ON THE STEINER TREE CONNECTING A FRACTAL SET

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Abstract. We construct an example of an infinite planar embedded self-similar binary tree Σ which is the essentially unique solution to the Steiner problem of finding the shortest connection of a given planar self-similar fractal set C of positive Hausdorff dimension. Moreover this Steiner tree is, in fact, universal, in the sense that it contains, as a subtree, a Steiner tree (finite or infinite) of any given combinatorial structure. The set C can be considered the set of *leaves*, or the "boundary", of the tree Σ, so that Σ is an *indecomposable* solution to the Steiner problem with datum C (i.e. $\Sigma \setminus C$ is connected).

1. Introduction

Let C be a subset of $\mathbb{R}^2$. The Steiner problem with datum C consists of finding a set $S \subset \mathbb{R}^2$ such that $S \cup C$ is connected and $\mathcal{H}^1(S)$ is minimal, where $\mathcal{H}^1$ stands for the 1-dimensional Hausdorff measure (i.e. the length of the set). Usually this problem is considered for the case when C is a finite set and, in this setting, it becomes a problem of combinatorial geometry. The Steiner problem goes back to Jarník and Kössler [10] (although some of its particular cases are much more ancient, for instance when C is the set of the three vertices of a triangle the problem was stated already by Fermat and solved by Torricelli). The problem is computationally hard [7]. Only few explicit examples of indecomposable solutions with an arbitrarily large number of points are known, namely when the set C is either contained in a circle [15] (in particular when it is the set of vertices of a regular polygon [6]), or in the case of the so called *ladders* [3, 2, 5].

However the problem is important for the more general setting of possibly infinite sets C. Moreover finding explicit solutions for an infinite C may shed light on the solutions to finite sets with an arbitrarily large number of points. One natural question that has arosen in this context is whether there exists a planar indecomposable Steiner tree S connecting an uncountable and possibly fractal set of points C (we say that S is *indecomposable* if $S \setminus C$ is connected). A very similar question, namely the existence of indecomposable Steiner sets with the topology of a binary tree for a finite set of points C with arbitrarily large number of points, has been answered positively by Ivanov and Tuzhilin [9] by means of an abstract construction which gives no information on the length of the branches of the tree. In this

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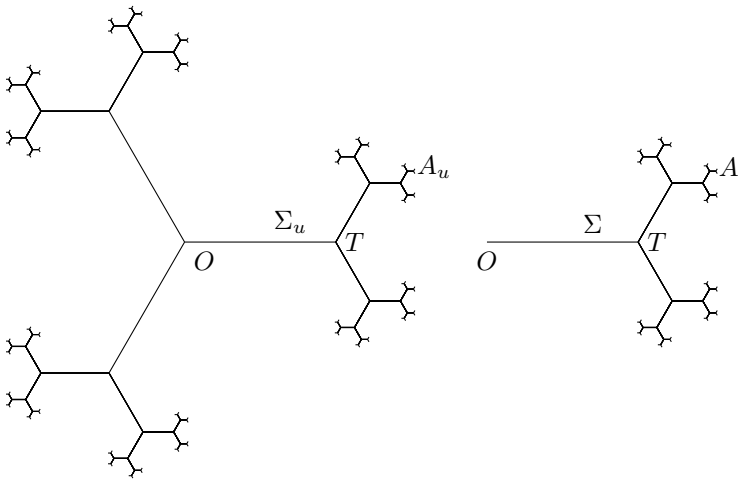


FIGURE 1. The self-similar tree with a countable number of triple junctions and an uncountable number of leaves which are the limit points of the triple junctions. On the left the *universal* tree Σ_u, on the right a single branch Σ, with a fixed root in the origin.

paper we provide a very natural construction of an infinite binary tree connecting a fractal set C of positive Hausdorff dimension, which is the unique solution to the Steiner problem with datum C. Moreover the tree is universal in the sense that it contains, as a subtree, any finite or infinite Steiner tree of any given combinatorial structure (Theorem 3.8).

The first explicit construction of an optimal Steiner tree connecting an uncountable set of vertices, has been developed in [14], were the set C consists of a *root* O and uncountably many points (*leaves*) which together give a totally disconnected perfect set (i.e. every connected component is a point, but the points are not isolated). No segment of S touches the leaves, while every leaf is an accumulation point of segments of S. The infinite tree S is composed by a *trunk* of some length ℓ which splits at angles of 120 degrees into two branches of length $\lambda_1\ell$ both of which split further at the same angles into two branches of length $\lambda_1\lambda_2\ell$ and so on (see Figure 1 right side, where the case $\lambda_k = \lambda$ is depicted).

In [14] it has been proven that the suggested tree is in fact the solution (and even the unique one) to the Steiner problem when the sequence of coefficients λ_k is small and rapidly decreasing to zero (in particular summable). In such a situation the set C, though being uncountable, is not self-similar and has zero Hausdorff dimension. The natural, original question, whether this construction is also valid for fractal self-similar C (i.e. when the coefficients are constant, i.e. $\lambda_j = \lambda$ for all j) was left open. In the attempt to solve it, a completely new and different method, based on special group valued calibrations, to check whether the given set S is a solution to the Steiner problem, has been developed in [12]. In [11] the authors also show an example of an infinite *irrigation* tree which is a solution to a ramified transportation problem similar to the Steiner problem. The method itself is very useful and can be applied in many situations. However it failed in this particular problem. A breakthrough has been achieved in the paper [4] where the authors

provide a much simpler, symmetry based argument, to show that for sufficiently small but constant coefficients the above construction, for a single branch with a fixed root, gives in fact an indecomposable Steiner tree for a set C of positive Hausdorff dimension. In this paper we develop a method which in particular uses a further simplification and a generalization of the idea developed in [4] to obtain the minimality and also uniqueness not only for the *rooted tree* (Figure 1, right side) but also for the universal fractal tree (Figure 1, left side), see Theorem 3.6 and 3.7 below.

Let us make some more precise statements. Let $\lambda > 0$ be fixed. Consider the affine functions $f_\pm \colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$f_\pm(x,y) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} \frac{1}{2} & \frac{\mp\sqrt{3}}{2} \\ \frac{\pm\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

or, by identifying $\mathbb{C}$ with $\mathbb{R}^2$, $f_\pm(z) = 1 + \lambda e^{\pm i\frac{\pi}{3}} z$. The function f_+ is the subsequent composition of: a counter-clockwise rotation of 60 degrees, a scaling by a factor λ, and a unit translation along the positive x-axis. Let $O = (0,0) \in \mathbb{R}^2$ be the origin, and let $A \subset \mathbb{R}^2$ be the set of the limits of all the sequences obtained starting from O and iterating f_+ and f_-. What we obtain is a compact set A which is the only non-empty compact set such that $A = f_+(A) \cup f_-(A)$. If instead we start from the unit segment $[O,T]$ where $T = (1,0)$, and consider the union of all images of $[O,T]$ under the composition of iterations of the functions f_+ and f_-, we obtain the only compact set Σ which satisfies $\Sigma = [O,T] \cup f_+(\Sigma) \cup f_-(\Sigma)$. The set Σ is depicted in Figure 1, right side: there is a trunk $[O,T]$ which splits into two branches $f_\pm([O,T])$, which in turn split into four branches $f_\pm(f_\pm([O,T]))$ and so on. The set A is composed by the *leaves* of this branching tree. If $\lambda < \frac{1}{2}$ (as we will assume from now on) one can check that A is homeomorphic to the Cantor set, it is totally disconnected, perfect, uncountable and it is a self-similar fractal with Hausdorff dimension $-\log_\lambda 2 < 1$ (see [8]) and hence $\mathcal{H}^1(A) = 0$ (the latter is in fact a necessary condition for A to be the set of endpoints of topological tree in $\mathbb{R}^n$, as shown in [14, proposition 3.3]). The set Σ is instead a topological tree (i.e. connected and simply connected), compact, with finite positive length: $\mathcal{H}^1(\Sigma) = \frac{1}{1-2\lambda}$. The set $C = \{O\} \cup A$ is the set of *endpoints* of the tree Σ i.e. the set of points $x \in \Sigma$ such that $\Sigma \setminus \{x\}$ remains connected.

The universal tree Σ_u, depicted in Figure 1, left side, is the union of three rotated copies of Σ: $\Sigma_u = \Sigma \cup \phi(\Sigma) \cup \phi^2(\Sigma)$ where ϕ is the counter-clockwise rotation of 120 degrees centered in O. Analogously we define $A_u = A \cup \phi(A) \cup \phi^2(A)$ the set of endpoints of Σ_u.

In Theorem 3.6 and 3.7, for λ sufficiently small, we show that the sets Σ and Σ_u are the shortest sets connecting A and A_u respectively.

2. Notation and preliminaries

For $C \subset \mathbb{R}^2$ let $\mathrm{St}(C)$ (Steiner competitors with datum C) be the family of all sets S such that $S \cup C$ is connected and let $\mathcal{M}(C)$ (minimal sets) be the subset of $\mathrm{St}(C)$ of the sets S with minimal (possibly infinite) length $\mathcal{H}^1(S)$.

For a set $S \subset \mathbb{R}^2$ we denote by $\bar{S}$ its closure and by ∂S its topological boundary. If $x \in \mathbb{R}^2$ and $\rho > 0$ then we let $B_\rho(x) \subset \mathbb{R}^2$ be the open ball of radius ρ centered

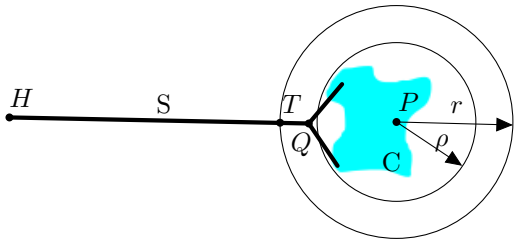


FIGURE 2. The situation in Lemma 2.3.

at x. When $x \in \mathbb{R}^2$, and S, S_1, S_2 are subsets of $\mathbb{R}^2$ we also write

$$\operatorname{dist}(x, S) := \inf \{|x - y| : y \in S\},$$
$$\operatorname{dist}(S_1, S_2) := \inf \{|x - y| : x \in S_1, y \in S_2\}.$$

For $P, Q \in \mathbb{R}^2$ we denote by $[P, Q]$ the closed line segment with endpoints P and Q, with $[P, Q)$ and $(P, Q]$ we denote the half-open segments, and with (P, Q) we denote the open segment.

2.1. **Preparatory results.** In the following proposition we state some known facts about solutions to the Steiner problem collected from [13] and [9].

Proposition 2.1 (known facts about minimizers)**.** *Let $C \subset \mathbb{R}^n$ be a compact set. Then $\mathcal{M}(C) \neq \emptyset$. If $S \in \mathcal{M}(C)$ and $\mathcal{H}^1(S) < +\infty$ then the following statements hold:*

(i) *$S \cup C$ is compact, and $\bar{S} \setminus S \subset C$;*
(ii) *the closure of every connected component of $S \setminus C$ is a topological tree (contains no subset homeomorphic to $\mathbb{S}^1$) with endpoints[1] on C and with at most one endpoint on each connected component of C;*
(iii) *$S \setminus C$ is a locally finite Steiner tree in the sense that for all $x \in S \setminus C$ and almost all $\rho < \operatorname{dist}(x, C)$ the set $S \cap \bar{B}_\rho(x)$ is the union of a finite number of line segments which meet in triple points with equal angles of 120 degrees and have endpoints in $\partial B_\rho(x)$.*

The following lemma is also taken from [13, proposition 2.2 and lemma 2.6].

Lemma 2.2. *If S is connected and $\mathcal{H}^1(S) < +\infty$ then $\mathcal{H}^1(S) = \mathcal{H}^1(\bar{S})$. If S is compact, connected, and $\mathcal{H}^1(S) < +\infty$ then S is pathwise connected.*

Lemma 2.3. *Let $C \subset \mathbb{R}^2$ be a compact set, $H \notin C$ a point, $C^* := C \cup \{H\}$. Let $S \in \mathcal{M}(C^*)$. Then there exists a point $Q \neq H$ such that $[H, Q] \subset \bar{S}$ and either $Q \in C$ or Q is a triple point of $S \setminus C^*$ and there are no other triple points of $S \setminus C^*$ on $[H, Q]$.*

Moreover if $C \subset \overline{B_\rho(P)}$ for some point P and radius $\rho > 0$, if $r \geq 2\rho/\sqrt{3}$, and $r < |P - H|$, then, there exists a point $T \in [H, Q] \cap \partial B_r(P)$ such that $\bar{S} \setminus B_r(P) = [H, T]$ (see Figure 2).

[1]When S is a compact topological tree, then $x \in S$ is an endpoint of S if and only if $S \setminus \{x\}$ is connected.

Proof. Since $H \notin C$ there exists $\varepsilon > 0$ such that $C \cap B_\varepsilon(H) = \emptyset$. Clearly $\mathcal{H}^1(S \cap B_\varepsilon(H)) < +\infty$ because otherwise we could find a better competitor to S by removing $B_\varepsilon(H)$ from S and adding the circle $\partial B_\varepsilon(H)$ and a single radius from H to a point on $\partial B_\varepsilon(H)$. This means that the connected components of $S \setminus C^*$ are a finite number because each of them has length at least ε in $B_\varepsilon(H)$. Let S_1 be a connected component of $S \setminus C^*$ such that $H \in \bar{S}_1$. Recall that, by Proposition 2.1(iii), S_1 is a locally finite Steiner tree composed by segments and triple points. We have two cases depending on whether S_1 contains a triple point.

If S_1 contains triple points then let Q be the triple point closest to H in the intrinsic (geodesic) distance on S_1. Such a point exists. In fact otherwise there would exist a sequence of triple points in $\bar{S}_1$ converging to H, because branching points can only accumulate on C^* by Proposition 2.1(ii). But this is not possible because in $B_\varepsilon(H)$ there are no triple points. Clearly the arc from H to Q in $\bar{S}_1$ is a line segment (by Proposition 2.1(iii)) hence $[H, Q] \subset \bar{S}$.

If S_1 contains no triple point then $\bar{S}_1 = [H, Q]$ for some Q, and, necessarily, $Q \in C$. In both cases the first claim of the lemma is proven.

For the second claim suppose that $|P - H| > r$. This means that the convex hull of $\{H\} \cup B_\rho(P)$ has an angle in H which is smaller than 120 degrees. The set S must be included in this convex hull and if T is any triple point of S then $T \in \bar{B}_r(P)$ because otherwise at least one of the three segments meeting in T would have a direction which is not going to meet the ball $\bar{B}_\rho(P)$ and is not going towards H. Along this segment one could hence find a further triple point T' and repeat the reasoning finding infinitely many triple points. Since any sequence of triple points must converge to a point of C^*, this is impossible.

We conclude that $S \setminus \bar{B}_r(P)$ does not contain triple points and hence is composed by a disjoint union of segments each of which has an endpoint in H and an endpoint in $\partial B_r(P)$. But we can have at most one such segment because otherwise we would have, in H, an angle smaller than 120 degrees, which is not possible. This means that $\bar{S} \setminus B_r(P)$ is contained in the segment $[H, T]$ and hence is equal to a segment $[H, Q]$ with $Q \in \partial B_r(P)$. □

Lemma 2.4. *Let C be a compact subset of $\mathbb{R}^2$ and ℓ be a line. Then $\mathcal{M}(\ell \cup C) \neq \emptyset$. If C is also totally disconnected, $C \cap \ell = \emptyset$ and $S \in \mathcal{M}(\ell \cup C)$ then $\bar{S} \supset C$. If moreover $\mathcal{H}^1(S) < +\infty$ and $\mathcal{H}^1(C) = 0$ then $\bar{S} \in \mathcal{M}(\ell \cup C)$.*

Proof. Take a compact connected segment $\sigma \subset \ell$ such that the orthogonal projection of C on ℓ is contained in σ. Then $\sigma \cup C$ is compact and by [13] we know that $\mathcal{M}(\sigma \cup C) \neq \emptyset$. For all $\Sigma \in \mathrm{St}(\ell \cup C)$ we have that its projection $\tilde{\Sigma}$ onto the convex hull of $\sigma \cup C$ is in $\mathrm{St}(\sigma \cup C)$ and $\mathcal{H}^1(\tilde{\Sigma}) \leq \mathcal{H}^1(\Sigma)$. Hence $\mathcal{M}(\ell \cup C) \supset \mathcal{M}(\sigma \cup C) \neq \emptyset$ as claimed.

To prove the second part note that if C is totally disconnected then for every $x \in C$ one has $x \in \bar{S}$ otherwise $S \cup C \cup \ell$ cannot be connected. Therefore $C \subset \bar{S}$.

For the last claim notice that $\bar{S} \cap \ell$ is a finite set. In fact the number of connected components of S arriving at ℓ is finite since every such component must touch also points of C and hence has length at least $\mathrm{dist}(C, \ell)$. Hence these components cannot be infinitely many, since otherwise we would have $\mathcal{H}^1(S) = +\infty$. In view of Proposition 2.1(ii) the closure of every connected component of $S \in \mathcal{M}(C \cup \sigma)$ has at most one point on σ, showing that the set $W := \bar{S} \cap \sigma = \bar{S} \cap \ell$ is finite.

Moreover $\bar S \setminus S \subset C \cup W$ because $S \cup C \cup \sigma$ is compact by Proposition 2.1(i). When $\mathcal{H}^1(C) = 0$ this implies $\mathcal{H}^1(\bar S) = \mathcal{H}^1(S)$ and hence $\bar S \in \mathcal{M}(\ell \cup C)$. □

Lemma 2.5. *Let ℓ be a line, $\rho > 0$, $r = 2\rho/\sqrt{3}$. Let P be a point such that $\operatorname{dist}(P, \ell) > 3r$ and let C be a compact subset of $\bar B_\rho(P)$. Let $S \in \mathcal{M}(\ell \cup C)$ (such S exists in view of Lemma 2.4) and suppose $\mathcal{H}^1(S) < +\infty$. Then there exists a point $Q \in \partial B_r(P)$ and a point $H \in \ell$ such that $\bar S \setminus B_r(P) = [H, Q]$. Moreover $[H, Q]$ is perpendicular to ℓ.*

Proof. We claim that $S \setminus B_r(P)$ belongs to a single connected component of S. In fact, if not, there are two different connected components S_1 and S_2 of S which have some points outside of $B_r(P)$. Their closures must have points in ℓ otherwise they would be contained in the convex hull of $C \subset B_r(P)$. By Proposition 2.1(ii) we know that $\bar S_1$ and $\bar S_2$ each have at most one endpoint on ℓ (hence exactly one). Let H_1 and H_2 be such endpoints. Note that $S_j \in \mathcal{M}(\{H_j\} \cup A_j)$ with $A_j := \bar S_j \cap C$, $j = 1, 2$, compact subsets of C. By Lemma 2.3 below with A_j and H_j in place of C and H respectively, one has that the set $S_j \setminus B_r(P)$ is a line segment $[H_j, T_j]$ with $T_j \in \partial B_r(P)$, $j = 1, 2$. Since $\bar S_1 \cap \bar S_2 \subset \ell \cup C$ the line segments $[H_1, T_1]$ and $[H_2, T_2]$ may only intersect at their endpoints. Let $S' := (S \setminus [H_2, T_2]) \cup [T_1, T_2]$ so that $S' \in \mathrm{St}(\ell \cup C)$ and

$$\mathcal{H}^1(S) - \mathcal{H}^1(S') = |H_2 - T_2| - |T_1 - T_2| \geq (d(P, \ell) - r) - 2r > (3r - r) - 2r = 0.$$

Hence we would have $\mathcal{H}^1(S') < \mathcal{H}^1(S)$ contrary to the minimality of S.

Having proven that $S \setminus B_r(P)$ belongs to a single connected component of S, denoting by H its endpoint on ℓ, and applying Lemma 2.3, we have that $S \setminus B_r(P)$ is a line segment $[H, T]$. Finally, if $[H, T]$ were not perpendicular to ℓ by moving H along ℓ we could decrease the length of $[H, T]$ contrary to the minimality of S. □

Lemma 2.6. *Let ν_1, ν_2, ν_3 be three unit vectors in $\mathbb{R}^2$ with $\nu_1 + \nu_2 + \nu_3 = 0$. Let Y_j be a line perpendicular to ν_j for $j = 1, 2, 3$. Let T be any point in $\mathbb{R}^2$ and let H_j be the orthogonal projection of T on Y_j. Define $d_j(T) := (T - H_j) \cdot \nu_j$ be the signed distance of T from Y_j. Then $d_1(T) + d_2(T) + d_3(T)$ is constant (i.e. independent of T).*

In particular suppose that Y_1, Y_2, Y_3 are three lines in $\mathbb{R}^2$ forming angles of 60 degrees so that the three pairwise intersections identify an equilateral triangle with side length ℓ. Let T be any point and let $d_i(T)$ be the signed distance of T from Y_i with positive sign when T is inside the triangle. Then $d_1(T) + d_2(T) + d_3(T) = \frac{\sqrt{3}}{2}\ell$.

Proof. If we move one of the lines parallel to itself by an amount δ then the sum $d_1(T) + d_2(T) + d_3(T)$ changes by δ, independent of T. Therefore, without loss of generality, we may assume that the three lines intersect in the origin. In this case $d_j(T) = T \cdot \nu_j$ and hence $d_1(T) + d_2(T) + d_3(T) = T \cdot (\nu_1 + \nu_2 + \nu_3) = 0$ concluding the proof of the first claim.

For the second claim just notice that $d_1(T) + d_2(T) + d_3(T)$ is constant (by the first claim) and hence is equal to the value obtained when T is one of vertices of the triangle, in which case two distances are 0 and the third one is equal to the height of the triangle, which is the claimed value. □

Lemma 2.7. *Let $S \in \mathcal{M}(\{T\} \cup C)$ for some $T \in \mathbb{R}^2$ and a compact set C contained in a horizontal strip $E = \{(x, y) \colon |y| \leq \delta\}$ of width $2\delta > 0$. Suppose also that there is some $T' \in S$, $T' \neq T$ such that $[T, T']$ is horizontal. Then $T \in E$ and $S \subset E$.*

Proof. Suppose by contradiction that $T = (x_T, y_T)$ is outside E. For example $y_T > \delta$. Then the convex hull of $\{T\} \cup C$ has a single point, which is T, on the line $\{y = y_T\}$. This is in contradiction with the fact that $T' \in S$ has the same y-coordinate as T. This proves $T \in E$ and hence shows that the whole convex hull of $\{T\} \cup C$ is contained in E. Therefore, S must be contained in E as well. □

Lemma 2.8. *Let $v_1, v_2, v_3 \in \mathbb{R}^2$ be the three vertices of an equilateral triangle of side length ℓ. Take $\rho \leq \frac{\ell}{51}$. Let $U = B_\rho(v_1) \cup B_\rho(v_2) \cup B_\rho(v_3)$ and let $A \subset U$ be a compact set such that $A_k = A \cap B_\rho(v_k) \neq \emptyset$ for $k = 1, 2, 3$.*

Then given any $S \in \mathcal{M}(A)$ there exists a point T such that $S = S_1 \cup S_2 \cup S_3$ with $S_k \in \mathcal{M}(A_k \cup \{T\})$. Moreover $T \in \bar{B}_\rho(O)$ where O is the center of the triangle.

Proof. Let C be the convex envelope of U. Since A is a compact subset of C, given any $S \in \mathcal{M}(A)$ we have that S does not touch ∂C and in particular for all $k = 1, 2, 3$, one has, $S \cap \partial B_\rho(v_k) \subset \partial B_\rho(v_k) \cap C$ which is an arc of length $4\pi\rho/3$. So, if we define W_k to be the point on $\partial B_\rho(v_k)$ closest to O, and define

$$S' = (S \cap U) \cup \bigcup_{k=1}^{3} [O, W_k] \cup \overline{\partial B_\rho(v_k) \cap C}$$

we clearly have $S' \in \mathrm{St}(A)$, hence

$$\mathcal{H}^1(S) \leq \mathcal{H}^1(S') = \mathcal{H}^1(S \cap U) + 3 \cdot \left(\frac{\ell}{\sqrt{3}} - \rho + \frac{4}{3}\pi\rho \right) \tag{1}$$

$$= \mathcal{H}^1(S \cap U) + \sqrt{3}\ell + (4\pi - 3)\rho \tag{2}$$

We claim that $S \setminus \bar{U}$ contains a branching point. Otherwise, it would contain two disjoint paths each one connecting two of the three circles composing ∂U, hence, we would have

$$\mathcal{H}^1(S \setminus \bar{U}) \geq 2(\ell - 2\rho).$$

Using (1) we would have

$$2(\ell - 2\rho) \leq \mathcal{H}^1(S) - \mathcal{H}^1(S \cap U) \leq \sqrt{3}\ell + (4\pi - 3)\rho$$

which gives is $\frac{\rho}{\ell} \geq \frac{2-\sqrt{3}}{4\pi+1} > \frac{1}{51}$, against our assumptions.

We conclude that S' contains a branching point T. Let S_1, S_2, S_3 be the three connected components of $S \setminus \{T\}$. It is clear that, up to renumbering, one has $S_k \in \mathrm{St}(A_k \cup \{T\})$. □

3. Main results

Let $Y := \left\{(x, y) \in \mathbb{R}^2 \colon x = 0\right\}$ be the y-axis, and $Y_j := f_j(Y)$ for $j = 1, 2$. Consider the point $P := (1 + \frac{\lambda}{2}, 0)$. Clearly $\Sigma \in \mathrm{St}(Y \cup A)$. We need the following lemmata.

Lemma 3.1. *For every $d \in \mathbb{R}$ the family $\mathcal{M}(\{x = d\} \cup A)$ is nonempty. If $\lambda < \frac{1}{2}$ one has $\mathcal{H}^1(\Sigma) < +\infty$, $\mathcal{H}^1(A) = 0$ hence A is totally disconnected. Moreover for any $S \in \mathcal{M}(\{x = d\} \cup A)$ one has that $\mathcal{H}^1(S) < \infty$, $\bar{S} \in \mathcal{M}(\{x = d\} \cup A)$ and $\bar{S}$ contains A.*

Proof. The family of minimizers is nonempty in view of Lemma 2.4.

We claim that since $\lambda < \frac{1}{2}$ then $\mathcal{H}^1(A) = 0$. Since $A \subset \Sigma$ and $\mathcal{H}^1(\Sigma) = \frac{1}{1-2\lambda}$ is finite if $\lambda < \frac{1}{2}$ then $\mathcal{H}^1(A) < +\infty$. By the self-similarity property of A we have that

$$\mathcal{H}^1(A) = 2\lambda\mathcal{H}^1(A).$$

If $0 < \mathcal{H}^1(A) < +\infty$ we would have $\lambda = \frac{1}{2}$. Hence, if $\lambda < \frac{1}{2}$ we conclude that $\mathcal{H}^1(A) = 0$. This also means that A is totally disconnected because each connected component of A is pathwise connected (by Lemma 2.2) and hence if it were not a single point it would have positive length.

Now the conclusion follows from Lemma 2.4 with $C := A$ minding that

$$\mathcal{H}^1(S) \le \mathcal{H}^1(\Sigma) = \frac{1}{1-2\lambda} < +\infty.$$

□

Lemma 3.2. *Let $T_0 = (1, 0)$ and A be defined as above, $\lambda \le \frac{1}{8}$, $\rho = \frac{\lambda}{1-\lambda}$, $r = \frac{2\rho}{\sqrt{3}}$ Then $A \subset B_\rho(T_0)$ and given any $d \le \frac{1}{2}$ and any $S \in \mathcal{M}(\{x = d\} \cup A)$ there exists $T \in \partial B_r(T_0)$ such that we have that $S \setminus B_r(T_0)$ is a line segment $[H, T]$ perpendicular to $\{x = d\}$ with $H \in \{x = d\}$. In particular $\bar{S}$ is connected.*

Proof. The claim $A \subset B_\rho(T_0)$ can be easily checked by noticing that any point of A has distance from T_0 smaller than

$$\lambda + \lambda^2 + \lambda^3 + \cdots = \rho.$$

Lemma 2.5 with $C := A$, $P := T_0$, $\ell := \{x = d\}$ can be applied since the assumption $\lambda \le 1/8$ assures that $3r < 1/2 \le 1 - d = \operatorname{dist}(T_0, \{x = d\})$. So the point T exists with the desired properties and $\bar{S} \cap \{x = d\} = \{H\}$. Since, by Lemma 3.1, $\bar{S} \supset A$ it follows that $\bar{S} \supset \{H\} \cup A$ which is totally disconnected and hence $\bar{S}$ must be connected. □

Lemma 3.3. *One has*

$$\operatorname{dist}(Y, f_j(A)) = \operatorname{dist}(Y, A) \ge 1 + \frac{\lambda}{2} - \frac{\lambda^2(1+2\lambda)}{2(1-\lambda^2)}, \qquad j = 1, 2,$$

$$\operatorname{dist}(f_1(A), f_2(A)) \ge \sqrt{3}\lambda - \sqrt{3}\frac{\lambda^3}{1-\lambda}.$$

Proof. If we follow the tree Σ starting from the root at the origin and trying to get as close as possible to the line Y we first must follow the trunk of length 1 then at the first branching nothing changes if we turn left or right and then we go further along the oblique branch of length λ so that the distance from Y increases by $\lambda/2$. After that the path leading towards Y is made of alternating branches with horizontal and oblique directions. The computation gives

$$\begin{aligned}
\operatorname{dist}(Y, f_j(A)) &\ge 1 + \frac{\lambda}{2} - \left(\frac{\lambda^2}{2} + \lambda^3 + \frac{\lambda^4}{2} + \lambda^5 + \frac{\lambda^6}{4} + \dots\right) \\
&= 1 + \frac{\lambda}{2} - \frac{1}{2}\frac{\lambda^2}{1-\lambda^2} - \frac{\lambda^3}{1-\lambda^2} \\
&= 1 + \frac{\lambda}{2} - \frac{\lambda^2(1+2\lambda)}{2(1-\lambda^2)}.
\end{aligned}$$

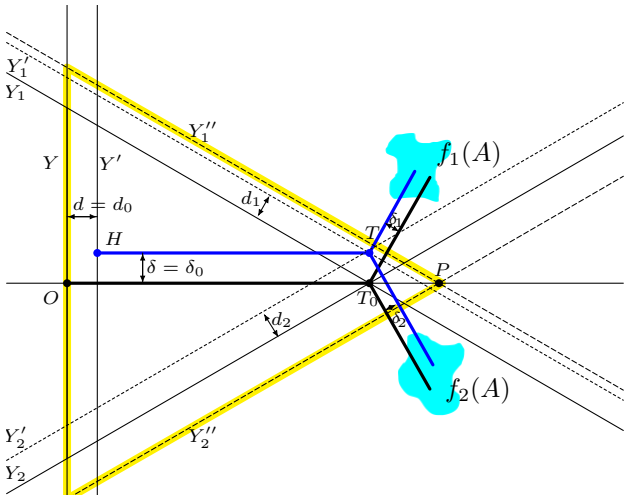


FIGURE 3. Constructions and notation in the proof of Lemma 3.5 and Theorem 3.6. The triangle Δ from the proof of Lemma 3.5 is highlighted.

Analogously,

$$\begin{aligned}\operatorname{dist}(f_1(A), f_2(A)) &\geq 2\operatorname{dist}(f_1(A), \{y=0\}) \\ &\geq 2\left(\frac{\sqrt{3}}{2}\lambda - \frac{\sqrt{3}}{2}\left(\lambda^3+\lambda^4+\dots\right)\right) = \sqrt{3}\lambda - \sqrt{3}\frac{\lambda^3}{1-\lambda},\end{aligned}$$

as claimed. □

Remark 3.4. Notice that if the right hand side of the inequalities in the claim of Lemma 3.3 are positive then in fact those inequalities become equalities as it is easy to see from the proof. This happens for λ sufficiently small, for instance when $\lambda < \frac{1}{2}$.

Lemma 3.5. *Let $Y' := \{x = d\}$ for some $d \leq \frac{1}{2}$, be a line parallel to $Y = \{x = 0\}$. If $S \in \mathcal{M}(Y' \cup A)$ and $\lambda < \frac{1}{25}$ then there is a branching point $T \in S$ such that $S \setminus \{T\}$ is composed of three connected sets, the closures of which are $[H, T]$, S_1 and S_2 respectively, with $H \in Y'$ and $[H, T]$ perpendicular to Y'. Moreover,*

$$\operatorname{dist}(H, \{y=0\}) \leq 2\lambda\rho \qquad \text{and} \qquad T \in B_{2\lambda\rho}(T_0), \tag{3}$$

where $\rho := \frac{\lambda}{1-\lambda}$ and $T_0 := (1, 0)$. Finally if Y_j' is the line parallel to $Y_j := f_j(Y)$ passing through T then $S_j \in \mathcal{M}(Y_j' \cup f_j(A))$ and $Y_j' \subset f_j(\{x \leq \frac{1}{2}\})$, for $j = 1, 2$.

Proof. Step 1. We first claim that $\bar{S}$ cannot contain two compact connected sets σ and γ with $\mathcal{H}^1(\sigma \cap \gamma) = 0$ such that σ touches both Y' and $f_1(A)$ while γ touches both $f_1(A)$ and $f_2(A)$. In fact, otherwise we would have

$$\mathcal{H}^1(\bar{S}) \geq \mathcal{H}^1(\sigma) + \mathcal{H}^1(\gamma) \geq \operatorname{dist}(Y', f_1(A)) + \operatorname{dist}(f_1(A), f_2(A)),$$

but $\mathrm{dist}(Y', f_1(A)) \geq \mathrm{dist}(Y, f_1(A)) - d$, hence, by Lemma 3.3, we would get

$$\mathcal{H}^1(\bar S) \geq 1 - d + \frac{\lambda}{2} - \frac{\lambda^2(1+2\lambda)}{2(1-\lambda^2)} + \sqrt{3}\lambda - \sqrt{3}\frac{\lambda^3}{1-\lambda}. \tag{4}$$

Let $O' = (d, 0)$ and $\Sigma' := [O', T_0] \cup f_1(\Sigma) \cup f_2(\Sigma)$ be the tree obtained by adding or removing a segment of length $|d|$ from Σ so that $\Sigma' \in \mathrm{St}(Y' \cup A)$. We have

$$\mathcal{H}^1(\Sigma') = \mathcal{H}^1(\Sigma) - d = 1 - d + 2\lambda + 4\lambda^2 + \cdots = \frac{1}{1-2\lambda} - d \tag{5}$$

and one can check that the rhs of (4) is strictly greater than the rhs of (5) for $\lambda < \frac{1}{25}$. Hence $\mathcal{H}^1(S) = \mathcal{H}^1(\bar S) > \mathcal{H}^1(\Sigma')$ contrary to the optimality of S.

Step 2. By Proposition 2.1(ii) we know that S touches Y' in a single point H. Recall that $\bar S$ is connected in view of Lemma 3.2. Consider an injective arc $\theta\colon [0,1] \to \bar S$ such that $\theta(0) = H$, $\theta([0,1)) \cap A = \emptyset$ and $\theta(1) \in A$. Let

$$t := \sup\big\{s \in [0,1]\colon \bar S \setminus \theta([0,s]) \text{ is connected}\big\}$$

and let $T := \theta(t)$, $S_0 := \theta([0,t])$. Observe that $(\bar S \setminus S_0) \cup \{T\}$ is compact and arcwise connected. In fact for every pair of points $P, Q \in (\bar S \setminus S_0) \cup \{T\}$ there exists a unique injective arc Γ in $\bar S$ connecting P and Q (this follows by Proposition 2.1(ii) since $\bar S$ is connected and having finite length is an arcwise connected topological tree). It is enough to note now that $\Gamma \cap \theta([0,s]) = \emptyset$ for all $s < t$ and hence $\Gamma \subset \bar S \setminus \theta([0,t)) = (S \setminus S_0) \cup \{T\}$.

Step 3. We are going to define S_1 and S_2 closed connected subsets of $\bar S$ such that $\bar S = S_0 \cup S_1 \cup S_2$ and S_j, $j = 0, 1, 2$, have T as the only common point, and moreover, S_j contains $f_j(A)$ for $j = 1, 2$. To this aim we are going to consider two cases which will be excluded.

Case 1. Suppose T is a point of A. Without loss of generality suppose $T \in f_1(A)$. The set $(\bar S \setminus S_0) \cup \{T\}$ is connected and contains points of both $f_1(A)$ and $f_2(A)$ because $(S_0 \setminus \{T\}) \cap A = \emptyset$ by construction hence there exists an arc γ in $(\bar S \setminus S_0) \cup \{T\}$. The claim of Step 1 with $\sigma := S_0$ implies that this cannot happen.

Case 2. Suppose $T \notin A$. In this case S is a finite tree in a neighbourhood of T by Proposition 2.1(iii) and therefore T is a triple point of S. Hence $\bar S \setminus S_0$ has two connected components S_1' and S_2' (recall that $\bar S$ contains no loops). Let $S_j := S_j' \cup \{T\}$, $j = 1, 2$. Each S_j must contain at least one point of A because otherwise $\bar S \setminus S_j' \in \mathrm{St}(Y \cup A)$ will be strictly shorter than $S \in \mathcal{M}(Y \cup A)$. Moreover each point of A is contained in either S_1 or S_2. Without loss of generality suppose that S_1 contains at least one point of $f_1(A)$.

Case 2a. Suppose that S_1 contains also points of $f_2(A)$. Then we can apply the claim of Step 1 with $\sigma := S_0 \cup S_2$ and $\gamma := S_1$ and exclude that this case can happen.

Case 2b. If S_2 contains points of both $f_1(A)$ and $f_2(A)$ we proceed as in Case 2a with S_1 and S_2 interchanged.

The only remaining possibility is that S_1 only touches points of $f_1(A)$ and S_2 only touches points of $f_2(A)$. Hence $S_1 \supset f_1(A)$ and $S_2 \supset f_2(A)$ since $A \subset \bar S$. Therefore, $S_0 \in \mathcal{M}(Y' \cup \{T\})$ and $S_j \in \mathcal{M}(\{T\} \cup f_j(A))$ for $j = 1, 2$: otherwise, by substituting S_0 with an element of $\mathcal{M}(Y' \cup \{T\})$ and S_j with an element of $\mathcal{M}(\{T\} \cup f_j(A))$, we could construct a better competitor to $S \in \mathrm{St}(Y' \cup A)$.

Step 4. Clearly $S_0 \in \mathcal{M}(Y' \cup \{T\})$ implies that S_0 is the straight line segment $[H, T]$ perpendicular to Y'.

Step 5. By Lemma 3.2 we have $A \subset B_\rho(T_0)$, with $\rho = \frac{\lambda}{1-\lambda}$ hence $f_j(A) \subset B_{\lambda\rho}(f_j(T_0))$ for $j = 1, 2$. As noticed in Step 3 we know that S is a regular tripod in a small neighbourhood of T composed by three line segments forming equal angles of 120 degrees. Since S_0 is perpendicular to Y it follows that S_j contains a small segment perpendicular to $Y_j = f_j(Y)$. Since $S_j \in \mathcal{M}(\{T\} \cup f_j(A))$ for $j = 1, 2$, by Lemma 2.7 (applied to S_j in place of S, $f(A_j)$ in place of C and coordinate y along the axis Y_j and x perpendicular to Y_j), we obtain that S_j is contained in the strip perpendicular to Y_j, centered in T_0 and containing $f_j(A)$. Since $f_j(A) \subset B_{\lambda\rho}(f_j(T_0))$ such a strip has width $2\lambda\rho$ (notice that $f_j(T_0)$ lies on the line passing through T_0 and perpendicular to Y_j).

The intersection between the two strips for $j = 1, 2$ is the union of two equilateral triangles each with height $2\lambda\rho$. Hence T is contained in the ball $B_{2\lambda\rho}(T_0)$, and thus, in particular,

$$\operatorname{dist}(H, \{y = 0\}) = \operatorname{dist}(T, \{y = 0\}) \le |TT_0| \le 2\lambda\rho,$$

proving (3). Moreover, also the distance of the line Y_j' from T_0 is less than $2\lambda\rho$ hence $Y_j' \subset f_j(\{x < 2\rho\}) \subset f_j(\{x < \frac{1}{2}\})$, for $j = 1, 2$, since $\lambda < \frac{1}{5}$.

Step 6. We claim that each S_j, $j = 1, 2$, has no branching points inside the triangle delimited by Y, Y_1 and Y_2 (to avoid confusion, notice that T is not a branching point of S_j).

If T is itself outside of this triangle there is nothing to prove because $S_j \subset \overline{\text{co}}(\{T\} \cup A_j)$ is also outside of the triangle. Otherwise, since $f_j(A) \subset B_{\lambda\rho}(f_j(P))$, then, by Lemma 2.5 all branching points of S_j are inside $B_{\frac{2\lambda\rho}{\sqrt{3}}}(f_j(T_0))$ while $d(f_j(T_0), Y_j) = \lambda d(T_0, Y) = \lambda > \frac{2\lambda\rho}{\sqrt{3}}$ since $\lambda < \frac{3}{2\sqrt{3}+3} \approx 0.46$.

Step 7. Let Y_j'' be the line parallel to Y_j and passing through P. Denote with Δ the triangle delimited by Y, Y_1'' and Y_2'', and notice that T is inside Δ because $T \in B_{\lambda^2}(T_0)$ (as proven in Step 5) while $\operatorname{dist}(T_0, Y_j'') = \frac{\lambda}{4} > 2\lambda\rho$ for $j = 1, 2$. Let T_j'' be the point on Y_j'' such that $[T, T_j'']$ is perpendicular to Y_j. Recall that S_j in a neighbourhood of T is a segment perpendicular to Y_j because S, in a neighbourhood of T, is a regular tripod with equal angles of 120 degrees and S_0 is perpendicular to Y. We claim that $[T, T_j'']$ is contained in S_j: otherwise there would be a branching point of S_j on such segment, which is not the case because, as already stated, all branching points of S_j are inside the ball $B_{\frac{2\lambda\rho}{\sqrt{3}}}(f_j(T_0))$, while

$$\begin{aligned}
\operatorname{dist}(f_j(A), Y_j'') &\ge \operatorname{dist}(f_j(A), Y_j) - \operatorname{dist}(Y_j, Y_j'') \ge \lambda \operatorname{dist}(A, Y) - \frac{\lambda}{4} \\
&\ge \lambda\left(1 + \frac{\lambda}{2} + \frac{\lambda^2}{1-\lambda}\right) - \frac{\lambda}{4} \\
&\ge \frac{3}{4}\lambda > \frac{2}{\sqrt{3}}\lambda\rho \qquad \text{for } \lambda < \frac{9}{8\sqrt{3}+9} \approx 0.39.
\end{aligned}$$

Hence $\tilde{S}_j := S_j \setminus [T, T_j'']$ is connected and $\mathcal{H}^1(\tilde{S}_j) = \mathcal{H}^1(S_j) - \operatorname{dist}(T, Y_j'')$.

Step 8: conclusion. We now show the last claim, $S_j \in \mathcal{M}(Y_j' \cup f_j(A))$, $j = 1, 2$. To this aim take any $S_j'' \in \mathcal{M}(Y_j'' \cup f_j(A))$ and let H_j'' be the endpoint of S_j'' on the line Y_j'' (S_j'' touches Y_j'' in a single point in view of Lemma 2.3). Note that H_j'' is on the boundary of the triangle Δ in view of Lemma 2.7 because the orthogonal projection of $f_j(A)$ onto Y_j'' belongs to the respective side of this triangle. Define

$S'_j := S''_j \cup [H''_j, H'_j]$ where H'_j is the projection of H''_j onto the line Y'_j, $j = 1, 2$ and let $\Gamma \in \mathcal{M}(Y \cup \{H''_1, H''_2\})$ be the regular tripod with endpoints in H''_1, H''_2 and a third point on the side of Δ contained in Y. Clearly $S'_j \in \mathcal{M}(Y'_j \cup f_j(A))$ and $\mathcal{H}^1(S'_j) = \mathcal{H}^1(S''_j) + \left|H_j H''_j\right|$ while $\mathcal{H}^1(\Gamma) = 1 + \frac{\lambda}{2}$ in view of Lemma 2.6.

Therefore, if we consider $\tilde{S} := \Gamma \cup S''_1 \cup S''_1$ we have $\tilde{S} \in \mathrm{St}(Y \cup A)$ hence $\mathcal{H}^1(\tilde{S}) \geq \mathcal{H}^1(S)$. On the other hand, we have

$$\begin{aligned}
\mathcal{H}^1(\tilde{S}) &= \mathcal{H}^1(\Gamma) + \mathcal{H}^1(S''_1) + \mathcal{H}^1(S''_2) \\
&\leq 1 + \frac{\lambda}{2} + \mathcal{H}^1(S'_1) - |H'_1 H''_1| + \mathcal{H}^1(S'_2) - |H'_2 H''_2| \\
&\leq 1 + \frac{\lambda}{2} + \mathcal{H}^1(S_1) - |H'_1 H''_1| + \mathcal{H}^1(S_2) - |H'_2 H''_2| \\
&= 1 + \frac{\lambda}{2} + \mathcal{H}^1(S_1) - |T T''_1| + \mathcal{H}^1(S_2) - |T T''_2|
\end{aligned}$$

but $\mathcal{H}^1(S_0) + |T T''_1| + |T T''_2| = 1 + \frac{\lambda}{2}$ by Lemma 2.6 and hence

$$\mathcal{H}^1(\tilde{S}) \leq \mathcal{H}^1(S_0) + \mathcal{H}^1(S_1) + \mathcal{H}^1(S_2) = \mathcal{H}^1(S).$$

Hence, all the above inequalities are, in fact, equalities, which means in particular that $\mathcal{H}^1(S'_j) = \mathcal{H}^1(S_j)$ and thus $S_j \in \mathcal{M}(Y'_j \cup f_j(A))$, $j = 1, 2$ as claimed. □

Finally we are in the position to prove the minimality of the branching tree Σ with a fixed root, or, more generally, with the root moving along the line Y.

Theorem 3.6. *Suppose $\lambda < \frac{1}{25}$. Then one has $\Sigma \in \mathcal{M}(\{0\} \cup A) = \mathcal{M}(Y \cup A)$. Moreover $S \in \mathcal{M}(Y \cup A)$ if and only if $\Sigma \setminus (\{0\} \cup A) \subset S \subset \Sigma$, which in particular implies $\bar{S} = \Sigma$.*

Proof. By Lemma 3.1 we know that $\mathcal{M}(\{x = d\} \cup A)$ is nonempty for every $d \in \mathbb{R}$ and that for any $S \in \mathcal{M}(\{x = d\} \cup A)$ one has that $\bar{S} \in \mathcal{M}(\{x = d\} \cup A)$ and $\bar{S} \supset A$. Moreover, by Lemma 3.2, $\bar{S}$ is connected. Thus we may consider an arbitrary closed $S \in \mathcal{M}(\{x = d\} \cup A)$. If $d < \frac{1}{2}$, by Lemma 3.5, we are able to define S_1 and S_2, H, T such that $S = [H, T] \cup S_1 \cup S_2$, $[H, T]$ is perpendicular to Y. If we define the sets $b_1(S)$ and $b_2(S)$ (the "*branches*" of S) by $b_j(S) := f_j^{-1}(S_j)$, $j = 1, 2$. We notice that $b_j(S) \in \mathcal{M}(\{x = d_j\} \cup A)$ for some $d_j < \frac{1}{2}$, as claimed in the Lemma 3.5, $j = 1, 2$. Let $d(S)$ and $\delta(S)$ be the two coordinates of the point H so that $d(S)$ is the distance of H from the line Y and $\delta(S)$ is the distance of the point H (or, which is the same, the point T) from the line $X := \{y = 0\}$. Define $\mathcal{S}^0 := \left\{\tilde{S}\right\}$ where $\tilde{S} \in \mathcal{M}(\{x = 0\} \cup A)$ is fixed, compact set, and define inductively $\mathcal{S}^k$ as the family of the rescaled branches of $\tilde{S}$ at level k:

$$\mathcal{S}^{k+1} := \bigcup_{S \in \mathcal{S}^k} \{b_1(S), b_2(S)\}.$$

Notice that Lemma 3.5 can be applied to $\tilde{S}$ and hence inductively on the two rescaled branches $b_1(S)$ and $b_2(S)$ for every $S \in \mathcal{S}^k$ since the properties $S \in \mathcal{M}(\{x = d(S)\} \cup A)$ and $d(S) < \frac{1}{2}$ are preserved by the operators b_1 and b_2 as stated in Lemma 3.5 itself.

We claim that for all $S \in \mathcal{S}^k$, one has

$$\delta(S) \leq 2\lambda \max\{\delta(b_1(S)), \delta(b_2(S))\}.$$

We know that $S \in \mathcal{S}^k$ touches the vertical line $\{x = d(S)\}$ in a single point H. By Lemma 3.5 we know that S is composed by a segment $[H, T]$ and two trees S_1, S_2 with $S_j \in \mathcal{M}(Y_j' \cup f_j(A))$, $j = 1, 2$ where Y_j' is the line passing through T and parallel to Y_j. Let δ_j be the distance of T from $f_j(X)$, $j = 1, 2$. By Lemma 3.5 one has $\delta_j \leq 2\lambda\rho$. By Lemma 2.6 applied to the three lines X, $f_1(X)$ and $f_2(X)$ (all lines passing through T_0) we know that the sum of the signed distances of T from the three lines passing through T_0 is equal to 0. The absolute values of these three distances are, respectively, $\delta(S)$, δ_1 and δ_2, so that

$$\begin{aligned}\delta(S) &\leq \delta_1 + \delta_2 \leq 2\max\{\delta_1, \delta_2\} \\ &= 2\max\{\lambda\delta(b_1(S)), \lambda\delta(b_2(S))\}\end{aligned}$$

because $b_j(S) = f_j^{-1}(S_j)$, showing the claim.

The proven claim now gives

$$\Delta_k := \max\left\{\delta(S) \colon S \in \mathcal{S}^k\right\} \leq 2\lambda \max\left\{\delta(S) \colon S \in \mathcal{S}^{k+1}\right\} = 2\lambda\Delta_{k+1}.$$

We have $\Delta_0 = 0$ since otherwise for $\Delta_0 > 0$ we would have $\Delta_k \to +\infty$ (because $2\lambda < 1$), while we know, by Lemma 3.5, that $\Delta_k \leq 2\lambda\rho$ for all k. But $\Delta_0 = 0$ implies $\Delta_k = 0$ for all k. In particular for $d = 0$ this implies $\tilde{S} = \Sigma$ (recall that $\tilde{S}$ has been choosen to be closed).

Note that we have proven that for any arbitrary $S \in \mathcal{M}(Y \cup A)$ one has $\bar{S} = \Sigma$ and in particular $S \in \mathcal{M}(\{0\} \cup A)$. But we also have that $\bar{S} \setminus S \subset \{0\} \cup A$ by Proposition 2.1(i), that is $\Sigma \setminus S \subset \{0\} \cup A$. This implies that $S \supset \Sigma \setminus (\{0\} \cup A)$.

It remains to prove that $\mathcal{M}(\{0\} \cup A) = \mathcal{M}(Y \cup A)$. To this aim note that we have already proven that $\mathcal{M}(Y \cup A) \subset \mathcal{M}(\{0\} \cup A)$ but for every $S \in \mathcal{M}(\{0\} \cup A)$ one necessarily has $S \in \mathrm{St}(Y \cup A)$ and $\mathcal{H}^1(S) = \mathcal{H}^1(\Sigma)$ while $\Sigma \in \mathcal{M}(Y \cup A)$ which implies the claim.

□

We can now prove the main result for the universal tree Σ_u.

Let A, Σ, A_u, Σ_u, be the sets defined in the introduction (see Figure 1) $A_u = A \cup \phi(A) \cup \phi^2(A)$ and $\Sigma_u = \Sigma \cup \phi(\Sigma) \cup \phi^2(\Sigma)$ where ϕ is the counter-clockwise rotation of angle of 120 degrees.

Theorem 3.7 (universal minimizer)**.** *Suppose $\lambda < \frac{1}{29}$. Then, $S \in \mathcal{M}(A_u)$ if and only if $S \cup A_u = \Sigma_u$.*

Proof. Let $v_1 = (1, 0) \in \mathbb{R}^2$, $v_2 = \phi(v_1)$, $v_3 = \phi^2(v_1)$ be the vertices of an equilateral triangle of side length $\ell = \sqrt{3}$ centered in the origin. Let $S \in \mathcal{M}(A_u)$. Let $\rho = \frac{\lambda}{1-\lambda}$, so that $A \subset B_\rho(v_1)$ as stated in Lemma 3.2. By Lemma 2.8 we know that there exists a point $T \in B_\rho(0)$ such that $S = S_1 \cup S_2 \cup S_3$ with $S_k \in \mathcal{M}(A_k \cup \{T\})$. Consider the triangle Δ with vertices $\left(\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right)$ and $(-1, 0)$. Let Y be the line with equation $x = \frac{1}{2}$. Clearly $\phi(\Delta) = \Delta$ and Δ has a side contained in the line Y. By Lemma 2.3 we know that $S_1 \setminus B_r(V_1)$ is a single line segment, for $r = \frac{2}{\sqrt{3}}\rho$, and hence $S_1 \cap \partial\Delta$ is composed by a single point, which we call $H_1 \in Y$. Let $S_1' := \overline{S_1 \setminus \Delta}$, and notice that S_1' is obtained from S_1 by removing the segment $[T, H_1)$. Hence, from $S_1 \in \mathcal{M}(A_1 \cup \{T\})$, we get $S_1' \in \mathcal{M}(A_1 \cup \{H_1\})$. Since

$H_1 \in \Delta$, we have also $S_1' \in \mathrm{St}(A_1 \cup \Delta)$. If we define $\Sigma_1' = \overline{\Sigma \setminus \Delta}$, by Theorem 3.6 we know that $\Sigma_1' \in \mathcal{M}(A_1 \cup Y)$ and hence $\Sigma_1' \in \mathcal{M}(A_1 \cup \Delta)$ because the projection of A_1 onto the line Y is contained in Δ. Hence $\mathcal{H}^1(\Sigma_1') \le \mathcal{H}^1(S_1')$. Repeat the reasoning for $k = 2, 3$ to define H_k, S_k' and Σ_k' as we did for $k = 1$. We obtain

$$\mathcal{H}^1(S) = \mathcal{H}^1(S \cap \Delta) + \sum_{k=1}^{3} \mathcal{H}^1(S_k'),$$
$$\mathcal{H}^1(\Sigma_u) = \mathcal{H}^1(\Sigma_u \cap \Delta) + \sum_{k=1}^{3} \mathcal{H}^1(\Sigma_k'),$$
$$\mathcal{H}^1(\Sigma_k') \le \mathcal{H}^1(S_k') \qquad \text{for } k = 1, 2, 3.$$

Now notice that $S \cap \Delta = [T, H_1] \cup [T, H_2] \cup [T, H_3]$ and hence, by Lemma 2.6, $\mathcal{H}^1(S \cap \Delta) \ge \mathcal{H}^1(\Sigma_u \cap \Delta)$, since $|H_1 - T| \ge \mathrm{dist}(T, Y)$, and analogously for $k = 2, 3$. Putting all together, we obtain $\mathcal{H}^1(S) \ge \mathcal{H}^1(\Sigma_u)$ which means that $\Sigma_u \in \mathcal{M}(A_u)$, and actually $H^1(S) = \mathcal{H}^1(\Sigma_u)$.

Since we have the equality $\mathcal{H}^1(S) = \mathcal{H}^1(\Sigma_u)$ the inequalities $\mathcal{H}^1(S \cap \Delta) \le \mathcal{H}^1(\Sigma_u \cap \Delta)$ and $\mathcal{H}^1(S_k') \ge \mathcal{H}^1(\Sigma_k')$, $k = 1, 2, 3$, must all be equalities. It means that $S_k' \in \mathcal{M}(A_k \cup \Delta)$ for $k = 1, 2, 3$, and, by Theorem 3.6, we might conclude that $S_k' \cup A_k = \Sigma_k'$. As a consequence we have also $S \cap \Delta = \Sigma_u \cap \Delta$ because the tripod $S \cap \Delta$ is uniquely determined by the points of S on the boundary of Δ. □

The Theorem below shows that, in fact, the constructed set Σ_u is a universal Steiner tree, in the sense that it contains homeomorphic copies of all Steiner trees.

Theorem 3.8. *Let Σ_u be the universal tree as defined in the introduction with such $\lambda > 0$ that $\Sigma_u \in \mathcal{M}(A_u)$. Then the following properties hold.*

(i) *Given any S connected subset of Σ_u, letting $D = \partial_{\Sigma_u}(S \setminus A_u)$ (the topological boundary of $S \setminus A_u$ relative to Σ_u), one has $S \in \mathcal{M}(D)$.*

(ii) *Given any abstract, possibly infinite, binary tree (i.e. a graph such that every pair of vertices has a unique finite path connecting them and all vertices have order at most 3) there exists a subset $S \subset \Sigma_u \setminus A_u$ such that S is homeomorphic to the topological space given by the abstract tree (and hence, by the previous point, $S \in \mathcal{M}(D)$ is a Steiner tree for the set of its endpoints D).*

(iii) *Let X be any metric space with the Heine-Borel property (for example $X = \mathbb{R}^d$), let $D \subset X$ be a compact set and $S \in \mathcal{M}(D)$ with $\mathcal{H}^1(S) < +\infty$. Then every connected component of $S \setminus D$ is homeomorphic to a subtree of Σ_u.*

Remark 3.9. Notice that the set Σ is also universal in the weaker sense of containing homeomerphic copies of every *finite* Steiner tree, but, for example, it does not contain a homeomorphic copy of Σ_u. This weaker form of universality has been used, e.g., in [1].

Proof of Theorem 3.8. Clearly $\mathcal{H}^1(S) \le \mathcal{H}^1(\Sigma_u) < +\infty$. To prove (i) take any set S' such that $S' \cup D$ is connected and $\mathcal{H}^1(S') < +\infty$. Consider

$$\Sigma' := (\Sigma_u \setminus S) \cup S' \cup D.$$

Step 1. We claim that Σ' is connected. After fixing $x_0 \in D$ we are going to show that for any $x \in \Sigma'$ there exists a connected set containing x and x_0 and contained

in $\Sigma' \cup D$. If $x \in S' \cup D$ then just take $S' \cup D$ and we are done. Otherwise

$$x \in \Sigma' \setminus (S' \cup D) = (\Sigma_u \setminus S) \setminus (S' \cup D) = \Sigma_u \setminus (S \cup S' \cup D),$$

in particular $x \notin S$ and hence $x \notin S \setminus A_u$. By Lemma 3.10 applied to the connected component C of $\Sigma_u \setminus (S \setminus A_u)$ containing x, since Σ_u is locally connected, we know that $\bar{C} \cap \partial(S \setminus A_u)$ is non empty. Hence there is $x_1 \in \bar{C} \cap \partial(S \setminus A_u)$ and $\bar{C}$ is connected. Therefore $\bar{C} \cup (S' \cup D)$ is a connected set containing both x and x_0, showing the claim.

Step 2. We claim that $\mathcal{H}^1(D) = 0$. In fact the points of D which are not elements of A_u are contained in one of the countably many segments which are the arcs of the tree $\Sigma_u \setminus A_u$. If one of these segments contains two points of D then S would be completely contained in the segment and hence these are the only two points of D. Otherwise the number of points of $D \setminus A_u$ is countable and hence $\mathcal{H}^1(D \setminus A_u) = 0$. On the other hand $\mathcal{H}^1(D \cap A_u) \leq \mathcal{H}^1(A_u) = 0$ which is true when $\lambda < \frac{1}{2}$ i.e. when Σ_u is a tree.

Step 3. It is easy to check that $A_u \subset \Sigma'$. Since Σ' is connected, we hence have that Σ' is in fact a competitor to the minimality of $\Sigma_u \in \mathcal{M}(A_u)$ and thus $\mathcal{H}^1(\Sigma') \leq \mathcal{H}^1(\Sigma_u)$. But since

$$\mathcal{H}^1(\Sigma') \leq \mathcal{H}^1(\Sigma_u) - \mathcal{H}^1(S) + \mathcal{H}^1(S') + \mathcal{H}^1(D)$$

we obtain $\mathcal{H}^1(S) \leq \mathcal{H}^1(S')$ which proves $S \in \mathcal{M}(D)$, concluding the proof of (i).

Step 4. For the proof of (ii) we take any abstract binary tree T and take an arbitrary vertex v of T. We map v to the point $(0,0) \in \Sigma_u$. Then we consider the $k \leq 3$ edges of T meeting in v and map them to k of the three segments of Σ_u meeting in $(0,0)$. We then proceed by induction on the graph distance of the vertices from v: once we have mapped all the vertices up to distance n, for each of these vertices we have at most two adjacent vertices at distance $n+1$ which we can map to one or both of the two adjacent vertices of the tree in Σ_u. Since both T and $\Sigma_u \setminus A_u$ are trees, the additional vertices discovered in the induction process were not already used. And since T is connected, all its vertices are reached by the induction.

Step 5. For the point (iii), given any compact set $D \subset \mathbb{R}^d$ and any $S \in \mathcal{M}(D)$, we recall that (by Proposition 2.1) every connected component S_0 of $S \setminus D$ is a locally finite binary tree. Hence, by (ii), S_0 is homeomorphic to a subset of Σ_u. □

Lemma 3.10. *Let X be a locally connected metric space, $A \subset X$, and C a connected component of A. Then $\partial C \subset \partial A$.*

Proof. Clearly $\partial C \subset \bar{C} \subset \bar{A}$. Take any $x \in \partial C$ and U any neighbourhood of x in X. Let $V \subset U$ be a connected neighbourhood of x. Since $C \cup \{x\}$ is connected then $C \cup V$ is also connected. But C is a maximal connected subset of A and therefore we conclude that $V \setminus A$ is non empty, hence $U \setminus A$ is non empty which means that $x \in \partial A$. □

Remark 3.11. Notice that if X is not locally connected, Lemma 3.10 is not true. In fact if X is the so called Knaster-Kuratowski fan, which is even connected, there exist $x \in X$ such that $A := X \setminus \{x\}$ is totally disconnected hence every connected component C of A is a singleton $C = \{c\}$. Hence $\partial C = \{c\}$ while $\partial A = \{x\}$ and clearly $c \neq x$.

References

[1] Mikhail Basok, Danila Cherkashin, Nikita Rastegaev, and Yana Teplitskaya. On uniqueness in Steiner problem. *International Mathematics Research Notices*, pages 8819–8838, 2024.

[2] Rainer E. Burkard and Tibor Dudás. Steiner minimum trees for equidistant points on two sides of an angle. *Acta Cybernetica*, 12(3):313–324, 1996.

[3] Rainer E. Burkard, Tibor Dudás, and Thomas Maier. Cut and patch Steiner trees for ladders. *Discrete Mathematics*, 161(1-3):53–61, 1996.

[4] Danila Cherkashin and Yana Teplitskaya. A self-similar infinite binary tree is a solution to the Steiner problem. *Fractal and Fractional*, 7(5), 2023.

[5] F. R. K. Chung and R. L. Graham. Steiner trees for ladders. *Annals of Discrete Mathematics*, 2:173–200, 1978.

[6] Ding-Zhu Du, Frank K. Hwang, and J. F. Weng. Steiner minimal trees for regular polygons. *Discrete & Computational Geometry*, 2:65–84, 1987.

[7] Michael R. Garey, Ronald L. Graham, and David S. Johnson. The complexity of computing Steiner minimal trees. *SIAM Journal on Applied Mathematics*, 32(4):835–859, 1977.

[8] John E. Hutchinson. Fractals and self-similarity. *Indiana Univ. Math. J.*, 30(5):713–747, 1981.

[9] A. O. Ivanov and A. A. Tuzhilin. *Minimal networks: the Steiner problem and its generalizations*. CRC Press, 1994.

[10] V. Jarník and M. Kössler. O minimálních grafech, obsahujících n daných bodů. *Časopis pro pěstovánímatematiky a fysiky*, 063(8):223–235, 1934.

[11] A. Marchese and A. Massaccesi. An optimal irrigation network with infinitely many branching points. *ESAIM Control Optim. Calc. Var.*, 22(2):543–561, 2016.

[12] A. Marchese and A. Massaccesi. The Steiner tree problem revisited through rectifiable G-currents. *Adv. Calc. Var.*, 9(1):19–39, 2016.

[13] E. Paolini and E. Stepanov. Existence and regularity results for the Steiner problem. *Calc. Var. Partial Diff. Equations*, 2012.

[14] Emanuele Paolini, Eugene Stepanov, and Yana Teplitskaya. An example of an infinite Steiner tree connecting an uncountable set. *Adv. Calc. Var.*, 8(3):267–290, 2015.

[15] Joachim H. Rubinstein, Doreen A. Thomas, and Nicholas C. Wormald. Steiner trees for terminals constrained to curves. *SIAM Journal on Discrete Mathematics*, 10(1):1–17, 1997.

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