Variational problems with percolation: dilute spin systems at zero temperature

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Abstract. We study the asymptotic behaviour of dilute spin lattice energies by exhibiting a continuous interfacial limit energy computed using the notion of Γ-convergence and techniques mixing Geometric Measure Theory and Percolation while scaling to zero the lattice spacing. The limit is not trivial above a percolation threshold. Since the lattice energies are not equi-coercive a suitable notion of limit magnetization must be defined, which can be characterized by two phases separated by an interface. The macroscopic surface tension at this interface is characterized through a first-passage percolation formula, which highlights interesting connections between variational problems and percolation issues. A companion result on the asymptotic description on energies defined on paths in a dilute environment is also given.

Keywords. Dilute spins, lattice energies, first-passage percolation, variational probems, Gamma-convergence.

1 Introduction

The study of continuous limits of spin systems in a variational setting (Γ-convergence) is linked to recent progress in the understanding of phase segregation and the validity of the Wulff construction for Ising-type models through a L^1 approach (see, e.g., [1, 4, 5, 6, 7, 8, 15, 16, 17]). The advantage of this approach is in that it can be implemented in high dimension, even though it provides only L^1 estimates, contrary to other approaches giving sharper controls of the phase boundaries but limited to two dimensions (see eg. [19, 23, 24, 25, 29, 30]).

The variational counterpart (as a zero-temperature approximation) of the L^1 approach is the asymptotic analysis of some lattice energies, which shows phase segregation through the description of their Γ-limit as a surface energy between two phases and the identification of a surface tension

between such phases. In the case of dilute spin systems the energy under examination is of the form

$$-\sum_{ij}\sigma_{ij}^{\omega}u_iu_j,$$

where $u_i \in \{\pm 1\}$ is a spin variable indexed on the lattice $\mathbb{Z}^d$, the sum runs on nearest neighbors in a given portion $\mathbf{D}\cap\mathbb{Z}^d$ of $\mathbb{Z}^d$, the coefficients σ_{ij}^{ω} depend on the realization ω of an i.i.d. random variable, and

$$\sigma_{ij}^{\omega}=\begin{cases}1 & \text{with probability } p\\ 0 & \text{with probability } 1-p\,,\end{cases}$$

with $p\in[0,1]$ fixed. In order to describe the behaviour as the size of $\mathbf{D}$ diverges we introduce a scaled problem, as is customary in the passage from lattice systems to continuous variational problems, in which, on the contrary, $\mathbf{D}$ is kept fixed, but scaled energies are defined as follows. A small parameter $\varepsilon>0$ is introduced, the lattice is scaled accordingly to $\varepsilon\mathbb{Z}^d$, and the energies are scaled (after adding proper random constants and multiplying by 2) to

$$\sum_{ij}\varepsilon^{d-1}\sigma_{ij}^{\omega}(u_i-u_j)^2\,.$$

Note that considering $(u_i-u_j)^2$ in place of $-u_iu_j$ is merely technical and amounts to the translation of the energies so that uniform states (which are pointwise minimizers of the 'integrand') have zero energy; moreover, the 'surface scaling' ε^{d-1} is driven by the knowledge that for $p=1$ (i.e., for ferromagnetic interactions) the Γ-limit with that scaling is not trivial (as shown e.g. by Alicandro, Braides and Cicalese [2]). After this scaling, the sum is taken on nearest neighbors in $\mathbf{D}\cap\varepsilon\mathbb{Z}^d$, and the normalization allows also to consider $\mathbf{D}=\mathbb{Z}^d$.

The corse graining of these energies corresponds to a general approach in the theory of Γ-convergence for lattice system where the discrete functions $u=\{u_i\}$ are identified with their piecewise-constant extensions, and the scaled lattice energies with energies on the continuum whose asymptotic behaviour is described by taking L^1-limits in the u variable and applying a mesoscopic homogenization process to the energies. The comparison with the case $p=1$ ensures that the limit is finite (but possibly trivial) on u with $\partial\{u=1\}$ of finite area in $\mathbf{D}$. A general theory for interfacial energies by Ambrosio and Braides [3] suggests the identification with functionals of the form

$$\int_{\mathbf{D}\cap\partial\{u=1\}}\varphi(x,\nu)d\mathcal{H}^{d-1}\,,$$

with ν the normal to $\partial\{u=1\}$. In the dilute case, however, neither the existence of an average macroscopic magnetization (the L^1 limit of the u) nor the definition of a limit surface tension follow from this general theory. They can instead be translated in almost-sure properties of the corresponding Bernoulli bond percolation model, and completely described in dimension two. Below the percolation threshold the energy is indeed trivial (the Γ-limit identically vanishing on its domain), since interfaces with zero energy are asymptotically L^1-dense. Above the percolation threshold instead the coarse graining leads first to showing that indeed we may define a limit magnetization u taking values in $\{\pm1\}$. This u is obtained as a L^1-limit on the scaled infinite strong cluster, thus neglecting the values u_i on nodes i isolated from that cluster. It must be noted that this limit variable can be alternatively thought as a renormalization of the 'effective magnetization' (the one obtained by local averages; i.e., as the weak L^1 limit of the spins on the scaled lattices). This effective magnetization does not take only the values ±1 but may take all values u with $|u|\in[m_{\rm eff},1]$,

where m_{eff} is the limit (almost sure) deterministic average (depending only on p) of the function taking the value 1 on points connected to the strong cluster, and -1 elsewhere. The surface tension is obtained by optimizing the almost sure contribution of the interfaces, and showing that it can be expressed as a first-passage percolation problem, so that the limit is of the form

$$\int_{\mathbf{D}\cap\partial\{u=1\}} \varphi_p(\nu) d\mathcal{H}^1 \,.$$

This type of variational percolation result can be linked to an earlier paper by the authors [13] where discrete fracture is studied and linked to large deviations for the chemical distance in supercritical Bernoulli percolation, thus showing a stimulating interaction between Variational Calculus and Percolation theory.

The paper is organized as follows. After briefly setting notation in Section 2, in Section 3.1 we prove some asymptotic properties of connected subsets of points in the underlying percolation model, and deduce the coerciveness of energies in the supercritical case $p > 1/2$. The convergence theorem is then proved in Section 3.2 by a blow-up argument, which corresponds to a coarse graining at the interfaces, which uses geometric measure theoretical properties and the description of (optimal) interfaces through a first-passage percolation formula. Section 4 deals briefly with the subcritical and critical regimes. Finally, in Section 5 we give a 'dual' result for the asymptotic behaviour of paths whose energy is the counterpart of the interfacial energy above. We give an almost-sure representation for the limit as an integral on continuous paths of the form

$$\int_0^L \psi_p(\gamma')\, dt$$

for all $0 < p < 1$. The shape of ψ_p is linked to properties of first-passage percolation in the supercritical case and of the chemical distance in supercritical Bernoulli percolation in the subcritical regime.

2 Setting of the problem

We use the notation for bond-percolation problems in [13] Section 2.4, and introduce coefficients

$$\sigma^\omega_{\hat{z}} = \begin{cases} 1 & \text{if } \xi_{\hat{z}}(\omega) = 1 \\ 0 & \text{otherwise,} \end{cases}$$

where

$$\xi_{\hat{z}} = \begin{cases} 0 \text{ ('weak')} & \text{with probability } 1-p, \\ 1 \text{ ('strong')} & \text{with probability } p \end{cases} \tag{1}$$

We also write $\sigma^\omega_{\hat{z}} = \sigma^\omega_{ij}$, after identifying each $\hat{z}$ with a pair of nearest neighbours in $\mathbb{Z}^2$.

Correspondingly, we consider the energies

$$E^\omega_\varepsilon(u) = \frac{1}{8} \sum_{i,j\in\mathbf{D}_\varepsilon} \varepsilon\, \sigma^\omega_{ij} (u_i - u_j)^2$$

defined on $u : \mathbf{D}_\varepsilon \to \{\pm 1\}$, where we use the notation $\mathbf{D}_\varepsilon = \mathbf{D} \cap \varepsilon\mathbb{Z}^2$, and $\mathbf{D}$ is an open subset of $\mathbb{R}^2$. The factor 8 is a normalization factor due to the fact that each bond is accounted for twice and $(u_i - u_j)^2 \in \{0, 4\}$.

The case $p = 1$ corresponds to a *ferromagnetic spin system*, which can be described approximately as $\varepsilon \to 0$ by the anisotropic perimeter energy (see [2])

$$F^1(u) = \int_{\partial^*\{u=1\}\cap \mathbf{D}} \|\nu_u\|_1 d\mathcal{H}^1$$

defined on $u \in BV(\mathbf{D}; \{\pm 1\})$ ($\partial^*\{u = 1\}$ denotes the measure-theoretical reduced boundary of the set of finite perimeter $\{u = 1\}$ and ν_u its inner normal; see e.g. [9]). In this approximation we identify each function $u : \mathbf{D}_\varepsilon \to \{\pm 1\}$ with the set $A = \bigcup\{\varepsilon i + \varepsilon Q : i \in \mathbf{D}_\varepsilon : u_i = 1\}$ or the function $u \in BV(\mathbf{D}; \{\pm 1\})$ given by $u = 2\chi_A - 1$.

3 The supercritical regime: p > 1/2

If $p > 1/2$ the *strong cluster* is denoted by $\mathcal{S}^\omega$. We define

$$\mathcal{Z}^\omega = \{i \in \mathbb{Z}^2 : \exists j \in \mathbb{Z}^2 \text{ such that } \hat{z}(i, j) \in \mathcal{S}^\omega\}$$

and

$$\mathcal{W} = \mathcal{W}^\omega = \bigcup\Big\{i + Q : i \in \mathcal{Z}^\omega\},$$

where Q denotes the coordinate (semi-open) unit square in $\mathbb{R}^2$ centered in 0.

We will use the following terminology:

• a *path* of points in $\mathbb{Z}^2$ is a finite or infinite sequence $\{i_k : k = 0, 1, \ldots\}$ such that $|i_k - i_{k+1}| = 1$ for all $k = 0, 1, \ldots$

• the *boundary* of a set $I \subset \mathbb{Z}^2$ is $\{i \in I : \exists j \in \mathbb{Z}^2 \setminus I : |i - j| = 1\}$;

• the *interior* of a bounded set $I \subset \mathbb{Z}^2$ is the set of points i such that there is no unbounded path with starting point i (i.e., such that $i_0 = i$) not intersecting the boundary of I. Note that the interior of I may contain also points not in I;

• the *size* of a bounded subset $I \subset \mathbb{Z}^2$ is the cardinality of its interior.

Note that the definition of "interior" of a discrete set I given here (the reader will excuse the abuse of notation with the topological notion) corresponds to the complement of the infinite connected component of $\mathbb{Z}^2$ not containing I. Loosely speaking, it is the portion of lattice enclosed by the "external boundary" of I.

3.1 Definition of the convergence of spins and compactness

Lemma 3.1. *Let* $\mathbf{D}$ *be a bounded Lipschitz open set. For a set of* ω *of full probability, if* (u_ε) *is a sequence such that* $\sup_\varepsilon E^\omega_\varepsilon(u_\varepsilon) < +\infty$*, then there exists a sequence* $(\widetilde{u}_\varepsilon)$ *such that* $E^\omega_\varepsilon(\widetilde{u}_\varepsilon) \le E^\omega_\varepsilon(u_\varepsilon)$*,*

$$\|(u_\varepsilon - \widetilde{u}_\varepsilon)\chi_{\mathbf{D}\cap\varepsilon\mathcal{W}}\|_{L^1(\mathbf{D})} = o(1) \tag{2}$$

as $\varepsilon \to 0$*, and there are no connected components of the sets* $\{i : \widetilde{u}_\varepsilon = 1\}$ *and* $\{i : \widetilde{u}_\varepsilon = -1\}$ *with size not exceeding* $1/\varepsilon$*.*

Proof. We extend each function as $u_\varepsilon = 1$ on $\mathbb{Z}^2 \setminus \mathbf{D}_\varepsilon$.

We first consider all the connected components of the complement of $\mathcal{Z}^\omega$. If $u_\varepsilon = 1$ identically on the boundary of one such component we set $\widetilde{u}_\varepsilon = 1$ on its interior. In the remaining cases, we set

$\widetilde{u}_\varepsilon = -1$ if $u_\varepsilon = -1$ on the boundary. With this operation we do not change the value of u_ε on $\mathcal{Z}^\omega$ and we have $E^\omega_\varepsilon(\widetilde{u}_\varepsilon) \le E^\omega_\varepsilon(u_\varepsilon)$. We can therefore assume from the beginning that u_ε is constant on each such connected component.

We can now subdivide $\mathbb{Z}^2$ into connected components $(I^{\varepsilon,+}_m)_{m\in M^+_\varepsilon}$ and $(I^{\varepsilon,-}_m)_{m\in M^-_\varepsilon}$ defined as the maximal connected components where $u_\varepsilon = 1$ and $u_\varepsilon = -1$, respectively.

Note that we have

$$\sum_{i,j:i\in I^{\varepsilon,+}_m} \sigma^\omega_{ij}(u_i - u_j)^2 \ge 1, \qquad \sum_{i,j:i\in I^{\varepsilon,-}_m} \sigma^\omega_{ij}(u_i - u_j)^2 \ge 1 \tag{3}$$

for all m since otherwise we would have $\sigma^\omega_{ij} = 0$ identically on the boundary of such connected components, which contradicts the construction above. We then have

$$\#M^+_\varepsilon \le \frac{C}{\varepsilon}, \qquad \#M^-_\varepsilon \le \frac{C}{\varepsilon}, \tag{4}$$

where $\#M^+_\varepsilon$ and $\#M^-_\varepsilon$ are the number of maximal connected components of the set where $u_\varepsilon = 1$ and $u_\varepsilon = -1$, respectively.

We fix $\delta > 0$ and consider a component $I^{\varepsilon,-}_m$ with interior of size not more than $\varepsilon^{-1+\delta}$. We denote by $M^-_\varepsilon(\delta) \subset M^-_\varepsilon$ the set of the corresponding indices m. If we identify each $I^{\varepsilon,-}_m$ with a subset of $\mathbb{R}^2$, as usual taking the union of the corresponding ε-squares, we estimate the measure of $I^{\varepsilon,-}_m$ by

$$|I^{\varepsilon,-}_m| \le \varepsilon^2 \cdot \varepsilon^{-1+\delta} = \varepsilon^{1+\delta}$$

The total volume of such components is then

$$\Big|\bigcup\{I^{\varepsilon,-}_m : m \in M^-_\varepsilon(\delta)\}\Big| \le \frac{C}{\varepsilon}\varepsilon^{1+\delta} = C\varepsilon^\delta$$

by (4). We can then set $\widetilde{u}_\varepsilon = 1$ on the interior of this sets. This change is compatible with (2) and decreases the energy. We may repeat the corresponding process with the components $I^{\varepsilon,+}_m$ with interior of size not exceeding $\varepsilon^{-1+\delta}$.

By what just proved, up to substituting u_ε with $\widetilde{u}_\varepsilon$ we then may suppose that there is no connected components of the sets $\{i : u_\varepsilon = 1\}$ and $\{i : u_\varepsilon = -1\}$ with size not exceeding $\varepsilon^{-1+\delta}$.

We consider now the components $I^{\varepsilon,-}_m$ with interior of size in the interval $(\varepsilon^{-1+\delta}, \varepsilon^{-1}]$. We denote by $N^-_\varepsilon(\delta) \subset M^-_\varepsilon$ the set of the corresponding indices m. In particular each their measure is greater than $\varepsilon^{1+\delta}$, so that their perimeter is then estimated as

$$\mathcal{H}^1(\partial I^{\varepsilon,-}_m) \ge C\varepsilon^{(1+\delta)/2}.$$

Since the maximum size of a connected component with $\sigma^\omega_{\hat{z}} = -1$ is of order $|\log\varepsilon|$ (see e.g. [26]) then the number of $\hat{z}$ along the boundary of $I^{\varepsilon,-}_m$ such that $\sigma^\omega_{\hat{z}} = 1$ is at least

$$C\frac{1}{|\log\varepsilon|^2}\frac{1}{\varepsilon}\varepsilon^{(1+\delta)/2} = C\frac{\varepsilon^{-\frac{1}{2}+\frac{\delta}{2}}}{|\log\varepsilon|^2}.$$

We then deduce that the energy contribution of each such component is at least

$$\sum_{i,j:i\in I^{\varepsilon,-}_m} \varepsilon\sigma^\omega_{ij}(u_i - u_j)^2 \ge C\varepsilon\frac{\varepsilon^{-\frac{1}{2}+\frac{\delta}{2}}}{|\log\varepsilon|^2} = C\frac{\varepsilon^{\frac{1}{2}+\frac{\delta}{2}}}{|\log\varepsilon|^2}. \tag{5}$$

In particular, by the boundedness of the energy, we deduce that

$$\#\, N_\varepsilon^-(\delta) \le C|\log\varepsilon|^2\varepsilon^{-\frac{1}{2}-\frac{\delta}{2}}$$

The measure of each such $I_m^{\varepsilon,-}$ is at most ε, so that the total measure of the union of these components is

$$\left|\bigcup\{I_m^{\varepsilon,-} : m \in N_\varepsilon^-(\delta)\}\right| \le C|\log\varepsilon|^2\varepsilon^{\frac{1}{2}-\frac{\delta}{2}} = o(1).$$

We can therefore again change the value setting $\widetilde{u}_\varepsilon = 1$ in each $I_m^{\varepsilon,-}$, and reason similarly for the analogous $I_m^{\varepsilon,+}$.

At the end of the process above we obtain a sequence $(\widetilde{u}_\varepsilon) \in BV(\mathbf{D};\{\pm 1\})$ with the desired properties. □

Theorem 3.2 (percolation animal)**.** *For a set of ω of full probability, there exist a deterministic positive constant α and $\varepsilon_0 = \varepsilon_0(\omega) > 0$ such that for all connected sets contained in a cube $\{\|x\|_1 \le M/\varepsilon\}$ and of size larger than $1/\sqrt{\varepsilon}$ with $\varepsilon < \varepsilon_0$, the proportion of strong links (such that $\sigma_{ij}^\omega = 1$) in each such a set is at least α.*

Proof. Denote $n = \lfloor 1/\sqrt{\varepsilon}\rfloor$, and let $\mathcal{Z}^2$ be the lattice dual to $\mathbb{Z}^2$. Our aim is to prove that almost surely, for sufficiently large n, any connected subset of $[-Mn^2, Mn^2]^2 \cap \mathcal{Z}^2$ of size n contains at least μn strong edges with a non-random $\mu = \mu(p, M) > 0$.

We begin by proving the result for probabilities p close enough to 1. First we recall the estimate for the number of connected sets of size n in $\mathbb{Z}^2$ which contain the origin. It reads (see [21], p.81–82)

$$\begin{aligned} \#\{A \subset \mathbb{Z}^2 \,:\, |A|_b = n, 0 \in A\} &\le (C)^n, \\ \#\{A \subset \mathbb{Z}^2 \,:\, |A| = n, 0 \in A\} &\le (C)^n, \end{aligned} \tag{6}$$

where $|\cdot|_b$ stands for the number of edges in a subset of $\mathbb{Z}^2$ and $|\cdot|$ for the number of vertices. For a fixed set A with $|A|_b = n$ and any $\mu \in (0,1)$, the probability that A contains less than μn strong edges admits the upper bound

$$\mathbf{P}\{A \text{ contains less than } \mu n \text{ strong edges}\} \le (1-p)^{(1-\mu)n} \sum_{k=\lfloor(1-\mu)n\rfloor}^{n} \left(\begin{array}{c} k \\ n \end{array} \right) \le (1-p)^{(1-\mu)n} 2^n.$$

Denote by $\mathcal{G}_\mu(n)$ the event

$\mathcal{G}_\mu(n) = \{$there is a connected set $A \subset [-Mn^2, Mn^2]^2$ of size n which contains less than μn strong edges$\}$.

From the last two estimates we deduce the inequality

$$\mathbf{P}\{\mathcal{G}_\mu(n)\} \le (Mn^2)^2 (C)^n 2^n (1-p)^{(1-\mu)n} \tag{7}$$

Therefore, there is $p_0 = p_0(\mu) < 1$ such that for all $p \in (p_0, 1)$ the inequality holds

$$\mathbf{P}\{\mathcal{G}_\mu(n)\} \le C(M)(1/2)^n. \tag{8}$$

With the help of the Borel-Cantelli lemma this yields the desired statement for $p \in (p_0, 1)$.

In order to extend the result to all values of $p > 1/2$, we are going to use the renormalization technique.

Remark 3.3. Consider in $\mathbb{Z}^2$ the set of pairs of adjacent vertices. We say that two pairs are p-adjacent is they do not intersect and contain at least two vertices, one from the first pair and another from the second one, which are adjacent. The sequence of pairs $\zeta_1, \dots \zeta_k$ forms a path if the pairs $\{\zeta_j\}\big|_{j=1}^k$ do not intersect, and any two consecutive elements of the sequence are p-adjacent. The set of pairs is said to be connected if they do not intersect, and for any two of them, say ζ_1 and ζ_2, there is a path consisting of elements of the set, which goes from ζ_1 to ζ_2.

To each pair in $\mathbb{Z}^2$ we assign a random variable which takes on the value "strong" with probability p and "weak" with probability $1-p$. Moreover, we assume that these random variables are independent for nonintersecting pairs.

In exactly the same way as above one can obtain an exponential estimate for the number of connected animals of pairs consisting of n elements and containing zero. In fact, it is easy to verify that this number does not exceed $C^{2n}4^n$ with the same constant C as in (6). The estimate (7) remains valid in the case of pairs for p sufficiently close to 1.

Now, consider the set of cubes $Q_y^N = Ny + [0, N-1]^2$ with $y \in \mathbb{Z}^2$ and integer $N > 1$, and denote $R^N_{y^-,y^+} = \big(Ny^- + [0,N-1]^2\big) \cup \big(Ny^+ + [0,N-1]^2\big) = Q^N_{y^-} \cup Q^N_{y^+}$ with $|y^- - y^+|_1 = 1$.

Proposition 3.4. *For any $p > p_{\rm cr} = 1/2$ and any $p_1 < 1$ there is $N_0 = N_0(p, p_1) > 0$ such that for each $N > N_0$ and y^-, $y^+ \in \mathbb{Z}^2$ with $|y^- - y^+|_1 = 1$ it holds*

$$\mathbf{P}\big\{\textit{any connected subset } A \textit{ of } R^N_{y^-,y^+} \textit{ with } |A| \ge N \textit{ contains at least ons strong link }\big\} > p_1. \quad (9)$$

Proof. The statement of Proposition is a straightforward consequence of the exponential estimates for the size of a weak cluster in the case $p > p_{\rm cr}$. □

We proceed by applying the renormalization arguments. Given $p > p_{\rm cr} = 1/2$, we choose $p_1 < 1$ and $\mu > 0$ so that (8) holds, and then choose N such that (9) holds true. We then partition the big cube $[-Mn^2, Mn^2]^2$ into the cubes Q_y^N, $y \in [-Mn^2/N, Mn^2/N]^2 \cap \mathbb{Z}^n$, consider the pairs $R^N_{y^-,y^+}$ of such cubes, and introduce the connectedness relation for pairs $\{y^-, y^+\}$ as in Remark 3.3.

Given a connected set $A \subset \mathbb{Z}^2$, we will say that a rectangle $R^N_{y^-,y^+}$ is good if it contains a connected subset of A of size at least N.

Let $A \subset [-Mn^2, Mn^2]^2$ be a connected set with $|A| \ge n$. Our aim is to build a connected set $\tilde{\mathcal{A}}$ of pairs of y such that

$$|\tilde{\mathcal{A}}| \ge \nu n/N^2, \qquad \nu > 0, \quad (10)$$

and for each $\{y^-, y^+\} \in \tilde{\mathcal{A}}$ the rectangle $R^N_{y^-,y^+}$ is good. To this end we choose first an arbitrary cube $Q^N_{y^0}$ having a nontrivial intersection with A. We denote by $\partial_y\{y^0\}$ the set of $y \in Z^2$ satisfying the estimate $|y - y^0|_\infty = 1$, and

$$\widehat{Q^N_{y^0}} = \bigcup_{y \in \{y_0\} \cup \partial_y\{y^0\}} Q^N_y.$$

We also denote

$$\partial_z \widehat{Q^N_{y^0}} = \{z \in \mathbb{Z}^2 \,:\, z \notin \widehat{Q^N_{y^0}},\, |z - \tilde{z}|_1 = 1 \text{ for some } \tilde{z} \in \widehat{Q^N_{y^0}}\}.$$

Similarly, for any set $\mathcal{B} \subset \mathbb{Z}^2$ we denote

$$\partial_y \mathcal{B} = \{y \in \mathbb{Z}^2 \setminus \mathcal{B} \,:\, |y - \tilde{y}|_\infty = 1 \text{ for some } \tilde{y} \in \mathcal{B}\}, \quad \widehat{Q^N_{\mathcal{B}}} = \bigcup_{y \in \mathcal{B} \cup \partial_y \mathcal{B}} Q^N_y.$$

Since A is connected, for sufficiently large n there is a path starting inside $Q^N_{y^0}$ which belongs to A and has a final point at $\partial_z\widehat{Q^N_{y^0}}$. It is then easy to conclude that at least one of the rectangles $\left\{R^N_{y^-,y^+} : y^-, y^+ \in \partial_y\{y^0\}, \quad |y^- - y^+|_1 = 1\right\}$ is good. We denote the corresponding indices by $y^{-,1}$, $y^{+,1}$ and set $\mathcal{A}^1 = \{y^{-,1}, y^{+,1}\}$, $\mathcal{B}^1 = \{y^{-,1}\} \cup \{y^{+,1}\}$.

In the same way one can show that there is a pair $y^{-,2}, y^{+,2} \in \partial_y\mathcal{B}^1$, $|y^{-,2} - y^{+,2}|_1 = 1$, such that the cube $R^N_{y^{2,-},y^{2,+}}$ is good. We set $\mathcal{A}^2 = \{y^{2,-}, y^{2,+}\} \cup \mathcal{A}^1$, $\mathcal{B}^2 = \{y^{2,-}\} \cup \{y^{2,+}\} \cup \mathcal{B}^1$.

At the next step we define

$$Q_{\mathcal{A}^2} = \bigcup_{y\in\mathcal{A}^2} Q^N_y, \qquad \widehat{Q_{\mathcal{B}^2}} = \bigcup_{y\in\mathcal{B}^2\cup\partial_y\mathcal{B}^2} Q^N_y$$

By the similar arguments, one of the rectangles $\left\{R^N_{y^-,y^+} : y^-, y^+ \in \partial_y\mathcal{B}^2, \ |y^- - y^+|_1 = 1\right\}$ is good. We denote the corresponding indices by $y^{3,-}$, $y^{3,+}$ and set $\mathcal{A}^3 = \{y^{3,-}, y^{3,+}\} \cup \mathcal{A}^2$, $\mathcal{B}^3 = \{y^{3,-}\} \cup \{y^{3,+}\} \cup \mathcal{B}^2$.

Iterating this process we construct the sequence of connected sets of pairs $\{\mathcal{A}^m\}$ with $|\mathcal{A}^m| = m$ and the sequence of connected sets $\{\mathcal{B}^m\}$ with $|\mathcal{B}^m| = 2m$.

It is easy to see that $|\mathcal{B}^m \cup \partial\mathcal{B}^m| \le 10m$. Therefore,

$$\Big|\bigcup_{y\in\mathcal{A}^m\cup\partial\mathcal{A}^m} Q^N_y\Big| \le 10N^2 m.$$

Hence, we will be able to iterate the process until $m \ge n/(10N^2)$. It remains to set $\nu = 1/10$ and $\tilde{\mathcal{A}} = \mathcal{A}^m$ with $m = \lfloor n/(10N^2)\rfloor$, and (10) follows.

Since every good cube contains a strong edge with probability greater than p_1, then for sufficiently large n the number of strong edges in the set A is at least $\dfrac{\mu n}{10N^2}$, as desired. □

Lemma 3.5. *For a set of ω of full probability, if $\sup_\varepsilon E^\omega_\varepsilon(u_\varepsilon) < +\infty$ and all connected components of the sets $\{i : u_\varepsilon = 1\}$ and $\{i : u_\varepsilon = -1\}$ have size greater than $1/\varepsilon$, then $\{u_\varepsilon = 1\}$ has equi-bounded perimeter in $\mathbf{D}$, and in particular (u_ε) is pre-compact in the weak topology of $BV(\mathbf{D};\{\pm1\})$.*

Proof. Each connected component of $\{i : u_\varepsilon = 1\}$ and $\{i : u_\varepsilon = -1\}$ has perimeter at least $1/\sqrt{\varepsilon}$. By Theorem 3.2 we then have

$$\mathcal{H}^1(\partial\{u_\varepsilon = 1\}) \le \frac{C}{\alpha}E_\varepsilon(u_\varepsilon) + C\mathcal{H}^1(\partial\mathbf{D}),$$

which proves the desired statement. □

We can collect the previous lemmas in the following one, which will define the convergence with respect to which energies E^ω_ε are equi-coercive.

Lemma 3.6. *Let $\mathbf{D}$ be a bounded Lipschitz open set. For a set of ω of full probability, if (u_{ε_j}) is a sequence such that $\sup_j E^\omega_{\varepsilon_j}(u_{\varepsilon_j}) < +\infty$, then there exists a function $u \in BV(\mathbf{D},\{\pm1\})$ and a subsequence, still denoted by (u_{ε_j}), such that*

$$\lim_j \|(u_{\varepsilon_j} - u)\chi_{\mathbf{D}\cap\varepsilon_j\mathcal{W}}\|_1 = 0. \tag{11}$$

Proof. It suffices to apply Lemma 3.5 to the sequence $(\widetilde{u}_{\varepsilon_j})$ obtained from Lemma 3.1. In this way we have $u \in BV(\mathbf{D}, \{\pm 1\})$ such that, up to subsequences, $\widetilde{u}_{\varepsilon_j} \to u$ in $L^1(\mathbf{D})$. We then get

$$\lim_j \|(u_{\varepsilon_j} - u)\chi_{\mathbf{D}\cap(\varepsilon_j \mathcal{W})}\|_{L^1} \leq \lim_j \|(u_{\varepsilon_j} - \widetilde{u}_{\varepsilon_j})\chi_{\mathbf{D}\cap(\varepsilon_j \mathcal{W})}\|_{L^1} + \lim_j \|\widetilde{u}_{\varepsilon_j} - u\|_{L^1} = 0,$$

as desired. □

By this last lemma, we can define a convergence for which the functionals E_ε are equicoercive, as

$$u_\varepsilon \to u \quad \Longleftrightarrow \quad \lim_{\varepsilon \to 0} \|(u_\varepsilon - u)\chi_{\mathbf{D}\cap(\varepsilon \mathcal{W})}\|_{L^1} = 0\,. \tag{12}$$

3.2 Definition of surface tension and convergence of the energies

For any vector $\tau \in \mathbb{R}^2$, $m \in \mathbb{N}$ and $\omega \in \Sigma$ we denote

$$\psi^\omega(x, y) = \min\left\{ \sum_{n=1}^{K} \sigma^\omega_{i_n i_{n-1}} : i_0 = x,\ i_K = y,\ K \in \mathbb{N} \right\}, \tag{13}$$

where the minimum is taken over all paths joining x and $y \in \mathbb{Z}^2$. The following statement holds.

Lemma 3.7 (Garet-Marchand)**.** *For any $\tau \in \mathbb{R}^2$ the following limit exists almost surely and does not depend on ω*

$$\psi_p(\tau) = \lim_m \frac{1}{m} \psi^\omega(0, \lfloor m\tau \rfloor), \tag{14}$$

where $\lfloor m\tau \rfloor_k = \lfloor m\tau_k \rfloor$ is the integer part of the k-th component of $m\tau$. Moreover, ψ defines a norm in $\mathbb{R}^2$.

Our main result is the following. For the definition and properties of Γ-convergence we refer to [10, 11, 12, 18].

Theorem 3.8. *Let $\mathbf{D}$ be a bounded Lipschitz open set and $p > 1/2$, then $\mathbf{P}$-almost surely there exists the Γ-limit of E^ω_ε with respect to the convergence in* (12)*, it is deterministic, and is given by*

$$F_p(u) = \int_{\partial^* \{u=1\} \cap \mathbf{D}} \psi_p(\nu)\, d\mathcal{H}^1 \tag{15}$$

with domain $BV(\mathbf{D}; \{\pm 1\})$.

Proof. We begin with the proof of the lower bound (liminf inequality), and fix a family u_ε such that $u_\varepsilon \to u$ as in (12), and $\liminf_{\varepsilon \to 0} E^\omega_\varepsilon(u_\varepsilon) < +\infty$. By Lemmas 3.1 and 3.5 we can find $\widetilde{u}_\varepsilon$ converging weakly* in $BV(\mathbf{D})$ to u and such that

$$\liminf_{\varepsilon \to 0} E^\omega_\varepsilon(u_\varepsilon) \geq \liminf_{\varepsilon \to 0} E^\omega_\varepsilon(\widetilde{u}_\varepsilon).$$

Up to subsequences, we may suppose that such liminf is actually a limit.

For all ε we consider the set in the dual lattice $\varepsilon\mathcal{Z}$ of $\varepsilon\mathbb{Z}^2$ defined by

$$S_\varepsilon = \left\{ \varepsilon k : k = \frac{i+j}{2},\ \varepsilon i, \varepsilon j \in \mathbf{D}_\varepsilon, |i-j| = 1, \widetilde{u}_\varepsilon(\varepsilon i) = 1, \widetilde{u}_\varepsilon(\varepsilon j) = -1 \right\}$$

and the measure

$$\mu_\varepsilon = \sum_{\varepsilon k \in S_\varepsilon} \varepsilon \sigma_k^\omega \delta_{\varepsilon k}.$$

Note that $E_\varepsilon^\omega(u_\varepsilon) = \mu_\varepsilon(\mathbf{D})$ so that the family of measures μ_ε is equibounded. Hence, up to further subsequences we can assume that μ_ε converges weakly* to a finite measure μ.

Let $A = \{u = 1\}$ and $A_\varepsilon = \{u_\varepsilon = 1\}$. With fixed $h \in \mathbb{N}$ we can consider the collection $\mathcal{Q}_h$ of squares $Q_\rho^\nu(x)$ such that the following conditions are satisfied:

(i) $x \in \partial^* A$ and $\nu = \nu(x)$;

(ii) $\left|(Q_\rho^\nu(x) \cap A) \triangle \Pi^\nu(x)\right| \leq \frac{1}{h}\rho^2$, where $\Pi^\nu(x) = \{y \in \mathbb{R}^2 : \langle y - x, \nu\rangle \geq 0\}$;

(iii) $\left|\frac{\mu(Q_\rho^\nu(x))}{\rho} - \frac{d\mu}{d\mathcal{H}^1 \llcorner \partial^* A}(x)\right| \leq \frac{1}{h}$;

(iv) $\left|\frac{1}{\rho}\int_{Q_\rho^\nu(x) \cap \partial^* A} \psi_p(\nu(y)) d\mathcal{H}^1(y) - \psi_p(\nu(x))\right| \leq \frac{1}{h}$;

(v) $\mu(Q_\rho^\nu(x)) = \mu(\overline{Q_\rho^\nu(x)})$.

Note that for fixed $x \in \partial^* A$ and for ρ small enough (ii) is satisfied by the definition of reduced boundary (see [9]), (iii) follows from the Besicovitch Derivation Theorem provided that

$$\frac{d\mu}{d\mathcal{H}^1 \llcorner \partial^* A}(x) < +\infty;$$

(iv) holds by the same reason, and (v) is satisfied for almost all $\rho > 0$ since μ is a finite measure. We deduce that $\mathcal{Q}_h$ is a fine covering of the set

$$\partial^* A_\mu = \left\{x \in \partial^* A : \frac{d\mu}{d\mathcal{H}^1 \llcorner \partial^* A}(x) < +\infty\right\},$$

so that (by Morse's lemma, see [28]) there exists a countable family of disjoint closed cubes $\{\overline{Q_{\rho_j}^{\nu_j}(x_j)}\}$ still covering $\partial^* A_\mu$. Note that we have

$$\mathcal{H}^1(\partial^* A \setminus \partial^* A_\mu) = 0$$

since $\mu(\partial^* A) < +\infty$.

We now fix one of such cubes $Q_\rho^\nu(x)$. Since $|A_\varepsilon \triangle A| \to 0$, for ε small enough we have

$$\left|(Q_\rho^\nu(x) \cap A_\varepsilon) \triangle \Pi^\nu(x)\right| \leq \frac{2}{h}\rho^2 \tag{16}$$

by (ii) above.

For simplicity of notation we can suppose that $\nu = e_2$ and $x = 0$. With fixed $\delta < 1/2$, from (16) we have in particular

$$\left|\left((Q_\rho^\nu(x) \cap A_\varepsilon) \triangle \Pi^\nu(x)\right) \cap \left\{y : \rho\frac{\delta}{2} \leq \operatorname{dist}(y, \partial Q_\rho^\nu(x)) \leq \rho\delta\right\}\right| \leq \frac{2}{h}\rho^2\,. \tag{17}$$

We deduce that there exists

$$t \in \left[\frac{\rho\delta}{2}, \rho\delta\right]$$

such that

$$\mathcal{H}^1\Big(((Q^\nu_\rho(x)\cap A_\varepsilon)\triangle \Pi^\nu(x))\cap\{y:\mathrm{dist}\,(y,\partial Q^\nu_\rho(x))=t\}\Big)\le \frac{4}{h\delta}\rho. \tag{18}$$

We can then define the subset $A^1_\varepsilon\subset Q^\nu_\rho(x)$ by

$$A^1_\varepsilon=\begin{cases} A_\varepsilon & \text{on } Q^\nu_{\rho-t}(x)\\ \Pi^\nu(x) & \text{otherwise.}\end{cases} \tag{19}$$

In this way the set A^1_ε has the same trace as $\Pi^\nu(x)$ on $\partial Q^\nu_\rho(x)$ and

$$\mathcal{H}^1\Big((\partial A^1_\varepsilon\setminus\partial A_\varepsilon)\cap Q^\nu_\rho(x)\Big)\le\frac{4}{h\delta}\rho+\frac{\delta}{2}\rho. \tag{20}$$

We can then find points $x_\varepsilon,y_\varepsilon\in\mathcal{Z}$ such that $\varepsilon x_\varepsilon,\varepsilon y_\varepsilon\in\partial A^1_\varepsilon$ and $|\varepsilon x_\varepsilon+\frac{\rho}{2}e_1|\le 2\varepsilon$, $|\varepsilon y_\varepsilon-\frac{\rho}{2}e_1|\le 2\varepsilon$ (recall that $\nu=e_2$), and a path $\{k^\varepsilon_n:0\le n\le K_\varepsilon: k_0=x_\varepsilon,\ k_{K_\varepsilon}=y_\varepsilon\}$ in $\frac{1}{\varepsilon}(\partial A^1_\varepsilon\cap Q^\nu_\rho(x))\cap\mathcal{Z}$ joining x_ε to y_ε. By the estimate (20) we have

$$\begin{aligned}\mu_\varepsilon(Q^\nu_\rho(x)) &\ge \sum_{n=0}^{K_\varepsilon}\varepsilon\sigma^\omega_{k_n}--\Big(\frac{4}{h\delta}+\frac{\delta}{2}\Big)\rho\\ &\ge \varepsilon\psi^\omega(x_\varepsilon,y_\varepsilon)-\Big(\frac{4}{h\delta}+\frac{\delta}{2}\Big)\rho\end{aligned}$$

Since $|(y_\varepsilon-x_\varepsilon)-\frac{\rho}{\varepsilon}e_1|\le 4$, by the arbitrariness of h and δ we then get

$$\liminf_{\varepsilon\to0}\mu_\varepsilon(Q^\nu_\rho(x))\ge\rho\,\psi_p(e_1)=\rho\,\psi_p(e_2)=\rho\,\psi_p(\nu).$$

By (iv) above we then have

$$\liminf_{\varepsilon\to0}\mu_\varepsilon(Q^\nu_\rho(x))\ge\int_{Q^\nu_\rho(x)\cap\partial^*A}\psi_p(\nu(y))\,d\mathcal{H}^1(y)-\frac{1}{h}\rho,$$

and we finally deduce that

$$\begin{aligned}\liminf_{\varepsilon\to0}\mu_\varepsilon(\mathbf{D}) &\ge \sum_j\liminf_{\varepsilon\to0}\mu_\varepsilon(Q^{\nu_j}_{\rho_j}(x_j)\cap\partial^*A)\\ &\ge \sum_j\int_{Q^{\nu_j}_{\rho_j}(x_j)\cap\partial^*A}\psi_p(\nu(y))\,d\mathcal{H}^1(y)-\frac{C}{h}\\ &= \int_{\mathbf{D}\cap\partial^*A}\psi_p(\nu(y))\,d\mathcal{H}^1(y)-\frac{C}{h},\end{aligned}$$

which gives the liminf inequality.

The construction of a recovery sequence giving the upper bound can be performed just for polyhedral sets, since they are dense in energy in the class of sets of finite perimeter. We only give the construction when the set is of the form $\Pi^\nu(x)\cap\mathbf{D}$ since it is easily generalized to each face of a polyhedral boundary.

It is no restriction to suppose that $\Pi^\nu(x) = \Pi^\nu(0) =: \Pi^\nu$, that ν is a rational direction (i.e., there exits S such that $S\nu \in \mathbb{Z}^2$), and that

$$\mathcal{H}^1(\partial \mathbf{D} \cap \partial \Pi^\nu) = 0, \tag{21}$$

since also with these restrictions we obtain a dense class of sets. We will compute the Γ-limsup for $u = 2\chi_{\Pi^\nu} - 1$.

We fix $\eta > 0$ and set $K_\varepsilon^\eta = \lfloor \frac{\eta}{S\varepsilon} \rfloor$. We also consider $M > 0$ large enough so that $\mathbf{D} \subset\subset Q_M^\nu(0)$. We can therefore consider the points

$$x_j^\varepsilon = jSK_\varepsilon^\eta \tau, \qquad j \in \mathbb{Z},\ |j| < \frac{M}{\eta}$$

with $\tau = \nu^\perp$. For each such j we consider a path $\{k_n^{\varepsilon,j} : 0 \le n \le N_{\varepsilon,j}\}$ joining x_j^ε and x_{j+1}^ε such that

$$\sum_n \sigma_{k_n}^\omega = \psi^\omega(x_j^\varepsilon, x_{j+1}^\varepsilon),$$

and the resulting path γ_ε obtained as the union of all these paths. Note that this final path is included in the strip $\{x : |\langle x, \nu\rangle| \le \eta/\varepsilon\}$ and, after identifying it with a curve in $\mathbb{R}^2$, for ε small enough γ_ε disconnects $\frac{1}{\varepsilon}\mathbf{D}$. We can therefore consider $\mathbf{D}_\varepsilon^+$ the maximal connected component of $\frac{1}{\varepsilon}\mathbf{D} \setminus \gamma_\varepsilon$ containing $\mathbf{D} \cup \{\langle x, \nu\rangle \ge \eta/\varepsilon\}$, and define

$$u_\varepsilon^\eta(\varepsilon i) = \begin{cases} 1 & \text{if } i \in \mathbb{Z}^2 \cap \mathbf{D}_\varepsilon^+ \\ -1 & \text{otherwise.} \end{cases}$$

Note that

$$E_\varepsilon^\omega(u_\varepsilon^\eta) = \psi_p(\tau)\mathcal{H}^1(\partial\Pi^\nu \cap \mathbf{D}) + o(1) + O(\eta)$$

as $\varepsilon \to 0$, and therefore, up to subsequences, there exists $u^\eta \in BV(\mathbf{D}; \{\pm 1\})$ such that $u_\varepsilon^\eta \to u^\eta$ in the sense of convergence (12).

Note that again the limit satisfies

$$S(u^\eta) \subset \mathbf{D} \cap \{x : |\langle x, \nu\rangle| \le \eta\}; \tag{22}$$

moreover, $\mathcal{H}^1(S(u^\eta)) \le C$ by Lemma 3.5. We have

$$\begin{aligned} F''_\omega(u^\eta) &:= \Gamma\text{-}\limsup_{\varepsilon\to 0^+} E_\varepsilon^\omega(u^\eta) \\ &\le \limsup_{\varepsilon\to 0^+} E_\varepsilon^\omega(u_\varepsilon^\eta) \\ &\le \psi_p(\tau)\mathcal{H}^1(\partial\Pi^\nu \cap \mathbf{D}) + O(\eta). \end{aligned}$$

By (22) we have that $u^\eta \to u$, and by the lower semicontinuity of the functional F''_ω we deduce then that

$$F''_\omega(u) \le \liminf_{\eta\to 0^+} F''_\omega(u^\eta) \le \psi_p(\tau)\mathcal{H}^1(\partial\Pi^\nu \cap \overline{\mathbf{D}})$$

Eventually, we obtain the desired inequality recalling that $\mathcal{H}^1\big(\overline{\mathbf{D}} \cap \partial\Pi^\nu\big) = \mathcal{H}^1\big(\mathbf{D} \cap \partial\Pi^\nu\big)$ by (21). □

4 The subcritical and critical regimes: $\mathbf{p \leq 1/2}$

In this regime the overall behaviour is degenerate; furthermore, if $p < 1/2$ the Γ-limit degenerates at all orders.

Theorem 4.1. *Let* $\mathbf{D}$ *be a bounded Lipschitz set.*

(i) (critical regime) *if* $p = 1/2$ *then* $\mathbf{P}$*-almost surely there exists the* Γ*-limit of* E^ω_ε *with respect to the weak* L^1 *convergence. The limit functional is given by*

$$F_0(u) = \begin{cases} 0 & \text{if } \|u\|_\infty \leq 1 \\ +\infty & \text{otherwise.} \end{cases} \tag{23}$$

(ii) (subcritical regime) *if* $p < 1/2$*, then for all choices of scaling factors* $C_\varepsilon > 0$ $\mathbf{P}$*-almost surely there exists the* Γ*-limit of* $C_\varepsilon E^\omega_\varepsilon$ *with respect to the weak* L^1 *convergence and it coincides with the functional* F_0 *above.*

Proof. Since the domain of the energies E^ω_ε is composed of functions with $\|u\|_\infty = 1$ then we immediately get that $F(u) = +\infty$ if $\|u\|_\infty > 1$.

(i) By the lower-semicontinuity of the Γ-limsup it suffices to check that

$$F''(u) := \Gamma\text{-}\limsup_{\varepsilon \to 0} E^\omega_\varepsilon(u) = 0$$

for a L^1-strongly dense set of functions in $BV(\mathbf{D}; \{\pm 1\})$ since the latter is weakly dense in the unit ball of L^∞. This immediately follows by the construction of the limsup inequality in the previous section, after remarking that $\psi_{1/2} = 0$ (see [27]);

(ii) In this case, by the arbitrariness of C_ε we have to show that for all u in a dense set of functions in $BV(\mathbf{D}; \{\pm 1\})$ there exists a sequence $u_\varepsilon \rightharpoonup u$ in $L^1(\mathbf{D})$ such that $E^\omega_\varepsilon(u_\varepsilon) = 0$ for all ε. To this end we can use arguments similar to those used for the proof of the Γ-limsup inequality in the previous section. In this case the path γ_ε is any path in the weak cluster of the dual lattice $\mathcal{Z}$ contained in the strip $\{x : |\langle x, \nu\rangle| \leq \eta/\varepsilon\}$ and with the two endpoints lying at distance at most 2ε from the two sides $\{x : \langle x, \nu^\perp\rangle = \pm M/2\}$. The existence of such a path in the subcritical regime is well known (see [26])

□

5 Curves with 'dilute' length

We define a path γ in $\mathbf{D}_\varepsilon$ as an array of points

$$\varepsilon i_0, \varepsilon i_1, \ldots, \varepsilon i_{N-1}, \varepsilon i_N \in \mathbf{D}_\varepsilon, \qquad N \in \mathbb{N},$$

such that

$$|i_n - i_{n-1}| = 1.$$

Note that self-intersections are allowed by this definition. Each such path can be identified by the piecewise-affine continuous curve $\gamma : [0, \varepsilon N] \to \mathbb{R}^d$ satisfying $\gamma(\varepsilon n) = \varepsilon i_n$ for $n = 0, 1, \ldots, N$, parameterized by arc length. We say that a path γ joins x to y if $\gamma(0) = \varepsilon i_0 = x$ and $\gamma(\varepsilon N) = \varepsilon i_N = y$.

The energy of a path γ in $\mathbf{D}_\varepsilon$ is

$$F_\varepsilon^\omega(\gamma) = \sum_{n=1}^{N} \varepsilon\, c^\omega_{i_n i_{n-1}} \tag{24}$$

In order to study the behaviour of such energies we extend each path to $\gamma(t) = \gamma(0)$ if $t < 0$ and $\gamma(t) = \gamma(\varepsilon N)$ if $t > \varepsilon N$, so that we may define the convergence $\gamma_\varepsilon \to \gamma$ as the L^∞_{loc}-convergence of such extended curves.

When c^ω_{ij} satisfy

$$0 < \alpha \le c^\omega_{ij} \le \beta < +\infty \tag{25}$$

the homogenization of such energies has been studied in [14], remarking first that F_ε^ω are L^∞-equi-coercive, in the sense that if $F_\varepsilon^\omega(\gamma_\varepsilon) \le C < +\infty$, and γ_ε are parameterized on $\{0, \dots, N_\varepsilon\}$ then $\varepsilon N_\varepsilon$ is bounded, so that $(\gamma_\varepsilon - \gamma_\varepsilon(0))$ is bounded in L^∞. In particular, up to subsequences $\varepsilon N_\varepsilon \to L$; the Γ-limit is almost surely given by

$$F(\gamma) = \int_0^L \psi_p(\gamma')\, dt, \tag{26}$$

where for $\|\tau\|_1 < 1$ the energy density $\psi_p(\tau) = \psi_p^\omega(\tau)$ is a.s. independent of ω and defined as the first-passage percolation time constant defined by

$$\psi_p^\omega(\tau) = \lim_m \frac{1}{m} \inf \left\{ \sum_{n=1}^{m} c^\omega_{i_n i_{n-1}} : i_0 = 0,\ i_m = \lfloor m\tau \rfloor \right\}, \tag{27}$$

where $\lfloor m\tau \rfloor$ denotes the vector each component of which is the integer part of the corresponding componemt of $m\tau$, extended by continuity to $\|\tau\|_1 = 1$, while we set $\psi_p^\omega(\tau) = +\infty$ if $\|\tau\|_1 > 1$.

In the dilute case the system is not elliptic and the energies E_ε^ω are not a priori L^∞ equi-coercive; i.e., we may have $L = +\infty$. The E_ε^ω are trivially equicoercive with respect to the $W^{1,\infty}_{\mathrm{loc}}(0, +\infty)$ topology, and their limit can be described from the results in [14].

Remark 5.1. For all $0 \le p < 1$ we have a.s. $\psi_p^\omega(0) = 0$. Indeed it suffices to remark that for fixed ω we can choose i_ω, i'_ω with $\|i_\omega - i'_\omega\| = 1$ and $c^\omega_{i_\omega i'_\omega} = 0$, and for m large enough choose a path in the definition of $\psi_p(0)$ with only a finite number (independent of m) of pairs $\{i_n, i_{n-1}\}$ not equal to $\{i_\omega, i'_\omega\}$.

After this remark, we can state the convergence theorem, remarking that even though the Γ-limit is written as an integral on $(0, +\infty)$, it also comprises the case when $\varepsilon N_\varepsilon \to L$ after extending functions as constant for $t \ge L$.

Theorem 5.2. *Let $0 < p < 1$; then almost surely the energies E_ε^ω Γ-converge to the energy*

$$F_p(\gamma) = \int_0^{+\infty} \psi_p(\gamma')\, dt$$

defined on $W^{1,\infty}((0, +\infty); \mathbb{R}^d)$.

Proof. (i) we first check the liminf inequality. We will reduce to the Γ-convergence result of [14] with $\widetilde{c}^\omega_{ij} = c^\omega_{ij} + 1$, which then is an elliptic model. Note that correspondingly, we have energies $\widetilde{E}_\varepsilon^\omega$ whose limit is described by $\widetilde{\psi}_p(\tau) = \psi_p(\tau) + 1$.

Let $\gamma_\varepsilon \to \gamma$ be given, with γ_ε parameterized on $[0, \varepsilon N_\varepsilon]$. It suffices to consider the case $\varepsilon N_\varepsilon \to +\infty$. We fix $L > 0$ and $\widetilde{N}_\varepsilon$ such that $\varepsilon \widetilde{N}_\varepsilon \to L$, and we consider the paths γ_ε^L being the restriction of γ_ε to $[0, \varepsilon N_\varepsilon]$. By [14] we then have

$$\begin{aligned} \liminf_{\varepsilon\to 0} E_\varepsilon^\omega(\gamma_\varepsilon) &\geq \liminf_{\varepsilon\to 0} \widetilde{E}_\varepsilon^\omega(\gamma_\varepsilon^L) - L \\ &\geq \int_0^L \widetilde{\psi}_p(\gamma')\,dt - L = \int_0^L \psi_p(\gamma')\,dt. \end{aligned}$$

By letting $L \to +\infty$ we then obtain the desired lower bound. Note that if $\varepsilon N_\varepsilon$ is bounded then it is not restrictive to suppose that $\varepsilon N_\varepsilon \to L$ and the argument above keeps working without the passage to the limit as $L \to +\infty$.

(ii) we now prove the limsup inequality. Again we can use the elliptic result in [14].

Given γ such that $F_p(\gamma) < +\infty$, and given $L > 0$, we can find a recovery sequence γ_ε^L for $\widetilde{F}_p(\gamma; L) = \int_0^L \widetilde{\psi}_p(\gamma')\,dt$. After extending such γ_ε^L by a constant, we have $\gamma_\varepsilon^L \to \gamma^L$ where $\gamma^L = \gamma$ on $[0, L]$ and $\gamma^L(t) = \gamma(L)$ for $t > L$. Again, by [14] we have

$$\lim_{\varepsilon\to 0} E_\varepsilon^\omega(\gamma_\varepsilon^L) = \lim_{\varepsilon\to 0} (\widetilde{E}_\varepsilon^\omega(\gamma_\varepsilon^L) - L) \leq \int_0^L \widetilde{\psi}_p(\gamma')\,dt - L = \int_0^L \psi_p(\gamma')\,dt,$$

so that

$$F_p''(\gamma^L) := \Gamma\text{-}\limsup_{\varepsilon\to 0} E_\varepsilon^\omega(\gamma^L) \leq \int_0^L \psi_p(\gamma')\,dt.$$

Note that $\gamma^L \to \gamma$ in $W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}; \mathbb{R}^d)$ and then by the lower semicontinuity of the Γ-limsup

$$F_p''(\gamma) \leq \liminf_{L\to+\infty} F_p''(\gamma^L) \leq \lim_{L\to+\infty} \int_0^L \psi_p(\gamma')\,dt = \int_0^{+\infty} \psi_p(\gamma')\,dt,$$

as desired. □

In the supercritical case we have the two propositions below that follow from Theorem 3.2.

Proposition 5.3. *Let $p > 1/2$; then almost surely the function ψ^ω defined above exists, is deterministic and $\psi_p^\omega(\tau) \geq C_p|\tau|$ for some positive constant C_p.*

Proof. To check the lower bound $\psi^\omega(\tau) \geq C_p|\tau|$ it suffices to remark that given a minimal path γ for $\psi_p^\omega(\tau)$ we can find a non intersecting path in $\mathbb{Z}^d$ joining 0 and $\lfloor m\tau \rfloor$ contained in the image of γ, which then consists of at least $\|\lfloor m\tau \rfloor\|_1$ edges. For sufficiently large m then the number of edges k of this path such that $c_k^\omega = 1$ is at least $C_p\|\lfloor m\tau \rfloor\|_1$, which implies the desired estimate. □

Proposition 5.4. *Let $p > 1/2$ and let $\sup_\varepsilon E_\varepsilon^\omega(\gamma_\varepsilon) < +\infty$ with $\gamma_\varepsilon(0)$ equibounded. Then almost surely the sequence (γ_ε) is bounded in L^∞.*

Proof. This is a straightforward consequence of the existence of the positive time constant in supercritical first-passage percolation [26]. □

Finally, in the subcritical regime $p < 1/2$ the function ψ_p satisfies the following property.

Proposition 5.5. *Let $p < 1/2$; then we have $\psi_p(\tau) = 0$ if $|\tau| \leq \varphi_{1-p}(\tau/|\tau|)$, where φ_s is the asymptotic chemical distance as defined in* [13].

Proof. It suffices to remark that by the properties of the chemical distance for such τ there exists a.s. a path from 0 to $\lfloor m\tau \rfloor$ contained in the weak cluster (up to a $o(m)$ number of nodes). □

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