AN INTEGRABILITY RESULT FOR L^p-VECTORFIELDS IN THE PLANE

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ABSTRACT. We prove that if $p > 1$ then the divergence of a L^p-vectorfield V on a 2-dimensional domain Ω is the boundary of an integral 1-current, if and only if V can be represented as the rotated gradient $\nabla^\perp u$ for a $W^{1,p}$-map $u : \Omega \to S^1$. Such result extends to exponents $p > 1$ the result on distributional Jacobians of Alberti, Baldo, Orlandi [ABO03].

1. INTRODUCTION

Consider a vectorfield $V \in L^p(B^2, \mathbb{R}^2)$. If $\mathrm{div}V = 0$ then by the Poincaré Lemma we know that there exists a $W^{1,p}$-function ψ with

$$V = \nabla^\perp \psi. \tag{1.1}$$

The next case in which the situation is relatively standard, is when (in the sense of distributions)

$$\mathrm{div}V = 2\pi \sum_{i=1}^{N} n_i \delta_{x_i}, \quad \text{for some } n_i \in \mathbb{Z} \setminus \{0\} \text{ and } x_i \in B^2. \tag{1.2}$$

In this case we cannot have $V \in L^p$ unless $p < 2$ (consider the model case $V(x) = \frac{x}{|x|^2}$, corresponding to $N = 1, x_1 = (0,0), n_1 = 1$ in (1.2)). The representation (1.1) holds then just locally outside the points x_i, and the local representations do not lift to a global one. If $p > 1$ then we obtain that the function ψ is locally harmonic and V is locally holomorphic. Therefore, it is possible to find a representation of the form (1.1) for a function $\psi \in W^{1,p}(B^2, \mathbb{R}/2\pi\mathbb{Z})$, by taking $u = \mathrm{Arg(V)} + C$ for any constant C. Equivalently, one could use the Green function for the laplacian to obtain a harmonic solution of $\nabla g = V$, and then from the regularity of g the existence of ψ would follow. If we now consider the preimage $u^{-1}(y)$ of any regular value $y \in \mathbb{R}/2\pi\mathbb{Z}$ of u, then we see by Sard's theorem that this will be a rectifiable set, and with the orientation corresponding to the vectorfield ∇g, we can also consider this set as an integral current I_u on $\bar{B}^2$. The boundary of this current is precisely the sum of Dirac masses in (1.2) (without the "2π" factor):

$$\partial I_u \llcorner B^2 = \sum_{i=1}^{N} n_i \delta_{x_i} = \frac{1}{2\pi} \mathrm{div}V. \tag{1.3}$$

When passing to the case where we allow $N = \infty$ in (1.2), we have to face the new difficulty that not all the formal infinite sums of Dirac masses can be represented as the distributional divergence of an L^p-vectorfield. The most obvious restriction (depending on the Fubini theorem) is seen as follows. Let Σ

be a closed smooth Jordan curve and consider its perturbations $\Sigma(t), t \in [-\varepsilon, \varepsilon]$ via a family of diffeomorphisms. Then the flux $f(t)$ of V through $\Sigma(t)$ should satisfy again $f \in L^p([-\varepsilon, \varepsilon])$. In particular, it cannot happen that the algebraic sum of the Dirac masses inside Σ stays infinite for a set of times t of positive measure.

If we assume for a moment that a rectifiable 1-current I as in (1.3) exists, the above condition would translate by saying that the mass of the slice of I along $\Sigma(t)$ is a L^p-function of t. In this work we prove a *necessary and sufficient condition* for a representability property like (1.1) to hold. Consider a smooth domain $\Omega \subset \mathbb{R}^2$ or $\Omega = S^2 \simeq \mathbb{C} \cup \{\infty\}$. Our main result is then:

Main Theorem 1 (first version)**.** *Suppose we have a vector field $V \in L^p(\Omega, \mathbb{R}^2)$ with $p > 1$, whose divergence can be represented by the boundary of an integral 1-current I on Ω, i.e.*

$$\frac{1}{2\pi} \int V \cdot \nabla \phi = \langle I, d\phi \rangle \quad \forall \phi \in C_c^\infty(\Omega). \tag{1.4}$$

Then there exists a $W^{1,p}$-function $u : \Omega \to \mathbb{R}/2\pi\mathbb{Z}$ such that $V = \nabla^\perp u$ and $u|_{\partial\Omega}$ has zero degree. Viceversa, for any $u \in W^{1,p}(\Omega, \mathbb{R}/2\pi\mathbb{Z})$ with $\deg(u|_{\partial\Omega}) = 0$, the vector field $\nabla^\perp u$ belongs to L^p and has divergence equal to the bounday of a current in $\mathcal{I}_1(\Omega)$, in the sense of (1.4).

The zero degree condition on $\partial\Omega$ in the above theorem can be removed in the following way. Consider a L^p-vectorfield V such that

$$\frac{1}{2\pi}\mathrm{div} V = \partial I + \sum_{i=1}^N n_i \delta_{x_i} \quad \text{for some } n_i \in \mathbb{Z} \setminus \{0\} \text{ and } x_i \in \Omega. \tag{1.5}$$

Then we can find, via the Green function method sketched in the introduction, a vectorfield V' satisfying (1.2) and a function $\psi' \in W^{1,p}(\Omega, \mathbb{R}/2\pi\mathbb{Z})$ satisfying (1.1), with

$$\deg(\psi'|_{\partial\Omega}) = \sum_{i=1}^N n_i$$
$$\frac{1}{2\pi}\mathrm{div}(V - V') = \partial I,$$

and we can apply the Main Theorem to $V - V'$ obtaining a function $\psi \in W^{1,p}(\Omega, \mathbb{R}/2\pi\mathbb{Z})$ with degree zero on $\partial\Omega$ and which satisfies $\nabla^\perp \psi = V - V'$. Then $\psi + \psi'$ will satisfy

$$\nabla^\perp(\psi + \psi') = V$$
$$\deg((\psi + \psi')|_{\partial\Omega}) = \sum_{i=1}^N n_i.$$

With this construction we obtain the following generalization

Corollary 1.1. *Suppose we have a L^p-vector field V satisfying (1.5). Then there exists a $W^{1,p}$-function $u : \Omega \to \mathbb{R}/2\pi\mathbb{Z}$ such that $V = \nabla^\perp u$ and $u|_{\partial\Omega}$ has degree $\sum_{i=1}^N n_i$. Viceversa, for any $u \in W^{1,p}(\Omega, \mathbb{R}/2\pi\mathbb{Z})$ with $\deg(u|_{\partial\Omega}) = d \in \mathbb{Z}$, the vector field $\nabla^\perp u$ belongs to L^p and satisfies (1.5), where $d = \sum n_i$.*

In the case $p = 1$, a result similar to the Main Theorem above is a subcase of the result of [ABO03]. An equivalent statement of such result is (see also Section 1.1 where different notations are proposed):

Proposition 1.2 ([ABO03])**.** *For each integral* 1*-current* I *of finite mass on* Ω *there exists a map* $\psi \in W^{1,1}(\Omega, \mathbb{R}/2\pi\mathbb{Z})$ *such that (in the sense of distributions)*

$$\partial I = \frac{1}{2\pi}\mathrm{div}(\nabla^\perp \psi).$$

The distribution $\mathrm{div}(\nabla^\perp\psi)$ is called *distributional Jacobian of* ψ.

Remark 1.3. *As seen in Example 7.1, for* $p > 1$*, unlike the case* $p = 1$*, a large subclass of the boundaries of integral currents is* not *realized as a distributional Jacobian of any map in* $W^{1,p}(B^2, S^1)$*, therefore we must ask for a higher integrability condition for the current* I*: this is why the existence of the* L^p*-vectorfield* V *is imposed.*

1.1. **Different formulations of the Main Theorem.** We have at least three ways of looking at the manifold S^1, namely:

(1) as a submanifold of $\mathbb{R}^2$: $S^1 = \{(x, y) \in \mathbb{R}^2 : \ x^2 + y^2 = 1\}$,
(2) via a parametrization: $S^1 = \{(\cos(t), \sin(t)) : \ t \in \mathbb{R}\}$,
(3) as a quotient: $S^1 = \mathbb{R}/2\pi\mathbb{Z}$.

When considering $W^{1,p}$-maps on B^2 with values in S^1, these three points of view lead to three possible spaces:

(1) $W_1 = \{u \in W^{1,p}(B^2, \mathbb{R}^2) : \ u_1^2(x) + u_2^2(x) = 1, \text{ a. e. } x \in B^2\}$, which is just the usual definition of $W^{1,p}(B^2, S^1)$,
(2) $W_2 = \{(\cos(\psi), \sin(\psi)) : \ \psi \in W^{1,p}(B^2, \mathbb{R})\}$,
(3) $W_3 = \{u \in W^{1,p}(B^2, \mathbb{R})\}/\sim$, where $u_1 \sim u_2$ if $u_1 - u_2$ is a measurable map with values on $2\pi\mathbb{Z}$ a.e. We denote this space by $W^{1,p}(B^2, \mathbb{R}/2\pi\mathbb{Z})$.

W_1 is isomorphic as a (topological vector space) to W_3 via the diffeomorphism $\phi : \mathbb{R}/2\pi\mathbb{Z} \to S^1, \ t \mapsto (\cos(t), \sin(t))$. Instead, the space W_2 is different from W_1, W_3 because of the following result:

Theorem 1.4 ([Dem90])**.** *If* $1 \le p < 2$ *and* $u \in W^{1,p}(B^n, S^1)$ *then the following statements are equivalent:*

- u *can be strongly approximated by smooth maps* $u_k \in C^\infty(B^n, S^1)$
- $d(u^*\theta) = 0$ *in the sense of distributions*
- *There exists* $\tilde{u} \in W^{1,p}(B^n, \mathbb{R})$ *such that* $u = (\cos(\tilde{u}), \sin(\tilde{u}))$.

In our work, the space W_3 seems notationally lighter, but since W_1 is more common, we would like to reformulate the Main Theorem here:

Main Theorem 2 (second version)**.** *Let* $V \in L^p(\Omega, \mathbb{R}^2)$ *with* $p > 1$ *be a vectorfield satisfying* (1.4) *for an integral* 1*-current* I*. Then there exist a map* $u \in W^{1,p}(\Omega, S^1)$ *with degree zero on* $\partial\Omega$ *such that* $V = u_2\nabla^\perp u_1 - u_1\nabla^\perp u_2$*. Viceversa, for any map* $u \in W^{1,p}(\Omega, S^1)$ *with zero degree on the boundary, the vectorfield* $u_2\nabla^\perp u_1 - u_1\nabla^\perp u_2$ *is in* L^p *and has divergence equal to the boundary of an integral current.*

We describe how to pass from the first to the second version of the Main Theorem in Section 4.1.

Our result can be reformulated in somewhat more geometrical terms by identifying differential forms $\alpha \in L^p(\Omega, \wedge^1\Omega)$ with vectorfields [1] $V_\alpha \in L^p(\Omega, \mathbb{R}^2)$ by setting $V_\alpha = (\alpha_2, -\alpha_1)$ if $\alpha = \alpha_1\, dx + \alpha_2\, dy$, so that $d\alpha$ corresponds to $\mathrm{div} V_\alpha$. We also observe that if we consider the tangent space of $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ to be identified with $\mathbb{R}$ in the canonical way, then $V_{u^*\theta}$ can be identified with $\nabla^\perp u$. We obtain therefore the following alternative formulation:

Main Theorem 3 (third version)**.** *Let $p > 1$, let Ω be either a regular open domain in $\mathbb{R}^2$ or the sphere S^2, and let θ be the volume form of S^1. Then the following equality holds*

$$\{u^*\theta :\ u \in W^{1,p}(\Omega, S^1), \deg(u|_{\partial\Omega}) = 0\}$$
$$=$$
$$\{\alpha :\ \alpha \in L^p(\Omega, \wedge^1\mathbb{R}^2), \exists I \in \mathcal{I}_1(\Omega),\ [d\alpha] = \partial I\},$$

where $\mathcal{I}_1(\Omega)$ represents the finite mass integral rectifiable 1-currents on Ω and $[d\alpha]$ is the distribution associated to $d\alpha$ by imposing

$$\langle [d\alpha], \varphi\rangle = \int_\Omega d\alpha \wedge \varphi \quad \forall \varphi \in \mathcal{D}_0(\Omega).$$

1.2. **Ingredients of the proof.** The proof of the first part of our theorem follows from a density result: We prove that the class of L^p-vectorfields with finitely many topological singularities is dense in the class of vectorfields satisfying the condition (1.4). This fact is proved in Section 3, and the proof is in the spirit of the work [Bet91] of Bethuel (see also [Bet90, BCDH91, BCL86, HL03] for related results), inspired by the ideas present in [KR08] and in [Kes08]. Since it is easy to approximate vectorfields with finitely many singularities, we can pass to the limit the $W^{1,p}$-maps obtained in that case in order to achieve the representation result in the first part of the Main Theorem (see Section 4).

The second part of the theorem is a direct consequence of a coarea formula (see for example [MSZ03]), which is related to the Sard theorem for Sobolev spaces (for which see among others [BHS05, dP01, Fig08]). We state here just the result that we need:

Theorem 1.5. *If $f \in W^{1,p}_{loc}(M^m, N^n)$ for some manifolds M, N, then there exists a Borel representative of f such that $f^{-1}(y)$ is countably $(m-n)$-rectifiable and has finite $\mathcal{H}^{m-n}$-measure for almost all $y \in N$ and such that for every measurable function g there holds*

$$\int_M g(x)|J_f(x)|d\mathcal{H}^m(x) = \int_N \left(\int_{f^{-1}(y)} g(x) d\mathcal{H}^{m-n}(x)\right) d\mathcal{H}^n(y), \tag{1.6}$$

[1] This is a special instance of the identification of k-covectors β with $(n-k)$-vectors $*\beta$ in an n-dimensional oriented manifold M given by imposing

$$\langle \alpha, *\beta\rangle = \langle \alpha \wedge \beta, \vec{M}\rangle$$

for all $(n-k)$-covectors α, where $\vec{M}$ is an orientating vectorfield of M.

where $|J_f(x)| = \sqrt{\det(Df_x \cdot Df_x^T)}$.

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3. A density result

We start with a density result concerning vector fields as in the Main Theorem. We consider two classes of vector fields:

$$\mathcal{V}_{\mathbb{Z}} := \{V \in L^p(D, \mathbb{R}^2) : \text{ (1.4) holds}\},$$

and

$$\mathcal{V}_R := \{V \in \mathcal{V}_{\mathbb{Z}} : V \text{ is smooth outside a finite set } S \subset D\}.$$

Since $\mathcal{V}_{\mathbb{Z}}$ is closed in L^p, it is clear that $\overline{\mathcal{V}_R}^{L^p} \subset \mathcal{V}_{\mathbb{Z}}$. We want to prove the following result:

Proposition 3.1. *With the above notations,* $\overline{\mathcal{V}_R}^{L^p} = \mathcal{V}_{\mathbb{Z}}$ *holds.*

By the remarks about $\mathcal{V}_R$ and $\mathcal{V}_{\mathbb{Z}}$, we just have to prove that any $V \in \mathcal{V}_{\mathbb{Z}}$ can be approximated up to an arbitrary small error $\varepsilon > 0$ in L^p-norm, by some $V_\varepsilon \in \mathcal{V}_R$. The strategy of our proof is by first choosing a "grid of circles of radius r", on which we mollify appropriately V, and then to extend the mollified vector field inside each circle by creating finitely many singularities (which may however become unboundedly many as we let $r \to 0$), and by staying L^p-near the initial V. Finally, we will patch together the extensions on each of the balls bounded by these circles, obtaining the wanted approximant V_ε. The way in which we "fill the r-balls" will be by either radial or harmonic extension: we decide the method to apply depending on the degree of V_m on the respective ball (we are guided in this by the result of Demengel [Dem90] cited in Theorem 1.4).

3.1. Choice of a good covering.

Lemma 3.2. *Given* $r > 0$*, there exists a natural number* N*, a set of centers* $\{x_1, \dots, x_N\}$ *and a positive measure subset* $E \subset [3/4r, r]^N$ *such that for all* $(r_1, \dots, r_N) \in E$

- *The balls* $\{B_1, \dots, B_N\}$*, where* $B_i = B_{r_i}(x_i)$ *cover* B^2*.*
- *The smaller balls* $B_{\frac{3}{8}r_i}(x_i)$ *are disjoint.*
- *For some constant depending only on* p *and on the dimension, there holds*

$$\sum_{i=1}^{N} \int_{\partial B_i} |V \cdot n_{B_i}|^p dx \leq C_{2,p} r^{p-2} ||V||^p_{L^p(B^2)}, \tag{3.1}$$

where n_{B_i} *is the outer normal to the ball* B_i*.*

Proof. See Section 5 □

The next lemma will be needed in order to translate properties of the current I to the vector field V.

Lemma 3.3. *Given a piecewise smooth domain $\Omega \subset B^2$, for almost all $t \in [-\varepsilon, \varepsilon]$ the following properties hold:*

- *The slice $\langle I, \mathrm{dist}_{\partial\Omega}, t\rangle$ exists and is a rectifiable 0-current with multiplicity in $2\pi\mathbb{Z}$.*
- *The map $\int_{\partial\Omega_t} V(y)\cdot n_t(y) d\mathcal{H}^1(y)$ (where n_t is the unit normal to $\partial\Omega_t$) is well-defined and coincides with the number $\langle I, \mathrm{dist}_{\partial\Omega}, t\rangle(1) \in 2\pi\mathbb{Z}$.*

Proof. See Section 6 □

Combining the Lemmas 3.2 and 3.3 the following follows:

Lemma 3.4. *Given $r > 0$, there exists a set of balls $\{B_1, \dots, B_N\}$ with radiuses in $[3/4r, r]$ such that the thesis of Lemma 3.2 holds and that for any Ω which is the closure of a connected component of $B^2 \setminus \cup_{i=1}^N \partial B_i$ the slice $\langle I, \mathrm{dist}_{\partial\Omega}, 0\rangle$ exists, is a rectifiable 0-current with multiplicity in $2\pi\mathbb{Z}$ and*

$$\langle I, \mathrm{dist}_{\partial\Omega}, 0\rangle(1) = \int_{\partial\Omega} V(y)\cdot n_\Omega(y) d\mathcal{H}^1(y) \in 2\pi\mathbb{Z}.$$

Proof. We can use Lemma 3.2 first, obtaining a set $E \subset [3/4r, r]^N$. For a cover $\{B_1', \dots, B_N'\}$ corresponding to a density point of E, we can then apply Lemma 3.3 for all the closures of connected components of $B^2 \setminus \cup \partial B_i'$, and then consider the slices for $t \le 0$ only. □

3.2. **Mollification on the boundary and estimates on good and bad balls.**

Lemma 3.5. *For a choice of balls B_i as in Lemma 3.4, it is possible to find a vector field $V_m \in C^\infty(\cup_i \partial B_i, \mathbb{R}^2)$ such that for all the regions Ω as in Lemma 3.4 there holds*

$$\forall i,\ \int_{\partial\Omega} V_m \cdot n_\Omega d\mathcal{H}^1 = \int_{\partial\Omega} V\cdot n_\Omega d\mathcal{H}^1 \in 2\pi\mathbb{Z} \tag{3.2}$$

$$||V_m - V||_{L^p(\cup_i \partial B_i)} \le \varepsilon_m. \tag{3.3}$$

Proof. Suppose V_m satisfies (3.2) and (3.3), but defined only on $\cup_i \partial B_i \setminus \{x : \exists i \neq j, x \in \partial B_i \cap \partial B_j\} := \cup_i \partial B_i \setminus I$. One can then easily modify it on a neighborhood of I in $\cup_i \partial B_i$, defining a global smooth vector field and not affecting the requirements (3.2) and (3.3).
We now find V_m as described above. From Lemma 3.4 it follows that $\sum_i \chi_{\partial B_i} V\cdot n_{B_i} \in L^p(\cup_i \partial B_i)$ and has integral in $2\pi\mathbb{Z}$. Therefore we can take its mollification as a definition of the normal component of V_m, automatically satisfying (3.2) by the properties of the mollification. Then we can mollify the component of V parallel to $\cup \partial B_i$, and take the resulting function as the normal component of V_m, thereby easily verifying (3.3) too. □

Lemma 3.6. *Suppose $\mathcal{B}_n$ are families of finitely many balls which cover B^2 such that each point is not covered more than C times and*

$$\max_{B\in\mathcal{B}_n} \mathrm{diam} B \to 0 \ \ (n\to\infty)$$

Then there holds

$$\sum_{B\in\mathcal{B}_n} \|V-\bar{V}\|_{L^p(B)} \to 0 \ \ (n\to\infty). \tag{3.4}$$

Proof. We take a smooth approximant $W = W_\varepsilon$ such that

$$\|V-W\|_{L^p(B^2)} \le \varepsilon/4C.$$

Then, we can use Poincaré's inequality

$$\|W-\bar{W}\|_{L^p(B)} \le C r_B^{1/p} \|\nabla W\|_{L^p(B)},$$

and for n big enough there will hold

$$\sum_{B\in\mathcal{B}_n} \|W-\bar{W}\|_{L^p(B)} \le \varepsilon/2.$$

Putting together the above two estimates, we obtain

$$\begin{aligned}
&\sum_{B\in\mathcal{B}_n} \|V-\bar{V}\|_{L^p(B)} \le \\
&\quad\le \sum_{B\in\mathcal{B}_n} \|V-W\|_{L^p(B)} + \sum_{B\in\mathcal{B}_n} \|W-\bar{W}\|_{L^p(B)} + \sum_{B\in\mathcal{B}_n} \|\bar{V}-\bar{W}\|_{L^p(B)} \\
&\quad\le 2\sum_{B\in\mathcal{B}_n} \|V-W\|_{L^p(B)} + \varepsilon/2 \\
&\quad\le 2C\|V-W\|_{L^p(B^2)} + \varepsilon/2 \\
&\quad\le \varepsilon,
\end{aligned}$$

as wanted. □

We now distinguish the balls B_i based on the value of the integral $\int_{\partial B_i} V\cdot n_{B_i} d\mathcal{H}^1$: we call B_i a *good* ball in case such integral is zero, and a *bad* ball in case it is in $2\pi\mathbb{Z}\setminus\{0\}$.

Lemma 3.7. *There exists a constant $C>0$ such that if we have a cover as in Lemma 3.2 with radiuses not greater than $r := \varepsilon$, then the number of bad balls satisfies the following estimate:*

$$\#(\textit{bad balls}) \le C\varepsilon^{2p-3}\|V\|_{L^p}^p.$$

Proof. For a bad ball B we have

$$1 \le \left| \int_{\partial B} V\cdot n_B d\mathcal{H}^1 \right|,$$

whence we deduce successively

$$1 \le C\varepsilon^{p-1} \int_{\partial B} |V\cdot n_B| d\mathcal{H}^1$$

and (by summing and using Lemma 3.2)

$$\#(\text{bad balls}) \le C\varepsilon^{p-1} \sum_{B \text{ bad}} \int_{\partial B} |V\cdot n_B| d\mathcal{H}^1 \le C\varepsilon^{2p-3}\|V\|_{L^p}^p,$$

as wanted. □

Remark 3.8. *We observe that by Theorem 1.4, on a good ball the normal component $v_m - v : \partial B_i \to \mathbb{R}^2$, $v_m - v = n_{B_i}[(V_m - V)\cdot n_{B_i}]$ satisfies $v_m - v = \nabla^\perp a_m$ for some $W^{1,p}$-function $a_m : \partial B_i \to \mathbb{R}$.*

The following is a well-known result from the theory of elliptic PDEs.

Lemma 3.9. *Let $\tilde{a}$ be a function on the boundary of the unit 2-ball S^1 having zero mean. Consider the harmonic extension $\tilde{A}$ of $\tilde{a}$ over B_1 satisfying*

$$\begin{cases} \Delta \tilde{A} = 0 \\ \tilde{A} = \tilde{a} \text{ on } S^1 \end{cases} \tag{3.5}$$

Then the following estimate holds:

$$\|\nabla \tilde{A}\|_{L^p(B_1)} \leq C \|\nabla \tilde{a}\|_{L^p(S^1)}. \tag{3.6}$$

We will consider a'_m on the boundary ∂B of a small ball instead of $\tilde{a}$ on ∂B_1, and obtain a harmonic extended function, denoted by A'_m, satisfying the analogous of (3.9). Taking into account the scaling factors we then obtain the following estimate analogous to (3.6) (where r is the radius of B):

$$\|\nabla A'_m\|_{L^p(B)} \leq C r^{1/p} \|v_m - v\|_{L^p(\partial B)}. \tag{3.7}$$

We claim that extending $V_m := \nabla^\perp A'_m + \bar{V}$ inside B, we obtain the wanted approximation:

Lemma 3.10. *If B is a* good *ball of radius ε on whose boundary we have $\|V - V_m\|_{L^p(\partial B)} < \varepsilon$, then the extended smooth vector field V_m defined as above satisfies on B*

$$\|V - V_m\|_{L^p(B)} \leq C\varepsilon^{\frac{p-1}{p}} \|v_m - v\|_{L^p(\partial B)} + \left\|V - \bar{V}\right\|_{L^p(B)}.$$

Proof. We can then write

$$\|V - V_m\|_{L^p(B)} \leq \|V - \bar{V}\|_{L^p(B)} + \|\nabla^\perp A'_m\|_{L^p(B)}.$$

The second term above is estimated as in (3.7), by $C\varepsilon^{1/p}\|v_m - v\|_{L^p(\partial B)}$, and the estimate (3.3) gives then $\varepsilon \|v_m - v\|^p_{L^p(\partial B)} \leq C\varepsilon^{p-1}$, finishing the proof. □

Lemma 3.11. *If $B \subset B_1^2$ is a* bad *ball of radius ε and v_m is the smooth orthogonal vector field on ∂B related to V_m as in Lemma 3.5 and V'_r is the radial extension $V'_r(\theta, \rho) := \frac{\varepsilon}{\rho} v_m(\theta)$ (in polar coordinates centered in the center of B), then with the notation $V_r := V'_r - \bar{V}$, we have the estimate:*

$$\|V - V_r\|_{L^p(B)} \leq \left\|V - \bar{V}\right\|_{L^p(B)} + C\varepsilon.$$

Proof. There holds

$$\begin{aligned} \|V - V_r\|_{L^p(B)} &\leq \left\|V - \bar{V}\right\|_{L^p(B)} + \|V'_r\|_{L^p(B)}, \\ \|V'_r\|^p_{L^p(B)} &= \int_0^\varepsilon \int_0^{2\pi} \left(\frac{\varepsilon}{\rho}\right)^p |v_m(\theta)|^p d\theta \rho d\rho \\ &= C\varepsilon^2 \|v_m\|^p_{L^p(\partial B)}. \end{aligned}$$

From (3.1), (3.3) and the last equality above we conclude that $\|V'_r\|^p_{L^p(B)} \leq C\varepsilon^p$, as wanted. □

3.3. **End of proof of Proposition 3.1.**

3.3.1. *Application of Lemmas 3.10 and 3.11.* We will use the results of Section 3.2 in order to achieve a first global approximation V_1 of V. We again start with the ball B_1, where we will use Lemma 3.10 or 3.11, respectively when B_1 is a good or a bad ball. The new vector field V_1 obtained by replacing V with the so obtained local approximant satisfies the following properties:

- **Good approximation of V on B_1:** The approximation error in L^p-norm on the ball B_1 is bounded above by $C\varepsilon^{1/p} + \left\|V - \bar{V}\right\|_{L^p(B_1)}$.
- **Controlled behavior on the boundary:** The extension inside B_1 is equal to $\nabla^\perp A'_m + \bar{V}$ on the boundaries of the B_i's, and in particular it has degree equal either to the one of V_m or to zero on any of the boundaries of the domains Ω of Lemma 3.5. Indeed, A'_m is smooth, so $V_m|_{B_1}$ will have divergence either zero (for good balls) or a Dirac mass in the center of B_1 (for bad balls), while on $B_1 \setminus \cup_i \partial B_i$, $V_m = V_1$. Therefore V_1 also has the properties stated in Lemma 3.5.

This allows us to apply iteratively the above construction for the balls $B_j, j = 2, \dots, N$, in order to further modify V_1 obtaining successively approximants $V_2, \dots, V_N$ according to Lemmas 3.10, 3.11, and we are able to continue ensuring the smallness condition $\|V - V_m\|_{L^p(\partial B_j)}$.

Lemma 3.12. *For each $\bar{\varepsilon} > 0$ there exist a radius bound ε and an approximation error bound ε_m (in Lemma 3.5) such that the approximant V_N constructed above satisfies*

$$\|V - V_N\|_{L^p(B^2)} \leq \bar{\varepsilon}.$$

Proof. By Lemmas 3.10 and 3.11 we can estimate

$$\begin{aligned}
\|V - V_N\|_{L^p(B^2)} &\leq \sum_{\text{good } B} \left[C\varepsilon^{\frac{p-1}{p}} \|v_m - v\|_{L^p(\partial B)} + \left\|V - \bar{V}\right\|_{L^p(B)} \right] \\
&\quad + \sum_{\text{bad } B} \left[\left\|V - \bar{V}\right\|_{L^p(B)} + C\varepsilon \right] \\
&= \sum_{\text{all } B} \left\|V - \bar{V}\right\|_{L^p(B)} + C\varepsilon \#(\text{bad balls}) + C\varepsilon^{\frac{p-1}{p}} \varepsilon_m.
\end{aligned}$$

Consider now the expression in the last row above: the first term converges to zero by Lemma 3.6, and the last one is small for ε_m small. The middle term can be estimated using Lemma 3.7 and has thus a bound of the form $C\varepsilon^{2p-2}\|V\|^p_{L^p}$. Since $p > 1$ and $V \in L^p$, also this term is small for ε small. □

3.3.2. *Smoothing on the boundary.* The preceding iteration procedure gives us an L^p-approximant with error $C\varepsilon$ if the radius r of the balls was chosen to be equal to ε. Moreover it is easy to verify that

$$\mathrm{div} V_N = \sum_{i=1}^{N} \delta_{x_i} \int_{\partial B_i} V_m, \quad \text{locally outside } \cup_i \partial B_i \tag{3.8}$$

where x_i is the center of B_i. The resulting vector field V_N is however not in $\mathcal{V}_R$: for instance, it is not smooth on all of $\cup_i \partial B_i$. We will thus smoothen V_N as

follows. We observe that locally near $\cup_i \partial B_i$ on $B^2 \setminus \cup_i \partial B_i$, V_N is represented as $\nabla^\perp A_i := \nabla^\perp A_i' + \bar{V}_i$, where A_i' is smooth and $\bar{V}_i$ is a constant equal to the average of V on a particular B_i. We can take an open cover by small balls of a neighborhood of $\cup_i \partial B_i$ then mollify the functions A_i inside each of these small balls, then use a partition of unity to patch the mollifications into a single smooth function A_ε without losing more than an error of ε in L^p-norm. Then we can safely define $V_\varepsilon := \nabla^\perp A_\varepsilon$. $\square$

4. Proof of Theorem 1

Proof. We first show how to deduce the second part of Main Theorem 1 from Proposition 3.1.

The main idea is that, by Proposition 3.1, we can take a sequence $V_n \overset{L^p}{\to} V$ which belongs to $\mathcal{V}_R$ and construct u_n's such that $V_n = \nabla^\perp u_n$, and they will be constrained to converge to a u with the wanted property $\nabla^\perp u = V$. We remark that if V_n is smooth and divergence-free outside a discrete set Σ, then $V_n^\perp$ is locally holomorphic, and the fact that the divergence around any point of Σ is a Dirac mass with coefficient in $2\pi\mathbb{Z}$ translates into saying that $V_n^\perp$ has degree equal to that coefficient around that point. Consider the divisor D supported on Σ with residue corresponding to the divergence of V_n. Therefore $V_n^\perp$ is a meromorphic function with divisor D, so we can take $u_n := \arg V_n^\perp$, which is well-defined with values in $\mathbb{R}/2\pi\mathbb{Z}$ and satisfies $\nabla u_n = V_n^\perp$.
We have thus functions $u_n \in W^{1,p}(\Omega, S^1)$ satisfying $V_n = \nabla^\perp u_n$ and therefore $\nabla u_n \overset{L^p}{\to} V$. We can change the u_n by a constant so that $\frac{1}{|\Omega|}\int_\Omega u_n = 0 \in \mathbb{R}/2\pi\mathbb{Z}$. Then by Poincaré's inequality we have that u_n form a L^p-Cauchy sequence, converging therefore to $\bar{u} \in L^p(\Omega, \mathbb{R}/2\pi\mathbb{Z})$. After extracting a subsequence $u_n \overset{W^{1,p}}{\rightharpoonup} u \in W^{1,p}$. Since we have a.e.-convergence too, it must hold $u = \bar{u}$ and $\nabla^\perp u = V$, as wanted.

As above, $u_n \overset{W^{1,p}}{\to} u$ and $d(u_n^*\theta)$ are finite sums of Dirac masses with integer coefficients. The fact that for $u \in W^{1,p}(\Omega, \mathbb{R}/2\pi\mathbb{Z})$ the vectorfield $\nabla^\perp u$ has the properties required from the vectorfield V in the theorem follows from Theorem 1.5, by taking

$$I = I_z^u = \tau\left(u^{-1}(z), 1, \frac{\nabla^\perp u(x)}{|\nabla^\perp u(x)|}\right),$$

for a common regular value $z \in \mathbb{R}/2\pi\mathbb{Z}$ of all the u_n and of u. Then by the coarea formula (observing that in our case $|J_u| = |\nabla^\perp u|$) we have for all $f \in C_c^\infty(\Omega)$

$$\begin{aligned}\int_\Omega u^*\theta \wedge df &= \int_\Omega \nabla^\perp u \cdot \nabla f dx = \int_{S^1} dy \int_{u^{-1}(y)} \left\langle df, \frac{\nabla^\perp u}{|\nabla^\perp u|}\right\rangle d\mathcal{H}^1 \\ &= \int_{S^1} I_y^u(df)dy = \frac{1}{2\pi}\int_{S^1} \partial I_y^u(f)dy.\end{aligned}$$

Similarly we obtain for all n:

$$\int_\Omega u_n^*\theta \wedge df = \int_{S^1} \partial I_y^{u_n}(f)dy = 2\pi \partial I_z^{u_n}(f),$$

since for u_n with finitely many singularities $\partial I_y^{u_n}(f)$ does not depend on y. We have (since $C_c^\infty \subset (W^{1,p})^*$)

$$\int_{S^1} \partial I_y^{u_n}(f)dy \to \int_{S^1} \partial I_y^{u}(f)dy,$$

We may assume that the integrands converge pointwise in z, proving the condition (1.4), and thus finishing the proof. □

4.1. **Proof of the second version of the Main Theorem.**

Proof. We consider the diffeomorphism $\varphi : \mathbb{R}/2\pi\mathbb{Z} \to S^1 \subset \mathbb{R}^2$ given by $t \mapsto (\cos t, \sin t)$, and then instead of the map $u : \Omega \to \mathbb{R}/2\pi\mathbb{Z}$ obtained in the Main Theorem 1 we take the map $\bar{u} := \varphi \circ u : \Omega \to S^1 \subset \mathbb{R}^2$. We then obtain

$$\begin{aligned} \nabla \bar{u} &= \nabla u \otimes (\nabla \varphi \circ u) \\ &= \begin{pmatrix} -\partial_1 u \sin u & \partial_1 u \cos u \\ -\partial_2 u \sin u & \partial_2 u \cos u \end{pmatrix}, \end{aligned}$$

therefore

$$\bar{u}_1 \nabla^\perp \bar{u}_2 - \bar{u}_2 \nabla^\perp \bar{u}_1 = \cos^2 u \begin{pmatrix} -\partial_2 u \\ \partial_1 u \end{pmatrix} + \sin^2 u \begin{pmatrix} -\partial_2 u \\ \partial_1 u \end{pmatrix} = \nabla^\perp u.$$

This proves the wanted identifications, and we only need to prove that if $\bar{u} \in W^{1,p}(\Omega, S^1)$ then $\bar{u}_1 \nabla^\perp \bar{u}_2 - \bar{u}_2 \nabla^\perp \bar{u}_1 \in L^p(\Omega, \mathbb{R}^2)$. This follows using the relation $\bar{u}_1^2 + \bar{u}_2^2 = 1$ and its consequence $\bar{u}_1 \nabla^\perp \bar{u}_1 = -\bar{u}_2 \nabla^\perp \bar{u}_2$. We have indeed:

$$\begin{aligned} |\bar{u}_1 \nabla^\perp \bar{u}_2 - \bar{u}_2 \nabla^\perp \bar{u}_1|^2 &= \bar{u}_1^2 |\nabla^\perp \bar{u}_2|^2 - 2\bar{u}_1 \bar{u}_2 \nabla^\perp \bar{u}_2 \nabla^\perp \bar{u}_2 + \bar{u}_2^2 |\nabla^\perp \bar{u}_1|^2 \\ &= (\bar{u}_1^2 + \bar{u}_2^2)|\nabla^\perp \bar{u}_2|^2 + (\bar{u}_1^2 + \bar{u}_2^2)|\nabla^\perp \bar{u}_1|^2 \\ &= (\partial_2 \bar{u}_2)^2 + (\partial_1 \bar{u}_2)^2 + (\partial_2 \bar{u}_1)^2 + (\partial_1 \bar{u}_1)^2 \\ &= |\nabla \bar{u}|^2, \end{aligned}$$

and since $u \in W^{1,p}$, this proves the result. □

5. Proof of Proposition 3.2

Our aim here is to prove the following

Proposition 5.1. *Given $r > 0$, there exists a cover of B_1^2 by a finite set of balls $\{B_r(y_1), \dots, B_r(y_N)\}$ such that the balls $B_{r/2}(y_i)$ are disjoint and such that for some constant depending only on p and on the dimension,*

$$\sum_{i=1}^N \int_{\partial B_r(y_i)} |V \cdot n_{B_r(y_i)}|^p dx \leq C_{2,p} r^{p-2} ||V||^p_{L^p(B^2)}, \tag{5.1}$$

where $n_{B_r(y_i)}$ is the outer unit normal vector to the circle $\partial B_r(y_i)$.

Directly form the proof of Proposition 5.1 we can also obtain the more refined result:

Proposition 5.2. *Given $r > 0$, there exists a natural number N, a set of centers $\{x_1, \dots, x_N\}$ and a positive measure subset $E \subset [3/4r, r]^N$ such that for all $(r_1, \dots, r_N) \in E$*

- *The balls $\{B_1, \dots, B_N\}$, where $B_i = B_{r_i}(x_i)$ cover B^2.*
- *The smaller balls $B_{\frac{3}{8}r_i}(x_i)$ are disjoint.*
- *For some constant depending only on p and on the dimension, there holds*

$$\sum_{i=1}^{N} \int_{\partial B_i} |V \cdot n_{B_i}|^p dx \leq C_{2,p} r^{p-2} ||V||^p_{L^p(B^2)}.$$

5.1. **Equivalent definition of the pointwise norm of V.** $\langle V, \theta \rangle$ for a vector $\theta \in S^1 \subset \mathbb{R}^2$, can be expressed as $|V||\cos\gamma|$ where γ is the angle between θ and V. After noting

$$\int_{S^1} |\cos\gamma|^p d\theta =: c_p,$$

we can write

$$|V|^p = \frac{1}{c_p} \int_{S^1} |\langle V, \theta \rangle|^p d\theta. \tag{5.2}$$

We now pass to consider the circle $S_r(x) = \partial B_r(x)$. Then we can write

$$\int_{S_r(x)} V(y) \cdot n_{B_r}(y)\, dy = \int_{S_r(x)} \left\langle V(y), \left(\frac{y-x}{|y-x|} \right) \right\rangle dy = \int_{S^1} \langle V(x + r\theta), \theta \rangle r d\theta.$$

Given a positive number r, a point $x \in \mathbb{R}^2$ then belongs to $S_r(y)$ exactly for $y \in S_r(x)$, and we have by (5.2), that

$$\begin{aligned} \int_{S_r(x)} \left| V(x) \cdot n_{B_r(y)}(x) \right|^p \, dy &= \int_{S_r(x)} \left| \left\langle V(x), \left(\frac{x-y}{|x-y|} \right) \right\rangle \right|^p \, dy \\ &= \int_{S^1} |\langle V(x), \theta \rangle|^p r^p d\theta \\ &= c_p r^p |V(x)|^p. \end{aligned} \tag{5.3}$$

5.2. **Proposition 5.1 and an extension of it.**

Proof of Proposition 5.1: We observe that (5.3) can be integrated on $\mathbb{R}^2$ (after having extended V by zero outside B^2), to give

$$\begin{aligned} c_p r^p \int_{B^2} |V(x)|^p \, dx &= c_p r^p \int_{\mathbb{R}^2} |V(x)|^p \, dx \\ &= \int_{\mathbb{R}^2} \int_{S_r(x)} \left| V(x) \cdot n_{B_r(y)}(x) \right|^p \, dy \, dx \\ &= \int_{\mathbb{R}^2} \int_{S_r(z)} \left| V(x) \cdot n_{B_r(z)}(x) \right|^p \, dx \, dz \\ &= \int_{B^2_{1+r}} \int_{S_r(z)} \left| V(x) \cdot n_{B_r(z)}(x) \right|^p \, dx \, dz. \end{aligned} \tag{5.4}$$

We now define some systems of disjoint balls. We consider a set

$$S = \{x_1, \dots, x_N\} \subset B^2_{1+r} \text{ s.t. } \begin{cases} \min_{1\le i\neq j\le N} d(x_i, x_j) \ge r \\ S \text{ is maximal} \end{cases} \tag{5.5}$$

and the corresponding set of translates of the ball $B_r(0)$.

$$\mathcal{S} := S + B_r(0) = \{\{x_1 + y, \dots, x_N + y\} : \ y \in B_r(0)\}$$

Then $\mathcal{S}$ covers B_{1+r} (by maximality in the definition of S) at most C times, where C is a packing number (by the requirement on the mutual distances of elements of S). We can then bound the integral (5.4) from below as follows

$$\begin{aligned} c_p r^p \int_{B^2} |V(x)|^p \, dx &= \int_{B^2_{1+r}} \int_{S_r(z)} \left|V(x) \cdot n_{B_r(z)}(x)\right|^p \, dx \, dz \\ &\ge \frac{1}{C} \int_{B_r} \left(\sum_{i=1}^N \int_{S_r(x_i+z)} |V \cdot n|^p dy \right) dz \end{aligned}$$

and it follows that there exists $z \in B_r$ such that

$$\sum_{i=1}^N \int_{S_r(x_i+z)} |V \cdot n|^p dy \le \frac{C c_p r^p}{|B_r|} \int_{B^3} |V|^p \, dx = C_{2,p} r^{p-2} ||V||^p_{L^p(B^2)}.$$

This is enough to prove (5.1). Moreover, again by the maximality of S_0, the balls $\{B_r(x_i + z)\}_{i=1}^N$ cover B^2_1, and by the requirement on the distances of the centers in (5.5), the $B_{r/2}(x_i + z)$ are disjoint, proving Proposition 5.1. □

6. Proof of Proposition 3.3

We suppose here that we are given a vector field $V \in L^p(B^2, \mathbb{R}^2)$, for some $p \neq \infty$, such that for some integer multiplicity rectifiable current I we have $\mathrm{div} V = \partial I$. This means more precisely that

$$\int V \cdot \nabla \phi = \langle I, d\phi \rangle, \text{ for all functions } \phi \in C^\infty_0(B^2). \tag{6.1}$$

Here $\langle I, d\phi \rangle$ refers to the action of the current I on the 1-form $d\phi$. If Ω is a piecewise smooth domain, we will also call $\partial\Omega_t$ the set $\{x \text{ s.t. } \mathrm{dist}_{\partial\Omega}(x) = t\}$. By $\mathrm{dist}_{\partial\Omega}$ we here denote the *oriented* distance from $\partial\Omega$, i.e. the function defined on a small neighborhood of $\partial\Omega$ and equal to dist_Ω outside Ω and to $-\mathrm{dist}_{\Omega^c}$ inside Ω. Our aim in this section is to prove the following

Proposition 6.1. *Given a piecewise smooth domain $\Omega \subset B^2$, for almost all $t \in [-\varepsilon, \varepsilon]$ the following properties hold:*

- *The slice $\langle I, \mathrm{dist}_{\partial\Omega}, t\rangle$ exists and is a rectifiable 0-current with multiplicity in $2\pi\mathbb{Z}$.*
- *The map $\int_{\partial\Omega_t} V(y) \cdot n_t(y) d\mathcal{H}^1(y)$ (where n_t is the unit normal to $\partial\Omega_t$) is well-defined and coincides with the number $\langle I, \mathrm{dist}_{\partial\Omega}, t\rangle(1) \in 2\pi\mathbb{Z}$.*

Proof. We consider a family of symmetric mollifiers $\varphi_\varepsilon : \mathbb{R} \to \mathbb{R}^+$ supported in $[-\varepsilon, \varepsilon]$, and their primitives $\chi_\varepsilon(x) := \int_{-\infty}^x \varphi_\varepsilon dt$. We will consider a non negative function g which is C^∞_c-extensions to a neighborhood of $\partial\Omega$ of the constant function equal to 1 on all the Ω_t's with $t \in [-2\varepsilon, 2\varepsilon]$, and we write

the current I as $(\mathcal{M}_I, \theta_I, \tau_I)$, where $\mathcal{M}_I$ is a 1-rectifiable set supporting the current I, τ_I is the orienting vector of I and θ_I is the multiplicity of I. Then the currents approximating the slice $\langle I, f, t\rangle$ (for some Lipschitz function $f : B^2 \to [-2\varepsilon, 2\varepsilon]$), when it exists, satisfy:

$$\begin{aligned} I \llcorner f^{\#}(\varphi_\varepsilon(\cdot - t)d\tau)(g) &= \int_{\mathcal{M}_I} \langle \tau_I(x), g(x)\varphi_\varepsilon(f(x) - t)df_x\rangle d\mathcal{H}^1(x) \qquad (6.2)\\ &= \int_{\mathcal{M}_I} \langle \tau_I(x), g(x)d(\chi_\varepsilon(\cdot - t) \circ f)_x\rangle d\mathcal{H}^1(x)\\ &= \langle I \llcorner g, dF_\varepsilon\rangle \quad \text{where} \quad F_\varepsilon(x) := \chi_\varepsilon(f_\varepsilon(x) - t),\\ &= \langle I, dF_\varepsilon\rangle \quad \text{since} \quad \mathrm{spt} F_\varepsilon \subset \{g = 1\},\\ &= \int_{\{x:\, |f(x)-t|\le\varepsilon\}} V \cdot \nabla F_\varepsilon dx^2 \quad (\text{by (6.1)}). \end{aligned}$$

Now we take $f(x) := \mathrm{dist}_{\partial\Omega}(x)$, obtaining that a.e. on a tubular neighborhood

$$T(\Omega, 2\varepsilon) := \cup_{-2\varepsilon\le t\le 2\varepsilon}\partial\Omega_t,$$

∇f exists, and on each $\partial\Omega_\tau = \{f = \tau\}$ it is a.e. equal to the unit normal vector n_τ. Therefore we have

$$\begin{aligned} \nabla F_\varepsilon(x) &= \nabla(\chi_\varepsilon(\cdot - t) \circ f_\varepsilon)(x)\\ &= \varphi_\varepsilon(\cdot - t) \circ f(x)\nabla f(x) = [\varphi_\varepsilon(\cdot - t) \circ \mathrm{dist}(x, \partial\Omega)]\, n_{\mathrm{dist}_{\partial\Omega}(x)} \end{aligned}$$

and

$$\begin{aligned} \int_{\{|f-t|\le\varepsilon\}} V \cdot \nabla F_\varepsilon dx^2 &= \int_{T(\Omega,2\varepsilon)} \varphi_\varepsilon \circ \mathrm{dist}_{\partial\Omega_t}(x)\, V(x) \cdot \nabla(\mathrm{dist}_{\partial\Omega_t})(x)dx^2\\ &= \int_{-\varepsilon}^{\varepsilon} \varphi_\varepsilon(t) \left(\int_{\partial\Omega_t} V \cdot n_t d\mathcal{H}^1\right) dt. \end{aligned}$$

As in the usual theory of slicing, for almost all t's the currents $I \llcorner f^{\#}(\varphi_\varepsilon(\cdot - t)d\tau)$ converge weakly to the slice $\langle I, f, t\rangle$ as $\varepsilon \to 0$. Similarly, V being in L^p, a dominated convergence argument gives also for almost all $\bar t$ the convergence

$$\int_{-\varepsilon}^{\varepsilon} \varphi_\varepsilon(\tau - t) \left(\int_{\partial\Omega_\tau} V \cdot n_{\partial\Omega_\tau} d\mathcal{H}^1\right) dt \to \int_{\partial\Omega_t} V \cdot n_{\partial\Omega_t} d\mathcal{H}^1. \qquad (6.3)$$

The fact that almost all slices of an integer multiplicity rectifiable current are integer multiplicity rectifiable gives the first point of the Proposition, while the second point follows from (6.2) and (6.3). □

7. Further remarks concerning the Main Theorem

We want first to point out that not all boundaries of rectifiable integral currents ∂I are representable as $u^*\theta$ for $u \in W^{1,p}(\Omega, S^1)$, if $p > 1$, showing that this case is more subtle than the case $p = 1$ treated in Theorem 1.2. To do this, we use the second formulation of the Main Theorem, which says that such $u^*\theta$ would then be equal to $V^\perp$ for some vectorfield $V \in L^p$ satisfying $\mathrm{div} V = \partial I$. We will show that not all integral currents I have ∂I equal to a divergence of a L^p-vectorfield.

Suppose first that we have a vectorfield V on $B_\varepsilon(p)$ satisfying $\mathrm{div}V = \delta_p$ (where δ_p is the Dirac mass in p). Then for almost all $r \in [0, \varepsilon[$ we have

$$\int_{\partial B_r(p)} V \cdot n_{B_r(p)} \, d\mathcal{H}^1 = 1, \tag{7.1}$$

and we see that under the constraint (7.1), the minimal L^p-mass is achieved by the radial (in polar coordinates around p) vectorfield

$$V_{min}(\theta, r) = \frac{1}{2\pi r}\hat{r}$$

(by a rearrangement argument and by the convexity of the L^p-norm for $p > 1$). We therefore obtain (for some geopmetric constant C)

$$\|V\|^p_{L^p(B_\varepsilon(p))} \geq \|V_{min}\|^p_{L^p(B_\varepsilon(p))} = C\varepsilon^{2-p} \tag{7.2}$$

We see that such estimate on the norm of V is only dependent on the fact that $(\mathrm{div}V) \llcorner B_\varepsilon(p) = \delta_p$. We can now use a series of inequalities like (7.2) on a series of (disjoint) balls in order to find our counterexample.

Example 7.1. *Take a sequence of positive numbers $(a_i)_{i\in\mathbb{N}}$ such that*

$$\begin{aligned} \sup_i a_i &= \varepsilon \\ \sum_{i=1}^{\infty} a_i &= 2 \qquad (7.3) \\ \sum_{i=1}^{\infty} a_i^{2-p} &= = +\infty. \qquad (7.4) \end{aligned}$$

It is possible to achieve this for any $\varepsilon > 0$, since $p > 1$.

Now take a 2-dimensional domain Ω. It is possible to find a series of disjoint balls B_i of radiuses a_i for any sequence a_i as above, provided that ε is small enough (because $\mathcal{H}^1(\Omega) = \infty$ and for any set C, $\mathcal{H}^1(C) > 0$ implies $H^{2-p}(C) = \infty$). Inside each B_i one can insert two disjoint balls B_i^+, B_i^- of radius $\frac{a_i}{2}$. Call $x_i^\pm$ the center of $B_i^\pm$, and consider the current

$$I = \sum_{i=1}^{\infty} [x_i^-, x_i^+].$$

Using the estimate (7.4) and the estimates (7.2) on the disjoint balls $B_i^\pm$, we obtain that any vectorfield satisfying $\mathrm{div}V = \partial I$ must not be in L^p. By our Main Theorem (second version), we see that none of the currents constructed in this way can possibly have boundary equal to the distributional Jacobian of a map $u \in W^{1,p}(\Omega, S^1)$.

References

[ABO03] G. Alberti, S. Baldo, and G. Orlandi, *Functions with prescribed singularities*, J. Eur. Math. Soc. (JEMS) **5** (2003), no. 3, 275–311. MR MR2002215 (2004g:49068)

[BCDH91] F. Bethuel, J.-M. Coron, F. Demengel, and F. Hélein, *A cohomological criterion for density of smooth maps in Sobolev spaces between two manifolds*, Nematics (Orsay, 1990), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., vol. 332, Kluwer Acad. Publ., Dordrecht, 1991, pp. 15–23. MR MR1178083 (93g:58020)

[BCL86] Haïm Brezis, Jean-Michel Coron, and Elliott H. Lieb, *Harmonic maps with defects*, Comm. Math. Phys. **107** (1986), no. 4, 649–705. MR MR868739 (88e:58023)

[Bet90] F. Bethuel, *A characterization of maps in $H^1(B^3, S^2)$ which can be approximated by smooth maps*, Ann. Inst. H. Poincaré Anal. Non Linéaire **7** (1990), no. 4, 269–286. MR MR1067776 (91f:58013)

[Bet91] Fabrice Bethuel, *The approximation problem for Sobolev maps between two manifolds*, Acta Math. **167** (1991), no. 3-4, 153–206. MR MR1120602 (92f:58023)

[BHS05] Bogdan Bojarski, Piotr Hajłasz, and Paweł Strzelecki, *Sard's theorem for mappings in Hölder and Sobolev spaces*, Manuscripta Math. **118** (2005), no. 3, 383–397. MR MR2183045 (2007b:58014)

[Dem90] Françoise Demengel, *Une caractérisation des applications de $W^{1,p}(B^N, S^1)$ qui peuvent être approchées par des fonctions régulières*, C. R. Acad. Sci. Paris Sér. I Math. **310** (1990), no. 7, 553–557. MR MR1050130 (91f:46044)

[dP01] Luigi de Pascale, *The Morse-Sard theorem in Sobolev spaces*, Indiana Univ. Math. J. **50** (2001), no. 3, 1371–1386. MR MR1871360 (2002k:58018)

[Fig08] Alessio Figalli, *A simple proof of the Morse-Sard theorem in Sobolev spaces*, Proc. Amer. Math. Soc. **136** (2008), no. 10, 3675–3681. MR MR2415054 (2009b:58018)

[HL03] Fengbo Hang and Fanghua Lin, *Topology of Sobolev mappings. II*, Acta Math. **191** (2003), no. 1, 55–107. MR MR2020419 (2005m:58023)

[Kes08] Thiemo Kessel, *Singular bundles with l^2 bounded curvatures*, Ph.D. thesis, ETH Zürich, 2008.

[KR08] Thiemo Kessel and Tristan Rivière, *Singular bundles with bounded L^2-curvatures*, Boll. Unione Mat. Ital. (9) **1** (2008), no. 3, 881–901. MR MR2455351 (2009m:58026)

[MSZ03] Jan Malý, David Swanson, and William P. Ziemer, *The co-area formula for Sobolev mappings*, Trans. Amer. Math. Soc. **355** (2003), no. 2, 477–492 (electronic). MR MR1932709 (2004a:46037)