# Some examples of Riemannian metrics of curves

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*(This is the handout companion to the seminar held as by the title; this handout is a proper superset of the seminar.)*

—1. Thanks

I wish to thank the organizers, and the Mathematisches Institut of the Albert-Ludwigs-Universität Freiburg, for the kind invitation.

—2. Structure

- In the first part we will see basilar concepts;
- In the second part we will see an example of first order Sobolev metric;
- In the third part we will see an example of second order Sobolev metric.

# Part I
# Introduction

## 1 Introduction

### 1.1 Curves and shapes

—3. Curves

Suppose that $c$ is a **closed curve** in the plane; we represent it by a parameterization

$$c : S^1 \to \mathbb{R}^2$$

(where $S^1 = \{x \in \mathbb{R}^2 : |x| = 1\}$ is the circle).

We'll call $M$ the **"manifold"** of all immersed regular curves.

(We'll assume that $M$ is a *differentiable manifold*, although there are important technical issues hidden in this assumption).

*We will present some comments on **open curves** that are maps*

$$c : [0,1] \to \mathbb{R}^2$$

—4. Homotopies

A motion of the curve is a **homotopy** $C = C(t,\theta)$ with

$$C : [0,1] \times S^1 \to \mathbb{R}^2$$

such that $C(t,\cdot) \in M$ for all $t$.

We will write $C' = \partial_\theta C$ and $\dot{C} = \partial_t C$.

$C$ represents a **path** $\gamma : [0,1] \to M$ in the space of curves.

—5. What for?

"Curves" are a kind of "shapes". What do we study "shapes" for?

- Shape Optimization
- Shape Analysis

## 1.2 Shape Optimization

—6. A method for Shape Optimization: Active Contours

A variational approach to solving Computer Vision problems is known as *Active Contours*: minimizing energy functionals $E(c)$ where $c$ is an embedded curve enclosing the region of interest $R$.

To seek the minimum we may think of performing a *gradient descent flow*

$$\frac{\partial C}{\partial t} = -\nabla E(C) \quad .$$

Such "gradient" $\nabla E$ is meaningful iff the manifold $M$ is equipped with some sort of "Riemannian metric"

—7. Gradient and metric

Given a curve $c \in M$ (the manifold of curves) we consider a **"tangent vector"** $h \in T_c M$ to be $h : S^1 \to \mathbb{R}^2$, a vector field along the curve.

We'll use the symbol

$$\langle h, k \rangle_c$$

for the **metric tensor** at $c$ computed on $h, k \in T_c M$.

If the energy $E$ is reasonably smooth, then we may compute the **"Gâteaux Derivative"**, denoted by $DE(c;h)$ or $D_{c;h}E$, defined by

$$DE(c;h) = D_{c;h}E = \lim_{\varepsilon \to 0} \frac{E(c+\varepsilon h) - E(c)}{\varepsilon} \quad .$$

If $M$ is a Riemannian manifold then the **gradient** $\nabla E$ at $c$ is implicitely defined by

$$\langle \nabla E(c), h \rangle_c = DE(c;h) \quad \forall h \in T_c M \quad .$$

## 1.3 Shape Analysis

—8. Shape Analysis

- Distances between curves
- Averages for curves
- Principle Component Analysis for curves
- Probabilistic models of curves

For all the above, a Riemannian Metric gives a well-founded and principled approach.

## 1.4 Group actions

—9. Reparameterization

Coming back to the beginning: we decided to represent a curve using parameterization

$$c : S^1 \to \mathbb{R}^2 \quad .$$

Any other parameterization may be written as $c \circ \varphi$ where $\varphi : S^1 \to S^1$ is a *diffeomorphism.*

The group $\mathrm{Diff}(S^1)$ of diffeomorphisms acts on the manifold $M$ of curves by right composition; its actions is the **reparameterization**.

We will require that the Riemannian metric on the manifold $M$ be **invariant** for the group action of *reparameterizations.*

—10. Group action

More in general.

**Definition 1.** Let $\mathcal{G}$ be a group. We say that $\mathcal{G}$ **acts on** $M$ if there is a map

$$\begin{array}{ccc} \mathcal{G} \times M & \to & M \\ g, m & \mapsto & g \cdot m \end{array}$$

that respects the group operations , that is, such that, if $e \in \mathcal{G}$ is the identity element then $e \cdot m = m$, and for any $g, h \in \mathcal{G}, m \in M$

$$h \cdot (g \cdot m) = (h \cdot g) \cdot m \quad .$$

—11. Group action

There are other interesting groups acting on the plane $\mathbb{R}^2$, and hence on curves (by left composition).

- the group of rotations
- the group of translations
- the group of rescalings
- ...

We will call **shape** a curve up to the actions of (a preselected family of) groups.

The action of the groups change the **pose** of the shape.

—12. Quotient space

Consider the equivalence relation $c_1 \sim c_2$ iff there is an element $g \in \mathcal{G}$ *s.t.* $c_1 = g \cdot c_2$. We write

$$M/\mathcal{G} = M/\sim$$

for the *quotient space*; equivalence classes $[c]$ are called **orbits**.

(In some important cases unfortunately $M/\mathcal{G}$ is not a differentiable manifold...)

—13. Geometric distance

Given a Riemannian metric on $M$ that is independent of parameterizations: then it projects to $B = M/\text{Diff}$.

Let $d$ be the distance on $M$ induced by a Riemannian metric.

We then study the projected distance on $B$ , by defining the **geometric induced distance**

$$d_B(c_0, c_1) = \inf_{\phi} d(c_0, c_1 \circ \phi)$$

for $\phi \in \text{Diff}$ all possible reparameterizations.

Note that $d_B$ is defined on parametric curves, but is independent of reparameterization.

## 1.5 Short history

—14. Short history

There is a moltitude of metrics proposed in the literature; for lack of time we will concentrate on few examples: all variations on the idea of *Sobolev–type geometric metrics*.

—15. $H^0$ metric

The simplest metric may be as follows:

$$\langle h, k \rangle_{H^0} := \int_c h \cdot k \, \mathrm{d}s$$

where integration is by arc parameter along $c$; $h \cdot k$ is scalar product in $\mathbb{R}^2$.

This metric was used in most of the literature in *Active Contours*, at least since [Caselles et al., 1993],[Malladi et al., 1995],[Kichenassamy et al., 1995],[Caselles et al., 1995].

Surprisingly, the metric $H^0$ does not yield a well define metric structure, since the associated distance $d_B$ is identically zero. (This striking fact was first described in [Michor and Mumford, 2006]).

## 1.6 Sobolev–type metrics

—16. Sobolev–type metrics

Let $D_c h(\theta) := \frac{h'(\theta)}{|c'(\theta)|}$ be the derivative in arc parameter.

Many authors [1] studied metrics of the form

$$\langle h, k\rangle_G := a_0 \int_c h \cdot k \, \mathrm{d}s + \int_c D_c^N h \cdot D_c^N k \, \mathrm{d}s \quad ,$$

where $N \geq 1$ is the *degree*.

Sobolev metrics have very good properties in shape optimization; hence the name *Sobolev Active Contours*

Also, when the degree $N \geq 1$ the distance $d_B$ is not degenerate.

—17. The gradient problem

Consider again the problem of computing the gradient of an energy, were we have already represented

$$D_{c,h}E = \int_c h \cdot v \, \mathrm{d}s$$

for an appropriate vector field $v$.

($v$ is often precomputed in the literature; in this context we may call it the $H^0$-gradient)

We need to solve the **gradient problem**: find $f = \nabla E(c)$ in

$$\langle f, h\rangle_c = \int_c h \cdot v \, \mathrm{d}s \quad \forall h \in T_c M \quad .$$

By *designing* the metric, we can simplify this problem. Consider the Sobolev metric of degree $N = 1$.

- If the metric is

  $$\langle h, k\rangle_{H^1} \doteq a_0 \int_c h \cdot k \, \mathrm{d}s + \int_c D_c h \cdot D_c k \, \mathrm{d}s$$

  then we need to solve

  $$-D_c D_c f + a_0 f = v$$

  this needs a convolution or an iterative approximation.

- [Sundaramoorthi et al., 2005] considered the following geometric Sobolev-type metric

  $$\langle h, k\rangle_{\tilde{H}^1} \doteq a_0 \int_c h \, \mathrm{d}s \cdot \int_c k \, \mathrm{d}s + \int_c D_s h \cdot D_s k \, \mathrm{d}s, \tag{1}$$

  in this case

  $$-D_c D_c f = v + \text{constant}$$

  and this is just computation of a primitive (twice).

—19. Sobolev Active Contours

[Sundaramoorthi et al., 2005, 2006, 2007, 2009b] proved that *Sobolev Active Contour* has a beneficial effect: it regularizes otherwise ill-posed minimization flows; the flow is more robust against data noise.

[1] [Sundaramoorthi et al., 2005, 2006, 2007, 2008, 2009b,a, 2011] [Mennucci et al., 2008] [Charpiat et al., 2005] [Younes, 1998] [Michor and Mumford, 2007] [Younes et al., 2008] [Bruveris et al., 2014] [Bauer et al., 2014] [Bruveris, 2015] [Martin Bauer and Kolev, 2018] [Srivastava et al., 2011] [Tumpach and Preston, 2016]

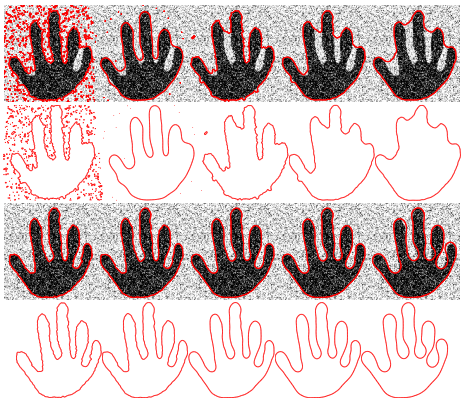


Figure 1: Top two rows: minima of $E$, with length penalty, by $H^0$ flow (top two rows). Left to right: $\alpha = 10000, 50000, 90000$. Bottom two rows: minima of $E$ , with elastic regularization, $H^1$ flow. Left to right: $\alpha = 0, 0.1, 5, 10, 25$. The second and fourth row show the same result as the row above them, but the image is removed for visibility.

In this experiment from [Sundaramoorthi et al., 2009b], we show a case when the scale-invariant elastic regularity term

$$E(c) = E_{\text{data}}(c) + \alpha \operatorname{len}(c) \int_c \kappa^2(s)\, \mathrm{d}s \quad . \tag{2}$$

is more beneficial than the using the traditional length penalty

$$E(c) = E_{\text{data}}(c) + \alpha \operatorname{len}(c) \quad . \tag{3}$$

In the following experiment, the image-based term $E_{\text{data}}$ is the Chan-Vese functional.

Note that the elastic regularizer does not generally have a length shrinking effect, but keeps the contour regular. This length shrinking effect may have a detrimental effect as shown in Fig. 1. Note that the length penalty restricts the curve from moving into the groves between the fingers. The elastic regularity term, on the other hand, has no such restriction, and makes the curve more smooth and rounded.

—21. Structure of $M$

What is the structure of the space of curves endowed with a Sobolev metric? [Mennucci et al., 2008] proved that when $N = 1$ then any $c$ that is Lispchitz can be approximated by smooth curves; whereas when $N \geq 2$ any limit of a Cauchy sequence of smooth curves will have a curvature $\kappa$ that is in $L^2$.

—22. Sobolev–type metrics of higher order

When the degree is $N \geq 2$, the space $M$ can be identified as the open subset $M = \mathcal{I}^N$ of immersed curves inside the standard Sobolev space $H^N$ of maps $c : S^1 \to \mathbb{R}^2$. In this case:

- [Bruveris et al., 2014] shown that the space of planar Sobolev immersions $\mathcal{I}^N$ is geodesically complete for a Sobolev metric with constant coefficients; and the metric is a smooth Riemannian metric on $\mathcal{I}^N$.

- [Bauer and Harms, 2015] noted that the same method also implies metric completeness of the space of Sobolev immersions $\mathcal{I}^N$;
- Thm. 5.2 in [Bruveris, 2015] shows that any two curves may be connected by a minimizing geodesic.

—23.

*(Let me recall that: The Hopf-Rinow theorem is false in infinite dimensions [Atkin, 1975].*

*Knowing that the Riemannian manifold is complete does not imply the existence of minimal geodesics — when it is true, we need to prove it, case by case).*

# 2 Designing a metric

What is discussed in this section is extracted from [Mennucci, 2018].

In the following for convenience we will represent the metrics using the **norm**

$$\|h\|_c = \sqrt{\langle h, h\rangle_c}$$

for $h \in T_c M$.

(The scalar product is uniquely identified by the norm, using *polarization*).

We can change point of view. We would like to build a metric satisfying some good properties; such as those seen before, and more.

Given a group, we would like to design a metric that factors according to the action of the group.

The key idea is in choosing a submanifold $M_0 \subset M$ that intersects each orbit of the action of $\mathcal{G}$ in only one point.

*(This implies that the bundle $\pi : M \to M/\mathcal{G}$ is trivial – important idea, skipped for lack of time).*

—27. The case of translations

Let's see a simple example: let $\mathcal{G} = \mathbb{R}^2$ be the group of translations. We may consider two choices for $M_0$, each associated to a map $\Phi : M_0 \times \mathcal{G} \to M$ .

- Let $M_0$ be the set of curves $c$ with $c(\theta_0) = 0$, where $\theta_0 \in S^1$ is fixed. The map is

$$\Phi(\tilde c, v) = \tilde c + v \quad , \quad \Phi^{-1}(c) = (\tilde c, v) = \big(c - c(\theta_0), c(\theta_0)\big) \quad .$$

- Let

$$\operatorname{avg}_c(c) = \fint_c c \, \mathrm{d}s = \frac{1}{\operatorname{len}(c)} \int_c c \, \mathrm{d}s$$

be the center of mass; let $M_0 = \{c : \operatorname{avg}_c(c) = 0\}$ then

$$\Phi(\tilde c, v) = \tilde c + v \quad , \quad \Phi^{-1}(c) = (\tilde c, v) = \big(c - \operatorname{avg}_c(c), \operatorname{avg}_c(c)\big) \quad .$$

In both cases $M_0$ is a representation for "*curves up to translation*". **The second map is better**, since the submanifold $M_0$ is invariant for reparameterizations.

—28. Factoring the space

The general scheme is as follows. Suppose that $\mathcal{G}$ acts freely. Suppose that there is a submanifold $M_0 \subset M$ that intersects each orbit of the action of $\mathcal{G}$ in only one point, and is transversal to the orbits.

This generates a map

$$\Phi : M_0 \times \mathcal{G} \to M$$
$$(\tilde{c}, g) \mapsto c$$

where $c \in M$ is associated to the unique $\tilde{c} \in M_0 \cap [c]$ and the unique $g \in \mathcal{G}$ such that $c = g \cdot \tilde{c}$.

This map $\Phi$ is a diffeomorphism.

Then the bundle

$$\pi : M \to {}^{M}/_{\mathcal{G}}$$

is trivialized: indeed $\pi_{|_{M_0}}$ will be a diffeomorphism of $M_0$ to $M/\mathcal{G}$.

The process of associating $c \in M$ to $\tilde{c}, g \in M_0 \times \mathcal{G}$ is often found in the literature, by the name of *normalization* or *registration*, to *factor out* the effect of a group action.

—29. Normalization and Length

**Lemma 2.** *For any smooth $\gamma : [0,1] \to M$ we can find another $\tilde{\gamma} : [0,1] \to M_0$ such that $\tilde{\gamma}(t)$ and $\gamma(t)$ are in the same orbit;* i.e. *there is a smooth path $g(t) \in G$ such that $\tilde{\gamma}(t) = g(t)\gamma(t)$.*

The length of a path $\tilde{\gamma}$ in $M_0$ is not necessarily the length of the path $\pi\gamma$ projected in $M/G$.

**Lemma 3.** *When $M_0$ is orthogonal to each orbit, then a minimal geodesic in $M/\mathcal{G}$ corresponds to a minimal geodesic in $M_0$ (up to normalization).*

Since we are actually designing metrics, we look at this the other way around: if we can find a $M_0$ as above, we will then design a metric such that $M_0$ is orthogonal to the orbits.

## 2.1 Path-wise and point-wise invariance

—30. Point-wise invariance

**Definition 4.** Let $g \in \mathcal{G}$, let

$$L_g : M \to M$$
$$L_g(c) = g \cdot c$$

be the action: then we will say that **the metric is (point-wise) invariant** for the action of $\mathcal{G}$ iff $L_g$ is an isometry (for any given $g \in \mathcal{G}$).

*By Noether's Theorem, for any isometric group action there is a **momentum**, a quantity that is conserved along geodesics.*

—31. Vertical space

For any group acting on $M$ we have a vertical bundle.

The **vertical space** $V_c$ is the vector space $V_c \subset T_cM$ that is tangent to the orbit of the action of $\mathcal{G}$ at the curve $c$.

*(Given a metric, the **horizontal space** $W_c$ is the orthogonal complement inside $T_cM$ to $V_c$. We will not use this concept in this presentation.).*

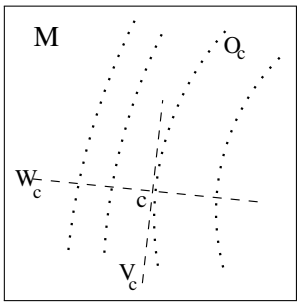



Figure 2: The orbits $O_c$ are dotted, the spaces $W_c$ and $V_c$ are dashed.

Let

$$\pi : M \to {}^{M}\!/_{\mathcal{G}} = B$$

be the canonical projection. The figure 3 may also help in understanding. The whole orbit $O_c$ is projected to $[c]$. The vertical space $V_c$ is the kernel of $D\pi_c$, so it is projected to 0.

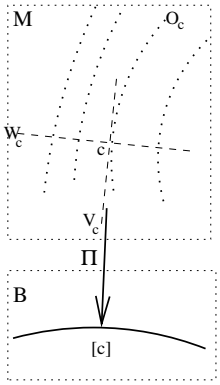



Figure 3: The vertical and horizontal spaces, and the projection $\pi$ from $M$ to $B$.

—33. Path-wise invariance

Let $\gamma \in H^1([0,1] \to M)$ be path. The **geodesic action**, or **geodesic energy**, of $\gamma$ is

$$\int_0^1 \|\dot\gamma\|_\gamma^2 \,\mathrm{d}t \quad .$$

**Definition 5.** We say that a semimetric is ***path-wise*** **invariant** for the action of the group $\mathcal{G}$ if

$$\int_0^1 \|\dot\gamma\|_\gamma^2 \,\mathrm{d}t = \int_0^1 \|\dot{\tilde\gamma}\|_{\tilde\gamma}^2 \,\mathrm{d}t$$

for any choice of smooth paths $\gamma : [0,1] \to M$ and $A : [0,1] \to \mathcal{G}$; where we define $\tilde\gamma(t) = A(t) \cdot \gamma(t)$.

This implies that $\|\|$ is a semimetric and not a metric.

More can be said.

**Proposition 6.** *These two facts are equivalent.*

- *the semimetric is path-wise invariant,*
- *the semimetric is point-wise invariant and the null space of $\|\cdot\|_c$ contains* ***the vertical space*** *$V_c$, namely, $\|v\|_c = 0$ for all $v \in V_c$. (Intuitively, $\|\cdot\|$ does not measure the infinitesimal action of $\mathcal{G}$).*

So a semimetric that is path-wise invariant cannot be a metric. So, when we will design a metric on $M$, then we will add other terms to $\|\cdot\|$ to create a true metric on the space.

## 2.2 Designing, for one group action

—35. Designing, for one group action

We use the map $\Phi$ that trivializes the bundle. We want to design a metric that splits orthogonally this map.

As a first step we define a metric $\|\cdot\|_{\mathcal{G}}$ on $\mathcal{G}$.

As a second step we define a metric $\|\cdot\|_0$ on $M_0$. To this end we define a semimetric $\|h\|_0$ on $M$ that is path-wise invariant, and projects to a metric in $M_0$. (This is equivalent to asking that the null space of $\|h\|_0$ at $c \in M$ be exactly the vertical space $V_c$).

*Note that we view $\|h\|_0$ at the same time as a metric in $M_0$ and as a semi metric in $M$. This largely simplifies the analysis and the applications.*

The full metric on $M$ is then defined by pullback as

$$\|h\| = \sqrt{\|\hat{h}\|_0^2 + \|\hat{g}\|_G^2} \tag{4}$$

where the decomposition

$$T_c M \to T_{\tilde{c}} M_0 \times T_g \mathcal{G} \quad , \quad h \mapsto (\hat{h}, \hat{g}) \tag{5}$$

is the derivative of the map $\Phi^{-1}$.

## 2.3 Geodesics

—36. Geodesics

There is a benefit to the scheme.

**Proposition 7.** *Let $c_0, c_1 \in M$, let $g \in \mathcal{G}$ and let $\tilde{c}_1 = g c_1$. Suppose that $C : [0,1] \to M$ is the geodesic connecting $c_0$ to $c_1$, and $\tilde{C} : [0,1] \to M$ is the geodesic connecting $c_0$ to $\tilde{c}_1$: then $\tilde{C}(t) = \xi(t) C(t)$ where $\xi(t)$ is the geodesic connecting the identity in $G$ to $g$; and vice-versa. In particular the projections of $C$ and $\tilde{C}$ onto the quotient space $M/G$ are identical.*

This is not true for generic metrics on $M$ (even if they are point-wise invariant).

—37. Minimal geodesics

We show how this strategy affects the computation of geodesics.

Let $c_0, c_1 \in M$. We want to find a geodesic $C : [0,1] \to M$ connecting $c_0$ to $c_1$.

We first decompose the endpoints using the map $\Phi$ so we find $g_0, g_1 \in \mathcal{G}$ and $\tilde{c}_0, \tilde{c}_1 \in M_0$ such that $c_0 = g_0 \tilde{c}_0$ and $c_1 = g_1 \tilde{c}_1$.

We compute a minimal geodesic $g(t)$ connecting $g_0$ to $g_1$. If we carefully chose the metric in $\mathcal{G}$, then this will be easy.

We then look for a geodesic $\xi(t)$ in $M_0$ connecting $\tilde{c}_0$ to $\tilde{c}_1$.

By the definition, the geodesic minimizes the *geodesic energy*

$$\min \left\{ \int_0^1 \|\dot{\xi}(t)\|^2_{0,\xi(t)} \, \mathrm{d}t : \xi : [0,1] \to M_0 \right\} \tag{6}$$

in the family of all smooth paths $\xi : [0,1] \to M_0$ connecting $\tilde{c}_0$ to $\tilde{c}_1$. Since $\|\cdot\|_0$ is path-wise invariant, then we can equivalently compute the minimum in the family of all smooth paths $\xi : [0,1] \to M$ connecting $\tilde{c}_0$ to $\tilde{c}_1$ — that is, dropping the constraint requiring that $\xi(t) \in M_0$ at all times.

Eventually $g(t)\tilde{\xi}(t)$ will be a minimal geodesic.

`TL;DR` We can compute the geodesic separately, in "pose" and in "shape".

### 2.4 Designing, for many group actions

—39. Designing, for many group actions

What if there are many groups $\mathcal{G}_1, \ldots, \mathcal{G}_k$ acting on $M$?

The "perfect metric" would be as follows:

$$\|h\|^2 = \|h\|^2_{\mathcal{G}_1} + \ldots + \|h\|^2_{\mathcal{G}_k} + \|h\|^2_S$$

where the null space of a semimetric $\|h\|^2_{\mathcal{G}_i}$ contains the direct sum of the vertical spaces of all other groups; whereas the kernel of the *"shape semimetric"* $\|h\|^2_S$ would be the direct sum of the vertical spaces of all groups.

This is currently not achieved.

There are many groups acting on curves.

Unfortunately some interfere.

For example, a rotation of a circle (around its center) is equivalent to a reparameterization.

For this reason, we cannot factor out all actions alltogether.

# Part II
# A first order example

## 3 A first order example: the Stiefel metric

### 3.1 History

—41. History

This idea first appeared in [Younes, 1998]

Then it was rewritten in [Younes et al., 2008] as follows.

—42. Square root representation

We always identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$.

Given two smooth functions $e, f : [0,1] \to \mathbb{R}$ we define the map $\Phi$ by

$$c = \Phi(e,f) \quad c(\theta) = c(0) + \frac{1}{2}\int_0^\theta (e+if)^2(\xi)\,\mathrm{d}\xi \tag{7}$$

where $i$ denotes the imaginary unit; this map uniquely identifies a curve $c : S^1 \to \mathbb{C}$ up to the choice of the base point $c(0)$, or equivalently, up to the choice of the centroid $\mathrm{avg}_c(c)$.

Vice versa given $c$ immersed smooth there are two such choices, $(e,f)$ and $(-e,-f)$.

—43. Stiefel manifold

We now require $c = \Phi(e,f)$ to be a closed curve of unit length.

For $c$ to be closed we must have that

$$0 = c(1) - c(0) = \frac{1}{2}\int_0^1 (e+if)^2(\theta)\,\mathrm{d}\theta = \frac{1}{2}\int_0^1 [e^2(\theta) - f^2(\theta) + 2ie(\theta)f(\theta)]\,\mathrm{d}\theta, \tag{8}$$

and for the curve to of unit length we must have that

$$1 = \int_0^1 |c'(\theta)|\,\mathrm{d}\theta = \frac{1}{2}\int_0^1 (e^2(\theta) + f^2(\theta))\,\mathrm{d}\theta. \tag{9}$$

By conditions (8) and (9), then the pair $(e,f)$ belongs to

$$\mathbf{St}(2, C^\infty) = \{(e,f) \in C^\infty \times C^\infty : \|e\|_{L^2} = \|f\|_{L^2} = 1,\ \langle e, f\rangle_{L^2} = 0\}$$

where the above $L^2$ norms and inner product are the standard ones on $L^2([0,1])$.

$$\mathbf{St}(2, C^\infty) \subset \mathbf{St}(2, L^2)$$

where it is dense;

$\mathbf{St}(2, L^2)$ is a closed smooth submanifold of $L^2 \times L^2$: hence it is a complete Riemannian manifold when we use the metric induced from the scalar product $L^2 \times L^2$.

$\mathbf{St}(2, C^\infty)$ and $\mathbf{St}(2, L^2)$ are known as a ***Stiefel* manifold**.

—45. Isometry

Let

$$M_d = \{c \in M : \mathrm{len}(c) = 1,\ \mathrm{avg}_c(c) = 0\} \quad , \tag{10}$$

be the space of curves with length 1 and center of mass in the origin.

The above square root transformation $\Phi$ is an isometry if the space $M_d$ is endowed with the metric

$$\int_c D_s h \cdot D_s k\,\mathrm{d}s$$

and the *Stiefel manifold* $\mathbf{St}(2, C^\infty)$ is endowed with the Riemannian metric induced from the ambient space $L^2 \times L^2$.

## 3.2 Stiefel metric

—46. Decomposition of tangent space

To define the metric $\mathbb{H}$, [Sundaramoorthi et al., 2011] defined the following decomposition for $c \in M$ and $h \in T_c M$:

$$h = h^t + h^l(c - \text{avg}_c(c)) + \text{len}(c)h^d \tag{11}$$

where $h^t$ is the component of $h$ that changes the centroid of $c$, $h^l(c - \text{avg}_c(c))$ is the component of $h$ that changes the scale (length) of $c$, and $h^d$ is the component of $h$ that deforms $c$. The components $h^t$ and $h^l$ of $h$ are defined as

$$h^t = D_{(c;h)}(\text{avg}_c(c)) \in \mathbb{R}^2 \tag{12}$$

$$h^l = D_{(c;h)}(\log \text{len}(c)) \in \mathbb{R} \tag{13}$$

$$h^d = \frac{1}{\text{len}(c)}[h - h^t - h^l(c - \text{avg}_c(c))] \tag{14}$$

—47. Stiefel metric

If $h, k \in T_c M$ are decomposed as above, then [Sundaramoorthi et al., 2011] defined the Riemannian metric $\mathbb{H}$ as

$$\langle h, k \rangle_{\mathbb{H}} \doteq h^t \cdot k^t + \lambda_l h^l k^l + \lambda_d \,\text{len}(c)^2 \oint_c D_s h^d \cdot D_s k^d \,\mathrm{d}s, \tag{15}$$

where the first two products are the Euclidean dot products, the last term is a normalized geometric Sobolev metric, and $\lambda_l, \lambda_d > 0$ are (constant) weights.

## 3.3 Explanation

The third term of the metric may be rewritten directly as a function of $h \in T_c M$ by using the identity

$$\oint_c D_s h^d \cdot D_s k^d \,\mathrm{d}s = \oint_c D_s h \cdot D_s k \,\mathrm{d}s - \oint_c D_s h \cdot D_s c \,\mathrm{d}s \oint_c D_s c \cdot D_s k \,\mathrm{d}s \ . \tag{16}$$

—49. Explanation *(ex post!)*

The third term in the metric

$$\oint_c D_s h^d \cdot D_s k^d \,\mathrm{d}s$$

is a semimetric whose null space coincides with the vertical space of translations and rescalings; so is path-wise invariant for those actions.

*(This was not the explanation given at the time of writing of [Sundaramoorthi et al., 2011])*

## 3.4 Isometry

—50. Isometric decomposition

Let again

$$M_d = \{c \in M : \text{len}(c) = 1,\, \text{avg}_c(c) = 0\} \quad , \tag{17}$$

We associate the Euclidean metric to $\mathbb{R}^n \times \mathbb{R}$ and the metric

$$\langle h, k \rangle_{M_d} = \int_{\tilde{c}} D_s h \cdot D_s k \ \mathrm{d}s \tag{18}$$

to $M_d$. This metric is the restriction of the metric $\mathbb{H}$ to $M_d$.

The metric $\mathbb{H}$ is associated to an isometry between the space of curves $M$ and the space $\mathbb{R}^2 \times \mathbb{R} \times M_d$.

**Theorem 8.** *Let $\lambda_l = \lambda_d = 1$ for simplicity. We define the map of $c \in M$ to $(v, l, \tilde{c}) \in \mathbb{R}^2 \times \mathbb{R} \times M_d$, and its inverse,*

$$c \in M \mapsto (v, l, \tilde{c}) = \left( avg_c(c) \ , \ \log \mathrm{len}(c) \ , \ \frac{c - avg_c(c)}{\mathrm{len}(c)} \right)$$
$$(v, l, \tilde{c}) \mapsto c = v + e^l \tilde{c} \in M$$

*This map is an isometry.*

## 3.5 Properties

—52. Momenta, geodesics

This new metric enjoys the following properties:

1. Centroid translations, scale changes and deformations of the curve are orthogonal. Moreover, the space of curves can be decomposed into a product space consisting of three components as shown in Thm. 8.

2. there is a fast and easy way to compute gradients of commonly used energies with respect to the new metric $\mathbb{H}$, that does not need convolutions nor iterative approximations.

3. Geodesics in this new metric can be numerically computed efficiently.

## 3.6 Geodesics

Classically, the Stiefel manifold $\mathbf{St}(p, \mathbb{R}^n)$ is defined as the set of all frames composed of $p$ orthonormal vectors in $\mathbb{R}^n$ (with $1 \leq p \leq n$); those frames are represented as $n \times p$ matrices. Geodesics in Stiefel manifolds $\mathbf{St}(p, \mathbb{R}^n)$ are known to have closed form solutions as demonstrated by [Edelman et al., 1998].[2]

The closed form formula for geodesic holds *mutatis mutandi* in $\mathbf{St}(p, L^2)$.

**Proposition 9** (Exponential Map in $\mathbf{St}(p, \mathbb{R}^n)$ )**.** *Suppose that $\mathbf{St}(p, \mathbb{R}^n)$ is endowed with the Euclidean metric, i.e.,*

$$\langle A, B \rangle = tr(A^T B) \ \ ,$$

*then the geodesic equation is*

$$\ddot{Y} + Y(\dot{Y}^T \dot{Y}) = 0 \ \ . \tag{19}$$

[2] [Edelman et al., 1998] credits a personal communication by R. A. Lippert for the final closed form formula (20).

*The solution is*

$$(Y(t)e^{At}, \dot{Y}(t)e^{At}) = (Y(0), \dot{Y}(0)) \exp t \begin{pmatrix} A & -S \\ Id & A \end{pmatrix} \tag{20}$$

*where* $A = Y^T(0)\dot{Y}(0)$, $S = \dot{Y}^T(0)\dot{Y}(0)$, *and* $Id$ *is the* $p \times p$ *identity matrix.*

The proof and discussion of these results is in Section 2.2.2 in [Edelman et al., 1998].

—55. Minimal geodesics

**Theorem 10.** *Any two points in* $\mathbf{St}(2, L^2)$ *are connected by a minimal geodesic [Harms and Mennucci, 2012]*

The [Edelman et al., 1998] closed form formula for geodesic *greatly* simplifies the computation of *minimal geodesics*.

The algorithm to compute a minimal geodesic reduces to compute an optimal problem with 5 real parameters (regardless of the dimensionality of the numerical approximation).

## 3.7 Applications

Having a well-defined Riemannian manifold, it is possible to define a Kalmann-type filtering.

We set up a hidden constant velocity model (by exponential map and parallel transport).

We will see some movies.

In the movies:

- green is observation, the contour extracted from the image;
- red is the estimated curve
- blue is the estimated velocity (vector field).

Some frames of the movie are in Fig. 4

## 3.8 Conclusions

—58. Metric completion

We started from smooth curves $c$, in this case $(e, f)$ is smooth; the family of such frame $\mathbf{St}(2, C^\infty)$ is dense in $\mathbf{St}(2, L^2)$. (Proved in [Deiala, 2010])

*Vice versa* $\mathbf{St}(2, L^2)$ represents curves that are absolutely continuous.

But an absolutely continous curve $c$ is represented by **infinitely many** frames with $e, f \in L^2$

—59. Conclusions

**good** This model can be completed to a well known Riemannian manifold: $\mathbf{St}(2, L^2)$.

**good** The computation of gradients is fast (no convolutions, no iterative approximation); the gradient regularizes ill conditioned and ill posed problems.

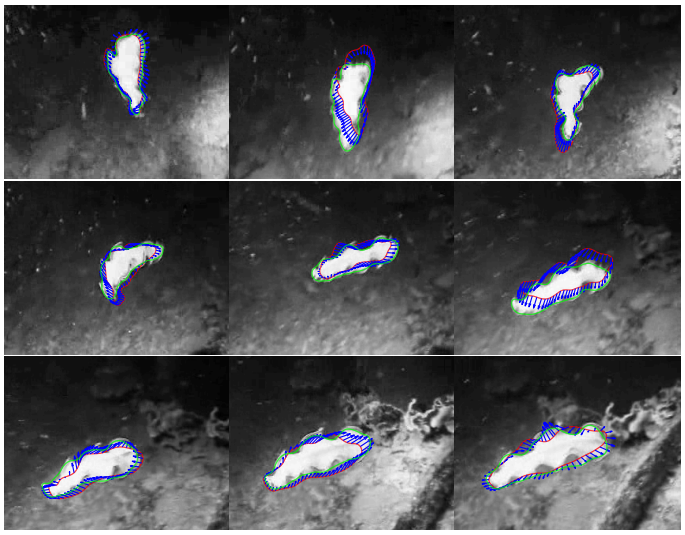


Figure 4: Tracking a flatworm (left to right, top to bottom) using the proposed filtering technique. Green is observation; red is the estimated curve; blue is the estimated vector field.

**good** Any two points in $\mathbf{St}(2, L^2)$ are connected by a minimal geodesic [Harms and Mennucci, 2012]

**good** The equation for geodesics is solved in closed forms, the algorithm just needs to compute the exponential of a $4 \times 4$ matrix.

**good** Simple fast numerical algorithm to compute a minimal geodesic in $\mathbf{St}(2, L^2)$

**good** In the quotient space $B = {}^{M}/_{\mathrm{Diff}}$ of *"curves up to parameterization"*, the distance $d_B$ between curves "up to parameterization" is not degenerate.

**bad** There may be curves that are not connected by a minimal geodesic "up to parameterization".

**bad** The basilar geometric concept of *index number* of the curve is lost in $\mathbf{St}(2, L^2)$. (Proved in [Deiala, 2010])

**bad** An absolutely continous curve $c$ is represented by **infinitely many** frames with $e, f \in L^2$

**bad** Any algorithm that computes minimal geodesics between curves should consider a very large number of possible representatives *(unless better analysis shows otherwise).*

# Part III
# A second order example

## 4 A second order example: the delta metric

(What follows is extracted from the second part of [Mennucci, 2018]).

### 4.1 Delta seminorm

—62. Delta operator

We always identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$. Recall that $D_c h := h'/|c'|$.

**Definition 11.** We propose the "delta" operator

$$\Delta_c h := h'/c' \tag{21}$$

where the division is in the sense of complex numbers.

The difference between $D_c h$ and $\Delta_c h$ is akin to the difference between Lagrangian coordinates and Eulerian coordinates (but transported to the level of first derivatives). When using $D_c h$ we are considering $h$ to be positioned in the ambient space $\mathbb{R}^2$, and we are just renormalizing $h'$ by $|c'|$, so that $D_c h$ will be reparameterization invariant. When using $\Delta_c h$ we are considering $h$ to be anchored to the curve, and so we are normalizing as above, and moreover we are interested in the relative angle between $h'$ and $c'$, not in the angle between $h'$ and a fixed reference versor in the space.

—63. Path-wise invariance

The kernel of $D_c D_c$ is given by constant vector fields (when we consider closed curves).

Instead if $\Delta_c \Delta_c h = 0$ then $h = \alpha c + \beta$ for two constants $\alpha, \beta \in \mathbb{C}$.

This kernel is exactly the vertical space of rotations, translations and rescalings.

—64. Delta semimetric

We then define the second order seminorm

$$\|h\|_{\Delta^2,c} := \sqrt{\int_c |\Delta_c^2 h|^2 \, \mathrm{d}s} = \sqrt{\int_{S^1} \left| \frac{1}{c'} \left( \frac{h'}{c'} \right)' \right|^2 |c'| \, \mathrm{d}\theta}$$

where products are in $\mathbb{C}$ and the absolute value $|\cdot|$ is the norm in $\mathbb{C}$. It is a seminorm that is path-wise invariant for rotations, translations and rescalings. Moreover it is point-wise invariant for reparameterization. It will be our *seminorm for shape*.

—65. Log coordinates

Consider a homotopy of curves $C : [0,1] \times S^1 \to \mathbb{C}$.

Recall that $C' = \partial_\theta C$ and $\dot{C} = \partial_t C$.

Then represent $C$ by a pair $E, F : [0,1] \times S^1 \to \mathbb{R}$ by the relation

$$C'(t,\theta) = e^{E(t,\theta) + iF(t,\theta)} \quad ;$$

we call this **representation in log–cordinates**.

If we know $E, F$ and the center of mass $\mathrm{avg}_c(C)(t)$ at all $t$, then $C$ is uniquely determined.

The representative $F$ is not unique, we can add multiples of $2\pi$ to it.

Note that $e^E \,\mathrm{d}\theta$ will replace $\mathrm{d}s$ in integration by arc parameter.

Deriving

$$C' = e^{E+iF}$$

in time we obtain

$$\dot{C}' = (\dot{E} + i\dot{F})e^{E+iF}$$

So we can express a curve $c \in M$ and its infinitesimal displacement $h \in T_cM$ using log-coordinates, as $(\tilde{e}, f)$ and respectively $(\hat{e}, \hat{f})$ satisfying

$$c'(\theta) = e^{\tilde{e}(\theta)+if(\theta)} \quad , \quad h'(\theta) = (\hat{e}(\theta) + i\hat{f}(\theta))e^{\tilde{e}(\theta)+if(\theta)} \quad .$$

—67. Delta seminorm in Log coordinates

The Delta seminorm has a simple representation in log-coordinates.

$$\|h\|^2_{\Delta^2,c} = \int_c |\Delta^2_c h|^2 \,\mathrm{d}s = \int_{S^1} (|\hat{e}'|^2 + |\hat{f}'|^2)e^{-\tilde{e}} \,\mathrm{d}\theta$$

Or, for homotopies, writing again

$$C'(t,\theta) = e^{E(t,\theta)+iF(t,\theta)}$$

we have

$$\|\dot{C}\|^2_{\Delta^2,C} = \int_C |\Delta^2_C \dot{C}|^2 \,\mathrm{d}s = \int_{S^1} (|\dot{E}'|^2 + |\dot{F}'|^2)e^{-E} \,\mathrm{d}\theta$$

Note that this involves only first order derivatives in $\theta$ (!)

So this is a second order norm, but in analysis and numerics it has the complexity of a first order norm.

—68. Companion seminorms

We now need seminorms associated to group actions.

- For translations, starting from the map

  $$\Phi(\tilde{c}, v) = \tilde{c} + v \quad , \quad \Phi^{-1}(c) = \big(c - \mathrm{avg}_c(c), \mathrm{avg}_c(c)\big) \quad .$$

  we use the norm of Gâteaux Derivative of the center of mass

  $$\|h\|_{\mathbb{t},c} := |D_{c,h}\mathrm{avg}_c(c)|$$

  (the pullback thru $\Phi$ of the Euclidean norm for $\mathbb{R}^2$)

- For rescalings, we consider the map

  $$\Phi(\tilde{c}, l) = e^l\tilde{c} = c \quad , \quad \Phi^{-1}(c) = (\tilde{c}, l) = (c/\,\mathrm{len}(c), \log \mathrm{len}(c))$$

  where $\mathrm{len}(c)$ is the length of the curve. The pullback of the standard metric on $\mathbb{R}$ is then

  $$\|h\|_{\mathrm{len},c} := \big|D_{c,h} \log \mathrm{len}(c)\big| \tag{22}$$

- For rotations, we express $c \in M, h \in T_cM$ using log-coordinates

$$c'(\theta) = e^{\tilde{e}(\theta)+if(\theta)} \quad , \quad h'(\theta) = (\hat{e}(\theta) + i\hat{f}(\theta))e^{\tilde{e}(\theta)+if(\theta)}$$

then we design the seminorm

$$\|h\|_{\mathbb{r},c} = \left| \int_0^1 \hat{f} e^{\tilde{e}} \, \mathrm{d}\theta \right| \quad . \tag{23}$$

(that is not associated, and cannot be associated, to a map $\Phi$).

## 4.2 Delta metric

—70. Full delta metric

Let $m_l, m_r, m_t > 0$ be fixed. We define then the metric

$$\begin{aligned} \|h\|^2_{(l\Delta^2+\mathrm{len}+r/l+t),c} := \mathrm{len}(c)\|h\|^2_{\Delta^2,c} + m_l\|h\|^2_{\mathrm{len},c} + \\ + m_r\|h\|^2_{\mathbb{r},c}/\,\mathrm{len}(c)^2 + m_t\|h\|^2_{\mathbb{t},c} \end{aligned} \tag{24}$$

(There are some conformal terms based on the length of the curve).

The definition of this metric requires three constants $m_l, m_r, m_t$. This is common to many models in the literature. In this model though we have an important property: geodesics (and in particular minimal length geodesics) do not depend on the choice of $m_l, m_t$.

A similar metric for open curves has a different term for rotations, in that case geodesics do not depend on the choice of $m_l, m_t, m_r$.

This metric is modular: e.g. if we wish to study *"curves up to rotation"* we just need to drop the third term: and so on.

## 4.3 Properties

—71. Curling

In the manifold of *open curves* there is another interesting group acting: **curling**.

Curling means: deforming an open ended curve without changing the parameterization.

Given the log-representation

$$c'(\theta) = e^{\tilde{e}(\theta)+if(\theta)}$$

the group of curling is represented by $H^1(S^1)$, the action of $\varphi \in H^1(S^1)$ on the curve is just by addition

$$(\tilde{e}, f) \mapsto (\tilde{e}, f + \varphi) \quad .$$

—72. Decomposition of shape seminorm

We again express $c \in M, h \in T_cM$ using log-coordinates

$$c'(\theta) = e^{\tilde{e}(\theta)+if(\theta)} \quad , \quad h'(\theta) = (\hat{e}(\theta) + i\hat{f}(\theta))e^{\tilde{e}(\theta)+if(\theta)}$$

We decompose for a moment the *"shape seminorm"*

$$\|h\|^2_{l\Delta^2_{\tilde{e}},c} := \mathrm{len}(c) \int_0^1 \Big( (\hat{e}')^2 + (\hat{f}')^2 \Big) e^{-\tilde{e}} \, \mathrm{d}\theta$$

in two components:

$$\|h\|^2_{l\Delta^2_{\tilde{e}},c} := \text{len}(c)\int_0^1 (\hat{e}')^2 e^{-\tilde{e}}\,\mathrm{d}\theta \quad , \quad \|h\|^2_{l\Delta^2_{\hat{f}},c} := \text{len}(c)\int_0^1 (\hat{f}')^2 e^{-\tilde{e}}\,\mathrm{d}\theta$$

—73. Invariances

We have this table of invariances of the semimetrics wrt the group actions.

| | $\|h\|_{l\Delta^2_{\tilde{e}},c}$ | $\|h\|_{l\Delta^2_{\hat{f}},c}$ | $\|h\|_{\text{len},c}$ | $\|h\|_{\mathbb{r}/l,c}$ | $\|h\|_{\mathbb{t},c}$ |
|---|---|---|---|---|---|
| reparameterization | .w. | *.W!* | PW | *.W!* | PW |
| curling | PW | .w. | PW | *.W-* | PW |
| scaling | PW | PW | .w. | PW | PW |
| rotation | PW | PW | PW | .w. | PW |
| translation | PW | PW | PW | PW | .w. |

Legenda: ".W" means point-wise invariance, "PW" means path-wise invariance.

We expect that a semimetric be point-wise invariant for the group action that is related to it. So there are ".w." entries along the diagonal: these are spots where ".w" is the correct behavior. (Indeed for any group action (that is, a row in the table) there must be a semimetric that is not path-wise invariant for that action — otherwise the sum of them would not be a metric)

Outside of the diagonal, we would love to see only "PW" entries; any such entry means that a semimetric (say $\|h\|_{l\Delta^2,c}$) is path-wise invariant for an action (say, translations): then this semimetric is, as to say, completely blind for that action. Unfortunately we have some ".W" entries out of the diagonal, marked as ".W!".

The "W-" in particular is due to the fact that "rotations" is a subgroup of "curling" (the case when $\phi$ is constant).

**Theorem 12.** *The space of curves can be described as a smooth differentiable manifold, as follows: for any curve $c$ we factor out translation*

$$v = avg_c(c) \quad , \quad \tilde{c} = c - v$$

*we represent in log-coordinates*

$$\tilde{c}'(\theta) = e^{\tilde{e}(\theta)+if(\theta)}$$

*then the manifold of curves is*

$$(\tilde{e}, f, v) \in H^1(S^1) \times (H^1(S^1)/(2\pi)) \times \mathbb{R}^2$$

*and this is equivalent to saying that $c \in H^2(S^1)$.* [3]

$$H^1(S^1) \times (H^1(S^1)/(2\pi)) \times \mathbb{R}^2$$

is the space of all open immmersed curves.

For closed curves of index $k$, we restrict to the submanifold where

$$\int_0^1 e^{\tilde{e}+if}\,\mathrm{d}\theta = 0 \quad , \quad f(1) = f(0) + 2\pi k$$

[3] The representative $f$ is not unique, we can add multiples of $2\pi$ to it, so $f \in H^1(S^1)/(2\pi)$.

(the last one, up to continuous lifting $f : [0,1] \to \mathbb{R}$).

In any case, this is a **complete Riemannian manifold** when equipped with the full delta metric $\|\cdot\|_{(l\Delta^2+\mathrm{len}+r/l+t)}$ defined in (24).

Let

$$(e, f, v) \in H^1(S^1) \times (H^1(S^1)/(2\pi)) \times \mathbb{R}^2$$

be the representative of a curve. Let then

$$(\hat{e}, \hat{f}, \hat{v}) \in H^1(S^1) \times H^1(S^1) \times \mathbb{R}^2$$

be the representation of a tangent vector $h$. In this decomposition the metric takes this form. Let $l = \mathrm{len}(c) = \big(\int_0^1 e^{\tilde{e}}\,\mathrm{d}\theta\big)$.

$$\begin{aligned}
\mathrm{len}(c)\|h\|^2_{\Delta^2,c} &= l\int_0^1 (|\hat{e}'|^2 + |\hat{f}'|^2)e^{-\tilde{e}}\,\mathrm{d}\theta \\
m_l\|h\|^2_{\mathrm{len},c} &= m_l l^{-2}\left|\int_0^1 \hat{e}e^{\tilde{e}}\,\mathrm{d}\theta\right|^2 \\
m_r\|h\|^2_{\mathrm{r},c}/\,\mathrm{len}(c)^2 &= m_r l^{-2}\left|\int_0^1 \hat{f}e^{\tilde{e}}\,\mathrm{d}\theta\right|^2 \\
m_t\|h\|^2_{\mathrm{t},c} &= m_t|\hat{v}|
\end{aligned}$$

—77. Momenta

Suppose that $\gamma : [0,1] \to M$ is a geodesic, that we can view as a homotopy $C : [0,1] \times S^1 \to \mathbb{C}$ and represent in log-coordinates

$$C'(t,\theta) = e^{E(t,\theta)+iF(t,\theta)} \quad .$$

**Translation** The center of mass $\mathrm{avg}_c(\gamma(t))$ of the curve is an affine map of $t$.

**Rescaling** $\log \mathrm{len}\,\gamma$ is an affine map of $t$

**Rotation** In log-coordinates

$$\int_0^1 \dot{F}e^E\,\mathrm{d}\theta = ce^{tb} \tag{25}$$

for appropriate constants $c, b$ (where $b$ is as before).

—78. Curling momentum

The metric $\|(\hat{e},\hat{f})\|^2_{(l\Delta^2+\mathrm{len}+r/l),(\tilde{e},f)}$ does not depend on $f$; hence, for any fixed $q \in H^1$, along a geodesic the quantity

$$\begin{aligned}
&D_{\hat{f},q}\|(\dot{E},\dot{F})\|^2_{(l\Delta^2+\mathrm{len}+r/l),(E,F)} = \\
&= \big(\int_0^1 e^E\,\mathrm{d}\theta\big)\int_0^1 2q'\dot{F}'e^{-E}\,\mathrm{d}\theta + \frac{2m_r\int_0^1 qe^E\,\mathrm{d}\theta\int_0^1 \dot{F}e^E\,\mathrm{d}\theta}{\left|\int_0^1 e^E\,\mathrm{d}\theta\right|^2}
\end{aligned}$$

will be constant.

This holds only for geodesics of open immersed curves.

This is the conserved momentum for the action of *curling*.

**Corollary 13.** *Along a geodesic $\gamma$ the four "speeds"*

$$\sqrt{\operatorname{len}(\gamma)}\|\dot\gamma\|_{\Delta^2,\gamma} \quad , \quad \|\dot\gamma\|_{\operatorname{len},\gamma} \quad , \quad \|\dot\gamma\|_{\mathbb{r},\gamma}/\operatorname{len}(\gamma) \quad , \quad \|\dot\gamma\|_{\mathbb{t},\gamma} \tag{26}$$

*are all constant.*

**Theorem 14.** *Any two curves are connected by a minimal length geodesic.*

Computation of geodesics is easier than it seems, perusing the ideas we saw while designing the metric.

Given two curves $c_0$ and $c_1$ we normalize them so that they have unit length and center of mass in the origin.

Then we compute a homothopy of unit length curves that minimizes

$$\int_0^1 \|\dot C\|^2_{\Delta^2,C} + m_r\|\dot C\|^2_{\mathbb{r},C}\,\mathrm{d}t$$

In log coordinates this is just

$$\int_0^1\int_0^1 (|\dot E'|^2+|\dot F'|^2)e^{-E}\,\mathrm{d}\theta\,\mathrm{d}t + m_r\int_0^1\left|\int_0^1 \dot F e^E\,\mathrm{d}\theta\right|^2\,\mathrm{d}t$$

Eventually we adjust for the affine motion of center of mass, and of the logarithm of the length.

This is justified from the "magick" of translation and rescaling seminorms being path-wise invariant (and not only curve-wise invariant).

## 4.4 Conclusions

—82. Conclusions

**good** A model that is a complete Riemannian manifold.

**good** The computation of gradients is fast (no convolutions, no iterative approximation); the gradient regularizes ill conditioned and ill posed problems.

**good** Any two curves are connected by a minimal geodesic.

**good** The quotient space $B = {}^{M}/_{\mathrm{Diff}}$ of *"curves up to parameterization"* has similar properties: it is complete; the distance $d_B$ between curves "up to parameterization" is not degenerate; there is existence of minimal geodesics *(skipped due to lack of time).*

**good** The basilar geometric concept of *index number* of the curve is preserved.

**todo** The equation for geodesics is not known in closed forms, unknown if there is a fast way to compute it. ( Is there another metric that may be designed, such that the computation of geodesics is simplified?)

**todo** No numerical experiments are available: in the TODO list.

**todo** Would be nice to export these ideas to surfaces, volumes...

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