LIE BRACKETS OF NONSMOOTH VECTOR FIELDS AND COMMUTATION OF THEIR FLOWS

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ABSTRACT. It is well-known that the flows generated by two smooth vector fields commute, if the Lie bracket of these vector fields vanishes. This assertion is known to extend to Lipschitz continuous vector fields, up to interpreting the vanishing of their Lie bracket in the sense of almost everywhere equality. We show that this cannot be extended to general a.e. differentiable vector fields admitting a.e. unique flows. We show however that the extension holds when one field is Lipschitz continuous and the other one is merely Sobolev regular (but admitting a regular Lagrangian flow).

1. INTRODUCTION

One of the well-known basic facts of differential geometry, ultimately leading to the Frobenius theorem on integral manifolds, is that the flows of two smooth vector fields V^1, V^2 over a smooth finite dimensional manifold M commute, if and only if their Lie bracket $[V^1, V^2]$ vanishes. In a more formal way, denoting the flow of V^i by $F^i_t(x) := y(t)$, where $y(\cdot)$ is the integral curve of the differential equation

$$\dot{y} = V^i(y), \quad y(0) = x, \tag{1.1}$$

one has

$$F^1_t(F^2_s(x)) = F^2_s(F^1_t(x)) \quad \text{for all } x \in M, \text{ whenever } [V^1, V^2] = 0, \tag{1.2}$$

once $t \in \mathbb{R}$ and $s \in \mathbb{R}$ are such that the respective expressions are defined. What happens with this statement for possibly nonsmooth vector fields V^i, even when the underlying manifold M itself remains smooth (in the sequel we will for simplicity consider just the case of $M = \mathbb{R}^d$, a finite-dimensional Euclidean space)?

Of course, to be meaningful, the question posed has to be restricted to the cases when all the objects present in (1.2) are well-defined. This is however nontrivial already in the case when V^i are Lipschitz. Namely, in this case the flows of V^i are still defined because the Cauchy problem (1.1) has a unique solution for *every* initial datum $x \in \mathbb{R}^d$. However, there is a problem with the meaning of the Lie bracket $[V^1, V^2]$; in fact, the latter is defined for smooth vector fields by the formula

$$[V^1, V^2](x) := DV^2(x)V^1(x) - DV^1(x)V^2(x), \tag{1.3}$$

where DV^i are Jacobi matrices of V^i, $i = 1, 2$, which is meaningless when V^i are only Lipschitz, because then their derivatives are not necessarily defined for *all* $x \in \mathbb{R}^d$ but only for *almost all* with respect to the Lebesgue measure. The natural question would be then: does (1.2) hold for a.e. rather than for all, $x \in \mathbb{R}^d$, once the vanishing of the commutator $[V^1, V^2] = 0$ is also understood in the sense of a.e.

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equality? The answer is positive as shown in the seminal paper by F. Rampazzo and H.J. Sussmann [14].

The problem is therefore what happens when one descends in regularity of V^i further beyond Lipschitz continuity. In fact, there are many cases when Cauchy problems (1.1) admit unique solutions for every or just for a.e. intial datum. For instance, as shown in [5, corollary 5.2], for a vector field $V \in W^{1,p}(\mathbb{R}^d;\mathbb{R}^d)$ with $p > d$ and $\operatorname{div} V$ bounded, a unique flow $x \mapsto F_t(x)$ is defined for a.e. $x \in \mathbb{R}^d$ and all $t \in \mathbb{R}$ as $F_t(x) := y(t)$ where $y(\cdot)$ is the unique solution to the ODE

$$\dot y = V(y)$$

satisfying $y(0) = x$. Further, even if $p \le d$, then a solution to the latter ODE may be not unique for a.e. initial datum, but in this case there is a natural selection of such solutions called *regular Lagrangian flow* [9, 1]. Moreover, there are many other cases besides Sobolev regularity when the vector field V admits a regular Lagrangian flow. In view of the Rampazzo-Sussmann result for Lipschitz vector fields, it is tempting to conjecture that also in the cases in which the fields V^i generating unique flows for a.e. initial datum (or at least regular Lagrangian flows) F^i, it is true that

$$F^1_t(F^2_s(x)) = F^2_s(F^1_t(x)) \quad \text{for a.e. } x \in \mathbb{R}^d, \text{ whenever } [V^1, V^2] = 0 \quad \text{a.e.}, \tag{1.4}$$

of course when the Lie bracket $[V^1, V^2](x)$ is well defined for a.e. $x \in \mathbb{R}^d$. We will show however by means of a counterexample that in general this is false. Nevertheless, we are able to show (Theorem 4.1 and Theorem 4.3) that this conjecture is true, for Sobolev vector fields $V^i \in W^{1,p_i}(\mathbb{R}^d;\mathbb{R}^d)$ once one of them is Lipschitz (which generalizes the result of [14]).

Note that in a completely independent recent paper [7] it has been proven by means of a different technique that when both V^i are not only Sobolev but have also bounded divergence (which in fact, under just mild growth assumptions, e.g. when these vector fields are bounded, guarantees the existence and uniqueness of regular Lagrangian flows, but of course is not necessary for that, see Remark 2.3), then commutativity of their regular Lagrangian flows is in fact *equivalent* to a.e. vanishing of their Lie bracket together with weak differentiability with locally bounded derivative of one of the respective flow maps in direction of another vector field. Thus for the particular case of vector fields with bounded divergence our Theorem 4.1 follows from [7, theorem 1.1]. It is worth remarking that in our counterexample the vector fields are not Sobolev and, what is more important, their *distributional* Lie bracket (as opposed to an a.e. one) is a non trivial vectorial measure with vanishing absolutely continuous part. It is therefore a curious open question whether for general vector fields having vanishing distributional Lie bracket, the associated regular Lagrangian flows (assumed they exist, e.g. when the vector fields are Sobolev with, say, bounded divergence) commute.

We finally think it worth mentioning that another very interesting, completely different but in a certain sense more classical approach based on smooth approximations and Stokes' theorem has been used to obtain a generalization of Frobenius integrability theorem for nonsmooth data (in that particular case, differential forms) in [13]; however the relationship of the result obtained there to the question of commutativity of flows is not at all clear (and most probably, is not even natural to look for).

The main difficulty here is that when the vector fields V^i are just Sobolev, the respective Lagrangian flows have only very weak regularity properties: in particular, they are just summable but not Sobolev [11], so that their weak derivatives do not have any pointwise (even a.e.) meaning. This corresponds well to the recently established loss of regularity results for solutions of continuity equations with

Sobolev velocity field of bounded divergence or even divergence-free [**?**, **?**], and is in sharp contrast with the situation when V^i are Lipschitz, since in the latter case it is a textbook result that the respective flows are Lipschitz too. That is why the technique we adopt in this paper is essentially different from that used in [14]. In fact, in the latter paper a *set-valued Lie bracket* of Lipschitz vector fields is introduced and then it is shown, substantially, that the usual expansions (of course adapted now to the set-valued setting) of compositions of flow maps in each point are still valid for this case with the set-valued Lie bracket instead of the classical one. As a side remark, we mention that another quite natural extension of both the classical Lie bracket and its set-valued version, the measure valued Lie bracket, has been introduced recently in [6]. In the general case of Sobolev V^i, in view of the mentioned lack of regularity of flows, no technique based on pointwise expansions would work and therefore we are forced to develop a completely different PDE/measure theory-style approach.

2. Notation and preliminaries

2.1. **General notation.** The finite-dimensional space $\mathbb{R}^d$ is assumed to be equipped with the Euclidean norm $|\cdot|$; the notation $B_r(x) \subset \mathbb{R}^d$ stands for the usual open Euclidean ball of radius r centered at x. In general, the norm in the normed space E will be denoted by $\|\cdot\|_E$.

All the measures over a metric space considered in the sequel are positive Radon measures, not necessarily finite. The notation $\mathcal{L}^d$ stands for the Lebesgue measure in $\mathbb{R}^d$. If μ is a measure over a metric space X, then for a Borel map $T\colon X \to Y$ between metric spaces X and Y we denote by $T_\#\mu$ the push-forward of μ, i.e. the measure over Y defined by $(T_\#\mu)(B) := \mu(T^{-1}(B))$ for every Borel $B \subset Y$.

We use the notation $A \lesssim B$ for possibly vector valued functions A, B defined on a subset $\Omega \subseteq \mathbb{R}^d$, when there exists a constant $C \geq 0$ such that

$$|A|(x) \leq C|B|(x) \quad \text{for every } x \in \Omega$$

with some constant $C > 0$ depending possibly on parameters such as dimension of the space or integrability exponents, but not on A and B.

2.2. **Spaces.** For a measure μ in a metric space E we denote by $L^p(E, \mu; \mathbb{R}^m)$ the usual Lebesgue space of μ-integrable with exponent $p \geq 1$ functions $f\colon E \to \mathbb{R}^m$ (μ-essentially bounded when $p = +\infty$); the reference to μ will be omitted in the case $\mu = \mathcal{L}^d$. The reference to E may be also omitted when no confusion is possible. Analogously, $W^{k,p}(\mathbb{R}^d; \mathbb{R}^m)$ (resp. $W^{k,p}_{\mathrm{loc}}(\mathbb{R}^d; \mathbb{R}^m)$) will stand for the usual Sobolev (resp. locally Sobolev) class of functions over $\mathbb{R}^d$ with values in $\mathbb{R}^m$. In all the cases the reference to $\mathbb{R}^m$ will be omitted when $m = 1$, i.e. for real valued functions. The norm in $L^p(\mathbb{R}^d)$ is denoted for brevity just $\|\cdot\|_p$. The notation $C_0^\infty(\mathbb{R}^d)$ stands for the class of infinitely differentiable functions with compact support in $\mathbb{R}^d$ (usually called test functions)¡ and the action of a distribution u on a test function $\varphi \in C_0^\infty(\mathbb{R}^d)$ is denoted by $\langle \varphi, u \rangle$.

The space $C([a,b]; \mathbb{R}^d)$ of continuous curves in $\mathbb{R}^d$ parameterized over the interval $[a,b]$ is endowed with the usual supremum norm. For every $t \in [a,b]$ we let $e_t\colon C([a,b]; \mathbb{R}^d) \to \mathbb{R}^d$ stand for the evaluation map $e_t(\theta) := \theta(t)$.

2.3. **Variation.** Given a function

$$\omega\colon [-T,T]^2 \to \mathbb{R}, \quad (s,t) \mapsto \omega_{st},$$

we write, where $\pi = \{t_i\}_i$ denotes any partition of $[-T,T]$ and $|\pi| := \sup_i |t_i - t_{i-1}|$,

$$[\omega] := \limsup_{|\pi| \to 0} \sum_{t_i \in \pi} |\omega_{t_i t_{i+1}}|.$$

This generalizes the notion of total variation (which corresponds to the case $\omega_{st} := f_t - f_s$).

2.4. **Maximal functions.** We recall the usual definitions of the sharp maximal function $g_r^\sharp$ for an $r > 0$ and the maximal function g^* of a locally integrable function $g\colon \mathbb{R}^d \to \mathbb{R}$, namely,

$$g_r^\sharp(x) := \sup_{s\in[0,r]} \frac{1}{\mathcal{L}^d(B_s(x))} \int_{B_s(x)} |g(y) - g(x)| \mathsf{d}y,$$
$$g^*(x) := \sup_{s\geq 0} \frac{1}{\mathcal{L}^d(B_s(x))} \int_{B_s(x)} |g(y)| \mathsf{d}y,$$

so that in particular

$$g_r^\sharp \leq 2g^*. \tag{2.1}$$

Remark 2.1 (convergence of maximal functions). We notice that, if $g \in L^p(\mathbb{R}^d)$ with $p > 1$, then

$$g_r^\sharp, g^* \in L^p(\mathbb{R}^d), \tag{2.2}$$

and $g_r^\sharp(x) \searrow 0$ as $r \searrow 0$ for a.e. $x \in \mathbb{R}^d$ (this can be proven by density of continuous functions). In particular,

$$\left\|g_r^\sharp\right\|_p \searrow 0 \quad \text{as } r \searrow 0.$$

2.5. **Pointwise inequalities for Sobolev functions.** Recall that for Sobolev functions $f \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^d)$, one has the following pointwise inequality [4, corollary 1], valid for every x, $y \in \mathbb{R}^d \setminus N$, where N is Lebesgue-negligible:

$$|f(y) - f(x) - Df(x)(y-x)| \lesssim |y-x| \left(|Df|^\sharp_{|y-x|}(y) + |Df|^\sharp_{|y-x|}(x) \right). \tag{2.3}$$

From (2.1) we obtain

$$|f(y) - f(x)| \lesssim |y-x| \left(|Df|^*(y) + |Df|^*(x) \right). \tag{2.4}$$

2.6. **Regular Lagrangian flows.**

Definition 2.2. *We say that a Borel map $F\colon [a,b] \times \mathbb{R}^d \to \mathbb{R}^d$ is a regular Lagrangian flow for the (possibly time-dependent) vector field $V\colon [a,b] \times \mathbb{R}^d \to \mathbb{R}^d$ if*

(i) *for a.e. $x \in \mathbb{R}^d$ the map $t \in [a,b] \mapsto F(t,x)$ is an absolutely continuous solution of the ODE*
$$\dot y = V(t,y)$$
for $t \in [a,b]$, satisfying $y(a) = x$;

(ii) *there is a constant $C > 0$ independent of t (called the compressibility constant of F) such that*
$$F(t,\cdot)_\# \mathcal{L}^d \leq C\mathcal{L}^d$$
for every $t \in [a,b]$.

For flows we usually write $F_t(x)$ instead of $F(t,x)$.

Remark 2.3. The regular Lagrangian flows over any time interval $[-T,T]$ for the vector field $V\colon \mathbb{R}^d \to \mathbb{R}^d$ is known to exist and be unique when, say, V is Sobolev and essentially bounded, i.e. $V \in W^{1,p}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and has essentially bounded divergence $\operatorname{div} V \in L^\infty(\mathbb{R}^n)$. However these conditions are far from being necessary for the existence of regular Lagrangian flows. For instance, if $\Phi\colon \mathbb{R}^d \to \mathbb{R}^d$ is a biLipschitz map, and the vector field $V\colon \mathbb{R}^d \to \mathbb{R}^d$ admits a regular Lagrangian flow $F_t(\cdot)$, then the vector field $\tilde V\colon \mathbb{R}^d \to \mathbb{R}^d$ defined by

$$\tilde V(y) := (D\Phi)(\Phi^{-1}(y)) V(\Phi^{-1}(y))$$

admits a regular Lagrangian flow $\Phi(F_t(\Phi^{-1}(\cdot)))$. It is also worth mentioning that for a vector field V defined over some open set $\Omega \subset \mathbb{R}^d$, theorem 5.2 from [2] gives sufficient contitions for a vector field to a admit a unique regular Lagrangian flows defined over $(-\bar{T}(x), \bar{T}(x))$ for some measurable function $\bar{T}\colon \Omega \to (0, +\infty)$.

We will further use a couple of observations.

Remark 2.4. When ν is a measure over $\mathbb{R}^d$ with bounded density with respect to the Lebesgue measure, i.e.

$$\nu = f\mathcal{L}^d \quad \text{with } f \in L^\infty(\mathbb{R}^d),$$

and $F\colon [a,b] \times \mathbb{R}^d \to \mathbb{R}^d$ is a regular Lagrangian flow with compressibility constant C for some vector field, then one has

$$F(t,\cdot)_\# \nu \leq C\|f\|_\infty \mathcal{L}^d, \tag{2.5}$$

because $F(t,\cdot)_\#\nu = (f \circ F(t,\cdot)^{-1})F(t,\cdot)_\#\mathcal{L}^d$.

Lemma 2.5. *If $F\colon [a,b] \times \mathbb{R}^d \to \mathbb{R}^d$ is a regular Lagrangian flow and $N \subset \mathbb{R}^d$ is Lebesgue negligible, then there is a $B_0 \subset \mathbb{R}^d$ with $\mathcal{L}^d(B_0) = 0$ such that $F(t,x) \notin N$ for every $x \notin B_0$ and a.e. $t \in [a,b]$.*

Proof. For the set $G := \{(x,t) \in [a,b] \times \mathbb{R}^d\colon F(t,x) \in N\}$ one has $\mathcal{L}^{d+1}(G) = 0$ by Fubini theorem since for each fixed $t \in [a,b]$ one has

$$\mathcal{L}^d(G \cap (\{t\} \times \mathbb{R}^d)) = (F(t,\cdot)_\#\mathcal{L}^d)(N) \leq C\mathcal{L}^d(N) = 0.$$

Therefore the set

$$B_0 := \{x \in \mathbb{R}^d\colon \mathcal{L}^1(G \cap ([a,b] \times \{x\})) > 0\} \subset \mathbb{R}^d$$

is Lebesgue negligible as claimed. □

3. Vanishing Lie bracket does not imply commutativity of flows

In this section we show by means of the example below that when two vector fields have a.e. vanishing Lie bracket, it is in general not true that their flows commute a.e. even if we assume that they are a.e. uniquely defined, and even if they are regular Lagrangian. The main idea comes from an example [15, Nelson's example] of non-commutativity of groups with generators commuting on a common dense set. In our case, we consider a foliation of $\mathbb{R}^d$ into helix-like hypersurfaces so that the composition of the two flows yields a rotation along the axis of the helix.

Example 3.1. Let $d := 3$,

$$f(x,y) := \arctan\left(\frac{y}{x}\right)$$

and consider the vector fields

$$V^1 := \partial_x + (\partial_x f)\partial_z = \partial_x - \frac{y}{x^2+y^2}\partial_z,$$
$$V^2 := \partial_y + (\partial_y f)\partial_z = \partial_y + \frac{x}{x^2+y^2}\partial_z,$$

where

$$\partial_x f = -\frac{y}{x^2+y^2}, \partial_y f = \frac{x}{x^2+y^2} \tag{3.1}$$

stand for the classical derivatives of the function f, so that

$$[V^1, V^2] = \left(\partial^2_{xy} f - \partial^2_{yx} f\right)\partial_z = 0, \tag{3.2}$$

i.e. their Lie bracket vanishes (of course the function f, the vector fields and their Lie bracket are defined everywhere in $\mathbb{R}^d$ outside of the two-dimensional plane $\{x = 0\}$, hence a.e. in $\mathbb{R}^d$). We now verify the sequence of claims.

CLAIM 1. Both V^1 and V^2 are tangent to the level sets of the function $(x,y,z) \in \mathbb{R}^d \mapsto z - f(x,y) \in \mathbb{R}$, i.e. to the graphs of the functions $z = f(x,y) + C$ in $\mathbb{R}^d$, with C any constant, as observed by a direct calculation.

CLAIM 2. The flows F^i for the vector fields V^i, $i = 1,2$, are uniquely defined over $[-T,T]$, for every $T > 0$ and for every initial datum outside of the plane $\{x = 0\}$. In fact, this is quite immediate for F^2, since the solution to the ODE $\dot\theta = V^2(\theta)$ is uniquely defined for every initial datum $\theta(0) \in \mathbb{R}^d \setminus \{x = 0\}$ over all times.

As for F^1, we consider the solutions to the ODE $\dot\theta = V^1(\theta)$ for every initial datum outside of the plane $\{x = 0\}$. If $\bar x := \theta_1(0) < 0$ then the solution $\theta(\cdot)$ to the respective Cauchy problem is defined uniquely for all $t \in (-\infty, -\bar x)$, and, analogously, if $\bar x := \theta_1(0) > 0$ then it is defined uniquely for all $t \in (-\bar x, +\infty)$. In the first case, one has

$$\lim_{t\to -\bar x - 0} f(\theta_1(t),\theta_2(t)) = \begin{cases} \pi/2, & \text{if } \bar y < 0,\\ 0, & \text{if } \bar y = 0,\\ -\pi/2, & \text{if } \bar y > 0, \end{cases}$$

because $\theta_2(t) = \theta_2(0)$ for all $t \in (-\infty, -\bar x)$ and

$$\lim_{t\to -\bar x - 0} \theta_1(t) = 0^-.$$

Thus, denoting $\bar z := \theta_3(0)$ and recalling Claim 1, we get that

$$\begin{aligned}\lim_{t\to -\bar x - 0} \theta_3(t) &= \lim_{t\to -\bar x - 0} f(\theta_1(t),\theta_2(t)) + (\bar z - f(\bar x,\bar y))\\ &= \bar z - f(\bar x,\bar y) - \frac{\pi}{2}\operatorname{sign}\bar y.\end{aligned}$$

Symmetrically in the second case, $\theta_2(t) = \theta_2(0)$ for all $t \in (-\bar x, +\infty)$,

$$\lim_{t\to -\bar x + 0} \theta_1(t) = 0^+$$

and

$$\lim_{t\to -\bar x + 0} \theta_3(t) = \bar z - f(\bar x,\bar y) - \frac{\pi}{2}\operatorname{sign}\bar y.$$

Summing up, we have that absolutely continuous solutions $\theta(\cdot)$ to the respective Cauchy problem are defined uniquely for every initial datum outside of the plane $\{x = 0\}$ and for a.e. $t \in \mathbb{R}$, or, more precisely, for every $t \in \mathbb{R}$ except $t = -\bar x$, with

$$\lim_{t\to -\bar x} \theta(t) = \left(0, \bar y, \bar z - f(\bar x,\bar y) - \frac{\pi}{2}\operatorname{sign}\bar y\right). \tag{3.3}$$

This defines uniquely the flow F^1.

CLAIM 3. One has $\operatorname{div} V^i = 0$, and hence the flows F^i are regular Lagrangian (because $F^i_{t\#}\mathcal{L}^d = \mathcal{L}^d$ for all t for which F^i_t is defined, i.e. the compressibility constant is one), for $i = 1,2$.

CLAIM 4. Finally, the flows F^1 and F^2 do not commute despite (3.2). In fact, consider an arbitrary $p = (\bar x, \bar y, \bar z) \in \mathbb{R}^d$ with $x < 0$ and $y < 0$. Then for $\theta^1(t) := F^1_t(p)$ one has

$$\theta^1(t) = \left(\bar x + t, \bar y, \theta^1_3(t)\right) \tag{3.4}$$

with $\theta^1_3(-\bar x) = \bar z - f(\bar x,\bar y) + \pi/2$ in view of (3.3). By Claim 1 one has that $\theta^1(t)$ for $t > -\bar x$ belongs to the graph of function $z = f(x,y) + C$, which means

$$\theta^1_3(t) = f(\bar x + t, \bar y) + C$$

for all $t > -\bar x$. Letting $t \to -\bar x + 0$, we get

$$\bar z - f(\bar x,\bar y) + \pi/2 = \theta^1_3(-\bar x) = -\pi/2 + C,$$

which gives $C = \bar{z} - f(\bar{x},\bar{y}) + \pi$. Therefore, $F^2_s(F^1_t(p))$ belongs to the graph of function

$$z = f(x,y) + \bar{z} - f(\bar{x},\bar{y}) + \pi$$

for every $s \in \mathbb{R}$ and $t > -\bar{x}$.

On the other hand, for $\theta^2(s) := F^2_s(p)$ one has

$$\theta^2_2(s) = \left(\bar{x}, \bar{y} + s, , \theta^2_3(s)\right),$$

so that for every $s > -\bar{y}$ one has $\theta^2_2(s) > 0$. Then for $\sigma^1(t) := F^1_t(\theta^2(s))$ for $s > -\bar{y}$ one has

$$\sigma^1(t) = \left(\bar{x} + t, \bar{y} + s, \sigma^1_3(t)\right). \tag{3.5}$$

Since $\theta^2(s)$ belongs to the graph of the function

$$z = f(x,y) + \bar{z} - f(\bar{x},\bar{y})$$

by Claim 1, then so is $\sigma^1(t)$ for $t < -\bar{x}$. By (3.3) one has

$$\sigma^1(-\bar{x}) = \left(\bar{x}, \bar{y} + s, \bar{z} - f(\bar{x},\bar{y}) - \frac{\pi}{2}\right). \tag{3.6}$$

Hence for $t > -\bar{x}$ (and $s > -\bar{y}$) one has that $F^1_t(F^2_s(p)) = \sigma^1(t)$ belongs to the graph of function $z = f(x,y) + C$, that is,

$$\sigma^1_3(t) = f(\bar{x} + t, \bar{y}) + C$$

for all $t > -\bar{x}$, and the value of C can be found by letting $t \to -\bar{x} + 0$, since then by (3.6) we have

$$\bar{z} - f(\bar{x},\bar{y}) - \pi/2 = \sigma^1_3(-\bar{x}) = \pi/2 + C,$$

which gives $C = \bar{z} - f(\bar{x},\bar{y}) - \pi$. Therefore, $F^1_t(F^2_s(p))$ belongs to the graph of function

$$z = f(x,y) + \bar{z} - f(\bar{x},\bar{y}) - \pi$$

for every $s > -\bar{y}$ and $t > -\bar{x}$. In other words for all such pairs (s,t) one has

$$F^1_t(F^2_s(p)) \neq F^2_s(F^1_t(p))$$

as claimed.

Remark 3.2. One can easily check that both V^1, V^2 are not Sobolev (even locally) on $\mathbb{R}^3$. Moreover, for $f(x,y) := \arctan(y/x)$ we have $f \notin W^{1,1}_{\mathrm{loc}}(\mathbb{R}^2)$. We may in fact prove more, that is, any two vector fields on $\mathbb{R}^3$ of the form

$$V^1 = \partial_x + \partial_x f \partial_z, \quad V^2 = \partial_y + \partial_y f \partial_z,$$

with $f \in W^{1,1}_{loc}(\mathbb{R}^2)$ must have commuting flows F^1, F^2. Indeed, they are explicitly given by

$$F^1_s(x,y,z) := \left(x + s, y, z + \int_0^s \partial_x f(x + r, y) dr\right),$$
$$F^2_t(x,y,z) := \left(x, y + t, z + \int_0^t \partial_y f(x, y + r) dr\right).$$

If we introduce the differential 1-form on $\mathbb{R}^3$

$$\omega := df = \partial_x f dx + \partial_y f dy,$$

then $F^1_s \circ F^2_t = F^2_t \circ F^1_s$, if and only if

$$\int_{\partial R} df = 0$$

for "almost every" oriented rectangle $R \subset \mathbb{R}^2$ with parallel sides to the coordinate axes and side lengths s, t (namely, for every rectangle of the form $[x, x+s] \times [y, y+t]$ for a.e. $(x,y) \in \mathbb{R}^2$). Note that ω is independent on z, and, moreover, if $f \in$

$W^{1,1}_{\mathrm{loc}}(\mathbb{R}^2)$, then choosing a precise representative of f, one has that for almost every rectangle $R = [x_0, x_0+s] \times [y_0, y_0+t]$ the restriction of f to its sides is an absolutely continuous $\mathcal{H}^1$-a.e. differentiable function (where $\mathcal{H}^1$ stands for the one-dimensional Hausdorff measure), so that calculating explicitly, one gets

$$\begin{aligned}\int_{\partial R} \omega &= \int_{x_0}^{x_0+s} f_x(x, y_0)\,dx + \int_{y_0}^{y_0+t} f_y(x_0+s, y)\,dy \\ &\quad + \int_{x_0+s}^{x_0} f_x(x, y_0+t)\,dx + \int_{y_0+t}^{y_0} f_y(x_0, y)\,dy \\ &= f(x_0+s, y_0) - f(x_0, y_0) + f(x_0+s, y_0+t) - f(x_0+s, y_0) \\ &\quad + f(x_0, y_0+t) - f(x_0+s, y_0+t) + f(x_0, y_0) - f(x_0, y_0+t) = 0,\end{aligned}$$

thus the two flows commute, i.e. to be more precise

$$F^1_t(F^2_s(x, y, z)) = F^2_s(F^1_t(x, y, z))$$

for a.e. $(x, y) \in \mathbb{R}^2$, all $z \in \mathbb{R}$ and $(s, t) \in \mathbb{R}^2$.

Remark 3.3. Due to the lack of Sobolev regularity of the two vector fields in Example 3.1, it does not follow from the a.e. vanishing of the Lie bracket $[V^1, V^2] = 0$ that the same holds also in the distributional sense. However just the notion of a distributional Lie bracket is quite nontrivial and deserves some discussion. In fact, trying to give the distributional meaning directly to the formula

$$[V^1, V^2] = (DV^2)V^1 - (DV^1)V^2,$$

where DV^i stands for the Jacobian matrix of V^i, $i = 1, 2$, by trying to understand the derivatives in the latter in distributional sense, might lead in no reasonable direction because this amounts in general (i.e. unless the vector fields involved have some extra regularity) to dealing with products of distributions. However, for, say, divergence free vector fields V^1, V^2 one has

(3.7)
$$\begin{aligned}\int_{\mathbb{R}^3} \varphi(x)([V^1, V^2]\psi)(x)\,dx &= \int_{\mathbb{R}^3} \varphi\left(V^1(x)(V^2\psi)(x) - \varphi V^2(x)(V^1\psi)(x)\right)\,dx \\ &= -\int_{\mathbb{R}^3} \left((V^1\varphi)(x)(V^2\psi)(x) - (V^2\varphi)(x)(V^1\psi)(x)\right)\,dx\end{aligned}$$

for all compactly supported $\{\phi, \psi\} \subset C^\infty(\mathbb{R}^3)$ once both V^1 and V^2 are smooth, so that this can be used as a definition of a distributional Lie bracket for nonsmooth divergence free vector fields. Namely, for such vector fields we define the action of their distributional Lie bracket on a compactly supported test function $\psi \in C^\infty(\mathbb{R}^3)$ as, in its turn, a distribution acting on a test function $\phi \in C^\infty(\mathbb{R}^3)$ with compact support by the formula

$$\langle \phi, [V^1, V^2]\psi\rangle := -\int_{\mathbb{R}^3} \left((V^1\varphi)(x)(V^2\psi)(x) - (V^2\varphi)(x)(V^1\psi)(x)\right)\,dx \tag{3.8}$$

In our case we show

$$[V^1, V^2] = 2\pi\mathcal{H}^1 \llcorner \{(x, y) = 0\}\,\partial_z \tag{3.9}$$

in the sense of (3.7), i.e. that for every test function ψ the action of the distributional Lie bracket $[V^1, V^2]$ on ψ is defined by

$$[V^1, V^2]\psi = 2\pi\mathcal{H}^1 \llcorner \{(x, y) = 0\}\,\partial_z\psi,$$

in the distributional sense, or else in other words, that the action $\langle \phi, [V^1, V^2]\psi\rangle$ of the distribution $[V^1, V^2]\psi$ on the test function ϕ is given by

$$\langle \phi, [V^1, V^2]\psi\rangle = 2\pi \int_{\mathbb{R}} \varphi(0, 0, z)\partial_z\psi(0, 0, z)\,dz. \tag{3.10}$$

In yet another language one can read the distributional Lie bracket $[V^1, V^2]$ as a one-dimensional metric current of locally finite mass [12] (a version of Ambrosio-Kirchheim metric currents [3] or measurable derivations by N. Weaver [16]) acting on a "differential form" $\phi\, d\psi$ as

$$[V^1, V^2](\phi\, d\psi) = 2\pi \int_{\mathbb{R}} \varphi(0,0,z)\partial_z\psi(0,0,z)\, dz.$$

Let us point out however that since in our example the vector fields V^1, V^2 are not (locally) square integrable, a separate definition of the two integrals,

$$\int_{\mathbb{R}^3} (V^2\varphi)(V^1\psi) \quad \text{and} \quad \int_{\mathbb{R}^3} (V^1\varphi)(V^2\psi),$$

could be problematic for generic test functions ϕ and ψ. Nevertheless it turns out that the integral of the difference in the righthand side of (3.7) is well defined, since a simple cancellation occurs in view of the a.e. identity

$$\begin{aligned}(V^2\varphi)(V^1\psi) - (V^1\varphi)(V^2\psi) &= \partial_z\varphi(\partial_x\psi\partial_y f - \partial_y\psi\partial_x f)\\ &\quad - \partial_z\psi(\partial_x\varphi\partial_y f - \partial_y\varphi\partial_x f)\end{aligned} \tag{3.11}$$

where for brevity we use the notation (3.1) from Example 3.1 for the a.e. (but not distributional) derivatives $\partial_x f, \partial_y f$ of the function $f(x,y) := \arctan(y/x)$.

To prove (3.10), use (3.8) and (3.11) to obtain in cylindrical coordinates (r,θ,z) the relationship

$$\begin{aligned}\langle \phi, [V^1, V^2]\psi\rangle &= \int_{\mathbb{R}^3} \left(\partial_z\varphi(x\partial_x\psi + y\partial_y\psi) - \partial_z\psi(x\partial_x\varphi + y\partial_y\varphi)\right) \frac{dxdydz}{x^2+y^2}\\ &= \int_0^{2\pi} d\theta \int_{\mathbb{R}} dz \int_0^\infty dr\, (\partial_z\varphi\partial_r\psi - \partial_z\psi\partial_r\varphi)\,,\end{aligned} \tag{3.12}$$

since $x\partial_x + y\partial_y = r\partial_r$. Using the compact support assumption on φ and ψ, we integrate by parts in (3.12) yielding

$$\int_{\mathbb{R}} \partial_z\varphi\partial_r\psi dz = \varphi\partial_r\psi\big|_{z=-\infty}^{z=\infty} - \int_{\mathbb{R}} \varphi\partial_z\partial_r\psi dz = -\int_{\mathbb{R}} \varphi\partial_z\partial_r\psi dz,$$

and

$$\int_0^\infty \partial_z\psi\partial_r\varphi dr = \varphi\partial_z\psi\big|_{r=0}^{r=\infty} - \int_0^\infty \varphi\partial_r\partial_z\psi dr = -\varphi\partial_z\psi\big|_{r=0} - \int_0^\infty \varphi\partial_r\partial_z\psi\, dr$$

Since $\partial_r\partial_z\psi - \partial_z\partial_r\psi = 0$, a cancellation occurs in (3.12) and we conclude that

$$\begin{aligned}\int_0^{2\pi}\int_{\mathbb{R}}\int_0^\infty (\partial_z\varphi\partial_r\psi - \partial_z\psi\partial_r\varphi)\, drdzd\theta &= \int_0^{2\pi}\int_{\mathbb{R}} \varphi\partial_z\psi\big|_{r=0}\\ &= \int_{\mathbb{R}} \varphi(0,0,z)\partial_z\psi(0,0,z)\, dz,\end{aligned}$$

that is (3.10) as claimed.

In view of the above remark, it would be interesting to provide an example, if exists, of two vector fields having the Lie bracket vanishing in the distributional sense, yet the two associated Lagrangian flows being defined, but not commuting. It has to be remarked though that just the definition of the distributional Lie bracket requires some extra assumptions on the vector fields (e.g. for the integral in the definition (3.8) to be well defined the natural assumption would be that the vector fields be locally square integrable).

4. Vanishing Lie bracket and commutativity of regular Lagrangian flows for Sobolev vector fields

We prove now that regular Lagrangian flows F^1, F^2 of Sobolev vector fields V^1, V^2 respectively commute a.e., if the Lie bracket of the latter vanishes a.e., i.e. (1.4) holds, once one of the two is Lipschitz continuous.

Theorem 4.1. *Let $p \in [1,\infty]$, and assume that $V^1 \in W^{1,p}(\mathbb{R}^d)$ is such that its regular Lagrangian flow F^1_s is uniquely defined for s in some interval containing zero, and $V^2 \in W^{1,\infty}(\mathbb{R}^d)$. Let $(F^1_t)_{t\in\mathbb{R}}$, $(F^2_t)_{t\in\mathbb{R}}$ be the corresponding regular Lagrangian flows. If*

$$[V^1,V^2](x) := (DV^2)(x)V^1(x) - (DV^1)(x)V^2(x) = 0 \quad \text{for a.e. } x \in \mathbb{R}^d,$$

then the flows commute, i.e., for a.e. $x \in \mathbb{R}^d$

$$F^2_t\left(F^1_s(x)\right) = F^1_s\left(F^2_t(x)\right), \quad \text{for every } s,\, t \in \mathbb{R}.$$

Remark 4.2. Theorem 1.1 from [7] proven by means of a different technique shows, roughly speaking, that if both V^i are not just Sobolev but also have bounded divergence, then commutativity of their regular Lagrangian flows is in fact *equivalent* to a.e. vanishing of their Lie bracket together with weak differentiability with locally bounded derivative of one of the respective flow maps in direction of another vector field. Thus for the particular case of vector fields with bounded divergence the above Theorem 4.1 follows from the cited result.

Proof. Since V^2 is Lipschitz continuous, its regular Lagrangian flow coincides a.e. with the classical flow. Therefore, we may assume that $x \mapsto F^2_t(x)$ is defined everywhere and Lipschitz continuous. It is also sufficient to prove the thesis for s, $t \in [-T,T]$ for an arbitrary $T > 0$ such that both F^i_t are defined for $t \in [-T,T]$, $i = 1,2$.

Let ν stand for a finite measure on $\mathbb{R}^d$ with bounded and strictly positive density with respect to $\mathcal{L}^d$ (the L^∞ norm of the latter will be denoted by $\|\nu\|_\infty$), e.g. a standard Gaussian. For $x \in \mathbb{R}^d$ and $s,t \in [-T,T]$ we set

$$X_{s,t}(x) := F^2_t(F^1_s(x)). \tag{4.1}$$

Since the flow F^2 is Lipschitz continuous, we have that for a.e. $x \in \mathbb{R}^d$ and for every $s,t \in [-T,T]$, the map $(s,t) \mapsto X_{s,t}(x)$ is continuous. Moreover, for a.e. $x \in \mathbb{R}^d$ one has

$$\partial_t X_{s,t}(x) = V^2(X_{s,t}(x)) \quad \text{for a.e. } t \in (-T,T),$$
$$\partial_s X_{s,0}(x) = V^1(X_{s,0}(x)) \quad \text{for a.e. } s \in (-T,T).$$

Now, for s, s', $t \in [-T,T]$, we write

$$A_{s,s';t}(x) := X_{s',t}(x) - X_{s,t}(x), \quad B_{s,s';t}(x) := A_{s,s';t}(x) - (s'-s)V^1(X_{s,t}(x)),$$

frequently omitting the reference to x for brevity, and for every $t \in [-T,T]$ we consider the measure $\eta := \hat{X}_{t\#}\nu$ over $C([-T,T],\mathbb{R}^d)$, where $\hat{X}_t \colon \mathbb{R}^d \to C([-T,T],\mathbb{R}^d)$ is defined by the formula

$$\hat{X}_t(x) := X_{(\cdot,t)}(x).$$

Note that η is concentrated over curves $[-T,T] \ni s \mapsto X_{s,t}(x)$ for a set of full ν-measure of $x \in \mathbb{R}^d$. Further,

$$e_{s\#}\eta = X_{s,t\#}\nu = F^2_{t\#}(F^1_{s\#}\nu) \le C_2C_1\mathcal{L}^d,$$

where C_i are the compressibility constants for F^i, $i = 1,2$.

We will show that for some function $(\omega_{ss'})_{s,s'\in[-T,T]}$, with $[\omega] = 0$ one has

$$\|B_{s,s';t}(\cdot)\|_{L^1(\nu)} \lesssim \omega_{ss'}, \quad \text{for } s,s' \in [-T,T]. \tag{4.2}$$

This would give

$$\left\|\theta_{s'} - \theta_s - (s'-s)V^1(\theta_s)\right\|_{L^1(C([-T,T],\mathbb{R}^d);\eta)} \lesssim \omega_{ss'}$$

for every $s, s' \in [-T,T]$, which means in view of Proposition A.1 (applied with V^1 instead of V and C_1C_2 instead of C) that η-a.e. $\theta \in C([-T,T],\mathbb{R}^d)$ is an integral curve of V^1. In other words, for ν-a.e. (hence for $\mathcal{L}^d$-a.e., because the density of ν is strictly positive) $x \in \mathbb{R}^d$ the curve θ defined by $\theta(s) := X_{s,t}(x)$ satisfies $\partial_s\theta = V^1(\theta(s))$ for every $s \in [-T,T]$. Since for such curves one has $\theta(0) = F^2_t(x)$, then for a.e. $x \in \mathbb{R}^d$ one has

$$X_{s,t}(x) = F^1_s(F^2_t(x)),$$

which is the claim of the theorem being proven.

The rest of the proof is therefore dedicated to showing the claim (4.2). It will be done in six steps.

Step 1: Estimate on $A_{s,s';0}$. In the subsequent estimates we suppose without loss of generality that $s' \geq s$ (the case $s \geq s'$ is completely symmetric). One has

$$\begin{aligned}
|A_{s,s';0}(x)|^p &= |X_{s',0}(x) - X_{s,0}(x)|^p = |F^1_s(x) - F^1_{s'}(x)|^p \\
&\leq \left| \int_s^{s'} |V^1(y(\tau))|\, d\tau \right|^p \\
&\leq |s-s'|^{p/p'} \int_s^{s'} |V^1(y(\tau))|^p \, d\tau \quad \text{by Hölder inequality.}
\end{aligned}$$

Integrating with respect to $\tilde\eta := \nu \llcorner B \otimes \delta_{F^1(\cdot,x)}$, where $B \subset \mathbb{R}^d$ is an arbitrary fixed Borel set, gives

$$\begin{aligned}
\int_B |A_{s,s';0}(x)|^p \mathsf{d}\nu(x) &\leq |s-s'|^{p/p'} \left| \int_{C([-T,T];\mathbb{R}^d)} \mathsf{d}\tilde\eta(y) \int_s^{s'} |V^1(y(\tau))|^p \, \mathsf{d}\tau \right| \\
&= |s-s'|^{p/p'} \left| \int_s^{s'} \mathsf{d}\tau \int_{C([-T,T];\mathbb{R}^d)} \mathsf{d}\tilde\eta(y) |V^1(y(\tau))|^p \right| \\
&= |s-s'|^{p/p'} \left| \int_s^{s'} \mathsf{d}\tau \int_{\mathbb{R}^d} \mathsf{d} e_{\tau\#}\tilde\eta(x) |V^1(x)|^p \right| \\
&\leq C\|\nu\|_\infty |s-s'|^{1+p/p'} \int_{F^1_\tau(B)} |V^1(x)|^p \, \mathsf{d}x,
\end{aligned} \tag{4.3}$$

because $e_{\tau\#}\tilde\eta = F^1_{\tau\#}(\nu \llcorner B) = \mathbf{1}_{F^1_\tau(B)} F^1_{\tau\#}\nu$ and in view of (2.5). In particular, with $B := \mathbb{R}^d$ we get

$$\|A_{s,s';0}(x)\|_{L^p(\nu)} \lesssim |s-s'|^{1/p+1/p'} \left\|V^1\right\|_p = |s-s'| \left\|V^1\right\|_p. \tag{4.4}$$

Further, since $F^1_\tau(B) = (F^1_{-\tau})^{-1}(B)$, we get

$$\mathcal{L}^d(F^1_\tau(B)) = (F^1_{-\tau\#}\mathcal{L}^d)(B) \leq C\mathcal{L}^d(B),$$

and hence from (4.3) with $p = 1$ we get that the functions $A_{s,s';0}(\cdot)/|s-s'|$ are equiintegrable in $L^1(\nu)$, if $V^1 \in L^1(\mathbb{R}^d)$.

Step 2: Estimate on $A_{s,s';t}$. Lipschitz continuity of F^2_t gives, for a.e. $x \in \mathbb{R}^d$,

$$\partial_t |A_{s,s';t}| \leq |V^2(X_{s,t}) - V^2(X_{s',t})| \leq \left\|DV^2\right\|_\infty |A_{s,s';t}|, \quad \text{for a.e. } t \in [-T,T].$$

Hence, by Gronwall lemma,

$$\sup_{t\in[-T,T]} |A_{s,s';t}| \leq |A_{s,s';0}| \exp\left(\left\|DV^2\right\|_\infty T\right).$$

Integrating with respect to ν gives

$$\begin{aligned}\sup_{t\in[-T,T]}\||A_{s,s';t}|\|_{L^p(\nu)} &\le \left\|\sup_{t\in[-T,T]}|A_{s,s';t}|\right\|_{L^p(\nu)}\\ &\le \|A_{s,s';0}\|_{L^p(\nu)}\exp\left(\left\|DV^2\right\|_\infty T\right)\\ &\lesssim |s'-s|\left\|V^1\right\|_p\exp\left(\left\|DV^2\right\|_\infty T\right)\quad\text{in view of (4.4)}\\ &\lesssim |s'-s|.\end{aligned}\tag{4.5}$$

Step 3: Integral inequality for $B_{s,s';t}$. We first prove that

(i) for a.e. $x\in\mathbb{R}^d$ and for every $s\in[-T,T]$ the curve $t\mapsto V^1(X_{s,t}(x))$ is absolutely continuous, and
(ii) satisfies

$$\partial_t V^1(X_{s,t}(x)) = DV^1(X_{s,t}(x))V^2(X_{s,t}(x)).\tag{4.6}$$

In fact, for a.e. $x\in\mathbb{R}^d$ and for every $s\in[-T,T]$ the curve $t\mapsto X_{s,t}(x)$ is absolutely continuous (even Lipschitz). Letting

$$\nu^1_s := F^1_{s\#}\nu,\quad \eta^2_s := F^2_{\#}\nu^1_s,$$

we have that η^2_s is a measure on $C([-T,T];\mathbb{R}^d)$ concentrated over such curves, while

$$\nu^1_s \le C_1\|\nu\|_\infty\mathcal{L}^d$$

by Remark 2.4, and therefore

$$e_{t\#}\eta^2_s = F^2_{t\#}\nu^1_s \le C_2C_1\|\nu\|_\infty\mathcal{L}^d$$

by the same Remark. Therefore, (i) and (ii) follow from Lemma B.2 with η^2_s in place of η and V^1 in place of V.

Note that since $[V^1,V^2](x)=0$ for a.e. $x\in\mathbb{R}^d$, then by Lemma 2.5 there is a $B_0\subset\mathbb{R}^d$ with $\mathcal{L}^d(B_0)=0$ such that $[V^1,V^2](X_{s,t}(x))=0$ for every $x\notin B_0$ and a.e. $s,t\in[-T,T]$.

Using (4.6) we obtain

$$\begin{aligned}\partial_t B_{s,s';t} &= \partial_t A_{s,s';t} - (s'-s)DV^1(X_{s,t})V^2(X_{s,t}),\\ &= \partial_t A_{s,s';t} - (s'-s)DV^2(X_{s,t})V^1(X_{s,t})\quad(\text{since } [V^1,V^2](X_{s,t})=0 \text{ a.e.})\\ &= DV^2(X_{s,t})B_{s,s';t} + R_{s,s';t},\end{aligned}$$

where

$$R_{s,s';t} := V^2(X_{s',t}) - V^2(X_{s,t}) - DV^2(X_{s,t})(X_{s',t}-X_{s,t}).$$

At this point Gronwall lemma yields

$$\sup_{t\in[-T,T]}|B_{s,s';t}(x)| \le \left(|B_{s,s';0}(x)| + \int_{-T}^T |R_{s,s';\tau}(x)|\mathsf{d}\tau\right)\exp\left(\left\|DV^2\right\|_\infty T\right),\tag{4.7}$$

which implies

$$\sup_{t\in[-T,T]}\|B_{s,s';t}\|_{L^1(\nu)} \le \left(\|B_{s,s';0}\|_{L^1(\nu)} + \int_{-T}^T \|R_{s,s';\tau}\|_{L^1(\nu)}\mathsf{d}\tau\right)\exp\left(\left\|DV^2\right\|_\infty T\right).\tag{4.8}$$

Step 4: Estimate on $B_{s,s';0}$. To estimate $\|B_{s,s';0}\|_{L^1(\nu)}$ we use Proposition A.1 with V^1 instead of V, $p_0:=p$, $p_1:=p'$, $q:=1$, obtaining

$$\|B_{s,s';0}\|_{L^1(\nu)} \lesssim |s'-s|^2.\tag{4.9}$$

Step 5: Estimate on $R_{s,s';\tau}$. We use (2.3) with the components of V^2 instead of f, getting

$$\left|V^2(y)-V^2(z)-DV^2(z)(y-z)\right| \lesssim |y-z|\left((DV^2)^\sharp_{|y-z|}(y)+(DV^2)^\sharp_{y-z}(z)\right),$$

and choose $z := X_{s,t}(x)$, $y := X_{s',t}(x)$, so that

$$R_{s,s';t}(x) \lesssim A_{s,s';t}(x)\left((DV^2)^\sharp_{|A_{s,s';t}(x)|}(X_{s',t}(x))+(DV^2)^\sharp_{|A_{s,s';t}(x)|}(X_{s,t}(x))\right).$$

Integrating the above inequality over $x\in\mathbb{R}^d$ with respect to ν and dividing by $|s-s'|$, we arrive at the estimate

(4.10)
$$\int_{-T}^{T}\frac{\|R_{s,s';\tau}\|_{L^1(\nu)}}{|s-s'|}\mathsf{d}\tau \lesssim \int_{-T}^{T}\mathsf{d}\tau\int_{\mathbb{R}^d}\frac{A_{s,s';\tau}(x)}{|s-s'|}D_{s,s';\tau}(x)\mathsf{d}\nu(x), \quad \text{where}$$
$$D_{s,s';\tau}(x) := \left((DV^2)^\sharp_{|A_{s,s';\tau}(x)|}(X_{s',\tau}(x))+(DV^2)^\sharp_{|A_{s,s';\tau}(x)|}(X_{s,\tau}(x))\right).$$

We will show that one has

$$\int_{-T}^{T}\mathsf{d}\tau\int_{\mathbb{R}^d}(DV^2)^\sharp_{|A_{s,s';\tau}(x)|}(X_{s',\tau}(x))\mathsf{d}\nu(x)\to 0, \tag{4.11}$$

$$\int_{-T}^{T}\mathsf{d}\tau\int_{\mathbb{R}^d}(DV^2)^\sharp_{|A_{s,s';\tau}(x)|}(X_{s,\tau}(x))\mathsf{d}\nu(x)\to 0, \tag{4.12}$$

which means in particular that the functions $\tilde{D}_{s,s'}$ defined by $(\tau,x)\mapsto D_{s,s';\tau}(x)$ converge to zero in $L^1(\mathbb{R}^d\times(-T,T),\nu\otimes\mathcal{L}^1)$, hence in measure $\nu\otimes\mathcal{L}^1\llcorner[-T,T]$ as $\delta := s'-s\to 0^+$, uniformly in $s\in[-T,T]$. Minding that these functions are uniformly bounded (since the maximal function $(DV^2)^\sharp$ is bounded by $\|DV^2\|_\infty$), and the functions $\tilde{A}_{s,s'}$ defined by $(\tau,x)\mapsto A_{s,s';\tau}(x)/|s-s'|$ are uniformly bounded in $L^1(\mathbb{R}^d\times(-T,T),\nu\otimes\mathcal{L}^1)$ and equiintegrable by Step 1, then choosing arbitrary sequences $s_k, s'_k\in[-T,T]$ with $\delta_k := s'_k-s_k\to 0^+$ from Lemma B.1 (i) (with $f_k := \tilde{A}_{s_k,s'_k}$, $g_k := \tilde{D}_{s_k,s'_k}$, $E := \mathbb{R}^d\times(-T,T)$, $\mu := \nu\otimes\mathcal{L}^1$) and from (4.10) one gets

$$\int_{-T}^{T}\|R_{s,s';\tau}\|_{L^1(\nu)}\,\mathsf{d}\tau \lesssim |s'-s|o(1). \tag{4.13}$$

Plugging (4.9) and (4.13) into (4.8), we get

$$\sup_{t\in[-T,T]}\|B_{s,s';t}\|_{L^1(\nu)}\lesssim o(|s'-s|)$$

as $s'-s\to 0^+$, which gives the thesis.

Step 6. It remains to prove (4.11) and (4.12). In order to verify (4.11), let ν_s be as in Step 3, i.e. $\nu_s := (F^1_s)_\sharp\nu$, so that a change of variables gives

$$\int_{\mathbb{R}^d}(DV^2)^\sharp_{|A_{s,s';\tau}(\cdot)|}(X_{s',\tau}(\cdot))\mathsf{d}\nu = \int_{\mathbb{R}^d}(DV^2)^\sharp_{|A_{0,s'-s;\tau}(\cdot)|}(X_{s'-s,\tau}(\cdot))\mathsf{d}\nu_s.$$

Thus we simply observe that

$$\int_{\mathbb{R}^d}(DV^2)^\sharp_{|A_{0,\delta;\tau}(x)|})(X_{\delta,\tau}(x))\mathsf{d}\nu_s(x)\to 0$$

as $\delta := s'-s\to 0^+$ uniformly in $s\in[-T,T]$ by Lemma B.1 (ii) with $g_\delta(x) := (DV^2)^\sharp_{|A_{0,\delta;\tau}(x)|}(X_{\delta,\tau}(x))$ (note that these functions are bounded by $\|DV^2\|_\infty$ since so are the maximal functions $(DV^2)^\sharp$ and $\lim_{\delta\to 0^+}g_\delta=0$ by Remark 2.1). Therefore by Lebesgue dominated convergence theorem we get

$$\int_0^T\mathsf{d}\tau\int_{\mathbb{R}^d}(DV^2)^\sharp_{|A_{0,s'-s;\tau}(x)|}(X_{s'-s,\tau}(x))\mathsf{d}\nu_s(x)\to 0$$

uniformly in $s \in [-T,T]$ as $s'-s \to 0^+$. The proof of (4.12) is completely analogous, and we can conclude the proof of the theorem. □

We give also the local version of the above Theorem 4.1.

Theorem 4.3. *For $p \in [1,\infty]$ assume that $V^1 \in W^{1,p}_{\mathrm{loc}}(\Omega;\mathbb{R}^d)$ for some open set $\Omega \subset \mathbb{R}^d$ is such that its regular Lagrangian flow $F^1_s(x)$ is uniquely defined over $(-\bar T(x),\bar T(x))$ for some measurable function $\bar T\colon \Omega \to (0,+\infty)$, and $V^2 \in W^{1,\infty}_{\mathrm{loc}}(\Omega;\mathbb{R}^d)$. If*

$$[V^1,V^2](x) = 0 \quad \text{for a.e. } x \in \Omega,$$

then for a.e. $x \in \Omega$ there is a $T = T(x) > 0$ such that the regular Lagrangian flows F^1, F^2 satisfy

$$F^2_t\left(F^1_s(x)\right) = F^1_s\left(F^2_t(x)\right) \quad \text{for every } s,\, t \in [-T,T]. \tag{4.14}$$

Proof. For every ball $B := B_R(x_0)$ such that $\bar B \subset \Omega$ and every $T_1 > 0$ we find a $\rho > R$ with the property that $B_\rho(x_0) \subset \Omega$. Let $M_{B,T_1} \subset B$ stand for the (possibly empty) Borel set of points $x \in B$ such that $F^1_s(x)$ is defined for all $s \in [-T_1,T_1]$ and stays in B. By Lemma 4.5 applied with V^2 in place of V there is a $T_2 > 0$ (depending on B) such that for all $x \in B$ the classical flow induced by V^2 (which is equal to the regular Lagrangian flow F^2) is defined for all $t \in [-T_2,T_2]$ and stays inside $B_\rho(x_0)$. We set then $T := T_1 \wedge T_2$, so that clearly $T = T(B,T_1)$. In this way the map (4.1) is defined over M_{B,T_1} for all $t,s \in [-T,T]$, and takes values in $B_\rho(x_0)$. Note now that in the proof of Theorem 4.1 one only evaluates the Lie bracket $[V_1,V_2]$ along the trajectories of ODEs with right-hand sides V_1, V_2. The latter, if started at a point of M_{B,T_1} remain in $B_\rho(x_0)$, so that $[V_1,V_2]$ vanishes along them. Thus reiterating the proof of Theorem 4.1 with $\nu := \mathcal{L}^d(M_{B,T_1})$ we get that for a.e. $x \in M_{B,T_1}$ the regular Lagrangian flows F^1, F^2 commute, i.e. (4.14) holds. Representing now Ω as a disjoint union of balls $\Omega := \sqcup_j B_j$ such that $\bar B_j \subset \Omega$ for all $j \in \mathbb{N}$, one has that (4.14) holds for a.e. $x \in M_{B_j,1/k}$ with $T = T(B_j,1/k)$, and since for all j the sets $\{M_{B_j,1/k}\}_{k\in\mathbb{N}}$ cover almost all B_j, the claim follows. □

Remark 4.4. If under conditions of Theorem 4.3 one has additionally that $V^1 \in L^\infty_{\mathrm{loc}}(\Omega)$, then one can assume that $T \in L^\infty_{loc}(\Omega)$. In fact, for every $x_0 \in \Omega$, $R > 0$, $\rho > R$ as in the above proof, letting $\rho_1 \in (R,\rho)$ apply Lemma 4.5 with V^1 instead of V and ρ_1 instead of ρ to get a number $T_1 > 0$ such that $F^1_s(x)$ is defined for $s \in [-T_1,T_1]$ and $x \in B_R(x_0)$ and maps almost all of $B := B_R(x_0)$ into $B_{\rho_1}(x_0)$ for every $s \in [0,T_1]$. Applying again Lemma 4.5 now with V^2 in place of V and ρ_1 in place of R we get the existence of a $T_2 > 0$ be such that for all $x \in B_{\rho_1}(x_0)$ the classical flow induced by V^2 (which is equal to the regular Lagrangian flow F^2) is defined for all $t \in [-T_2,T_2]$ and stays inside $B_\rho(x_0)$. We set then $T := T_1 \wedge T_2$, so that clearly $T = T(B)$. One has then that the regular Lagrangian flows $(F^1_t)_{t\in[-T,T]}$, $(F^2_t)_{t\in[-T,T]}$ defined for a.e. $x \in B$ commute, i.e., (4.14) holds for every s, $t \in [-T,T]$. The claim follows by representing Ω as a disjoint union of balls $\Omega := \sqcup_j B_j$ such that $\bar B_j \subset \Omega$ for all $j \in \mathbb{N}$: in fact, every relatively compact subset of Ω is covered by only a finite number of B_j.

The following lemma has been used in the above proof.

Lemma 4.5. *Let $\Omega \subset \mathbb{R}^d$ be an open set and the vector field $V \in L^\infty_{\mathrm{loc}}(\Omega;\mathbb{R}^d)$ be such that the respective regular Lagrangian flow $(t,x) \mapsto F_t(x)$ is defined for all $t \in (-\bar T,\bar T)$ whenever $x \in B$, $\bar B \subset \Omega$, for some $T = T(B) > 0$. Then for every $B_\rho(x_0)$, and $0 < R < \rho$ there is a $T \in (0,\bar T(B_R(x_0)))$ such that $F_t(x) \in B_\rho(x_0)$ for a.e. $x \in B_R(x_0)$ and all $t \in [-T,T]$.*

Proof. Take a positive

$$T < \frac{\rho - R}{\|V\|_{L^\infty(B_\rho(x_0))}} \wedge \bar{T}(B_R(x_0)).$$

We have then that F_t maps almost all of $B_R(x_0)$ into $B_\rho(x_0)$ for every $t \in [-T, T]$. We show this for an arbitrary $t \in [0, T]$, the remaining case being completely symmetric. In fact, otherwise there is an $x \in B_\rho(x_0)$ and $y \in C([0, \bar{t}]; \mathbb{R}^d)$, $y(t) := F_t(x)$ such that $y(t) \in B_\rho(x_0)$ for all $t \in [0, \bar{t})$, $|y(\bar{t}) - x_0| = \rho$ and $\bar{t} \leq T$. But

$$F_t(x) = x + \int_0^t V(x(\tau))\mathsf{d}\tau,$$

so that

$$|F_{\bar{t}}(x) - x_0| \leq R + \bar{t}\|V\|_{L^\infty(B_\rho(x_0))} \leq R + T\|V\|_{L^\infty(B_\rho(x_0))} < \rho,$$

which is impossible because $|F^1_{\bar{t}}(x) - x_0| = |y(\bar{t}) - x_0| = \rho$ by assumption. □

Appendix A. Concentration on integral curves

We prove here our main technical tool, namely, a criterion of when a measure over continuous curves is concentrated over on integral curves of a given vector field V. Let us remark that it is based on a discrete formulation of the equation for integral curves.

Proposition A.1. *Let p_0, $p_1 \in (1, +\infty)$ with $1/p_0 + 1/p_1 = 1/q \leq 1$. Assume that $V \in L^1\left([a, b]; L^{p_0}(\mathbb{R}^d)\right)$, $DV \in L^1\left([a, b]; L^{p_1}(\mathbb{R}^d)\right)$, where $a < 0 < b$ and η be a finite measure on $C([a, b]; \mathbb{R}^d)$ with $e_{t\#}\eta \leq C\mathcal{L}^d$ for all $t \in [a, b]$. Then, η is concentrated on integral curves of V, if and only if there exists a variation function $(\omega_{st})_{st\in[a,b]}$ with $[\omega] = 0$ such that*

$$\left\|\theta_t - \theta_s - \left(\int_s^t V_\tau \mathsf{d}\tau\right)(\theta_s)\right\|_{L^q(C([a,b];\mathbb{R}^d),\eta)} \leq \omega_{st} \quad \text{for every } s, t \in [a, b]. \tag{A.1}$$

In such a case one can always choose

$$\omega_{st} \lesssim C^{1/q}\left(\int_s^t \|V_\tau\|_{p_0}\,\mathsf{d}\tau\right)\left(\int_s^t \|DV_\tau\|_{p_1}\,\mathsf{d}\tau\right). \tag{A.2}$$

Remark A.2. If (A.1) holds, then we can always represent $\eta = F_\#(e_{0\#}\eta)$, where F denotes the regular Lagrangian flow for V.

Proof. Assume first that η is concentrated on integral curves of V. Then

$$\|\theta_t - \theta_s\|_{p_0} = \left\|\int_s^t V_\tau(\theta_\tau)\mathsf{d}\tau\right\|_{p_0} \leq C^{1/p_0}\int_s^t \|V_\tau\|_{p_0}\,\mathsf{d}\tau. \tag{A.3}$$

It follows that

(A.4)
$$\begin{aligned}
&\left\| \int_s^t V_\tau(\theta_r) - \left(\int_s^t V_\tau \mathsf{d}\tau \right)(\theta_s) \right\|_{L^q(C([a,b];\mathbb{R}^d),\eta)} \\
&\leq \int_s^t \|V_\tau(\theta_\tau) - V_\tau(\theta_s)\|_{L^q(C([a,b];\mathbb{R}^d),\eta)} \, \mathsf{d}\tau \\
&\lesssim \int_s^t \| |\theta_\tau - \theta_s| \left(|DV_\tau|^*(\theta_\tau) + |DV_\tau|^*(\theta_s) \right) \|_{L^q(C([a,b];\mathbb{R}^d),\eta)} \, \mathsf{d}\tau \qquad \text{by (2.4)}, \\
&\leq \int_s^t \|\theta_\tau - \theta_s\|_{L^{p_0}(C([a,b];\mathbb{R}^d),\eta)} \, \| |DV_\tau|^*(\theta_\tau) + |DV_\tau|^*(\theta_s) \|_{L^{p_1}(C([a,b];\mathbb{R}^d),\eta)} \, \mathsf{d}\tau \\
&\lesssim \int_s^t C^{1/p_1} \|DV_\tau\|_{p_1} \, C^{1/p_0} \int_s^\tau \|V_u\|_{p_0} \, \mathsf{d}u \mathsf{d}\tau \quad \text{by (A.3) and (2.2)}, \\
&\leq C^{1/q} \int_s^t \|DV_\tau\|_{p_1} \, \mathsf{d}\tau \int_s^t \|V_\tau\|_{p_0} \, \mathsf{d}\tau
\end{aligned}$$

as claimed. Conversely, assume that (A.1) holds. Then

$$\|\theta_t - \theta_s\|_{L^{p_0}(\eta)} \leq \omega_{st} + \left\| \int_s^t V_\tau(\theta_\tau) \mathsf{d}\tau \right\|_{L^{p_0}(\eta)} \leq \omega_{st} + C^{1/p_0} \int_s^t \|V_\tau\|_{p_0} \, \mathsf{d}\tau. \tag{A.5}$$

Applying the triangle inequality along any partition π of $[s,t]$, we obtain that

$$\|\theta_t - \theta_s\|_{L^{p_0}(\eta)} \leq \sum_{t_i \in \pi} \omega_{t_i t_{i+1}} + C^{1/p_0} \int_s^t \|V_\tau\|_{p_0} \, \mathsf{d}\tau.$$

Therefore, (A.5) self-improves to (A.3) and we can argue then exactly as in (A.4). It follows that

$$\left\| \theta_t - \theta_s - \int_s^t V_\tau(\theta_\tau) \mathsf{d}\tau \right\|_{L^q(C([a,b];\mathbb{R}^d),\eta)} \leq \omega_{st} + C^{1/q} \int_s^t \|DV_\tau\|_{p_1} \, \mathsf{d}\tau \int_s^t \|V_\tau\|_{p_0} \, \mathsf{d}\tau.$$

This entails that $f(t) := \theta_t - \theta_0 - \int_0^t V_\tau(\theta_\tau) \mathsf{d}\tau$ has zero total variation, i.e., it must be η-a.e. constant (and null since $f(0) = 0$). □

As curious consequence of Proposition A.1, we obtain the following quantitative stability estimate for regular Lagrangian flows, which is of interest itself though not used elsewhere in this paper.

In particular it quantifies the qualitative statement that weak convergence of vector fields in time and strong convergence in space leads to convergence of the respective flows, see e.g. [8, remark 2.11] or [10] for a Trotter-type formula.

To state it we write

$$\Phi^\delta(x) := \log(1 + \delta^{-1}|x|)$$

for $x \in \mathbb{R}^d$, $\delta > 0$, and notice that, for every x, $y \in \mathbb{R}^d$,

$$\Phi^\delta(y) \leq \Phi^\delta(x) + \frac{|y-x|}{\delta + |x|}. \tag{A.6}$$

Indeed,

$$\begin{aligned}
\log(1 + \delta^{-1}|y|) &\leq \log(1 + \delta^{-1}(|x| + |y|)) = \log\left((1 + \delta^{-1}|x|) \left(1 + \frac{|y-x|}{\delta + |x|} \right) \right) \\
&\leq \log\left(1 + \delta^{-1}|x|\right) + \frac{|y-x|}{\delta + |x|}.
\end{aligned}$$

Corollary A.3. *For $i = 1, 2$ let*

$$V^i \in L^1\left([a,b]; L^{p_0}(\mathbb{R}^d)\right), \quad DV^i \in L^1\left([a,b]; L^{p_1}(\mathbb{R}^d)\right),$$

with p_0, p_1, a, b as in Proposition A.1, η^i be finite measures on $\Theta := C([a,b];\mathbb{R}^d)$ with $e_{t\#}\eta^i \le C_i\mathcal{L}^d$ for all $t \in [a,b]$ concentrated on integral curves of V^i. Then, for every $\delta > 0$ one has

$$\left\| \sup_i \Phi^\delta(\theta^1_{t_i} - \theta^2_{t_i}) \right\|_{L^q(\Theta\times\Theta,\eta)} \lesssim \left\|\Phi^\delta(\theta^1_0 - \theta^2_0)\right\|_{L^q(\Theta\times\Theta,\eta)} + \sum_i \left\| D\int_{t_i}^{t_{i+1}} V^1_\tau \mathsf{d}\tau \right\|_{p_1}$$
$$+ \frac{1}{\delta}\sum_i \left\| \int_{t_i}^{t_{i+1}} (V^1_\tau - V^2_\tau)\mathsf{d}\tau \right\|_{p_0} + \frac{1}{\delta}\sum_i \left(\omega^1_{t_it_{i+1}} + \omega^2_{t_it_{i+1}}\right),$$

whenever $a = t_0 < t_1 \ldots < t_n \le b$, where ω^i_{st} are as in (A.2) (with C_i, V^i instead of C, V), and η is any coupling between η^1 and η^2 (i.e. a Borel measure over $\Theta\times\Theta$ with marginals η^1 and η^2).

Proof. Write

$$R^i_{st} := \theta^i_t - \theta^i_s - \left(\int_s^t V_\tau \mathsf{d}\tau\right)(\theta^i_s),$$

so that $\left\|R^i_{st}\right\|_q \le \omega^i_{st}$. We use (A.6) with $y = \theta^1_t - \theta^2_t$, $x = \theta^1_s - \theta^2_s$, obtaining

$$\Phi^\delta(\theta^1_t - \theta^2_t) \le \Phi^\delta(\theta^1_s - \theta^2_s) + \frac{\left|\int_s^t V^1_\tau \mathsf{d}\tau(\theta^1_s) - \int_s^t V^2_\tau \mathsf{d}\tau(\theta^2_s)\right| + \left|R^1_{st}\right| + \left|R^2_{st}\right|}{\delta + |\theta^1_s - \theta^2_s|}$$
$$\le \Phi^\delta(\theta^1_s - \theta^2_s) + \frac{\left|\int_s^t V^1_\tau \mathsf{d}\tau(\theta^1_s) - \int_s^t V^1_\tau \mathsf{d}\tau(\theta^2_s)\right|}{|\theta^1_s - \theta^2_s|}$$
$$+ \delta^{-1}\left(\left|\int_s^t (V^1_\tau - V^2_\tau)\mathsf{d}\tau(\theta^2_s)\right| + \left|R^1_{st}\right| + \left|R^2_{st}\right|\right).$$

To conclude, it is sufficient to choose $s := t_i$, $t := t_{i+1}$, proceed recursively, and finally take the L^q norms (recall that we are considering finite measures, so that L^p spaces are nested). □

Remark A.4. To see why the above result provides quantitative estimates on convergence of regular Lagrangian flows associated to Sobolev vector fields $(V^n)_n$ weakly converging in time and strongly in space to a field $\bar{V}$, one needs a rate of convergence for the terms

$$\left\| \int_{t_i}^{t_{i+1}} (V^n_\tau - \bar{V}_\tau)\mathsf{d}\tau \right\|_{p_0},$$

as well as the variation functions $\omega^n_{t_it_{i+1}}$, $\bar{\omega}_{t_it_{i+1}}$, e.g. uniform in the chosen sequence of times $\{t_i\}_i$. Once this is known, we may choose optimally $\delta = \delta(n) \to 0$ and obtain some quantitative rate for the trajectories, when evaluated at times $\{t_i\}_i$. Choosing properly $\{t_i\}_i$ and using some control on the modulus of continuity (e.g. if the vector fields are bounded) yields rates on the entire trajectories. We do not provide more details here, since this would be outside the scope of this paper.

Appendix B. Auxiliary lemmata

The lemma below provides a useful version of the Lebesgue dominated convergence theorem.

Lemma B.1. *Let μ be a σ-finite Borel measure on a metric space (E,d), let $\{f_k\} \subset L^1(E,\mu)$ be a bounded sequence of nonnegative functions, and $\{g_k\}$ be a sequence of uniformly bounded positive functions. If*

(i) *either* $g_k \to 0$ *in measure* μ, *i.e.,* $\lim_{k\to\infty} \mu(|g_k| > \varepsilon) = 0$ *for every* $\varepsilon > 0$ *and* $\{f_k\}$ *are* $\mu-$*equiintegrable,*
(ii) *or* $g_k \to 0$ *locally in measure* μ, *i.e.,* $\lim_{k\to\infty} \mu(K \cap \{|g_k| > \varepsilon\}) = 0$ *for every* $\varepsilon > 0$ *and* $K \subseteq E$ *compact, and moreover the sequence of measures* $\nu_k = f_k\mu$ *is tight and* $\{f_k\}$ *are uniformly bounded in* $L^\infty(E, \mu)$

then

$$\int_E f_k g_k \mathsf{d}\mu \to 0.$$

Proof. In case (i), for every $\varepsilon > 0$ let $E_{k,\varepsilon} := \{x \in E\colon |g_k(x)| > \varepsilon\}$. Then

$$\begin{aligned}\int_E f_k g_k \mathsf{d}\mu &= \int_{E_{k,\varepsilon}} f_k g_k \mathsf{d}\mu + \int_{E\setminus E_{k,\varepsilon}} f_k g_k \mathsf{d}\mu \\ &\le \sup_k \|g_k\|_{L^\infty(E,\mu)} \int_{E_{k,\varepsilon}} f_k \mathsf{d}\mu + \varepsilon \|f_k\|_{L^1(E,\mu)},\end{aligned}$$

the first term vanishing as $k \to \infty$ because $\lim_k \mu(E_{k,\varepsilon}) = 0$ and in view of equiintegrability of f_k. Since $\varepsilon > 0$ is arbitrary, one gets the claim.

Similarly, in case (ii), for any $\varepsilon > 0$, let $K \subseteq \mathbb{R}^d$ be compact with $\sup_k \nu_k(K^c) < \varepsilon$. Then,

$$\begin{aligned}\int_{\mathbb{R}^d} g_k \mathsf{d}\nu_k &= \int_K g_k f_k \mathsf{d}x + \int_{K^c} g_k f_k \mathsf{d}x \\ &\le \sup_k \|f_k\|_{L^\infty(E,\mu)} \int_K g_k \mathsf{d}x + \sup_k \|g_k\|_{L^\infty(E,\mu)}\, \nu_k(K^c) \\ &\le C\left(\int_K g_k \mathsf{d}x + \varepsilon\right),\end{aligned}$$

for some constant $C > 0$. By Lebesgue dominated convergence theorem one has $\lim_{k\to 0} \int_K g_k \mathsf{d}x = 0$, and hence the thesis since $\varepsilon > 0$ is arbitrary. □

The following lemma gives a chain rule for Sobolev functions along "almost every" integral curve of an ODE.

Lemma B.2. *Let* $p, q \ge 1$ *with* $\frac{1}{p} + \frac{1}{q} \le 1$, *let* $f \in W^{1,p}(\mathbb{R}^d)$, $V \in L^1([a,b]; L^q(\mathbb{R}^d))$ *and* η *be a finite measure on* $C([a,b];\mathbb{R}^d)$ *with bounded compression and concentrated on integral curves of* V. *Then, for any a.e. representative of* f *one has that for* η*-a.e.* θ *the curve* $t \mapsto f(\theta_t)$ *is absolutely continuous and*

$$\partial_t f(\theta_t) = V(\theta_t)\nabla f(\theta_t) \quad \text{for a.e. } t \in [a,b].$$

Proof. We approximate f with a fast converging sequence $(f_n)_{n\ge 1} \subseteq C^1(\mathbb{R}^d)$, i.e., such that

$$\sum_{n=1}^\infty \|f - f_n\|_p + \|\nabla f - \nabla f_n\|_p < \infty.$$

The thesis clearly holds for every f_n in place of f. We have

$$\|f_n \circ e_t - f \circ e_t\|_{L^p(\eta)} \lesssim \|f_n - f\|_p \quad \text{for every } t \in [a,b],$$

so that for every $t \in [a,b]$, η-a.e., $\lim_{n\to\infty} f_n(\theta_t) = f(\theta_t)$. In fact the limit holds η-a.e. and uniformly with respect to $t \in [a,b]$. To show this, we notice that

$$\begin{aligned}\left\| \sup_{t\in[a,b]} (f_{n+1} - f_n) \circ e_t \right\|_{L(\eta)} &\leq \|(f_{n+1} - f_n) \circ e_0\|_{L^1(\eta)} \\ &\quad + \left\| \int_a^b |\nabla f_{n+1} - \nabla f_{n+1}||V_\tau| \circ e_\tau \mathsf{d}\tau \right\|_{L^1(\eta)} \\ &\lesssim \|f_{n+1} - f_n\|_p \\ &\quad + \int_a^b \|\nabla f_{n+1} - \nabla f_n\|_p \|V_\tau\|_q \, \mathsf{d}\tau.\end{aligned}$$

The last quantity is summable with respect to $n \geq 1$. It follows that η-a.e. the sequence $(f_n(\theta_s)_{s\in[a,b]}$ is Cauchy, hence convergent in $C([a,b])$, towards $(f(\theta_s))_{s\in[a,b]}$, that is in particular η-a.e. continuous.

To argue that it is indeed absolutely continuous, we use the bound so that, η-a.e., for every $t \in [a,b]$,

$$\lim_{n\to\infty} \int_0^t \nabla f_n(\theta_\tau) V_\tau(\theta_\tau) \mathsf{d}\tau = \int_s^t \nabla f(\theta_\tau) V_\tau(\theta_\tau) \mathsf{d}\tau.$$

To conclude, we simply pass to the limit in the η-a.e. identity

$$f_n(\theta_t) - f_n(\theta_0) = \int_a^t \nabla f_n(\theta_\tau) V_\tau(\theta_\tau) \mathsf{d}\tau. \qquad \square$$

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