A LIOUVILLE THEOREM IN THE HEISENBERG GROUP

GIOVANNI CATINO, YANYAN LI, DARIO D. MONTICELLI, ALBERTO RONCORONI

ABSTRACT. In this paper we classify positive solutions to the critical semilinear elliptic equation in the Heisenberg group $\mathbb{H}^1$. We prove that they are the Jerison-Lee's bubbles. We do not require any finite energy or a priori symmetry assumptions. The proof is based on a classical Jerison-Lee's differential identity and on new pointwise/integral estimates, in the spirit of recent developments for critical semilinear and quasilinear elliptic equations in $\mathbb{R}^n$. In particular, our result is in $\mathbb{H}^1$ the analogue of the celebrated Caffarelli-Gidas-Spruck classification theorem in $\mathbb{R}^n$.

Key Words: Heisenberg group, semilinear elliptic equation, critical exponent, Liouville theorem, CR Yamabe problem, CR Sobolev inequality.

AMS subject classification: 35J61, 32V20, 35B33

1. INTRODUCTION

In this paper, we consider positive solutions to the following *critical semilinear* elliptic equation

$$-\Delta_{\mathbb{H}^1} u = 2u^3 \quad \text{in } \mathbb{H}^1, \tag{1.1}$$

where $\mathbb{H}^1$ is the Heisenberg group. More in general, in higher dimensions, the problem reads as

$$-\Delta_{\mathbb{H}^n} u = 2n^2 u^{q^*} \quad \text{in } \mathbb{H}^n \tag{1.2}$$

where $\mathbb{H}^n$ is the Heisenberg group, u is a smooth, real and positive function defined in $\mathbb{H}^n$, $\Delta_{\mathbb{H}^n} u$ is the Heisenberg Laplacian (or sub-Lapacian) of u (see the definition in Section 2) and

$$q^* := \frac{Q+2}{Q-2}$$

with $Q = 2n+2$ the homogeneous dimension of $\mathbb{H}^n$. It is well known that every positive weak solution to (1.2) is smooth on $\mathbb{H}^n$.

Equation (1.2) has been deeply studied since it is connected with the CR Yamabe problem in $\mathbb{H}^n$ and with the CR Sobolev inequality. The CR Yamabe problem on $\mathbb{H}^n$ is the following: given $(\mathbb{H}^n, \Theta)$ the sub-Riemannian manifold with standard contact form Θ, consider the conformal contact form $\tilde{\Theta} = u^{\frac{2}{n}}\Theta$ on $\mathbb{H}^n$, then the pseudo-Hermitian scalar curvature associated to $\tilde{\Theta}$ is a positive constant, $R \equiv 4n(n+1)$, if and only if u solves equation (1.2). The CR Yamabe problem has been studied in [26, 27, 28] and has been partially solved in [14, 19, 20]; we also refer to the recent papers [13, 44] for further developments. Moreover, the number

$$q^* + 1 = \frac{2Q}{Q-2},$$

is the critical exponent for the CR Sobolev embedding (or Folland-Stein inequality [17]). Thanks to the work [26] we know that there are (nontrivial) positive solutions of (1.2) given by

$$\mathcal{U}_{\lambda,\mu}(z,t) := \frac{C}{|t+i|z|^2+z\cdot\mu+\lambda|^n}\,, \tag{1.3}$$

for some $\lambda\in\mathbb{C}$, $\mu\in\mathbb{C}^n$ such that $\mathrm{Im}(\lambda) > \frac{|\mu|^2}{4}$ and for some explicit $C = C(n,\lambda) > 0$. The functions in (1.3) are the only extremals of the Folland-Stein inequality in $\mathbb{H}^n$ and are usually called *Jerison-Lee's bubbles*. Moreover, in [27] the authors obtained that (1.3) are the unique positive solutions of (1.2), satisfying the finite energy assumption $u\in L^{\frac{2Q}{Q-2}}(\mathbb{H}^n)$. We also refer to [21] where the authors obtained a uniqueness result under the assumption of cylindrical symmetry on groups of Heisenberg type.

Since it is well known that all nonnegative solutions of (1.2) are either strictly positive and smooth or identically zero, as the sub-Laplacian on the Heisenberg group satisfies the strong maximum principle (see e.g. [6]), we will only focus on positive solutions. We also recall that there exist infinitely many nonradial sign-changing solutions with finite energy to

$$-\Delta_{\mathbb{H}^n} u = 2n^2|u|^{q^*-1}u \quad \text{in } \mathbb{H}^n,$$

as proved in [33].

In the subcritical case, i.e.

$$-\Delta_{\mathbb{H}^n} u = 2n^2 u^q \quad \text{in } \mathbb{H}^n$$

where $1 < q < q^*$ it is known that the only nonnegative solution is the trivial one (see [32], and [4, 5, 45] for previous partial results).

The analogue of (1.2) in the Euclidean space is the so-called critical Laplace equation

$$-\Delta u = \frac{n(n-2)}{4}u^{2^*-1} \quad \text{in } \mathbb{R}^n\,, \tag{1.4}$$

where $2^* = \frac{2n}{n-2}$ is the critical Sobolev exponent. Equation (1.4) is related to the Yamabe problem in Riemannian geometry (see the survey [30]) and to the extremals in the Sobolev inequality (see the survey [37]). From [1, 36, 41], we know that the following class of solutions to (1.4)

$$\mathcal{V}_{\lambda,x_0}(x) := \left(\frac{\lambda}{\lambda^2+|x-x_0|^2}\right)^{\frac{n-2}{2}}\,, \quad \lambda>0\,,\ x_0\in\mathbb{R}^n \tag{1.5}$$

are the unique minimizers for the Sobolev inequality. Moreover, from the seminal paper [8] (see also [22, 34] for previous important results) we know that (1.5) are the only positive solutions to (1.4) (see also [12] and [31]). The proof of the classification result is based on the technique of moving planes and on the Kelvin transform. An alternative proof, based on integral estimates that can be applied also in the Riemannian setting when the Ricci curvature is non-negative, has been recently obtained in [9] when the dimension is $n = 3$, and was extended to dimensions $n = 4, 5$ in [35, 43] respectively.

In the subcritical case, i.e.

$$-\Delta u = u^q \quad \text{in } \mathbb{R}^n \tag{1.6}$$

with $1 < q < 2^* - 1$, it is well-known that the only nonnegative solution is the trivial one (see [23]). Recently, also the critical $p-$Laplace equation has been considered, we refer the interested reader to [10, 15, 35, 38, 40, 42, 43].

Our main result is a classification of *all* positive solutions to (1.1) in $\mathbb{H}^1$ (see Theorem 1.1). We do not require any finite energy or a priori symmetry assumptions on the solutions. For other important classification results on the Heisenberg group $\mathbb{H}^n$ that do not require a priori symmetry but assume finite energy on the solutions (besides [27]) we refer to [7, 24, 18, 46, 47] where the authors classify extremals of different functional inequalities.

Indeed we have

Theorem 1.1. *Let u be a positive solution to (1.1) in $\mathbb{H}^1$. Then u is a Jerison-Lee bubble, that is*

$$u \equiv \mathcal{U}_{\lambda,\mu},$$

for some $\lambda, \mu \in \mathbb{C}$ such that $\mathrm{Im}(\lambda) > \frac{|\mu|^2}{4}$.

The proof of our result rely on a remarkable differential identity proved in [27] (see also [32]), which involves a vector field depending on the solution u and its derivatives, that has nonnegative divergence whose vanishing implies that u is actually one of the Jerison-Lee bubbles (1.3). Inspired by [9, 10], through a test function argument we are able to obtain integral estimates which imply that such divergence must vanish identically, thus giving the desired classification result. In order to obtain Theorem 1.1 we need to suitably adapt the technique used in [35, 43] to our setting; this allows us to obtain the full classification result without any extra assumption in $\mathbb{H}^1$.

In the last section, as a further application of our techniques we also show the following classification result for positive solutions of (1.2) on the higher dimensional Heisenberg group $\mathbb{H}^n$, with $n \geq 2$, under a suitable decay condition at infinity, which do not imply that the solution has finite energy.

Theorem 1.2. *Let u be a positive solution to (1.2) in $\mathbb{H}^n$, $n \geq 2$ such that*

$$u(\xi) \leq \frac{C}{1 + |\xi|^{\frac{Q-2}{2}}} \quad \forall \xi \in \mathbb{H}^n,$$

for some $C > 0$. Then u is a Jerison-Lee bubble, that is

$$u \equiv \mathcal{U}_{\lambda,\mu}$$

for some $\lambda \in \mathbb{C}$, $\mu \in \mathbb{C}^n$ such that $\mathrm{Im}(\lambda) > \frac{|\mu|^2}{4}$.

In the proof of Theorem 1.2 the assumption on the behavior of the solution at infinity is needed to prove the validity of a useful gradient estimate on the solution u, that gives us the desired decay in the integral estimates, which may have an independent interest.

The analogue of Theorems 1.1 and 1.2 should hold and yield a classification result for positive solutions of the critical sub-Laplace equation also in the Sasakian setting and in the context of Carnot groups, under appropriate geometric conditions (such as nonnegative pseudo-Hermitian Ricci curvature).

Organization of the paper. In Section 2 we collect some preliminaries and notations, in Section 3 we prove Theorem 1.1. In Section 4 we prove a gradient estimate (see Proposition 4.1), which is then used in the proof of Theorem 1.2.

Added note. After this paper was submitted, further developments on problem (1.2) on $\mathbb{H}^n$ with $n \geq 2$ appeared in the interesting preprint [16], which was uploaded on the arXiv.

2. Preliminaries and Notations

We first give a brief introduction to the Heisenberg group $\mathbb{H}^n$ with some notations (for further details we refer to [2, 26, 27, 29]). We consider

$$\mathbb{H}^n := \mathbb{C}^n \times \mathbb{R}$$

with coordinates $\xi = (z,t) = (z_1, \dots, z_n, t) \in \mathbb{H}^n$ and with the group law $\circ$: given $\xi = (z,t)$ and $\zeta = (w,s)$

$$(z,t) \circ (w,s) = (z + w, t + s + 2\mathrm{Im}(z^\alpha \bar{w}^\alpha)) \,,$$

where here and in the sequel we use the Einstein notation sum for the Greek indices $1 \leq \alpha, \beta, \gamma \leq n$. We define, for $\xi = (z,t) \in \mathbb{H}^n$, the norm

$$|\xi| = \left(|z|^4 + t^2\right)^{\frac{1}{4}} \,,$$

with the associated distance function

$$d(\xi, \zeta) = |\zeta^{-1} \circ \xi| \quad \text{for } \xi, \zeta \in \mathbb{H}^n \,,$$

where ζ^{-1} denotes the inverse of ζ with respect to $\circ$.

We use the notation $B_R(\xi)$ for the metric ball centred at $\xi \in \mathbb{H}^n$ with radius $R > 0$, i.e.

$$B_R(\xi) = \{\zeta \in \mathbb{H}^n \,:\, d(\xi, \zeta) < R\} \,.$$

If $\xi = 0$, we will write $B_R := B_R(0)$. It is well-known and it is important to recall that the volume of a metric ball is given by

$$|B_r(\xi)| = C r^Q \,, \tag{2.1}$$

where $C > 0$ is a positive constant, $Q = 2n + 2$ and $|\cdot|$ denotes the Lebesgue measure. The (even) integer Q is called the homogeneous dimension of $\mathbb{H}^n$.

We define the following left-invariant (with respect to $\circ$) vector fields in $\mathbb{H}^n$

$$Z_\alpha = \frac{\partial}{\partial z^\alpha} + i\bar{z}_\alpha \frac{\partial}{\partial t} \quad \text{and} \quad Z_{\bar{\alpha}} = \frac{\partial}{\partial \bar{z}^\alpha} - i z_\alpha \frac{\partial}{\partial t} \quad \text{for } \alpha = 1, \dots, n.$$

For a smooth function $f : \mathbb{H}^n \to \mathbb{C}$ we denote its derivatives by

$$f_\alpha = Z_\alpha f \,, \quad f_{\bar{\alpha}} = Z_{\bar{\alpha}} f \,, \quad f_0 = \frac{\partial f}{\partial t} \,, \quad f_{\alpha\bar{\beta}} = Z_{\bar{\beta}} \left(Z_\alpha f\right) \,, \quad f_{0\alpha} = Z_\alpha \left(\frac{\partial f}{\partial t}\right) \,,$$

and so on. Computing commutators [32] (cf. [27]) the following formulas hold

$$f_{\alpha\beta} - f_{\beta\alpha} = 0 \,, \quad f_{\alpha\bar{\beta}} - f_{\bar{\beta}\alpha} = 2i\delta_{\alpha\bar{\beta}} f_0 \,, \quad f_{0\alpha} - f_{\alpha 0} = 0 \,,$$

$$f_{\alpha\beta 0} - f_{\alpha 0\beta} = 0 \,, \quad f_{\alpha\beta\bar{\gamma}} - f_{\alpha\bar{\gamma}\beta} = 2i\delta_{\beta\bar{\gamma}} f_{\alpha 0} \,.$$

Moreover, using the summation convention over repeated indexes, we define

$$|\partial f|^2 := \sum_{\alpha=1}^n f_\alpha f_{\bar{\alpha}} = f_\alpha f_{\bar{\alpha}} \quad \text{and} \quad \Delta_{\mathbb{H}^n} f := \sum_{\alpha=1}^n \left(f_{\alpha\bar{\alpha}} + f_{\bar{\alpha}\alpha}\right) = f_{\alpha\bar{\alpha}} + f_{\bar{\alpha}\alpha} \,.$$

We recall that the Heisenberg group is a strictly pseudoconvex CR manifold, where the CR structure is given by the bundle $\mathcal{H}$ spanned by the vector fields Z_α, for $\alpha = 1, \dots, n$, and where the standard contact form on $\mathbb{H}^n$ is given by

$$\Theta = dt + \sum_{\alpha=1}^{n} iz^\alpha d\bar{z}^\alpha - i\bar{z}^\alpha dz^\alpha.$$

We also recall that the Heisenberg group is a Carnot group, which can be viewed as a flat model in Sub-Riemannian geometry similar to the Euclidean space $\mathbb{R}^n$ in Riemannian geometry, where the family of vector fields $T = \frac{\partial}{\partial t}$, $X_\alpha = 2\,\mathrm{Re}\, Z_\alpha$, $Y_\alpha = 2i\,\mathrm{Im}\, Z_\alpha$, for $\alpha = 1, \dots, n$, form a base of the Lie algebra of vector fields on $\mathbb{H}^n$ which are left invariant with respect to the group action $\circ$.

Given $u > 0$ a solution of (1.2) we consider the auxiliary function f defined as follows

$$e^f = u^{\frac{1}{n}}\,, \tag{2.2}$$

then f solves

$$-\Delta_{\mathbb{H}^n} f = 2n|\partial f|^2 + 2ne^{2f} \quad \text{in } \mathbb{H}^n\,. \tag{2.3}$$

We also introduce the function $g : \mathbb{H}^n \to \mathbb{C}$ such that

$$g = |\partial f|^2 + e^{2f} - if_0\,, \tag{2.4}$$

then the equation (2.3) can be rewritten as

$$f_{\alpha\bar{\alpha}} = -ng \quad \text{in } \mathbb{H}^n\,. \tag{2.5}$$

As done in [27] and in [32] we define the following tensors

$$\begin{array}{ll} D_{\alpha\beta} = f_{\alpha\beta} - 2f_\alpha f_\beta & D_\alpha = D_{\alpha\beta} f_{\bar{\beta}} \\ E_{\alpha\bar{\beta}} = f_{\alpha\bar{\beta}} - \frac{1}{n} f_{\gamma\bar{\gamma}}\delta_{\alpha\bar{\beta}} & E_\alpha = E_{\alpha\bar{\beta}} f_{\bar{\beta}} \\ G_\alpha = if_{0\alpha} - if_0 f_\alpha + e^{2f} f_\alpha + |\partial f|^2 f_\alpha\,. & \end{array} \tag{2.6}$$

The above tensors will be important in our argument and we refer to [27] for the reason to introduce them. Moreover (see also [32]) we observe that

$$\begin{array}{ll} E_{\alpha\bar{\beta}} = f_{\alpha\bar{\beta}} + g\delta_{\alpha\bar{\beta}} & E_\alpha = f_{\alpha\bar{\beta}} f_\beta + g f_\alpha \\ D_\alpha = f_{\alpha\beta} f_{\bar{\beta}} - 2|\partial f|^2 f_\alpha & G_\alpha = if_{0\alpha} + g f_\alpha \\ |\partial f|^2_{\bar{\alpha}} = D_{\bar{\alpha}} + E_{\bar{\alpha}} + \bar{g} f_{\bar{\alpha}} - 2f_{\bar{\alpha}} e^{2f} & \\ g_{\bar{\alpha}} = D_{\bar{\alpha}} + E_{\bar{\alpha}} + G_{\bar{\alpha}} & \bar{g}_\alpha = D_\alpha + E_\alpha + G_\alpha. \end{array} \tag{2.7}$$

We are now in a position to recall the following differential identity obtained in [27, Formula (4.2)] and [32, Proposition 2.1] (with $p = 0$) which will be fundamental in our arguments.

Proposition 2.1. *With the notations above, we have*

$$\mathcal{M} = \mathrm{Re} Z_{\bar{\alpha}} \left\{ e^{2(n-1)f} \left[(g + 3if_0)\, E_\alpha + (g - if_0)\, D_\alpha - 3if_0 G_\alpha \right] \right\}\,,$$

where

$$\begin{aligned} \mathcal{M} = {} & e^{2nf} \left(|E_{\alpha\bar{\beta}}|^2 + |D_{\alpha\beta}|^2 \right) \\ & + e^{2(n-1)f} \left(|G_\alpha|^2 + |G_\alpha + D_\alpha|^2 + |G_\alpha - E_\alpha|^2 + |D_{\alpha\beta} f_{\bar{\gamma}} + E_{\alpha\bar{\gamma}} f_\beta|^2 \right)\,. \end{aligned}$$

From this proposition we obtain the next two lemmas: the first one in the case $n = 1$ and the second one for $n \geq 2$.

Lemma 2.2. *With the notations above, if $n = 1$, for every $s > 2$ and every $0 < \beta < \frac{1}{12}$ there exists a constant $C = C(s, \beta) > 0$ such that, for every (real) non-negative cut-off function η with compact support, we have*

$$\int_{\mathbb{H}^1} \mathcal{M}\Psi^{-\beta}\eta^s \leq C \left(\int_{supp|\partial\eta|} \mathcal{M}\Psi^{-\beta}\eta^s \right)^{1/2} \left(\int_{supp|\partial\eta|} |g|^2\Psi^{-\beta}|\partial\eta|^2\eta^{s-2} \right)^{1/2},$$

where

$$\Psi := g\bar{g}\, e^{-2f} = |g|^2\, e^{-2f}\,.$$

In particular

$$\int_{\mathbb{H}^1} \mathcal{M}\Psi^{-\beta}\eta^s \leq C \int_{\mathbb{H}^1} |g|^2\Psi^{-\beta}|\partial\eta|^2\eta^{s-2}\,.$$

Proof. We define

$$\mathcal{I}_1 := \int_{\mathbb{H}^1} \mathcal{M}\Psi^{-\beta}\eta^s\,.$$

From Proposition 2.1 we obtain

$$\begin{aligned}
\int_{\mathbb{H}^1} \mathcal{M}\Psi^{-\beta}\eta^s &= \int_{\mathbb{H}^1} \mathrm{Re} Z_{\bar{\alpha}} \left[(g + 3if_0)\, E_\alpha + (g - if_0)\, D_\alpha - 3if_0 G_\alpha \right] \Psi^{-\beta}\eta^s \\
&= \beta \int_{\mathbb{H}^1} \mathrm{Re} \left\{ \left[(g + 3if_0)\, E_\alpha + (g - if_0)\, D_\alpha - 3if_0 G_\alpha \right] \Psi_{\bar{\alpha}} \right\} \Psi^{-\beta-1}\eta^s \\
&\quad - s \int_{\mathbb{H}^1} \mathrm{Re} \left\{ \left[(g + 3if_0)\, E_\alpha + (g - if_0)\, D_\alpha - 3if_0 G_\alpha \right] \eta_{\bar{\alpha}} \right\} \Psi^{-\beta}\eta^{s-1} \qquad (2.8)
\end{aligned}$$

where we integrate by parts. We now observe that, on the one hand, from the definition of g (2.4) we have

$$\begin{aligned}
(g + 3if_0)\, E_\alpha + (g - if_0)\, D_\alpha - 3if_0 G_\alpha &= (D_\alpha + G_\alpha)\,(g - if_0) + (E_\alpha - G_\alpha)\,(g + 3if_0) + if_0 G_\alpha \\
&= (D_\alpha + G_\alpha)\left(|\partial f|^2 + e^{2f} - 2if_0 \right) \\
&\quad + (E_\alpha - G_\alpha)\left(|\partial f|^2 + e^{2f} + 2if_0 \right) + if_0 G_\alpha\,,
\end{aligned}$$

and so, from Cauchy-Schwarz inequality

$$\begin{aligned}
|\,(g + 3if_0)\, E_\alpha + (g - if_0)\, D_\alpha - 3if_0 G_\alpha| &\leq |D_\alpha + G_\alpha| \sqrt{|\partial f|^4 + e^{4f} + 2|\partial f|^2 e^{2f} + 4f_0^2} \\
&\quad + |E_\alpha - G_\alpha| \sqrt{|\partial f|^4 + e^{4f} + 2|\partial f|^2 e^{2f} + 4f_0^2} \\
&\quad + |f_0||G_\alpha| \\
&\leq 2|g|\,(|D_\alpha + G_\alpha| + |E_\alpha - G_\alpha| + |G_\alpha|) \\
&\leq 2\sqrt{3}|g|\sqrt{\mathcal{M}}
\end{aligned}$$

where we used the fact that

$$|g| = \sqrt{|\partial f|^4 + e^{4f} + 2|\partial f|^2 e^{2f} + f_0^2}\,. \qquad (2.9)$$

Summing up, we have obtained the following

$$|\,(g + 3if_0)\, E_\alpha + (g - if_0)\, D_\alpha - 3if_0 G_\alpha| \leq 2\sqrt{3}|g|\sqrt{\mathcal{M}}\,. \qquad (2.10)$$

On the other hand, from (2.7) we have

$$\begin{aligned}\Psi_{\bar\alpha} &= \left[(g\bar g)\, e^{-2f}\right]_{\bar\alpha} = e^{-2f}\,(\bar g g_{\bar\alpha} + g\bar g_{\bar\alpha}) - 2\,(g\bar g)\, f_{\bar\alpha} e^{-2f}\\ &= e^{-2f}\left[\bar g\,(D_{\bar\alpha} + E_{\bar\alpha} + G_{\bar\alpha}) + g\,(D_{\bar\alpha} + E_{\bar\alpha} - G_{\bar\alpha} + 2\bar g f_{\bar\alpha})\right] - 2\,(g\bar g)\, f_{\bar\alpha} e^{-2f}\\ &= e^{-2f}\left[D_{\bar\alpha}\,(g+\bar g) + E_{\bar\alpha}\,(g+\bar g) + G_{\bar\alpha}\,(\bar g - g)\right]\\ &= e^{-2f}\left[(D_{\bar\alpha} + G_{\bar\alpha})\,(g+\bar g) + (E_{\bar\alpha} - G_{\bar\alpha})\,(g+\bar g) + G_{\bar\alpha}\,(\bar g - g)\right]\\ &= 2e^{-2f}\left[(D_{\bar\alpha} + G_{\bar\alpha})\left(|\partial f|^2 + e^{2f}\right) + (E_{\bar\alpha} - G_{\bar\alpha})\left(|\partial f|^2 + e^{2f}\right) + iG_{\bar\alpha} f_0\right]\,,\end{aligned}$$

where we used the fact that

$$\bar g_{\bar\alpha} = D_{\bar\alpha} + E_{\bar\alpha} - G_{\bar\alpha} + 2\bar g f_{\bar\alpha}\,. \tag{2.11}$$

Indeed by (2.7) we have

$$G_{\bar\alpha} = -if_{0\bar\alpha} + \bar g f_{\bar\alpha}\,,$$

hence

$$\begin{aligned}\bar g_{\bar\alpha} &= \left(|\partial f|^2 + e^{2f} + if_0\right)_{\bar\alpha}\\ &= D_{\bar\alpha} + E_{\bar\alpha} + \bar g f_{\bar\alpha} + if_{0\bar\alpha}\\ &= D_{\bar\alpha} + E_{\bar\alpha} - G_{\bar\alpha} + 2\bar g f_{\bar\alpha}\,.\end{aligned}$$

Moreover, from Cauchy-Schwarz inequality

$$\begin{aligned}|\Psi_{\bar\alpha}| \le{}& 2e^{-2f}\left[|D_{\bar\alpha} + G_{\bar\alpha}|\,\sqrt{|\partial f|^4 + e^{4f} + 2|\partial f|^2 e^{2f}}\right.\\ &\left. + |E_{\bar\alpha} - G_{\bar\alpha}|\,\sqrt{|\partial f|^4 + e^{4f} + 2|\partial f|^2 e^{2f}} + |G_{\bar\alpha}||f_0|\right]\\ \le{}& 2e^{-2f}|g|\left[|D_{\bar\alpha} + G_{\bar\alpha}| + |E_{\bar\alpha} - G_{\bar\alpha}| + |G_{\bar\alpha}|\right]\\ \le{}& 2\sqrt{3}e^{-2f}|g|\sqrt{\mathcal M}\,,\end{aligned}$$

i.e.

$$|\Psi_{\bar\alpha}| \le\ 2\sqrt{3}e^{-2f}|g|\sqrt{\mathcal M}\,. \tag{2.12}$$

Hence, by substituting (2.10) and (2.12) in (2.8) we get, after a Cauchy-Schwarz inequality,

$$\begin{aligned}\mathcal I_1 = \int_{\mathbb H^1} \mathcal M \Psi^{-\beta}\eta^s \le{}& 12\beta \int_{\mathbb H^1} \mathcal M |g|^2 e^{-2f}\Psi^{-\beta-1}\eta^s + 2\sqrt{3}s\int_{\mathbb H^1} |g|\sqrt{\mathcal M}\Psi^{-\beta}|\partial\eta|\eta^{s-1}\\ \le{}& 12\beta\int_{\mathbb H^1}\mathcal M\Psi^{-\beta}\eta^s + 2\sqrt{3}s\int_{\mathbb H^1}\sqrt{\mathcal M}|g|\Psi^{-\beta}|\partial\eta|\eta^{s-1}\,.\end{aligned}$$

Since $0<\beta<\frac{1}{12}$ we find

$$\mathcal I_1 \le C\int_{\mathbb H^1}\sqrt{\mathcal M}|g|\Psi^{-\beta}|\partial\eta|\eta^{s-1}\,.$$

We now use Hölder's inequality and we have

$$\mathcal I_1 \le C\left(\int_{\mathrm{supp}|\partial\eta|}\mathcal M\Psi^{-\beta}\eta^s\right)^{\frac12}\left(\int_{\mathrm{supp}|\partial\eta|}|g|^2\Psi^{-\beta}|\partial\eta|^2\eta^{s-2}\right)^{\frac12}\,,$$

for some $C = C(s,\beta) > 0$. Now the conclusion easily follows. □

Lemma 2.3. *With the notations above, for every (real) non-negative cut-off function η with compact support and for every $s > 2$ we have*

$$\int_{\mathbb{H}^n} \mathcal{M}\eta^s \leq C \left(\int_{supp|\partial\eta|} \mathcal{M}\eta^s \right)^{1/2} \left(\int_{supp|\partial\eta|} e^{2(n-1)f} |g|^2 |\partial\eta|^2 \eta^{s-2} \right)^{1/2} .$$

In particular we also have

$$\int_{\mathbb{H}^n} \mathcal{M}\eta^s \leq C \int_{\mathbb{H}^n} |g|^2 e^{2(n-1)f} |\partial\eta|^2 \eta^{s-2} .$$

Proof. From Proposition 2.1, integrating by parts, using Cauchy-Schwarz and Young inequalities we obtain

$$\begin{aligned}
&\int_{\mathbb{H}^n} \mathcal{M}\eta^s = \int_{\mathbb{H}^n} \mathrm{Re} Z_{\bar\alpha} \left\{ e^{2(n-1)f} \left[(g + 3if_0) E_\alpha + (g - if_0) D_\alpha - 3if_0 G_\alpha \right] \right\} \eta^s \\
&= -s \int_{\mathbb{H}^n} \mathrm{Re} \left\{ e^{2(n-1)f} \left[(g + 3if_0) E_\alpha + (g - if_0) D_\alpha - 3if_0 G_\alpha \right] \eta_{\bar\alpha} \right\} \eta^{s-1} \\
&= -s \int_{\mathbb{H}^n} \mathrm{Re} \left\{ e^{2(n-1)f} \left[\left(|\partial f|^2 + e^{2f} + 2if_0 \right) E_\alpha + \left(|\partial f|^2 + e^{2f} - 2if_0 \right) D_\alpha - 3if_0 G_\alpha \right] \eta_{\bar\alpha} \right\} \eta^{s-1} \\
&\leq s \int_{\mathbb{H}^n} e^{2(n-1)f} \left[\left(|\partial f|^2 + e^{2f} + 2|f_0| \right) |E_\alpha| + \left(|\partial f|^2 + e^{2f} + 2|f_0| \right) |D_\alpha| + 3|f_0||G_\alpha| \right] |\partial\eta| \eta^{s-1} \\
&\leq C \left(\int_{\mathrm{supp}|\partial\eta|} e^{2(n-1)f} \left[|E_\alpha|^2 + |D_\alpha|^2 + |G_\alpha|^2 \right] \eta^s \right)^{\frac{1}{2}} \\
&\qquad \left(\int_{\mathrm{supp}|\partial\eta|} e^{2(n-1)f} \left[|\partial f|^4 + e^{4f} + |f_0|^2 \right] |\partial\eta|^2 \eta^{s-2} \right)^{\frac{1}{2}} \\
&\leq C \left(\int_{\mathrm{supp}|\partial\eta|} e^{2(n-1)f} \left[|G_\alpha - E_\alpha|^2 + |G_\alpha + D_\alpha|^2 + |G_\alpha|^2 \right] \eta^s \right)^{\frac{1}{2}} \\
&\qquad \left(\int_{\mathrm{supp}|\partial\eta|} e^{2(n-1)f} |g|^2 |\partial\eta|^2 \eta^{s-2} \right)^{\frac{1}{2}} \\
&\leq C \left(\int_{\mathrm{supp}|\partial\eta|} \mathcal{M}\eta^s \right)^{\frac{1}{2}} \left(\int_{\mathrm{supp}|\partial\eta|} e^{2(n-1)f} |g|^2 |\partial\eta|^2 \eta^{s-2} \right)^{\frac{1}{2}}
\end{aligned}$$

which immediately yields the conclusion. □

We conclude this section by recalling the following lower bound for positive superharmonic functions in $\mathbb{H}^n$ (see [3]).

Proposition 2.4. *Let u be a positive superharmonic function in $\mathbb{H}^n$, i.e $u \in C^2(\mathbb{H}^n)$ and*

$$\Delta_{\mathbb{H}^n} u \leq 0 \quad \text{in } \mathbb{H}^n.$$

Then, there exists a constant $C > 0$ such that

$$u(\xi) \geq \frac{C}{|\xi|^{Q-2}},$$

for any $\xi \in \mathbb{H}^n$ with $|\xi| > 1$, where $Q = 2n + 2$.

3. Proof of Theorem 1.1

In this section $n = 1$ and given $R > 1$, we choose a real cut-off function η such that $\eta \equiv 1$ in $B_{R/2}$, $\eta \equiv 0$ in B_R^c and $|\partial\eta| \leq \frac{C}{R}$ in $B_R \setminus B_{R/2}$, with $C > 0$ independent of R. Let $s > 6$ and $\beta > 0$ small enough.

From Lemma 2.2, for $\beta > 0$ small enough we have

$$\begin{aligned}
\mathcal{I}_1 = \int_{\mathbb{H}^1} \mathcal{M}\Psi^{-\beta}\eta^s \leq & C\int_{\mathbb{H}^1} |g|^2\Psi^{-\beta}|\partial\eta|^2\eta^{s-2} \\
\leq & \frac{C}{R^2}\int_{B_R\setminus B_{R/2}} |g|^2\Psi^{-\beta}\eta^{s-2} \qquad (3.1)\\
\leq & \frac{C}{R^2}\int_{B_R} (g\bar{g})\,\Psi^{-\beta}\eta^{s-2} \\
= & -\frac{C}{R^2}\int_{B_R} \mathrm{Re}(f_{\alpha\bar{\alpha}}\bar{g})\Psi^{-\beta}\eta^{s-2} \\
= & : \frac{C}{R^2}\mathcal{I}_2\,, \qquad (3.2)
\end{aligned}$$

where we used (2.5). By integrating by parts and using (2.11) we get

$$\begin{aligned}
\mathcal{I}_2 = & -\int_{B_R} \mathrm{Re}(f_{\alpha\bar{\alpha}}\bar{g})\Psi^{-\beta}\eta^{s-2} \\
= & \int_{B_R} \mathrm{Re}(f_\alpha\bar{g}_{\bar{\alpha}})\Psi^{-\beta}\eta^{s-2} - \beta\int_{B_R} \mathrm{Re}(f_\alpha\Psi_{\bar{\alpha}}\bar{g})\Psi^{-\beta-1}\eta^{s-2} \\
& + (s-2)\int_{B_R} \mathrm{Re}(f_\alpha\bar{g}\eta_{\bar{\alpha}})\Psi^{-\beta}\eta^{s-3} \\
= & \int_{B_R} \mathrm{Re}(f_\alpha\,(D_{\bar{\alpha}} + E_{\bar{\alpha}} - G_{\bar{\alpha}} + 2\bar{g}f_{\bar{\alpha}}))\Psi^{-\beta}\eta^{s-2} - \beta\int_{B_R} \mathrm{Re}(f_\alpha\Psi_{\bar{\alpha}}\bar{g})\Psi^{-\beta-1}\eta^{s-2} \\
& + (s-2)\int_{B_R} \mathrm{Re}(f_\alpha\bar{g}\eta_{\bar{\alpha}})\Psi^{-\beta}\eta^{s-3} \\
\leq & \int_{B_R} \mathrm{Re}(f_\alpha\,(D_{\bar{\alpha}} + E_{\bar{\alpha}} - G_{\bar{\alpha}}))\Psi^{-\beta}\eta^{s-2} + 2\int_{B_R} \mathrm{Re}(\bar{g})|\partial f|^2\Psi^{-\beta}\eta^{s-2} \\
& - \beta\int_{B_R} \mathrm{Re}(f_\alpha\Psi_{\bar{\alpha}}\bar{g})\Psi^{-\beta-1}\eta^{s-2} + C\int_{B_R} |\partial f||g||\partial\eta|\Psi^{-\beta}\eta^{s-3}.
\end{aligned}$$

Since

$$|D_{\bar{\alpha}} + E_{\bar{\alpha}} - G_{\bar{\alpha}}| = |(D_{\bar{\alpha}} + G_{\bar{\alpha}}) + (E_{\bar{\alpha}} - G_{\bar{\alpha}}) - G_{\bar{\alpha}}| \leq \sqrt{3}\sqrt{\mathcal{M}}, \qquad (3.3)$$

for every $\theta > 0$, using (2.12), the expressions of g and Ψ and Young's inequality, we obtain

$$\begin{aligned}\mathcal{I}_2 \leq & \theta \int_{B_R} \mathcal{M}\Psi^{-\beta}\eta^s + \frac{3}{4\theta}\int_{B_R} |\partial f|^2\Psi^{-\beta}\eta^{s-4} + 2\int_{B_R}\left(|\partial f|^4 + e^{2f}|\partial f|^2\right)\Psi^{-\beta}\eta^{s-2} \\ & + 2\sqrt{3}\beta\int_{B_R}|\partial f|\sqrt{\mathcal{M}}\Psi^{-\beta}\eta^{s-2} + \frac{C_\varepsilon}{R^2}\int_{B_R}|\partial f|^2\Psi^{-\beta}\eta^{s-4} + \varepsilon\mathcal{I}_2 \\ \leq & \left(\theta + \varepsilon R^2\right)\int_{B_R}\mathcal{M}\Psi^{-\beta}\eta^s + C_\varepsilon\left(\frac{1}{\theta} + \frac{1}{R^2} + \frac{\beta^2}{R^2}\right)\int_{B_R}|\partial f|^2\Psi^{-\beta}\eta^{s-4} \\ & + 4\int_{B_R}|\partial f|^4\Psi^{-\beta}\eta^{s-2} + \frac{1}{2}\int_{B_R} e^{4f}\Psi^{-\beta}\eta^{s-2} + \varepsilon\mathcal{I}_2.\end{aligned}$$

By choosing $\varepsilon < 1$ and since $\beta < 1$,

$$\begin{aligned}\mathcal{I}_2 \leq & C\left(\theta + \varepsilon R^2\right)\int_{B_R}\mathcal{M}\Psi^{-\beta}\eta^s + C_\varepsilon\left(\frac{1}{\theta} + \frac{1}{R^2} + \frac{\beta^2}{R^2}\right)\int_{B_R}|\partial f|^2\Psi^{-\beta}\eta^{s-4} \\ & + C\int_{B_R}|\partial f|^4\Psi^{-\beta}\eta^{s-2} + \frac{2}{3}\int_{B_R} e^{4f}\Psi^{-\beta}\eta^{s-2}.\end{aligned}$$

Since

$$\mathcal{I}_2 = \int_{B_R}\left(|\partial f|^4 + e^{4f} + 2e^{2f}|\partial f|^2 + f_0^2\right)\Psi^{-\beta}\eta^{s-2},$$

we obtain

$$\begin{aligned}\mathcal{I}_2 \leq & C\varepsilon R^2\int_{B_R}\mathcal{M}\Psi^{-\beta}\eta^s + \frac{C_\varepsilon}{R^2}\int_{B_R}|\partial f|^2\Psi^{-\beta}\eta^{s-4} + C\int_{B_R}|\partial f|^4\Psi^{-\beta}\eta^{s-2} \\ =: & C\varepsilon R^2\mathcal{I}_1 + \frac{C}{R^2}\mathcal{I}_3 + C\mathcal{I}_4,\end{aligned} \tag{3.4}$$

where we chose $\theta = \varepsilon R^2$. Now, since

$$\Psi^{-\beta} = |g|^{-2\beta}e^{2\beta f} \leq e^{2\beta f}\left(|\partial f|^4 + e^{4f}\right)^{-\beta} \leq e^{-2\beta f} \tag{3.5}$$

where we used (2.9), then using once again (2.5) and integration by parts

$$\begin{aligned}\mathcal{I}_3 = & \int_{B_R}|\partial f|^2\Psi^{-\beta}\eta^{s-4} \\ \leq & \int_{B_R}|\partial f|^2 e^{-2\beta f}\eta^{s-4} =: \mathcal{I}_3' \qquad (3.6) \\ \leq & \int_{B_R}\mathrm{Re}(g)e^{-2\beta f}\eta^{s-4} \\ = & -\int_{B_R}\mathrm{Re}(f_{\alpha\bar{\alpha}})e^{-2\beta f}\eta^{s-4} \\ = & -2\beta\int_{B_R}|\partial f|^2 e^{-2\beta f}\eta^{s-4} + (s-4)\int_{B_R}\mathrm{Re}(f_\alpha\eta_{\bar{\alpha}})e^{-2\beta f}\eta^{s-5} \\ \leq & -2\beta\mathcal{I}_3' + \varepsilon'\mathcal{I}_3' + \frac{C_{\varepsilon'}}{R^2}\int_{B_R} e^{-2\beta f} \qquad (3.7)\end{aligned}$$

for every $\varepsilon' > 0$ and for some $C_{\varepsilon'} > 0$. Now, we observe that from (2.2) and from Proposition 2.4 we obtain

$$e^{-2\beta f} = u^{-2\beta} \leq CR^{4\beta} \quad \text{in } B_R, \tag{3.8}$$

for $R \gg 1$. Hence, by choosing $\varepsilon' = 2\beta$ in (3.7) we find

$$\mathcal{I}_3 \leq \mathcal{I}'_3 \leq CR^{2+4\beta}\,, \tag{3.9}$$

where we used (3.8), (3.6) and (2.1).

We argue in the same way in order to estimate $\mathcal{I}_4$; from (3.5), (2.4), (2.5) and integration by parts we get

$$\begin{aligned}
\mathcal{I}_4 &= \int_{B_R} |\partial f|^4 \Psi^{-\beta} \eta^{s-2} \\
&\leq \int_{B_R} |\partial f|^4 e^{-2\beta f} \eta^{s-2} =: \mathcal{I}'_4 \\
&\leq \int_{B_R} |\partial f|^2 \mathrm{Re}(g) e^{-2\beta f} \eta^{s-2} \\
&= -\int_{B_R} |\partial f|^2 \mathrm{Re}(f_{\alpha\bar\alpha}) e^{-2\beta f} \eta^{s-2} \\
&= \int_{B_R} \mathrm{Re}(f_\alpha |\partial f|^2_{\bar\alpha}) e^{-2\beta f} \eta^{s-2} - 2\beta \int_{B_R} |\partial f|^4 e^{-2\beta f} \eta^{s-2} \\
&\quad + (s-2) \int_{B_R} \mathrm{Re}(f_\alpha \eta_{\bar\alpha}) |\partial f|^2 e^{-2\beta f} \eta^{s-3} \\
&= \int_{B_R} \mathrm{Re}(f_\alpha (D_{\bar\alpha} + E_{\bar\alpha})) e^{-2\beta f} \eta^{s-2} + \int_{B_R} |\partial f|^2 \mathrm{Re}(\bar g) e^{-2\beta f} \eta^{s-2} - 2\int_{B_R} |\partial f|^2 e^{2(1-\beta)f} \eta^{s-2} \\
&\quad - 2\beta \int_{B_R} |\partial f|^4 e^{-2\beta f} \eta^{s-2} + (s-2) \int_{B_R} \mathrm{Re}(f_\alpha \eta_{\bar\alpha}) |\partial f|^2 e^{-2\beta f} \eta^{s-3}\,,
\end{aligned} \tag{3.10}$$

where we used (2.7). Summing up

$$\begin{aligned}
\mathcal{I}'_4 &\leq \int_{B_R} \mathrm{Re}(f_\alpha (D_{\bar\alpha} + E_{\bar\alpha})) e^{-2\beta f} \eta^{s-2} + \int_{B_R} |\partial f|^4 e^{-2\beta f} \eta^{s-2} - \int_{B_R} |\partial f|^2 e^{2(1-\beta)f} \eta^{s-2} \\
&\quad - 2\beta \int_{B_R} |\partial f|^4 e^{-2\beta f} \eta^{s-2} + (s-2) \int_{B_R} \mathrm{Re}(f_\alpha \eta_{\bar\alpha}) |\partial f|^2 e^{-2\beta f} \eta^{s-3} \\
&= \int_{B_R} \mathrm{Re}(f_\alpha (D_{\bar\alpha} + E_{\bar\alpha})) e^{-2\beta f} \eta^{s-2} + \mathcal{I}'_4 - \int_{B_R} |\partial f|^2 e^{2(1-\beta)f} \eta^{s-2} \\
&\quad - 2\beta \mathcal{I}'_4 + (s-2) \int_{B_R} \mathrm{Re}(f_\alpha \eta_{\bar\alpha}) |\partial f|^2 e^{-2\beta f} \eta^{s-3}
\end{aligned}$$

i.e.

$$\begin{aligned}
2\beta \mathcal{I}'_4 &\leq \int_{B_R} \mathrm{Re}(f_\alpha (D_{\bar\alpha} + E_{\bar\alpha})) e^{-2\beta f} \eta^{s-2} - \int_{B_R} |\partial f|^2 e^{2(1-\beta)f} \eta^{s-2} \\
&\quad + (s-2) \int_{B_R} \mathrm{Re}(f_\alpha \eta_{\bar\alpha}) |\partial f|^2 e^{-2\beta f} \eta^{s-3}\,.
\end{aligned} \tag{3.11}$$

Now, we tackle the first integral on the right-hand side of the (3.11),

$$\begin{aligned}\int_{B_R} \mathrm{Re}(f_\alpha(D_{\bar\alpha}+E_{\bar\alpha}))e^{-2\beta f}\eta^{s-2} \leq& \sqrt{2}\int_{B_R}\sqrt{\mathcal{M}}|\partial f|e^{-2\beta f}\eta^{s-2}\\ =&\sqrt{2}\int_{B_R}\sqrt{\mathcal{M}}|\partial f||g|^{2\beta}e^{-4\beta f}\Psi^{-\beta}\eta^{s-2}\end{aligned}$$

where we used

$$|D_{\bar\alpha}+E_{\bar\alpha}| = |(D_{\bar\alpha}+G_{\bar\alpha})+(E_{\bar\alpha}-G_{\bar\alpha})| \leq \sqrt{2}\sqrt{\mathcal{M}}.$$

By using Young inequality,

$$\int_{B_R} \mathrm{Re}(f_\alpha(D_{\bar\alpha}+E_{\bar\alpha}))e^{-2\beta f}\eta^{s-2} \leq \varepsilon R^2\mathcal{I}_1 + \frac{C}{R^2}\int_{B_R}|\partial f|^2|g|^{4\beta}e^{-8\beta f}\Psi^{-\beta}\eta^{s-4}$$

for every $\varepsilon>0$ and for some $C>0$. From (3.8) we obtain

$$\int_{B_R} \mathrm{Re}(f_\alpha(D_{\bar\alpha}+E_{\bar\alpha}))e^{-2\beta f}\eta^{s-2} \leq \varepsilon R^2\mathcal{I}_1 + \frac{C}{R^2}\mathcal{I}_2 + \frac{C}{R^2}\int_{B_R}|\partial f|^{\frac{2}{1-2\beta}}e^{-\frac{8\beta f}{1-2\beta}}\Psi^{-\beta}\eta^{s-\frac{4-4\beta}{1-2\beta}},$$

where we used the following Young inequality

$$|\partial f|^2|g|^{4\beta}e^{-8\beta f}\eta^{s-4} \leq 2\beta|g|^2\eta^{s-2} + (1-2\beta)|\partial f|^{\frac{2}{1-2\beta}}e^{-\frac{8\beta f}{1-2\beta}}\eta^{s-\frac{4-4\beta}{1-2\beta}}.$$

Hence,

$$\begin{aligned}\int_{B_R} \mathrm{Re}(f_\alpha(D_{\bar\alpha}+E_{\bar\alpha}))e^{-2\beta f}\eta^{s-2} \leq& \varepsilon R^2\mathcal{I}_1 + \frac{C}{R^2}\mathcal{I}_2 + \frac{C}{R^2}\int_{B_R}|\partial f|^{\frac{2}{1-2\beta}}e^{-\frac{8\beta f}{1-2\beta}}\Psi^{-\beta}\eta^{s-\frac{4-4\beta}{1-2\beta}}\\ \leq& \varepsilon R^2\mathcal{I}_1 + \frac{C}{R^2}\mathcal{I}_2 + \frac{C}{R^{2-36\beta}}\mathcal{I}_4 + \frac{C}{R^2}\int_{B_R}\eta^{s-\frac{6-8\beta}{1-4\beta}},\end{aligned}$$

where we used the following Young inequality

$$|\partial f|^{\frac{2}{1-2\beta}}e^{-\frac{8\beta f}{1-2\beta}}\Psi^{-\beta}\eta^{s-\frac{4-4\beta}{1-2\beta}} \leq \frac{1}{2(1-2\beta)}|\partial f|^4e^{-16\beta f}\Psi^{-2\beta(1-2\beta)}\eta^{s-2} + \frac{1-4\beta}{2-4\beta}\eta^{s-\frac{6-8\beta}{1-4\beta}}$$

together with (3.5) and (3.8), which yield

$$e^{-16\beta f}\Psi^{-\beta(1-4\beta)} \leq e^{-18\beta f+8\beta^2 f} \leq CR^{36\beta-16\beta^2} \leq CR^{36\beta}$$

since $0<\beta<\frac{1}{4}$. From (2.1) we obtain

$$\int_{B_R} \mathrm{Re}(f_\alpha(D_{\bar\alpha}+E_{\bar\alpha}))e^{-2\beta f}\eta^{s-2} \leq \varepsilon R^2\mathcal{I}_1 + \frac{C}{R^2}\mathcal{I}_2 + \frac{C}{R^{2-36\beta}}\mathcal{I}_4 + CR^2. \tag{3.12}$$

By using (3.12) in (3.11) we get

$$\begin{aligned}2\beta\mathcal{I}'_4 \leq& \varepsilon R^2\mathcal{I}_1 + \frac{C}{R^2}\mathcal{I}_2 + \frac{C}{R^{2-36\beta}}\mathcal{I}_4 + CR^2 + \varepsilon\mathcal{I}'_4 + \frac{C_\varepsilon}{R^2}\mathcal{I}'_3\\ \leq& \varepsilon R^2\mathcal{I}_1 + \frac{C}{R^{2-36\beta}}\left(\mathcal{I}_2+\mathcal{I}'_4\right) + \varepsilon\mathcal{I}'_4 + \frac{C_\varepsilon}{R^2}\mathcal{I}'_3 + CR^2,\end{aligned} \tag{3.13}$$

where we used (3.6), (3.10) and the following Young inequality

$$\mathrm{Re}(f_\alpha\eta_{\bar\alpha})|\partial f|^2e^{-2\beta f}\eta^{s-3} \leq |\partial f||\partial\eta||\partial f|^2e^{-2\beta f}\eta^{s-3} \leq \varepsilon|\partial f|^4e^{-2\beta f}\eta^{s-2} + C_\varepsilon|\partial f|^2e^{-2\beta f}|\partial\eta|^2\eta^{s-4}.$$

Hence, by choosing ε small enough and R large enough in (3.13) from (3.10) we have

$$\mathcal{I}_4 \leq \mathcal{I}'_4 \leq C\varepsilon R^2\mathcal{I}_1 + \frac{C}{R^{2-36\beta}}\left(\mathcal{I}_2+\mathcal{I}'_3\right) + CR^2. \tag{3.14}$$

Coming back to (3.4) we obtain, by using (3.14),

$$\begin{aligned}\mathcal{I}_2 \leq & C\varepsilon R^2\mathcal{I}_1 + \frac{C}{R^2}\mathcal{I}_3 + C\mathcal{I}_4\\ \leq & C\varepsilon R^2\mathcal{I}_1 + \frac{C}{R^2}\mathcal{I}_3' + 2\varepsilon R^2\mathcal{I}_1 + \frac{C}{R^{2-36\beta}}\left(\mathcal{I}_2 + \mathcal{I}_3'\right) + CR^2\\ \leq & C\varepsilon R^2\mathcal{I}_1 + \frac{C}{R^{2-36\beta}}\left(\mathcal{I}_2 + \mathcal{I}_3'\right) + CR^2\,,\end{aligned}$$

i.e. for R large enough and β small enough,

$$\mathcal{I}_2 \leq C\varepsilon R^2\mathcal{I}_1 + \frac{C}{R^{2-36\beta}}\mathcal{I}_3' + CR^2\,. \tag{3.15}$$

By using (3.9) in (3.15) we get

$$\mathcal{I}_2 \leq C\varepsilon R^2\mathcal{I}_1 + CR^{40\beta} + CR^2 \leq C\varepsilon R^2\mathcal{I}_1 + CR^2\,. \tag{3.16}$$

Finally, from (3.2) and (3.16) we obtain

$$\mathcal{I}_1 \leq C\varepsilon\mathcal{I}_1 + C\,,$$

i.e. for $\varepsilon > 0$ small enough

$$\mathcal{I}_1 = \int_{B_R} \mathcal{M}\Psi^{-\beta}\eta^s \leq C,$$

and, from (3.16),

$$\mathcal{I}_2 = \int_{B_R} |g|^2\Psi^{-\beta}\eta^{s-2} \leq CR^2.$$

Then

$$\int_{\mathbb{H}^1} \mathcal{M}\Psi^{-\beta} \leq C.$$

In particular, from Lemma 2.2, it follows that

$$\begin{aligned}\int_{\mathbb{H}^1} \mathcal{M}\Psi^{-\beta}\eta^s \leq & \frac{C}{R}\left(\int_{\mathrm{supp}|\partial\eta|} \mathcal{M}\Psi^{-\beta}\eta^s\right)^{1/2}\left(\int_{\mathrm{supp}|\partial\eta|} |g|^2\Psi^{-\beta}\eta^{s-2}\right)^{1/2}\\ \leq & C\left(\int_{A_R} \mathcal{M}\Psi^{-\beta}\right)^{1/2} \longrightarrow 0\end{aligned}$$

as $R \to \infty$, i.e.

$$\mathcal{M} \equiv 0.$$

The conclusion follows arguing as in [27, Section 3].

4. Gradient estimates and proof ot Theorem 1.2

In this section we first prove some gradient estimates on positive solutions of equation (1.2), that we then use to prove Theorem 1.2.

Proposition 4.1. *Let $n \in \mathbb{N}$, $\xi_0 \in \mathbb{H}^n$, $R \geq 1$ and let u be a positive solution of (1.2) in $B_{3R}(\xi_0)$. If*

$$u(\xi) \leq \frac{a}{R^{\frac{Q-2}{2}}} \quad \forall \xi \in B_{3R}(\xi_0), \tag{4.1}$$

for some $a > 0$, then there exists $C = C(a) > 0$ independent of R such that

$$\sup_{B_{2R}(\xi_0)\setminus B_R(\xi_0)} \frac{|\partial u|}{u} \leq \frac{C}{R}. \tag{4.2}$$

Remark 4.2. *We explicitly note that, by Proposition 4.1, if u is a positive solution of (1.2) in $B_{3R}(\xi_0)$ satisfying (4.1) and if $f = \frac{1}{n}\log u$ then*

$$\sup_{B_{2R}(\xi_0)\setminus B_R(\xi_0)} |\partial f|^2 \leq \frac{C}{R^2}$$

for some $C > 0$.

Proof of Proposition 4.1. The proof follows arguments similar in spirit to [25], where the authors consider harmonic functions for the sub-Laplacian. Let $f = \frac{1}{n}\log u$ and let

$$\nabla f := f_{\bar\alpha} Z_\alpha + f_\alpha Z_{\bar\alpha}.$$

Then we have, see also (2.3),

$$\begin{aligned} |\nabla f|^2 &= 2 f_{\bar\alpha} f_\alpha = 2|\partial f|^2, \\ \Delta_{\mathbb{H}^n} f &= -n|\nabla f|^2 - 2ne^{2f}, \\ \nabla \Delta_{\mathbb{H}^n} f &= -n\nabla|\nabla f|^2 - 4ne^{2f}\nabla f. \end{aligned}$$

For simplicity, let $\xi_0 = 0$. Now if η is any nonnegative smooth cutoff function on B_{3R} we define for every $t \in (0,1]$

$$F := t(|\nabla f|^2 + \gamma t \eta f_0^2)\,,$$

where $\gamma > 0$ is to be chosen later. Then we have

$$\Delta_{\mathbb{H}^n} F = t\Delta_{\mathbb{H}^n}|\nabla f|^2 + \gamma t^2 \eta \Delta_{\mathbb{H}^n} f_0^2 + 2\gamma t^2 \langle \nabla \eta, \nabla f_0^2\rangle + \gamma t^2 f_0^2 \Delta_{\mathbb{H}^n}\eta. \tag{4.3}$$

We recall the following Bochner inequality for the sub-Laplacian on the Heisenberg group (see [39] and references therein and see also [11], [25, Lemma 2.3]): for every function $\nu : B_{3R} \to (0, +\infty)$ we have

$$\Delta_{\mathbb{H}^n}|\nabla f|^2 \geq \frac{1}{n}(\Delta_{\mathbb{H}^n} f)^2 + n f_0^2 + 2\langle \nabla f, \nabla \Delta_{\mathbb{H}^n} f\rangle - \frac{2}{\nu}|\nabla f|^2 - 2\nu|\nabla f_0|^2. \tag{4.4}$$

Moreover for every $\varepsilon \in (0, \frac{1}{4})$ we have

$$\begin{aligned} (\Delta_{\mathbb{H}^n} f)^2 &= \Big(-\frac{\varepsilon}{t}F - (n-\varepsilon)|\nabla f|^2 - 2ne^{2f} + \varepsilon\gamma t\eta f_0^2\Big)^2 \\ &= \frac{\varepsilon^2}{t^2}F^2 + \frac{2\varepsilon(n-\varepsilon)}{t}F|\nabla f|^2 - 2\varepsilon^2\gamma f_0^2 F\eta \\ &\quad + \frac{4n\varepsilon}{t}e^{2f}F + \big(-(n-\varepsilon)|\nabla f|^2 - 2ne^{2f} + \varepsilon\gamma t\eta f_0^2\big)^2 \\ &\geq \frac{\varepsilon^2}{t^2}F^2 + \frac{2\varepsilon(n-\varepsilon)}{t}F|\nabla f|^2 - 2\varepsilon^2\gamma f_0^2 F\eta, \end{aligned} \tag{4.5}$$

and

$$\Delta_{\mathbb{H}^n} f_0^2 = -8ne^{2f} f_0^2 - 2n\langle \nabla f, \nabla f_0^2\rangle + 2|\nabla f_0|^2. \tag{4.6}$$

By (4.3)-(4.6) we obtain

$$\begin{aligned}\Delta_{\mathbb{H}^n}F \geq{}& \frac{\varepsilon^2}{nt}F^2 + 2\varepsilon(1-\tfrac{\varepsilon}{n})F|\nabla f|^2 - \frac{2}{n}t\varepsilon^2\gamma\eta f_0^2 F + ntf_0^2\\ &- 8nte^{2f}|\nabla f|^2 - 2nt\langle\nabla|\nabla f|^2,\nabla f\rangle - \frac{2t}{\nu}|\nabla f|^2 - 2\nu t|\nabla f_0|^2\\ &+ t^2\gamma\eta(-8ne^{2f}f_0^2 - 2n\langle\nabla f,\nabla f_0^2\rangle + 2|\nabla f_0|^2)\\ &+ 4t^2\gamma f_0\langle\nabla\eta,\nabla f_0\rangle + t^2\gamma f_0^2\Delta_{\mathbb{H}^n}\eta.\end{aligned}$$

Since

$$\langle\nabla|\nabla f|^2,\nabla f\rangle = \langle\tfrac{1}{t}\nabla F - t\gamma\eta\nabla f_0^2 - t\gamma f_0^2\nabla\eta,\nabla f\rangle$$

then we have

$$\begin{aligned}\Delta_{\mathbb{H}^n}F \geq{}& \frac{\varepsilon^2}{nt}F^2 + 2\varepsilon(1-\tfrac{\varepsilon}{n})F|\nabla f|^2 - \frac{2}{n}t\varepsilon^2\gamma\eta f_0^2 F + ntf_0^2\\ &- 8nte^{2f}|\nabla f|^2 - 2n\langle\nabla F,\nabla f\rangle + 2nt^2\gamma f_0^2\langle\nabla\eta,\nabla f\rangle - \frac{2t}{\nu}|\nabla f|^2 - 2\nu t|\nabla f_0|^2\\ &+ t^2\gamma\eta(-8ne^{2f}f_0^2 + 2|\nabla f_0|^2) + 4t^2\gamma f_0\langle\nabla\eta,\nabla f_0\rangle + t^2\gamma f_0^2\Delta_{\mathbb{H}^n}\eta.\end{aligned}$$

We remark that in what follows we are going to use the above inequality evaluated only at a specific point ξ such that $\eta(\xi)\neq 0$. For this reason, from now on, we will go on with our computations only at those points, neglecting points ξ such that $\eta(\xi)=0$. Therefore, for every ξ such that $\eta(\xi)\neq 0$ there holds

$$|4t^2\gamma f_0\langle\nabla\eta,\nabla f_0\rangle| \leq t^2\gamma\eta|\nabla f_0|^2 + 4t^2\gamma\frac{|\nabla\eta|^2}{\eta}f_0^2,$$

then

$$\begin{aligned}\Delta_{\mathbb{H}^n}F \geq{}& \frac{\varepsilon^2}{nt}F^2 + 2\varepsilon(1-\tfrac{\varepsilon}{n})F|\nabla f|^2 - \frac{2}{n}t\varepsilon^2\gamma\eta f_0^2 F + ntf_0^2\\ &- 8nte^{2f}|\nabla f|^2 - 2n\langle\nabla F,\nabla f\rangle + 2nt^2\gamma f_0^2\langle\nabla\eta,\nabla f\rangle - \frac{2t}{\nu}|\nabla f|^2 - 2\nu t|\nabla f_0|^2\\ &+ t^2\gamma\eta(-8ne^{2f}f_0^2 + 2|\nabla f_0|^2) - t^2\gamma\eta|\nabla f_0|^2 - 4t^2\gamma\frac{|\nabla\eta|^2}{\eta}f_0^2 + t^2\gamma f_0^2\Delta_{\mathbb{H}^n}\eta.\end{aligned}$$

For every ξ such that $\eta(\xi)\neq 0$ we choose $\nu = \frac{t\gamma\eta}{2}$, then

$$\begin{aligned}\Delta_{\mathbb{H}^n}F \geq{}& \frac{\varepsilon^2}{nt}F^2 + 2\varepsilon(1-\tfrac{\varepsilon}{n})F|\nabla f|^2 - \frac{2}{n}t\varepsilon^2\gamma\eta f_0^2 F + ntf_0^2\\ &- 8nte^{2f}|\nabla f|^2 - 2n\langle\nabla F,\nabla f\rangle + 2nt^2\gamma f_0^2\langle\nabla\eta,\nabla f\rangle - \frac{4}{\gamma\eta}|\nabla f|^2\\ &- 8nt^2\gamma\eta e^{2f}f_0^2 - 4t^2\gamma\frac{|\nabla\eta|^2}{\eta}f_0^2 + t^2\gamma f_0^2\Delta_{\mathbb{H}^n}\eta.\end{aligned}$$

Now note that for every ξ such that $\eta(\xi)\neq 0$

$$|2nt^2\gamma f_0^2\langle\nabla\eta,\nabla f\rangle| \leq \varepsilon^2 nt\gamma\eta f_0^2 F + \frac{nt^2\gamma}{\varepsilon^2}f_0^2\frac{|\nabla\eta|^2}{\eta},$$

hence

$$\begin{aligned}\Delta_{\mathbb{H}^n} F \geq\ & \frac{\varepsilon^2}{nt}F^2 + 2\varepsilon(1-\tfrac{\varepsilon}{n})F|\nabla f|^2 - \left(n+\tfrac{2}{n}\right)t\varepsilon^2\gamma\eta f_0^2 F + ntf_0^2\\ & - 8nte^{2f}|\nabla f|^2 - 2n\langle\nabla F,\nabla f\rangle - \frac{nt^2\gamma}{\varepsilon^2}\frac{|\nabla\eta|^2}{\eta}f_0^2 - \frac{4}{\gamma\eta}|\nabla f|^2\\ & - 8nt^2\gamma\eta e^{2f}f_0^2 - 4t^2\gamma\frac{|\nabla\eta|^2}{\eta}f_0^2 + t^2\gamma f_0^2\Delta_{\mathbb{H}^n}\eta.\end{aligned}$$

Let $H=\eta F$, then $H\geq 0$ and for every ξ such that $H(\xi)\neq 0$

$$\begin{aligned}\Delta_{\mathbb{H}^n} H \geq\ & F\Delta_{\mathbb{H}^n}\eta + 2\langle\nabla\eta,\nabla F\rangle + \eta\Bigg[\frac{\varepsilon^2}{nt}F^2 + 2\varepsilon(1-\tfrac{\varepsilon}{n})F|\nabla f|^2 - \left(n+\tfrac{2}{n}\right)t\varepsilon^2\gamma\eta f_0^2 F\\ & + ntf_0^2 - 8nte^{2f}|\nabla f|^2 - 2n\langle\nabla F,\nabla f\rangle - \frac{nt^2\gamma}{\varepsilon^2}\frac{|\nabla\eta|^2}{\eta}f_0^2 - \frac{4}{\gamma\eta}|\nabla f|^2\\ & - 8nt^2\gamma\eta e^{2f}f_0^2 - 4t^2\gamma\frac{|\nabla\eta|^2}{\eta}f_0^2 + t^2\gamma f_0^2\Delta_{\mathbb{H}^n}\eta\Bigg].\end{aligned}$$

Let $\eta=\eta(|\xi|)$ be a radial, nonnegative, smooth cutoff function such that $\eta=0$ in $B_{\frac{R}{2}}\cup B^c_{\frac{5}{2}R}$, $\eta=1$ on $A_{2R}=B_{2R}\setminus B_R$ and satisfying

$$|\nabla\eta|\leq\frac{C_1}{R}\sqrt{\eta},\quad \Delta_{\mathbb{H}^n}\eta\geq-\frac{C_1}{R^2}$$

on B_{3R}, for some universal constant $C_1>1$. Since u satisfies (4.1) we have

$$e^{2f}=u^{\frac{2}{n}}\leq\frac{a^{\frac{2}{n}}}{R^2}$$

in $B_{\frac{5}{2}R}\setminus B_{\frac{R}{2}}$. Since $t\in(0,1)$ and $0\leq\eta\leq 1$, at any ξ such that $H(\xi)>0$, which in particular must satisfy $\xi\in B_{\frac{5}{2}R}\setminus B_{\frac{R}{2}}$, we thus obtain

$$\begin{aligned}t\eta\Delta_{\mathbb{H}^n}H \geq\ & -\frac{C_1 t}{R^2}H + 2t\langle\nabla\eta,\nabla H\rangle - 2tF|\nabla\eta|^2 + \frac{\varepsilon^2}{n}H^2\\ & + t\eta|\nabla f|^2\left(2\varepsilon(1-\tfrac{\varepsilon}{n})H - 8ne^{2f} - \frac{4}{\gamma}\right)\\ & + t^2\eta^2 f_0^2\left(n-(n+\tfrac{2}{n})\varepsilon^2\gamma H - 8n\gamma e^{2f} - \frac{C_1^2(n+5)\gamma}{\varepsilon^2R^2}\right)\\ & - 2nt\eta\langle\nabla H,\nabla f\rangle + 2nt\langle\nabla\eta,\nabla f\rangle H\\ \geq\ & \frac{\varepsilon^2}{n}H^2 - \frac{3C_1^2}{\varepsilon R^2}H + 2t\langle\nabla H,\nabla\eta - n\eta\nabla f\rangle\\ & + t\eta|\nabla f|^2\left(2\varepsilon(1-\tfrac{\varepsilon}{n})H - \frac{8na^{\frac{2}{n}}}{R^2} - \frac{4}{\gamma}\right)\\ & + t^2\eta^2 f_0^2\left(n-(n+\tfrac{2}{n})\varepsilon^2\gamma H - \frac{8n\gamma a^{\frac{2}{n}}}{R^2} - \frac{C_1^2(n+5)\gamma}{\varepsilon^2R^2}\right) + 2nt\langle\nabla\eta,\nabla f\rangle H.\end{aligned}$$

Now we note that

$$|2nt\langle\nabla\eta,\nabla f\rangle H|\leq\varepsilon tH|\nabla f|^2\eta + \frac{n^2t}{\varepsilon}\frac{|\nabla\eta|^2}{\eta}H\leq\varepsilon tH|\nabla f|^2\eta + \frac{C_1^2n^2}{\varepsilon R^2}H.$$

Therefore we have

$$\begin{aligned} t\eta\Delta_{\mathbb{H}^n}H - 2t\langle\nabla H, \nabla\eta - n\eta\nabla f\rangle \geq{}& \frac{\varepsilon^2}{n}H^2 - \frac{C_1^2(n^2+3)}{\varepsilon R^2}H \\ &+ t\eta|\nabla f|^2\left(\varepsilon(1-\tfrac{2\varepsilon}{n})H - \frac{8na^{\frac{2}{n}}}{R^2} - \frac{4}{\gamma}\right) \\ &+ t^2\eta^2 f_0^2\left(n - (n+\tfrac{2}{n})\varepsilon^2\gamma H - \frac{8n\gamma a^{\frac{2}{n}}}{R^2} - \frac{C_1^2(n+5)\gamma}{\varepsilon^2R^2}\right). \end{aligned}$$

Choosing $\gamma = \varepsilon^{\frac{5}{2}}R^2$ we obtain

$$\begin{aligned} t\eta\Delta_{\mathbb{H}^n}H - 2t\langle\nabla H, \nabla\eta - n\eta\nabla f\rangle \geq{}& \left(\varepsilon^2 H - \frac{C_1^2 n(n^2+3)}{\varepsilon R^2}\right)\frac{H}{n} \\ &+ t\eta|\nabla f|^2\left(\frac{\varepsilon}{2}H - \frac{8na^{\frac{2}{n}}+4}{\varepsilon^{\frac{5}{2}}R^2}\right) \\ &+ t^2\eta^2 f_0^2\left[n - (n+\tfrac{2}{n})\varepsilon^{\frac{9}{2}}R^2H - \left(C_1^2(n+5) + 8na^{\frac{2}{n}}\right)\varepsilon^{\frac{1}{2}}\right] \\ \geq{}& \left(\varepsilon^2 H - \frac{C_0}{\varepsilon R^2}\right)\frac{H}{n} \\ &+ t\eta|\nabla f|^2\left(\frac{\varepsilon}{2}H - \frac{C_0}{\varepsilon^{\frac{5}{2}}R^2}\right) \\ &+ t^2\eta^2 f_0^2\left[n - (n+\tfrac{2}{n})\varepsilon^{\frac{9}{2}}R^2H - C_0\varepsilon^{\frac{1}{2}}\right], \end{aligned}$$

where

$$C_0 := \max\left\{C_1^2 n(n^2+3), 8na^{\frac{2}{n}}+4, C_1^2(n+5) + 8na^{\frac{2}{n}}\right\}.$$

We observe that H achieves its maximum on $\overline{B}_{\frac{5}{2}R} \setminus B_{\frac{R}{2}}$ at some point p. If $H(p) = 0$, then $H = 0$ on A_R and $F = 0$ on A_R, since $\eta \equiv 1$ on A_R; therefore $|\nabla f| = 0$ on A_R and the proof is complete. If $H(p) > 0$, then $\eta(p) > 0$; in this case we denote by $H_* = H(p)$ and $\eta_* = \eta(p)$, then

$$\begin{aligned} 0 \geq{}& \left(\varepsilon^2 H_* - \frac{C_0}{\varepsilon R^2}\right)\frac{H_*}{n} + t\eta_*|\nabla f|^2\left(\frac{\varepsilon}{2}H_* - \frac{C_0}{\varepsilon^{\frac{5}{2}}R^2}\right) \\ &+ t^2\eta_*^2 f_0^2\left(n - (n+\tfrac{2}{n})\varepsilon^{\frac{9}{2}}R^2H_* - C_0\varepsilon^{\frac{1}{2}}\right). \end{aligned} \tag{4.7}$$

We now fix $\varepsilon > 0$ such that

$$\varepsilon < \frac{1}{C_0}, \quad \varepsilon < \frac{1}{4C_0^2}, \quad \text{and} \quad \varepsilon < \frac{n^2}{\left(n+\frac{2}{n}+C_0\right)^2}. \tag{4.8}$$

We explicitly note that ε and C_0 depend on u only trough a. We claim that for $t = 1$

$$H_* < \frac{2}{\varepsilon^4 R^2}.$$

If not, by contradiction, for $t = 1$ we have

$$H_* \geq \frac{2}{\varepsilon^4 R^2}.$$

Now note that H (and thus also H_*) are continuous functions in $t \in [0,1]$, satisfying $H_* = 0$ for $t = 0$ and $H_* \geq \frac{2}{\varepsilon^4 R^2}$ for $t = 1$. Then, there exists $t_1 \in (0,1)$ such that at $t = t_1$

$$H_* = \frac{1}{\varepsilon^4 R^2}.$$

Then at $t = t_1$ we have from (4.7) and (4.8)

$$0 \geq \left(\frac{1}{\varepsilon^2 R^2} - \frac{C_0}{\varepsilon R^2}\right)\frac{H_*}{n} + t_1\eta_*|\nabla f|^2\left(\frac{1}{2\varepsilon^3 R^2} - \frac{C_0}{\varepsilon^{\frac{5}{2}} R^2}\right) + t_1^2\eta_*^2 f_0^2\left(n - (n + \tfrac{2}{n} + C_0)\varepsilon^{\frac{1}{2}}\right) > 0\,,$$

thus reaching a contradiction. We conclude that for $C = \frac{2}{\varepsilon^4}$, which is independent of R and which depends on u only through a, at $t = 1$ we have

$$H_* = \max_{B_{\frac{5}{2}R}\setminus B_{\frac{R}{2}}} H \leq \frac{C}{R^2}$$

for every $R \geq 1$, which in turn implies that

$$\sup_{B_{2R}\setminus B_R} |\nabla f|^2 \leq \max_{B_{\frac{5}{2}R}\setminus B_{\frac{R}{2}}} H \leq \frac{C}{R^2}$$

for every $R \geq 1$. Recalling the definition of f, we conclude that (4.2) holds. □

Proof of Theorem 1.2. Let $n \geq 2$ and $R > 1$, we choose, as in the proof of Theorem 1.1, a real cut-off function η such that $\eta \equiv 1$ in $B_{R/2}$, $\eta \equiv 0$ in B_R^c and $|\partial\eta| \leq \frac{c}{R}$ in $A_R := B_R \setminus B_{R/2}$, and $s > 4$.

From Lemma 2.3 we have

$$\begin{aligned}\int_{\mathbb{H}^n} \eta^s \mathcal{M} \leq & C\int_{\mathbb{H}^n} e^{2(n-1)f}|\partial\eta|^2\eta^{s-2}|g|^2 \\ \leq & \frac{C}{R^2}\int_{A_R} e^{2(n-1)f}\eta^{s-2} g\bar{g} \\ = & -\frac{C}{R^2}\int_{A_R} e^{2(n-1)f}\eta^{s-2}\mathrm{Re}\left(f_{\alpha\bar{\alpha}}\bar{g}\right)\end{aligned} \tag{4.9}$$

where we used (2.5). Integrating by parts we get

$$\begin{aligned}-\frac{C}{R^2}\int_{A_R} e^{2(n-1)f}\eta^{s-2}\mathrm{Re}\left(f_{\alpha\bar{\alpha}}\bar{g}\right) = & \frac{C}{R^2}\int_{A_R} e^{2(n-1)f}\mathrm{Re}\left(\eta^{s-2} f_\alpha \bar{g}_{\bar{\alpha}}\right) \\ & + \frac{C(s-2)}{R^2}\int_{A_R} e^{2(n-1)f}\mathrm{Re}\left(\eta^{s-3} f_\alpha \eta_{\bar{\alpha}}\bar{g}\right) \\ & + \frac{C}{R^2}\int_{A_R} e^{2(n-1)f}\eta^{s-2}|\partial f|^2\mathrm{Re}(\bar{g}) \\ =: & \frac{C}{R^2}\left(\mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3\right).\end{aligned}$$

In particular, we have

$$\int_{A_R} e^{2(n-1)f}\eta^{s-2}|g|^2 \leq C(\mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3). \tag{4.10}$$

From (2.11), using Young and Cauchy-Schwartz inequalities, we obtain

$$\begin{aligned}
\mathcal{J}_1 &= \int_{A_R} e^{2(n-1)f}\mathrm{Re}\left(\eta^{s-2} f_\alpha \bar{g}_{\bar{\alpha}}\right)\\
&= \int_{A_R} e^{2(n-1)f}\mathrm{Re}\left(\eta^{s-2} f_\alpha (D_{\bar{\alpha}} + E_{\bar{\alpha}} - G_{\bar{\alpha}})\right) + 2\int_{A_R} e^{2(n-1)f}\mathrm{Re}\left(\eta^{s-2}|\partial f|^2 \bar{g}\right)\\
&\leq \frac{1}{2}\int_{A_R} e^{2(n-1)f}\eta^s |D_{\bar{\alpha}} + E_{\bar{\alpha}} - G_{\bar{\alpha}}|^2 + \frac{1}{2}\int_{A_R} e^{2(n-1)f}\eta^{s-4}|\partial f|^2\\
&\quad + 2\int_{A_R} e^{2(n-1)f}\left(|\partial f|^4 + e^{2f}|\partial f|^2\right)\eta^{s-2}\\
&\leq \frac{1}{2}\int_{A_R} e^{2(n-1)f}\eta^s \left(|D_{\bar{\alpha}} + G_{\bar{\alpha}}| + |E_{\bar{\alpha}} - G_{\bar{\alpha}}| + |G_{\bar{\alpha}}|\right)^2 + \frac{1}{2}\int_{A_R} e^{2(n-1)f}\eta^{s-4}|\partial f|^2\\
&\quad + \frac{1}{2}\int_{A_R} e^{2(n+1)f}\eta^{s-2} + 4\int_{A_R} e^{2(n-1)f}|\partial f|^4\eta^{s-2}\\
&\leq \frac{3}{2}\int_{A_R} \eta^s \mathcal{M} + \frac{1}{2}\int_{A_R} e^{2(n-1)f}\eta^{s-4}|\partial f|^2\\
&\quad + \frac{1}{2}\int_{A_R} e^{2(n+1)f}\eta^{s-2} + 4\int_{A_R} e^{2(n-1)f}|\partial f|^4\eta^{s-2}
\end{aligned}$$

where we also used the definitions of g in (2.4) and of $\mathcal{M}$ in Proposition 2.1.

Now, we prove that

$$\int_{A_R} e^{2(n-1)f}\eta^{s-4}|\partial f|^2 \leq CR^2\,, \tag{4.11}$$

indeed, by Proposition 4.1, we have

$$\sup_{A_R}|\partial f|^2 \leq \frac{C}{R^2}.$$

Moreover, by assumption, in A_R we have

$$e^{2(n-1)f} = u^{\frac{2(n-1)}{n}} = u^{\frac{2(Q-4)}{Q-2}} \leq \frac{C}{R^{Q-4}}.$$

Therefore, using the volume estimate (2.1),

$$\int_{A_R} e^{2(n-1)f}\eta^{s-4}|\partial f|^2 \leq \frac{C}{R^{Q-2}}|B_R| \leq CR^2\,,$$

and (4.11) follows. Then from (4.11) we have

$$\int_{A_R} e^{2(n-1)f}\eta^{s-2}|\partial f|^4 \leq \frac{C}{R^2}\int_{A_R} e^{2(n-1)f}\eta^{s-2}|\partial f|^2 \leq C\,, \tag{4.12}$$

Thus, from the previous computation, (4.11) and (4.12), we obtain

$$\mathcal{J}_1 = \int_{A_R} e^{2(n-1)f}\mathrm{Re}\left(\eta^{s-2} f_\alpha \bar{g}_{\bar{\alpha}}\right) \leq C\int_{A_R}\eta^s\mathcal{M} + CR^2 + \frac{1}{2}\int_{A_R} e^{2(n+1)f}\eta^{s-2} + C\,.$$

On the other hand

$$\begin{aligned}\mathcal{J}_2 = \int_{A_R} e^{2(n-1)f}\mathrm{Re}\left(\eta^{s-3} f_\alpha \eta_{\bar\alpha}\bar g\right) \leq & \frac{1}{2}\int_{A_R} e^{2(n-1)f}\eta^{s-4}|\partial f|^2 + \frac{1}{2}\int_{A_R} e^{2(n-1)f}\eta^{s-2}|g|^2|\partial\eta|^2\\ \leq & CR^2 + \frac{C}{R^2}\int_{A_R} e^{2(n-1)f}\eta^{s-2}|g|^2\end{aligned}$$

and, for every $\delta > 0$ small enough,

$$\begin{aligned}\mathcal{J}_3 = \int_{A_R} e^{2(n-1)f}\eta^{s-2}|\partial f|^2\mathrm{Re}(\bar g) \leq & \frac{1}{4\delta}\int_{A_R} e^{2(n-1)f}\eta^{s-2}|\partial f|^4 + \delta\int_{A_R} e^{2(n-1)f}\eta^{s-2}|g|^2\\ \leq & \frac{C}{\delta} + \delta\int_{A_R} e^{2(n-1)f}\eta^{s-2}|g|^2\end{aligned}$$

Thus, from (4.10) and (4.11), we obtain

$$\int_{A_R} e^{2(n-1)f}\eta^{s-2}|g|^2 \leq C\int_{A_R}\eta^s\mathcal{M} + \frac{C}{\delta}R^2 + \frac{1}{2}\int_{A_R} e^{2(n+1)f}\eta^{s-2} + \left(\frac{C}{R^2}+\delta\right)\int_{A_R} e^{2(n-1)f}\eta^{s-2}|g|^2\,.$$

For δ small enough and R large enough, we get

$$\int_{A_R} e^{2(n-1)f}\eta^{s-2}|g|^2 \leq C\int_{A_R}\eta^s\mathcal{M} + CR^2 + \frac{2}{3}\int_{A_R} e^{2(n+1)f}\eta^{s-2}\,.$$

Recalling (2.9), we have

$$\int_{A_R} e^{2(n-1)f}\eta^{s-2}\left(|\partial f|^4 + 2|\partial f|^2 e^{2f} + \frac{1}{3}e^{4f} + f_0^2\right) \leq C\int_{A_R}\eta^s\mathcal{M} + CR^2$$

and therefore

$$\int_{A_R} e^{2(n-1)f}\eta^{s-2}|g|^2 \leq C\int_{A_R}\eta^s\mathcal{M} + CR^2\,. \tag{4.13}$$

Going back to (4.9), for R large enough, we obtain

$$\int_{\mathbb{H}^n}\mathcal{M}\eta^s \leq \frac{C}{R^2}\int_{A_R} e^{2(n-1)f}\eta^{s-2}|g|^2 \leq C\,. \tag{4.14}$$

Hence,

$$\int_{\mathbb{H}^n}\mathcal{M} \leq C \quad \text{and} \quad \int_{A_R} e^{2(n-1)f}\eta^{s-2}|g|^2 \leq CR^2.$$

In particular, from Lemma 2.3, it follows that

$$\begin{aligned}\int_{\mathbb{H}^n}\mathcal{M}\eta^s \leq & \frac{C}{R}\left(\int_{\mathrm{supp}|\partial\eta|}\mathcal{M}\eta^s\right)^{1/2}\left(\int_{\mathrm{supp}|\partial\eta|} e^{2(n-1)f}|g|^2\eta^{s-2}\right)^{1/2}\\ \leq & C\left(\int_{A_R}\mathcal{M}\right)^{1/2} \longrightarrow 0\end{aligned}$$

as $R\to\infty$, i.e.

$$\mathcal{M} \equiv 0.$$

The conclusion follows arguing as in [27, Section 3]. □

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G. Catino, Dipartimento di Matematica, Politecnico di Milano, Piazza Leonardo da Vinci 32, 20133, Milano, Italy.
Email address: giovanni.catino@polimi.it

Y.Y. Li, Department of Mathematics, Rutgers University, 110 Frelinghuysen Road, Piscataway, NJ, USA
Email address: yyli@math.rutgers.edu

D. Monticelli, Dipartimento di Matematica, Politecnico di Milano, Piazza Leonardo da Vinci 32, 20133, Milano, Italy.
Email address: dario.monticelli@polimi.it

A. Roncoroni, Dipartimento di Matematica, Politecnico di Milano, Piazza Leonardo da Vinci 32, 20133, Milano, Italy.
Email address: alberto.roncoroni@polimi.it