A GAP THEOREM FOR α-HARMONIC MAPS BETWEEN TWO-SPHERES

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Abstract. In this paper we consider approximations introduced by Sacks-Uhlenbeck of the harmonic energy for maps from S^2 into S^2. We continue the analysis in [7] about limits of α-harmonic maps with uniformly bounded energy. Using a recent energy identity in [8], we obtain an optimal gap theorem for the α-harmonic maps of degree $-1, 0$ or 1.

1. Introduction

Given a compact Riemannian manifold (N^n, h), the *energy functional* on the space of smooth closed curves $\gamma\colon S^1 \to N$ is defined by

$$E(\gamma) := \frac{1}{2}\int_0^1 h(\dot\gamma, \dot\gamma)(t)\, dt\,. \tag{1.1}$$

(Here we are viewing S^1 as $\mathbb{R}/\mathbb{Z}$, i.e., as the circle of length 1.) It is closely related to the length functional, but it has the advantage to select not only curves that extremize length but also their parameterization, namely one of constant speed.

It is easy to prove that there exists $\varepsilon > 0$, depending only on N and h, such that if γ is a critical point of E and $E(\gamma) < \varepsilon$ then γ is constant. Indeed, $\varepsilon = \frac{1}{2}\ell^2$ where ℓ is the length of the shortest closed geodesic in (N^n, h) ($\varepsilon = +\infty$ if (N^n, h) contains no closed geodesics).

This is one of the simplest examples of a gap theorem in Geometric Analysis. More significant gap theorems, just to name a few, have been established in minimal surface theory, the study of Willmore surfaces, Yang-Mills theory and conformal geometry. A selection of such theorems can be found in [15], [11], [13], [2], [5] and *many others* in the literature.

Let now (M^2, g) be a compact Riemannian surface without boundary. As a natural generalization of (1.1) one can consider the *Dirichlet energy* $E(u)$ defined for every $u \in W^{1,2}(M, N)$ by

$$E(u) = \frac{1}{2}\int_M |\nabla u|^2\, dA_M = \int_M e(u)\, dA_M, \tag{1.2}$$

with $e(u) = \frac{1}{2}|\nabla u|^2$ the *energy density* of u.

While (1.1) enjoys nice compactness properties, this is not the case for (1.2). For example, if (M^2, g) is the standard 2-sphere then the Dirichlet energy is invariant under composition on the right with Möbius maps. To deal with this issue, a *regularization* of the following form was considered in [14]. For $\alpha > 1$ and $u \in$

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$W^{1,2\alpha}(M,N)$,

$$E_\alpha(u) := \frac{1}{2}\int_M (2+|\nabla u|^2)^\alpha \, dA_M \tag{1.3}$$

(indeed in [14] the integrand was $(1+|\nabla u|^2)^\alpha$ but it is convenient for our purposes to choose this equivalent variant). The functional E_α has the advantage of satisfying the Palais-Smale compactness condition, and hence critical points can be found in every homotopy class. Critical points of E_α are called α-harmonic maps. They are smooth and, if (N,h) is embedded isometrically into $\mathbb{R}^k$ with second fundamental form A, they satisfy

$$\Delta u + A(u)(\nabla u, \nabla u) = -2(\alpha-1)(2+|\nabla u|^2)^{-1}\langle \nabla^2 u, \nabla u\rangle \nabla u. \tag{1.4}$$

Of course, a sequence of solutions with uniformly bounded E_α-energy might still develop blow-ups as α tends to one, but in [14] it was shown that, after proper rescaling, it would form a finite number of *bubbles* (i.e., non-trivial harmonic maps from S^2 to N).

Theorem 3.3 in [14] is the gap theorem for α-harmonic maps that is analogous to the gap theorem for geodesics stated at the beginning of this Introduction.

Theorem 1.1. *([14]) There exists $\varepsilon > 0$ and $\alpha_0 > 1$ such that if $E(u) < \varepsilon$, $1 < \alpha < \alpha_0$ and u is a critical map of E_α, then u is constant.*

From now on, (M^2, g) and (N^n, h) will both be (S^2, g_{S^2}), the standard 2-sphere isometrically embedded in $\mathbb{R}^3$. For $u\colon S^2 \to S^2$, its *Jacobian* is $J(u) = u\cdot e_1(u) \wedge e_2(u)$ and its *energy density* is $\frac{1}{2}|\nabla u|^2$, where (e_1, e_2) stands for a local oriented orthonormal frame of TS^2. The *degree* of u is then

$$\deg(u) = \frac{1}{4\pi}\int_{S^2} J(u)\, dA_{S^2}. \tag{1.5}$$

In our previous paper ([7], Proposition 7.1) we improved the gap theorem above for maps of degree zero.

Proposition 1.2. *Fix $\eta > 0$. Then there exists $\overline{\alpha} - 1 > 0$ small, $\overline{\alpha}$ depending only on η, such that if $1 < \alpha \leqslant \overline{\alpha}$ and $u : S^2 \to S^2$ is α-harmonic, of degree zero and $E(u) \leqslant 8\pi - \eta$, then u is constant.*

While there exist α-harmonic maps of degree zero whose energy converges from above to 8π as $\alpha \searrow 1$, see Theorem 1.5 below, it is an interesting question to understand whether the convergence of the energies might be from below. We have preliminary numerical evidence that this somewhat unexpected phenomenon may, in fact, occur.

If $u \in W^{1,2\alpha}(S^2, S^2)$ has degree one, we have

$$\begin{aligned} 8\pi &= \int_{S^2} (1+J(u))\, dA_{S^2} \\ &\leqslant \int_{S^2} (1+e(u))\, dA_{S^2} \\ &\leqslant \frac{1}{2}\,(2E_\alpha(u))^{\frac{1}{\alpha}}\,(4\pi)^{\frac{\alpha-1}{\alpha}}, \end{aligned} \tag{1.6}$$

which implies

$$E_\alpha(u) \geqslant 4^\alpha 2\pi. \tag{1.7}$$

We have equality if and only if u is conformal with constant energy density equal to one, which happens only if u is a rotation. In [7] we proved the following result which is again an extension of the gap theorem 1.1 to this special setting.

Theorem 1.3. *([7]) There exists $\varepsilon > 0$ and $\overline{\alpha} - 1 > 0$ small such that the only critical points u_α of E_α which satisfy $E_\alpha(u_\alpha) \leqslant 4^\alpha 2\pi + \varepsilon$ and $\alpha \leqslant \overline{\alpha}$ are the constant maps and the maps of the form $u^R(x) = Rx$ with $R \in O(3)$.*

Since the only minimizers of E among maps from (S^2, g_{S^2}) to itself of degree one are rational (see [4]), it is natural to expect that the functions u_α as in Theorem 1.3 are close in $W^{1,2}(S^2)$ to a rational map. Indeed, in [7] this is proved in stronger norms and for specific rational maps (which are of Möbius type, the degree being equal to 1). A finite-dimensional reduction of the form in [1], together with refined asymptotic expansions then yields Theorem 1.3. It is somehow natural to expect the above result, since the α-energy *breaks* the Möbius symmetry and penalizes functions with large gradients.

In this paper, exploiting some bubbling estimates of new type from [8], we are able to quantitatively and optimally improve Theorem 1.3 in order to obtain a result which resembles the one obtained in Proposition 1.2. This is our result.

Theorem 1.4. *Fix $\eta > 0$. Then there exists $\overline{\alpha} - 1 > 0$ so that the only critical points u_α of E_α of degree ± 1 which satisfy $E_\alpha(u_\alpha) \leqslant 8^\alpha 2\pi - \eta$ and $1 < \alpha \leqslant \overline{\alpha}$ are maps of the form $u^R(x) = Rx$ with $R \in O(3)$.*

In the final section of this paper we shall show that Proposition 1.2 is optimal in the following sense.

Theorem 1.5. *For every $\varepsilon > 0$, there exists $\alpha_0 > 1$ which depends only on ε such that, if $1 < \alpha < \alpha_0$ there exists an α-harmonic map $u_\alpha \colon S^2 \to S^2$ with $\deg(u_\alpha) = 0$ and $6^\alpha 2\pi \leqslant E_\alpha(u_\alpha) < 6^\alpha 2\pi + \varepsilon$.*

The α-harmonic map in Theorem 1.5 is produced by the methods of Section 8 in [7] where we constructed an example of an α-harmonic map u_α of degree one with $E_\alpha(u_\alpha) > 8^\alpha 2\pi$. We believe that this degree 1 example is also optimal in a sense similar to that in Theorem 1.5, i.e., $E_\alpha(u_\alpha)$ is close to $8^\alpha 2\pi$ if α is close to 1.

It would be interesting to understand whether one could replace the energy bounds in Theorem 1.4 by Morse-index estimates as in [12], [16]).

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2. Proof of Theorem 1.4

Without loss of generality, we assume that u_α is a critical point of E_α with $\deg(u_\alpha) = 1$ and

$$E_\alpha(u_\alpha) \leqslant 8^\alpha 2\pi - \eta.$$

It then follows from Hölder's inequality that

$$\begin{aligned} E(u_\alpha) &= \int_{S^2} (1 + e(u_\alpha))\, dA_{S^2} - 4\pi \\ &\leqslant \frac{1}{2} \left(2E_\alpha(u_\alpha)\right)^{\frac{1}{\alpha}} (4\pi)^{\frac{\alpha-1}{\alpha}} - 4\pi \\ &\leqslant 12\pi - \tilde{\eta} \end{aligned} \tag{2.1}$$

where $\tilde{\eta} > 0$.

We claim that for $(\alpha - 1)$ small enough all these critical points have to be of the form $u_\alpha(x) = Rx$ with $R \in SO(3)$.

Assume, for a contradiction, that there exists a sequence $\alpha_k \searrow 1$ and a sequence $u_k := u_{\alpha_k}$ of critical points of $E_k := E_{\alpha_k}$ with $\deg(u_k) = 1$ and $E_k(u_k) \leqslant 8^{\alpha_k} 2\pi - \eta$ which are not of the form $u_k = R_k x$ with $R_k \in SO(3)$. It follows from Theorem 1.1 in [8] that, for a subsequence (which we still denote by u_k), the energy identity

$$12\pi \ > \lim_{k\to\infty} E(u_k) = \sum_{i=1}^{m} E(\omega^i) = 4\pi \sum_{i=1}^{m} |\deg(\omega^i)|,$$

where $m \in \mathbb{N}$ and the $\omega^i \colon S^2 \to S^2$, $1 \leqslant i \leqslant m$, are non-trivial harmonic maps, holds true. More precisely, the authors show that $\lim_{k\to\infty} r_k^{1-\alpha_k} = 1$ (see equation (4.9) in [8]), where r_k is an energy concentration radius on which a bubble forms. This is precisely the information which is needed in [6] (the entropy condition in this paper was only used in order to obtain the same limiting relation between α_k and the concentration radius) or [9] in order to conclude the even stronger energy identity

$$\lim_{k\to\infty} E_k(u_k) = 4\pi + \sum_{i=1}^{m} E(\omega^i) = 4\pi(1 + \sum_{i=1}^{m} |\deg(\omega^i)|). \tag{2.2}$$

Note that these results rely crucially on the assumption that the target manifold is a round sphere since it is known that the energy identity is not true in general for a sequence of α-harmonic maps; see [10]. Finally, it follows from Theorem 2 in [3] that we have the decomposition

$$1 = \deg(u_k) = \sum_{i=1}^{m} \deg(\omega^i).$$

Now we get from the above estimates that

$$\sum_{i=1}^{m} |\deg(\omega^i)| < 3 \quad \text{and} \quad \sum_{i=1}^{m} \deg(\omega^i) = 1.$$

Since $\deg(\omega^i) \neq 0$, the first inequality gives $m = 1$ or 2 and the second equality then only allows $m = 1$ and $\deg(\omega^1) = 1$. Looking back at (2.2) we see that, for every $\delta > 0$ there exists k large enough so that

$$E_k(u_k) < 8\pi + \delta \leqslant 4^{\alpha_k} 2\pi + \delta/2.$$

Hence it follows from Theorem 1.3 that we get a contradiction to the fact that u_k is not a map of the form described in this Theorem.

Remark 2.1. *Here we want to highlight that the above arguments for maps of degree one also yield a new proof of Proposition 1.2.*

Namely, by following the same contradiction argument as above, we obtain a sequence $\alpha_k \searrow 1$ and a sequence of non-constant α_k-harmonic maps u_k with $\deg(u_k) = 0$ and finitely many non-trivial harmonic maps $\omega^i \colon S^2 \to S^2$, $1 \leqslant i \leqslant m$, so that

$$8\pi > \lim_{k\to\infty} E(u_k) = 4\pi \sum_{i=1}^{m} |\deg(\omega^i)|,$$

$$\lim_{k\to\infty} E_k(u_k) = 4\pi(1 + \sum_{i=1}^{m} |\deg(\omega^i)|) \quad \textit{and} \quad \deg(u_k) = 0 = \sum_{i=1}^{m} \deg(\omega^i).$$

It follows that $m = 1$, $\deg(\omega^1) = 0$ which contradicts the fact that ω^i is supposed to be non-constant.

3. Proof of Theorem 1.5

3.1. **Construction of a non-constant α-harmonic map of degree zero.** As in Section 8 in our paper [7], we construct the desired map as a minimizer of the α-energy in a certain class of co-rotationally symmetric maps. More precisely, we consider the parameterisation

$$(r, \theta) \mapsto (\sin r \, \cos\theta, \ \sin r \, \sin\theta, \ \cos r)$$

of S^2, where $n \in \mathbb{N}$, $r \in [n\pi, (n+1)\pi]$ and $\theta \in [0, 2\pi]$. We are then interested in maps $u_f \colon S^2 \to S^2$ such that

$$(r, \theta) \mapsto (\sin(f(r)) \cos\theta, \ \sin(f(r)) \sin\theta, \ \cos(f(r)))$$

with

$$f \colon [0, \pi] \to \mathbb{R}, \ f(0) = 0, \ f(\pi) = 2\pi.$$

Such a map u_f has degree zero and for

$$X := \{f \colon [0, \pi] \to \mathbb{R} : u_f \in W^{1,2\alpha}(S^2, \mathbb{R}^3), \quad f(0) = 0, \ f(\pi) = 2\pi\}$$

the infimum $\Lambda := \inf_{f\in X} I(f)$, where

$$I(f) := E_\alpha(u_f) = \pi \int_0^\pi \left(2 + (f')^2 + \frac{(\sin f)^2}{(\sin r)^2}\right)^\alpha \sin r \, dr$$

is attained by a map $f^* \in X$ as was shown in [7]. (Note that the argument given there trivially extends to the case considered here.) Moreover, setting $u^* := u_{f^*}$, one estimates

$$\begin{aligned} E_\alpha(u^*) &= \pi \int_0^\pi \left(2 + (f^{*\prime})^2 + \frac{(\sin f^*)^2}{(\sin r)^2}\right)^\alpha \sin r \, dr \\ &\geqslant \pi \left(\int_0^\pi \left(2 + (f^{*\prime})^2 + \frac{(\sin f^*)^2}{(\sin r)^2}\right) \sin r \, dr\right)^\alpha \left(\int_0^\pi \sin r \, dr\right)^{1-\alpha} \\ &\geqslant 2^{1-\alpha}\pi \left(\int_0^\pi (2\sin r + 2|f^{*\prime}(\sin f^*)|) \ dr\right)^\alpha. \end{aligned}$$

Now there exists $r_1 \in (0,\pi)$ such that $f^*(r_1) = \pi$ and hence

$$\begin{aligned}\int_0^\pi |f^{*\prime}(\sin f^*)|\,dr &\geqslant \int_0^{r_1} f^{*\prime}(\sin f^*)\,dr - \int_{r_1}^\pi f^{*\prime}(\sin f^*)\,dr \\ &= -\cos f^*(r)|_0^{r_1} + \cos f^*(r)|_{r_1}^\pi \\ &= 4\end{aligned}$$

which implies $E_\alpha(u^*) \geqslant 6^\alpha 2\pi$.

Remark 3.1. *If we minimize $I(f)$ among maps $f\colon [0,\pi] \to [0, 2k\pi]$ then we get an α-harmonic map of degree 0 and alpha-energy at least $(4k+2)^\alpha 2\pi$ and if we minimize $I(f)$ among maps $f\colon [0,\pi] \to [0,(2k+1)\pi]$ then we get an α-harmonic map of degree 1 and alpha-energy at least $(4k+4)^\alpha 2\pi$.*

In what follows we shall rename the map u^* constructed above as u_α, to indicate its dependence on α.

3.2. **An upper bound for $E_\alpha(u_\alpha)$.**

Proposition 3.2. *There exists $C > 0$ universal such that, for $1 < \alpha < \frac{5}{4}$,*

$$E_\alpha(u_\alpha) < 6^\alpha 2\pi + C(\alpha-1)^{1/2}. \tag{3.1}$$

Proof. For $\lambda > 1$, let $f(r) := 2\arctan(\lambda \tan r),\ r \in [0,\pi]$ with arctan also taking values in $[0,\pi]$. Then as r increases from 0 to $\pi/2$, $f(r)$ increases from 0 to π, i.e., u_f maps the upper hemisphere to the full sphere with the equator being all mapped to the South Pole $(0,0,-1)$. The requirement that arctan takes values between 0 and π forces $f(r)$ to increase from π to 2π as r increases from $\pi/2$ to π, i.e., u_f maps the lower hemisphere to the full sphere again but with the opposite orientation (which confirms that u_f has degree zero). Furthermore, $u_f \in X$.

In §8.1 of [7], we showed that the energy density $e(u_f)$ is given by

$$e(u_f) = \frac{1}{2}\left((f')^2 + \frac{(\sin f)^2}{(\sin r)^2}\right).$$

So we calculate

$$f'(r) = \frac{2\lambda}{\cos^2 r + \lambda^2 \sin^2 r}, \qquad \sin(f(r)) = \frac{2\lambda \tan r}{1+\lambda^2\tan^2 r} = (\sin r)(\cos r) f'(r).$$

Therefore,

$$e(u_f) = 2\lambda^2 \frac{1+\cos^2 r}{(\lambda^2 - (\lambda^2-1)\cos^2 r)^2} \tag{3.2}$$

and

$$E_\alpha(u_f) = 2^\alpha \pi \int_0^\pi (1+e(u_f))^\alpha (\sin r)\,dr = 2^{\alpha+1}\pi \int_0^{\pi/2} (1+e(u_f))^\alpha (\sin r)\,dr, \tag{3.3}$$

where we have used the even symmetry of $\cos^2 r$ and $\sin r$ about $\pi/2$. We shall use the inequality

$$(1+x)^\alpha \leqslant 1 + \alpha x + (\alpha-1)x^2, \quad \alpha \in [1,2],\ x \geqslant 0 \tag{3.4}$$

and therefore, we need to estimate the integrals

$$I_1 := \int_0^{\pi/2} \frac{1+\cos^2 r}{(\lambda^2-(\lambda^2-1)\cos^2 r)^2}\,(\sin r)\,dr \tag{3.5}$$

$$I_2 := \int_0^{\pi/2} \frac{(1+\cos^2 r)^2}{(\lambda^2-(\lambda^2-1)\cos^2 r)^4}\,(\sin r)\,dr. \tag{3.6}$$

We make the change of variables $t = \cos r$ and set $a^2 := 1-\lambda^{-2}$. Then (3.5) and (3.6) become

$$I_1 := \lambda^{-4}\int_0^1 \frac{1+t^2}{(1-a^2t^2)^2}\,dt \tag{3.7}$$

$$I_2 := \lambda^{-8}\int_0^1 \frac{(1+t^2)^2}{(1-a^2t^2)^4}\,dt. \tag{3.8}$$

To estimate (3.7) we use

$$\frac{1+t^2}{(1-a^2t^2)^2} < \frac{1}{2a^2}\left(\frac{1}{(1-at)^2}+\frac{1}{(1+at)^2}\right)$$

and note that

$$\int_0^1 \left(\frac{1}{(1-at)^2}+\frac{1}{(1+at)^2}\right)dt = \frac{2}{1-a^2} = 2\lambda^2.$$

Therefore,

$$\int_0^{\pi/2} e(u_f)\,(\sin r)\,dr = 2\lambda^2 I_1 < \frac{2}{a^2} = \frac{2\lambda^2}{\lambda^2-1}. \tag{3.9}$$

To estimate (3.8) we use

$$\frac{(1+t^2)^2}{(1-a^2t^2)^4} < \frac{1}{a^4(1-at)^4}$$

and note that

$$\frac{1}{a^4}\int_0^1 \frac{1}{(1-at)^4}\,dt = \frac{1}{3a^5}\left(\frac{1}{(1-a)^3}-1\right) < \frac{1}{a^4(1-a)^3}.$$

Now

$$\frac{1}{a^4(1-a)^3} = \frac{(1+a)^3}{a^4(1-a^2)^3} < 8\lambda^6 a^{-4}$$

and therefore, if $\lambda^2 > 2$ so that $a^2 > \frac{1}{2}$ we have that $I_2 < 32\lambda^{-2}$. It follows that

$$\int_0^{\pi/2} \big(e(u_f)\big)^2\,(\sin r)\,dr = 4\lambda^4 I_2 < 128\lambda^2. \tag{3.10}$$

Using (3.4) in (3.3) we get

$$E_\alpha(u_f) \leqslant 2^{\alpha+1}\pi\left(1+\alpha\int_0^{\pi/2} e(u_f)\,(\sin r)\,dr + (\alpha-1)\int_0^{\pi/2}\big(e(u_f)\big)^2\,(\sin r)\,dr\right)$$

from which it follows that

$$E_\alpha(u_f) < 2^{\alpha+1}\pi\left(1+\frac{2\alpha\lambda^2}{\lambda^2-1}+128(\alpha-1)\lambda^2\right).$$

Still assuming that $\lambda^2 > 2$ we have $\frac{\lambda^2}{\lambda^2-1} < 1 + 2\lambda^{-2}$ and therefore,

$$E_\alpha(u_f) < 2^{\alpha+1}\pi\left(3 + \frac{4\alpha}{\lambda^2} + 2(\alpha-1) + 128(\alpha-1)\lambda^2\right).$$

By choosing $\lambda = (\alpha-1)^{-(1/4)}$ (with $\alpha \in (1, \frac{5}{4})$ so that $\lambda^2 > 2$) we obtain

$$E_\alpha(u_f) < 2^{\alpha+1}\pi\big(3 + C(\alpha-1)^{1/2}\big) < 6^\alpha 2\pi + C(\alpha-1)^{1/2}.$$

Finally, since u_α minimizes I among maps in X, we have that

$$E_\alpha(u_\alpha) < E_\alpha(u_f) < 6^\alpha 2\pi + C(\alpha-1)^{1/2},$$

namely, (3.1). □

3.3. **Completion of Proof of Theorem 1.5.**

Proof. Given $\varepsilon > 0$ choose $\frac{5}{4} > \alpha_0 > 1$ so that $C(\alpha_0-1)^{1/2} < \varepsilon$. Then, by Theorem 1.4 and Proposition 3.2, for $1 < \alpha < \alpha_0$ we have

$$6^\alpha 2\pi \leqslant E_\alpha(u_\alpha) < 6^\alpha 2\pi + C(\alpha-1)^{1/2} < 6^\alpha 2\pi + \varepsilon,$$

which concludes the proof. □

Remark 3.3. *The map u_f constructed above with $f(r) = 2\arctan(\lambda \tan r)$ can also be written as*

$$u_\lambda(z) = \frac{2\lambda z}{1-|z|^2}, \quad z \in \mathbb{C} \cup \{\infty\},$$

where S^2 is identified with $\mathbb{C} \cup \{\infty\}$ via stereographic projection from the South Pole. The calculations seemed to us somewhat more messy in this representation of S^2. However, this representation suggests the following example for proving the optimality of the degree 1 example referred to just after Theorem 1.5:

$$u_\varepsilon(z) := \frac{z(1-\varepsilon^2|z|^2)}{(\varepsilon^2-|z|^2)}, \quad 0 < \varepsilon \ll 1, \text{ depending on } \alpha.$$

As $\varepsilon \downarrow 0$, u_ε develops a holomorphic bubble at 0 and also at ∞ and, on compact subsets of $0 < |z| < \infty$, $u_\varepsilon(z)$ converges smoothly to $\frac{1}{\bar{z}}$. It is straightforward to check that the corresponding $f_\varepsilon\colon [0,\pi] \to [0,3\pi]$ is given by

$$f_\varepsilon(r) := 2\arctan\left(\frac{(\tan(r/2))(1-\varepsilon^2\tan^2(r/2))}{\varepsilon^2-\tan^2(r/2)}\right).$$

(The values taken by arctan *in the vicinity of $\tan(r/2) = \varepsilon$ and $\tan(r/2) = 1/\varepsilon$ have to be chosen appropriately so as to make f_ε continuous.) This example shows how the weak limit of degree 1 α-harmonic maps may have degree -1.*

References

[1] A. Ambrosetti and A. Malchiodi. *Perturbation Methods and Semilinear Elliptic Problems on $\mathbb{R}^n$*, volume 240 of *Progress in Mathematics*, Birkhäuser Verlag, Basel, Switzerland, 2006.

[2] S.-Y. A. Chang, M. Gursky and S. Zhang. A conformally invariant gap theorem characterizing $\mathbb{CP}^2$ via the Ricci flow. arXiv:1809.05918 [math.DG], 2018.

[3] F. Duzaar and E. Kuwert. Minimization of conformally invariant energies in homotopy classes. *Calc. Var.*, 6:285–313, 1998.

[4] J. Eells and L. Lemaire. A report on harmonic maps, *Bull. London Math. Soc.*, 10:1–68, 1978.

[5] M. Gursky, C.L. Kelleher and J. Streets. A conformally invariant gap theorem in Yang-Mills theory. *Comm. Math. Phys.*, 361:1155–1167, 2018.

[6] T. Lamm. Energy identity for approximations of harmonic maps from surfaces. *Trans. Am. Math. Soc.*, 362:4077–4097, 2010.

[7] T. Lamm, A. Malchiodi and M. Micallef. Limits of α-harmonic maps. *J. Diff. Geom.*, 116:321–348, 2020.
[8] J. Li and X. Zhu. Energy identity and necklessness for a sequence of Sacks-Uhlenbeck maps to a sphere. *Ann. I. H. Poincaré, AN*, 36:103–118, 2019.
[9] Y. Li and Y. Wang. A weak energy identity and the length of necks for a sequence of Sacks-Uhlenbeck α-harmonic maps. *Adv. Math.*, 225:1134–1184, 2010.
[10] Y. Li and Y. Wang. A counterexample to the energy identity for sequences of α-harmonic maps. *Pacific J. Math.*, 274:107–123, 2015.
[11] F.C. Marques and A. Neves. Min-max theory and the Willmore conjecture. *Ann. of Math.*, 179:683–782, 2014.
[12] M. Micallef and J.D. Moore. Minimal two-spheres and the topology of manifolds with positive curvature on totally isotropic two-planes. *Ann. of Math.*, 127:199–227, 1988.
[13] M. Min-Oo. An L_2-isolation theorem for Yang-Mills fields. *Compositio Mathematica.*, 47:153–163, 1982.
[14] J. Sacks and K. Uhlenbeck. The existence of minimal immersions of 2-spheres. *Annals of Math.*, 113:1–24, 1981.
[15] J. Simons. Minimal varieties in riemannian manifolds. *Annals of Math.*, 88:62–105, 1968.
[16] K. Uhlenbeck. Morse theory on Banach manifolds. *J. Funct. Anal.*, 10:430–445, 1972.

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