SURFACTANTS IN A NON-LOCAL MODEL FOR PHASE TRANSITIONS

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Abstract. We investigate the influence of surfactants on stabilizing the formation of interfaces in non-local anisotropic two-phase fluid at equilibrium. The analysis focuses on singularly perturbed non-local van der Waals–Cahn–Hillard-type energies, supplemented with a term that accounts for the interaction between the surfactant and the fluid. We derive by Γ-convergence the effective surface tension model as the thickness of the transition layer vanishes and show that it decreases when the surfactant segregates to the interface.

Keywords: Phase transitions; Surfactants; Γ-convergence.

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Introduction

Within the framework of the gradient theory of fluid-fluid phase transitions, surfactants (surface active agents) can be incorporated into the model following the ideas proposed by Perkins, Sekerka, Warren and Langer. The resulting energy functional, which represents one of the easiest possible toy models for foam stability, is obtained as a modification of the classical van der Waals-Cahn-Hillard energy. The latter is a phase-field approximation of the classical perimeter functional and it has been analyzed by Modica and Mortola in terms of Γ-convergence in the limit as the transition length vanishes in [23, 24]. Then same kind of analysis has been carried out in [20] for the surfactant model. In particular an integral term accounting for the fluid-surfactant interaction is added to the classical Cahn-Hillard functional to model the adsorption of surface active molecules onto phase interfaces. As a result, the effective surface tension energy of a phase transition decreases as the density of the surfactant at the phase interface increases. The possibility of modulating phase transitions by changing the distribution of surfactant has found many practical applications in enhanced oil recovery, emulsion formation or foam stability to cite a few (see for instance [22] and the references therein).

In this paper, we revisit the aforementioned theory and the mathematical results obtained in [20] (see also [4, 9] for extensions to more general models of fluid-fluid or multiphase-fluid-fluid phase transitions in the presence of surfactants, and [15] for a surfactant model applied to solid-solid phase transitions) when the local model is replaced by a non-local one. By this, we mean that spatial inhomogeneities in the order parameter describing the fluid phases are captured through long-range differential operators, rather than traditional gradients. Energy functionals as the one we consider here arise for instance as continuum limits of Ising spin systems with Kac potentials in equilibrium statistical mechanics (see e.g. [3]).

It is noteworthy that our research aligns with a growing interest in non-local variational problems, which has emerged over the past two decades. For an overview of this field, we direct the reader to the survey [18] and the book [13]. Additionally, for a general variational framework dealing with convolution-type energies in Sobolev spaces, we highlight the recent work [5]. From the perspective of applied analysis, significant interest has also developed around the theory of peridynamics, which has seen various applications in this context (see, for example, [11, 17, 19, 25]).

Given a smooth bounded open set $\Omega \subset \mathbb{R}^N$ (the region occupied by the fluid and the surfactant), one considers a scalar function $u : \Omega \to \mathbb{R}$ and a non-negative function $\rho : \Omega \to [0, +\infty)$ representing the order parameter of the fluid and the density of the surfactant. For $u, \rho \in L^1(\Omega)$ we set $\mu = \rho\, dx$ and define for $\varepsilon > 0$ the family of energy functionals

$$F_\varepsilon(u,\mu) := \frac{1}{\varepsilon}\int_\Omega W(u(x))dx + \varepsilon \int_\Omega\int_\Omega \frac{1}{\varepsilon^N} J\left(\frac{y-x}{\varepsilon}\right)\left(\frac{|u(y)-u(x)|}{\varepsilon}\right)^2 dy\, dx$$
$$+ \varepsilon \int_\Omega \left(\int_\Omega \frac{1}{\varepsilon^N} J\left(\frac{y-x}{\varepsilon}\right)\frac{|u(y)-u(x)|}{\varepsilon}\, dy - \rho(x)\right)^2 dx.$$

Assuming $W : \mathbb{R} \to [0, +\infty)$ to be a double-well potential with wells at -1 and $+1$ like $W(z) = (z^2-1)^2$ and the kernel $J \in L^1(\mathbb{R}^N, [0,\infty))$ to be even and such that $\int_{\mathbb{R}^N} J(h)|h|\, dh < \infty$ (see Section 1.2 for the precise assumptions on W and J), one is interested in the asymptotic behaviour as ε tends to zero of $F_\varepsilon(u,\mu)$ in the sense of Γ-convergence. The first two integral terms in $F_\varepsilon(u,\mu)$ define the functional NCH_ε, a non-local version of the classical Cahn-Hillard functional, namely

$$NCH_\varepsilon(u) := \frac{1}{\varepsilon}\int_\Omega W(u(x))dx + \varepsilon \int_\Omega\int_\Omega \frac{1}{\varepsilon^N} J\left(\frac{y-x}{\varepsilon}\right)\left(\frac{|u(y)-u(x)|}{\varepsilon}\right)^2 dy\, dx.$$

The variational analysis of NCH_ε as ε vanishes has been carried out in [1] (see also [2] for fine properties of the minimizers). In particular in [1] it has been proved the pre-compactness in $BV(\Omega;\{-1,+1\})$ of sequences of phase-fields u_ε with uniformly bounded NCH_ε energy and it has been computed the Γ-limit of NCH_ε as ε tends to zero with respect to the L^1 convergence. Roughly speaking, the limit u of a converging subsequence of energy bounded u_ε is forced to take the values -1 and 1 almost everywhere, partitioning Ω in the two sets $\{u=-1\}$ and $\{u=1\}$. These two sets are interpreted as the pure phases of the fluid while their common boundary, corresponding to the jump set S_u of the function u, as the phase interface. The effective asymptotic energy of the system, captured by the Γ-limit of NCH_ε, is proved to be proportional to the surface measure of the interface S_u weighted by the anisotropic surface tension σ which depends on the shape of W. If one fixes the measure of the set $\{u_\varepsilon = -1\} = m$ to be strictly smaller than the measure of Ω, both phases will be non empty. Moreover as ε vanishes the minimal non-local Cahn-Hillard energy corresponds to the partition of Ω in two sets $\{u=-1\}$ (having measure m) and $\{u=1\}$ having the least anisotropic perimeter of the common boundary. Such an energy is achieved along a sequence u_ε of phase fields whose non-local spatial inhomogeneity $\int_\Omega J_\varepsilon(y-x)\frac{|u_\varepsilon(y)-u_\varepsilon(x)|}{\varepsilon}\, dy$ concentrates on $x \in S_u$. In this perspective, one can understand the role of the third term in F_ε as modelling the interaction between the surfactant and the fluid and favouring the phases of minimizers of NCH_ε to separate where the surfactant is present. The main result of this paper is stated in Theorem 2.2 and it is obtained combining Theorem 2.5 and Theorem 2.10. It shows that carrying out the Γ-limit of (an extension of) F_ε with respect to the strong L^1 convergence of the phase fields and the weak$*$-convergence of the surfactant measures, one obtains a limit functional which is finite for $u \in BV(\Omega,\{-1,1\})$ and $\mu \in \mathcal{M}(\Omega)$ (the space of positive Radon measures) where it takes the form

$$F(u,\mu) := \int_{S_u} \sigma\left(\nu_u, \frac{d\mu}{d\,\mathcal{H}^{\text{N-1}}\llcorner S_u}\right) d\,\mathcal{H}^{\text{N-1}}\,. \tag{0.1}$$

In the formula above ν_u represents the measure-theoretic normal to the phase interface, S_u denotes the jump set of u, and $\sigma : S^{N-1} \times [0,+\infty) \to [0,+\infty)$ is the anisotropic surface tension per unit length and it is obtained by the blow-up formula in Definition 1.3. For each direction $\nu \in S^{N-1}$, the function $\sigma(\nu,\cdot)$ decreases as a function of the relative density of the surfactant measure with respect to the surface measure of the interface. Consequently, increasing the surfactant density at the interface lowers the energy cost of the phase boundary, a phenomenon characteristic of immiscible fluids in the presence of surface-active agents. It is important to note that the limit energy in (0.1) differs from the isotropic one

derived in [20], where the surfactant model was obtained within the classical van der Waals-Cahn-Hilliard gradient theory of phase transitions. Additional examples of anisotropic surface tension energies can be found in [6, 7], where these are obtained as discrete-to-continuum Γ-limits, modeling the coarse-graining process originating from the microscopic Blume-Emery-Griffiths ternary surfactant model. In that model, the anisotropy arises due to the underlying lattice structure, as in classical Ising models, where particle interactions occur only along the lattice directions.

1. Notation and preliminaries

1.1. **Notation.** We denote by $\mathcal{L}^N$ and $\mathcal{H}^{N-1}$ the N-dimensional Lebesgue measure and the $(N-1)$-dimensional Hausdorff measure in $\mathbb{R}^N$, respectively. Given a Lebesgue measurable set $A \subset \mathbb{R}^N$ we denote its Lebesgue measure by $\mathcal{L}^N(A)$ or by $|A|$. We denote the open ball centred at x with radius r in $\mathbb{R}^N$ as $B_r(x)$ or by $B(x,r)$ and set $\omega_N = |B_1(0)|$. In the whole paper $\Omega \subset \mathbb{R}^N$ is assumed to be a bounded open set with regular boundary. We write $\mathcal{M}(\Omega)$ for the space of positive finite Radon measures on Ω, which we endow with the weak$*$ convergence. Given vectors $x, e \in \mathbb{R}^N$ where $e \neq 0$, we write x_e for the component of x in direction e. Given a function $u \in L^1(\Omega, \mathbb{R})$, we denote by S_u the approximate discontinuity set of u, i.e., the set of those points $x \in \Omega$ for which no $z \in \mathbb{R}$ exists such that $\lim_{r\to 0^+} |B_r(x)|^{-1} \int_{B_r(x)} |u(y) - z| dy = 0$ holds. We denote by $BV(\Omega)$ the set of functions of bounded variation in Ω. We say that a measurable set $E \subset \mathbb{R}^N$ is a set of finite perimeter in Ω if $\chi_E \in BV(\Omega)$. Denoting by $P(E, \Omega)$ the De Giorgi's perimeter of E in Ω, if E is a set of finite perimeter we also write that $P(E, \Omega) = \mathcal{H}^{N-1}(\partial^* E \cap \Omega) < +\infty$ where $\partial^* E$ stands for the reduced boundary of E. If $u \in BV(\Omega; \{a, b\})$ is a function of bounded variation in Ω taking only the two distinct values $a, b \in \mathbb{R}$, the $(N-1)$-Hausdorff measure of S_u equals the perimeter of the level set $\{u = a\}$ (and $\{u = b\}$) in Ω or in formula $\mathcal{H}^{\text{N-1}}(S_u) = P(\{u = a\}, \Omega)$. In what follows we will denote by ν_u the measure theoretic normal to S_u. For all properties of functions of bounded variations and of sets of finite perimeter needed in this paper we refer the reader to [8]. Finally we denote by c and C generic real positive constants that may vary from line to line and expression to expression within the same formula.

1.2. **The model.** In this section we introduce the energy functional whose asymptotic variational behaviour we are going to analyze in the rest of the paper. We consider potential functions W that satisfy the following set of assumptions

$$\begin{aligned} &W : \mathbb{R} \to [0, \infty) \text{ is continuous,} \\ &W \text{ has exactly two zeroes at the points } -1 \text{ and } 1, \\ &W \text{ has at least linear growth at inifinity} \end{aligned} \tag{1.1}$$

and kernels J such that

$$\begin{aligned} &J \in L^1(\mathbb{R}^N, [0, \infty)), \\ &J \text{ is even, i.e. } J(-h) = J(h), h \in \mathbb{R}^N \\ &\int_{\mathbb{R}^N} J(h)|h|\, dh < \infty. \end{aligned} \tag{1.2}$$

In what follows we set $J_\varepsilon(h) := \frac{1}{\varepsilon^N} J(h/\varepsilon)$.

Definition 1.1. Given a measurable set $A \subset \mathbb{R}^N$, $u \in L^1(A)$ and $\rho \in L^1(A, [0, \infty))$ we define

$$\begin{aligned} F_\varepsilon(u, \rho, A) :=& \frac{1}{\varepsilon} \int_A W(u(x)) dx + \varepsilon \int_A \int_A J_\varepsilon(y - x) \left(\frac{|u(y) - u(x)|}{\varepsilon} \right)^2 dy\, dx \\ &+ \varepsilon \int_A \left(\int_A J_\varepsilon(y - x) \frac{|u(y) - u(x)|}{\varepsilon}\, dy - \rho(x) \right)^2 dx. \end{aligned}$$

For $u \in L^1(\mathbb{R}^N)$ and $\rho \in L^1(A, [0, \infty))$ we define

$$\begin{aligned} \mathcal{F}_\varepsilon(u, \rho, A) :=& \frac{1}{\varepsilon} \int_A W(u(x)) dx + \varepsilon \int_A \int_{\mathbb{R}^N} J_\varepsilon(y - x) \left(\frac{|u(y) - u(x)|}{\varepsilon} \right)^2 dy\, dx \\ &+ \varepsilon \int_A \left(\int_{\mathbb{R}^N} J_\varepsilon(y - x) \frac{|u(y) - u(x)|}{\varepsilon} dy - \rho(x) \right)^2 dx \end{aligned}$$

and

$$\mathcal{F}(u,\rho,A) := \int_A W(u(x))dx + \int_A \int_{\mathbb{R}^N} J(y-x)\left(|u(y)-u(x)|\right)^2 dy\,dx + \int_A \left(\int_{\mathbb{R}^N} J(y-x)|u(y)-u(x)|dy - \rho(x)\right)^2 dx$$

In order to state the Γ-convergence result later on we extend the definition of F_ε to $L^1(\Omega)\times\mathcal{M}(\Omega)$.

Definition 1.2. Given a measurable set $A\subset\mathbb{R}^N$ we define $F_\varepsilon(\cdot,\cdot,A): L^1(A)\times\mathcal{M}(A)\to[0,\infty]$ as

$$F_\varepsilon(u,\mu,A) = \begin{cases} F_\varepsilon(u,\rho,A), & \text{if } \mu=\rho\mathcal{L}^N \\ +\infty, & \text{otherwise.}\end{cases} \tag{1.3}$$

Here below we define the effective energy density of our limit model. The latter will depend on the density of the limit surfactant measure on the jump set S_u of the order parameter u and on the normal to S_u.

Definition 1.3. Given $e\in\mathbb{R}^N$ we denote by $\mathcal{C}_e$, the set of all $(N-1)$-dimensional cubes centered at 0 which lie on the hyperplane orthogonal to e. For $C\in\mathcal{C}_e$ we say that $u:\mathbb{R}^N\to\mathbb{R}$ is C-periodic, if

$$u(x+ra)=u(x),$$

where a is any of the $N-1$ axes of C and r denotes the side length of C. For any $C\in\mathcal{C}_e$ we also set

$$T_C := \{y+te : y\in C, t\in\mathbb{R}\}.$$

Given $C\in\mathcal{C}_e$ and $\gamma\geq 0$ we define the set of admissible (u,ρ) pairs $\mathcal{A}(e,\gamma)$ as

$$\mathcal{A}(e,\gamma) := \Big\{(u,\rho)|\, u:\mathbb{R}^N\to[-1,1] \text{ is } C\text{-periodic}, \lim_{x_e\to\pm\infty} u(x)=\pm1, \rho:\mathbb{R}^N\to[0,\infty) \text{ is } C\text{-periodic}, \rho\in L^1(T_C), \int_{T_C}\rho\,dx\leq\gamma\,\mathcal{H}^{\text{N-1}}(C)\Big\}$$

and the limit energy density $\sigma(e,\gamma)$ as

$$\sigma(e,\gamma) := \inf\{\mathcal{F}(u,\rho,T_C)(\mathcal{H}^{\text{N-1}}(C))^{-1} : (u,\rho)\in\mathcal{A}(e,\gamma)\}.$$

Remark 1.4. We notice that for $\gamma_1\leq\gamma_2$ it holds that $\mathcal{A}(e,\gamma_1)\subset\mathcal{A}(e,\gamma_2)$. Hence by the very definition of $\sigma(e,\gamma)$, we obtain that $\sigma(e,\gamma_1)\geq\sigma(e,\gamma_2)$.

The effective limit energy of our model $F(u,\mu,\Omega): L^1(\Omega)\times\mathcal{M}(\Omega)\to[0,+\infty]$ is then defined as

$$F(u,\mu,\Omega) := \begin{cases} \displaystyle\int_{S_u}\sigma\left(\nu_u, \frac{d\mu}{d\,\mathcal{H}^{\text{N-1}}\llcorner S_u}\right)d\,\mathcal{H}^{\text{N-1}}, & \text{if } u\in \mathrm{BV}(\Omega,\{-1,1\}) \\ +\infty, & \text{otherwise,}\end{cases} \tag{1.4}$$

where σ is as in Definition 1.3.

1.3. **Preliminary results.** In this section we collect some preliminary definitions and results from [1].
We start with the definition of polyhedral functions.

Definition 1.5. *Polyhedral sets* $P\subset\mathbb{R}^N$ are defined as open sets with a Lipschitz boundary that is contained in the union of finitely many affine hyperplanes in $\mathbb{R}^N$. The intersection of any such hyperplane with the boundary of P is called a *face* of P. A *polyhedral set in* Ω is given as the intersection of a polyhedral set in $\mathbb{R}^N$ wit Ω. A function $u\in\mathrm{BV}(\Omega,\{-1,1\})$ is said to be a *polyhedral function* corresponding to a polyhedral set $A\subset\mathbb{R}^N$, if $\mathcal{H}^{\text{N-1}}(\partial A\cap\partial\Omega)=0$ and $\{u=1\}=\Omega\cap A$ (and $\{u=-1\}=\Omega\setminus A$). A subset Σ of an affine hyperplane H in $\mathbb{R}^N$ that is a polyhedral set in $\mathbb{R}^{N-1}$ is said to be a $(N-1)$-*dimensional polyhedral set* in $\mathbb{R}^N$. Let $E\subset\mathbb{R}^N$ and let Σ be a $(N-1)$-dimensional polyhedral set in $\mathbb{R}^N$. Denoting by ν_Σ the normal to the hyperplane H, we define the *projection* of Σ to E as the set $E_\Sigma := \{x\in E\,|\,\exists\, t\in\mathbb{R}, y\in\Sigma : x=y+t\nu_\Sigma\}$.

We continue by defining a notion of convergence of a non-local notion of traces of a sequence of functions.

Definition 1.6. Let $A \subset \mathbb{R}^N$ be an open set, $\Sigma \subset \mathbb{R}^N$ a Lipschitz hypersurface and $v : \Sigma \to [-1,1]$. For $h \in \mathbb{N}$ let $(\varepsilon_h)_h$ be such that $\varepsilon_h \to 0$ and let $(u_h)_h$ be a sequence of functions $u_h : A \to [-1,1]$. We say that the *ε_h-traces of u_h relative to A converge on Σ* to v, if

$$\int_\Sigma \int_{\{\frac{1}{\varepsilon_h}(A-y)\}} \left(\int_0^1 J\Big(\frac{x}{t}\Big) \frac{|x|}{t} t^{-N}\, dt |u_h(y+\varepsilon_h x) - v(y)| \right) dx\, d\mathcal{H}^{\text{N-1}}(y) \to_{h\to\infty} 0.$$

The next two propositions are proven in [1, Lemma 2.7] and [1, Proposition 2.5], respectively.

Proposition 1.7. *Let $A, A' \subset \mathbb{R}^N$ be disjoint sets and let $\Sigma \subset \mathbb{R}^N$ be a Lipschitz hypersurface such that for any $x \in A, x' \in A'$, the segment $[x,x']$ intersects Σ. Let moreover sequences $\varepsilon_h \to 0^+$ and $u_h : A \cup A' \to [-1,1]$ be given. If the ε_h-traces of u_h relative to A and A' converge on Σ to $v : \Sigma \to [-1,1]$ and $v' : \Sigma \to [-1,1]$ respectively, then there exists a constant C depending only on J such that*

$$\limsup_{h\to\infty} \frac{1}{\varepsilon_h} \int_A \int_{A'} J_{\varepsilon_h}(y-x)(u_h(y)-u_h(x))^2\, dy\, dx \le C \int_\Sigma |v'(x) - v(x)|\, d\mathcal{H}^{\text{N-1}}(x).$$

Remark 1.8. Under the assumptions of the previous proposition, in [1] the following stronger result has been proved:

$$\limsup_{h\to\infty} \frac{1}{\varepsilon_h} \int_A \int_{A'} J_{\varepsilon_h}(y-x)|u_h(y)-u_h(x)|\, dy\, dx \le C \int_\Sigma |v'(x) - v(x)|\, d\mathcal{H}^{\text{N-1}}(x).$$

Proposition 1.9. *Let sequences $\varepsilon_h \to 0^+$ and $u_h : A \to \mathbb{R}$ be given such that $u_h \to u \in L^1(A)$. Given a Lipschitz function $g : A \to \mathbb{R}$, then, up to subsequences, the ε_h-traces of u_h relative to A converge to u on $\{g = t\}$ for almost every $t \in \mathbb{R}$.*

2. Compactness and Γ-convergence

This section contains the main results of the paper. In what follows we state the compactness of sequences $(u_h), (\rho_h)$ with equibounded energy $F_{\varepsilon_h}(u_h, \rho_h, \Omega)$ as well as the Γ-convergence result for F_ε as $\varepsilon \to 0^+$. We recall that, as it is customary in this framework (see [12, 16]), we say that a family of functionals E_ε depending on a real parameter $\varepsilon > 0$ Γ-converge to a functional E and we write $\Gamma\text{-}\lim_{\varepsilon\to 0^+} E_\varepsilon = E$ if for any sequence of positive numbers $(\varepsilon_h)_h$ such that $\varepsilon_h \to 0^+$, the Γ-limit of E_{ε_h} exists and equals E. The same is intended for $\Gamma\text{-}\limsup$ and $\Gamma\text{-}\liminf$.

Theorem 2.1 (Compactness)**.** *Given sequences $\varepsilon_h \to 0^+$, $u_h \in L^1(\Omega, [-1,1])$ and $\rho_h \in L^1(\Omega, [0,\infty))$ such that $\sup_h \|\rho_h\|_{L^1(\Omega)} < \infty$ and $\sup_h F_{\varepsilon_h}(u_h, \rho_h, \Omega) < \infty$, where F_{ε_h} is as in Definition 1.1, then up to subsequences u_h converge in $L^1(\Omega)$ to some $u \in \mathrm{BV}(\Omega, \{-1,1\})$ and $\rho_h \mathcal{L}^N$ converge in the weak$*$ sense to some $\mu \in \mathcal{M}(\Omega)$.*

Proof. The result is a direct consequence of [1, Theorem 3.1] and of the weak$*$ compactness of $\mathcal{M}(\Omega)$. □

In the following theorem we states our Γ-convergence result. As it is customary in this setting (see [20, 15]) the Γ-convergence will be understood with respect to the product topology given by the strong L^1 topology of the functions and the weak$*$ topology of the measures. The proof of the result is a consequence of Theorem 2.5 and Theorem 2.10 that will be proven in the next two subsections.

Theorem 2.2. *Let W be a potential as in (1.1), let J be a kernel as in* 1.2*, let F_ε be as in Definition 1.2 and let F be defined as in (1.4). Then the following* Γ*-convergence result holds true.*

$$\Gamma\text{-}\lim_{\varepsilon\to 0^+} F_\varepsilon(u, \mu, \Omega) = F(u, \mu, \Omega). \tag{2.1}$$

2.1. Γ-liminf inequality.

In the following proposition we show that we can truncate the functions u and ρ without increasing the energy $F_\varepsilon(u, \rho, \Omega)$. This property of the energy functional will be needed both in the proof of the $\Gamma\text{-}\liminf$ and in the proof of the $\Gamma\text{-}\limsup$ inequality.

Proposition 2.3. *Let us consider $\varepsilon > 0$, $A \subset \Omega$, $u_\varepsilon \in L^1(A)$ and $\rho_\varepsilon \in L^1(A, [0,\infty))$. Denoting $u_\varepsilon^T(x) := (u_\varepsilon(x) \wedge 1) \vee -1$ and $\rho_\varepsilon^T(x) := \rho_\varepsilon(x) \wedge \int_A J_\varepsilon(y-x) \frac{|u_\varepsilon^T(y) - u_\varepsilon^T(x)|}{\varepsilon} dy$ it holds*

$$F_\varepsilon(u_\varepsilon, \rho_\varepsilon, A) \ge F_\varepsilon(u_\varepsilon^T, \rho_\varepsilon^T, A).$$

Moreover, if u_ε and ρ_ε are such that $\sup_\varepsilon F_\varepsilon(u_\varepsilon,\rho_\varepsilon,A)\le C$, then

$$\lim_{\varepsilon\to 0}\|u_\varepsilon^T-u_\varepsilon\|_{L^1(A)}=0$$

Proof. The proof of the first statement is straightforward and is left to the reader. The proof of the second statement follows the same lines of [1, Lemma 1.14]. We rewrite it here for the reader's convenience: Given $\delta>0$, we have $W(t)>c>0$ on $\{1+\delta<|t|<M\}$ and $W(t)>C|t|$ on $\{|t|>M\}$ for some $M>0$ and therefore

$$\begin{aligned}\|u_\varepsilon-u_\varepsilon^T\|_{L^1}&\le\delta|\Omega|+\frac{M}{c}\int_{\{1+\delta<u_\varepsilon(x)<M\}}W(u_\varepsilon(x))\,dx+\frac{1}{C}\int_{\{|u_\varepsilon(x)>M|\}}W(u_\varepsilon(x))\,dx\\&\le\delta|\Omega|+C\int_A W(u_\varepsilon(x))\,dx,\end{aligned}$$

which implies the claim using that δ is arbitrary and F_ε is bounded as $\varepsilon\to 0$. □

The next elementary Lemma will be used in the proof of the liminf inequality below.

Lemma 2.4. *Given a sequence of functions (f_h), where $f_h\in L^1([0,1])$, that satisfies*

$$\sum_{h=0}^{\infty}\|f_h\|_{L^1([0,1])}<\infty,$$

it holds that

$$\lim_{h\to\infty}f_h(t)=0\quad\text{for almost every } t\in[0,1].$$

Proof. By the monotone convergence theorem we can compute

$$\int_0^1\sum_{h=0}^{\infty}|f_h(t)|\,dt=\sum_{h=0}^{\infty}\|f_h\|_{L^1([0,1])}<\infty,$$

which implies that up to a set of measure zero it holds that $\sum_{h=0}^{\infty}|f_h(t)|<\infty$, hence the claim. □

Theorem 2.5 (lim-inf inequality)**.** *Given $u\in \mathrm{BV}(\Omega,\{-1,1\})$, $\rho\in\mathcal{M}(\Omega)$ and given $\varepsilon\to 0^+$, $u_\varepsilon\in L^1(\Omega)$ and $\rho_\varepsilon\in L^1(\Omega,[0,\infty))$ such that $u_\varepsilon\to u$ in $L^1(\Omega)$ and $\rho_\varepsilon\overset{*}{\rightharpoonup}\mu$ in $\mathcal{M}(\Omega)$, it holds that*

$$\liminf_{\varepsilon\to 0}F_\varepsilon(u_\varepsilon,\rho_\varepsilon,\Omega)\ge\int_{S_u}\sigma\left(\nu_u,\frac{d\mu}{d\,\mathcal{H}^{\text{N-1}}\llcorner S_u}\right)d\,\mathcal{H}^{\text{N-1}},$$

where F_ε is the functional given by Definition 1.1.

Proof. Using Proposition 2.3 and the monotonicity of $\sigma(\nu,\cdot)$ observed in Remark 1.4, it is enough to show the claim for functions u_ε and ρ_ε such that

$$u_\varepsilon:\Omega\to[-1,1]\quad\text{and}\quad\rho_\varepsilon(x)\le\int_\Omega J_\varepsilon(y-x)\frac{|u_\varepsilon(y)-u_\varepsilon(x)|}{\varepsilon}\,dy.\tag{2.2}$$

Extracting subsequences, we may assume that

$$\liminf_{\varepsilon\to 0}F_\varepsilon(u_\varepsilon,\rho_\varepsilon,\Omega)=\lim_{\varepsilon\to 0}F_\varepsilon(u_\varepsilon,\rho_\varepsilon,\Omega)\tag{2.3}$$

and moreover we may assume that this limit is finite. We prove the liminf inequality by blow-up. We start by setting

$$g_\varepsilon(x):=\frac{1}{\varepsilon}W(u_\varepsilon(x))+\frac{1}{\varepsilon}\int_\Omega J_\varepsilon(y-x)(u_\varepsilon(y)-u_\varepsilon(x))^2dy+\varepsilon\left(\int_\Omega J_\varepsilon(y-x)\frac{|u_\varepsilon(y)-u_\varepsilon(x)|}{\varepsilon}\,dy-\rho_\varepsilon(x)\right)^2$$

and note that $\int_\Omega g_\varepsilon(x)\,dx=F_\varepsilon(u_\varepsilon,\rho_\varepsilon,\Omega)$. Hence, by (2.3), passing to a subsequence if necessary, we can assume that $g_\varepsilon\mathcal{L}^N\overset{*}{\rightharpoonup}\lambda$ for a measure $\lambda\in\mathcal{M}(\Omega)$. In order to prove the statement of the theorem, it is then enough to show that

$$\frac{d\lambda}{d\,\mathcal{H}^{\text{N-1}}\llcorner S_u}(x_0)\ge\sigma\left(\nu_u(x_0),\frac{d\mu}{d\,\mathcal{H}^{\text{N-1}}\llcorner S_u}(x_0)\right)\ \text{ for }\ \mathcal{H}^{\text{N-1}}\ \text{ a.e. }\ x_0\in S_u.$$

In what follows we refer to [8] for all the results about the blow-up properties of functions of bounded variations. Let us denote by $\nu_u(x_0)$ the generalized normal at $x_0\in S_u$ which exists at $\mathcal{H}^{\text{N-1}}$ a.e. point. We denote by $Q^l_{\nu_u(x_0)}$ the cube obtained by rotating $[-l,l]^N$ around the origin in such a way that one

axis is in direction $\nu_u(x_0)$ and assume that $l>0$ is chosen such that $x_0+Q^l_{\nu_u(x_0)}\subset\Omega$. Denoting by u_b the jump function

$$u_b(x):=\begin{cases}u^+(x_0), & (x,\nu_u(x_0))>0\\ u^-(x_0), & (x,\nu_u(x_0))<0,\end{cases}$$

it holds that

$$\lim_{\delta\to0^+}\int_{x_0+Q^l_{\nu_u(x_0)}}|u(\delta x)-u_b(x)|\,dx=0. \tag{2.4}$$

In what follows we fix any sequence $\varepsilon_h\to0^+$. We claim that there exists a vanishing sequence of positive real numbers (δ_m) and a set $S\subset[0,1]$ (depending on this sequence) with $|S|=1$ such that for all $t\in S$

$$\lim_{m\to\infty}\delta_m^{1-N}\int_{\partial Q^{l\delta_m t}_{\nu_u(x_0)}}|u(x)-u_b(x)|\,d\mathcal{H}^{\text{N-1}}(x)=\lim_{m\to\infty}\int_{\partial Q^{lt}_{\nu_u(x_0)}}|u(\delta_m x)-u_b(x)|\,d\mathcal{H}^{\text{N-1}}(x)=0 \tag{2.5}$$

and such that the ε_h-traces of $u_h(\delta_m\cdot)$ relative to $Q^l_{\nu_u(x_0)}$ converge to $u(\delta_m\cdot)|_{\partial Q^l_{\nu_u(x_0)}}$ as $h\to\infty$.

We start by observing that, thanks to (2.4) there exist $\delta_m\to0^+$ such that

$$(2l)^{N-1}\int_0^1\int_{x_0+\partial Q^{lt}_{\nu_u(x_0)}}|u(\delta_m x)-u_b(x)|\,d\mathcal{H}^{\text{N-1}}(x)dt=\int_{x_0+Q^l_{\nu_u(x_0)}}|u(\delta_m x)-u_b(x)|\,dx<2^{-m}.$$

Moreover, from Proposition 1.9 we know that, up to choosing a subsequence of ε_h, for all but at most countably many δ, the ε_h-traces of $u_h(\delta\cdot)$ relative to $x_0+Q^l_{\nu_u(x_0)}$ converge to $u(\delta\cdot)|_{x_0+\partial Q^l_{\nu_u(x_0)}}$ as $h\to\infty$ and therefore the δ_m can be chosen to have this property. The claim follows by an application of Lemma 2.4 to the functions $f_m(t):=\int_{x_0+\partial Q^{lt}_{\nu_u(x_0)}}|u(\delta_m x)-u_b(x)|\,d\mathcal{H}^{\text{N-1}}(x)$. For $\mathcal{H}^{\text{N-1}}$ a.e. x_0 and for all but at most countably many $t\in[0,1]$ it holds that $\mu(x_0+\partial Q^{l\delta_m t}_{\nu_u(x_0)})=\lambda(x_0+\partial Q^{l\delta_m t}_{\nu_u(x_0)})=0$. Therefore we can find a $t\in S$ such that

$$\mu(x_0+\partial Q^{l\delta_m t}_{\nu_u(x_0)})=\lambda(x_0+\partial Q^{l\delta_m t}_{\nu_u(x_0)})=0\quad\text{for all } m.$$

We fix such a t and write $Q_{\delta_m}:=x_0+Q^{l\delta_m t}_{\nu_u(x_0)}$ and we set $\omega:=(2lt)^{N-1}$ for the $\mathcal{H}^{\text{N-1}}$-measure of a midplane of $x_0+Q^{lt}_{\nu_u(x_0)}$, i.e., the measure of any face of $x_0+Q^{lt}_{\nu_u(x_0)}$. It holds

$$\begin{aligned}
&\mu(\partial Q_{\delta_m})=0\Rightarrow\lim_{h\to\infty}\int_{Q_{\delta_m}}\rho_h\,dx=\mu(Q_{\delta_m}),\\
&\lambda(\partial Q_{\delta_m})=0\Rightarrow\lim_{h\to\infty}F_{\varepsilon_h}(u_h,\rho_h,Q_{\delta_m})=\lambda(Q_{\delta_m}),\\
&\lim_{m\to\infty}\frac{\mathcal{H}^{\text{N-1}}\llcorner S_u(Q_{\delta_m})}{\omega\delta_m^{N-1}}=1,\\
&\lim_{m\to\infty}\frac{\mu(Q_{\delta_m})}{\mathcal{H}^{\text{N-1}}\llcorner S_u(Q_{\delta_m})}=\frac{d\mu}{d\,\mathcal{H}^{\text{N-1}}\llcorner S_u}(x_0),\\
&\lim_{m\to\infty}\frac{\lambda(Q_{\delta_m})}{\mathcal{H}^{\text{N-1}}\llcorner S_u(Q_{\delta_m})}=\frac{d\lambda}{d\,\mathcal{H}^{\text{N-1}}\llcorner S_u}(x_0).
\end{aligned}$$

In particular, given $s>0$, we have

$$\begin{aligned}
&\int_{Q_{\delta_m}}\rho_h\,dx\le\omega\delta_m^{N-1}\frac{d\mu}{d\,\mathcal{H}^{\text{N-1}}\llcorner S_u}(x_0)+s\\
&\frac{d\lambda}{d\,\mathcal{H}^{\text{N-1}}\llcorner S_u}(x_0)+s\ge\lim_{h\to\infty}F_{\varepsilon_h}(u_h,\rho_h,Q_{\delta_m})\omega^{-1}\delta_m^{1-N}
\end{aligned}\tag{2.6}$$

for all $m>m_0$ (depending on s). We now fix a $\delta\in\{\delta_m\,:\,m\in\mathbb{N}\}$, denote by C_δ the intersection of Q_δ with the hyperplane through x_0 and orthogonal to $\nu_u(x_0)$ and define C_δ-periodic functions $\tilde u_h:\mathbb{R}^N\to[-1,1]$, $\tilde u:\mathbb{R}^N\to[-1,1]$ and $\tilde\rho_h:\mathbb{R}^N\to\mathbb{R}$ which in $T_\delta:=T_{C_\delta}$ are given by

$$\tilde u_h(x)=\begin{cases}u_h(x), & x\in Q_\delta\\ u_b(x), & x\in T_\delta\setminus Q_\delta,\end{cases}\quad\tilde u(x)=\begin{cases}u(x), & x\in Q_\delta\\ u_b(x), & x\in T_\delta\setminus Q_\delta,\end{cases}\quad\tilde\rho_h(x)=\begin{cases}\rho_h(x), & x\in Q_\delta\\ 0, & x\in T_\delta\setminus Q_\delta.\end{cases}$$

Thanks to the C_δ-periodicity of $\tilde u_h$, as $h\to\infty$ the ε_h-traces of $\tilde u_h$ relative to T_δ (resp. relative to $\mathbb{R}^N\setminus T_\delta$) converge to $v:\partial T_\delta\to\mathbb{R}$ defined as

$$v(x)=\begin{cases} u_{|\partial T_\delta}(x), & x\in Q_\delta\\ u_{b|\partial T_\delta}(x), & x\in T_\delta\setminus Q_\delta\end{cases}$$

(resp. to $\bar v:\partial T_\delta\to\mathbb{R}$, $\bar v(x)=v(\bar x)$ where $\bar x$ is the point obtained reflecting x with respect to the hyperplane through x_0 that is parallel to the face on which x lies). The ε_h-traces of $\tilde u_h$ relative to Q_δ resp. relative to $T_\delta\setminus Q_\delta$ converge to $u_{|\partial Q_\delta}$ resp. to $u_{b|\partial T_\delta\setminus Q_\delta}$. By the very definition of the energy functionals given in Definition 1.1 we have that

$$0\le \mathcal{F}_{\varepsilon_h}(\tilde u_h,\tilde\rho_h,T_\delta)-F_{\varepsilon_h}(\tilde u_h,\tilde\rho_h,Q_\delta)=I_1+I_2+I_3,$$

where we have set

$$I_1=\varepsilon_h\int_{T_\delta}\int_{\mathbb{R}^N}J_{\varepsilon_h}(y-x)\bigg(\frac{|\tilde u_h(y)-\tilde u_h(x)|}{\varepsilon_h}\bigg)^2dy\,dx-\varepsilon_h\int_{Q_\delta}\int_{Q_\delta}J_{\varepsilon_h}(y-x)\bigg(\frac{|\tilde u_h(y)-\tilde u_h(x)|}{\varepsilon_h}\bigg)^2dy\,dx,$$

$$I_2=\varepsilon_h\int_{T_\delta\setminus Q_\delta}\bigg(\int_{\mathbb{R}^N}J_{\varepsilon_h}(y-x)\frac{|\tilde u_h(y)-\tilde u_h(x)|}{\varepsilon_h}\,dy-\tilde\rho_h(x)\bigg)^2dx,$$

$$\begin{aligned}I_3=\quad&\varepsilon_h\int_{Q_\delta}\bigg(\int_{\mathbb{R}^N}J_{\varepsilon_h}(y-x)\frac{|\tilde u_h(y)-\tilde u_h(x)|}{\varepsilon_h}\,dy-\tilde\rho_h(x)\bigg)^2dx\\&-\varepsilon_h\int_{Q_\delta}\bigg(\int_{Q_\delta}J_{\varepsilon_h}(y-x)\frac{|\tilde u_h(y)-\tilde u_h(x)|}{\varepsilon_h}\,dy-\tilde\rho_h(x)\bigg)^2dx.\end{aligned}$$

The term I_1 can be estimated using $|\tilde u_h|\le 1$ to get that

$$\begin{aligned}I_1=&\varepsilon_h\int_{T_\delta}\int_{\mathbb{R}^N\setminus T_\delta}J_{\varepsilon_h}(y-x)\left(\frac{|\tilde u_h(y)-\tilde u_h(x)|}{\varepsilon_h}\right)^2dy\,dx+\varepsilon_h\int_{T_\delta\setminus Q_\delta}\int_{T_\delta\setminus Q_\delta}J_{\varepsilon_h}(y-x)\left(\frac{|\tilde u_h(y)-\tilde u_h(x)|}{\varepsilon_h}\right)^2dy\,dx\\&+2\varepsilon_h\int_{T_\delta\setminus Q_\delta}\int_{Q_\delta}J_{\varepsilon_h}(y-x)\left(\frac{|\tilde u_h(y)-\tilde u_h(x)|}{\varepsilon_h}\right)^2dy\,dx\\\le& C\int_{T_\delta}\int_{\mathbb{R}^N\setminus T_\delta}J_{\varepsilon_h}(y-x)\frac{|\tilde u_h(y)-\tilde u_h(x)|}{\varepsilon_h}\,dy\,dx+C\int_{T_\delta\setminus Q_\delta}\int_{T_\delta\setminus Q_\delta}J_{\varepsilon_h}(y-x)\frac{|\tilde u_h(y)-\tilde u_h(x)|}{\varepsilon_h}\,dy\,dx\\&+C\int_{T_\delta\setminus Q_\delta}\int_{Q_\delta}J_{\varepsilon_h}(y-x)\frac{|\tilde u_h(y)-\tilde u_h(x)|}{\varepsilon_h}\,dy\,dx,\end{aligned}$$

To estimate the other two terms I_2 and I_3 one uses $|\tilde u_h|\le 1$, (2.2) and Jensen's inequality to find out that

$$\begin{aligned}I_2&=\varepsilon_h\int_{T_\delta\setminus Q_\delta}\bigg(\int_{\mathbb{R}^N}J_{\varepsilon_h}(y-x)\frac{|\tilde u_h(y)-\tilde u_h(x)|}{\varepsilon_h}\,dy-\tilde\rho_h(x)\bigg)^2dx\\&\le\varepsilon_h\int_{T_\delta\setminus Q_\delta}\bigg(\int_{\mathbb{R}^N}J_{\varepsilon_h}(y-x)\frac{|\tilde u_h(y)-\tilde u_h(x)|}{\varepsilon_h}\,dy\bigg)^2dx\\&\le\|J\|_{L^1(\mathbb{R}^d)}\varepsilon_h\int_{T_\delta\setminus Q_\delta}\int_{\mathbb{R}^N}J_{\varepsilon_h}(y-x)\left(\frac{|\tilde u_h(y)-\tilde u_h(x)|}{\varepsilon_h}\right)^2dy\,dx\le\|J\|_{L^1(\mathbb{R}^N)}I_1.\end{aligned}$$

and that

$$
\begin{aligned}
I_3 &= \varepsilon_h \int_{Q_\delta} \left(\int_{\mathbb{R}^N \setminus Q_\delta} J_{\varepsilon_h}(y-x) \frac{|\tilde{u}_h(y)-\tilde{u}_h(x)|}{\varepsilon_h}\, dy \right)^2 dx \\
&\quad + 2\varepsilon_h \int_{Q_\delta} \int_{\mathbb{R}^N \setminus Q_\delta} J_{\varepsilon_h}(y-x) \frac{|\tilde{u}_h(y)-\tilde{u}_h(x)|}{\varepsilon_h}\, dy \left(\int_{Q_\delta} J_{\varepsilon_h}(y-x) \frac{|\tilde{u}_h(y)-\tilde{u}_h(x)|}{\varepsilon_h}\, dy - \tilde{\rho}_h(x) \right) dx \\
&\leq \|J\|_{L^1(\mathbb{R}^N)} \varepsilon_h \int_{Q_\delta} \int_{\mathbb{R}^N \setminus Q_\delta} J_{\varepsilon_h}(y-x) \left(\frac{|\tilde{u}_h(y)-\tilde{u}_h(x)|}{\varepsilon_h} \right)^2 dy\, dx \\
&\quad + 2\varepsilon_h \int_{Q_\delta} \int_{\mathbb{R}^N \setminus Q_\delta} J_{\varepsilon_h}(y-x) \frac{|\tilde{u}_h(y)-\tilde{u}_h(x)|}{\varepsilon_h}\, dy \int_{Q_\delta} J_{\varepsilon_h}(y-x) \frac{|\tilde{u}_h(y)-\tilde{u}_h(x)|}{\varepsilon_h}\, dy\, dx \\
&\leq C \int_{Q_\delta} \int_{\mathbb{R}^N \setminus Q_\delta} J_{\varepsilon_h}(y-x) \frac{|\tilde{u}_h(y)-\tilde{u}_h(x)|}{\varepsilon_h}\, dy\, dx \\
&\leq C \int_{T_\delta} \int_{\mathbb{R}^N} J_{\varepsilon_h}(y-x) \frac{|\tilde{u}_h(y)-\tilde{u}_h(x)|}{\varepsilon_h}\, dy\, dx - C \int_{Q_\delta} \int_{Q_\delta} J_{\varepsilon_h}(y-x) \frac{|\tilde{u}_h(y)-\tilde{u}_h(x)|}{\varepsilon_h}\, dy\, dx \\
&\leq C \int_{T_\delta} \int_{\mathbb{R}^N \setminus T_\delta} J_{\varepsilon_h}(y-x) \frac{|\tilde{u}_h(y)-\tilde{u}_h(x)|}{\varepsilon_h}\, dy\, dx + C \int_{T_\delta \setminus Q_\delta} \int_{T_\delta \setminus Q_\delta} J_{\varepsilon_h}(y-x) \frac{|\tilde{u}_h(y)-\tilde{u}_h(x)|}{\varepsilon_h}\, dy\, dx \\
&\quad + C \int_{T_\delta \setminus Q_\delta} \int_{Q_\delta} J_{\varepsilon_h}(y-x) \frac{|\tilde{u}_h(y)-\tilde{u}_h(x)|}{\varepsilon_h}\, dy\, dx
\end{aligned}
$$

By an application of Proposition 1.7 (see Remark 1.8) we also have that

$$
\begin{aligned}
&\lim_{h\to\infty} \int_{T_\delta} \int_{\mathbb{R}^N \setminus T_\delta} J_{\varepsilon_h}(y-x) \frac{|\tilde{u}_h(y)-\tilde{u}_h(x)|}{\varepsilon_h}\, dy\, dx \leq C \int_{\partial T_\delta} |v-\bar{v}|\, d\mathcal{H}^{\text{N-1}}, \\
&\lim_{h\to\infty} \int_{T_\delta \setminus Q_\delta} \int_{Q_\delta} J_{\varepsilon_h}(y-x) \frac{|\tilde{u}_h(y)-\tilde{u}_h(x)|}{\varepsilon_h}\, dy\, dx \leq C \int_{T_\delta \cap \partial Q_\delta} |u_{|\partial Q_\delta} - u_b|\, d\mathcal{H}^{\text{N-1}}, \\
&\lim_{h\to\infty} \int_{T_\delta \setminus Q_\delta} \int_{T_\delta \setminus Q_\delta} J_{\varepsilon_h}(y-x) \frac{|\tilde{u}_h(y)-\tilde{u}_h(x)|}{\varepsilon_h}\, dy\, dx \\
&\qquad = \lim_{h\to\infty} 2 \int_{(T_\delta \setminus Q_\delta) \cap \{(x,\nu_u(x_0))>0\}} \int_{(T_\delta \setminus Q_\delta) \cap \{(x,\nu_u(x_0))<0\}} J_{\varepsilon_h}(y-x) \frac{|\tilde{u}_h(y)-\tilde{u}_h(x)|}{\varepsilon_h}\, dy\, dx = 0,
\end{aligned} \tag{2.7}
$$

where in the last equation we used that the sets $(T_\delta \setminus Q_\delta) \cap \{(x, \nu_u(x_0)) > 0\}$ and $(T_\delta \setminus Q_\delta) \cap \{(x, \nu_u(x_0)) < 0\}$ have positive distance, which implies that the limit of the double integral is zero (this can be shown by extending $\tilde{u}_h$ by 0 on Q_δ). Defining $\bar{\rho}_h : \mathbb{R}^N \to \mathbb{R}$ as the C_δ-periodic function such that

$$
\bar{\rho}_h(x) = \begin{cases} \rho_h(x) - \dfrac{s}{|Q_\delta|}, & x \in Q_\delta \\ 0, & x \in T_\delta \setminus Q_\delta, \end{cases}
$$

we have that

$$
\begin{aligned}
&|\mathcal{F}_{\varepsilon_h}(\tilde{u}_h, \bar{\rho}_h, Q_\delta) - \mathcal{F}_{\varepsilon_h}(\tilde{u}_h, \tilde{\rho}_h, Q_\delta)| \\
&\qquad \leq \varepsilon_h \left| 2 \int_{Q_\delta} \int_{\mathbb{R}^N} J_{\varepsilon_h}(y-x) \frac{|\tilde{u}_h(y)-\tilde{u}_h(x)|}{\varepsilon_h}\, dy \frac{s}{|Q_\delta|}\, dx \right| + \varepsilon_h \left| \int_{Q_\delta} \left(\frac{s}{|Q_\delta|} \right)^2 dx \right| \\
&\qquad \leq \sqrt{\varepsilon_h} C(\delta) + \varepsilon_h \frac{Cs^2}{|Q_\delta|} \to_{h\to\infty} 0,
\end{aligned} \tag{2.8}
$$

where we used the boundedness of $(\mathcal{F}_{\varepsilon_h}(\tilde{u}_h, 0, Q_\delta))_h$ which is a consequence of the second inequality in (2.2) and the boundedness of $(\mathcal{F}_{\varepsilon_h}(\tilde{u}_h, \rho_h, Q_\delta))_h$. By the very definition of $\bar{\rho}_h$, Definition 1.3 and (2.6), it holds true that

$$
(\tilde{u}_h, \bar{\rho}_h) \in \mathcal{A}\left(\nu_u(x_0), \frac{d\mu}{d\,\mathcal{H}^{\text{N-1}} \llcorner S_u}(x_0) \right), \quad \mathcal{F}_{\varepsilon_h}(\tilde{u}_h, \bar{\rho}_h, Q_\delta) \omega^{-1} \delta^{1-N} \geq \sigma\left(\nu_u(x_0), \frac{d\mu}{d\,\mathcal{H}^{\text{N-1}} \llcorner S_u}(x_0) \right)
$$

Therefore, collecting the above results,

$$\begin{aligned}\frac{d\lambda}{d\,\mathcal{H}^{\text{N-1}}\llcorner S_u}(x_0)+s &\geq \lim_{h\to\infty} F_{\varepsilon_h}(u_h,\rho_h,Q_{\delta_m})\omega^{-1}\delta_m^{1-N}\\ &= \lim_{h\to\infty} F_{\varepsilon_h}(\tilde u_h,\tilde\rho_h,Q_{\delta_m})\omega^{-1}\delta_m^{1-N}\\ &\geq \liminf_{h\to\infty} \mathcal{F}_{\varepsilon_h}(\tilde u_h,\bar\rho_h,Q_{\delta_m})\omega^{-1}\delta_m^{1-N}\\ &\quad - \limsup_{h\to\infty} |\mathcal{F}_{\varepsilon_h}(\tilde u_h,\tilde\rho_h,Q_{\delta_m})\omega^{-1}\delta_m^{1-N} - \mathcal{F}_{\varepsilon_h}(\tilde u_h,\bar\rho_h,Q_{\delta_m})\omega^{-1}\delta_m^{1-N}|\\ &\quad - \limsup_{h\to\infty} |F_{\varepsilon_h}(\tilde u_h,\tilde\rho_h,Q_{\delta_m})\omega^{-1}\delta_m^{1-N} - \mathcal{F}_{\varepsilon_h}(\tilde u_h,\tilde\rho_h,Q_{\delta_m})\omega^{-1}\delta_m^{1-N}|\\ &\geq \sigma\left(\nu_u(x_0),\frac{d\mu}{d\,\mathcal{H}^{\text{N-1}}\llcorner S_u}(x_0)\right) - C\delta_m^{1-N}\int_{\partial T_{\delta_m}} |v-\bar v| d\,\mathcal{H}^{\text{N-1}}\\ &\quad - C\delta_m^{1-N}\int_{T_{\delta_m}\cap\,\partial Q_{\delta_m}} |u_{|\partial Q_{\delta_m}} - u_b| d\,\mathcal{H}^{\text{N-1}}\,.\end{aligned}$$

Here in the last estimate we have used (2.8) and (2.7). Passing to the limit as $m\to\infty$ in the estimate above by (2.5) we eventually get that for $\mathcal{H}^{\text{N-1}}$ a.e. $x_0\in S_u$

$$\frac{d\lambda}{d\,\mathcal{H}^{\text{N-1}}\llcorner S_u}(x_0)+s \geq \sigma\left(\nu_u(x_0),\frac{d\mu}{d\,\mathcal{H}^{\text{N-1}}\llcorner S_u}(x_0)\right),$$

which proves the claim thanks to the arbitrariness of s. □

2.2. **Γ-limsup inequality.** In this section we complete the proof of the Γ-convergence result by proving the limsup inequality for the energy functionals F_ε in Definition 1.1. The proof consists of several steps. At each step we assume u and μ to be of increasing generality and provide for them a recovery sequence. We start with the case of polyhedral functions (see Definition 1.5), the proof of which draws some inspiration from [1, Theorem 5.2] and is given in Proposition 2.6. We further generalize it in Proposition 2.9. Therein, we consider functions of bounded variation u whose jump set need not be polyhedral, while we further keep restrictions on the limit measure μ. Finally, in Theorem 2.10, we show the statement in full generality.

Proposition 2.6. *Let $u\in \mathrm{BV}(\Omega,\{-1,1\})$ be a polyhedral function corresponding to a polyhedral set P as in Definition 1.5. For $n\in\mathbb{N}$ and $i\in\{1,2,\dots,n\}$ let $x_i\in\Omega\setminus S_u$, g be a piecewise constant function on polyhedral subsets of the faces of S_u and let $\mu=\sum_{i=1}^n \beta_i\delta_{x_i}+g\,\mathcal{H}^{\text{N-1}}\llcorner S_u$. Then there exist functions $u_\varepsilon\in L^1(\Omega)$ and $\rho_\varepsilon\in L^1(\Omega,[0,\infty))$, such that $u_\varepsilon\to u$ in $L^1(\Omega)$ and $\rho_\varepsilon\mathcal{L}^N \overset{*}{\rightharpoonup}\mu$ as $\varepsilon\to 0^+$ which satisfy*

$$\limsup_{\varepsilon\to 0^+} F_\varepsilon(u_\varepsilon,\rho_\varepsilon,\Omega)\leq \int_{S_u}\sigma\left(\nu_u,\frac{d\mu}{d\,\mathcal{H}^{\text{N-1}}\llcorner S_u}\right) d\,\mathcal{H}^{\text{N-1}}\,. \tag{2.9}$$

Proof. Let us assume without loss of generality (we may reduce to this case by the argument of step 3 below) that

$$\mu=\beta\delta_{x_1}+\gamma\,\mathcal{H}^{\text{N-1}}\llcorner\Sigma_g,$$

where Σ_g is a polyhedral subset (the subscript g points out the dependence of this set on the function g) of a face of P.

Step 1. We first consider the case of a polyhedral set A in Ω that does not intersect the boundary of P (see Figure 1) and define sequences $u_\varepsilon\to u$ in $L^1(A)$ and $\rho_\varepsilon\mathcal{L}^N\overset{*}{\rightharpoonup}\mu$ in A that satisfy the (local) energy estimate

$$\limsup_{\varepsilon\to 0^+} F_\varepsilon(u_\varepsilon,\rho_\varepsilon,A)\leq \int_{S_u|_A}\sigma\left(\nu_u,\frac{d\mu}{d\,\mathcal{H}^{\text{N-1}}\llcorner S_u}\right) d\,\mathcal{H}^{\text{N-1}}\,. \tag{2.10}$$

In this case we notice that $u_{|A}$ is constant and that the r.h.s. of (2.10) is zero. Hence it suffices to define $u_\varepsilon:=u_{|A}$ and $\rho_\varepsilon:=\beta\,\omega_N^{-1}\varepsilon^{-1/2}\chi_{B(x_1,\varepsilon^{1/(2N)})}$, if $x_1\in A$ and $\rho_\varepsilon:=0$ if instead $x_1\notin A$ to have

$$u_\varepsilon\to u,\quad \rho_\varepsilon\mathcal{L}^N\overset{*}{\rightharpoonup}\mu.$$

The limsup inequality 2.10 holds true since

$$F_\varepsilon(u_\varepsilon,\rho_\varepsilon,A)\leq \varepsilon\int_{B(x_1,\varepsilon^{1/(2N)})}\frac{\beta^2}{\omega_N^2\varepsilon}dx\leq C\varepsilon^{1/2}.$$

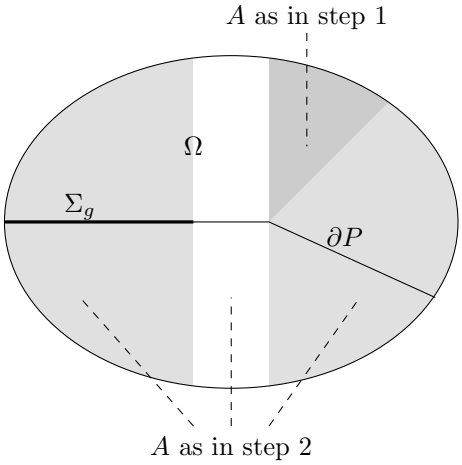


FIGURE 1. Sketch of the situation of Proposition 2.6

Step 2. We now take a polyhedral set A in Ω such that it intersects exactly one face of P and the orthogonal projection of A on the hyperplane which contains this face of P equals the intersection $\partial P \cap A =: \Sigma$. In what follows, we define sequences $u_\varepsilon \to u$ in $L^1(A)$ and $\rho_\varepsilon \mathcal{L}^N \overset{*}{\rightharpoonup} \mu$ in A such that the (local) energy estimate (2.10) holds true. Considering step 3 below, we can assume without loss of generality that $\Sigma \subset \Sigma_g$ or $\Sigma \cap \Sigma_g = \emptyset$. We now assume that $\Sigma \subset \Sigma_g$, the other case can be treated similarly. In order to simplify the notation we also assume that $\Sigma \subset \{x_N = 0\}$. Given any $\eta > 0$, we find $(v, \tilde{\rho}) \in \mathcal{A}(e_N, \gamma)$ such that for a $(N-1)$-dimensional cube $C \subset \{x_N = 0\}$ it holds $\mathcal{H}^{\text{N-1}}(C)^{-1} \mathcal{F}(v, \tilde{\rho}, T_C) \leq \sigma(e_N, \gamma) + \eta$. We define

$$u_\varepsilon(x) := v\left(\frac{x}{\varepsilon}\right), \quad \tilde{\rho}_\varepsilon(x) := \frac{1}{\varepsilon}\tilde{\rho}\left(\frac{x}{\varepsilon}\right)$$

and

$$r_\varepsilon(x) := \frac{\beta}{\omega_N \varepsilon^{1/2}} \chi_{B_{\varepsilon^{1/(2N)}}(x_1)}(x) + \frac{\gamma - (\mathcal{H}^{\text{N-1}}(C))^{-1} \int_{T_C} \tilde{\rho} dx}{\varepsilon^{1/2}} \chi_{\{x : 0 < x_N < \varepsilon^{1/2}\}}(x)$$
$$\rho_\varepsilon(x) := \tilde{\rho}_\varepsilon(x) + r_\varepsilon(x).$$

Remark that by the Riemann-Lebesgue Lemma it holds

$$\tilde{\rho}_\varepsilon \mathcal{L}^N \llcorner A \overset{*}{\rightharpoonup} (\mathcal{H}^{\text{N-1}}(C))^{-1} \int_{T_C} \tilde{\rho}\, dx\, \mathcal{H}^{\text{N-1}} \llcorner \Sigma,$$

hence the convergence $\rho_\varepsilon \mathcal{L}^N \llcorner A \overset{*}{\rightharpoonup} \mu \llcorner A$ holds true. We note that, by the very definition of $\mathcal{A}(e_N, \gamma)$, one has that $r_\varepsilon \geq 0$. Since $|u_\varepsilon| \leq 1$, using Proposition 2.3, we may assume that $\int_A J_\varepsilon(y-x) \frac{|u_\varepsilon(y) - u_\varepsilon(x)|}{\varepsilon} dy - \tilde{\rho}_\varepsilon(x) \geq 0$ and have that

$$\begin{aligned} 0 &\leq F_\varepsilon(u_\varepsilon, \rho_\varepsilon, A) - F_\varepsilon(u_\varepsilon, \tilde{\rho}_\varepsilon, A) \\ &= 2\varepsilon \int_A \left(\int_A J_\varepsilon(y-x) \frac{|u_\varepsilon(y) - u_\varepsilon(x)|}{\varepsilon}\, dy - \tilde{\rho}_\varepsilon(x) \right) (-r_\varepsilon(x))\, dx + \varepsilon \int_A r_\varepsilon(x)^2\, dx \\ &\leq \varepsilon \int_A r_\varepsilon(x)^2\, dx \leq C\varepsilon^{1/2}. \end{aligned}$$

The estimate above implies that

$$\limsup_{\varepsilon \to 0^+} F_\varepsilon(u_\varepsilon, \rho_\varepsilon, A) \leq \limsup_{\varepsilon \to 0^+} F_\varepsilon(u_\varepsilon, \tilde{\rho}_\varepsilon, A). \tag{2.11}$$

We notice that

$$\mathcal{F}_\varepsilon(u_\varepsilon, \tilde{\rho}_\varepsilon, T_{\varepsilon C}) = \varepsilon^{N-1} \mathcal{F}(u, \tilde{\rho}, T_C).$$

Given a covering (depending on ε) of Σ made of M_ε translated copies of εC, this equality implies

$$F_\varepsilon(u_\varepsilon, \tilde{\rho}_\varepsilon, A) \leq M_\varepsilon \varepsilon^{N-1} \mathcal{F}(u, \tilde{\rho}, T_C).$$

Further choosing the covering such that $M_\varepsilon \mathcal{H}^{\text{N-1}}(C)\varepsilon^{N-1} \to \mathcal{H}^{\text{N-1}}(\Sigma)$ as $\varepsilon \to 0^+$, we have shown that

$$\limsup_{\varepsilon\to 0^+} F_\varepsilon(u_\varepsilon, \tilde\rho_\varepsilon, A) \le \mathcal{H}^{\text{N-1}}(\Sigma)(\mathcal{H}^{\text{N-1}}(C))^{-1}\mathcal{F}(v,\tilde\rho, T_C) \le \mathcal{H}^{\text{N-1}}(\Sigma)\sigma(e_N,\gamma) + \mathcal{H}^{\text{N-1}}(\Sigma)\eta.$$

The latter estimate together with (2.11) eventually gives

$$\limsup_{\varepsilon\to 0^+} F_\varepsilon(u_\varepsilon, \rho_\varepsilon, A) \le \mathcal{H}^{\text{N-1}}(\Sigma)\sigma(e_N,\gamma) + C\eta,$$

where η can be chosen arbitrarily small. Since $u(x) \to \pm 1$ as $x \to \pm\infty$, also u_ε converges to u in $L^1(A)$ and the proof of step 2 is completed.

Step 3. We can now prove the statement of the theorem observing that Ω is a finite union of sets of the type considered in the previous two steps, In fact, let us denote by A_1, A_2 two sets, each given by the finite unions of sets of the type considered in the previous steps and let $(u^1_\varepsilon, \rho^1_\varepsilon)$ (resp. $(u^2_\varepsilon, \rho^2_\varepsilon)$) satisfy the stated convergence properties and energy estimates for A_1 (resp. A_2). We define the sequence of pairs $(u_\varepsilon, \rho_\varepsilon)$ as follows:

$$u_\varepsilon: \quad x \in A_1 \cup A_2 \mapsto \begin{cases} u^1_\varepsilon(x), & x \in A_1 \\ u^2_\varepsilon(x), & x \in A_2. \end{cases}$$

$$\rho_\varepsilon: \quad x \in A_1 \cup A_2 \mapsto \begin{cases} \rho^1_\varepsilon(x), & x \in A_1 \\ \rho^2_\varepsilon(x), & x \in A_2. \end{cases}$$

It holds that $u_\varepsilon \to u$ and $\rho_\varepsilon \mathcal{L}^N \overset{*}{\rightharpoonup} \mu$ on $A_1 \cup A_2$. It remains to show that

$$\limsup_{\varepsilon\to 0^+} F_\varepsilon(u_\varepsilon,\rho_\varepsilon, A_1\cup A_2) - F_\varepsilon(u_\varepsilon,\rho_\varepsilon,A_1) - F_\varepsilon(u_\varepsilon,\rho_\varepsilon,A_2) = 0.$$

As already observed in Step 2, we may assume that $|u_\varepsilon(x)| \le 1$ for a.e. $x \in A_i$. Exploiting such a property, expanding the squares in the definition of F_ε, thanks to Jensen's inequality we obtain

$$\begin{aligned}
0 &\le F_\varepsilon(u_\varepsilon,\rho_\varepsilon, A_1\cup A_2) - F_\varepsilon(u_\varepsilon,\rho_\varepsilon,A_1) - F_\varepsilon(u_\varepsilon,\rho_\varepsilon,A_2) \\
&= \frac{2}{\varepsilon}\int_{A_1}\int_{A_2} J_\varepsilon(y-x)|u^2_\varepsilon(y) - u^1_\varepsilon(x)|^2\,dy\,dx \\
&\quad + 2\varepsilon\int_{A_1}\left(\int_{A_1} J_\varepsilon(y-x)\frac{|u^1_\varepsilon(y)-u^1_\varepsilon(x)|}{\varepsilon}dy - \rho^1_\varepsilon(x)\right)\int_{A_2} J_\varepsilon(y-x)\frac{|u^2_\varepsilon(y)-u^1_\varepsilon(x)|}{\varepsilon}\,dy\,dx \\
&\quad + \varepsilon\int_{A_1}\left(\int_{A_2} J_\varepsilon(y-x)\frac{|u^2_\varepsilon(y)-u^2_\varepsilon(x)|}{\varepsilon}dy\right)^2 dx \\
&\quad + 2\varepsilon\int_{A_2}\left(\int_{A_2} J_\varepsilon(y-x)\frac{|u^2_\varepsilon(y)-u^2_\varepsilon(x)|}{\varepsilon}dy - \rho^2_\varepsilon(x)\right)\int_{A_1} J_\varepsilon(y-x)\frac{|u^1_\varepsilon(y)-u^2_\varepsilon(x)|}{\varepsilon}\,dy\,dx \\
&\quad + \varepsilon\int_{A_2}\left(\int_{A_1} J_\varepsilon(y-x)\frac{|u^1_\varepsilon(y)-u^1_\varepsilon(x)|}{\varepsilon}dy\right)^2 dx \\
&\le \frac{C}{\varepsilon}\int_{A_1}\int_{A_2} J_\varepsilon(y-x)|u_\varepsilon(y)-u_\varepsilon(x)|\,dy\,dx,
\end{aligned}$$

Since u_ε converges to u in $L^1(A_1\cup A_2)$, we can use Proposition 1.7 and Remark 1.8 to get that for any sequence $\varepsilon_h \to 0^+$ the last term (with ε replaced with ε_h) converges to zero as $h \to \infty$. The proof of the Proposition follows by the arbitrariness of $(\varepsilon_h)_h$.

□

Using the above results we can further characterize the properties of the limit energy density σ as stated in the propositions below.

Proposition 2.7. *In the definition 1.3 of $\mathcal{A}(e,\gamma)$ we may replace the constraint $\int_{T_c}\rho\,dx \le \gamma\,\mathcal{H}^{\text{N-1}}(C)$ by $\int_{T_c}\rho\,dx = \gamma\,\mathcal{H}^{\text{N-1}}(C)$. Moreover, we may also assume that for every $(u,\rho) \in \mathcal{A}(e,\gamma)$ there exists a constant $C > 0$ such that $u(x) = 1$ if $(x,e) \ge C$ and $u(x) = -1$ if $(x,e) \le -C$.*

Proof. In order to simplify notation we assume that $e = e_N$. The first assertion follows directly from the proof of Proposition 2.6. For the second assertion, we notice that, again by Proposition 2.6, taking Q a unit cube centered at zero with one face orthogonal to e_N, and $\varepsilon_h \to 0^+$, there exist sequences (u_h)

and (ρ_h), such that $u_h \in L^1(Q)$, $\rho_h \in L^1(Q,[0,\infty))$, $\rho_h \mathcal{L}^N \overset{*}{\rightharpoonup} \gamma\, \mathcal{H}^{\mathrm{N-1}} \llcorner \{x_N = 0\}$ on Q, (u_h) converges in $L^1(Q)$ to the function

$$u : x \in Q \mapsto \begin{cases} 1, & x_N > 0 \\ -1, & x_N \leq 0, \end{cases}$$

and

$$\sigma(e_N, \gamma) \geq \limsup_{h\to\infty} F_{\varepsilon_h}(u_h, \rho_h, Q).$$

To conclude one can repeat the proof of Theorem 2.5 and show that $\sigma(e_N,\gamma)$ can also be attained as an infimum of functions u_h that equal ± 1 for large values of $|x_N|$. □

Proposition 2.8. *For any $\gamma \geq 0$, the function $\sigma(\cdot,\gamma) : S^{N-1} \to \mathbb{R}$ is upper semi-continuous.*

Proof. Given $\delta > 0$ and $\nu \in S^{N-1}$, there exists an $(N-1)$-dimensional cube C orthogonal to ν and a pair $(u,\rho) \in \mathcal{A}(\nu,\gamma)$ such that $u(x) = 1$ if $(x,e) \geq D$ and $u(x) = -1$ if $(x,e) \leq -D$ for some $D > 0$ (here we used Proposition 2.7) and such that $\mathcal{F}(u,\rho,T_C) \leq \sigma(\nu,\gamma) + \delta$. Thanks to Proposition 2.3 we can also assume $|u(x)| \leq 1$ and $\int_{\mathbb{R}^N} J(h)|u(y) - u(x)|\, dy - \rho(x) \geq 0$ for a.e. $x \in T_C$. In order to simplify notation, and without loss of generality, in what follows we moreover assume that $\nu = e_N$ and that C is of volume one and centered at zero. Let $(\nu_h)_h \subset S^{N-1}$ be such that $\nu_h \to e_N$ and let $(R_h) \subset SO(N)$ be such that $R_h e_N = \nu_h$, hence $R_h \to I$ where $I \in SO(N)$ denotes the identity matrix. We notice that $(u \circ R_h^{-1}, \rho \circ R_h^{-1}) \in \mathcal{A}(R_h(e_N),\gamma)$ and that $\mathcal{F}(u \circ R_h^{-1}, \rho \circ R_h^{-1}, T_{R_h C})$ equals the functional $\mathcal{F}(u,\rho,T_C)$ but with J replaced by $J \circ R_h$. Therefore we can estimate, using the boundedness of u and the finiteness of the L^1-norm of J,

$$\begin{aligned} |\mathcal{F}(u \circ R_h^{-1}, \rho \circ R_h^{-1}, T_{R_h C}) - \mathcal{F}(u,\rho,T_C)| &\leq C \int_{T_C}\int_{\mathbb{R}^N} |J - J\circ R_h|(y-x)|u(y) - u(x)|\, dy\, dx \\ &\quad + C \int_{T_C}\int_{\mathbb{R}^N} |J - J\circ R_h|(y-x)\rho(x)\, dy\, dx. \end{aligned}$$

Exploiting again $|u(x)| \leq 1$ and $u(x) = \pm 1$ if $|x_N| \geq D$ the following estimate holds true

$$\begin{aligned} \int_{T_C}\int_{\mathbb{R}^N} |J - J\circ R_h|(y-x)|u(y) - u(x)|\, dy\, dx &= \int_{\mathbb{R}^N} |J - J\circ R_h|(z) \int_{T_C} |u(x+z) - u(x)|\, dx\, dz \\ &\leq \int_{\mathbb{R}^N} |J - J\circ R_h|(z) \int_{T_C \cap \{|x_N| < D + |z_N|\}} |u(x+z) - u(x)|\, dx\, dz \\ &\leq C \int_{\mathbb{R}^N} |J - J\circ R_h|(z)(|z| + D)\, dz \\ &= C\left(\int_{\mathbb{R}^N} |J - J\circ R_h|(z)\, dz + \int_{\mathbb{R}^N} |J - J\circ R_h|(z)|z|\, dz \right). \end{aligned}$$

This last term converges to zero as $h \to \infty$ because $J \in L^1(\mathbb{R}^N)$ and the map $h \in \mathbb{R}^N \mapsto J(h)|h| \in L^1(\mathbb{R}^N)$ by (1.2). Moreover, it holds that

$$\int_{T_C}\int_{\mathbb{R}^N} |J - J\circ R_h|(y-x)\rho(x)\, dy\, dx \leq |\rho|_{L^1(T_C)} \int_{\mathbb{R}^N} |J - J\circ R_h|(z)\, dz.$$

As a result we can write that

$$\begin{aligned} \sigma(\nu,\gamma) + \delta &\geq \mathcal{F}(u,\rho,T_C) \\ &\geq \limsup_{h\to\infty} \mathcal{F}(u \circ R_h^{-1}, \rho \circ R_h^{-1}, T_{R_h C}) \geq \limsup_{h\to\infty} \sigma(R_h(e_N),\gamma), \end{aligned}$$

which shows the claim by the arbitrariness of δ. □

Making use of the previous proposition, we now extend the proof of the limsup to the case of a function $u \in \mathrm{BV}(\Omega,\{-1,1\})$ and of a measure μ which concentrates its $\mathcal{H}^{\mathrm{N-1}}$-absolutely continuous part on projections of finitely many $(N-1)$ -dimensional polyhedral sets on the jump set of u.

Proposition 2.9. *Let $u \in \mathrm{BV}(\Omega,\{-1,1\})$, $n \in \mathbb{N}$ and for $i \in \{1,2,\dots,n\}$ let $x_i \in \Omega \backslash S_u$, g be a piecewise constant function on the projections on S_u of finitely many $(N-1)$-dimensional polyhedral sets in $\mathbb{R}^N$*

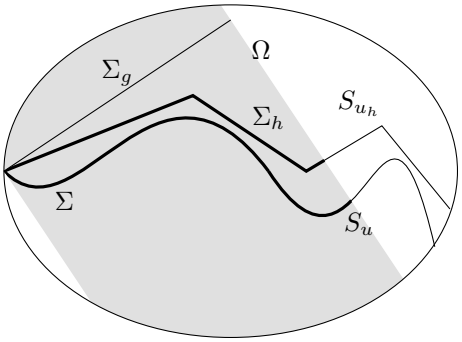


FIGURE 2. Sketch of the situation of Proposition 2.9

and let $\mu = \sum_{i=1}^n \beta_i \delta_{x_i} + g\,\mathcal{H}^{\text{N-1}}\llcorner S_u$. *Then there exist functions* $u_\varepsilon \in L^1(\Omega)$ *and* $\rho_\varepsilon \in L^1(\Omega,[0,\infty))$, *such that* $u_\varepsilon \to u$ *in* $L^1(\Omega)$ *and* $\rho_\varepsilon \mathcal{L}^N \overset{*}{\rightharpoonup} \mu$ *as* $\varepsilon \to 0^+$ *which satisfy*

$$\limsup_{\varepsilon\to 0^+} F_\varepsilon(u_\varepsilon,\rho_\varepsilon,\Omega) \le \int_{S_u} \sigma\left(\nu_u, \frac{d\mu}{d\,\mathcal{H}^{\text{N-1}}\llcorner S_u}\right) d\,\mathcal{H}^{\text{N-1}}\,. \tag{2.12}$$

Proof. Let us assume without loss of generality that

$$\mu = \beta\delta_{x_1} + \gamma\,\mathcal{H}^{\text{N-1}}\llcorner\Sigma,$$

where Σ is the projection of $\Sigma_g \subset \mathbb{R}^N$, a finite union of $(N-1)$-dimensional polyhedral subsets, on S_u. There exists ([1] section 5.4) a sequence of polyhedral functions $u_h \in \mathrm{BV}(\Omega,\{-1,1\})$ such that

$$u_h \to u \text{ in } L^1(\Omega), \quad |Du_h| \to |Du|.$$

If we write Σ_h for the projection of Σ_g to S_{u_h}, we have that Σ_h is a polyhedral subset of S_{u_h} and that

$$\mu_h := \beta\delta_{x_1} + \gamma\,\mathcal{H}^{\text{N-1}}\llcorner\Sigma_h \overset{*}{\rightharpoonup} \mu.$$

We apply Proposition 2.6 to u_h and μ_h and use the lower semicontinuity of Γ-lim sup to obtain

$$\begin{aligned}
\Gamma\text{-}\limsup\nolimits_{\varepsilon\to0^+} F_\varepsilon(u,\mu) &\le \liminf_{h\to\infty} \Gamma\text{-}\limsup\nolimits_{\varepsilon\to0^+} F_\varepsilon(u_h,\mu_h)\\
&\le \liminf_{h\to\infty} \int_{S_{u_h}} \sigma\left(\nu_{u_h}, \frac{d\mu_h}{d\,\mathcal{H}^{\text{N-1}}\llcorner S_{u_h}}\right) d\,\mathcal{H}^{\text{N-1}}\\
&\le \limsup_{h\to\infty} \int_{S_{u_h}\cap\Sigma_h} \sigma(\nu_{u_h},\gamma)\,d\,\mathcal{H}^{\text{N-1}} + \limsup_{h\to\infty} \int_{S_{u_h}\setminus\Sigma_h} \sigma(\nu_{u_h},0)\,d\,\mathcal{H}^{\text{N-1}}\,.
\end{aligned}$$

Note that for any fixed $\gamma \ge 0$, $A \subset \mathbb{R}^N$ open and $v \in \mathrm{BV}(A,\{-1,1\})$ we know that $\sigma(\cdot,\gamma)$ is upper semi-continuous and that

$$\int_{S_v\cap A} \sigma(\nu_{S_v},\gamma)\,d\,\mathcal{H}^{\text{N-1}} = \frac{1}{2}\int_A \sigma\left(\frac{Dv}{|Dv|},\gamma\right) d|Dv|.$$

Approximating $\sigma(\cdot,\gamma)$ by bounded and continuous functions (e.g., via a sup-convolution Yosida transform) and exploiting the Reshetnyak continuity theorem (see [8]) one obtains that

$$\limsup_{h\to\infty} \int_{S_{u_h}\cap\Sigma_h} \sigma(\nu_{u_h},\gamma)\,d\,\mathcal{H}^{\text{N-1}} \le \int_\Sigma \sigma(\nu_u,\gamma)\,d\,\mathcal{H}^{\text{N-1}}$$

and

$$\limsup_{h\to\infty} \int_{S_{u_h}\setminus\Sigma_h} \sigma(\nu_{u_h},0)\,d\,\mathcal{H}^{\text{N-1}} \le \int_{S_u\setminus\Sigma} \sigma(\nu_u,0)\,d\,\mathcal{H}^{\text{N-1}},$$

i.e. the assertion. □

It remains to show the Γ-lim sup-inequality in the case of a general measure $\mu \in \mathcal{M}(\Omega)$.

Theorem 2.10 (lim-sup inequality)**.** *Given* $u \in \mathrm{BV}(\Omega,\{-1,1\})$ *and a Radon measure* $\mu \in \mathcal{M}(\Omega)$, *there are functions* $u_\varepsilon \in L^1(\Omega)$, $u_\varepsilon \to u$ *and* $\rho_\varepsilon \in L^1(\Omega,[0,\infty))$, $\rho_\varepsilon \overset{*}{\rightharpoonup} \mu$ *for* $\varepsilon \to 0^+$, *such that*

$$\limsup_{\varepsilon\to0^+} F_\varepsilon(u_\varepsilon,\rho_\varepsilon,\Omega) \le \int_{S_u} \sigma\left(\nu_u, \frac{d\mu}{d\,\mathcal{H}^{\text{N-1}}\llcorner S_u}\right) d\,\mathcal{H}^{\text{N-1}}\,. \tag{2.13}$$

Proof. Let us set

$$g := \frac{d\mu}{d\,\mathcal{H}^{\text{N-1}}\llcorner S_u} \in L^1(S_u, \mathcal{H}^{\text{N-1}}).$$

and notice that $g \geq 0$. We can use the equiintegrability of g and Chebyshev's inequality to show that the functions $g \wedge h \in L^\infty(S_u, \mathcal{H}^{\text{N-1}})$ converge to g in $L^1(S_u, \mathcal{H}^{\text{N-1}})$ as $h \to \infty$. Moreover, since $\mathcal{H}^{\text{N-1}}(S_u) < \infty$ we have that

$$\int_{S_u} g \wedge h \; d\,\mathcal{H}^{\text{N-1}} = \inf\left\{ \int_{S_u} s \; d\,\mathcal{H}^{\text{N-1}} : s \geq g \wedge h, s \text{ simple} \right\}.$$

Therefore, there exist simple functions $g \wedge h \leq s_h \leq h$ such that $\lim_{h\to\infty} s_h = g$ in L^1 and $\mathcal{H}^{\text{N-1}}(\{s_h < g\}) < C_h$ hold, where $\lim_{h\to\infty} C_h = 0$. Using the inner regularity of $\mathcal{H}^{\text{N-1}}$, we can find simple functions $g_h \leq s_h$ supported on (finitely many) pairwise disjoint compact subsets of S_u that can be chosen as projections of polyhedral sets on S_u such that $\mathcal{H}^{\text{N-1}}(\{g_h < s_h\}) < \frac{1}{h^2}$, which also implies that $\int_{S_u} |g_h - s_h| \leq \frac{2}{h}$. To sum up, we observe that there exist functions

$$g_h = \sum_{i=0}^{n_h} \gamma_{h,i} \chi_{K'_{h,i}}, \quad K'_{h,i} \subset S_u \text{ compact, projections of polyhedral sets on } S_u, \text{ pairwise disjoint}$$

which satisfy

$$\lim_{h\to\infty} g_h = g \text{ in } L^1(S_u, \mathcal{H}^{N-1}), \quad \lim_{h\to\infty} \mathcal{H}^{N-1}(\{g_h < g\}) = 0.$$

By the boundedness of μ, we find measures $\sum_{i=0}^{m_h} \beta_{h,i}\delta_{x_{h,i}}$ supported outside S_u such that

$$\sum_{i=0}^{m_h} \beta_{h,i}\delta_{x_{h,i}} \overset{*}{\rightharpoonup} \mu - g\mathcal{H}^{N-1}\llcorner S_u.$$

It follows that

$$\mu_h := g_h \mathcal{H}^{N-1}\llcorner S_u + \sum_{i=0}^{m_h} \beta_{h,i}\delta_{x_{h,i}} \overset{*}{\rightharpoonup} \mu.$$

We can finally combine the lower-semicontinutiy of the Γ-lim sup with the result of Proposition 2.9 to obtain that

$$\begin{aligned}
\Gamma\text{-}\limsup_{\varepsilon\to 0^+} F_\varepsilon(u,\mu,\Omega) &\leq \liminf_{h\to\infty} \Gamma\text{-}\limsup_{\varepsilon\to 0^+} F_\varepsilon(u,\mu_h,\Omega) \\
&\leq \liminf_{h\to\infty} \int_{S_u} \sigma(\nu_u, g_h)\, d\mathcal{H}^{N-1} \\
&\leq \liminf_{h\to\infty} \int_{S_u} \sigma(\nu_u, g)\, d\mathcal{H}^{N-1} + C \lim_{h\to\infty} \mathcal{H}^{N-1}(\{g_h < g\}) \\
&= \int_{S_u} \sigma(\nu_u, g)\, d\mathcal{H}^{N-1}.
\end{aligned}$$

where we have exploited the fact that σ is non-increasing in the second variable and bounded thanks to its upper-semicontinuity on S^{N-1}. This concludes the proof of the theorem. □

References

[1] Alberti, G. and Bellettini G., A non-local anisotropic model for phase transitions: asymptotic behaviour of rescaled energies, European Journal of Applied Mathematics vol. 9, 1998.

[2] Alberti, G. and Bellettini G., A non-local anisotropic model for phase transitions. I. The optimal profile problem, Math. Ann., **310**, (1998), 527–560.

[3] Alberti, G., Bellettini, G., Cassandro, G. and Presutti, E. Surface tension in Ising systems with Kac potentials. J. Stat. Phys., **82**, 743–796.

[4] Acerbi, E. and Bouchitté, G., A general class of phase transition models with weighted interface energy, Ann. Inst. H. Poincaré C Anal. Non Linéaire, **25** no. 6, (2008), 1111–1143.

[5] Alicandro, R., Ansini, N., Braides, A., Piatnitski, A. and Tribuzio, A., A variational theory of convolution-type functionals, SpringerBriefs on PDEs and Data Science, Springer, Singapore, 2023, viii+116.

[6] Alicandro, R., Braides A., Cicalese, M. and Solci, M., Discrete variational problems with interfaces, Cambridge Monographs on Applied and Computational Mathematics, **40**, Cambridge University Press, Cambridge, 2024.

[7] Alicandro, R., Cicalese, M. and Sigalotti, L., Phase transitions in presence of surfactants: from discrete to continuum, Interfaces Free Bound., **14** no. 1, (2012), 65–103.

[8] Ambrosio, L., Fusco, N. and Pallara, D., Functions of Bounded Variation and Free Discontinuity Problems, Clarendon Press, Oxford, 2000.

[9] Baía, M., Barroso, A. C. and Matias, J., A model for phase transitions with competing terms, Q. J. Math., **68** no. 3, (2017), 957–1000.
[10] Baldo, S., Minimal interface criterion for phase transitions in mixtures of Cahn-Hilliard fluids, Ann. Inst. H. Poincaré C Anal. Non Linéaire, **7** no. 2, (1990), 67–90.
[11] Bobaru, F., Foster, J.T., Geubelle, P.H. and Silling, S.A., Handbook of Peridynamic Modeling. Advances in Applied Mathematics. CRC press, Boca Raton 2016.
[12] Braides, A., Γ-convergence for beginners, Oxford Lecture Series in Mathematics and its Applications, **22**, Oxford University Press, Oxford, 2002, xii+218.
[13] Bucur, C., Valdinoci, E., Nonlocal Diffusion and Applications. Lecture Notes of the Unione Matematica Italiana, vol. 20. Springer, Cham 2016.
[14] Caroccia, M., Chambolle, A. and Slepčev, D., Mumford-Shah functionals on graphs and their asymptotics. Nonlinearity **33**, (2020), 3846–3888.
[15] Cicalese, M., Heilmann, T., Surfactants in the Two Gradient Theory of Phase Transitions, preprint, (2024), available online at https://arxiv.org/abs/2401.03840
[16] Dal Maso, G., An introduction to Γ-convergence, Progress in Nonlinear Differential Equations and their Application, 8, Birkhäuser Boston, 1993 xiv+340.
[17] Diehl, P., Lipton, R., Wick, T. and Tyagi, M., A comparative review of peridynamics and phase- field models for engineering fracture mechanics. Comput. Mech. **69**, (2022), 1259–1293.
[18] Di Nezza, E., Palatucci, G., Valdinoci, E., Hitchhiker's guide to the fractional Sobolev spaces. Bull. Sci. Math. **136**, (2012), 521–573.
[19] Du, Q. and Zhou, K., Mathematical analysis for the peridynamic nonlocal continuum theory. ESAIM: Math. Model. Numer. Anal. **45**, (2011), 217–234.
[20] Fonseca I., Morini M. and Slastikov, V., Surfactants in Foam Stability: A Phase-Field Model, Arch. Rational Mech. Anal. **183** no. 3, (2007), 411–456.
[21] García Trillos, N. and Slepčev, D., Continuum limit of total variation on point clouds. Arch. Ration. Mech. Anal. **220**, (2016), 193–241.
[22] Myers, D., Surfactant science and technology, (2020), John Wiley & Sons.
[23] Modica L., The Gradient Theory of Phase Transitions and the Minimal Interface Criterion, Arch. Rational Mech. Anal. **98**, (1987) 123–142.
[24] Modica, L. and Mortola, S., Un esempio di Γ^--convergenza, Boll. Un. Mat. Ital. B (5), **14**, (1977), 285–299.
[25] Silling, S.A. and Lehoucq, R.B., Peridynamic theory of solid mechanics. Adv. Appl. Mech. **44**, (2010), 73–168.

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