A BOURGAIN–BREZIS–MIRONESCU FORMULA ACCOUNTING FOR NONLOCAL ANTISYMMETRIC EXCHANGE INTERACTIONS

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Abstract. The present study concerns the nonlocal-to-local convergence of a family of exchange energy functionals in the limit of very short-range interactions. The analysis accounts for both symmetric and antisymmetric exchange. Our result is twofold. First, we extend the Bourgain-Brezis-Mironescu formula to encompass the scenario where antisymmetric contributions are encoded into the energy. Second, we prove that, under physically relevant assumptions on the families of exchange kernels, the family of nonlocal functionals Gamma-converges to their local counterparts. As a byproduct of our analysis, we obtain a rigorous justification of Dzyaloshinskii–Moriya interactions in chiral magnets under the form commonly adopted in the variational theory of micromagnetism when modeling antisymmetric exchange interactions.

1. Introduction and motivation

The present paper investigates the short-range interaction limit of a family of nonlocal exchange energies of the form

$$\mathcal{E}_\varepsilon(m) := \mathcal{F}_\varepsilon(m) + \mathcal{H}_\varepsilon(m), \tag{1.1}$$

where the energy functionals $\mathcal{F}_\varepsilon$ and $\mathcal{H}_\varepsilon$ are given by

$$\mathcal{F}_\varepsilon(m) := \iint_{\Omega\times\Omega} \rho_\varepsilon(x-y)\frac{|m(x)-m(y)|^2}{|x-y|^2}\mathrm{d}x\,\mathrm{d}y, \tag{1.2}$$

$$\mathcal{H}_\varepsilon(m) := \iint_{\Omega\times\Omega} \nu_\varepsilon(x-y)\cdot\frac{(m(x)\times m(y))}{|x-y|}\,\mathrm{d}x\,\mathrm{d}y, \tag{1.3}$$

and are both defined on a suitable (metric) subspace of $L^2(\Omega;\mathbb{S}^2)$. Here, $L^2(\Omega;\mathbb{S}^2)$ denotes the space of vector-valued maps $m:\Omega\to\mathbb{S}^2$, where Ω is a bounded Lipschitz domain of $\mathbb{R}^3$ and $\mathbb{S}^2$ the unit sphere of $\mathbb{R}^3$.

The energy functional $\mathcal{E}_\varepsilon$ is the result of two different types of interactions. The term $\mathcal{F}_\varepsilon$ in (1.2) accounts for the so-called *symmetric* exchange interactions, whereas the energy term $\mathcal{H}_\varepsilon$ in (1.3) accounts for *antisymmetric* exchange interactions. The scalar kernel $\rho_\varepsilon:\mathbb{R}^3\to\mathbb{R}_+$ and the vector-valued kernel $\nu_\varepsilon:\mathbb{R}^3\to\mathbb{R}^3$ model the strength and positional configuration of the exchange interactions at spatial scale $\varepsilon>0$. They will be referred to as symmetric and antisymmetric exchange kernels, respectively.

The main aim of this paper is to show that, under physically relevant assumptions on the families of exchange kernels $(\rho_\varepsilon)_\varepsilon$ and $(\nu_\varepsilon)_\varepsilon$, the family $\mathcal{E}_\varepsilon$ converges, in the sense of Γ-convergence in $L^2(\Omega;\mathbb{S}^2)$, and up to constant factors, to the local energy functional

$$\mathcal{E}(m) := \mathcal{F}(m)+\mathcal{H}(m) := \frac{1}{2}\int_\Omega |\nabla m(x)|^2 + \sum_{i=1}^3\int_\Omega m(x)\cdot(d_i\times\partial_i m(x))\mathrm{d}x, \tag{1.4}$$

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provided that $m \in H^1(\Omega;\mathbb{S}^2)$. Here, the quantities $d_i \in \mathbb{R}^3$ denote constant vectors referred to as *Dzyaloshinskii vectors* in the Physics literature; their expression strongly depends on the limiting behavior of the family $(\nu_\varepsilon)_\varepsilon$.

Rigorous statements will be given in Section 2, where we formulate explicit assumptions on the exchange kernels.

1.1. Outline. The paper is organized as follows. In the rest of this Section, we present a brief overview of the physical framework that motivated our investigation and elucidate the significance of our findings in relation to prior research on the subject. In Section 2, we present the precise formulation of the problem and provide a detailed account of the contributions of our work. Proofs of our main results are given in Sections 3 and 4. Specifically, Section 3 focuses on the pointwise convergence of the antisymmetric exchange interactions (1.3), while Section 4 establishes the Γ-convergence of the nonlocal family $(\mathcal{E}_\varepsilon)$ in (1.1) to the local energy functional $\mathcal{E}$ in (1.4).

1.2. Physics context: symmetric and antisymmetric exchange interactions in micromagnetics. Reliable theoretical models for studying magnetic phenomena must depend on the relevant length scales. At the mesoscopic scale, there is a well-established and effective variational theory of micromagnetism, whose roots may be found in the works of Landau–Lifshitz [18] and Brown [6,7] on fine ferromagnetic particles. In this theory, for a rigid ferromagnetic particle occupying a region $\Omega \subseteq \mathbb{R}^3$, the order parameter is the magnetization field $M : \Omega \to \mathbb{R}^3$. The modulus of M, $M_s := |M|$, is called *spontaneous magnetization* and is a function of the temperature that vanishes above the so-called *Curie point* T_c: a critical value strongly depends on the specific crystal structure of the ferromagnet. When the specimen is at a fixed temperature well below T_c, the function M_s can be assumed constant in Ω, and the magnetization can be conveniently written as $M := M_s m$, where $m : \Omega \to \mathbb{S}^2$ is a vector field with values in the unit sphere of $\mathbb{R}^3$ (cf. [7,15]).

Despite $|M|$ being constant in Ω, this is generally not the case for the direction of M, and according to the variational theory of micromagnetism, the observable magnetization patterns are the local minimizers of the *micromagnetic energy functional*, which, after normalization, reads as[1]

$$\mathcal{G}(m) := \mathcal{F}(m) + \mathcal{W}(m) := \frac{1}{2}\int_\Omega |\nabla m|^2 + \frac{1}{2}\int_{\mathbb{R}^3} |h_{\mathsf{d}}[m\chi_\Omega]|^2. \tag{1.5}$$

Here, $m \in H^1(\Omega;\mathbb{S}^2)$, and $m\chi_\Omega$ is the extension by zero of m to $\mathbb{R}^3$. The *Dirichlet energy* $\mathcal{F}$, i.e., the first term in (1.5), penalizes nonuniformities in the magnetization orientation, whereas the *magnetostatic self-energy* $\mathcal{W}$, i.e., the second term in (1.5), is the energy associated with the demagnetizing field h_{d} generated by $m\chi_\Omega$, which describes the long-range dipole interaction of the magnetic moments: for $v \in L^2(\mathbb{R}^3,\mathbb{R}^3)$, the demagnetizing field $-h_{\mathsf{d}}[v] \in L^2(\mathbb{R}^3,\mathbb{R}^3)$ can be characterized as the L^2-projection of v on the space of gradients $\nabla\dot{H}^1(\mathbb{R}^3) := \{\nabla v : v \in \mathcal{D}'(\mathbb{R}^3), \nabla v \in L^2(\mathbb{R}^3,\mathbb{R}^3)\}$, see [10,25] for details.

Much of the pattern observed in ferromagnetic materials is explained by the competition between the two contributions in (1.5); in particular, the formation of almost uniform magnetization regions (*magnetic domains*) separated by thin transition layers (*domain walls*), as predicted by the Weiss theory of ferromagnetism (cf. [7,15]).

[1] In writing (1.5) *is non-convex, non-local, and contains multiple length scales*, we neglected the *magnetocrystalline anisotropy* and *Zeeman* energy, but only to shorten the notation. Indeed, although these contributions are of fundamental importance in ferromagnetism [6,7], from the variational point of view, they behave like continuous perturbations, their analysis is usually straightforward, and in our specific context, they play no role.

However, recent advancements in nanotechnology have led to the discovery of *magnetic skyrmions*: chiral spin textures that carry a nontrivial topological charge. Unlike conventional magnetic domains, magnetic skyrmions exhibit unusual swirling textures, and they arise in ferromagnetic materials with low crystallographic symmetry. The primary mechanism behind their formation and stability is weak antisymmetric exchange interactions, also known as *Dzyaloshinskii–Moriya interactions* (DMI), which result from the combination of spin-orbit and superexchange interactions [12, 20].

In the continuum theory of micromagnetism, DMI is accounted through the so-called *chirality tensor* $\nabla m \times m := (\partial_i m \times m)_i$, whose components are the Lifshitz invariants of m (see, e.g., [14, Supplementary information]). The *bulk* DMI energy density corresponds to the trace of the chirality tensor: in the presence of the bulk DMI, the micromagnetic energy functional (1.5) has to be remodeled under the form

$$\mathcal{G}(m) := \mathcal{F}(m) + \mathcal{H}_{\rm bulk}(m) + \mathcal{W}(m) := \frac{1}{2}\int_\Omega |\nabla m|^2 + \gamma \int_\Omega \operatorname{curl} m \cdot m + \frac{1}{2}\int_{\mathbb{R}^3} |h_{\mathsf{d}}[m\chi_\Omega]|^2. \quad (1.6)$$

The normalized constant $\gamma \in \mathbb{R}$ is the bulk DMI constant, and its sign affects the chirality of the ferromagnetic system [2, 21] (see [11, Sec. 4] for further forms of DMI). Note that the bulk DMI $\mathcal{H}_{\rm bulk}$ in (1.6) is what the energy term $\mathcal{H}$ in (1.4) reduces to when the Dzyaloshinskii vectors have the form $d_i := \gamma e_i$, with e_i being the i-th element of the standard basis of $\mathbb{R}^3$ (observe that $\operatorname{curl} m = \sum_{i=1}^3 e_i \times \partial_i m$).

1.3. The L^2-theory of symmetric and antisymmetric exchange interactions. The Heisenberg setting. Ferromagnetism occurs in materials where the spins tend to align with each other, thereby generating an observable magnetic field outside of the media. *Symmetric* exchange interactions are the primary mechanism behind this effect, and in the isotropic Heisenberg model their behavior is described by the Hamiltonian (cf. [17])

$$\mathcal{F}_\Lambda = -S^2 \sum_{(i,j)\in\Lambda\times\Lambda} J_{ij} m_i \cdot m_j. \quad (1.7)$$

Here, $J_{ij} = J_{ji}$ is the symmetric exchange constant between the spins $m_i, m_j \in \mathbb{S}^2$, occupying the i-th and j-th site of the crystal lattice Λ, and S is the magnitude of the spin. The exchange constant J_{ij} is a positive quantity in ferromagnetism; it depends on the ferromagnet's crystal structure and weighs the intensity of the interaction among different spins.

When antisymmetric interactions cannot be neglected because, e.g., the ferromagnetic crystal lacks inversion symmetry, the DMI induces a spin canting of the magnetic moments, and the Hamiltonian (1.7) has to be remodeled under the form (cf. [17])

$$\mathcal{E}_\Lambda := \mathcal{F}_\Lambda + \mathcal{H}_\Lambda := -S^2 \sum_{(i,j)\in\Lambda\times\Lambda} J_{ij} m_i \cdot m_j + \sum_{(i,j)\in\Lambda\times\Lambda} d_{ij} \cdot (m_i \times m_j). \quad (1.8)$$

The *Dzyaloshinskii vector* $d_{ij} = -d_{ji}$ is an axial vector that depends, other than from the relative distance between the spins m_i and m_j, on the symmetry of the crystal lattice; its precise form has to be determined following Moriya's rules [20]. We stress that while the term $\mathcal{F}_\Lambda$ is symmetric, in the sense that magnetic moments with right-handed ($\nwarrow\nearrow$) or left-handed ($\nearrow\nwarrow$) alignment give the same contributions, the discrete antisymmetric exchange energy $\mathcal{H}_\Lambda$ distinguishes between those two states via the local chirality imposed by the Dzyaloshinskii vectors d_{ij}.

In the limit of a continuous distribution of lattice sites occupying a region $\Omega \subseteq \mathbb{R}^3$, the exchange energy (1.8), up to a constant term, can be expressed as

$$\mathcal{E}_{J,d}(m) := \mathcal{F}_J(m) + \mathcal{H}_d(m) := \frac{1}{2}\int_{\Omega\times\Omega} J(x-y)|m(x)-m(y)|^2 \mathrm{d}x\,\mathrm{d}y + \int_{\Omega\times\Omega} d(x-y)\cdot(m(y)\times m(x))\mathrm{d}x\,\mathrm{d}y, \tag{1.9}$$

with $m : \Omega \to \mathbb{S}^2$ the normalized magnetization density, $J \geqslant 0$ a *symmetric* exchange kernel with even symmetry, i.e., such that $J(z) = J(-z)$ for every $z \in \mathbb{R}^3$, and d an *antisymmetric* exchange kernel with odd symmetry, i.e., such that $d(-z) = -d(z)$ for every $z \in \mathbb{R}^3$.

The energy functional $\mathcal{E}_\varepsilon$ in (1.1) is a faithful analog of the energy $\mathcal{E}_{J,d}$ in (1.9), and we found it more convenient to work with $\mathcal{E}_\varepsilon$ because of scaling reasons. Specifically, we set $J(z) := \rho(z)/|z|^2$ and $d(z) := \nu_\varepsilon(z)/|z|$. Overall, the exchange terms $\mathcal{F}_\varepsilon$, $\mathcal{H}_\varepsilon$, in (1.2) and (1.3), are the continuous counterparts, respectively, of the symmetric Heisenberg Hamiltonian associated with a many-electron system and of the antisymmetric exchange interactions due to the spin-orbit coupling between neighboring magnetic spins.

1.4. State of the art and contributions of the present work. At first glance, the relationship between the nonlocal energies $\mathcal{F}_\varepsilon$, $\mathcal{H}_\varepsilon$, in (1.2), (1.3), and the terms $\mathcal{F}$ and $\mathcal{H}$ in (1.4) is not evident, but it can be formally revealed through a first-order asymptotic expansion of the magnetization $m(x)$ in a neighborhood of $x \in \Omega$, i.e., by setting $m(y) = m(x)+\mathrm{D}m(x)(y-x)+\mathcal{O}(|y-x|)$. The asymptotic analysis becomes more reliable the more one can neglect variations of m around $x \in \Omega$, i.e., the more the kernels ρ_ε and ν_ε concentrate their mass around the origin, i.e., the more the exchange interactions act on a very short range only. The question then is whether and in which sense the exchange energy $\mathcal{E}$ in (1.1) is a short-range approximation of the Heisenberg L^2-description in (1.9). In this paper, we give an affirmative answer to this question under physically relevant hypotheses on the kernels.

When only symmetric exchange interactions are considered, the (affirmative) answer is already known because of the Γ-convergence result established in [24]. Surprisingly, the motivation for the results in [24] came from very different reasons: a question left open in the seminal paper [3] where a new characterization of Sobolev spaces is presented, and the Bourgain–Brezis–Mironescu (BBM) formula made its first appearance. In [3], *pointwise* convergence of $\mathcal{F}_\varepsilon$ to $\mathcal{F}$ is established when $(\rho_\varepsilon)_{\varepsilon\in\mathbb{R}_+}$ is a family of radial mollifiers concentrating their mass at the origin as $\varepsilon \to 0$; also, a technical lemma provides upper and lower bounds on the *variational* convergence of $\mathcal{F}_\varepsilon$, but not sufficient to deduce Γ-convergence, finally obtained in [24]. In summary, in the absence of DMI, the classical symmetric exchange energy can be considered as the very short-rage limit of the family of nonlocal energies $(\mathcal{F}_\varepsilon)$, and the main aim of this paper is to extend the result to the case in which also antisymmetric interactions are present and under legitimate hypotheses on the antisymmetric exchange kernels ν_ε.

Over the years, several papers have presented new BBM-type formulas that expanded the original results in [3] into various directions [1,5,9,16,19,24] (see also [22,26] for some applications to the magnetic Schrödinger operator). Also, formal asymptotics predicts that the choice of appropriate exchange kernels is crucial for $\mathcal{E}_\varepsilon$ to reduce to $\mathcal{E}$, and we refer the reader to [8,13] for some results about the class of admissible kernels.

In contrast to the symmetric energy functional $\mathcal{F}_\varepsilon$, the variational convergence of $\mathcal{H}_\varepsilon$ in the asymptotic regime of very short-range interactions had not been investigated so far. To the best of the authors' knowledge, no variational analysis involving $\mathcal{H}_\varepsilon$ has been carried out thus far; this includes basic questions like the existence of $\mathbb{S}^2$-valued minimizers of $\mathcal{E}_\varepsilon$ or the qualitative

behaviors of minimizing sequences; these are significant issues because of their direct bearing on the emergence of magnetic skyrmions, and will be the subject of a forthcoming paper.

Notation. In what follows, we will adopt standard notation for the spaces of Lebesgue and Sobolev functions, as well as for the space of functions with bounded variation (BV). Balls with center x and radius r will be denoted by $B_r(x)$ and characteristic functions of sets A by χ_A, with the convention that $\chi_A(x) = 1$ if $x \in A$ and $\chi_A(x) = 0$ otherwise. Whenever not explicitly mentioned otherwise, $C > 0$ will alway denote a generic constant, only dependent on the data of the problem, and whose value might, in principle, change from line to line.

2. Statement of main results

For a given $\varepsilon > 0$, we denote by $D_\varepsilon(\Omega;\mathbb{S}^2)$ the subspace of $L^2(\Omega;\mathbb{S}^2)$ where $\mathcal{E}_\varepsilon$ is finite:

$$D_\varepsilon(\Omega;\mathbb{S}^2) = \{m \in L^2(\Omega;\mathbb{S}^2) : \mathcal{E}_\varepsilon(m) < +\infty\}. \tag{2.1}$$

Also, since many of our results still hold when m is not constrained to take values on $\mathbb{S}^2$, and even if Ω is unbounded, it is convenient for us to denote by $D_\varepsilon(\Omega;\mathbb{R}^3)$ the analog unconstrained subspace of $L^2(\Omega;\mathbb{R}^3)$. Of course, the space $D_\varepsilon(\Omega;\mathbb{S}^2)$ on which the energy $\mathcal{E}_\varepsilon$ is finite depends on the particular choice of the kernels ρ_ε and ν_ε. In what follows, we will make the following assumptions.

The symmetric exchange kernels ρ_ε. Driven by formal asymptotics, we consider a family of kernels $(\rho_\varepsilon)_\varepsilon$ satisfying the following hypotheses already adopted in [24] —which are more general than the ones initially proposed in [3] (see also [8] for an extensive discussion).

(G1) For every $\varepsilon > 0$, $\rho_\varepsilon \geqslant 0$ in $\mathbb{R}^3$ and $\rho_\varepsilon \in L^1(\mathbb{R}^3) \cap BV_{\mathrm{loc}}(\mathbb{R}^3)$ with $\|\rho_\varepsilon\|_{L^1(\mathbb{R}^3)} = 1$.

(G2) For every $\delta > 0$

$$\lim_{\varepsilon\to 0} \int_{\mathbb{R}^3\setminus B_\delta(0)} \rho_\varepsilon(y)\,\mathrm{d}y = 0. \tag{2.2}$$

(G3) There exist linearly independent directions $v_1, v_2, v_3 \in \mathbb{S}^2$ and $\delta > 0$ such that $C_\delta(v_i) \cap C_\delta(v_j) = \emptyset$ for $i \neq j$, and for any $i = 1, 2, 3$

$$\limsup_{\varepsilon\to 0} \int_{C_\delta(v_i)} \rho_\varepsilon(y)\mathrm{d}y > 0. \tag{2.3}$$

Here, for $v \in \mathbb{S}^2$ and $\delta > 0$, $C_\delta(v) := \mathbb{R}_+ E_\delta(v)$ denotes the cone centered at the origin, whose projection on $\mathbb{S}^2$ is given by $E_\delta(v) := \{w \in \mathbb{S}^2 : w \cdot v > (1-\delta)\}$.

(G4) There exists a constant $0 < \kappa \leqslant 1$ such that, denoting by $\rho_\varepsilon^{\mathrm{rad}}$ the radial functions $\rho_\varepsilon^{\mathrm{rad}} : \mathbb{R}_+ \to \mathbb{R}_+$, defined for every $\varepsilon > 0$ as

$$\rho_\varepsilon^{\mathrm{rad}}(t) := \operatorname{ess\,inf}\{\rho_\varepsilon(x) :\ x \in \mathbb{R}^3 \text{ with } |x| = t\} \quad \text{for every } t \in \mathbb{R}_+, \tag{2.4}$$

there holds

$$\inf_{\varepsilon>0} \int_{\mathbb{R}^3} \rho_\varepsilon^{\mathrm{rad}}(|x|)\mathrm{d}x \geqslant \kappa. \tag{2.5}$$

Remark 2.1 (On Hypotheses (G1), (G3), and (G4)). A few words about Hypotheses (G1),(G3) and (G4) are in order. Hypothesis (G3) is a weaker condition than the *radiality* assumed in [3]. Roughly speaking, it assures that in the limit of very short-range interactions ($\varepsilon \to 0$), the family $(\rho_\varepsilon)_\varepsilon$ has nontrivial support at least around three linearly independent directions. Condition (G3) has been introduced in [24] in order to prove a characterization of $W^{1,p}(\Omega)$, $1 < p < \infty$. Our work uses it to prove a regularity result for our nonlocal functional (1.1) (see Theorem 2.2). In view of (G1), Hypothesis (G4) is automatically satisfied when the kernels $(\rho_\varepsilon)_\varepsilon$ are radial. If this is not the case, Hypothesis (G4) provides a non-degeneracy condition for suitable radial

lower envelopes of the kernels. In particular, in the prototypical case in which the family $(\rho_\varepsilon)_\varepsilon$ is given by

$$\rho_\varepsilon(x) := \frac{1}{\varepsilon^3}\rho\Big(\frac{x}{\varepsilon}\Big),$$

for every $\varepsilon > 0$ and $x \in \mathbb{R}^3$, where $\rho \in C_c^\infty(\mathbb{R}^3)$ is such that $0 \leqslant \rho \leqslant 1$ in $\mathbb{R}^3$ and $\|\rho\|_{L^1(\mathbb{R}^3)} = 1$, Hypothesis (G4) is directly satisfied as long as ρ is strictly positive at the origin. In fact, assume that $\rho(0) = 2\delta > 0$. By the regularity of ρ, there exists a ball of radius $r > 0$ such that $\rho(x) \geqslant \delta$ for every $x \in \overline{B_r(0)}$. In particular, $\rho_\varepsilon^{\mathrm{rad}}(|x|) \geqslant \frac{\delta}{\varepsilon^3}$ for every $\varepsilon > 0$ and $x \in \overline{B_{r\varepsilon}(0)}$, so that (2.5) holds with $\kappa = \delta|B_r(0)|$.

Finally, the regularity of the kernels in (G1) can be weakened to just $L^1(\mathbb{R}^3)$ in the case in which the kernels are radial. The further BV_{loc}- regularity is only needed to provide a meaning in the sense of traces to the restriction of the kernels ρ_ε to spheres of radius r, so that the definition in (2.4) is well-posed.

The antisymmetric exchange kernels ν_ε. Motivated by formal asymptotics, we consider a family of vector-valued kernels $(\nu_\varepsilon)_\varepsilon$ satisfying the following assumptions.

(H1) For every $\varepsilon > 0$, ν_ε is odd, i.e., $\nu_\varepsilon(-y) = -\nu_\varepsilon(y)$ for each $y \in \mathbb{R}^3$, and $\nu_\varepsilon \in L^1(\mathbb{R}^3;\mathbb{R}^3)$ with $\|\nu_\varepsilon\|_{L^1(\mathbb{R}^3)} = 1$.

(H2) For every $\delta > 0$ there holds

$$\lim_{\varepsilon\to 0} \int_{\mathbb{R}^3\setminus B_\delta(0)} |\nu_\varepsilon(y)|\,\mathrm{d}y = 0. \tag{2.6}$$

(H3) For $i = 1, 2, 3$, the following limit exists

$$\lim_{\varepsilon\to 0} \int_{\mathbb{R}^3} \nu_\varepsilon(y)\frac{y_i}{|y|}\,\mathrm{d}y =: d_i \in \mathbb{R}^3. \tag{2.7}$$

Motivated by their physical significance, we will refer to the vectors d_i as the Dzyaloshinskii vectors. Note that $|d_i| \leqslant 1$ for every $i = 1, 2, 3$.

Later on, we will need the following condition (A1), which connects the integrability of the two families of kernels $(\rho_\varepsilon)_\varepsilon$ and $(\nu_\varepsilon)_\varepsilon$.

(A1) There exist a constant $C > 0$ such that

$$\sup_{\varepsilon>0} \left\| \frac{\nu_\varepsilon}{\rho_\varepsilon} \right\|_{L^\infty(\mathbb{R}^3)} \leqslant C. \tag{2.8}$$

Throughout the work, we specify in which cases assumption (A1) is needed.

Remark 2.2 (On the odd symmetry of ν_ε). We emphasize that the requirement of ν_ε being odd causes no loss of generality. Indeed, one can always decompose a vector-valued kernel into its even and odd parts, and an immediate symmetry argument shows that the presence of an even component in ν_ε would give no contribution to $\mathcal{H}_\varepsilon$ due to the antisymmetric nature of the functional $\mathcal{H}_\varepsilon$ and its linearity with respect to the kernel. Also, our results still hold when one relaxes (H3), requiring that (2.7) holds up to the extraction of a subsequence, but we do not insist on these refinements.

The main result of this paper consists of a compactness and Γ-convergence analysis for our nonlocal exchange functionals in (1.1).

Theorem 2.1. (*Compactness and Γ-convergence*) *Let $\Omega \subseteq \mathbb{R}^3$ be a bounded Lipschitz domain. Assume (G1)–(G4), (H1)–(H3), and (A1). The following assertions hold.*

(i) (Compactness) *If* $(m_\varepsilon)_\varepsilon \subset L^2(\Omega;\mathbb{S}^2)$ *is such that*

$$\liminf_{\varepsilon\to 0} \mathcal{E}_\varepsilon(m_\varepsilon) < +\infty, \tag{2.9}$$

then, there exists $m \in H^1(\Omega;\mathbb{S}^2)$ *such that, possibly up to a non-relabeled subsequence,* $m_\varepsilon \to m$ *strongly in* $L^2(\Omega;\mathbb{S}^2)$.

(ii) (Γ-convergence) *There exists a finite Radon measure* $\mu \in \mathcal{M}(\mathbb{S}^2)$, *with* $\mu(\mathbb{S}^2) = 1$, *such that, possibly up to a non-relabeled subsequence,*

$$\Gamma\text{-}\lim_{\varepsilon\to 0} \mathcal{E}_\varepsilon = \mathcal{E}_\mu, \quad \text{strongly in } L^2(\Omega;\mathbb{S}^2), \tag{2.10}$$

with $\mathcal{E}_\mu(m) = +\infty$ *if* $m \in L^2(\Omega;\mathbb{S}^2)\backslash H^1(\Omega;\mathbb{S}^2)$ *and*

$$\mathcal{E}_\mu(m) = \int_\Omega \Big(\int_{\mathbb{S}^2} |\partial_\sigma m(x)|^2 \mathrm{d}\mu(\sigma) \Big) \mathrm{d}x + \sum_{i=1}^3 \int_\Omega m(x) \cdot (d_i \times \partial_i m(x))\, \mathrm{d}x \tag{2.11}$$

if $m \in H^1(\Omega;\mathbb{S}^2)$.

Remark 2.3 (The micromagnetic case). We note that choosing kernels $(\nu_\varepsilon)_\varepsilon$ such that $d_i = \gamma e_i$ for $i = 1,2,3$ in (H3), and kernels $(\rho_\varepsilon)_\varepsilon$ radial, from Theorem 2.1 we obtain that with respect to the strong topology of L^2, there holds $\Gamma\text{-}\lim_{\varepsilon\to 0} \mathcal{E}_\varepsilon = \mathcal{E}$ with

$$\mathcal{E}(m) = \frac{1}{3}\int_\Omega |\nabla m(x)|^2 \mathrm{d}x + \gamma \int_\Omega \mathrm{curl} m(x) \cdot m(x)\, \mathrm{d}x \tag{2.12}$$

if $m \in H^1(\Omega;\mathbb{S}^2)$, and $\mathcal{E}(m) = +\infty$ otherwise (if $m \in L^2(\Omega;\mathbb{S}^2)\backslash H^1(\Omega;\mathbb{S}^2)$). Up to a constant factor, this is nothing but the micromagnetic energy functional (1.6) thoroughly studied in recent years as a theoretical foundation for the analysis of magnetic skyrmions emerging from bulk DMI.

Remark 2.4 (Examples of kernels ρ_ε and ν_ε). The conditions imposed on the families of kernels $(\rho_\varepsilon)_\varepsilon$ and $(\nu_\varepsilon)_\varepsilon$ are not meant to be sharp but rather to cover most situations of interest in applications, in particular, for the analysis of magnetic skyrmions in chiral ferromagnetic materials. An explicit example of kernels satisfying the hypotheses above are presented below. Let $\nu(y) := \frac{4}{|\mathbb{S}^2|} y \chi_{B_1(0)}(y)$ and $\rho(y) := |\nu(y)|$ for every $y \in \mathbb{R}^3$, where we have denoted by $\chi_{B_1(0)}(y)$ the characteristic function of the unit ball in $\mathbb{R}^3$. Note that ν is odd, $\|\nu\|_{L^1(\mathbb{R}^3)} = \|\rho\|_{L^1(\mathbb{R}^3)} = 1$, and ρ is radial. Then, setting

$$\nu_\varepsilon(y) := \frac{1}{\varepsilon^3}\nu\Big(\frac{y}{\varepsilon}\Big) \quad \text{and } \rho_\varepsilon(y) := \frac{1}{\varepsilon^3}\rho\Big(\frac{y}{\varepsilon}\Big)$$

for every $\varepsilon > 0$ and $y \in \mathbb{R}^3$, Hypotheses (G1)–(G4), as well as (H1)–(H2) and (A1) are directly fulfilled. Additionally, (H3) holds with $d_i = \frac{1}{3}e_i$, $i = 1,2,3$.

Remark 2.5 (On the measure μ). Note that, on the one hand, as for the purely symmetric case, cf. [24, Lemma 8], the measure μ in Theorem 2.1 (ii) is, in principle, dependent on the choice of the extracted subsequence. For some specific choices of the kernels $(\rho_\varepsilon)_\varepsilon$, on the other hand, for example when they are radial, the limiting functional is uniquely identified independently of the extracted subsequence, see also Remark 2.9 below, so that the Gamma-convergence result actually holds for the entire sequence $(\mathcal{E}_\varepsilon)_\varepsilon$.

The proof of Theorem 2.1 is provided in Section 4. The argument relies upon the following results, which are of interest in their own rights. The first assures that the nonlocal functional (1.1) is well-defined on $H^1(\Omega;\mathbb{S}^2)$.

Theorem 2.2. *Assume (G1) and (H1). For every $m \in H^1(\Omega;\mathbb{R}^3)$ there holds*

$$\sup_{\varepsilon>0} \mathcal{E}_\varepsilon(m) \leqslant C \, \|m\|^2_{H^1(\Omega)}, \tag{2.13}$$

for some constant $C > 0$ depending only on Ω.

Viceversa, assume (G1)–(G3), as well as (H1) and (A1). If $m \in L^2(\Omega;\mathbb{R}^3)$ is such that

$$\sup_{\varepsilon>0} \mathcal{E}_\varepsilon(m) < +\infty, \tag{2.14}$$

then $m \in H^1(\Omega;\mathbb{R}^3)$.

Remark 2.6 (The domain of the nonlocal functionals). Estimate (2.13) gives, in particular (cf. (2.1)), that $H^1(\Omega;\mathbb{R}^3) \subseteq D_\varepsilon(\Omega;\mathbb{R}^3)$ and $H^1(\Omega;\mathbb{S}^2) \subseteq D_\varepsilon(\Omega;\mathbb{S}^2)$. Also, it will be evident from the proof that $H^1(\Omega;\mathbb{R}^3) \subseteq D_\varepsilon(\Omega;\mathbb{R}^3)$ even when Ω is unbounded. In particular, $H^1(\mathbb{R}^3;\mathbb{R}^3) \subseteq D_\varepsilon(\mathbb{R}^3;\mathbb{R}^3)$.

Remark 2.7 (On the assumptions of Theorem 2.2). Estimate (2.13) guarantees that if (G1) and (H1) hold and $m \in H^1(\Omega;\mathbb{R}^3)$, then $\sup_{\varepsilon>0} \mathcal{E}_\varepsilon(m) < +\infty$. The converse statement holds under the additional assumptions (G2), (G3), (H1), and (A1). Assumptions (G2)–(G3) are the main requirements in [24, Theorem 5]. Roughly speaking, (G3) prevents degenerate mass behavior of the family $(\rho_\varepsilon)_\varepsilon$, so that the partial derivatives of m can be controlled over all directions. The further assumption (A1) expresses an integrability relation between the two families of kernels $(\rho_\varepsilon)_\varepsilon$ and $(\nu_\varepsilon)_\varepsilon$ which, nevertheless, naturally arise when modeling antisymmetric exchange interactions in the variational theory of micromagnetism.

Another crucial ingredient for the proof of Theorem 2.1 is a uniform convergence result for the antisymmetric exchange energies $\mathcal{H}_\varepsilon$, contained in the following Theorem 2.3.

Theorem 2.3. (Pointwise and uniform convergence of the antisymmetric exchange term) *Assume (H1)–(H3) and let Ω be bounded. For every $m \in H^1(\Omega;\mathbb{R}^3)$ there holds*

$$\lim_{\varepsilon\to 0} \mathcal{H}_\varepsilon(m) = \mathcal{H}(m), \tag{2.15}$$

where, we recall (see (1.2) and (1.4))

$$\mathcal{H}_\varepsilon(m) = \iint_{\Omega\times\Omega} \frac{\nu_\varepsilon(x-y)}{|x-y|} \cdot (m(x)\times m(y))\,\mathrm{d}x\,\mathrm{d}y\,, \quad \mathcal{H}(m) = \sum_{i=1}^{3} \int_\Omega m(x)\cdot(d_i \times \partial_i m(x))\,\mathrm{d}x,$$

and the $d_i \in \mathbb{R}^3$ are the Dzyaloshinskii vectors defined in (H3).

Moreover, if $(m_\varepsilon)_{\varepsilon>0}$ is a family in $C^2(\bar\Omega;\mathbb{R}^3)$ such that $m_\varepsilon \to m$ in $C^2(\bar\Omega;\mathbb{R}^3)$, then

$$\lim_{\varepsilon\to 0} \mathcal{H}_\varepsilon(m_\varepsilon) = \mathcal{H}(m). \tag{2.16}$$

Remark 2.8 (The case of unbounded Ω). It will be evident from the proof that (2.15) holds even when Ω is unbounded. In particular, when $\Omega = \mathbb{R}^3$. Also, for simplicity, we required (H3) to ensure the convergence of the whole sequence in (2.7). It will be clear from the proof that this requirement can be weakened by only requiring that the convergence in (H3) holds up to the extraction of a subsequence.

As a direct consequence of Theorem 2.3, and [24, Theorem 1], we infer the following result about the pointwise asymptotic behavior of the total nonlocal energy (1.1).

Corollary 2.1. (Pointwise convergence of the total energy) *Assume (G1)–(G3), as well as (H1)–(H3) and (A1). Then, there exists a finite Radon measure $\mu \in \mathcal{M}(\mathbb{S}^2)$, with*

$\mu(\mathbb{S}^2) = 1$, *such that, possibly up to a non-relabeled subsequence, for every* $m \in H^1(\Omega;\mathbb{R}^3)$, *there holds*

$$\lim_{\varepsilon\to 0} \mathcal{E}_\varepsilon(m) = \int_\Omega \Big(\int_{\mathbb{S}^2} |\partial_\sigma m|^2 \mathrm{d}\mu(\sigma) \Big) \mathrm{d}x + \sum_{i=1}^3 \int_\Omega m(x) \cdot (d_i \times \partial_i m(x))\, \mathrm{d}x, \tag{2.17}$$

with the $d_i \in \mathbb{R}^3$ *being the Dzyaloshinskii vectors introduced in* (H3).

Moreover, if the family $(\rho_\varepsilon)_\varepsilon$ *consists of radial kernels, then for every* $m \in H^1(\Omega;\mathbb{R}^3)$ *we have*

$$\lim_{\varepsilon\to 0} [\mathcal{F}_\varepsilon(m) + \mathcal{H}_\varepsilon(m)] = \frac{1}{3} \int_\Omega |\nabla m|^2 \, \mathrm{d}x + \sum_{i=1}^3 \int_\Omega m(x) \cdot (d_i \times \partial_i m(x))\, \mathrm{d}x.$$

Remark 2.9 (Structure of the limiting symmetric term). Since we are in the quadratic setting, one has $|\partial_\sigma m|^2 = \sum_k (\sigma \otimes \sigma) \nabla m_k \cdot \nabla m_k$ and, therefore, one can rewrite the first term in (2.17) under the form

$$\lim_{\varepsilon\to 0} \mathcal{F}_\varepsilon(m) = \sum_k \int_\Omega A \nabla m_k \cdot \nabla m_k \mathrm{d}x \tag{2.18}$$

where A is the anisotropic matrix given by $A := \int_{\mathbb{S}^2} (\sigma \otimes \sigma) \mathrm{d}\mu(\sigma)$. If the kernels are radial, then μ is isotropic, in the sense that the resulting matrix A is given by $(1/3)I$ with I being the 3×3 identity matrix.

3. Pointwise and uniform behavior of the energy (proofs of Theorem 2.2 and Theorem 2.3)

Proof of Theorem 2.2. We split the proof in two steps. In Step 1, we derive the estimate (2.13), while in Step 2, we prove that if (2.14) holds, then $m \in H^1(\Omega;\mathbb{R}^3)$.

Step 1. Recall that, by definition, $\mathcal{E}_\varepsilon(m) = \mathcal{F}_\varepsilon(m) + \mathcal{H}_\varepsilon(m)$. For ease of computation, we treat the two energy terms separately. For $m \in H^1(\Omega;\mathbb{R}^3)$, we denote by $\tilde{m} \in H^1(\mathbb{R}^3;\mathbb{R}^3)$ the extension of m to the whole of $\mathbb{R}^3$ given by the classical extension operator on $H^1(\Omega;\mathbb{R}^3)$.

For the symmetric exchange term $\mathcal{F}_\varepsilon$ we have

$$\begin{aligned}
\mathcal{F}_\varepsilon(m) &\leqslant \iint_{\mathbb{R}^3\times\mathbb{R}^3} \rho_\varepsilon(x-y) \frac{|\tilde{m}(x) - \tilde{m}(y)|^2}{|x-y|^2} \, \mathrm{d}x \, \mathrm{d}y \\
&= \int_{\mathbb{R}^3} \rho_\varepsilon(h) \Big(\int_{\mathbb{R}^3} \frac{|\tilde{m}(x-h) - \tilde{m}(x)|^2}{|h|^2} \, \mathrm{d}x \Big) \mathrm{d}h \qquad (3.1) \\
&= \int_{\mathbb{R}^3} \rho_\varepsilon(h) \, \frac{\|\tau_{-h}\tilde{m} - \tilde{m}\|^2_{L^2(\mathbb{R}^3)}}{|h|^2} \, \mathrm{d}h \\
&\leqslant \|\rho_\varepsilon\|_{L^1(\mathbb{R}^3)} \, \|\nabla \tilde{m}\|^2_{L^2(\mathbb{R}^3)} \\
&\leqslant C \, \|m\|^2_{H^1(\Omega)}, \qquad (3.2)
\end{aligned}$$

for some constant $C > 0$ depending only on Ω through the linear extension operator. The equality (3.1) is the result of the change of variables $y \mapsto x - h$ for fixed $x \in \mathbb{R}^3$, whereas to get (3.2), we used (G1) and standard properties of Sobolev spaces (cf., e.g., [3, Thm. 1] or [4, Prop. 9.3]).

Similarly, for the antisymmetric exchange term, we obtain

$$
\begin{aligned}
|\mathcal{H}_\varepsilon(m)| &\leqslant \iint_{\Omega\times\Omega} \frac{|\nu_\varepsilon(x-y)|}{|x-y|}\cdot |m(x)\times[m(y)-m(x)]|\,\mathrm{d}x\,\mathrm{d}y\\
&\leqslant \iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{|\nu_\varepsilon(x-y)|}{|x-y|}|\tilde{m}(x)||\tilde{m}(y)-\tilde{m}(x)|\,\mathrm{d}x\,\mathrm{d}y\\
&= \int_{\mathbb{R}^3}|\nu_\varepsilon(h)|\Big(\int_{\mathbb{R}^3}|\tilde{m}(x)|\frac{|\tilde{m}(x+h)-\tilde{m}(x)|}{|h|}\,\mathrm{d}x\Big)\mathrm{d}h, \qquad (3.3)
\end{aligned}
$$

where in (3.3) we performed the change of variables $y\mapsto x+h$ for fixed $x\in\mathbb{R}^3$ and then used the odd symmetry of the kernel ν_ε. Applying Hölder's inequality as well as classical properties of Sobolev spaces (see [4, Prop. 9.3]), from (3.3) and (H1) we infer that

$$
\begin{aligned}
|\mathcal{H}_\varepsilon(m)| &\leqslant \int_{\mathbb{R}^3}|\nu_\varepsilon(h)|\|\tilde{m}\|_{L^2(\mathbb{R}^3)}\frac{\|\tau_h\tilde{m}-\tilde{m}\|_{L^2(\mathbb{R}^3)}}{|h|}\,\mathrm{d}h\\
&\leqslant \|\nu_\varepsilon\|_{L^1(\mathbb{R}^3)}\|\tilde{m}\|_{L^2(\mathbb{R}^3)}\|\nabla\tilde{m}\|_{L^2(\mathbb{R}^3)}\\
&\leqslant C\,\|m\|^2_{H^1(\Omega)},
\end{aligned}
$$

for some constant $C>0$ depending only on Ω. This completes the proof of (2.13).

Step 2. We claim that it is sufficient to show that if $\sup_{\varepsilon>0}\mathcal{E}_\varepsilon(m)<+\infty$, then $\sup_{\varepsilon>0}\mathcal{F}_\varepsilon(m)<+\infty$. Indeed, as soon as we show that $\sup_{\varepsilon>0}\mathcal{F}_\varepsilon(m)<+\infty$, we can then invoke [24, Theorem 5] which shows that if $\sup_{\varepsilon>0}\mathcal{F}_\varepsilon(m)<+\infty$ and the family $(\rho_\varepsilon)_\varepsilon$ satisfies (G1)–(G3), then $m\in H^1(\Omega;\mathbb{R}^3)$ and there exists $\alpha>0$ (depending only on the family of kernels $(\rho_\varepsilon)_\varepsilon$) such that

$$
\alpha^2\int_\Omega|\nabla m|^2\mathrm{d}x\leqslant\limsup_{\varepsilon\to0}\iint_{\Omega\times\Omega}\rho_\varepsilon(x-y)\frac{|m(x)-m(y)|^2}{|x-y|^2}\,\mathrm{d}x\,\mathrm{d}y. \qquad (3.4)
$$

Since $\mathcal{H}_\varepsilon(m)$ is a quantity without a definite sign, we use Young's inequality for products to estimate the term $\mathcal{H}_\varepsilon(m)$ from below. We get that for every $\delta>0$ there holds

$$
\begin{aligned}
\mathcal{H}_\varepsilon(m) &= \iint_{\Omega\times\Omega}\frac{m(x)-m(y)}{|x-y|}\cdot(m(y)\times\nu_\varepsilon(x-y))\,\mathrm{d}x\,\mathrm{d}y\\
&= \iint_{\Omega\times\Omega}\Big(\sqrt{\rho_\varepsilon(x-y)}\frac{m(x)-m(y)}{|x-y|}\Big)\cdot\Big(m(y)\times\frac{\nu_\varepsilon(x-y)}{\sqrt{\rho_\varepsilon(x-y)}}\Big)\,\mathrm{d}x\,\mathrm{d}y\\
&\geqslant -\delta^2\mathcal{F}_\varepsilon(m)-\frac{1}{4\delta^2}\iint_{\Omega\times\Omega}\frac{|m(y)\times\nu_\varepsilon(x-y)|^2}{\rho_\varepsilon(x-y)}\mathrm{d}x\,\mathrm{d}y. \qquad (3.5)
\end{aligned}
$$

Adding $\mathcal{F}_\varepsilon(m)$ to both sides of the previous estimate, we get that for every $\delta>0$ there holds

$$
(1-\delta^2)\mathcal{F}_\varepsilon(m)-\frac{1}{4\delta^2}\iint_{\Omega\times\Omega}\frac{|m(y)\times\nu_\varepsilon(x-y)|^2}{\rho_\varepsilon(x-y)}\mathrm{d}x\,\mathrm{d}y\leqslant\mathcal{E}_\varepsilon(m). \qquad (3.6)
$$

Now, by (A1), and taking into account that $\|\nu_\varepsilon\|_{L^1(\mathbb{R}^3)}=1$, we infer

$$
\begin{aligned}
\iint_{\Omega\times\Omega}\frac{|m(y)\times\nu_\varepsilon(x-y)|^2}{\rho_\varepsilon(x-y)}\mathrm{d}x\,\mathrm{d}y &\leqslant \int_\Omega|m(y)|^2\Big(\int_{\mathbb{R}^3}\frac{|\nu_\varepsilon(z)|^2}{\rho_\varepsilon(z)}\mathrm{d}z\Big)\mathrm{d}y\\
&\leqslant \frac14\Big(\int_\Omega|m(y)|^2\mathrm{d}y\Big)\|\nu_\varepsilon\|_{L^1(\mathbb{R}^3)}\Big\|\frac{\nu_\varepsilon}{\rho_\varepsilon}\Big\|_{L^\infty(\mathbb{R}^3)}\\
&\leqslant C\,\|m\|^2_{L^2(\Omega)}, \qquad (3.7)
\end{aligned}
$$

with $C > 0$ proportional to the constant from (A1). Hence, from (3.6) and (3.7) we deduce

$$(1-\delta^2)\mathcal{F}_\varepsilon(m) \leqslant \mathcal{E}_\varepsilon(m) + C\|m\|^2_{L^2(\Omega)} \tag{3.8}$$

By taking $\delta^2 := 1/2$ we conclude that if $\sup_{\varepsilon>0} \mathcal{E}_\varepsilon(m) < +\infty$ then $\sup_{\varepsilon>0} \mathcal{F}_\varepsilon(m) < +\infty$ and hence, by [24, Theorem 5], that $m \in H^1(\Omega;\mathbb{R}^3)$. □

Proof of Theorem 2.3. We split the proof into two main steps. In Step 1, we prove (2.16), which, in particular, gives us (2.15) under the additional hypothesis that $m \in C^2(\bar\Omega;\mathbb{R}^3)$. Then, in Step 2, we use a density argument to show that if (2.15) holds for every $m \in C^2(\bar\Omega;\mathbb{R}^3)$ then it holds for every $m \in H^1(\Omega;\mathbb{R}^3)$.

Step 1. Let $y \in \Omega$. For $R > 0$ such that $R \leqslant \operatorname{dist}(y, \partial\Omega)$, we split $\mathcal{H}_\varepsilon(m_\varepsilon)$ as

$$\begin{aligned}\mathcal{H}_\varepsilon(m_\varepsilon) &= \int_\Omega \left(\int_{B_R(y)} \frac{\nu_\varepsilon(x-y)}{|x-y|} \cdot (m_\varepsilon(x) \times m_\varepsilon(y))\,\mathrm{d}x \right) \mathrm{d}y \\ &\quad + \int_\Omega \left(\int_{\Omega\setminus B_R(y)} \frac{\nu_\varepsilon(x-y)}{|x-y|} \cdot (m_\varepsilon(x) \times m_\varepsilon(y))\,\mathrm{d}x \right) \mathrm{d}y \\ &=: \int_\Omega I_{\varepsilon,R}(m_\varepsilon, y) + J_{\varepsilon,R}(m_\varepsilon, y)\mathrm{d}y \end{aligned} \tag{3.9}$$

Outside of the ball $B_R(y)$, we note that

$$\begin{aligned}|J_{\varepsilon,R}(m_\varepsilon, y)| &\leqslant \int_{\Omega\setminus B_R(y)} \frac{|\nu_\varepsilon(x-y)|}{|x-y|} |m_\varepsilon(x)||m_\varepsilon(y)|\,\mathrm{d}x \\ &\leqslant \frac{1}{R}\|m_\varepsilon\|^2_{L^\infty(\Omega)} \int_{\mathbb{R}^3\setminus B_R(0)} |\nu_\varepsilon(x)|\,\mathrm{d}x.\end{aligned}$$

Sending $\varepsilon \to 0$ (for fixed $R > 0$), by assumption (H2) we find that $\lim_{\varepsilon\to 0} \|\nu_\varepsilon\|_{L^1(\mathbb{R}^3\setminus B_R(0))} = 0$ and, therefore, that

$$\lim_{\varepsilon\to 0} |J_{\varepsilon,R}(m_\varepsilon, y)| = 0. \tag{3.10}$$

We now take into account the contribution to $\mathcal{H}_\varepsilon(m_\varepsilon)$ inside of the ball $B_R(y)$, i.e., the term $I_\varepsilon(m_\varepsilon, y)$ in (3.9). Given that $m_\varepsilon \in C^2(\bar\Omega;\mathbb{R}^3)$, we perform a first-order Taylor expansion inside $B_R(y)$ of the form $m_\varepsilon(x) = m_\varepsilon(y) + \mathrm{D}m_\varepsilon(y)(x-y) + R_\varepsilon(x,y)$ with (up to a constant factor)

$$|R_\varepsilon(x,y)| \leqslant \|D^{(2)}m_\varepsilon\|_{L^\infty} \cdot |x-y|^2$$

to find

$$\begin{aligned}I_{\varepsilon,R}(m_\varepsilon, y) &= \int_{B_R(y)} \frac{\nu_\varepsilon(x-y)}{|x-y|} \cdot (\mathrm{D}m_\varepsilon(y)(x-y) \times m_\varepsilon(y))\,\mathrm{d}x \\ &\quad + \int_{B_R(y)} \frac{\nu_\varepsilon(x-y)}{|x-y|} \cdot (R_\varepsilon(x,y) \times m_\varepsilon(y))\,\mathrm{d}x \\ &=: I^{(1)}_{\varepsilon,R}(m_\varepsilon, y) + I^{(2)}_{\varepsilon,R}(m_\varepsilon, y).\end{aligned} \tag{3.11}$$

Let us focus on the first term on the right-hand side of (3.11). Simple algebra gives that

$$\begin{aligned}I^{(1)}_{\varepsilon,R}(m_\varepsilon, y) &= m_\varepsilon(y) \cdot \int_{B_R(y)} \left(\frac{\nu_\varepsilon(x-y)}{|x-y|} \times \partial_i m_\varepsilon(y)(x_i - y_i) \right) \mathrm{d}x \\ &= \sum_{i=1}^3 \left(\int_{B_R(y)} \frac{(x_i - y_i)}{|x-y|} \nu_\varepsilon(x-y)\,\mathrm{d}x \right) \cdot (\partial_i m_\varepsilon(y) \times m_\varepsilon(y)).\end{aligned} \tag{3.12}$$

Next, we observe that as a consequence of (H2) we have

$$\lim_{\varepsilon\to 0}\left|\int_{\mathbb{R}^3\setminus B_R(y)}\frac{\nu_\varepsilon(x-y)}{|x-y|}(x_i-y_i)\,\mathrm{d}x\right|\leqslant\lim_{\varepsilon\to 0}\int_{\mathbb{R}^3\setminus B_R(y)}|\nu_\varepsilon(x-y)|\,\mathrm{d}x=0. \tag{3.13}$$

Therefore, combining (3.13) and (H3) we infer that for $i=1,2,3$, there holds

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_{B_R(y)}\frac{(x_i-y_i)}{|x-y|}\nu_\varepsilon(x-y)\,\mathrm{d}x&=\lim_{\varepsilon\to 0}\int_{\mathbb{R}^3}\frac{(x_i-y_i)}{|x-y|}\nu_\varepsilon(x-y)\,\mathrm{d}x\\&=\lim_{\varepsilon\to 0}\int_{\mathbb{R}^3}\frac{x_i}{|x|}\nu_\varepsilon(x)\,\mathrm{d}x\\&=d_i.\end{aligned} \tag{3.14}$$

As a consequence, from (3.12) and (3.14) we conclude

$$\lim_{\varepsilon\to 0}I^{(1)}_{\varepsilon,R}(m_\varepsilon,y)=\sum_{i=1}^3 m(y)\cdot(d_i\times\partial_i m(y)). \tag{3.15}$$

Note that (3.15) does not depend on R.

For the term $I^2_{\varepsilon,R}(m_\varepsilon,y)$, we observe that

$$\begin{aligned}\left|I^{(2)}_{\varepsilon,R}(m_\varepsilon,y)\right|&\leqslant|m_\varepsilon(y)|\int_{B_R(y)}|\nu_\varepsilon(x-y)|\frac{|R_\varepsilon(x,y)|}{|x-y|}\,\mathrm{d}x\\&\leqslant R|m_\varepsilon(y)|\|D^{(2)}m_\varepsilon\|_{L^\infty}\cdot\int_{\mathbb{R}^3}|\nu_\varepsilon(x)|\,\mathrm{d}x\\&=R|m_\varepsilon(y)|\|D^{(2)}m_\varepsilon\|_{L^\infty}.\end{aligned}$$

For the latter equality we used assumption (H1). Thus, since $m_\varepsilon\to m$ strongly in $C^2(\overline{\Omega};\mathbb{R}^3)$,

$$\lim_{R\to 0}\lim_{\varepsilon\to 0}\left|I^{(2)}_{\varepsilon,R}(m_\varepsilon,y)\right|=0. \tag{3.16}$$

Overall, combining estimates (3.10), (3.12), (3.14), (3.15), and (3.16), we infer

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_\Omega\frac{\nu_\varepsilon(x-y)}{|x-y|}\cdot(m_\varepsilon(x)\times m_\varepsilon(y))\,\mathrm{d}x&=\lim_{R\to 0}\lim_{\varepsilon\to 0}(I_{\varepsilon,R}(m_\varepsilon,y)+J_{\varepsilon,R}(m_\varepsilon,y))\\&=\lim_{R\to 0}\lim_{\varepsilon\to 0}I_{\varepsilon,R}(m_\varepsilon,y)\\&=\lim_{R\to 0}\lim_{\varepsilon\to 0}\Big(I^{(1)}_{\varepsilon,R}(m_\varepsilon,y)+I^{(2)}_{\varepsilon,R}(m_\varepsilon,y)\Big)\\&=\lim_{\varepsilon\to 0}I^{(1)}_{\varepsilon,R}(m_\varepsilon,y)\\&=\sum_{i=1}^3 m(y)\cdot(d_i\times\partial_i m(y)).\end{aligned} \tag{3.17, 3.18}$$

In view of (3.18), to prove

$$\lim_{\varepsilon\to 0}\mathcal{H}_\varepsilon(m_\varepsilon)=\sum_{i=1}^3\int_\Omega m(y)\cdot(d_i\times\partial_i m(y))\mathrm{d}y, \tag{3.19}$$

it is sufficient to show that the family of functions on the left-hand side of (3.17) falls under the hypotheses of the Dominated Convergence Theorem. But this is indeed the case because of the

estimate

$$\left|\int_\Omega \frac{\nu_\varepsilon(x-y)}{|x-y|}\cdot(m_\varepsilon(x)\times m_\varepsilon(y))\,\mathrm{d}x\right| \leqslant \|m_\varepsilon\|_{L^\infty(\Omega)}\int_\Omega |\nu_\varepsilon(x-y)|\frac{|m_\varepsilon(x)-m_\varepsilon(y)|}{|x-y|}\,\mathrm{d}x$$
$$\leqslant L_{m_\varepsilon}\,\|m_\varepsilon\|_{L^\infty(\Omega)}\int_{\mathbb{R}^3}|\nu_\varepsilon(x)|\,\mathrm{d}x$$
$$= L_{m_\varepsilon}\,\|m_\varepsilon\|_{L^\infty(\Omega)}, \tag{3.20}$$

where L_{m_ε} is the Lipschitz constant of m_ε, which is uniformly bounded in ε.

Step 2. This second step concludes the proof by showing, through a density argument, that if (2.15) holds for every $m\in C^2(\bar\Omega;\mathbb{R}^3)$ then it holds for every $m\in H^1(\Omega;\mathbb{R}^3)$.

We note that, for every $m_1,m_2\in H^1(\Omega;\mathbb{R}^3)$, there holds

$$\mathcal{H}_\varepsilon(m_1)-\mathcal{H}_\varepsilon(m_2)=\iint_{\Omega\times\Omega}\frac{\nu_\varepsilon(x-y)}{|x-y|}\cdot((m_1(x)-m_1(y))\times m_1(y))\mathrm{d}x\,\mathrm{d}y$$
$$-\iint_{\Omega\times\Omega}\frac{\nu_\varepsilon(x-y)}{|x-y|}\cdot((m_2(x)-m_2(y))\times m_2(y))\mathrm{d}x\,\mathrm{d}y.$$

Thus, adding and subtracting the quantity $(m_1(x)-m_1(y))\times m_2(y)$, we infer

$$|\mathcal{H}_\varepsilon(m_1)-\mathcal{H}_\varepsilon(m_2)|\leqslant\iint_{\Omega\times\Omega}\frac{|\nu_\varepsilon(x-y)|}{|x-y|}|m_1(x)-m_1(y)||m_1(y)-m_2(y)|\mathrm{d}x\,\mathrm{d}y$$
$$+\iint_{\Omega\times\Omega}\frac{|\nu_\varepsilon(x-y)|}{|x-y|}|(m_1(x)-m_1(y))-(m_2(x)-m_2(y))||m_2(y)|\mathrm{d}x\,\mathrm{d}y.$$
$$=: I_1+I_2. \tag{3.21}$$

Let now $\tilde m_1,\tilde m_2\in H^1(\mathbb{R}^3;\mathbb{R}^3)$ be extensions of m_1,m_2 from Ω to $\mathbb{R}^3$. For the first integral I_1 in (3.21) we have

$$I_1\leqslant\int_{\mathbb{R}^3}\frac{|\nu_\varepsilon(h)|}{|h|}\Big(\int_{\mathbb{R}^3}|\tilde m_1(x+h)-\tilde m_1(x)||\tilde m_1(x+h)-\tilde m_2(x+h)|\,\mathrm{d}x\Big)\mathrm{d}h$$
$$\leqslant\int_{\mathbb{R}^3}|\nu_\varepsilon(h)|\,\frac{\|\tau_h\tilde m_1-\tilde m_1\|_{L^2(\mathbb{R}^3)}}{|h|}\,\|\tilde m_1-\tilde m_2\|_{L^2(\mathbb{R}^3)}\,\mathrm{d}h$$
$$\leqslant\|\nabla\tilde m_1\|_{L^2(\mathbb{R}^3)}\|\tilde m_1-\tilde m_2\|_{L^2(\mathbb{R}^3)}$$
$$\leqslant C\,\|m_1\|_{H^1(\Omega)}\|m_1-m_2\|_{L^2(\Omega)}, \tag{3.22}$$

for some constant $C=C(\Omega)>0$, where we have applied Hölder inequality and classical properties of Sobolev spaces, cf. [4, Prop. 9.3]. Analogously, for the second integral I_2 in (3.21) we find

$$I_2\leqslant\int_{\mathbb{R}^3}|\nu_\varepsilon(h)|\int_{\mathbb{R}^3}|\tilde m_2(x+h)|\frac{|(\tilde m_1-\tilde m_2)(x)-(\tilde m_1-\tilde m_2)(x+h)|}{|h|}\,\mathrm{d}x\,\mathrm{d}h$$
$$\leqslant\int_{\mathbb{R}^3}|\nu_\varepsilon(h)|\|\tilde m_2\|_{L^2(\mathbb{R}^3)}\frac{\|\tau_h(\tilde m_1-\tilde m_2)-(\tilde m_1-\tilde m_2)\|_{L^2(\mathbb{R}^3)}}{|h|}\,\mathrm{d}h$$
$$\leqslant\|\nabla(\tilde m_1-\tilde m_2)\|_{L^2(\mathbb{R}^3)}\|\tilde m_2\|_{L^2(\mathbb{R}^3)}$$
$$\leqslant C\,\|m_1-m_2\|_{H^1(\Omega)}\|m_2\|_{L^2(\Omega)}, \tag{3.23}$$

for some constant $C=C(\Omega)>0$.

To sum up, estimating (3.21) through (3.22) and (3.23), we obtain that, for every $m_1,m_2\in H^1(\Omega;\mathbb{R}^3)$, there holds

$$|\mathcal{H}_\varepsilon(m_1)-\mathcal{H}_\varepsilon(m_2)|\leqslant C\,(\|m_1\|_{H^1(\Omega)}+\|m_2\|_{H^1(\Omega)})\|m_1-m_2\|_{H^1(\Omega)}. \tag{3.24}$$

Now, by density, for every $m \in H^1(\Omega;\mathbb{R}^3)$ there exists a sequence $(m_k)_{k\in\mathbb{N}}$ in $C^2_c(\mathbb{R}^3;\mathbb{R}^3)$ such that

$$\lim_{k\to\infty} \|m - m_k\|_{H^1(\Omega)} = 0. \tag{3.25}$$

Also, by Step 1, we know that for every $m_k \in C^2_c(\mathbb{R}^3;\mathbb{R}^3)$ one has

$$\lim_{\varepsilon\to 0} \mathcal{H}_\varepsilon(m_k) = \sum_{i=1}^3 \int_\Omega m_k(x)\cdot(d_i \times \partial_i m_k(x))\,\mathrm{d}x = \mathcal{H}(m_k) \tag{3.26}$$

Moreover, in view of (3.24), for every $k\in\mathbb{N}$, there holds

$$\limsup_{\varepsilon\to 0} |\mathcal{H}_\varepsilon(m) - \mathcal{H}_\varepsilon(m_k)| \leqslant C(\|m\|_{H^1(\Omega)} + \|m_k\|_{H^1(\Omega)})\|m - m_k\|_{H^1(\Omega)}. \tag{3.27}$$

Letting now $k\to+\infty$, by (3.25) and by (3.27),

$$\lim_{k\to+\infty} \limsup_{\varepsilon\to 0} |\mathcal{H}_\varepsilon(m) - \mathcal{H}_\varepsilon(m_k)| = 0, \tag{3.28}$$

thanks to the density (3.25) and the fact that $\|m_k\|_{H^1(\Omega)}$ is bounded in H^1.

In view of (3.25), we also have that $d_i \times \partial_i m_k \to d_i \times \partial_i m$ in $L^2(\Omega;\mathbb{R}^3)$ for $i = 1,2,3$. As a consequence, we have that

$$\lim_{k\to\infty} \mathcal{H}(m_k) = \mathcal{H}(m), \tag{3.29}$$

that is, the limit functional $\mathcal{H}$ is continuous with respect to the H^1-convergence.

To sum up, by (3.26), (3.28) and (3.29), we obtain that

$$\begin{aligned}\limsup_{\varepsilon\to 0} |\mathcal{H}_\varepsilon(m) - \mathcal{H}(m)| &\leqslant \limsup_{\varepsilon\to 0} \left(|\mathcal{H}_\varepsilon(m) - \mathcal{H}_\varepsilon(m_k)| + |\mathcal{H}_\varepsilon(m_k) - \mathcal{H}(m_k)|\right) + |\mathcal{H}(m_k) - \mathcal{H}(m)| \\ &= \left(\limsup_{\varepsilon\to 0} |\mathcal{H}_\varepsilon(m) - \mathcal{H}_\varepsilon(m_k)|\right) + |\mathcal{H}(m_k) - \mathcal{H}(m)|. \end{aligned} \tag{3.30}$$

Passing to the limit for $k\to\infty$, we conclude that for every $m\in H^1(\Omega;\mathbb{R}^3)$, there holds $\mathcal{H}_\varepsilon(m) \to \mathcal{H}(m)$ as $\varepsilon\to 0$, i.e.,

$$\lim_{\varepsilon\to 0} \iint_{\Omega\times\Omega} \frac{\nu_\varepsilon(x-y)}{|x-y|}\cdot(m(x)\times m(y))\mathrm{d}x\,\mathrm{d}y = \sum_{i=1}^3 \int_\Omega m(x)\cdot(d_i\times\partial_i m(x))\mathrm{d}x.$$

This completes the proof. □

4. Compactness and Γ-convergence (proof of Theorem 2.1)

Proof of Theorem 2.1.*i*, (Compactness). By the proof of Theorem 2.2 (cf. (3.8)) we know that for every $\varepsilon > 0$ there holds

$$\frac{1}{2}\mathcal{F}_\varepsilon(m_\varepsilon) \leqslant C\Big(\mathcal{E}_\varepsilon(m_\varepsilon) + \|m_\varepsilon\|^2_{L^2(\Omega)}\Big), \tag{4.1}$$

with $C>0$ depending only on the homonymous constant in (A1). Since $(m_\varepsilon)_\varepsilon$ consists of $\mathbb{S}^2$-valued vector fields, i.e., $|m_\varepsilon| = 1$ a.e. in Ω for every $\varepsilon>0$, from the assumption (2.9) and (4.1) we infer that

$$\liminf_{\varepsilon\to 0} \mathcal{F}_\varepsilon(m_\varepsilon) \leqslant C, \tag{4.2}$$

for some $C>0$ depending only on $\liminf_{\varepsilon\to 0}\mathcal{E}_\varepsilon(m_\varepsilon)$, Ω, and on the homonymous constant in (A1). In particular, in view of assumption (G4), recalling the definition of the kernels $(\rho^{\mathrm{rad}}_\varepsilon)_\varepsilon$ in (2.4) and setting

$$\tilde\rho_\varepsilon(x) := \frac{\rho^{\mathrm{rad}}_\varepsilon(|x|)}{\|\rho^{\mathrm{rad}}_\varepsilon(|\cdot|)\|_{L^1(\mathbb{R}^3)}},$$

from estimate (4.2) we infer

$$\liminf_{\varepsilon\to 0} \iint_{\Omega\times\Omega} \tilde{\rho}_\varepsilon(x-y)\frac{|m(x)-m(y)|^2}{|x-y|^2}\,\mathrm{d}x\,\mathrm{d}y \leqslant \frac{C}{\kappa}. \tag{4.3}$$

Estimate (4.3) allows us to invoke the compactness result for the symmetric exchange energy established in [23, Theorem 1.2] for the case of radial kernels, which implies that if $(m_\varepsilon)_\varepsilon$ is a bounded family in $L^2(\Omega;\mathbb{R}^3)$ such that $\mathcal{F}_\varepsilon(m_\varepsilon) \leqslant C$ for some positive constant C, then there exists $m \in H^1(\Omega;\mathbb{R}^3)$ and a subsequence $\varepsilon_j \to 0$ such that $m_{\varepsilon_j} \to m$ strongly in $L^2(\Omega;\mathbb{R}^3)$. In particular, we obtain that (up to a further subsequence) $m_{\varepsilon_j} \to m$ a.e. as $j \to \infty$ and, therefore, also the limiting function satisfies the constraint, that is, $m \in L^2(\Omega;\mathbb{S}^2)$. □

Proof of Theorem 2.1.*ii*, (Γ-convergence). We split the proof into two classical steps. First, we derive the Γ-limsup inequality, then we focus on the Γ-liminf inequality.

Limsup inequality. It is sufficient to note that the constant family $m_\varepsilon := m$ for every ε, is a recovery sequence. Indeed, on the one hand, if $m \in L^2(\Omega;\mathbb{S}^2)\setminus H^1(\Omega;\mathbb{S}^2)$ then Theorem 2.2 gives that $\mathcal{E}_\varepsilon(m) \to +\infty$; on the other hand, if $m \in H^1(\Omega;\mathbb{S}^2)$ then by Corollary 2.1, we conclude that

$$\lim_{\varepsilon\to 0} \mathcal{E}_\varepsilon(m) = \mathcal{E}(m).$$

Liminf inequality. Let $(m_\varepsilon)_\varepsilon \subset L^2(\Omega;\mathbb{S}^2)$ be such that $m_\varepsilon \to m$ strongly in $L^2(\Omega;\mathbb{S}^2)$ as $\varepsilon \to 0$. If $\liminf_{\varepsilon\to 0}\mathcal{E}_\varepsilon(m_\varepsilon) = +\infty$, there is nothing to prove. If $\liminf_{\varepsilon\to 0}\mathcal{E}_\varepsilon(m_\varepsilon) < +\infty$ then, by (4.1), we know that

$$\liminf_{\varepsilon\to 0} \mathcal{F}_\varepsilon(m_\varepsilon) \leqslant C\Big(1+\liminf_{\varepsilon\to 0}\mathcal{E}_\varepsilon(m_\varepsilon)\Big) < +\infty, \tag{4.4}$$

for some constant C depending only on Ω and the homonymous constant in (A1). Also, by the compactness result (Theorem 2.1.*i*), $m \in H^1(\Omega;\mathbb{S}^2)$, and from [24, Lemma 8] used in the case $\omega(t) = t^2$, we get the existence of a finite Radon measure $\mu \in \mathcal{M}(\mathbb{S}^2)$, with $\mu(\mathbb{S}^2) = 1$, such that, possibly up to the extraction of a non-relabelled subsequence, there holds

$$\liminf_{\varepsilon\to 0} \mathcal{F}_\varepsilon(m_\varepsilon) \geqslant \int_\Omega \Big(\int_{\mathbb{S}^2} |\partial_\sigma m(x)|^2 \mathrm{d}\mu(\sigma)\Big)\mathrm{d}x. \tag{4.5}$$

Therefore, it only remains to show the analogous liminf inequality for the nonlocal antisymmetric exchange term, i.e., that

$$\liminf_{\varepsilon\to 0} \mathcal{H}_\varepsilon(m_\varepsilon) \geqslant \mathcal{H}(m). \tag{4.6}$$

We show something more, as stated in the next claim.
Claim:

$$\lim_{\varepsilon\to 0} \mathcal{H}_\varepsilon(m_\varepsilon) = \mathcal{H}(m). \tag{4.7}$$

In order to prove our claim, we introduce a family of mollifiers $(\eta_k)_{k\in\mathbb{N}}$ in $C_c^\infty(\mathbb{R}^3)$ build in the usual way: we consider an element $\eta \in C_c^\infty(\mathbb{R}^3)$, $\eta \geqslant 0$, $\|\eta\|_{L^1(\mathbb{R}^3)} = 1$, $\mathrm{supp}\eta \subset B_1(0)$, and we set $\eta_k(x) := k^3\eta(kx)$. Also, for every sufficiently small $\delta > 0$ we set $\Omega_\delta := \{x \in \Omega : \mathrm{dist}(x,\partial\Omega) > \delta\}$. For every $k > 1/\delta$ the regularization of m_ε given by $m_{\varepsilon,k} := m_\varepsilon * \eta_k$ is well-defined on Ω_δ. By classical properties of mollifiers, we know that for a given $\varepsilon > 0$ there holds

$$m_{\varepsilon,k} := m_\varepsilon * \eta_k \xrightarrow{k\to+\infty} m_\varepsilon \quad \text{strongly in } L^2(\Omega_\delta;\mathbb{R}^3), \tag{4.8}$$

$$m_k := m * \eta_k \xrightarrow{k\to+\infty} m \quad \text{strongly in } H^1(\Omega_\delta;\mathbb{R}^3). \tag{4.9}$$

In writing the previous relations, we agreed (with a safe abuse of notation) that the $\mathbb{S}^2$-valued vector fields m and m_ε are extended by zero outside of Ω.

In what follows, it will be convenient to denote by $\mathcal{H}_{\varepsilon,\delta}$ and $\mathcal{H}_\delta$ the analog of the functionals $\mathcal{H}_\varepsilon$ and $\mathcal{H}$ when defined on space of functions defined on the compact set $\bar{\Omega}_\delta \subset \Omega$ instead of Ω. Thus, for example,

$$\mathcal{H}_{\varepsilon,\delta}(m_{\varepsilon,k}) = \iint_{\bar{\Omega}_\delta^2} \frac{\nu_\varepsilon(x-y)}{|x-y|} m_{\varepsilon,k}(x) \times m_{\varepsilon,k}(y) \mathrm{d}x\mathrm{d}y.$$

For the nonlocal antisymmetric energy $\mathcal{H}_{\varepsilon,\delta}$ we then observe that the uniform convergence result stated in Theorem 2.3 assures that

$$\lim_{\varepsilon\to 0} \mathcal{H}_{\varepsilon,\delta}(m_{\varepsilon,k}) = \mathcal{H}_\delta(m_k) = \sum_{i=1}^{3} \int_{\Omega_\delta} m_k(x) \cdot (d_i \times \partial_i m_k(x)) \mathrm{d}x. \tag{4.10}$$

Indeed, since $m_\varepsilon \to m$ in $L^2(\Omega;\mathbb{S}^2)$ as $\varepsilon \to 0$, we have that for every $k \in \mathbb{N}$ there holds $m_{\varepsilon,k} = m_\varepsilon * \eta_k \xrightarrow{\varepsilon\to 0} m_k = m * \eta_k$ in $C^2(\bar{\Omega}_\delta;\mathbb{R}^3)$, and therefore, we are in the hypotheses of (2.16) in Theorem 2.3 applied to $\bar{\Omega}_\delta$.

Note that, by (4.9) we have $\mathcal{H}(m) = \lim_{\delta\to 0}(\lim_{k\to+\infty} \mathcal{H}_\delta(m_k))$ and, therefore, our goal (4.7) boils down to showing the equality

$$\lim_{\varepsilon\to 0} \mathcal{H}_\varepsilon(m_\varepsilon) = \lim_{\delta\to 0} \lim_{k\to+\infty} \lim_{\varepsilon\to 0} \mathcal{H}_{\varepsilon,\delta}(m_{\varepsilon,k}), \tag{4.11}$$

because, by (4.10), the RHS of (4.11) is nothing but $\mathcal{H}(m)$. To show equality (4.11) we split $\mathcal{H}_\varepsilon(m_\varepsilon)$ under the form

$$\mathcal{H}_\varepsilon(m_\varepsilon) = \mathcal{H}_{\varepsilon,\delta}(m_{\varepsilon,k}) + (\mathcal{H}_\varepsilon(m_\varepsilon) - \mathcal{H}_{\delta,\varepsilon}(m_\varepsilon)) + (\mathcal{H}_{\delta,\varepsilon}(m_\varepsilon) - \mathcal{H}_{\varepsilon,\delta}(m_{\varepsilon,k})) \tag{4.12}$$

and then show that

$$\lim_{\delta\to 0} \lim_{k\to+\infty} \limsup_{\varepsilon\to 0} |\mathcal{H}_\varepsilon(m_\varepsilon) - \mathcal{H}_{\delta,\varepsilon}(m_\varepsilon)| = 0, \tag{4.13}$$

$$\lim_{\delta\to 0} \lim_{k\to+\infty} \limsup_{\varepsilon\to 0} |\mathcal{H}_{\delta,\varepsilon}(m_\varepsilon) - \mathcal{H}_{\varepsilon,\delta}(m_{\varepsilon,k})| = 0. \tag{4.14}$$

Indeed, as soon as we achieve (4.13)–(4.14), passing to the limit $\lim_{\delta\to 0} \lim_{k\to+\infty} \limsup_{\varepsilon\to 0}$ in (4.12) we conclude.

Step 1. Proof of (4.13). Using the fact that the kernel ν_ε is odd, we obtain that

$$\mathcal{H}_\varepsilon(m_\varepsilon) - \mathcal{H}_{\delta,\varepsilon}(m_\varepsilon) = \iint_{\Omega^2\setminus\Omega_\delta^2} \frac{\nu_\varepsilon(x-y)}{|x-y|} \cdot (m_\varepsilon(x) \times m_\varepsilon(y)) \mathrm{d}x\mathrm{d}y$$

and, therefore,

$$\begin{aligned} |\mathcal{H}_\varepsilon(m_\varepsilon) - \mathcal{H}_{\delta,\varepsilon}(m_\varepsilon)| &\leqslant 3\bigg(C \iint_{\mathbb{R}\times(\Omega\setminus\Omega_\delta)} |\nu_\varepsilon(x-y)| \mathrm{d}x\mathrm{d}y\bigg)^{1/2} \cdot (\mathcal{F}_\varepsilon(m_\varepsilon))^{1/2} \\ &= 3C^{1/2} (\mathcal{F}_\varepsilon(m_\varepsilon))^{1/2} |\Omega\setminus\Omega_\delta|^{1/2}. \end{aligned}$$

Passing to the limit in the previous inequality, we infer (4.13).

STEP 2. PROOF OF (4.14). We observe that

$$\begin{aligned}
&\mathcal{H}_{\delta,\varepsilon}(m_\varepsilon) - \mathcal{H}_{\varepsilon,\delta}(m_{\varepsilon,k}) \\
&\quad = \iint_{\Omega_\delta^2} \frac{\nu_\varepsilon(x-y)}{|x-y|} \cdot (m_\varepsilon(x) \times m_\varepsilon(y) - m_{\varepsilon,k}(x) \times m_{\varepsilon,k}(y)) \mathrm{d}x\mathrm{d}y \\
&\quad = \iint_{\mathbb{R}^3\times\mathbb{R}^3} \left(\iint_{\Omega_\delta^2} \frac{\nu_\varepsilon(x-y)}{|x-y|} \cdot (m_\varepsilon(x) \times m_\varepsilon(y)) \mathrm{d}x\mathrm{d}y \right) \eta_k(z)\eta_k(w) \mathrm{d}z\,\mathrm{d}w \\
&\quad - \iint_{\mathbb{R}^3\times\mathbb{R}^3} \left(\iint_{(-z+\Omega_\delta)\times(-w+\Omega_\delta)} \frac{\nu_\varepsilon(x-y)}{|x-y|} \cdot (m_\varepsilon(x) \times m_\varepsilon(y)) \mathrm{d}x\mathrm{d}y \right) \eta_k(z)\eta_k(w) \mathrm{d}z\,\mathrm{d}w \\
&\quad = \iint_{\mathbb{R}^3\times\mathbb{R}^3} \left[\left(\iint_{\Omega^2} \frac{\nu_\varepsilon(x-y)}{|x-y|} \cdot (m_\varepsilon(x) \times m_\varepsilon(y)) \chi_{(z,w)}(x,y) \mathrm{d}x\mathrm{d}y \right) \right] \eta_k(z)\eta_k(w) \mathrm{d}z\,\mathrm{d}w,
\end{aligned} \tag{4.15}$$

where we set $\chi_{(z,w)}(x,y) := 1_{\Omega_\delta}(x)1_{\Omega_\delta}(y) - 1_{(-z+\Omega_\delta)}(x)1_{(-w+\Omega_\delta)}(y)$. Before continuing estimating (4.15), we need the following inequality, whose proof is postponed to STEP 3 below:

$$\begin{aligned}
&\iint_{\mathbb{R}^3\times\mathbb{R}^3} \left(\iint_{\Omega^2} |\nu_\varepsilon(x-y)||\chi_{(z,w)}(x,y)| \mathrm{d}x\mathrm{d}y \right) \eta_k(z)\eta_k(w) \mathrm{d}z\mathrm{d}w \\
&\qquad\qquad \leqslant 2 \int_{\mathbb{R}^3} \int_{\Omega} |1_{\Omega_\delta}(x) - 1_{(-z+\Omega_\delta)}(x)| \mathrm{d}x \eta_k(z) \mathrm{d}z.
\end{aligned} \tag{4.16}$$

Having (4.16) at our disposal, by combining Hölder inequality, (A1), as well as (4.15), we deduce

$$\begin{aligned}
&|\mathcal{H}_{\delta,\varepsilon}(m_\varepsilon) - \mathcal{H}_{\varepsilon,\delta}(m_{\varepsilon,k})| \\
&\leqslant (2\mathcal{F}_\varepsilon(m_\varepsilon))^{1/2} \iint_{(\mathbb{R}^3)^2} \left(\iint_{\Omega^2} \frac{|\nu_\varepsilon(x-y)|^2}{\rho_\varepsilon(x-y)} |\chi_{(z,w)}(x,y)| \mathrm{d}x\mathrm{d}y \right)^{1/2} \eta_k(z)\eta_k(w) \mathrm{d}z\,\mathrm{d}w \\
&\leqslant C^{1/2} (2\mathcal{F}_\varepsilon(m_\varepsilon))^{1/2} \iint_{(\mathbb{R}^3)^2} \left(\iint_{\Omega^2} |\nu_\varepsilon(x-y)||\chi_{(z,w)}(x,y)| \mathrm{d}x\mathrm{d}y \right)^{1/2} \eta_k(z)\eta_k(w) \mathrm{d}z\mathrm{d}w \\
&\leqslant C^{1/2} (2\mathcal{F}_\varepsilon(m_\varepsilon))^{1/2} \left(\iint_{(\mathbb{R}^3)^2} \left(\iint_{\Omega^2} |\nu_\varepsilon(x-y)||\chi_{(z,w)}(x,y)| \mathrm{d}x\mathrm{d}y \right) \eta_k(z)\eta_k(w) \mathrm{d}z\,\mathrm{d}w \right)^{1/2} \\
&\leqslant 2C^{1/2} (\mathcal{F}_\varepsilon(m_\varepsilon))^{1/2} \left(\int_{\mathbb{R}^3} \int_{\Omega} |1_{\Omega_\delta}(x) - 1_{(-z+\Omega_\delta)}(x)| \eta_k(z) \mathrm{d}x\mathrm{d}z \right)^{1/2} \\
&= 2C^{1/2} (\mathcal{F}_\varepsilon(m_\varepsilon))^{1/2} \left(\int_{B_1} \|1_{\Omega_\delta}(\cdot) - 1_{\Omega_\delta}(\cdot - z/k)\|_{L^1(\Omega)} \mathrm{d}z \right)^{1/2} \xrightarrow{k\to\infty} 0,
\end{aligned} \tag{4.17}$$

the convergence of (4.17) to 0 being a consequence of Lebesgue's Dominated Convergence Theorem, as well as of the continuity of the L^2-norm with respect to translations.

STEP 3. PROOF OF (4.16). We first rewrite the function $\chi_{(z,w)}(x,y)$ as

$$\chi_{(z,w)}(x,y) = [1_{\Omega_\delta}(x) - 1_{(-z+\Omega_\delta)}(x)]1_{\Omega_\delta}(y) + 1_{(-z+\Omega_\delta)}(x)[1_{\Omega_\delta}(y) - 1_{(-w+\Omega_\delta)}(y)],$$

from which we infer the following estimate:

$$|\chi_{(z,w)}(x,y)| \leqslant |1_{\Omega_\delta}(x) - 1_{(-z+\Omega_\delta)}(x)| + |1_{\Omega_\delta}(y) - 1_{(-w+\Omega_\delta)}(y)|.$$

In particular,

$$\begin{aligned}\iint_{\Omega^2}|\nu_\varepsilon(x-y)||\chi_{(z,w)}(x,y)|\mathrm{d}x\mathrm{d}y \leqslant &\iint_{\Omega\times\mathbb{R}^3}|\nu_\varepsilon(x-y)||1_{\Omega_\delta}(x)-1_{(-z+\Omega_\delta)}(x)|\mathrm{d}x\mathrm{d}y\\ &+\iint_{\mathbb{R}^3\times\Omega}|\nu_\varepsilon(x-y)||1_{\Omega_\delta}(y)-1_{(-w+\Omega_\delta)}(y)|\mathrm{d}x\mathrm{d}y\\ &=\int_\Omega |1_{\Omega_\delta}(x)-1_{(-z+\Omega_\delta)}(x)|+|1_{\Omega_\delta}(x)-1_{(-w+\Omega_\delta)}(x)|\mathrm{d}x.\end{aligned}$$

The previous inequality, once integrated on $\mathbb{R}^3\times\mathbb{R}^3$ with respect to the measure $\eta_k(z)\eta_k(w)\mathrm{d}z\,\mathrm{d}w$ yields (4.16). □

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