat{n}+2}{\kappa}}} \cdot \exp\left(c\,\frac{\|f\|^{\Theta}_{L^{\hat{n}+2}\log^\alpha L(Q_r(z_1))}}{r^{\frac{n-\hat{n}}{\alpha}}}\right)\left[\fint_{Q_r(z_1)} (\tilde{\mathsf{H}}_\delta(Du_\varepsilon))^p\,dz\right]^{\frac{1}{\kappa}} \tag{6.118}$$

for some positive constants $c \equiv c(N, n, \hat{n}, \alpha, p, \delta, C_1, K)$ and $\Theta \equiv \Theta(\hat{n}, \alpha)$, and for

$$\kappa = \begin{cases} \frac{p(\hat{n}+2)-2\hat{n}}{2} & \text{if } \frac{2n}{n+2} < p < 2 \\ 2 & \text{if } p \geq 2. \end{cases} \tag{6.119}$$

6.6.4 Step 4: the iteration for $1 < p \leq \frac{2n}{n+2}$ and $f = 0$

We now start the iteration procedure for $1 < p \leq \frac{2n}{n+2}$, keeping both the assumptions and the notations used in Propositions 6.18 and 6.19. Note that, by the choice of β in (6.4), the condition $1 < p \leq \frac{2n}{n+2}$ implies that $n \geq 3$, and therefore $\hat{n} = n$. We will however restrict ourselves to the case $f = 0$ and $u \in L^\infty(Q_R(z_0), \mathbb{R}^N)$. We again define by induction a sequence $\{\gamma_k\}_{k\in\mathbb{N}_0}$ by setting $\gamma_0 = 0$ and

$$\gamma_k := \left(1 + \frac{2}{n}\right)\gamma_{k-1} + \frac{2p}{n}, \qquad k \in \mathbb{N}.$$

This sequence diverges to $+\infty$ as $k \to \infty$ and, by induction, we have

$$\gamma_k = p\left[\left(1 + \frac{2}{n}\right)^k - 1\right], \qquad k \in \mathbb{N}_0.$$

Moreover, one can easily check that

$$\gamma_k + p + \frac{2(\gamma_k + 2)}{n} > \gamma_{k+1} + p \qquad \forall\, k \in \mathbb{N}_0, \forall\, p \in \left(1, \frac{2n}{n+2}\right]. \tag{6.120}$$

Now, for $k \in \mathbb{N}_0$ and $s \in (0,1)$, we define r_k, $\mathcal{B}_k$, $\mathcal{Q}_k$ and the cut-off functions η_k and ω_k as in Section 6.6.3. Using inequality (6.120) and replacing γ, η and ω with γ_k, η_k and ω_k respectively,

from (6.98) we obtain the following *recursive reverse Hölder-type inequality*

$$\begin{aligned}
&\fint_{Q_r(z_1)} (\omega_k\eta_k)^{2+\frac{4}{n}}\,(\tilde{\mathsf{H}}_\delta(Du_\varepsilon))^{\gamma_{k+1}+p}\,dz\\
&\quad\le C(\gamma_k+p)^{4+\frac{6}{n}}\bigg[\frac{b^{\gamma_k+2}}{r^n}\,|\operatorname{spt}\eta_k|\\
&\qquad+r^2\fint_{Q_r(z_1)}(\tilde{\mathsf{H}}_\delta(Du_\varepsilon))^{\gamma_k+p}\big[(|D\eta_k|^2+\eta_k^2)\,\omega_k^2+\omega_k\,\partial_t\omega_k\,\eta_k^2+\|u_\varepsilon\|^2_{L^\infty(Q_r(z_1))}(\partial_t\omega_k)^2\,\eta_k^2\big]\,dz\bigg]^{1+\frac{2}{n}},
\end{aligned}$$

which holds for any $k\in\mathbb{N}_0$. Exploiting the properties of η_k and ω_k as well as the definition of $\mathcal{Q}_k$, for every $k\in\mathbb{N}_0$ we get

$$\begin{aligned}
&r^2\fint_{Q_r(z_1)}(\tilde{\mathsf{H}}_\delta(Du_\varepsilon))^{\gamma_k+p}\big[(|D\eta_k|^2+\eta_k^2)\,\omega_k^2+\omega_k\,\partial_t\omega_k\,\eta_k^2+\|u_\varepsilon\|^2_{L^\infty(Q_r(z_1))}(\partial_t\omega_k)^2\,\eta_k^2\big]\,dz\\
&\qquad\le\frac{4^{2k+3}\,r_k^{n+2}}{(1-s)^2\,r^{n+2}}\left[1+\frac{\|u_\varepsilon\|^2_{L^\infty(Q_r(z_1))}}{(1-s)^2\,r^2}\right]\fint_{\mathcal{Q}_k}(\tilde{\mathsf{H}}_\delta(Du_\varepsilon))^{\gamma_k+p}\,dz,
\end{aligned}$$

where, in the last line, we have used (6.103). Combining the two previous inequalities with (6.101) and (6.104), and using the fact that $r_k\le 2r_{k+1}$ and $r_k\le r$ for every $k\in\mathbb{N}_0$, we find

$$\begin{aligned}
\fint_{\mathcal{Q}_{k+1}}(\tilde{\mathsf{H}}_\delta(Du_\varepsilon))^{\gamma_{k+1}+p}\,dz\;&\le\;C(\gamma_k+p)^{4+\frac{6}{n}}\\
&\quad\cdot\left[\frac{16^k}{(1-s)^2}\left(1+\frac{\|u_\varepsilon\|^2_{L^\infty(Q_r(z_1))}}{(1-s)^2\,r^2}\right)\fint_{\mathcal{Q}_k}(\tilde{\mathsf{H}}_\delta(Du_\varepsilon))^{\gamma_k+p}\,dz\right]^{1+\frac{2}{n}}
\end{aligned}$$

for any $k\in\mathbb{N}_0$, where $C\equiv C(N,n,\alpha,p,\delta,C_1,K)>1$. To shorten our notation, we now set

$$c_\varepsilon:=\;C\left[\frac{1}{(1-s)^2}\left(1+\frac{\|u_\varepsilon\|^2_{L^\infty(Q_r(z_1))}}{(1-s)^2\,r^2}\right)\right]^{1+\frac{2}{n}}. \tag{6.121}$$

Thus, recalling the definition (6.106), the preceding inequality turns into

$$M_{k+1}\;\le\;c_\varepsilon(\gamma_k+p)^{4+\frac{6}{n}}\,16^{k(1+\frac{2}{n})}M_k^{1+\frac{2}{n}}$$

for $k\ge 0$. Note that the constant c_ε depends on ε and r through the quotient

$$\frac{\|u_\varepsilon\|^2_{L^\infty(Q_r(z_1))}}{r^2}\,.$$

Iterating the above estimate, we obtain

$$M_k\;\le\;\prod_{j=0}^{k-1}\Big[c_\varepsilon(\gamma_j+p)^{4+\frac{6}{n}}\,16^{j(1+\frac{2}{n})}\Big]^{(1+\frac{2}{n})^{k-1-j}}\;M_0^{(1+\frac{2}{n})^k}$$

for any $k \in \mathbb{N}$. This inequality can be rewritten as follows:

$$M_k^{\frac{1}{\gamma_k+p}} \leq \prod_{j=0}^{k-1} \left[c_\varepsilon (\gamma_j + p)^{4+\frac{6}{n}} 16^{j(1+\frac{2}{n})} \right]^{\frac{\left(1+\frac{2}{n}\right)^{k-1-j}}{\gamma_k+p}} M_0^{\frac{\left(1+\frac{2}{n}\right)^k}{\gamma_k+p}}. \tag{6.122}$$

Next, we observe that

$$\lim_{k\to\infty} \frac{\left(1+\frac{2}{n}\right)^k}{\gamma_k+p} = \lim_{k\to\infty} \frac{\left(1+\frac{2}{n}\right)^k}{p+p\left[\left(1+\frac{2}{n}\right)^k-1\right]} = \frac{1}{p}, \tag{6.123}$$

so that

$$\lim_{k\to\infty} M_0^{\frac{\left(1+\frac{2}{n}\right)^k}{\gamma_k+p}} = \left[\fint_{Q_r(z_1)} (\tilde{\mathsf{H}}_\delta(Du_\varepsilon))^p \, dz \right]^{\frac{1}{p}}. \tag{6.124}$$

Since $c_\varepsilon > 1$, we can apply inequality (6.29) to obtain the following estimate:

$$\prod_{j=0}^{k-1} c_\varepsilon^{\frac{\left(1+\frac{2}{n}\right)^{k-1-j}}{\gamma_k+p}} \leq c_\varepsilon^{\frac{n}{2p}}. \tag{6.125}$$

Similarly, by inequality (6.30) we get

$$\prod_{j=0}^{k-1} 16^{j\,\frac{\left(1+\frac{2}{n}\right)^{k-j}}{\gamma_k+p}} \leq 2^{\frac{n(n+2)}{p}}. \tag{6.126}$$

Now we use that $1 < \gamma_j + p = p(\frac{n+2}{n})^j$ for any $j \in \mathbb{N}_0$ to estimate the following product by means of (6.29) and (6.30):

$$\prod_{j=0}^{k-1} \left[(\gamma_j+p)^{4+\frac{6}{n}} \right]^{\frac{\left(1+\frac{2}{n}\right)^{k-1-j}}{\gamma_k+p}} \leq \prod_{j=0}^{k-1} \left[p \left(\frac{n+2}{n} \right)^j \right]^{4\,\frac{\left(1+\frac{2}{n}\right)^{k-j}}{\gamma_k+p}} \leq p^{\frac{2(n+2)}{p}} \left(\frac{n+2}{n} \right)^{\frac{n(n+2)}{p}}. \tag{6.127}$$

Combining estimates (6.125)−(6.127) and recalling the definition of c_ε in (6.121), from (6.122) we obtain that

$$M_k^{\frac{1}{\gamma_k+p}} \leq \frac{c}{(1-s)^{\frac{n+2}{p}}} \left[1 + \frac{\|u_\varepsilon\|^2_{L^\infty(Q_r(z_1))}}{(1-s)^2 r^2} \right]^{\frac{n+2}{2p}} M_0^{\frac{\left(1+\frac{2}{n}\right)^k}{\gamma_k+p}} \tag{6.128}$$

for any $k \in \mathbb{N}$, with $c \equiv c(N, n, \alpha, p, \delta, C_1, K) > 0$. Since the constant c is independent of $k \in \mathbb{N}$, recalling (6.106), (6.123) and (6.124), and passing to the limit as $k \to \infty$ in both sides of (6.128), we arrive at

$$\operatorname*{ess\,sup}_{Q_{sr}(z_1)} \tilde{\mathsf{H}}_\delta(Du_\varepsilon) \leq \frac{c}{(1-s)^{\frac{n+2}{p}}} \left[1 + \frac{\|u_\varepsilon\|^2_{L^\infty(Q_r(z_1))}}{(1-s)^2 r^2} \right]^{\frac{n+2}{2p}} \left[\fint_{Q_r(z_1)} (\tilde{\mathsf{H}}_\delta(Du_\varepsilon))^p \, dz \right]^{\frac{1}{p}}, \tag{6.129}$$

due to the fact that $r_k \searrow sr$ and $\gamma_k \nearrow \infty$ as $k \to \infty$. Notice that the right-hand side of (6.129) depends on $\|u_\varepsilon\|_{L^\infty(Q_r(z_1))}$. However, since we are assuming that $f = 0$ and $u \in L^\infty(Q_R(z_0), \mathbb{R}^N)$,

we can use the result from Section 6.4, thus obtaining

$$\operatorname*{ess\,sup}_{Q_{sr}(z_1)} \tilde{\mathsf{H}}_\delta(Du_\varepsilon) \leq c \frac{\left(1+\|u\|^2_{L^\infty(Q_R(z_0))}\right)^{\frac{n+2}{2p}}}{\left[(1-s)^2 r\right]^{\frac{n+2}{p}}} \left[\fint_{Q_r(z_1)} (\tilde{\mathsf{H}}_\delta(Du_\varepsilon))^p \, dz\right]^{\frac{1}{p}}.$$

In conclusion, we have so far proved the following theorem, which ensures the desired local L^∞-bound of Du_ε when $1 < p \leq \frac{2n}{n+2}$, $f = 0$ and $u \in L^\infty(Q_R(z_0), \mathbb{R}^N)$.

Theorem 6.21. *Let $1 < p \leq \frac{2n}{n+2}$, $f = 0$, $\delta > 0$ and $0 < \varepsilon < \min\{\frac{1}{2}, (\frac{\delta}{4})^{p-1}\}$. Moreover, assume that*

$$u_\varepsilon \in C^0\left([t_0 - R^2, t_0]; L^2(B_R(x_0), \mathbb{R}^N)\right) \cap L^p\left(t_0 - R^2, t_0; W^{1,p}(B_R(x_0), \mathbb{R}^N)\right)$$

is the unique energy solution of problem $(\mathcal{P}_\varepsilon)$ with $Q' = Q_R(z_0) \Subset \Omega_T$ and $u \in L^\infty(Q_R(z_0), \mathbb{R}^N)$ a weak solution of (6.1). Then, for any parabolic cylinder $Q_r(z_1) \Subset Q_R(z_0)$ with $r \in (0,1)$ and any $s \in (0,1)$, we have

$$\operatorname*{ess\,sup}_{Q_{sr}(z_1)} \tilde{\mathsf{H}}_\delta(Du_\varepsilon) \leq c \frac{\left(1+\|u\|^2_{L^\infty(Q_R(z_0))}\right)^{\frac{n+2}{2p}}}{\left[(1-s)^2 r\right]^{\frac{n+2}{p}}} \left[\fint_{Q_r(z_1)} (\tilde{\mathsf{H}}_\delta(Du_\varepsilon))^p \, dz\right]^{\frac{1}{p}} \tag{6.130}$$

for some positive constant $c \equiv c(N, n, \alpha, p, \delta, C_1, K)$.

6.7 Proof of Theorem 6.1

We are now in a position to prove Theorem 6.1. In fact, using the strong L^p-convergence of $G_\delta(Du_\varepsilon)$ to $G_\delta(Du)$ established in Lemma 6.12 as well as the results from Theorems 6.20 and 6.21, we will now show that estimates (6.118) and (6.130) also hold true with the weak solution u of (6.1) in place of u_ε. From this the desired local L^∞-bounds of Du will immediately follow. Therefore, in the next proof we will use the same assumptions and notations of Theorems 6.20 and 6.21.

Proof of Theorem 6.1. By virtue of Lemma 6.12, we have that $G_\delta(Du_\varepsilon) \to G_\delta(Du)$ strongly in $L^p(Q_R(z_0), \mathbb{R}^{Nn})$ as $\varepsilon \searrow 0$. Thus we conclude that there exists a sequence $\{\varepsilon_j\}_{j\in\mathbb{N}}$ such that:

- $0 < \varepsilon_j < \min\{\frac{1}{2}, (\frac{\delta}{4})^{p-1}\}$ for every $j \in \mathbb{N}$ and $\varepsilon_j \searrow 0$ as $j \to +\infty$;
- $|G_\delta(Du_{\varepsilon_j}(x,t))| \to |G_\delta(Du(x,t))|$ almost everywhere in $Q_R(z_0) \Subset \Omega_T$ as $j \to +\infty$.

If $p > \frac{2n}{n+2}$, then, using the definition of G_δ and Theorem 6.20, we have for almost every $\tilde{z} \in Q_{sr}(z_1)$ that

$$\begin{aligned} |Du(\tilde{z})| &\leq 1 + \delta + |G_\delta(Du(\tilde{z}))| \leq \lim_{j\to\infty} \tilde{\mathsf{H}}_\delta(Du_{\varepsilon_j}(\tilde{z})) \leq \limsup_{j\to\infty} \operatorname*{ess\,sup}_{Q_{sr}(z_1)} \tilde{\mathsf{H}}_\delta(Du_{\varepsilon_j}) \\ &\leq \limsup_{j\to\infty} \frac{c}{(1-s)^{\frac{\hat{n}+2}{\kappa}}} \cdot \exp\left(c \frac{\|f\|^\Theta_{L^{\hat{n}+2}\log^\alpha L(Q_r(z_1))}}{r^{\frac{n-\hat{n}}{\alpha}}}\right) \left[\fint_{Q_r(z_1)} (1+\delta+|G_\delta(Du_{\varepsilon_j})|)^p \, dz\right]^{\frac{1}{\kappa}} \end{aligned}$$

$$= \frac{c}{(1-s)^{\frac{\hat{n}+2}{\kappa}}} \cdot \exp\left(c\, \frac{\|f\|^{\Theta}_{L^{\hat{n}+2}\log^{\alpha} L(Q_r(z_1))}}{r^{\frac{n-\hat{n}}{\alpha}}} \right) \left[\fint_{Q_r(z_1)} (1+\delta+|G_\delta(Du)|)^p \, dz \right]^{\frac{1}{\kappa}}$$

$$= \frac{c}{(1-s)^{\frac{\hat{n}+2}{\kappa}}} \cdot \exp\left(c\, \frac{\|f\|^{\Theta}_{L^{\hat{n}+2}\log^{\alpha} L(Q_r(z_1))}}{r^{\frac{n-\hat{n}}{\alpha}}} \right) \left[\fint_{Q_r(z_1)} (\max\{|Du|, 1+\delta\})^p \, dz \right]^{\frac{1}{\kappa}},$$

where κ is defined by (6.119), c is a positive constant depending only on N, n, $\hat{n}$, α, p, δ, C_1 and K, and $\Theta \equiv \Theta(\hat{n}, \alpha) > 0$. We now choose $\delta = 1$. Since the above inequality holds for almost every $\tilde{z} \in Q_{sr}(z_1)$, we immediately obtain

$$\begin{aligned} \operatorname*{ess\,sup}_{Q_{sr}(z_1)} |Du| &\le \operatorname*{ess\,sup}_{Q_{sr}(z_1)} \tilde{\mathsf{H}}_1(Du) \\ &\le \frac{C}{(1-s)^{\frac{\hat{n}+2}{\kappa}}} \cdot \exp\left(C\, \frac{\|f\|^{\Theta}_{L^{\hat{n}+2}\log^{\alpha} L(Q_r(z_1))}}{r^{\frac{n-\hat{n}}{\alpha}}} \right) \left[\fint_{Q_r(z_1)} (|Du|+1)^p \, dz \right]^{\frac{1}{\kappa}} < +\infty, \end{aligned}$$

where $C \equiv C(N, n, \hat{n}, \alpha, p, C_1, K) > 0$.

Finally, let us consider the case $1 < p \le \frac{2n}{n+2}$, $f = 0$ and $u \in L^\infty(Q_R(z_0), \mathbb{R}^N)$. Arguing as above, but this time using Theorem 6.21 instead of Theorem 6.20, we get

$$\begin{aligned} \operatorname*{ess\,sup}_{Q_{sr}(z_1)} |Du| &\le \operatorname*{ess\,sup}_{Q_{sr}(z_1)} \tilde{\mathsf{H}}_1(Du) \\ &\le C\, \frac{\left(1+\|u\|^2_{L^\infty(Q_R(z_0))}\right)^{\frac{n+2}{2p}}}{\left[(1-s)^2\, r\right]^{\frac{n+2}{p}}} \left[\fint_{Q_r(z_1)} (|Du|+1)^p \, dz \right]^{\frac{1}{p}} < +\infty. \end{aligned}$$

This concludes the proof. □

Chapter 7

A physics-informed deep learning approach for solving strongly degenerate parabolic problems

In this chapter, we will show how to exploit a novel Artificial Intelligence (AI) methodology, known as *Physics-Informed Neural Networks* (PINNs), to predict solutions to Cauchy-Dirichlet problems of the type

$$\begin{cases} \partial_t u - \operatorname{div}\left((|Du|-1)_+ \frac{Du}{|Du|}\right) = f & \text{in } \Omega_T := \Omega \times (0,T), \\ u = w & \text{on } \partial_{\text{par}}\Omega_T, \end{cases} \tag{7.1}$$

where Ω is a bounded connected open subset of $\mathbb{R}^n$ $(2 \leq n \leq 3)$ with Lipschitz boundary, f and w are given real-valued functions defined over $\overline{\Omega} \times [0,T]$ and $\partial_{\text{par}}\Omega_T$ respectively, while Du denotes the spatial gradient of an unknown solution $u : \overline{\Omega} \times [0,T) \to \mathbb{R}$.

As for the methodology proposed here, the PINNs predict the solution to a PDE under prescribed initial-boundary conditions by training a neural network to minimize a cost function, called *loss function*, which penalizes some suitable terms on a set of admissible functions u. For more information, we refer the interested reader to [34] (see also Section 7.2 below).

The PINN-based approach we are about to present was introduced in [8]. There, we were the first to evaluate the effectiveness of the PINN framework in solving strongly degenerate parabolic problems of the type (7.1).

This chapter is organized as follows. Section 7.1 is devoted to the preliminaries: we provide information on the solutions to problem (7.1), drawing on the results in Chapter 4. In Section 7.2, we describe the PINN methodology on which our predictive model is based. Section 7.3 presents the results that were obtained. Finally, in Section 7.4 we provide the conclusions of our study.

7.1 Definitions and preliminary results

Let us start with a simple observation. Taking $\gamma = \lambda = 1$ into (1.6), we obtain the function

$$H(\xi) := \begin{cases} (|\xi| - 1)_+ \frac{\xi}{|\xi|} & \text{if } \xi \in \mathbb{R}^n \setminus \{0\}, \\ 0 & \text{if } \xi = 0. \end{cases} \tag{7.2}$$

Now let $f \in L^1_{loc}(\Omega_T)$. Then, setting $p = 2$ in Definition 1.9, we get the following definition.

Definition 7.1. A function

$$u \in C^0\left((0,T); L^2(\Omega)\right) \cap L^2\left(0,T; W^{1,2}(\Omega)\right)$$

is a *weak solution* of equation $(7.1)_1$ if and only if the condition

$$\int_{\Omega_T} \left(u \cdot \partial_t \varphi - \langle H(Du), D\varphi \rangle\right) dz = -\int_{\Omega_T} f\varphi \, dz \tag{7.3}$$

is satisfied for all $\varphi \in C_0^\infty(\Omega_T)$.

At this point, we are ready to give the definition of weak solution to problem (7.1).

Definition 7.2. Let $w \in C^0\left([0,T]; L^2(\Omega)\right) \cap L^2\left(0,T; W^{1,2}(\Omega)\right)$. We identify a function

$$u \in C^0\left([0,T]; L^2(\Omega)\right) \cap L^2\left(0,T; W^{1,2}(\Omega)\right)$$

as a *weak solution of the Cauchy-Dirichlet problem* (7.1) if and only if (7.3) holds and, moreover,

$$u \in w + L^2\left(0,T; W_0^{1,2}(\Omega)\right)$$

and $u(\cdot,0) = w(\cdot,0)$ in the L^2-sense, that is,

$$\lim_{t \searrow 0} \|u(\cdot,t) - w(\cdot,0)\|_{L^2(\Omega)} = 0. \tag{7.4}$$

Therefore, the initial condition $u = w$ on $\overline{\Omega} \times \{0\}$ has to be understood in the usual L^2-sense (7.4), while the condition $u = w$ on the lateral boundary $\partial\Omega \times (0,T)$ has to be meant in the sense of traces, i.e.

$$(u - w)(\cdot,t) \in W_0^{1,2}(\Omega)$$

for almost every $t \in (0,T)$.

Now, taking $p = 2$ and $\lambda = 1$ in Theorem 4.1, we immediately obtain the following spatial Sobolev regularity result.

Theorem 7.3. *Let $n \geq 2$, $\frac{2n+4}{n+4} \leq q < \infty$ and $f \in L^q\left(0,T; W^{1,q}(\Omega)\right)$. Moreover, assume that*

$$u \in C^0\left((0,T); L^2(\Omega)\right) \cap L^2\left(0,T; W^{1,2}(\Omega)\right)$$

is a weak solution of equation $(7.1)_1$. Then

$$H(Du) \in L^2_{loc}\left(0,T; W^{1,2}_{loc}(\Omega, \mathbb{R}^n)\right),$$

where the function H is defined by (7.2)*. Furthermore, the estimate*

$$\int\limits_{Q_{\rho/2}(z_0)} |DH(Du)|^2 \, dz \leq c\left(\|Df\|_{L^q(Q_{R_0})} + \|Df\|^2_{L^q(Q_{R_0})}\right) + \frac{c}{R^2}\left(\|Du\|^2_{L^2(Q_{R_0})} + 1\right)$$

holds true for any parabolic cylinder $Q_\rho(z_0) \subset Q_R(z_0) \subset Q_{R_0}(z_0) \Subset \Omega_T$ *and a positive constant* c *depending at most on* n, q *and* R_0.

From Theorem 7.3 one can easily deduce that any weak solution u of $(7.1)_1$ admits a weak time derivative u_t, which belongs to the local Lebesgue space $L^{\min\{2,\,q\}}_{loc}(\Omega_T)$. In fact, it is enough to argue as in the proof of Theorem 4.2, from which we derive the next result.

Theorem 7.4. *Under the assumptions of Theorem 7.3, the time derivative of the solution exists in the weak sense and satisfies*

$$\partial_t u \,\in\, L^{\min\{2,\,q\}}_{loc}(\Omega_T).$$

Furthermore, the estimate

$$\begin{aligned}\left(\,\int\limits_{Q_{\rho/2}(z_0)} |\partial_t u|^{\min\{2,\,q\}}\,dz\right)^{\frac{1}{\min\{2,\,q\}}} &\leq\, c\,\|f\|_{L^q(Q_{R_0})} \,+\, c\left(\|Df\|_{L^q(Q_{R_0})} \,+\, \|Df\|^2_{L^q(Q_{R_0})}\right)^{\frac12}\\ &\quad+\,\frac{c}{R}\left(\|Du\|^2_{L^2(Q_{R_0})}+1\right)^{\frac12}\end{aligned}$$

holds true for any parabolic cylinder $Q_\rho(z_0) \subset Q_R(z_0) \subset Q_{R_0}(z_0) \Subset \Omega_T$ *and a positive constant* c *depending on* n, q *and* R_0.

Now, let the assumptions of Theorem 7.3 be in force. For $\varepsilon \in [0,1]$ and a couple of standard, non-negative, radially symmetric mollifiers $\phi_1 \in C^\infty_0(B_1(0))$ and $\phi_2 \in C^\infty_0((-1,1))$, we define the function f_ε as in (4.13), where f is meant to be extended by zero outside Ω_T. Observe that $f_0 = f$ and $f_\varepsilon \in C^\infty(\Omega_T)$ for every $\varepsilon \in (0,1]$.

Next, we consider a domain in space-time denoted by $\Omega'_{1,2} := \Omega' \times (t_1,t_2)$, where $\Omega' \subseteq \Omega$ is a bounded domain with smooth boundary and $(t_1,t_2) \subseteq (0,T)$. In the sequel, we will need the following definitions.

Definition 7.5. Let $\varepsilon \in (0,1]$. A function

$$u_\varepsilon \,\in\, C^0\left((t_1,t_2);L^2(\Omega')\right) \cap L^2\left(t_1,t_2;W^{1,2}(\Omega')\right)$$

is a *weak solution* of the equation

$$\partial_t u_\varepsilon - \operatorname{div}\left(H(Du_\varepsilon) + \varepsilon\, Du_\varepsilon\right) = f_\varepsilon \quad \text{ in } \Omega'_{1,2}$$

if and only if, for any test function $\varphi \in C^\infty_0(\Omega'_{1,2})$, the following integral identity holds:

$$\int_{\Omega'_{1,2}} \left(u_\varepsilon \cdot \partial_t\varphi - \langle H(Du_\varepsilon) + \varepsilon\, Du_\varepsilon, D\varphi\rangle\right)\,dz \,=\, -\int_{\Omega'_{1,2}} f_\varepsilon\,\varphi\,dz. \tag{7.5}$$

Definition 7.6. Let $\varepsilon \in (0,1]$ and

$$u \,\in\, C^0\left([t_1,t_2];L^2(\Omega')\right) \cap L^2\left(t_1,t_2;W^{1,2}(\Omega')\right).$$

We identify a function

$$u_\varepsilon \in C^0\left([t_1,t_2];L^2(\Omega')\right)\cap L^2\left(t_1,t_2;W^{1,2}(\Omega')\right)$$

as a *weak solution of the Cauchy-Dirichlet problem*

$$\begin{cases} \partial_t u_\varepsilon - \operatorname{div}\left(H(Du_\varepsilon)+\varepsilon\, Du_\varepsilon\right) = f_\varepsilon & \text{in } \Omega'_{1,2}\,, \\ u_\varepsilon = u & \text{on } \partial_{\mathrm{par}}\Omega'_{1,2}\,, \end{cases} \tag{7.6}$$

if and only if (7.5) holds and, moreover,

$$u_\varepsilon \in u + L^2\left(t_1,t_2;W_0^{1,2}(\Omega')\right),$$

$u_\varepsilon(\cdot,t_1)=u(\cdot,t_1)$ in the usual L^2-sense and the condition $u_\varepsilon = u$ on the lateral boundary $\partial\Omega'\times(t_1,t_2)$ holds in the sense of traces, i.e.

$$(u_\varepsilon - u)(\cdot,t)\in W_0^{1,2}(\Omega')$$

for almost every $t\in(t_1,t_2)$.

Due to the strong degeneracy of equation $(7.1)_1$, in order to prove Theorems 7.3 and 7.4, we had to resort to the family of approximating parabolic problems (7.6) (cf. Chapter 4). These problems exhibit a milder degeneracy than (7.1) and the advantage of considering them stems from the fact that the existence of a unique energy solution u_ε satisfying the requirements of Definition 7.6 is ensured by the classic existence theory for parabolic equations (see Remark 4.11).

Moreover, if $\Omega'_{1,2}=Q_{R_0}(z_0):=B_{R_0}(x_0)\times(t_0-R_0^2,t_0)\Subset\Omega_T$, then, from (4.53) and (4.55) with $p=2$ and $\lambda=1$, we can easily deduce

$$\sup_{t\in(t_0-R_0^2,t_0)}\|u_\varepsilon(\cdot,t)-u(\cdot,t)\|^2_{L^2(B_{R_0}(x_0))}\to 0 \qquad \text{as } \varepsilon\to 0^+, \tag{7.7}$$

that is,

$$u_\varepsilon\to u \quad \text{in } L^\infty(t_0-R_0^2,t_0;L^2(B_{R_0}(x_0))) \qquad \text{as } \varepsilon\to 0^+.$$

Hence, we can conclude that there exists a sequence $\{\varepsilon_j\}_{j\in\mathbb{N}}$ such that:

- $0<\varepsilon_j\le 1$ for every $j\in\mathbb{N}$ and $\varepsilon_j\searrow 0$ monotonically as $j\to+\infty$;
- $u_{\varepsilon_j}(x,t)\to u(x,t)$ almost everywhere in $Q_{R_0}(z_0)$ as $j\to+\infty$.

7.2 Physics-informed methodology

PINNs are a type of SciML approach used in neural networks to solve PDEs. Unlike traditional neural networks, PINNs incorporate physics constraints into the model, resulting in predicted outcomes that adhere more closely to the natural laws governing the specific problem being addressed. The general form of the problem involves a PDE along with initial and/or boundary conditions.

In particular, we consider a (well-posed) problem of the type

$$\begin{cases} \mathcal{F}(u(\mathbf{x},t),\gamma) = f(\mathbf{x},t) & \text{if } (\mathbf{x},t) \in \Omega_T := \Omega \times (0,T), \\ \mathcal{B}(u(\mathbf{x},t)) = w(\mathbf{x},t) & \text{if } (\mathbf{x},t) \in \partial_{\text{par}}\Omega_T, \end{cases} \tag{7.8}$$

where Ω is a bounded domain in $\mathbb{R}^n$, $\mathcal{F}$ denotes a nonlinear differential operator, γ is a parameter associated with the physics of the problem, $\mathcal{B}$ is an operator defining arbitrary initial-boundary conditions, the functions f and w represent the problem data, while $u(\mathbf{x},t)$ denotes the unknown solution.

The objective of PINNs is to predict the solution to (7.8) by training the neural network to minimize a cost function. The neural network's architecture used for PINNs is typically a FeedForward fully-connected Neural Network (FF-DNN), also known as Multi-Layer Perceptron (MLP). In an FF-DNN, information flows only forward direction, in the sense that the neural network does not form a loop. Furthermore, all neurons are interconnected. Once the number N of hidden layers has been chosen, for any $i \in \{1,\dots,N\}$ and set $\mathbf{z} = (\mathbf{x},t)$ we define

$$\Gamma_i(\mathbf{z}_{i-1}) := W_i\,\mathbf{z}_{i-1} + \mathbf{b}_i\,,$$

where W_i is the weights matrix of the links between the layers $i-1$ and i, while $\mathbf{b}_i$ corresponds to the biases vector. Then, a generic layer of the neural network is defined by

$$h_i(\mathbf{z}_{i-1}; W_i, \mathbf{b}_i) := \varphi_i(\Gamma_i(\mathbf{z}_{i-1})), \qquad i \in \{1,\dots,N\},$$

for some nonlinear activation function φ_i. The output of the FF-DNN, denoted by $\hat{u}_\theta(\mathbf{z})$, can be expressed as a composition of these layers by

$$\hat{u}_\theta(\mathbf{z}) := (\Gamma_N \circ \varphi \circ \Gamma_{N-1} \circ \cdots \circ \varphi \circ \Gamma_1)(\mathbf{z})\,,$$

where θ represents the set of hyperparameters of the neural network and the activation function φ is assumed to be the same for all layers. To solve the differential problem (7.8) using PINNs, the exact solution is approximated by finding an optimal set θ^* of neural network hyperparameters that minimizes a loss function $\mathcal{L}$. This function consists of two components: the former, denoted by $\mathcal{L}_\mathcal{F}$, is related to the differential equation, while the latter, here denoted by $\mathcal{L}_\mathcal{B}$, is connected to the initial-boundary conditions (see Fig. 7.1). In particular, the loss function can be defined as follows:

$$\mathcal{L} := \omega_\mathcal{F} \underbrace{\int_{\Omega_T} [\mathcal{F}(u(\mathbf{z}),\gamma) - f]^2\, d\mathbf{z}}_{=:\mathcal{L}_\mathcal{F}} + \omega_\mathcal{B} \underbrace{\int_{\partial_{\text{par}}\Omega_T} [\mathcal{B}(u(\mathbf{z})) - w]^2\, d\mathcal{H}^n}_{=:\mathcal{L}_\mathcal{B}},$$

where $\omega_\mathcal{F}$ and $\omega_\mathcal{B}$ represent the weights that are usually applied to balance the importance of each component. Hence, we can write

$$\theta^* := \arg\min_\theta \mathcal{L}(\theta)\,.$$

The aim of this approach is to approximate the solution of the PDE satisfying the initial-boundary conditions. This is known in the literature as the *direct problem*, which is the only one we will address here.

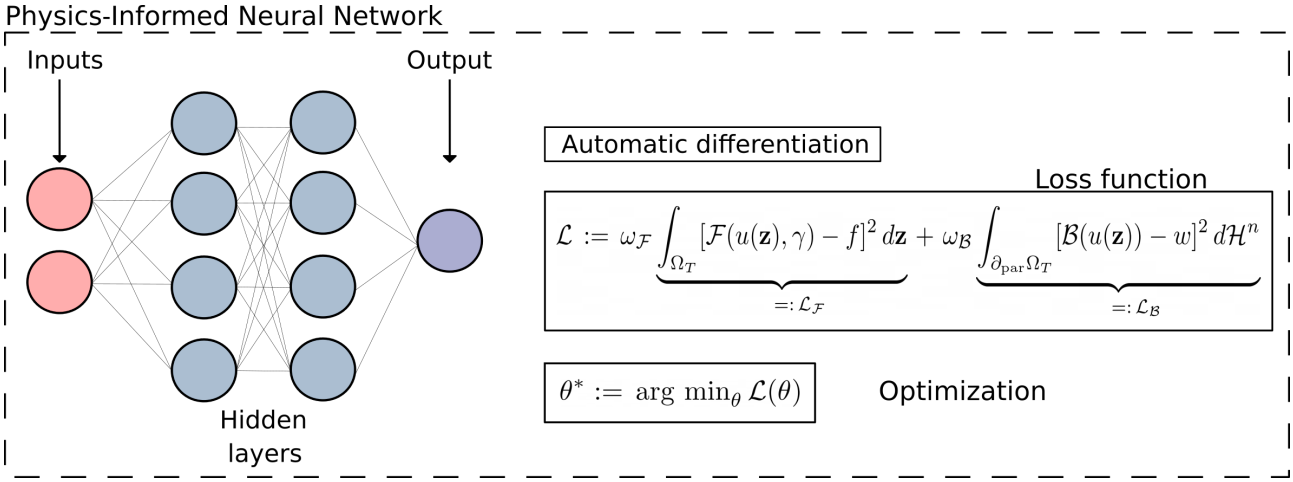


Figure 7.1: Overall structure of the proposed methodology. An FF-DNN serves as the neural network's architecture. Automatic differentiation is employed to calculate the loss terms associated with the model's dynamics. The loss function comprises two components: the physics loss, represented by $\mathcal{L}_{\mathcal{F}}$, and the boundary loss denoted by $\mathcal{L}_{\mathcal{B}}$. During the optimization phase, the objective is to minimize the loss function with respect to the set of hyperparameters θ.

7.3 Numerical results

In this section we evaluate the accuracy and effectiveness of our predictive method, by testing it with five problems of the type (7.1) whose exact solutions are known. For each problem, we will denote the *exact solution* by u, and the *predicted* (or *approximate*) *solution* by $\hat{u}$. Sometimes, by abuse of language, for a given time $t \geq 0$ we will refer to the partial maps $u(\cdot, t)$ and $\hat{u}(\cdot, t)$ as the exact solution and the predicted (or approximate) solution, respectively. The meaning will be clear from the context every time. We will deal with each test problem separately, so that no confusion can arise. In the first three problems, Ω will be a bounded domain of $\mathbb{R}^2$, while, in the last two problems, Ω will denote the open unit sphere of $\mathbb{R}^3$ centered at the origin.

In addition, for each of the test problems, we employed the same neural network architecture. This consists of four layers, each with 20 neurons. We utilized the hyperbolic tangent function as an activation function for both the input layer and the hidden layers, while a linear function served as an activation function for the output layer.

Lastly, to train the neural network, we conducted 80,000 epochs with a learning rate (lr) of 3×10^{-3} and employed the Adaptive Moment Estimation (ADAM) optimizer. The decision to set the lr to the constant value 3×10^{-3} was based on the observation that this specific hyperparameter led to the optimal convergence of our method. Experimentation with lr set to 1×10^{-1} highlighted the network's inability to achieve convergence, while using an lr of 1×10^{-5} allowed the method to converge, albeit requiring a significantly higher number of epochs. The latter scenario, while ensuring convergence, proved to be less computationally efficient.

The experiments were performed on a NVIDIA GeForce RTX 3080 GPU with AMD Ryzen 9 5950X 16-Core Processor and 128 GB of RAM.

7.3.1 First test problem

The first test problem that we consider is

$$\begin{cases} \partial_t v - \operatorname{div}\left((|Dv|-1)_+ \frac{Dv}{|Dv|}\right) = 1 & \text{in } \Omega_T, \\ v(x,y,0) = \frac{1}{2}(x^2+y^2) & \text{if } (x,y) \in \overline{\Omega}, \\ v(x,y,t) = \frac{1}{2} + t & \text{if } (x,y) \in \partial\Omega \wedge t \in (0,T), \end{cases} \tag{$\mathcal{P}1$}$$

where

$$\Omega = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}.$$

The exact solution of this problem is given by

$$u(x,y,t) = \frac{1}{2}(x^2+y^2) + t.$$

Therefore, for any fixed time $t \geq 0$ the graph of the function $u(\cdot,t)$ is an elliptic paraboloid. As time goes on, this paraboloid slides along an oriented vertical axis at a constant velocity, without deformation, since $\partial_t u \equiv 1$ over Ω_T (see Fig. 7.2).

To train the neural network, in each experiment we have initially used 441 points to suitably discretize the domain Ω and its boundary $\partial\Omega$, and 21 equispaced points in the time interval $[0,T]$. Once the network has been trained, we have made a prediction of the solution to problem ($\mathcal{P}1$) at different times t (Fig. 7.2).

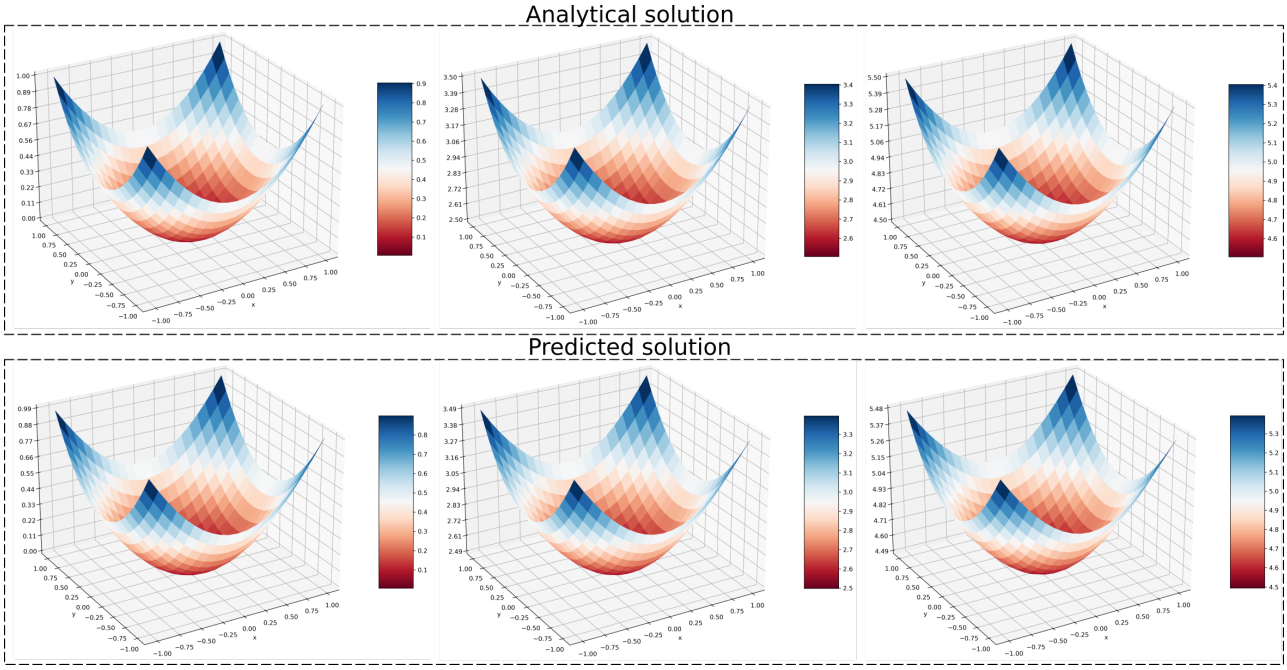


Figure 7.2: Plots of the exact solution to problem ($\mathcal{P}1$) (above) and the predicted solution $\hat{u}(\cdot,t)$ (below) for $t = 0$ (left), $t = 2.5$ (center) and $t = 4.5$ (right).

What has been verified is that the plot of the predicted solution $\hat{u}(\cdot, t)$ has precisely the same shape and geometric properties as the graph of the exact solution $u(\cdot, t)$, for both short and long times t. Moreover, the time evolution of the approximate solution $\hat{u}$ exactly mirrors the behavior described for the known solution u. A further interesting aspect that can be noticed is that the level curves of the approximate solution $\hat{u}(\cdot, t)$ overlap almost perfectly those of $u(\cdot, t)$, provided that t is not very large (see Fig. 7.3).

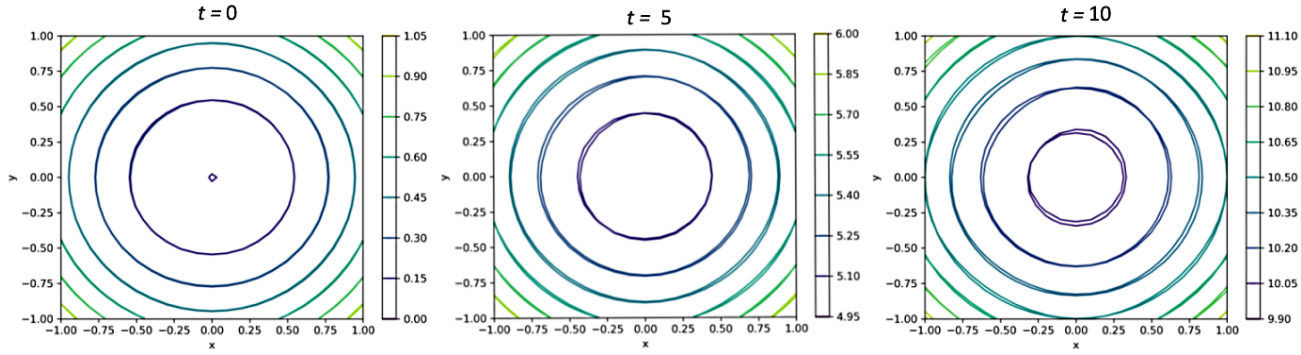


Figure 7.3: Superposition of the level curves of the exact solution $u(\cdot, t)$ and the predicted solution $\hat{u}(\cdot, t)$ for $t = 0$ (left), $t = 5$ (center) and $t = 10$ (right). The contour lines corresponding to the same level are almost indistinguishable for any fixed $t \in [0, 10]$.

We have also noted that, at time $t = 0$, the approximate solution is basically equal to zero in a very tiny region around the origin $(0, 0)$ of the xy-plane. This means that the said region is composed of "numerical zeros" of the solution predicted at time $t = 0$, while we know that $u(x, y, 0) = 0$ if and only if $(x, y) = (0, 0)$. However, this discrepancy is actually negligible, since the order of magnitude of $u(x, y, 0)$ is not greater than 10^{-6} within the above region.

To assess the accuracy of our predictive method and the numerical convergence of the solution $\hat{u}$ toward u in a more quantitative way, we now look at the time behavior of the L^2-error $\|\hat{u}(\cdot, t) - u(\cdot, t)\|_{L^2(\Omega)}$ by considering the natural quantities

$$\mathcal{E}(T) := \sup_{t \in (0,T)} \|\hat{u}(\cdot, t) - u(\cdot, t)\|^2_{L^2(\Omega)} \tag{7.9}$$

and

$$\mathcal{E}_{rel}(T) := \frac{\mathcal{E}(T)}{\sup\limits_{t \in (0,T)} \|u(\cdot, t)\|^2_{L^2(\Omega)}} . \tag{7.10}$$

Passing from Cartesian to polar coordinates, one can easily find that

$$\|u(\cdot, t)\|^2_{L^2(\Omega)} = \iint_{\Omega} \left[\frac{1}{2}(x^2 + y^2) + t\right]^2 dx\, dy = \pi \left(t^2 + \frac{t}{2} + \frac{1}{12}\right),$$

and therefore

$$\mathcal{E}_{rel}(T) = \frac{12 \cdot \mathcal{E}(T)}{\pi\,(12\,T^2 + 6\,T + 1)} .$$

During our numerical experiments, we have estimated both $\mathcal{E}(T)$ and $\mathcal{E}_{rel}(T)$ for

$$T \in \{1, 10, 100, 200, 300, 400, 500\}.$$

The results that we have obtained are shown in Table 7.1. The estimates of $\mathcal{E}(1)$ and $\mathcal{E}(10)$ are equal to 2.24×10^{-5} and 4.30×10^{-4} respectively, which is very satisfactory, especially considering that the order of magnitude of $\mathcal{E}_{rel}(1)$ and $\mathcal{E}_{rel}(10)$ is equal to 10^{-6}.

Final time T	**Estimate of $\mathcal{E}(T)$**	**Estimate of $\mathcal{E}_{rel}(T)$**
1	2.24×10^{-5}	4.50×10^{-6}
10	4.30×10^{-4}	1.30×10^{-6}
100	1.87×10^{-1}	5.92×10^{-6}
200	2.05×10^{-1}	1.63×10^{-6}
300	2.45×10^{-1}	8.65×10^{-7}
400	2.47×10^{-1}	4.91×10^{-7}
500	2.80×10^{-1}	3.56×10^{-7}

Table 7.1: Estimates of $\mathcal{E}(T)$ and $\mathcal{E}_{rel}(T)$ for $T \in \{1, 10, 100, 200, 300, 400, 500\}$.

In order to get more accurate estimates for larger values of T, for every fixed $T \geq 100$ we have used $2.1 \times T$ equispaced points (instead of the initial 21) to discretize the time interval $[0, T]$. By doing so, we have observed that the variation of the estimate of $\mathcal{E}(T)$ displays a monotonically increasing behavior, in accordance with the definition (7.9). However, even for $100 \leq T \leq 500$, the order of magnitude of $\mathcal{E}_{rel}(T)$ remains not greater than 10^{-6}. Therefore, for this first test problem, we can conclude that our predictive method is indeed very accurate and efficient, on both a short and long time scale.

Now, for $0 < \varepsilon \leq 1$ we consider the problem

$$\begin{cases} \partial_t v_\varepsilon - \operatorname{div}\left(H(Dv_\varepsilon) + \varepsilon\, Dv_\varepsilon\right) = 1 & \text{in } \Omega'_{1,2} := \Omega' \times (t_1, t_2), \\ v_\varepsilon(x, y, t_1) = \frac{1}{2}(x^2 + y^2) + t_1 & \text{if } (x, y) \in \overline{\Omega'}, \\ v_\varepsilon(x, y, t) = \frac{1}{2}(x^2 + y^2) + t & \text{if } (x, y) \in \partial\Omega' \wedge t \in (t_1, t_2), \end{cases} \tag{7.11}$$

which is nothing but the approximating problem (7.6) associated with ($\mathcal{P}1$). In what follows, we will denote the exact solution of (7.11) by u_ε, while its predicted solution will be denoted by $\hat{u}_\varepsilon$.

Throughout our tests, for $10^{-9} \leq \varepsilon \leq 10^{-3}$, for $\Omega' = \Omega$ and for $(t_1, t_2) = (0, T)$, we have observed that the plots of the predicted solution $\hat{u}_\varepsilon(\cdot, t)$ and the exact solution $u(\cdot, t)$ share the same configurations and geometric peculiarities, on both a short and long time scale (see, e.g. Fig. 7.4). Furthermore, we have seen that the evolution over time of $\hat{u}_\varepsilon$ reflects the behavior depicted for the solution u quite faithfully.

In addition, the contour lines of $\hat{u}_\varepsilon(\cdot, t)$ perfectly overlap those of $u(\cdot, t)$, at least for not very long times t (see Fig. 7.5).

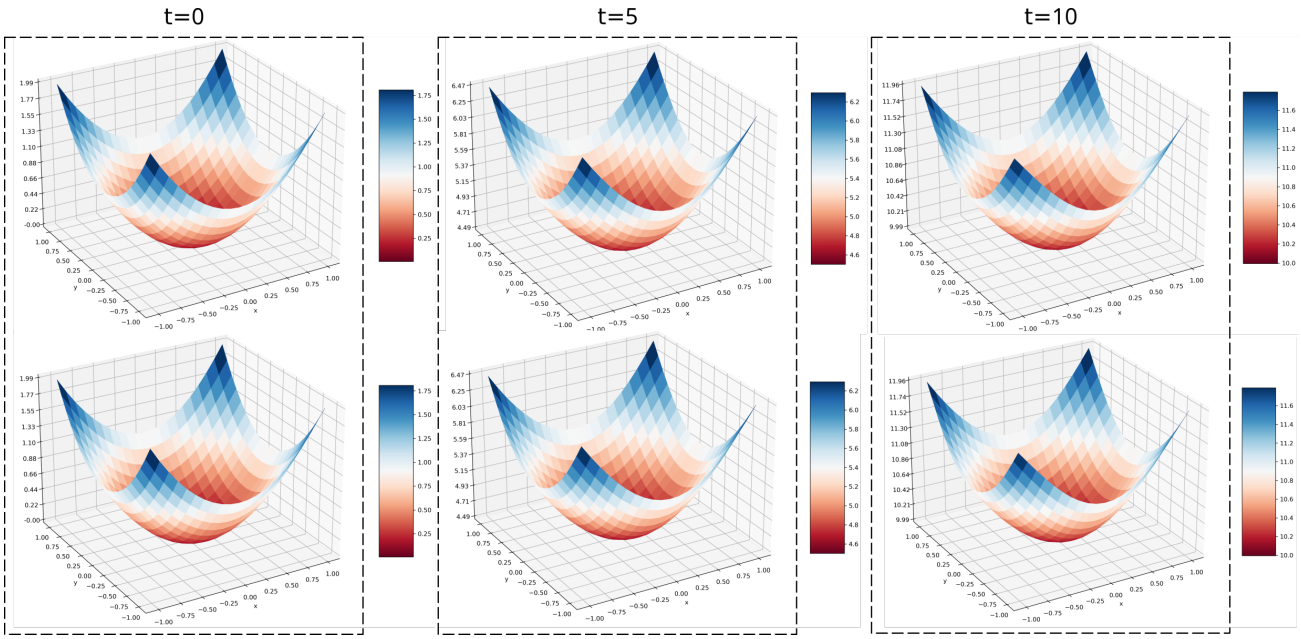


Figure 7.4: Plots of the approximate solution $\hat{u}_\varepsilon(\cdot, t)$ for $\varepsilon = 10^{-3}$ (above) and $\varepsilon = 10^{-9}$ (below), at times $t = 0$ (left), $t = 5$ (center) and $t = 10$ (right). Here $\Omega' = \Omega$ and $[t_1, t_2] = [0, 10]$.

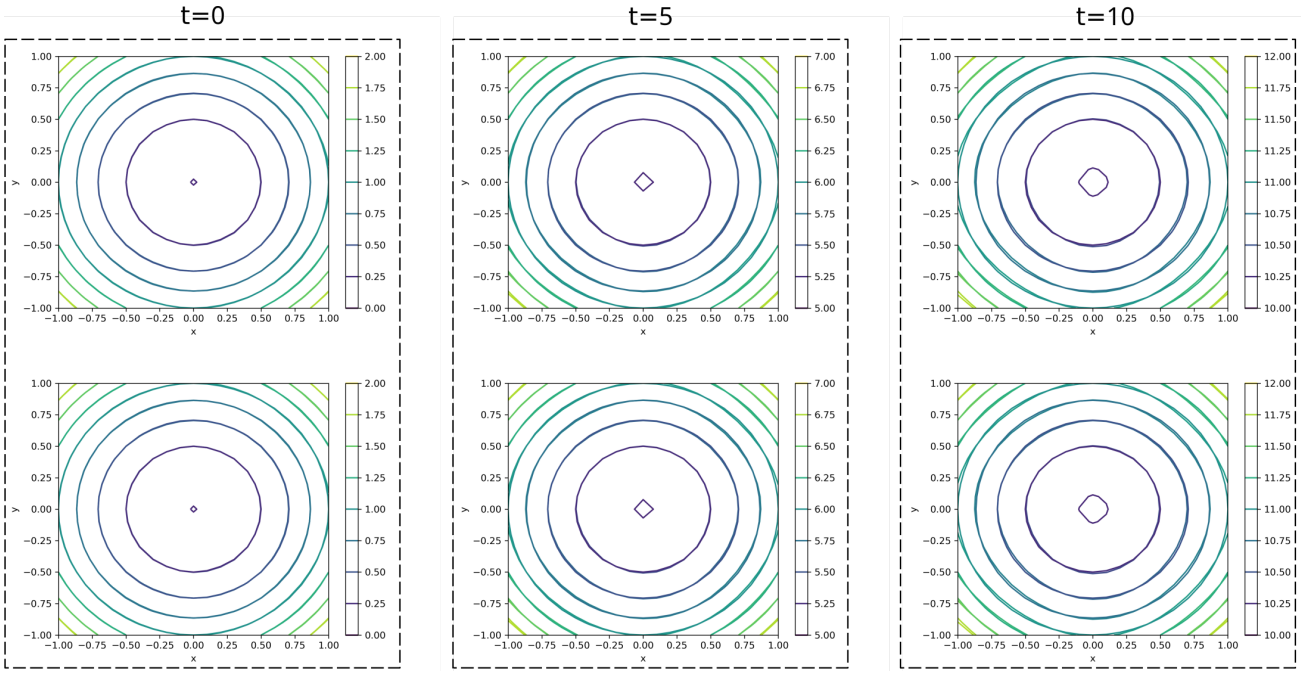


Figure 7.5: Superposition of the level curves of the exact solution $u(\cdot, t)$ and the predicted solution $\hat{u}_\varepsilon(\cdot, t)$ for $\varepsilon = 10^{-3}$ (above) and $\varepsilon = 10^{-9}$ (below), at times $t = 0$ (left), $t = 5$ (center) and $t = 10$ (right). Here $\Omega' = \Omega$ and $[t_1, t_2] = [0, 10]$. For every $\varepsilon \in [10^{-9}, 10^{-3}]$, the contour lines of the approximate solution $\hat{u}_\varepsilon(\cdot, t)$ perfectly overlap those of $u(\cdot, t)$ for any fixed $t \in [0, 10]$.

Let us now assume that

$$\Omega'_{1,2} \,=\, Q_{\frac{1}{2}}(z_0) := \, B_{\frac{1}{2}}(\mathbf{0}) \times \left(\frac{7}{4}, 2\right),$$

where $z_0 = (\mathbf{0}, 2) = (0, 0, 2)$. Then, the limit in (7.7) suggests that $\hat{u}_\varepsilon$ should numerically converge to u as $\varepsilon \searrow 0$. To obtain a numerical evidence of such convergence, we have chosen

$$\Omega' = B_{1/2}(\mathbf{0}) \qquad \text{and} \qquad (t_1, t_2) = (\frac{7}{4}, 2)$$

into (7.11) and examined the time behavior of the L^2-error $\|\hat{u}_\varepsilon(\cdot, t) - u(\cdot, t)\|_{L^2(\Omega')}$, by evaluating the quantities

$$E_\varepsilon := \sup_{t \in (\frac{7}{4}, 2)} \|\hat{u}_\varepsilon(\cdot, t) - u(\cdot, t)\|^2_{L^2(B_{1/2}(\mathbf{0}))}$$

and

$$E_{rel} \,\equiv\, E_{rel}(\varepsilon) \,:=\, \frac{E_\varepsilon}{\sup\limits_{t \in (\frac{7}{4}, 2)} \|u(\cdot, t)\|^2_{L^2(B_{1/2}(\mathbf{0}))}}\,.$$

Switching from Cartesian to polar coordinates, one can easily compute

$$\|u(\cdot, t)\|^2_{L^2(B_{1/2}(\mathbf{0}))} \,=\, \iint\limits_{B_{\frac{1}{2}}(\mathbf{0})} \left[\frac{1}{2}(x^2 + y^2) + t\right]^2 dx\, dy \,=\, \frac{\pi}{4}\left(t^2 + \frac{t}{8} + \frac{1}{192}\right),$$

from which it immediately follows that

$$E_{rel} \,=\, \frac{768\, E_\varepsilon}{817\, \pi}\,.$$

In the testing phase, we have estimated both E_ε and E_{rel} for

$$\varepsilon \in \{10^{-15}, 10^{-14}, \ldots, 10^{-1}, 1\}.$$

Table 7.2 shows the results obtained and reveals that the predicted solution $\hat{u}_\varepsilon$ converges to u as ε tends to zero, although not very quickly. In fact, the estimates of E_ε and $E_{rel}(\varepsilon)$ approach zero with a convergence rate much lower than that of ε. Furthermore, they seem to start decreasing monotonically, i.e. without oscillations, for $\varepsilon \leq 10^{-10}$.

ε	Estimate of E_ε	Estimate of E_{rel}
1	5.816×10^{-4}	1.74×10^{-4}
10^{-1}	1.658×10^{-5}	4.96×10^{-6}
10^{-2}	2.485×10^{-5}	7.44×10^{-6}
10^{-3}	9.280×10^{-6}	2.78×10^{-6}
10^{-4}	5.331×10^{-6}	1.60×10^{-6}
10^{-5}	9.681×10^{-6}	2.90×10^{-6}
10^{-6}	7.175×10^{-6}	2.15×10^{-6}
10^{-7}	2.027×10^{-6}	6.06×10^{-7}
10^{-8}	8.205×10^{-6}	2.45×10^{-6}
10^{-9}	3.749×10^{-6}	1.12×10^{-6}
10^{-10}	4.254×10^{-6}	1.27×10^{-6}
10^{-11}	3.611×10^{-6}	1.08×10^{-6}
10^{-12}	2.769×10^{-6}	8.28×10^{-7}
10^{-13}	2.399×10^{-6}	7.18×10^{-7}
10^{-14}	2.017×10^{-6}	6.03×10^{-7}
10^{-15}	1.649×10^{-6}	4.94×10^{-7}

Table 7.2: Estimates of E_ε and $E_{rel}(\varepsilon)$ for $\varepsilon \in \{10^{-15}, 10^{-14}, \dots, 10^{-1}, 1\}$.

7.3.2 Second test problem

Let $\alpha > 0$. As a second test problem, we consider

$$\begin{cases} \partial_t v - \operatorname{div}\left((|Dv| - 1)_+ \frac{Dv}{|Dv|}\right) = f & \text{in } \Omega_T, \\ v(x, y, 0) = 0 & \text{if } (x, y) \in \overline{\Omega}, \\ v(x, y, t) = t & \text{if } (x, y) \in \partial\Omega \wedge t \in (0, T), \end{cases} \tag{$\mathcal{P}2$}$$

where $\Omega = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ again and

$$f(x, y, t) := \begin{cases} (x^2 + y^2)^\alpha & \text{if } 2\alpha t\,(x^2 + y^2)^{\alpha - \frac{1}{2}} \leq 1, \\ (x^2 + y^2)^\alpha - 4\alpha^2 t\,(x^2 + y^2)^{\alpha - 1} + \frac{1}{\sqrt{x^2 + y^2}} & \text{if } 2\alpha t\,(x^2 + y^2)^{\alpha - \frac{1}{2}} > 1. \end{cases}$$

The exact solution of problem ($\mathcal{P}2$) is given by

$$u(x, y, t) \equiv u_\alpha(x, y, t) := t\,(x^2 + y^2)^\alpha.$$

At any fixed time $t > 0$, the shape and geometric properties of the graph of $u(\cdot, t)$ strongly depend on the value of the parameter α.

If $\alpha = \frac{1}{2}$, then the graph of $u(\cdot, t)$ is a cone whose vertex coincides with the origin $(0, 0, 0)$ at any given positive time t. As time goes on, the cone in question gets narrower and narrower around the vertical axis. In this case, the plot of the approximate solution $\hat{u}(\cdot, t)$ has the same form as the graph of the exact solution $u(\cdot, t)$ for both short and long times $t > 0$, except near the origin, where the tip of the cone appears to have been smoothed out (see Fig. 7.6, center).

However, this is not a surprise at all, since we already know that for $t > 0$ the function

$$(x, y) \in \Omega \mapsto t\,(x^2 + y^2)^{\frac{1}{2}}$$

is not differentiable at the center $(0, 0)$ of Ω.

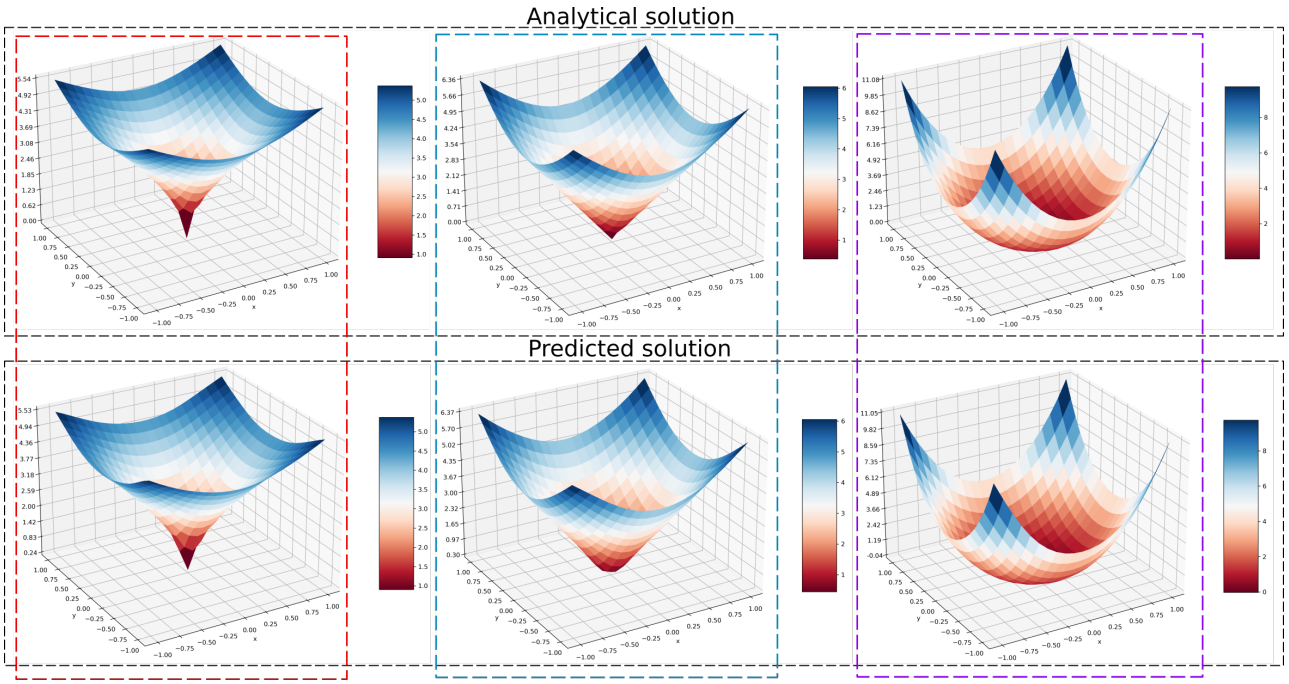


Figure 7.6: Plots of the exact solution (above) and the predicted solution (below) to problem ($\mathcal{P}2$) at time $t = 4.5$ for $\alpha = 0.3$ (left), $\alpha = 0.5$ (center) and $\alpha = 1.3$ (right).

When $0 < \alpha < \frac{1}{2}$, the graph of $u(\cdot, t)$ is cusp-shaped for any fixed time $t > 0$, the origin now being a cusp for all positive times. In this case, a loss on convexity occurs, which is also observed in the plot of the predicted solution $\hat{u}(\cdot, t)$ for all times $t > 0$ (see, e.g. Fig. 7.6, left).

Lastly, when $\alpha > \frac{1}{2}$ the graph of $u(\cdot, t)$ is no longer cusp-shaped and becomes increasingly narrow around the vertical axis as t increases. Furthermore, for any fixed $t > 0$ the exact solution $u(\cdot, t)$ is convex again and its graph gets flatter and flatter near the origin when $\alpha >> 1$ (see Fig. 7.7).

In all three of the above cases, we have noticed that the plot of $\hat{u}(\cdot, t)$ is basically identical in its shape and geometry to the graph of the exact solution $u(\cdot, t)$, for both short and long periods t.

Moreover, also for problem ($\mathcal{P}2$) we have verified that the time evolution of the predicted solution faithfully reflects the trend described for the exact solution in all three previous cases. Therefore, we may conclude that $\alpha = \frac{1}{2}$ represents a critical value for the global behavior of both the exact and the predicted solution.

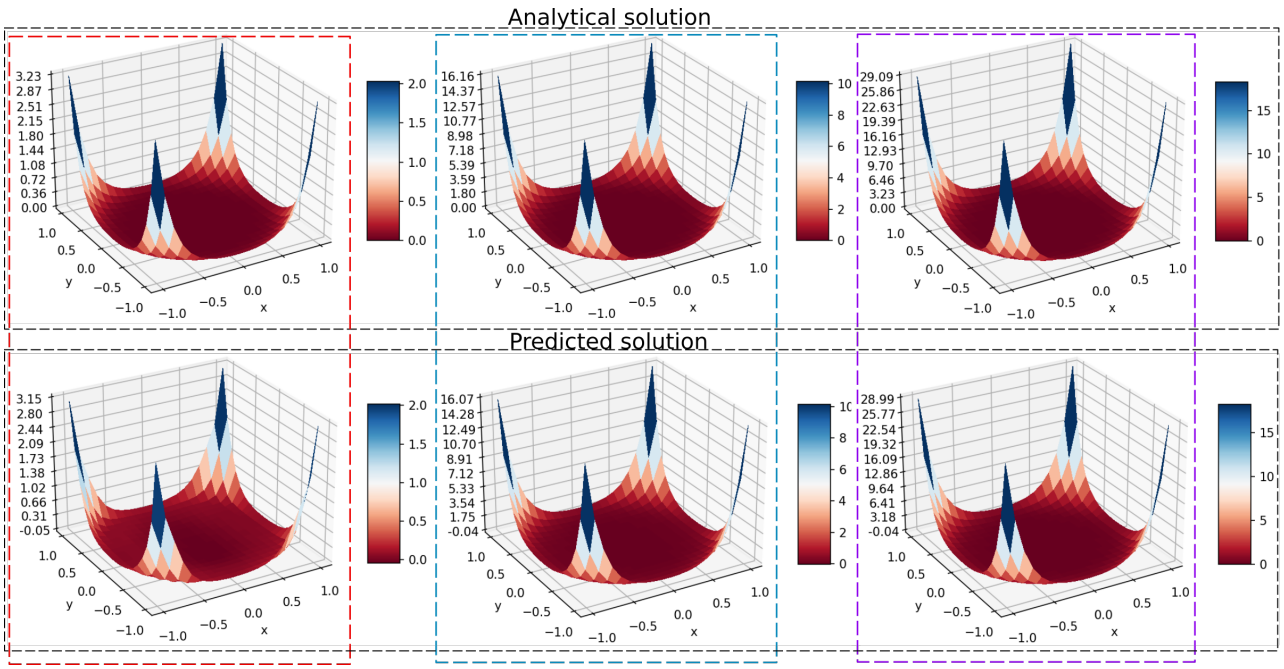


Figure 7.7: Plots of the exact solution to problem $(\mathcal{P}2)$ (above) and the predicted solution $\hat{u}(\cdot,t)$ (below) for $\alpha = 5$, at times $t = 0.5$ (left), $t = 5$ (center) and $t = 10$ (right).

Later, we have examined the contour lines of $\hat{u}_\alpha(\cdot,t)$ for $\alpha \in \{0.3, 0.5, 1.3, 5\}$ and for not very large times $t > 0$. For every fixed $\alpha \in \{0.3, 0.5, 1.3\}$, the level curves of $\hat{u}_\alpha(\cdot,t)$ overlap quite well those of $u_\alpha(\cdot,t)$, with some small differences between one case and the other. More precisely, for each $\alpha \in \{0.5, 1.3\}$ the contour lines corresponding to the same level are almost indistinguishable, at least for not very long times t (see, for example, Fig. 7.8, where $t = 0.5$).

For $\alpha = 5$ and $t > 0$, we have also noted that the approximate solution is essentially equal to zero in a fairly large region Σ_t around the origin $(0,0)$ of the xy-plane (see Fig. 7.9). As already said for problem $(\mathcal{P}1)$, this means that such region consists of numerical zeros of $\hat{u}_5(\cdot,t)$, while for $t > 0$ we know that $u_5(x,y,t) = 0$ if and only if $(x,y) = (0,0)$. However, this discrepancy is reasonably small for short times, since the order of magnitude of $u_5(x,y,t)$ does not exceed 10^{-2} within Σ_t for $0 < t \leq 10$.

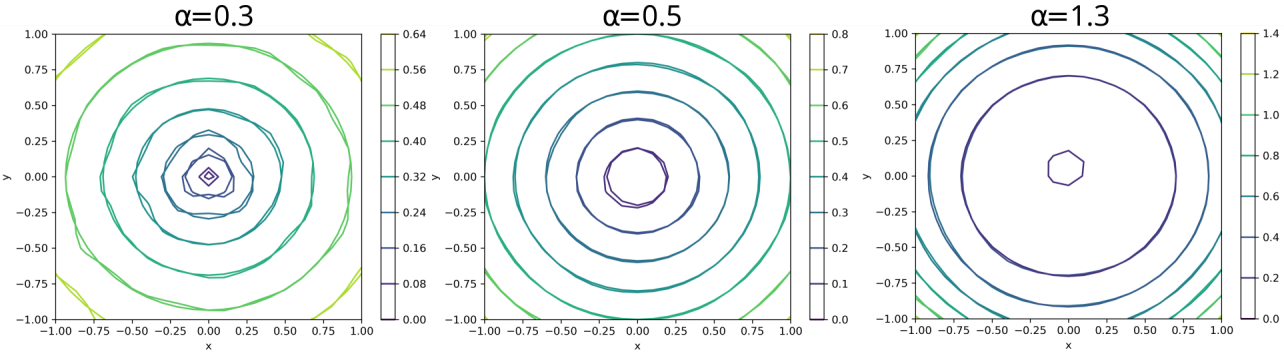


Figure 7.8: Superposition of the level curves of the exact solution $u_\alpha(\cdot,t)$ and the predicted solution $\hat{u}_\alpha(\cdot,t)$ at time $t = 0.5$, for $\alpha = 0.3$ (left), $\alpha = 0.5$ (center) and $\alpha = 1.3$ (right).

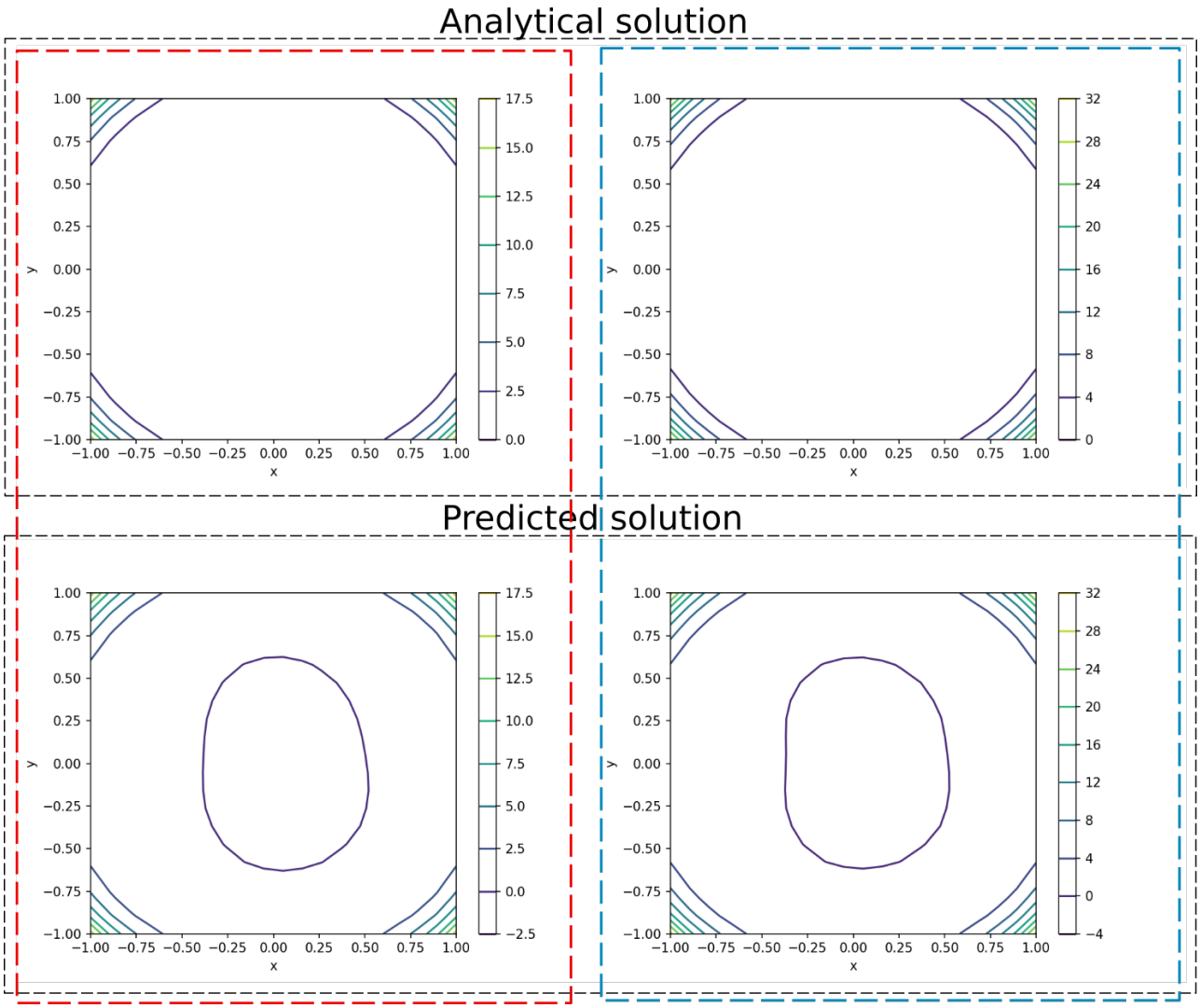


Figure 7.9: Level curves of the exact solution $u_\alpha(\cdot,t)$ (above) and the predicted solution $\hat{u}_\alpha(\cdot,t)$ (below) for $\alpha = 5$, at times $t = 0.5$ (left) and $t = 1$ (right).

To evaluate in a more quantitative manner the accuracy of our method in solving problem ($\mathcal{P}2$) and the numerical convergence of the solution $\hat{u}$ toward u, we may now consider again the quantities (7.9) and (7.10). Passing from Cartesian to polar coordinates, we find

$$\|u_\alpha(\cdot,t)\|^2_{L^2(\Omega)} \;=\; t^2 \iint\limits_{\Omega} (x^2+y^2)^{2\alpha}\, dx\, dy \;=\; \frac{\pi\, t^2}{2\alpha+1}\,,$$

so that we now have

$$\mathcal{E}_{rel}(T) \;:=\; \frac{\mathcal{E}(T)}{\sup\limits_{t\,\in\,(0,T)} \|u_\alpha(\cdot,t)\|^2_{L^2(\Omega)}} \;=\; \frac{2\alpha+1}{\pi\, T^2} \sup_{t\,\in\,(0,T)} \|\hat{u}_\alpha(\cdot,t) - u_\alpha(\cdot,t)\|^2_{L^2(\Omega)}\,.$$

During the experimental phase, we have estimated $\mathcal{E}(T)$ and $\mathcal{E}_{rel}(T)$ for $\alpha \in \{0.3, 0.5, 1.3, 5\}$ and $T \in \{1, 10, 20, 40, 100\}$. The results that we have obtained are reported in Tables 7.3−7.6 and show that, for any fixed value of α, the estimate of $\mathcal{E}(T)$ follows an increasing trend, as prescribed by (7.9). Furthermore, by analyzing the orders of magnitude of $\mathcal{E}(T)$ and $\mathcal{E}_{rel}(T)$, we

may affirm that our approach provides very accurate predictions, on both a short and long-term scale.

$\boldsymbol{\alpha = 0.3}$		
Final time $\boldsymbol{T}$	**Estimate of $\boldsymbol{\mathcal{E}(T)}$**	**Estimate of $\boldsymbol{\mathcal{E}_{rel}(T)}$**
1	2.42×10^{-5}	1.23×10^{-5}
10	4.23×10^{-4}	2.15×10^{-6}
20	2.20×10^{-3}	2.80×10^{-6}
40	1.48×10^{-2}	4.71×10^{-6}
100	2.54×10^{-1}	1.29×10^{-5}

Table 7.3: Estimates of $\mathcal{E}(T)$ and $\mathcal{E}_{rel}(T)$ for $\alpha = 0.3$ and $T \in \{1, 10, 20, 40, 100\}$.

$\boldsymbol{\alpha = 0.5}$		
Final time $\boldsymbol{T}$	**Estimate of $\boldsymbol{\mathcal{E}(T)}$**	**Estimate of $\boldsymbol{\mathcal{E}_{rel}(T)}$**
1	2.81×10^{-5}	1.79×10^{-5}
10	3.41×10^{-4}	2.17×10^{-6}
20	1.26×10^{-3}	2.01×10^{-6}
40	9.41×10^{-2}	3.74×10^{-5}
100	2.25×10^{-1}	1.43×10^{-5}

Table 7.4: Estimates of $\mathcal{E}(T)$ and $\mathcal{E}_{rel}(T)$ for $\alpha = 0.5$ and $T \in \{1, 10, 20, 40, 100\}$.

$\boldsymbol{\alpha = 1.3}$		
Final time $\boldsymbol{T}$	**Estimate of $\boldsymbol{\mathcal{E}(T)}$**	**Estimate of $\boldsymbol{\mathcal{E}_{rel}(T)}$**
1	2.85×10^{-5}	3.27×10^{-5}
10	1.76×10^{-4}	2.02×10^{-6}
20	1.26×10^{-3}	3.61×10^{-6}
40	1.51×10^{-3}	1.08×10^{-6}
100	1.36×10^{-2}	1.56×10^{-6}

Table 7.5: Estimates of $\mathcal{E}(T)$ and $\mathcal{E}_{rel}(T)$ for $\alpha = 1.3$ and $T \in \{1, 10, 20, 40, 100\}$.

$\boldsymbol{\alpha = 5}$		
Final time $\boldsymbol{T}$	**Estimate of $\boldsymbol{\mathcal{E}(T)}$**	**Estimate of $\boldsymbol{\mathcal{E}_{rel}(T)}$**
1	4.09×10^{-4}	1.43×10^{-3}
10	1.01×10^{-3}	3.53×10^{-5}
20	1.19×10^{-1}	1.04×10^{-3}
40	7.51×10^{-1}	1.64×10^{-3}
100	8.78×10^{-1}	3.07×10^{-4}

Table 7.6: Estimates of $\mathcal{E}(T)$ and $\mathcal{E}_{rel}(T)$ for $\alpha = 5$ and $T \in \{1, 10, 20, 40, 100\}$.

7.3.3 Third test problem

We shall now consider the problem

$$\begin{cases} \partial_t v - \operatorname{div}\left((|Dv|-1)_+ \frac{Dv}{|Dv|}\right) = 1 & \text{in } \Omega_T, \\ v(x,y,0) = g(x,y) & \text{if } (x,y) \in \overline{\Omega}, \\ v(x,y,t) = h(x,y,t) & \text{if } (x,y) \in \partial\Omega \wedge t \in (0,T), \end{cases} \tag{$\mathcal{P}3$}$$

where $\Omega = (-1,1) \times (-1,1)$,

$$g(x,y) := \begin{cases} 1 & \text{if } -1 \le x \le 0 \wedge -1 \le y \le 1, \\ 1-x & \text{if } 0 < x \le 1 \wedge -1 \le y \le 1, \end{cases}$$

and

$$h(x,y,t) := \begin{cases} 1+t & \text{if } -1 \le x \le 0 \wedge y = -1, \\ 1+t & \text{if } -1 \le x \le 0 \wedge y = 1, \\ 1+t & \text{if } x = -1 \wedge -1 \le y \le 1, \\ t & \text{if } x = 1 \wedge -1 \le y \le 1, \\ 1-x+t & \text{if } 0 < x \le 1 \wedge y = -1, \\ 1-x+t & \text{if } 0 < x \le 1 \wedge y = 1. \end{cases}$$

The exact solution of this problem is given by

$$u(x,y,t) = \begin{cases} 1+t & \text{if } -1 \le x \le 0 \wedge -1 \le y \le 1, \\ 1-x+t & \text{if } 0 < x \le 1 \wedge -1 \le y \le 1. \end{cases} \tag{7.12}$$

Therefore, for any fixed time $t \ge 0$, the graph of the function $u(\cdot,t)$ is given by the union of the horizontal region

$$\mathcal{H}_t := \{(x,y,1+t) : -1 \le x \le 0,\ -1 \le y \le 1\}$$

and the sliding plane

$$\mathcal{I}_t := \{(x,y,1-x+t) : 0 \le x \le 1,\ -1 \le y \le 1\}.$$

Let us denote by $\mathcal{G}_t := \mathcal{H}_t \cup \mathcal{I}_t$ the graph of $u(\cdot,t)$. Then, as time goes by, the set $\mathcal{G}_t$ slides along a vertical axis with a constant velocity and no deformation, since $\partial_t u \equiv 1$ over Ω_T (see Fig. 7.10, above).

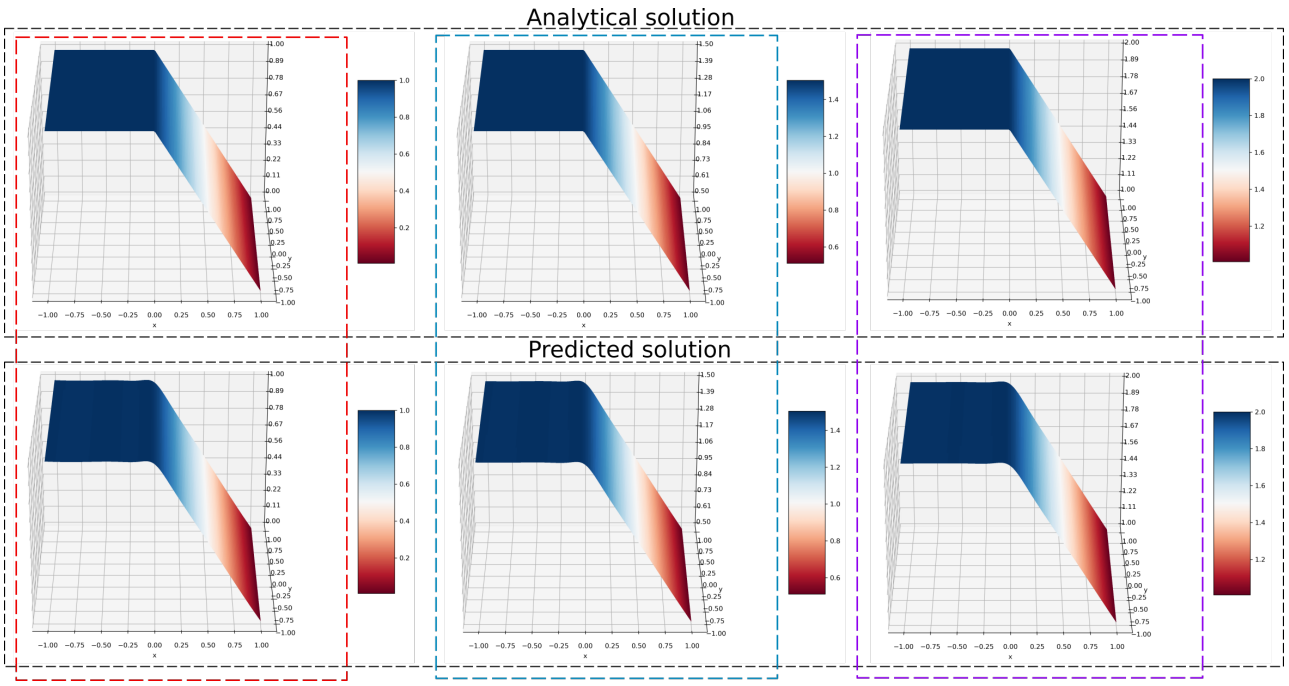


Figure 7.10: Plots of the exact solution to problem ($\mathcal{P}3$) (above) and the predicted solution $\hat{u}(\cdot,t)$ (below) for $t=0$ (left), $t=0.5$ (center) and $t=1$ (right).

The plot of the approximate solution $\hat{u}(\cdot,t)$ roughly resembles that of $u(\cdot,t)$ for both short and long times $t \geq 0$, except near the joining line $\mathcal{H}_t \cap \mathcal{I}_t$, where the graph of the solution appears to have been slightly smoothed (see Fig. 7.10, below). However, this is not surprising at all, since we already know that, for any fixed $t \geq 0$, the function $u(\cdot,t) : \overline{\Omega} \to \mathbb{R}$ defined by (7.12) is not differentiable at any point of the open segment $S_0 := \{(x,y) \in \Omega : x = 0\}$. This fact also has repercussions in the comparison between the level curves of $u(\cdot,t)$ and $\hat{u}(\cdot,t)$, whose superposition is far from being perfect on approaching the segment S_0 from the right, i.e. for $x > 0$ (see Fig. 7.11).

Furthermore, we have also observed that the evolution of $\hat{u}$ over time accurately reflects the evolution of the set $\mathcal{G}_t$ described above.

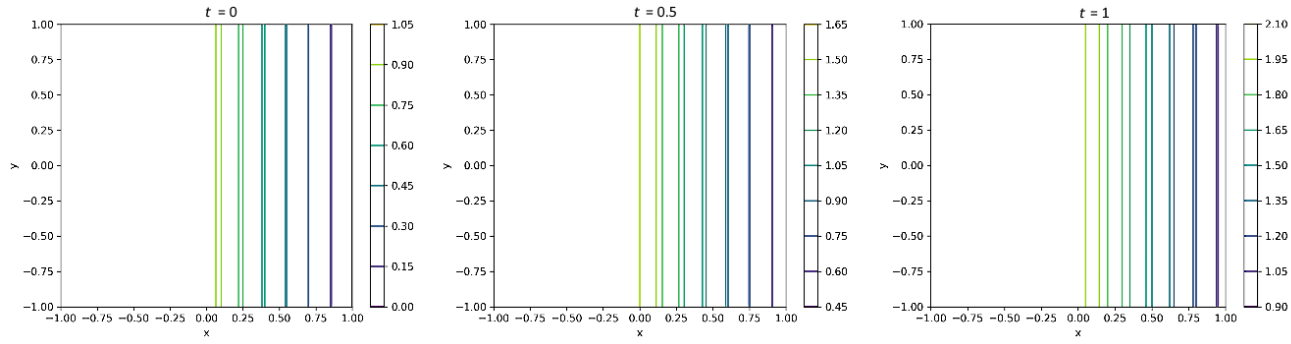


Figure 7.11: Superposition of the level curves of the exact solution $u(\cdot,t)$ and the predicted solution $\hat{u}(\cdot,t)$ for $t=0$ (left), $t=0.5$ (center) and $t=1$ (right).

In order to assess in a more quantitative way the accuracy of our method in solving ($\mathcal{P}3$) and the distance between the solutions u and $\hat{u}$, we resort again to the quantities defined in (7.9)

and (7.10). Through an easy calculation, we get

$$\|u(\cdot,t)\|^2_{L^2(\Omega)} \;=\; 4\,t^2 + 6\,t + \frac{8}{3}\,,$$

so that we now have

$$\mathcal{E}_{rel}(T) \;:=\; \frac{\mathcal{E}(T)}{\sup\limits_{t\,\in\,(0,T)} \|u(\cdot,t)\|^2_{L^2(\Omega)}} \;=\; \frac{3}{12\,T^2 + 18\,T + 8} \sup_{t\,\in\,(0,T)} \|\hat{u}(\cdot,t) - u(\cdot,t)\|^2_{L^2(\Omega)}\,. \tag{7.13}$$

Proceeding as for the previous problems, we have estimated $\mathcal{E}(T)$ and $\mathcal{E}_{rel}(T)$ for

$$T \in \{1, 10, 100, 200, 300\}.$$

Table 7.7 contains the results obtained and reveals that the estimate of $\mathcal{E}(T)$ exhibits again an increasing behavior, as expected from (7.9). Furthermore, from this table, it seems that the asymptotic trend of the estimate of $\mathcal{E}_{rel}(T)$ may encounter a sort of plateau at $T = 100$, after which convergence sensibly slows down. We do not know whether this is a typical behavior, since we cannot draw information from (7.13) in this sense. In fact, from the definition of $\mathcal{E}_{rel}(T)$ it is not possible to predict what the combined effect of $\mathcal{E}(T)$ and $\sup\limits_{t\,\in\,(0,T)} \|u(\cdot,t)\|^2_{L^2(\Omega)}$ is, since $\mathcal{E}_{rel}(T)$ is the ratio of two functions which are both increasing with respect to T and we cannot determine a priori the growth rate of $\mathcal{E}(T)$. Nevertheless, by carefully examining the orders of magnitude of both $\mathcal{E}(T)$ and $\mathcal{E}_{rel}(T)$, we can conclude that our method produces accurate results also in this case, in both short and long-term predictions.

Final time T	**Estimate of $\mathcal{E}(T)$**	**Estimate of $\mathcal{E}_{rel}(T)$**
1	1.67×10^{-1}	1.32×10^{-2}
10	1.69×10^{-1}	3.65×10^{-4}
100	3.72×10^{-1}	9.16×10^{-6}
200	6.92×10^{-1}	4.29×10^{-6}
300	7.21×10^{-1}	1.99×10^{-6}

Table 7.7: Estimates of $\mathcal{E}(T)$ and $\mathcal{E}_{rel}(T)$ for $T \in \{1, 10, 100, 200, 300\}$.

7.3.4 Fourth test problem

We now consider the problem

$$\begin{cases} \partial_t v - \operatorname{div}\left((|Dv| - 1)_+ \frac{Dv}{|Dv|}\right) = 1 & \text{in } \Omega_T, \\ v(x,y,z,0) = \frac{1}{2}(x^2 + y^2 + z^2) & \text{if } (x,y,z) \in \overline{\Omega}, \\ v(x,y,z,t) = \frac{1}{2} + t & \text{if } (x,y,z) \in \partial\Omega \wedge t \in (0,T), \end{cases} \tag{$\mathcal{P}4$}$$

where $\Omega = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 < 1\}$. This problem is the three-dimensional version of problem ($\mathcal{P}1$) and its exact solution is given by

$$u(x, y, z, t) \;=\; \frac{1}{2}(x^2 + y^2 + z^2) + t.$$

To evaluate the accuracy of our method in solving problem ($\mathcal{P}4$) and the distance between the predicted solution $\hat{u}$ and the exact solution u, we confined ourselves to considering the quantities (7.9) and (7.10). Passing from Cartesian to spherical coordinates, one can easily find that

$$\|u(\cdot, t)\|^2_{L^2(\Omega)} \;=\; \iiint\limits_{\Omega} \left[\frac{1}{2}(x^2 + y^2 + z^2) + t\right]^2 dx\, dy\, dz \;=\; \pi\left(\frac{4}{3}\,t^2 + \frac{4}{5}\,t + \frac{1}{7}\right),$$

and therefore

$$\mathcal{E}_{rel}(T) \;:=\; \frac{\mathcal{E}(T)}{\sup\limits_{t\,\in\,(0,T)} \|u(\cdot, t)\|^2_{L^2(\Omega)}} \;=\; \frac{105}{\pi\,(140\,T^2 + 84\,T + 15)} \sup_{t\,\in\,(0,T)} \|\hat{u}(\cdot, t) - u(\cdot, t)\|^2_{L^2(\Omega)}\,.$$

Proceeding as for problem ($\mathcal{P}1$), we have estimated both $\mathcal{E}(T)$ and $\mathcal{E}_{rel}(T)$ for

$$T \in \{1, 10, 20, 30, 40, 50, 100\}.$$

The data that we have obtained are reported in Table 7.8 and show that the estimate of $\mathcal{E}(T)$ is monotonically increasing, in agreement with the definition (7.9). From Table 7.8 it also emerges that the trend of the estimate of $\mathcal{E}_{rel}(T)$ has a sort of plateau between $T = 30$ and $T = 40$, after which there is a slight rise. In this regard, the same considerations made for Table 7.7 apply. However, for every $T \leq 100$ the order of magnitude of $\mathcal{E}_{rel}(T)$ is not greater than 10^{-5}. Therefore, we may affirm that our predictive method is very accurate and efficient in this case, on both a short and long time scale.

Final time T	**Estimate of $\mathcal{E}(T)$**	**Estimate of $\mathcal{E}_{rel}(T)$**
1	6.59×10^{-4}	9.21×10^{-5}
10	1.21×10^{-3}	2.73×10^{-6}
20	1.73×10^{-3}	1.00×10^{-6}
30	2.90×10^{-3}	7.54×10^{-7}
40	6.13×10^{-3}	9.02×10^{-7}
50	4.89×10^{-2}	4.61×10^{-6}
100	1.44×10^{-1}	3.42×10^{-6}

Table 7.8: Estimates of $\mathcal{E}(T)$ and $\mathcal{E}_{rel}(T)$ for $T \in \{1, 10, 20, 30, 40, 50, 100\}$.

7.3.5 Fifth test problem

Let $\alpha > 0$ and $\omega = 2\alpha(2\alpha+1)$. The last problem that we consider is

$$\begin{cases} \partial_t v - \operatorname{div}\left((|Dv|-1)_+ \frac{Dv}{|Dv|}\right) = f & \text{in } \Omega_T, \\ v(x,y,z,0) = 0 & \text{if } (x,y,z) \in \overline{\Omega}, \\ v(x,y,z,t) = t & \text{if } (x,y,z) \in \partial\Omega \wedge t \in (0,T), \end{cases} \tag{$\mathcal{P}5$}$$

where $\Omega = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 < 1\}$ and

$$f := \begin{cases} (x^2+y^2+z^2)^{\alpha} & \text{if } 2\alpha t\,(x^2+y^2+z^2)^{\alpha-\frac{1}{2}} \le 1, \\ (x^2+y^2+z^2)^{\alpha-1}(x^2+y^2+z^2-\omega t) + \frac{2}{\sqrt{x^2+y^2+z^2}} & \text{if } 2\alpha t\,(x^2+y^2+z^2)^{\alpha-\frac{1}{2}} > 1. \end{cases}$$

This problem is nothing but the three-dimensional version of ($\mathcal{P}2$) and its exact solution is given by

$$u(x,y,z,t) \equiv u_\alpha(x,y,z,t) := t\,(x^2+y^2+z^2)^{\alpha}.$$

In order to assess the accuracy of our method in solving ($\mathcal{P}5$) and the distance between the approximate solution $\hat{u}$ and the exact solution u, we again limited ourselves to estimating the quantities defined in (7.9) and (7.10). Switching from Cartesian to spherical coordinates, we can easily obtain

$$\|u_\alpha(\cdot,t)\|^2_{L^2(\Omega)} = t^2 \iiint_{\Omega} (x^2+y^2+z^2)^{2\alpha}\,dx\,dy\,dz = \frac{4\,\pi\,t^2}{4\,\alpha+3}.$$

This yields

$$\mathcal{E}_{rel}(T) = \frac{4\,\alpha+3}{4\,\pi\,T^2} \sup_{t\,\in\,(0,T)} \|\hat{u}(\cdot,t) - u(\cdot,t)\|^2_{L^2(\Omega)}.$$

Proceeding as for problem ($\mathcal{P}2$), we have estimated $\mathcal{E}(T)$ and $\mathcal{E}_{rel}(T)$ for $\alpha \in \{0.3, 0.5, 1.3, 5\}$ and $T \in \{1, 10, 20, 40, 100\}$. The results that have been obtained are shown in Tables 7.9−7.12 and reveal that, for any fixed value of α, the estimate of $\mathcal{E}(T)$ is again monotonically increasing, as expected from (7.9). Nevertheless, by carefully analyzing the orders of magnitude of both $\mathcal{E}(T)$ and $\mathcal{E}_{rel}(T)$, we can deduce that our method provides accurate solutions also in this case, in both short and long-term predictions.

$\boldsymbol{\alpha = 0.3}$		
Final time T	**Estimate of $\mathcal{E}(T)$**	**Estimate of $\mathcal{E}_{rel}(T)$**
1	1.27×10^{-5}	4.25×10^{-6}
10	6.93×10^{-4}	2.31×10^{-6}
20	4.64×10^{-3}	3.88×10^{-6}
40	2.37×10^{-2}	4.95×10^{-6}
100	2.58×10^{-1}	8.63×10^{-6}

Table 7.9: Estimates of $\mathcal{E}(T)$ and $\mathcal{E}_{rel}(T)$ for $\alpha = 0.3$ and $T \in \{1, 10, 20, 40, 100\}$.

$\alpha = 0.5$		
Final time T	**Estimate of $\mathcal{E}(T)$**	**Estimate of $\mathcal{E}_{rel}(T)$**
1	1.02×10^{-5}	4.05×10^{-6}
10	4.60×10^{-4}	1.83×10^{-6}
20	2.25×10^{-3}	2.24×10^{-6}
40	3.44×10^{-2}	8.54×10^{-6}
100	4.47×10^{-1}	1.78×10^{-5}

Table 7.10: Estimates of $\mathcal{E}(T)$ and $\mathcal{E}_{rel}(T)$ for $\alpha = 0.5$ and $T \in \{1, 10, 20, 40, 100\}$.

$\alpha = 1.3$		
Final time T	**Estimate of $\mathcal{E}(T)$**	**Estimate of $\mathcal{E}_{rel}(T)$**
1	3.21×10^{-5}	2.09×10^{-5}
10	9.83×10^{-4}	6.42×10^{-6}
20	9.73×10^{-3}	1.59×10^{-5}
40	2.61×10^{-2}	1.06×10^{-5}
100	1.56×10^{-1}	1.02×10^{-5}

Table 7.11: Estimates of $\mathcal{E}(T)$ and $\mathcal{E}_{rel}(T)$ for $\alpha = 1.3$ and $T \in \{1, 10, 20, 40, 100\}$.

$\alpha = 5$		
Final time T	**Estimate of $\mathcal{E}(T)$**	**Estimate of $\mathcal{E}_{rel}(T)$**
1	9.72×10^{-4}	1.78×10^{-3}
10	1.19×10^{-3}	2.19×10^{-5}
20	3.61×10^{-2}	1.65×10^{-4}
40	2.45×10^{-1}	2.80×10^{-4}
100	6.83×10^{-1}	1.25×10^{-4}

Table 7.12: Estimates of $\mathcal{E}(T)$ and $\mathcal{E}_{rel}(T)$ for $\alpha = 5$ and $T \in \{1, 10, 20, 40, 100\}$.

7.4 Conclusions

In this chapter, we have explored the ability of PINNs to accurately predict the solutions of some strongly degenerate parabolic problems arising in gas filtration through porous media. Since there are no general methods for finding analytical solutions to such problems, it is essential to use efficient and accurate numerical methods. One of the prevalent methods for addressing these problems is the Finite Difference Method (FDM), wherein PDEs are discretized into a system of algebraic equations to be solved numerically. However, the FDM requires the discretization of the domain into a grid of cells or nodes, which can become computationally expensive for large and intricate systems. Although the primary objective of our investigation is not to prove the effectiveness of a PINN compared to a classical numerical method, we engaged in a comparison with the FDM. As established in the literature, for problems characterized by a less complex domain, the FDM typically exhibits a higher level of accuracy compared to PINNs.

Nevertheless, in our study, the advantage of using a PINN lies in the ability to test the model on various presented problems (varying the initial/boundary functions and the α parameter), once it has been trained. Additionally, the FDM can be utilized as a benchmark in cases where the solution to the problem is unknown, ensuring a fair comparison under equivalent accuracy conditions.

For the test problems discussed here, whose exact solutions are known, we have compared the plots of the predicted solutions with those of the analytical solutions. Moreover, to evaluate the accuracy of our predictive method in a purely quantitative way, we have also analyzed the error trends over time. The proposed approach provides accurate results in line with expectations, at least in short-term predictions. However, some issues remain open, such as how to obtain fully reliable plots for the predicted solution when the exact (unknown) one is not differentiable somewhere, and how to reduce or eliminate some slight discrepancies between the contour lines of the predicted solution and those of the analytical solution in the case $n = 2$.

To the best of our knowledge, our study is the first to demonstrate the effectiveness of the PINN framework in solving strongly degenerate parabolic problems with asymptotic structure of Laplacian type.

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