KORN AND POINCARÉ-KORN INEQUALITIES: A DIFFERENT PERSPECTIVE

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Abstract. We present a concise point of view on the first and the second Korn's inequality for general exponent p and for a class of domains that includes Lipschitz domains. Our argument is conceptually very simple and, for $p = 2$, uses only the classical Riesz representation theorem in Hilbert spaces. Moreover, the argument for the general exponent $1 < p < \infty$ remains the same, the only change being invoking now the q-Riesz representation theorem (with q the harmonic conjugate of p). We also complement the analysis with elementary derivations of Poincaré-Korn inequalities in bounded and unbounded domains, which are essential tools in showing the coercivity of variational problems of elasticity but also propedeutic to the proof of the first Korn inequality.

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1. Introduction and motivation

Korn's inequality is a fundamental result in mathematical analysis and a cornerstone of linear elasticity theory. For an elastic body occupying a regular region $\Omega \subset \mathbb{R}^3$, or more generally for a smooth bounded domain $\Omega \subset \mathbb{R}^N$, $N \geqslant 1$, the inequality, in its more classical form, states the existence, for every $1 < p < \infty$, of a positive constant $K_{\Omega,p} > 0$, depending only on Ω and p, such that

$$\int_\Omega |\boldsymbol{u}|^p + |\nabla \boldsymbol{u}|^p \leqslant K_{\Omega,p} \int_\Omega |\boldsymbol{u}|^p + |\nabla_{\mathrm{sym}} \boldsymbol{u}|^p \qquad \forall \boldsymbol{u} \in C^\infty(\overline{\Omega}, \mathbb{R}^N), \tag{1.1}$$

with $\nabla_{\mathrm{sym}} \boldsymbol{u}$ being the symmetric part of the Jacobian matrix $\nabla \boldsymbol{u}$, $|\nabla_{\mathrm{sym}} \boldsymbol{u}|$ and $|\nabla \boldsymbol{u}|$ the corresponding Frobenius norms (see sec. 2.1).

In the language of the theory of elasticity, the symmetric part of the gradient represents the strain experienced by an elastic body when subjected to deformation. Inequality (1.1) allows controlling the L^p-norm of the gradient of the displacement field with the L^p-norm of the *linearized* strain tensor. It is a fundamental tool for establishing the existence, uniqueness, and stability of solutions to linear and nonlinear elastic deformation problems. However, it has applications in diverse areas of mathematics and physics. The family of open sets where the inequality (1.1) holds, sometimes referred to as *Korn domains* [1], allows for a well-posed theory of linear elasticity and, therefore, for reliable reduced models, for example, for the study of elastic plates and shells [11,12].

The inequality (1.1) is usually referred to as the *second* Korn inequality [10, 50, 55] to distinguish it from its variant in the restricted class $C_c^\infty(\Omega, \mathbb{R}^N)$ of vector fields with compact support in Ω, usually known as the *first* Korn inequality, which states the existence of a positive constant $K_{\Omega,p} > 0$, such that

$$\int_\Omega |\nabla \boldsymbol{u}|^p \leqslant K_{\Omega,p} \int_\Omega |\nabla_{\mathrm{sym}} \boldsymbol{u}|^p \qquad \forall \boldsymbol{u} \in C_c^\infty(\Omega, \mathbb{R}^N). \tag{1.2}$$

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The first Korn inequality, for $p = 2$, immediately follows from the equality

$$\int_\Omega |\nabla_{\mathrm{sym}} \boldsymbol{u}|^2 = \frac{1}{2} \int_\Omega |\nabla \boldsymbol{u}|^2 + |\operatorname{div} \boldsymbol{u}|^2 \qquad \forall \boldsymbol{u} \in C_c^\infty(\Omega, \mathbb{R}^N). \tag{1.3}$$

Note that, when $p = 2$, the constant in (1.2) can be chosen independently of Ω (for example, $K_{\Omega,2} = 2$). This turns out to be a general remark. Indeed, as long as one is not interested in optimal constants, a simple scaling argument on balls gives that the constant $K_{\Omega,p}$ in equation (1.2) can always be taken to be independent of Ω.

Despite their significance, available proofs of Korn inequalities are pretty involved. Their proof strategies typically rely on approximation and compactness arguments combined with Calderón-Zygmund estimates for singular integral operators. As described in subsection 1.1, over the years, there has been a sporadic yet constant scientific production of different points of view that often led to simplifications or extensions.

The primary objective of this paper is to present a concise point of view on the first and the second Korn's inequality, for general exponent p, and for a class of domains that includes Lipschitz domains. Our proof is conceptually very simple and, for $p = 2$, gives a proof of the second Korn's inequality that uses only the classical Riesz representation theorem in Hilbert spaces. Moreover, conceptually, the argument for the general exponent p remains the same, the only change being invoking now the q-Riesz representation theorem (with q the harmonic conjugate of p) instead of the classical 2-Riesz representation theorem. We stress that the chief outcomes of the article are known, the novelty being in the alternative take on the subject we propose. Further details on the approach are given in section 2.

1.1. State of the art. Named after the German scientist Arthur Korn[1], Korn inequalities are a subject with more than a century of history. Their extensions to more general settings continue to be an active area of research (see, e.g., [5, 6, 8, 13, 15, 23, 29, 38, 40, 53]). The literature on the topic is enormous, and in what follows, we only focus on the (still vast) literature that is most pertinent to our study. For complementary aspects of the questions, we refer the reader to the excellent and comprehensive treatments [10, 50, 55], [20, secs. XI.4 and XII.4], [27, sec. 3.3], and to [39, sec. 7] for a quick survey on the applications of Korn inequalities to continuum mechanics. As far as possible, we present the relevant literature in historical chronological order.

The first and the second Korn's inequality, for $p = 2$, were formulated and proved by Korn himself (cf. [44, 46]). While the original proof in [44] for the first case is clear and essentially based on the identity (1.3), the proof of the second Korn inequality given by

[1]Arthur Korn was a mathematician, physicist, and inventor. Korn was born on May 20, 1870, in Breslau, Germany (now Wrocław, Poland). He studied physics and mathematics first at the University of Freiburg/Breisgau (with Emil Warburg) and then at the University of Leipzig (with Carl Neumann), where he graduated in 1890. Afterward, he studied in Berlin, Paris, London, and Würzburg, benefiting from the guidance of other prominent figures like Henri Poincaré during his time in Paris. In 1895, he became Priv.-Doz. in physics at the University of Munich, where he was appointed professor in 1903. In 1914, he accepted the chair of physics at the Berlin Institute of Technology. He emigrated to the USA with his family in 1939, where he became a professor of mathematics and physics at the Stevens Institute of Technology in Hoboken (New Jersey) in the same year. In 1945, he received American citizenship. He died at the age of 75 in the Jersey City Medical Center [63]. To the general public, Arthur Korn is primarily recognized for his pioneering contributions to telecommunications engineering. In 1902, he patented the first apparatus that enabled the transmission of images through telegraph lines, and with this system, in 1904, as part of a public demonstration, a portrait of Kaiser Wilhelm II from Munich to Nuremberg. In 1907, The New York Times published an article headlined "Photographs by Telegraph: Television Next?" that boldly declared that the «successful test of Prof. Korn's remarkable invention indicates the possibility of another field for scientific discoverer» [62]. For a comprehensive account of Arthur Korn's life, including his groundbreaking contributions to telephotography, we refer to [47].

Korn in [46] is, in the words of Friedrichs [30], *very complicated*, to the level that Friedrichs himself had to admit he was unable to verify Korn's proof for the second case. The reason for that, according to Friedrichs, other than the length of the involved arguments, was their nested structure: the proof given in [46] refers to previous results presented in [45], which still refer to earlier papers of Korn. Giving direct proof of Korn's inequality in the second case was what motivated Friedrichs contribution [46].

Starting from the work of Friedrichs, several authors proposed alternative proofs and generalizations. The work of Friedrichs assumes that $\langle \mathbf{curl} \boldsymbol{u} \rangle_\Omega = 0$, i.e., the curl of the displacement vector field $\boldsymbol{u}$ in (1.1) is null-average in Ω. Also, it assumes that the region of integration is what he calls a Ω-domain, a more stringent notion than the one of a Lipschitz domain. In [57], Payne and Weinberger investigate the best possible Korn constants $K_{\Omega,2}$ in balls and show that if Korn's inequality holds on a finite number of domains, each of which is C^2-diffeomorphic to balls, it also holds on their union.

The extension to domains with the cone property and without the null-average assumption on the curl of $\boldsymbol{u}$ comes with the work of Gobert [37], who fruitful uses some previous strategies sketched by Smith to investigate formally positive integro-differential forms [60]. Later on, Nitsche [54], who, like Friedrichs, also labels Korn's original proof as *doubtful*, provides an alternative argument for Korn's second inequality based on the idea of strain-preserving extension operators which reduces the study of Korn inequalities on bounded Lipschitz domains to the one on (not necessarily bounded) C^1-domains. All the cited references are pretty technical and involved. Only with the works of Kondratiev and Oleinik [43], but still, for $p = 2$, a much shorter, simpler, yet not elementary proof appears. The relatively recent work of Ciarlet [9] proposes a new proof of the second Korn inequality based on a distributional version of the Saint-Venant lemma. In the same paper, the author also recalls an elegant proof of the second Korn's inequality from [20], yet based on what he refers to as a *remarkably difficult-to-prove* lemma of J.L. Lions (for the proof of Lions' lemma, see, e.g., [49, Lemma 11.2, p. 316], [27, Thm 3.2, p. 111], and [41]). A more refined version of this lemma is given in [19, Lemma 2.1], and used to prove a generalized version of Korn's inequality where the linearized strain tensor is replaced by its trace free part. Such an inequality holds, however, only for $n \geqslant 3$.

All the references cited so far cover only the quadratic setting $p = 2$. The generalization to the general exponent $p \in]1, \infty[$, i.e., inequalities (1.1) and (1.2), introduces new severe difficulties even in the proof of the first Korn's inequality. The proofs of (1.1) and (1.2) can be found in Mikhlin [51] and, under weaker hypotheses, in Mosolov and Mjasnikov [52]. Both [51] and [52] use Calderón-Zygmund estimates. A simpler and self-contained proof that does not rely on the theory of singular integrals is given by Kondratiev and Oleinik [42].

Another proof [14] of the second Korn's inequality can be given by adapting the arguments used for a nonlinear counterpart thereof, the so-called geometric rigidity inequality [31]. One may indeed derive the equivalent statement[2] that for all $\boldsymbol{u} \in C^\infty(\overline{\Omega}, \mathbb{R}^N)$ there exists a skew-symmetric matrix A such that

$$\int_\Omega |\nabla \boldsymbol{u} - A|^p \leqslant K_{\Omega,p} \int_\Omega |\nabla_{\mathrm{sym}} \boldsymbol{u}|^p \,. \tag{1.4}$$

The inequality is first proved on a square in the case of *harmonic* functions, to which one is reconducted by considering an auxiliary elliptic problem and the related regularity estimates. Then, a fine local-to-global construction making use of a Whitney covering with dyadic squares and of a weighted Poincaré inequality (for which some regularity of the boundary is needed) entails the general result. Let us also mention that if the domain is

[2] A proof that (1.4) and (1.1) are equivalent is given in [9, Thm 2.3]: although stated for $p = 2$, that proof indeed applies for general $p \neq 1$.

star-shaped, inequality (1.4) has been known since the work of Reshetnyak [58], which is based on a representation of a vector function as the sum of a skew-symmetric affine motion and a potential-type integral only depending on the symmetric gradient.

We recall that for $p = 1, \infty$ inequalities (1.1) and (1.2) do *not* hold. When $p = 1$, this is a consequence of a famous result of Ornstein [56] showing that a priori estimates for elliptic operators, known to hold in L^p norm ($1 < p < \infty$), are no longer true in the L^1-norm (see also [16] for a counterexample based on laminates). Similar results of de Leeuw and Mirkil [22], antecedent to the ones of Ornstein, imply that Korn's inequality cannot be true also if $p = \infty$ (but counterexamples for the case $p = \infty$ can be easily constructed). From a different perspective, the result follows from the fact that the so-called ∇_2- and Δ_2- conditions are sharp requirements for the first and second Korn inequalities to hold in Orlicz spaces; since the Φ-function associated with the L^1-norm does not satisfy the ∇_2-condition, and the Φ-function associated with the L^∞-norm does not satisfy the Δ_2-condition [5, Remark 1.3], we get that Korn's inequality cannot hold when $p = 1, \infty$. The results in [5] have been further extended in [8] where different, suitably balanced Orlicz norms are considered to cope with the possible failure of either the ∇_2- or Δ_2- conditions; in [4], where an optimal trace-free inequality is given for $n \geqslant 3$; and in [18, Prop 4.1], which establishes a general Korn-type inequality for elliptic differential operators. Let us also mention that the even broader class of incompatible Korn inequalities, that is, for vector fields that are not necessarily gradient fields, has been the subject of many recent contributions [17, 33, 34, 36, 48, 53] that find applications in elastoplasticity models, fluid-mechanical problems, and the analysis of numerical schemes based on the finite element method.

Finally, we recall that Korn domains appear extremely difficult to characterize. What is known is that not every open set is a Korn domain. Indeed, as shown by Geymonat and Gilardi in [32] and by Weck in [64], it is possible to construct domains that satisfy the segment property, where Korn's second inequality does not hold. However, in [26], it is shown that Korn's inequality holds on a class of domains that is much larger than the class of Lipschitz domains and contains sets that may possess fractal boundaries or internal cusps (external cusps are excluded), the so-called John domains. For finitely connected planar domains, being a John domain is even a necessary condition for a Korn inequality to hold [40].

1.2. Outline. The rest of the paper is structured as follows. Section 2 presents three Poincaré-Korn inequalities, valid even for $p = 1$. One of them is the classical one needed to show that the symmetric gradient seminorm is a norm in $W^{1,p}$) and holds for any bounded open subset of $\mathbb{R}^N$. After that, we state the first and second Korn's inequality in our geometric setting.

The proofs of the result are given in section 3. Following the ideas of [25], we first give elementary proofs of the Poincaré-Korn inequalities, whose unique ingredient is the divergence theorem. Then we focus on our Korn inequalities. Our proofs for the first and second inequality are essentially the same and based on a lemma (see Lemma 1) that we state and prove in the same section 3. The lemma allows us to prove the first Korn inequality on general bounded domains and the second Korn's inequality on *extension domains* which, in particular, include the class of Lipschitz domains —in fact, the class of (ε, δ)-domains in the sense of Jones (cf. Remark 1).

2. CONTRIBUTIONS OF THE PRESENT WORK

The main results of the present work are concise proofs of first and second Korn's inequality, as well as Poincaré-Korn inequalities. To state our results precisely, we need to set up the framework, the mathematical notation, and the terminology used throughout the paper.

2.1. Notation. In what follows, we denote by $e_1, \dots, e_N$ the standard basis of $\mathbb{R}^N$ and, for $u \in \mathbb{R}^N$ we denote by $|u|$ the euclidean norm, i.e., $|u|^2 := (u \cdot e_1)^2 + \cdots + (u \cdot e_N)^2$. Coherently, the notation $|u|^p$ will stand for the p-power of the Euclidean norm, i.e.,

$$|u|^p = ((u \cdot e_1)^2 + \cdots + (u \cdot e_N)^2)^{p/2}. \tag{2.1}$$

For matrices $\Phi, \Psi \in \mathbb{R}^{N\times N}$, their Frobenius inner product is defined by $(\Phi : \Psi) := \mathrm{tr}(\Phi^{\mathsf{T}}\Psi)$, and the associated norm is given by $|\Phi|^2 = \sum_{i,j=1}^N (\Phi e_i \cdot e_j)^2$. Coherently, the notation $|\Phi|^p$ will stand for the p-power of $|\Phi|$, i.e.,

$$|\Phi|^p = \left(\sum_{i,j=1}^N (\Phi e_i \cdot e_j)^2 \right)^{p/2}. \tag{2.2}$$

We denote by I the identity matrix in $\mathbb{R}^{N\times N}$. Observe that $|I| = \sqrt{N}$.

For $\Omega \subseteq \mathbb{R}^N$ open set, we denote by $C_c^\infty(\Omega, \mathbb{R}^N)$ the space of infinitely differentiable vector fields with compact support, by $C^\infty(\overline{\Omega}, \mathbb{R}^N)$ the space of infinitely differentiable vector fields that are smooth up to the boundary, and by $C^\infty(\Omega, \mathbb{R}^N)$ the space of infinitely differentiable vector fields that are smooth *in* Ω. We denote by $\mathcal{D}'(\Omega, \mathbb{R}^N)$ the space of $\mathbb{R}^N$-valued distributions on Ω. For $\boldsymbol{u} \in \mathcal{D}'(\Omega, \mathbb{R}^N)$ and $\boldsymbol{\varphi} \in C_c^\infty(\Omega, \mathbb{R}^N)$ we denote by $\langle \boldsymbol{u}, \boldsymbol{\varphi} \rangle$ the value of $\boldsymbol{u}$ on $\boldsymbol{\varphi}$.

Similar notation is used when the target space is, e.g., the algebra $\mathbb{R}^{N\times N}$ of square matrices of order N.

For classical differential operators, the distributional counterparts are the familiar ones. For $u \in \mathcal{D}'(\Omega, \mathbb{R})$, $\boldsymbol{u} \in \mathcal{D}'(\Omega, \mathbb{R}^N)$, $\varphi \in C_c^\infty(\Omega)$ and $\boldsymbol{\varphi} \in C_c^\infty(\Omega, \mathbb{R}^N)$ we set

$$\langle \nabla u, \boldsymbol{\varphi} \rangle := -\langle u, \mathrm{div}\boldsymbol{\varphi} \rangle, \quad \langle \mathrm{div}\boldsymbol{u}, \varphi \rangle := -\langle u, \nabla\varphi \rangle, \quad \langle \mathbf{curl}\boldsymbol{u}, \boldsymbol{\varphi} \rangle := \langle \boldsymbol{u}, \mathbf{curl}\boldsymbol{\varphi} \rangle. \tag{2.3}$$

Also, we set

$$\langle \Delta u, \varphi \rangle := \langle \mathrm{div}\nabla u, \varphi \rangle = \langle u, \Delta\varphi \rangle, \quad \langle \Delta\boldsymbol{u}, \boldsymbol{\varphi} \rangle := \langle \boldsymbol{u}, \Delta\boldsymbol{\varphi} \rangle. \tag{2.4}$$

We need to introduce further differential operators. For $\boldsymbol{u} \in C^\infty(\Omega, \mathbb{R}^N)$, the *gradient* of $\boldsymbol{u}$ is defined by $\nabla\boldsymbol{u} := (\nabla\boldsymbol{u} \cdot e_1 \,|\, \dots \,|\, \nabla\boldsymbol{u} \cdot e_N)$. In other words, $\nabla\boldsymbol{u}$ is the matrix in $\mathbb{R}^{N\times N}$ whose columns are the gradients of the components of $\boldsymbol{u}$. Consistently, we denote by $\nabla^{\mathsf{T}}\boldsymbol{u} := (\partial_1\boldsymbol{u} \,|\, \dots \,|\, \partial_N\boldsymbol{u})$ the Jacobian matrix of $\boldsymbol{u}$. The *symmetric* and *skew-symmetric* gradient of $\boldsymbol{u}$ are, respectively, defined by

$$\nabla_{\mathrm{sym}}\boldsymbol{u} := \frac{1}{2}(\nabla\boldsymbol{u} + \nabla^{\mathsf{T}}\boldsymbol{u}), \quad \nabla_{\mathrm{skw}}\boldsymbol{u} := \frac{1}{2}(\nabla\boldsymbol{u} - \nabla^{\mathsf{T}}\boldsymbol{u}). \tag{2.5}$$

Given a matrix-valued field $\boldsymbol{\Phi} := (\boldsymbol{\varphi}_1 \,|\, \dots \,|\, \boldsymbol{\varphi}_N) \in C^\infty(\Omega, \mathbb{R}^{N\times N})$, we agree that the *divergence* of $\boldsymbol{\Phi}$, denoted by $\mathbf{div}\boldsymbol{\Phi}$, is the vector field

$$\mathbf{div}\boldsymbol{\Phi} := \sum_{i=1}^N (\mathrm{div}\boldsymbol{\varphi}_i)e_i = \sum_{j=1}^N \partial_j \boldsymbol{\Phi}^{\mathsf{T}} e_j, \tag{2.6}$$

obtained by applying the classical divergence to the columns of $\boldsymbol{\Phi}$. With the previous definition, the operator $\mathbf{div}$ is the formal adjoint of $-\nabla$, in the sense that if $\boldsymbol{u} \in C^\infty(\Omega, \mathbb{R}^N)$

and $\boldsymbol{\Phi} \in C_c^\infty(\Omega, \mathbb{R}^{N\times N})$, then the classical integration by parts formula holds:

$$\int_\Omega \nabla \boldsymbol{u} : \Phi = -\int_\Omega \boldsymbol{u} \cdot \mathbf{div}\boldsymbol{\Phi}.$$

The *symmetric* and *skew-symmetric* divergence of $\boldsymbol{\Phi}$ are defined by

$$\mathbf{div}_{\rm sym}\boldsymbol{\Phi} := \frac{1}{2}\mathbf{div}(\boldsymbol{\Phi} + \boldsymbol{\Phi}^{\mathsf{T}}) = \frac{1}{2}\sum_{j=1}^N \partial_j(\boldsymbol{\Phi}^{\mathsf{T}} + \boldsymbol{\Phi})e_j, \tag{2.7}$$

$$\mathbf{div}_{\rm skw}\boldsymbol{\Phi} := \frac{1}{2}\mathbf{div}(\boldsymbol{\Phi} - \boldsymbol{\Phi}^{\mathsf{T}}) = \frac{1}{2}\sum_{j=1}^N \partial_j(\boldsymbol{\Phi}^{\mathsf{T}} - \boldsymbol{\Phi})e_j. \tag{2.8}$$

Note that the operators $\mathbf{div}_{\rm sym}$ and $\mathbf{div}_{\rm skw}$ are the formal adjoints of $-\nabla_{\rm sym}$ and $-\nabla_{\rm skw}$. Consistently, if $\boldsymbol{u} \in \mathcal{D}'(\Omega, \mathbb{R}^N)$, the distributions $\nabla_{\rm sym}\boldsymbol{u} \in \mathcal{D}'(\Omega, \mathbb{R}^{N\times N})$ and $\nabla_{\rm skw}\boldsymbol{u} \in \mathcal{D}'(\Omega, \mathbb{R}^{N\times N})$ are, respectively, defined for every $\boldsymbol{\Phi} \in C_c^\infty(\Omega, \mathbb{R}^{N\times N})$ by $\langle \nabla_{\rm sym}\boldsymbol{u}, \boldsymbol{\Phi}\rangle := -\langle \boldsymbol{u}, \mathbf{div}_{\rm sym}\boldsymbol{\Phi}\rangle$ and $\langle \nabla_{\rm skw}\boldsymbol{u}, \boldsymbol{\Phi}\rangle := -\langle \boldsymbol{u}, \mathbf{div}_{\rm skw}\boldsymbol{\Phi}\rangle$. The definitions of $\nabla_{\rm sym}\boldsymbol{u}$ and $\nabla_{\rm skw}\boldsymbol{u}$ are compatible with the smooth setting. If $\boldsymbol{u} \in C^\infty(\Omega, \mathbb{R}^N)$ is a regular distributions and $\boldsymbol{\Phi} \in C_c^\infty(\Omega, \mathbb{R}^{N\times N})$ then

$$\langle \nabla_{\rm sym}\boldsymbol{u}, \boldsymbol{\Phi}\rangle = \int_\Omega \nabla_{\rm sym}\boldsymbol{u} : \boldsymbol{\Phi}, \qquad \langle \nabla_{\rm skw}\boldsymbol{u}, \boldsymbol{\Phi}\rangle = \int_\Omega \nabla_{\rm skw}\boldsymbol{u} : \boldsymbol{\Phi}. \tag{2.9}$$

Similar definitions apply to the distributional version of the operator $\mathbf{div}_{\rm sym}$ and $\mathbf{div}_{\rm skw}$, meaning that if $\boldsymbol{U} \in \mathcal{D}'(\Omega, \mathbb{R}^{N\times N})$, the distributions $\mathbf{div}_{\rm sym}\boldsymbol{U} \in \mathcal{D}'(\Omega, \mathbb{R}^N)$ and $\mathbf{div}_{\rm skw}\boldsymbol{U} \in \mathcal{D}'(\Omega, \mathbb{R}^N)$ are defined for every $\boldsymbol{\varphi} \in C_c^\infty(\Omega, \mathbb{R}^N)$ by $\langle \mathbf{div}_{\rm sym}\boldsymbol{U}, \boldsymbol{\varphi}\rangle := -\langle \boldsymbol{U}, \nabla_{\rm sym}\boldsymbol{\Phi}\rangle$ and $\langle \mathbf{div}_{\rm skw}\boldsymbol{U}, \boldsymbol{\varphi}\rangle := -\langle \boldsymbol{U}, \nabla_{\rm skw}\boldsymbol{\Phi}\rangle$. Simple computations show that for $\boldsymbol{u} \in C^\infty(\Omega, \mathbb{R}^N)$ there holds

$$-\Delta\boldsymbol{u} = -\nabla{\rm div}\boldsymbol{u} - 2\mathbf{div}_{\rm skw}\nabla_{\rm skw}\boldsymbol{u}, \tag{2.10}$$

$$-\Delta\boldsymbol{u} = +\nabla{\rm div}\boldsymbol{u} - 2\mathbf{div}_{\rm sym}\nabla_{\rm sym}\boldsymbol{u}, \tag{2.11}$$

and very same relations hold if $\boldsymbol{u} \in \mathcal{D}'(\Omega, \mathbb{R}^N)$. Note that if $N = 3$, we have $-2\mathbf{div}_{\rm skw}\nabla_{\rm skw}\boldsymbol{u} = \mathbf{curlcurl}\boldsymbol{u}$ and (2.10) reduces to the classical Helmholtz decomposition in $\mathbb{R}^3$.

2.2. Main results. Our first result concerns Poincaré-Korn inequalities in bounded and unbounded domains. These inequalities are essential tools in showing the coercivity of variational problems of elasticity. One of them (cf. (2.13)) is a propedeutic tool for the proof of the first Korn's inequality: it shows that the symmetric gradient seminorm is actually a norm on $C_c^\infty(\Omega, \mathbb{R}^N)$.

Theorem 1 (POINCARÉ-KORN INEQUALITIES)**.** *Let $\Omega \subseteq \mathbb{R}^N$ be an open set and $1 \leqslant p < \infty$. The following Poincaré-Korn inequalities hold.*

i. There exists a constant $C_{p,N} > 0$ depending only on p and N, such that

$$\int_\Omega |\boldsymbol{u}|^p \leqslant C_{p,N}^p \int_\Omega |x|^p |\nabla_{\rm sym}\boldsymbol{u}|^p \qquad \forall \boldsymbol{u} \in C_c^\infty(\Omega, \mathbb{R}^N). \tag{2.12}$$

In particular, if Ω is also bounded, then there exists a constant $\kappa_{p,\Omega} > 0$ depending only on p and Ω, such that

$$\int_\Omega |\boldsymbol{u}|^p \leqslant \kappa_{p,\Omega}^p \int_\Omega |\nabla_{\rm sym}\boldsymbol{u}|^p \qquad \forall \boldsymbol{u} \in C_c^\infty(\Omega, \mathbb{R}^N). \tag{2.13}$$

ii. If Ω is a bounded Lipschitz domain, then there exist constants $\kappa_{p,\Omega}, \kappa_{p,\partial\Omega} > 0$, depending only on p and Ω such that

$$\int_\Omega |\boldsymbol{u}|^p \leqslant \kappa_{p,\Omega}^p \int_\Omega |\nabla_{\rm sym}\boldsymbol{u}|^p + \kappa_{p,\partial\Omega} \int_{\partial\Omega} |\boldsymbol{u}|^p \qquad \forall \boldsymbol{u} \in C^\infty(\overline{\Omega}, \mathbb{R}^N). \tag{2.14}$$

We give an elementary proof of Theorem 1 in section 4. Here, instead, we make some remarks. First, we stress that in contrast to Korn's inequality which does not hold for $p = 1$, the Poincaré-Korn inequality holds even for $p = 1$ and *arbitrary* open sets. Second, we note that *ii.* implies (2.13). Indeed, if Ω is bounded but *not* Lipschitz, one can extend by zero an element $\boldsymbol{u} \in C_c^\infty(\Omega, \mathbb{R}^N)$ to an element of $\boldsymbol{u}_0 \in C_c^\infty(B, \mathbb{R}^N)$ where B is a ball containing Ω, and evaluate (2.14) on $\boldsymbol{u}_0$ (and with Ω replaced by B) to get back to (2.13). We opted to present (2.13) as a particular case of (2.12). Finally, it will be apparent from the proof that one can take

$$C_{p,N} := (2+|p-2|+\sqrt{N})\cdot\frac{p}{p+N}, \quad \kappa_{p,\Omega} := (\operatorname{diam}\Omega)C_{p,N}, \quad \kappa_{p,\partial\Omega} := p\left(\frac{p+1}{p+N}\right)(\operatorname{diam}\Omega). \tag{2.15}$$

However, there is no pretense of optimality in the previous constants.

Our second contribution is the main objective of our paper and concerns a concise point of view on the first and second Korn inequality. To state the results in the generality we want, we need to introduce the proper functional setting.

For every $1 \leqslant p < \infty$, we denote by $E^{1,p}(\Omega, \mathbb{R}^N)$ the vector space of L^p-vector-fields, whose symmetric gradient, in the distributional sense, is still in L^p, i.e.,

$$E^{1,p}(\Omega, \mathbb{R}^N) := \{\boldsymbol{u} \in L^p(\Omega, \mathbb{R}^N) : \nabla_{\mathrm{sym}}\boldsymbol{u} \in L^p(\Omega, \mathbb{R}^{N\times N})\}. \tag{2.16}$$

It is standard to show (see, e.g., [61]) that $E^{1,p}(\Omega)$ is a Banach space when endowed with the norm

$$\|\boldsymbol{u}\|_{E^{1,p}} := (\|\boldsymbol{u}\|_{L^p}^p + \|\nabla_{\mathrm{sym}}\boldsymbol{u}\|_{L^p}^p)^{1/p} = \left(\int_\Omega |\boldsymbol{u}|^p + |\nabla_{\mathrm{sym}}\boldsymbol{u}|^p\right)^{1/p}. \tag{2.17}$$

Also, the Poincaré-Korn inequality (2.13) assures that on the Banach subspace of $E^{1,p}(\Omega, \mathbb{R}^N)$, given by

$$E_0^{1,p}(\Omega, \mathbb{R}^N) := \overline{C_c^\infty(\Omega, \mathbb{R}^N)}, \tag{2.18}$$

i.e., defined as the closure of $C_c^\infty(\Omega, \mathbb{R}^N)$ in $E^{1,p}(\Omega, \mathbb{R}^N)$, the seminorm $\|\nabla_{\mathrm{sym}}\boldsymbol{u}\|_{L^p}$ is actually a norm equivalent to (2.17). It is standard to show that, as happens for the space $W_0^{1,p}(\Omega, \mathbb{R}^N)$, even $E_0^{1,p}(\Omega, \mathbb{R}^N)$ has the extension-by-zero property. Namely, if $\boldsymbol{u} \in E_0^{1,p}(\Omega, \mathbb{R}^N)$ then

$$\tilde{\boldsymbol{u}} \in E^{1,p}(\mathbb{R}^N, \mathbb{R}^N) \quad \text{and} \quad \nabla_{\mathrm{sym}}\tilde{\boldsymbol{u}} = \widetilde{\nabla_{\mathrm{sym}}\boldsymbol{u}}, \tag{2.19}$$

where we denoted by $\tilde{\boldsymbol{u}}$ and $\widetilde{\nabla_{\mathrm{sym}}\boldsymbol{u}}$ the extensions by zero of $\boldsymbol{u}$ and $\nabla_{\mathrm{sym}}\boldsymbol{u}$ to the whole of $\mathbb{R}^N$. Indeed, if $\boldsymbol{u} \in E_0^{1,p}(\Omega, \mathbb{R}^N)$ and $(\boldsymbol{\varphi}_n)_{n\in\mathbb{N}}$ is a sequence in $C_c^\infty(\Omega, \mathbb{R}^N)$ such that $\boldsymbol{\varphi}_n \to \boldsymbol{u}$ in $E_0^{1,p}(\Omega, \mathbb{R}^N)$, then for every $\boldsymbol{\Phi} \in C_c^\infty(\mathbb{R}^N, \mathbb{R}^{N\times N})$ we have that

$$\begin{aligned}\langle \nabla_{\mathrm{sym}}\tilde{\boldsymbol{u}}, \boldsymbol{\Phi}\rangle_{\mathbb{R}^N} &= -\int_\Omega \boldsymbol{u}\cdot\mathbf{div}_{\mathrm{sym}}\boldsymbol{\Phi} = -\lim_{n\to\infty}\int_\Omega \boldsymbol{\varphi}_n\cdot\mathbf{div}_{\mathrm{sym}}\boldsymbol{\Phi} \;=\; \lim_{n\to\infty}\int_\Omega \nabla_{\mathrm{sym}}\boldsymbol{\varphi}_n : \boldsymbol{\Phi}\\ &= \int_{\mathbb{R}^N} \widetilde{\nabla_{\mathrm{sym}}\boldsymbol{u}} : \boldsymbol{\Phi} = \langle \widetilde{\nabla_{\mathrm{sym}}\boldsymbol{u}}, \boldsymbol{\Phi}\rangle_{\mathbb{R}^N}.\end{aligned} \tag{2.20}$$

But the extension-by-zero property is not the only feature that the spaces $E_0^{1,p}(\Omega, \mathbb{R}^N)$ and $W_0^{1,p}(\Omega, \mathbb{R}^N)$ share. Indeed, as we show in Theorem 2 below, when $1 < p < \infty$ and Ω is a bounded open set, then $E_0^{1,p}(\Omega, \mathbb{R}^N) = W_0^{1,p}(\Omega, \mathbb{R}^N)$ and their associated norms are equivalent. Also, one has $E^{1,p}(\Omega, \mathbb{R}^N) = W^{1,p}(\Omega, \mathbb{R}^N)$ when $1 < p < \infty$ and Ω is what we call an *extension domain*.

Definition 1. *We say that a bounded open set Ω is an extension domain (for $E^{1,p}$) when any element $\boldsymbol{u} \in E^{1,p}(\Omega, \mathbb{R}^N)$ can be extended to an element of $E^{1,p}(\mathbb{R}^N, \mathbb{R}^N)$.*

Remark 1. Generally speaking, $E^{1,p}$-extension domains can be investigated by transposing the known techniques from $W^{1,p}$ to $E^{1,p}$. For example, the technique in [54], which suitably adapts the standard extension by reflection to the setting of symmetrized gradients, shows that Lipschitz domains are $E^{1,p}$-extension domains. Recently, it has been shown that (ε,δ)-domains in the sense of Jones are $E^{1,p}$-extension domains. For that, we refer the reader to [3, Thm 6.2] which generalizes previous results contained in [35]. Finally, we observe that a notion of $E^{1,p}_0$-extension domain would not be interesting because, as we have shown in (2.20), every bounded open set would be of this type.

Theorem 2 (FIRST AND SECOND KORN'S INEQUALITY)**.** *For $1<p<\infty$, the following assertions hold:*

i. If $\Omega\subseteq\mathbb{R}^N$ is an extension domain, then $E^{1,p}(\Omega,\mathbb{R}^N)=W^{1,p}(\Omega,\mathbb{R}^N)$ and there exists a constant $K_{\Omega,p}>0$ such that

$$\int_\Omega |\boldsymbol{u}|^p+|\nabla\boldsymbol{u}|^p\leqslant K_{\Omega,p}\int_\Omega |\boldsymbol{u}|^p+|\nabla_{\mathrm{sym}}\boldsymbol{u}|^p\quad\forall\boldsymbol{u}\in W^{1,p}(\Omega,\mathbb{R}^N). \tag{2.21}$$

ii. If $\Omega\subseteq\mathbb{R}^N$ is an open set, then $E^{1,p}_0(\Omega,\mathbb{R}^N)=W^{1,p}_0(\Omega,\mathbb{R}^N)$ and there exists a constant $K_{\Omega,p}>0$ such that

$$\int_\Omega |\nabla\boldsymbol{u}|^p\leqslant K_{\Omega,p}\int_\Omega |\nabla_{\mathrm{sym}}\boldsymbol{u}|^p\quad\forall\boldsymbol{u}\in W^{1,p}_0(\Omega,\mathbb{R}^N). \tag{2.22}$$

Remark 2. As already remarked in the introduction, as long as one is not interested in optimal constants, a simple scaling argument on balls gives that the constant $K_{\Omega,p}$ in equation (2.22) can always be taken to be independent of Ω. Also, as already recalled in section 1.1, Korn inequalities are no longer true when $p=1$ and the space $W^{1,1}(\Omega,\mathbb{R}^N)$ and $E^{1,1}(\Omega,\mathbb{R}^N)$ are different. The space $E^{1,1}(\Omega,\mathbb{R}^N)$ is often denoted by $\mathrm{LD}(\Omega,\mathbb{R}^N)$ and plays a pivotal role in the mathematical theory of plasticity. We refer the reader to [61, Chap. II] for some of the main properties of the space $\mathrm{LD}(\Omega,\mathbb{R}^N)$.

The concise proof of Theorem 2 is given at the beginning of section 3 and relies on the following lemma, whose proof is deferred to the end of section 3.

Lemma 1. *Let Ω be an open set and $1<p<\infty$. If $\boldsymbol{u}\in E^{1,p}(\Omega,\mathbb{R}^N)$, then $\boldsymbol{u}\in W^{1,p}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$.*

The approach we follow to prove the first and second Korn's inequality appeared concise and effective in showing the regularity of distributional solutions of the Poisson equation [24]. Part of our arguments has a nonempty intersection with the one Duvaut and Lions gave in [27, Thm 3.2, p. 111]. However, our approach avoids the *remarkably difficult-to-prove* lemma of J.L. Lions [20] and, for $p=2$, one only needs the classical Riesz representation theorem in Hilbert space and the Weyl lemma on the C^∞-regularity of harmonic distribution (for which short and elementary proofs are available [2]). Even for general $1<p<\infty$, we rely on the same ingredients: the q-Riesz representation theorem [59, pp. 10-11], with q such that $1/q+1/p=1$, and Weyl lemma. We are unaware of a simple proof of the q-Riesz representation theorem when $q\neq 2$. However, our proof requires invoking the result only on balls.

3. FIRST AND SECOND KORN'S INEQUALITY: PROOFS OF THEOREM 2 AND LEMMA 1

3.1. Proof of Theorem 2.*i*. We first show that if $1 < p < \infty$ then $W^{1,p}(\Omega,\mathbb{R}^N) = E^{1,p}(\Omega,\mathbb{R}^N)$. By the very definition, we have $W^{1,p}(\Omega) \subseteq E^{1,p}(\Omega,\mathbb{R}^N)$, even for $p = 1$ and general open sets. Vice versa, if $1 < p < \infty$, Ω is an extension domain, and $\boldsymbol{u} \in E^{1,p}(\Omega,\mathbb{R}^N)$, then its extension $\tilde{\boldsymbol{u}} \in E^{1,p}(\mathbb{R}^N,\mathbb{R}^N)$ is, by Lemma 1, in $W^{1,p}_{\mathrm{loc}}(\mathbb{R}^N,\mathbb{R}^N)$. Therefore, $\boldsymbol{u} \in W^{1,p}(\Omega,\mathbb{R}^N)$.

Next, we observe that the inclusion map

$$\jmath : \boldsymbol{u} \in W^{1,p}(\Omega,\mathbb{R}^N) \mapsto \boldsymbol{u} \in E^{1,p}(\Omega,\mathbb{R}^N) \tag{3.1}$$

is continuous for any $1 \leqslant p < \infty$, because of the inequality $\|\nabla_{\mathrm{sym}}\boldsymbol{u}\|_{L^p} \leqslant \|\nabla\boldsymbol{u}\|_{L^p}$, and, if $p > 1$, also a surjection of $W^{1,p}(\Omega,\mathbb{R}^N)$ onto $E^{1,p}(\Omega)$. Therefore, by the open mapping theorem, $\jmath$ is a topological isomorphism of $W^{1,p}(\Omega,\mathbb{R}^N)$ onto $E^{1,p}(\Omega,\mathbb{R}^N)$, and this implies (2.21). □

3.2. Proof of Theorem 2.*ii*. It is sufficient to show that if $1 < p < \infty$ then $W^{1,p}_0(\Omega,\mathbb{R}^N) = E^{1,p}_0(\Omega,\mathbb{R}^N)$. Indeed, after that, inequality (2.22) follows from the fact that the inclusion map $\jmath : \boldsymbol{u} \in W^{1,p}_0(\Omega,\mathbb{R}^N) \mapsto \boldsymbol{u} \in E^{1,p}_0(\Omega,\mathbb{R}^N)$ is a continuous surjection of the Banach space $(W^{1,p}_0(\Omega,\mathbb{R}^N), \|\nabla\cdot\|_{L^p})$ onto the Banach space $(E^{1,p}_0(\Omega,\mathbb{R}^N), \|\nabla_{\mathrm{sym}}\cdot\|_{L^p})$.

The inclusion $W^{1,p}_0(\Omega,\mathbb{R}^N) \subseteq E^{1,p}_0(\Omega,\mathbb{R}^N)$ is trivial and holds even for $p = 1$. Indeed, with $\jmath$ given by (3.1), we get that

$$W^{1,p}_0(\Omega,\mathbb{R}^N) = \jmath(\overline{C^\infty_c(\Omega,\mathbb{R}^N)} \subseteq \overline{\jmath(C^\infty_c(\Omega,\mathbb{R}^N))} = E^{1,p}_0(\Omega,\mathbb{R}^N),$$

where the first closure is meant with respect to the topology induced by the $W^{1,p}$-norm, while the second refers to the topology induced by the $E^{1,p}$-norm. For the inclusion $E^{1,p}_0(\Omega,\mathbb{R}^N) \subseteq W^{1,p}_0(\Omega,\mathbb{R}^N)$ we proceed in two steps. In STEP$_1$, we assume that Ω is bounded and of class C^1, and then, in STEP$_2$, we show how to remove this hypothesis.

STEP$_1$. If $\boldsymbol{u} \in E^{1,p}_0(\Omega,\mathbb{R}^N)$ then its extension by zero to the whole of $\mathbb{R}^N$, let us denote it by $\tilde{\boldsymbol{u}}$, is in $E^{1,p}(\mathbb{R}^N,\mathbb{R}^N)$ and has distributional support in Ω. Hence, by Lemma 1, we have that

$$\tilde{\boldsymbol{u}} \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^N,\mathbb{R}^N), \quad \operatorname{supp}\tilde{\boldsymbol{u}} \subseteq \overline{\Omega}. \tag{3.2}$$

Since Ω is a domain of class C^1, we conclude[3] that $\boldsymbol{u} \in W^{1,p}_0(\Omega,\mathbb{R}^N)$.

STEP$_2$. Let Ω be a bounded open set, $\boldsymbol{u} \in E^{1,p}_0(\Omega,\mathbb{R}^N)$, and $(\boldsymbol{u}_n)_{n\in\mathbb{N}}$ a sequence in $C^\infty_c(\Omega,\mathbb{R}^N)$ converging to $\boldsymbol{u}$ in $E^{1,p}_0(\Omega,\mathbb{R}^N)$. It is sufficient to prove that $(\boldsymbol{u}_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $W^{1,p}_0(\Omega,\mathbb{R}^N)$ because, after that, we have $\boldsymbol{u}_n \to \boldsymbol{v}$ in L^p for some $\boldsymbol{v} \in W^{1,p}_0(\Omega,\mathbb{R}^N)$ and, therefore, $\boldsymbol{u} = \boldsymbol{v}$. To that end, let $\tilde{\boldsymbol{u}}_n \in C^\infty_c(B,\mathbb{R}^N)$ denote the trivial extensions by zero of $\boldsymbol{u}_n$ to a ball B that includes Ω. By STEP$_1$, Korn's inequality (2.22) holds for domains of class C^1 (in particular, for B). Therefore, for every $n, m \in \mathbb{N}, n > m$

[3]The motivation for a first step under the assumption that Ω is of class C^1 comes from the fact that such domains satisfy the property that if $\tilde{\boldsymbol{u}} \in W^{1,p}(\mathbb{R}^N,\mathbb{R}^N)$ and $\operatorname{supp}\tilde{\boldsymbol{u}} \subseteq \overline{\Omega}$, then $\tilde{\boldsymbol{u}} \in W^{1,p}_0(\Omega,\mathbb{R}^N)$. The result is proved, e.g., in [28, Thm. 2, p. 273]. It is also known that domains of class C^0 still assure the same result (see [21, sec. 2.1] and [7] for details). However, to the best of our knowledge, it is unknown how general this class can be, and that is why we have to rely on a second step. The idea of arguing in two steps to have a result valid for a general open set was suggested to us by the anonymous referee. We thank the anonymous referee for this and other valuable suggestions.

there holds

$$\begin{aligned}\lim_{n\to\infty}\sup_{m>n}\|\boldsymbol{u}_n-\boldsymbol{u}_m\|_{W_0^{1,p}(\Omega)} &= \lim_{n\to\infty}\sup_{m>n}\|\tilde{\boldsymbol{u}}_n-\tilde{\boldsymbol{u}}_m\|_{W_0^{1,p}(B)}\\ &\leqslant K_{B,p}\cdot\lim_{n\to\infty}\sup_{m>n}\|\tilde{\boldsymbol{u}}_n-\tilde{\boldsymbol{u}}_m\|_{E_0^{1,p}(B)}\\ &= K_{B,p}\cdot\lim_{n\to\infty}\sup_{m>n}\|\boldsymbol{u}_n-\boldsymbol{u}_m\|_{E_0^{1,p}(\Omega)}=0.\end{aligned}$$

This completes the proof if the case where Ω is bounded. The case of Ω being unbounded follows from the scale invariance of Korn's constant on balls. □

3.3. Proof of Lemma 1. The argument we present is inspired by the one in [24]. We first give the proof for $N=3$, and show afterward how to modify the proof for general N. The proof for general N is as easy as in the case $N=3$, but for $N=3$, we have the familiar **curl** operator at our disposal, and we don't need further notation.

Proof of Lemma 1 for $N=3$. Let $\boldsymbol{u}\in E^{1,p}(\Omega,\mathbb{R}^N)$ and consider a ball $B\subseteq\Omega$. By the q-exponent version of the Riesz representation theorem [59, pp. 10-11] there exists an element $\boldsymbol{v}\in W_0^{1,p}(B,\mathbb{R}^N)$, $p>1$, such that

$$-\frac{1}{2}\langle\Delta\boldsymbol{v},\boldsymbol{\varphi}\rangle=\langle\nabla_{\mathrm{sym}}\boldsymbol{u},\nabla_{\mathrm{sym}}\boldsymbol{\varphi}\rangle\quad\forall\boldsymbol{\varphi}\in C_c^\infty(B,\mathbb{R}^N).\tag{3.3}$$

Indeed, the right-hand side of (3.3) defines a bounded linear functional on $W_0^{1,q}(B,\mathbb{R}^N)$ — note that for $q=p=2$, we only need the classical Riesz representation theorem in Hilbert spaces.

Also, simple algebra and integration by parts formula, give that $2\langle\nabla_{\mathrm{sym}}\boldsymbol{u},\nabla_{\mathrm{sym}}\boldsymbol{\varphi}\rangle=-\langle\Delta\boldsymbol{u}+\nabla\mathrm{div}\boldsymbol{u},\boldsymbol{\varphi}\rangle$ for every $\boldsymbol{\varphi}\in C_c^\infty(B,\mathbb{R}^N)$. Therefore, by combining this equality with (3.3), we get that there exists $\boldsymbol{v}\in W_0^{1,p}(B,\mathbb{R}^N)$, $p>1$, such that

$$-\Delta\boldsymbol{v}=-(\Delta\boldsymbol{u}+\nabla\mathrm{div}\boldsymbol{u})\quad\text{in }\mathcal{D}'(B,\mathbb{R}^N).\tag{3.4}$$

In particular, applying the divergence operator to both sides of (3.4) we infer that the distribution $\mathrm{div}(\boldsymbol{u}-\boldsymbol{v}/2)$ is harmonic in B, and this, by Weyl's lemma [2], assures that

$$\mathrm{div}(\boldsymbol{u}-\boldsymbol{v}/2)\in C^\infty(B).\tag{3.5}$$

Similarly, applying the **curl** operator to both sides of (3.4) we deduce that the distribution $\mathbf{curl}(\boldsymbol{u}-\boldsymbol{v})$ is harmonic in B and, therefore, that

$$\mathbf{curl}(\boldsymbol{u}-\boldsymbol{v})\in C^\infty(B,\mathbb{R}^N).\tag{3.6}$$

But then

$$\begin{aligned}-\Delta(\boldsymbol{u}-\boldsymbol{v}) &= \mathbf{curlcurl}(\boldsymbol{u}-\boldsymbol{v})-\nabla\mathrm{div}(\boldsymbol{u}-\boldsymbol{v}/2)+\nabla\mathrm{div}(\boldsymbol{v}/2)\\ &\in C^\infty(B,\mathbb{R}^N)+C^\infty(B,\mathbb{R}^N)+\nabla\mathrm{div}(W_0^{1,p}(B,\mathbb{R}^N))\\ &\in C^\infty(B,\mathbb{R}^N)+C^\infty(B,\mathbb{R}^N)+W^{-1,q'}(B,\mathbb{R}^N)\end{aligned}\tag{3.7}$$

and, therefore, given that $-\Delta\boldsymbol{v}\in W^{-1,q'}(B,\mathbb{R}^N)$ we obtain that

$$-\Delta\boldsymbol{u}\in C^\infty(B,\mathbb{R}^N)+W^{-1,q'}(B,\mathbb{R}^N).\tag{3.8}$$

The previous relation completes the proof. Indeed, shrinking eventually the ball B, we get that $-\Delta\boldsymbol{u}\in C^\infty(\bar{B},\mathbb{R}^N)+W^{-1,q'}(B,\mathbb{R}^N)=W^{-1,q'}(B,\mathbb{R}^N)$. Applying again the q-exponent version of the Riesz representation theorem, we deduce the existence of a vector field $\boldsymbol{w}\in W_0^{1,p}(B,\mathbb{R}^N)$ such that $-\Delta(\boldsymbol{u}-\boldsymbol{w})=0$ in $\mathcal{D}'(B,\mathbb{R}^N)$. But then, as before, harmonicity implies that $\boldsymbol{u}\in C^\infty(B,\mathbb{R}^N)+W_0^{1,p}(B,\mathbb{R}^N)\subseteq W^{1,p}_{\mathrm{loc}}(B,\mathbb{R}^N)$ and we conclude.

Proof of Lemma 1 for general N. We can resume from (3.5). Applying the $\nabla_{\rm skw}$ operator to both sides of (3.4), we deduce that the distribution $\nabla_{\rm skw}(\boldsymbol{v}-\boldsymbol{u})$ is harmonic in B and, therefore, that $\nabla_{\rm skw}(\boldsymbol{v}-\boldsymbol{u})$ is smooth. But then

$$\begin{aligned}-\Delta(\boldsymbol{u}-\boldsymbol{v}) &= -2\mathbf{div}_{\rm skw}\nabla_{\rm skw}(\boldsymbol{u}-\boldsymbol{v}) - \nabla{\rm div}(\boldsymbol{u}-\boldsymbol{v}/2) + \nabla{\rm div}(\boldsymbol{v}/2)\\ &\in C^\infty(B,\mathbb{R}^N) + C^\infty(B,\mathbb{R}^N) + \nabla{\rm div}(W_0^{1,p}(B,\mathbb{R}^N))\\ &\in C^\infty(B,\mathbb{R}^N) + C^\infty(B,\mathbb{R}^N) + W^{-1,q'}(B,\mathbb{R}^N) \end{aligned} \tag{3.9}$$

and, therefore, given that $-\Delta\boldsymbol{v}\in W^{-1,q'}(B,\mathbb{R}^N)$, we are back to (3.8), and from that point on, the proof is the same as for $N=3$. □

4. Proof of Theorem 1: Poincaré-Korn inequalities

Following the ideas of [25], we first give elementary proofs of the Poincaré-Korn inequalities, whose unique ingredient is the divergence theorem.

Proof of Theorem 1. For every $\varepsilon>0$, we consider the vector field $\boldsymbol{u}_\varepsilon:\Omega\to\mathbb{R}^{N+1}$ defined by $\boldsymbol{u}_\varepsilon:=(\boldsymbol{u},\varepsilon)$. Note that $|\boldsymbol{u}_\varepsilon|=(|\boldsymbol{u}|^2+\varepsilon^2)^{1/2}$ and, therefore, $|\boldsymbol{u}_\varepsilon|^\alpha\in C^\infty(\Omega,\mathbb{R}^N)$ for every $\alpha\in\mathbb{R}$. Here, with a convenient abuse of notation, we used the same symbol to denote the Euclidean norms in $\mathbb{R}^{N+1}$ and $\mathbb{R}^N$. It is easy to check that the following equalities hold

$$|\boldsymbol{u}_\varepsilon|^{p-2}\boldsymbol{u}\cdot(\nabla^{\mathsf{T}}\boldsymbol{u}x) = \frac{1}{p}{\rm div}(|\boldsymbol{u}_\varepsilon|^p x) - \frac{N}{p}|\boldsymbol{u}_\varepsilon|^p, \tag{4.1}$$

$$|\boldsymbol{u}_\varepsilon|^{p-2}\boldsymbol{u}\cdot(\nabla\boldsymbol{u}x) = {\rm div}((\boldsymbol{u}\cdot x)|\boldsymbol{u}_\varepsilon|^{p-2}\boldsymbol{u}) - (\boldsymbol{u}\cdot x){\rm div}(|\boldsymbol{u}_\varepsilon|^{p-2}\boldsymbol{u}) - |\boldsymbol{u}|^2|\boldsymbol{u}_\varepsilon|^{p-2}, \tag{4.2}$$

from which, summing term by term, we get

$$\begin{aligned}\left(\frac{N}{p}+\frac{|\boldsymbol{u}|^2}{|\boldsymbol{u}_\varepsilon|^2}\right)|\boldsymbol{u}_\varepsilon|^p &= -2|\boldsymbol{u}_\varepsilon|^{p-1}\frac{\boldsymbol{u}}{|\boldsymbol{u}_\varepsilon|}\cdot(\nabla_{\rm sym}\boldsymbol{u}\,x) - (\boldsymbol{u}\cdot x){\rm div}(|\boldsymbol{u}_\varepsilon|^{p-2}\boldsymbol{u})\\ &\quad + \frac{1}{p}{\rm div}(|\boldsymbol{u}_\varepsilon|^p x) + {\rm div}\left(|\boldsymbol{u}_\varepsilon|^p\left(\frac{\boldsymbol{u}}{|\boldsymbol{u}_\varepsilon|}\cdot x\right)\frac{\boldsymbol{u}}{|\boldsymbol{u}_\varepsilon|}\right).\end{aligned} \tag{4.3}$$

The previous relation is the key ingredient in the proof of Theorem 1.

Proof of Theorem 1.*i.* Proof of (2.12). If $\boldsymbol{u}\in C_c^\infty(\Omega,\mathbb{R}^N)$, integrating on Ω both sides (4.3), and then passing to the limit for $\varepsilon\to0$, we get that

$$\left(\frac{p+N}{p}\right)\int_\Omega|\boldsymbol{u}|^p \leqslant 2\int_\Omega|\boldsymbol{u}|^{p-1}|\nabla_{\rm sym}\boldsymbol{u}||x| + \liminf_{\varepsilon\to0}\int_\Omega|\boldsymbol{u}||{\rm div}(|\boldsymbol{u}_\varepsilon|^{p-2}\boldsymbol{u})||x|, \tag{4.4}$$

and, we are left to evaluate the last term on the right-hand side of the previous relation (4.4). For that, we observe that

$${\rm div}(|\boldsymbol{u}_\varepsilon|^{p-2}\boldsymbol{u}) = |\boldsymbol{u}_\varepsilon|^{p-2}\left[(p-2)(\nabla_{\rm sym}\boldsymbol{u})\frac{\boldsymbol{u}}{|\boldsymbol{u}_\varepsilon|}\cdot\frac{\boldsymbol{u}}{|\boldsymbol{u}_\varepsilon|} + {\rm div}\boldsymbol{u}\right] \tag{4.5}$$

and, therefore, given that $|{\rm div}\boldsymbol{u}| = |\nabla_{\rm sym}\boldsymbol{u}:I| \leqslant \sqrt{N}|\nabla_{\rm sym}\boldsymbol{u}|$, we have the estimate

$$\liminf_{\varepsilon\to0}\int_\Omega|\boldsymbol{u}||{\rm div}(|\boldsymbol{u}_\varepsilon|^{p-2}\boldsymbol{u})||x| \leqslant (|p-2|+\sqrt{N})\int_\Omega|\boldsymbol{u}|^{p-1}|\nabla_{\rm sym}\boldsymbol{u}|. \tag{4.6}$$

Estimating (4.4) with (4.6) we get that

$$\left(\frac{N+p}{p}\right)\int_\Omega|\boldsymbol{u}|^p \leqslant (2+|p-2|+\sqrt{N})\int_\Omega|\boldsymbol{u}|^{p-1}|\nabla_{\rm sym}\boldsymbol{u}||x|\,,$$

which establishes (2.12) when $p=1$. For $p>1$, Hölder inequality gives the Poincaré-Korn inequality under the weighted form (2.12).

PROOF OF (2.13). It is sufficient to note that the proof of (2.12) works verbatim if one replaces everywhere x with $x - x_0$, x_0 being an arbitrary point of $\mathbb{R}^N$. But then, if Ω is bounded, one gets

$$\|\boldsymbol{u}\|_{L^p(\Omega)} \leqslant C_{p,N}\left(\inf_{x_0\in\mathbb{R}^N}\sup_{x\in\Omega}|x-x_0|\right)\|\nabla_{\mathrm{sym}}\boldsymbol{u}\|_{L^p(\Omega)} \tag{4.7}$$

from which (2.13) follows as a special case. □

Proof of Theorem 1.*ii.* PROOF OF (2.14). If $\boldsymbol{u} \in C^\infty(\overline{\Omega},\mathbb{R}^N)$ and Ω is a bounded Lipschitz domain, then integrating on Ω both sides of (4.3), passing to the limit for $\varepsilon \to 0$, and taking into account (4.6), we get that

$$\int_\Omega |\boldsymbol{u}|^p \leqslant C_{p,N}\int_{x\in\Omega}|\boldsymbol{u}|^{p-1}|\nabla_{\mathrm{sym}}\boldsymbol{u}||x| + \left(\frac{p+1}{p+N}\right)\int_{\xi\in\partial\Omega}|\boldsymbol{u}|^p|\xi|. \tag{4.8}$$

Again, the previous computation works verbatim if one replaces everywhere x with $x - x_0$, with x_0 an arbitrary point of $\mathbb{R}^N$. Therefore, we have

$$\int_\Omega |\boldsymbol{u}|^p \leqslant (\operatorname{diam}\Omega)C_{p,N}\int_\Omega|\boldsymbol{u}|^{p-1}|\nabla_{\mathrm{sym}}\boldsymbol{u}| + \left(\frac{p+1}{p+N}\right)(\operatorname{diam}\Omega)\int_{\partial\Omega}|\boldsymbol{u}|^p\,. \tag{4.9}$$

Now, if $p = 1$, we are done. Otherwise, by Young's inequality for products applied to the first term on the right-hand side of (4.9), we obtain that

$$\frac{1}{p}\int_\Omega|\boldsymbol{u}|^p \leqslant \frac{(\operatorname{diam}\Omega)^p C_{p,N}^p}{p}\int_\Omega|\nabla_{\mathrm{sym}}\boldsymbol{u}|^p + \left(\frac{p+1}{p+N}\right)(\operatorname{diam}\Omega)\int_{\partial\Omega}|\boldsymbol{u}|^p\,, \tag{4.10}$$

from which (2.14) follows. □

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