A COMPACTNESS RESULT FOR BV FUNCTIONS IN METRIC SPACES

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Abstract. We prove a compactness result for bounded sequences $(u_j)_j$ of functions with bounded variation in metric spaces (X, d_j) where the space X is fixed but the metric may vary with j. We also provide an application to Carnot-Carathéodory spaces.

1. Introduction

One of the milestones in the theory of functions with bounded variation (BV) is the following Rellich-Kondrachov-type theorem: given a bounded open set $\Omega \subseteq \mathbb{R}^n$ with Lipschitz regular boundary, the space $BV(\Omega)$ of functions with bounded variation in Ω compactly embeds in $L^q(\Omega)$ for any $q \in [1, \frac{n}{n-1}[$. One notable consequence is the following property: if $(u_j)_j$ is a sequence of functions in $BV_{loc}(\mathbb{R}^n)$ that are locally uniformly bounded in BV, then for any $q \in [1, \frac{n}{n-1}[$ a subsequence $(u_{j_h})_h$ converges in $L^q_{loc}(\mathbb{R}^n)$.

Sobolev and BV functions in metric measure spaces have recently received a great deal of attention; to this regard we only mention the celebrated paper [7], where the authors show how the validity of Poincaré-type inequalities and a doubling property of the reference measure are enough to prove fundamental properties like Sobolev inequalities, Sobolev embeddings, Trudinger inequality, etc. We also point out a Rellich-Kondrachov-type result [7, Theorem 8.1]: if a sequence $(u_j)_j$ is bounded in some $W^{1,p}$, then a subsequence converges in some L^q.

In this paper we study similar compactness properties for sequences $(u_j)_j$ of locally uniformly bounded BV functions in metric measure spaces (X, λ, d_j) where the underlying measure space (X, λ) is fixed but the metric d_j varies with j. In our main result we prove that, if d_j converges locally uniformly to some distance d on X such that (X, λ, d) is a (locally) doubling separable metric measure space, and if the functions $u_j : X \to \mathbb{R}$ are locally uniformly (in j) bounded with respect to a BV-type norm in (X, d_j) and satisfy some local Poincaré inequality (with constant independent of j), then a subsequence of u_j converges in some $L^q_{loc}(X, \lambda)$. See Theorem 2.1 for a precise statement. To our knowledge, the strategy we adopt to prove Theorem 2.1 is novel even when the metric on X is not varying (i.e., when $d_j = d$ for any j); in particular, we are able to provide a different proof of the case $p = 1$ in [7, Theorem 8.1] for separable metric spaces.

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The motivation that led us to Theorem 2.1 comes from an application to the study of BV functions in *Carnot-Carathéodory* (CC) spaces. In Theorem 3.6 we indeed prove that, if $X^j = (X_1^j, \dots, X_m^j)$ are families of smooth vector fields in $\mathbb{R}^n$ that, as $j \to \infty$, converge in $C^\infty_{loc}(\mathbb{R}^n)$ to a family $X = (X_1, \dots, X_m)$ satisfying the Chow-Hörmander condition, and if $u_j : \mathbb{R}^n \to \mathbb{R}$ are locally uniformly bounded in $BV_{X^j,loc}$, then a subsequence u_{j_h} converges in $L^1_{loc}(\mathbb{R}^n)$ to some $u \in BV_{X,loc}(\mathbb{R}^n)$. Theorem 3.6 directly follows from Theorem 2.1 once we show that the CC distances induced by X^j converge locally uniformly to the one induced by X, and that (locally) a Poincaré inequality holds for BV_{X^j} functions with constant independent of j; these two results (Theorems 3.4 and 3.5, respectively) use in a crucial way some outcomes of the papers [1, 11].

Our interest in Theorem 3.6, in turn, was originally motivated by the study of fine properties of BV_X functions in CC spaces and, in particular, of their local properties. Here, one often needs to perform a blow-up procedure around a fixed point p: it is well-known that this produces a sequence of CC metric spaces $(\mathbb{R}^n, X^j)$ that converges to (a quotient of) a *Carnot group* structure $\mathbb{G}$. In this blow-up, the original BV_X function u_0 gives rise to a sequence $(u_j)_j$ of functions in BV_{X^j} which, up to a subsequence, will converge in L^1_{loc} to a $BV_{\mathbb{G},loc}$ function u in $\mathbb{G}$. The function u (typically: a linear map, or a *jump map* taking two different values on complementary halfspaces of $\mathbb{G}$) will then provide some information on u_0 around p. We refer to [3] for more details.

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2. The main result

This section is devoted to the statement and the proof of our main result. See e.g. [10] for a definition of BV functions in metric spaces.

Theorem 2.1. *Let X be a set, $q \geq 1$, $\delta > 0$ and let d, d_j $(j \in \mathbb{N})$ be metrics on X such that (X, d) is locally compact and separable. Let λ, μ_j $(j \in \mathbb{N})$ be Radon measures on X and consider a sequence $(u_j)_j$ in $L^q_{loc}(X; \lambda)$. Suppose that the following assumptions hold.*

(i) *The sequence $(d_j)_j$ converges to d in $L^\infty_{loc}(X \times X)$.*

(ii) *(X, d, λ) is a locally doubling metric measure space, i.e., for any compact set $K \subseteq X$ there exist $C_D \geq 1$ and $R_D > 0$ such that*

$$\forall\ x \in K,\ \forall r \in (0, R_D) \qquad \lambda(B(x, 2r)) \leq C_D \lambda(B(x, r)).$$

(iii) *For every compact set $K \subseteq X$ there exist $C_P, R_P > 0$ and $\alpha \geq 1$ such that*

$$\forall x \in K,\ \forall j \in \mathbb{N},\ \forall r \in (0, R_P) \qquad \|u_j - u_j(B^j)\|_{L^q(B^j)} \leq C_P\, r^\delta \mu_j(\alpha B^j),$$

where $B^j := B^j(x, r)$ denotes a ball in (X, d_j), $\alpha B^j := B^j(x, \alpha r)$ and $u_j(B^j) := \fint_{B^j} u_j d\lambda$.

(iv) *For every compact set $K \subseteq X$ there exists $M_K > 0$ such that*

$$\forall j \in \mathbb{N} \qquad \|u_j\|_{L^1(K;\lambda)} + \mu_j(K) \leq M_K.$$

Then there exist $u \in L^q_{loc}(X; \lambda)$ and a subsequence $(u_{j_h})_h$ of $(u_j)_j$ such that $(u_{j_h})_h$ converges to u in $L^q_{loc}(X; \lambda)$ as $h \to +\infty$.

Concerning the classical Euclidean case when $(X, d_j, \lambda) = (X, d, \lambda) = (\mathbb{R}^n, |\cdot|, \mathscr{L}^n)$, we invite the reader to compare the assumption in (iii) with the well-known Poincaré inequality

$$\|u - u(B_r)\|_{L^q(B_r)} \le C r^\delta |Du|(B_r) \qquad \forall\, q \in [1, \tfrac{n}{n-1}[\text{ with } \delta := \tfrac{n}{q} + 1 - n > 0$$

valid for for any BV function u on any ball $B_r \subseteq \mathbb{R}^n$ of radius r and where $u(B_r)$ denotes the mean value $\mathscr{L}^n(B_r)^{-1} \int_{B_r} u \, d\mathscr{L}^n$ of u in B_r, $C > 0$ is a geometric constant, and $|Du|$ denotes the total variation measure associated with u (i.e., the total variation of the distributional derivatives of u).

Proof. Let $K \subseteq X$ be a fixed compact set and let $\varepsilon > 0$. We first prove that there exists a subsequence $(u_{j_h})_h$ such that

$$\limsup_{h,k \to +\infty} \|u_{j_h} - u_{j_k}\|_{L^q(K;\lambda)} \le 2C_0 \varepsilon, \tag{1}$$

for some $C_0 > 0$ depending on K only.

Consider an open set $U_1 \subseteq X$ such that $K \subseteq U_1$, $\overline{U}_1$ is compact and

$$\lambda(U_1 \setminus K) \le \frac{1}{4C_D^{\beta+3}} \lambda(K). \tag{2}$$

where β is an integer such that $2^\beta > 2\alpha$ and α is given by condition (iii). By the $5r-$covering Theorem (see e.g. [8, Theorem 1.2]) we can find a family $\{B(x_\ell, r_\ell) : \ell \in \mathbb{N}\}$ of pairwise disjoint balls such that $x_\ell \in K$, $0 < r_\ell < \min\{\varepsilon^{1/\delta}, R_D/4, 2\alpha R_P\}$, $\overline{B(x_\ell, 5r_\ell)} \subseteq U_1$ and

$$K \subseteq \bigcup_{\ell=0}^{\infty} \overline{B(x_\ell, 5r_\ell)}.$$

Denote for shortness $B_\ell := B(x_\ell, r_\ell)$; then

$$\lambda(K) \le \sum_{\ell=0}^{\infty} \lambda(5\overline{B}_\ell) \le \sum_{\ell=0}^{\infty} \lambda(8B_\ell) \le C_D^{\beta+3} \sum_{\ell=0}^{\infty} \lambda(\tfrac{1}{2^\beta} B_\ell) = C_D^{\beta+3} \lambda\left(\bigcup_{\ell=0}^{\infty} \tfrac{1}{2^\beta} B_\ell \right).$$

Hence we can choose $L \in \mathbb{N}$ such that

$$\lambda\left(\bigcup_{\ell=0}^{L} \tfrac{1}{2^\beta} B_\ell \right) \ge \frac{1}{2C_D^{\beta+3}} \lambda(K).$$

Taking into account (2) we easily get that $A_1 := K \cap \bigcup_{\ell=0}^{L} \frac{1}{2^\beta} B_\ell$ satisfies

$$\lambda(A_1) \ge \frac{1}{4C_D^{\beta+3}} \lambda(K).$$

For $j \in \mathbb{N}$ and $\ell = 1, \dots, L$ set for shortness $B_\ell^j := B^j(x_\ell, r_\ell)$. By assumption (i) there exists $J \in \mathbb{N}$ such that for every $j \ge J$, and for every $\ell = 0, \dots, L$

$$\tfrac{1}{2^\beta} B_\ell \subseteq \tfrac{1}{2\alpha} B_\ell^j \qquad \text{and} \qquad \tfrac{1}{2} B_\ell^j \subseteq B_\ell. \tag{3}$$

Hence for every $j \ge J$ one has

$$\left| u_j\left(\tfrac{1}{2\alpha} B_\ell^j\right) \right| \le \lambda\left(\tfrac{1}{2\alpha} B_\ell^j\right)^{-1} \|u_j\|_{L^1(U_1;\lambda)} \le M_{\overline{U_1}} \max\{\lambda\left(\tfrac{1}{2^\beta} B_\ell\right)^{-1} : \ell = 0, \dots, L\} < +\infty.$$

By Bolzano-Weierstrass Theorem we get an increasing function $\nu_1 : \mathbb{N} \to \mathbb{N}$ such that

$$\text{the sequence } \left(u_{\nu_1(j)}\left(\tfrac{1}{2\alpha} B_\ell^{\nu_1(j)}\right)\right)_j \text{ is convergent for every } \ell = 0, \dots, L. \tag{4}$$

Then

$$\begin{aligned}
&\limsup_{h,k\to+\infty} \|u_{\nu_1(h)} - u_{\nu_1(k)}\|_{L^q(A_1;\lambda)} \\
\leq &\limsup_{h,k\to+\infty} \sum_{\ell=0}^{L} \Bigg(\left\| u_{\nu_1(h)} - u_{\nu_1(h)} \left(\tfrac{1}{2\alpha} B_\ell^{\nu_1(h)} \right) \right\|_{L^q\left(\frac{1}{2^\beta} B_\ell;\lambda\right)} \\
&\qquad + \left\| u_{\nu_1(k)} - u_{\nu_1(k)} \left(\tfrac{1}{2\alpha} B_\ell^{\nu_1(k)} \right) \right\|_{L^q\left(\frac{1}{2^\beta} B_\ell;\lambda\right)} \\
&\qquad + \left\| u_{\nu_1(h)} \left(\tfrac{1}{2\alpha} B_\ell^{\nu_1(h)} \right) - u_{\nu_1(k)} \left(\tfrac{1}{2\alpha} B_\ell^{\nu_1(k)} \right) \right\|_{L^q\left(\frac{1}{2^\beta} B_\ell;\lambda\right)} \Bigg)
\end{aligned}$$

and, using (3) and (4),

$$\begin{aligned}
\leq &\limsup_{h,k\to+\infty} \sum_{\ell=0}^{L} \Bigg(\left\| u_{\nu_1(h)} - u_{\nu_1(h)} \left(\tfrac{1}{2\alpha} B_\ell^{\nu_1(h)} \right) \right\|_{L^q\left(\frac{1}{2\alpha} B_\ell^{\nu_1(h)};\lambda\right)} \\
&\qquad + \left\| u_{\nu_1(k)} - u_{\nu_1(k)} \left(\tfrac{1}{2\alpha} B_\ell^{\nu_1(k)} \right) \right\|_{L^q\left(\frac{1}{2\alpha} B_\ell^{\nu_1(k)};\lambda\right)} \Bigg) \\
\leq &\limsup_{h,k\to+\infty} \sum_{\ell=0}^{L} \frac{C_P\, r_\ell^\delta}{(2\alpha)^\delta} \left(\mu_{\nu_1(h)} \left(\tfrac{1}{2} B_\ell^{\nu_1(h)} \right) + \mu_{\nu_1(k)} \left(\tfrac{1}{2} B_\ell^{\nu_1(k)} \right) \right) \\
\leq &\limsup_{h,k\to+\infty} \frac{C_P\, \varepsilon}{(2\alpha)^\delta} \left(\mu_{\nu_1(h)} \left(\overline{U}_1 \right) + \mu_{\nu_1(k)} \left(\overline{U}_1 \right) \right) \leq C_0 \varepsilon,
\end{aligned}$$

where C_0 depends only on U_1 and thus only on K.
We proved that there exist $A_1 \subseteq K$ and a subsequence $(u_{\nu_1(h)})_h$ of $(u_j)_j$ such that

$$\begin{aligned}
&\lambda(K \setminus A_1) \leq \left(1 - \frac{1}{4C_D^{\beta+3}} \right) \lambda(K), \\
&\limsup_{h,k\to+\infty} \|u_{\nu(h)} - u_{\nu(k)}\|_{L^q(A_1;\lambda)} \leq C_0 \varepsilon.
\end{aligned}$$

Since the set $K_2 = K \setminus A_1$ is compact we can repeat the same argument on K_2, with $\frac{\varepsilon}{2}$ in place of ε, and paying attention to choose an open set $U_2 \subseteq U_1$ so that C_0 can be left unchanged. By a recursive argument, for every $j \in \mathbb{N}$ we get pairwise disjoint sets $A_j \subseteq K$ and subsequences $(u_{\nu_j(h)})_h$ such that for every $j \geq 1$

(a) $(u_{\nu_{j+1}(h)})_h$ is a subsequence of $(u_{\nu_j(h)})_h$;
(b) $\lambda\left(K \setminus \bigcup_{i=1}^{j} A_i \right) \leq \left(1 - \frac{1}{4C_D^{\beta+3}} \right)^j \lambda(K)$;
(c) $\limsup_{h,k\to+\infty} \|u_{\nu_j(h)} - u_{\nu_j(k)}\|_{L^q(A_j;\lambda)} \leq C_0 2^{1-j} \varepsilon$.

Inequality (b) immediately implies that $\lambda\left(K \setminus \bigcup_{i=1}^{\infty} A_i\right) = 0$. Working on the diagonal subsequence $(u_{\nu_h(h)})_h$ we can conclude that

$$\begin{aligned}
\limsup_{h,k\to+\infty} \|u_{\nu_h(h)} - u_{\nu_k(k)}\|_{L^q(K;\lambda)} &= \limsup_{h,k\to+\infty} \|u_{\nu_h(h)} - u_{\nu_k(k)}\|_{L^q\left(\bigcup_{i=1}^{\infty} A_i;\lambda\right)} \\
&\leq \sum_{i=1}^{\infty} \limsup_{h,k\to+\infty} \|u_{\nu_h(h)} - u_{\nu_k(k)}\|_{L^q(A_i;\lambda)} \leq 2C_0\varepsilon.
\end{aligned} \tag{5}$$

This proves (1).

Let us denote for simplicity $(u_h)_h$ instead of $(u_{\nu_h(h)})_h$. We now prove that for every compact set $K \subseteq X$ there exists a subsequence $(u_{j_h})_h$ of $(u_h)_h$ such that

$$\lim_{h,k\to+\infty} \|u_{j_h} - u_{j_k}\|_{L^q(K;\lambda)} = 0. \tag{6}$$

By (5), for every $i \in \mathbb{N}$, we can recursively build a subsequence $(u_{\nu_{i+1}(h)})_h$ of $(u_{\nu_i(h)})_h$ such that

$$\limsup_{h,k\to+\infty} \|u_{\nu_i(h)} - u_{\nu_i(k)}\|_{L^q(K;\lambda)} \leq \tfrac{2}{i+1} C_0.$$

Then the diagonal sequence $(u_{\nu_h(h)})$ satisfies (6).

Eventually, take a sequence (K_j) of compact sets such that $K_j \subseteq \mathrm{int}(K_{j+1})$ and $\bigcup_{j\in\mathbb{N}} K_j = X$. By (6), for every $i \in \mathbb{N}$ we can recursively build a subsequence $(u_{\nu_i(h)})_h$ such that $(u_{\nu_{i+1}(h)})_h$ is a subsequence of $(u_{\nu_i(h)})_h$ and

$$\lim_{h,k\to+\infty} \|u_{\nu_i(h)} - u_{\nu_i(k)}\|_{L^q(K_i;\lambda)} = 0.$$

The diagonal subsequence $(u_{\nu_h(h)})_h$ will then converge to some u in $L^q_{loc}(X;\lambda)$. This concludes the proof. □

Remark 2.2. The careful reader will easily notice that Theorem 2.1 holds also when assumption (iii) is replaced by the following weaker one:

(iii') For every compact set $K \subseteq X$ there exist $R_P > 0, \alpha \geq 1$ and $f : (0,+\infty) \to (0,+\infty)$ such that $\lim_{r\to 0^+} f(r) = 0$ and

$$\forall x \in K,\ \forall j \in \mathbb{N},\ \forall r \in (0, R_P) \qquad \|u_j - u_j(B^j)\|_{L^q(B^j)} \leq f(r)\, \mu_j(\alpha B^j).$$

3. An application to Carnot-Carathéodory spaces

Let Ω be an open set in $\mathbb{R}^n$ and let $X = (X_1, \dots, X_m)$ be an m-tuple of smooth and linearly independent vector fields on $\mathbb{R}^n$, with $2 \leq m \leq n$. We say that an absolutely continuous curve $\gamma : [0,T] \to \mathbb{R}^n$ (briefly denoted by $\gamma \in AC([0,T];\mathbb{R}^n)$) is an X-subunit path joining x and y in $\mathbb{R}^n$ if $\gamma(0) = x$, $\gamma(T) = y$ and there exist $h_1, \dots, h_m : [0,T] \to \mathbb{R}$ with $\sum_{j=1}^m h_j^2 \leq 1$ such that

$$\dot\gamma(t) = \sum_{j=1}^m h_j(t) X_j(\gamma(t)) \qquad \text{for a.e. } t \in [0,T]. \tag{7}$$

Moreover, for every $x, y \in \mathbb{R}^n$ we define the quantity

$$d(x,y) := \inf\big\{T \in (0,+\infty) : \exists \gamma \in AC([0,T];\mathbb{R}^n)\ X\text{-subunit joining } x \text{ and } y\big\}, \tag{8}$$

where we agree that $\inf \emptyset = +\infty$.

We will suppose in the following that the Chow-Hörmander condition holds, i.e., that for every $x \in \mathbb{R}^n$ the vector space spanned by $X_1, \dots, X_m$ and their commutators of any order computed at x is the whole $\mathbb{R}^n$. By the Chow-Rashevsky Theorem, if the Chow-Hörmander condition holds, the function d defined above is a distance and the couple $(\mathbb{R}^n, X)$ (or equivalently $(\mathbb{R}^n, d)$) is called *Carnot-Carathéodory space* (CC space for short). It is well known that d and the Euclidean distance d_e induce on $\mathbb{R}^n$ the same topology (see [13]).

We denote balls induced by d by $B(x,r)$ and Euclidean balls by $B_e(x,r)$. As customary in the literature, in what follows we also suppose that the metric balls $B(x,r)$ are bounded

with respect to the Euclidean metric. One consequence of this assumption is the existence of geodesics, i.e., for any $x, y \in \mathbb{R}^n$ the infimum in (8) (as well as the one in (9) below) is indeed a minimum; see e.g. [12, Theorem 1.4.4].

For $j \in \mathbb{N}$ let $X^j = (X_1^j, \dots, X_m^j)$ be a family of linearly independent vector fields such that, for every fixed $i = 1, \dots, m$, X_i^j converges to X_i in $C^\infty_{loc}(\mathbb{R}^n)$ as $j \to \infty$. We denote by d_j, $j \in \mathbb{N}$, the CC distance associated with X^j. If $h \in L^\infty([0,T];\mathbb{R}^m)$ with $\|h\| \le 1$, $T > 0$ and $x \in \mathbb{R}^n$, it is convenient to define $\gamma_{h,x}, \gamma^j_{h,x} : [0,T] \to \mathbb{R}^n$ as the AC curves such that $\gamma_{h,x}(0) = \gamma^j_{h,x}(0) = x$ and for almost every $t \in [0,T]$

$$\dot\gamma_{h,x}(t) = \sum_{i=1}^m h_i(t) X_i(\gamma_{h,x}(t)), \qquad \dot\gamma^j_{h,x}(t) = \sum_{i=1}^m h_i(t) X^j_i(\gamma^j_{h,x}(t)).$$

With this notation, an equivalent definition of the CC distance is

$$d(x,y) = \inf\{\|h\|_{L^\infty(0,1)} : h \in L^\infty([0,1];\mathbb{R}^m) \text{ and } \gamma_{h,x}(1) = y\}. \tag{9}$$

The boundedness of metric balls implies that, for every $T > 0$ and $h \in L^\infty([0,T];\mathbb{R}^m)$, the curve $\gamma_{h,x}$ is well-defined on $[0,T]$.

It can be easily seen that, if the Chow-Hörmander condition holds, then for every compact set $K \subseteq \mathbb{R}^n$ there exists an integer $s(K)$ such that the following holds: for any $x \in K$, $X_1, \dots, X_m$ and their commutators up to order $s(K)$ computed at x span the whole $\mathbb{R}^n$. The following theorem gives a sort of quantitative version of some of the celebrated results of [13]. The proof of Theorem 3.1 follows fairly easily from [1, 11] (see in particular [1, Proposition 5.8 and Claim 3.3]) and from the following observation: for any compact set $K \subseteq \mathbb{R}^n$ there exists $J \in \mathbb{N}$ such that, for any $x \in K$ and $j \ge J$, the vector fields $X_1^j, \dots, X_m^j$ and their commutators up to order $s(K)$ computed at x span the whole $\mathbb{R}^n$.

Theorem 3.1. *For every compact set $K \subseteq \mathbb{R}^n$ there exist $J_0 \in \mathbb{N}$ and $C_K > 0$ such that for every $x, y \in K$ and $j \ge J_0$*

$$\frac{1}{C_K}|x-y| \le d(x,y) \le C_K |x-y|^{1/s(K)}$$
$$\frac{1}{C_K}|x-y| \le d_j(x,y) \le C_K |x-y|^{1/s(K)}.$$

We aim at proving that the sequence of distances d_j converges to d locally uniformly; we need some preparatory lemmata.

Lemma 3.2. *Let K be a compact set in $\mathbb{R}^n$. Then for every $T > 0$, there exist $J_1 = J_1(K,T) \in \mathbb{N}$ and $R = R(K,T) > 0$ such that for every $x \in K$, $h \in L^\infty([0,T];\mathbb{R}^m)$ with $\|h\| \le 1$ and any $j \ge J_1$ the following hold:*

(a) *the curve $\gamma^j_{h,x}$ is well defined on $[0,T]$;*
(b) *$\gamma^j_{h,x}([0,T]) \subseteq B_e(0,R)$.*

Proof. Define first

$$K' := \{\gamma_{h,x}(T) : x \in K, h \in L^\infty([0,T];\mathbb{R}^m), \|h\| \le 1\} = \bigcup_{x \in K} \overline{B(x,T)}.$$

Since metric balls are bounded, also K' is bounded. We can therefore find $R > 0$ such that $K' \subseteq B_e(0,R)$ and $d_e(K', \mathbb{R}^n \setminus B_e(0,R)) > 1$. Choose $J_1 \in \mathbb{N}$ such that for every $j \geq J_1$

$$T\left(\sum_{i=1}^m \sup_{B_e(0,R)} |X_i^j - X_i|\right) e^{mCT} \leq \frac{1}{2},$$

where $C > 0$ will be determined later. Let $h \in L^\infty([0,T];\mathbb{R}^m)$ and $j \geq J_1$ be fixed; define

$$t_j := \sup\{t > 0 : \gamma_{h,x}^j \text{ is well-defined on } [0,t] \text{ and } \gamma_{h,x}^j([0,t]) \subseteq B_e(0,R)\}$$

and suppose by contradiction that $t_j < T$. Then $\gamma_{h,x}^j(t_j) \in \partial B_e(0,R)$ and for every $\tau < t_j$ one has

$$\begin{aligned}
\left|\gamma_{h,x}^j(\tau) - \gamma_{h,x}(\tau)\right| &\leq \int_0^\tau \sum_{i=1}^m \left|h_i(s)X_i^j(\gamma_{h,x}^j(s)) - h_i(s)X_i(\gamma_{h,x}(s))\right| ds \\
&\leq \int_0^\tau \sum_{i=1}^m \left|X_i^j(\gamma_{h,x}^j(s)) - X_i^j(\gamma_{h,x}(s))\right| ds \\
&\quad + \int_0^\tau \sum_{i=1}^m \left|X_i^j(\gamma_{h,x}(s)) - X_i(\gamma_{h,x}(s))\right| ds.
\end{aligned}$$

Notice that, since X_i^j is converging to X_i locally in C^1, and since $\gamma_{h,x}^j(s), \gamma_{h,x}(s) \in B_e(0,R)$, the Lipschitz constants

$$c_i^j := \sup_{x,y \in B_e(0,R)} \frac{|X_i^j(x) - X_i^j(y)|}{|x-y|}$$

are converging to the Lipschitz constant $c_i := \sup_{x,y\in B_e(0,R)} \frac{|X_i(x)-X_i(y)|}{|x-y|}$. Therefore there exists $C > 0$ such that $c_i^j, c_i \leq C$ for any $j \in \mathbb{N}$ and $i = 1, \dots, m$, which gives

$$\left|\gamma_{h,x}^j(\tau) - \gamma_{h,x}(\tau)\right| \leq \int_0^\tau \left(mC\left|\gamma_{h,x}^j(s) - \gamma_{h,x}(s)\right| + \sum_{i=1}^m \sup_{B_e(0,R)} |X_i^j - X_i|\right) ds.$$

We can therefore apply Grönwall's Lemma (see [6]) to get

$$\left|\gamma_{h,x}^j(t_j) - \gamma_{h,x}(t_j)\right| \leq t_j \left(\sum_{i=1}^m \sup_{B_e(0,R)} |X_i^j - X_i|\right) e^{mCt_j} \leq \frac{1}{2}.$$

Notice that $\gamma_{h,x}(t_j) \in K'$ and $\gamma_{h,x}^j(t_j) \in \partial B_e(0,R)$: this contradicts the definition of R, giving $t_j = T$. The lemma is proved. □

Lemma 3.3. *Fix $\varepsilon \in (0,1)$ and a compact set K in $\mathbb{R}^n$. Then, for every $T > 0$ there exists $J_2 = J_2(K,T,\varepsilon) \in \mathbb{N}$ such that for every $x \in K$, $j \geq J_2$, $h \in L^\infty([0,T];\mathbb{R}^m)$ with $\|h\| \leq 1$ and $t \in [0,T]$ one has*

$$|\gamma_{h,x}^j(t) - \gamma_{h,x}(t)| \leq \varepsilon$$

Proof. Let $J_1 = J_1(K,T)$ and $R = R(K,T)$ be given by Lemma 3.2 and let $C > 0$ be the constant appearing in its proof. We can reason as in Lemma 3.2 above and use Grönwall's Lemma to get, for any x, j, h, t as in the statement, that

$$\left|\gamma_{h,x}^j(t) - \gamma_{h,x}(t)\right| \leq t\left(\sum_{i=1}^m \sup_{B_e(0,R)} |X_i^j - X_i|\right) e^{mCt}.$$

The proof is then accomplished by choosing $J_2 \geq J_1$ sufficiently large to have

$$T\left(\sum_{i=1}^{m} \sup_{B_e(0,R)} |X_i^j - X_i|\right) e^{mCT} < \varepsilon.$$

□

Clearly, J_2 can be chosen with the additional property that $J_2(K, T_1, \varepsilon) \leq J_2(K, T_2, \varepsilon)$ whenever $0 < T_1 \leq T_2$.

Theorem 3.4. *Let $X = (X_1, \dots, X_m)$ and $X^j = (X_1^j, \dots, X_m^j)$, $j \in \mathbb{N}$, be m-tuples of linearly independent smooth vector fields on $\mathbb{R}^n$ such that X satisfies the Chow-Hörmander condition and its CC balls are bounded in $\mathbb{R}^n$; assume that, for every $i = 1, \dots, m$, $X_i^j \to X_i$ in $C^\infty_{loc}(\mathbb{R}^n)$ as $j \to \infty$. Then the sequence $(d_j)_j$ converges to d in $L^\infty_{loc}(\mathbb{R}^n \times \mathbb{R}^n)$ as $j \to +\infty$.*

Proof. Let $K \subseteq \mathbb{R}^n$ be a fixed compact set.

We first prove that for every $\varepsilon \in (0,1)$ there exists $J_3 = J_3(K, \varepsilon) \in \mathbb{N}$ such that for every $x, y \in K$ and $j \geq J_3$ one has

$$d_j(x, y) \leq d(x, y) + \varepsilon.$$

Consider $x, y \in K$; by the existence of geodesics, there exists $h \in L^\infty([0,1]; \mathbb{R}^m)$ such that $\|h\|_{L^\infty} = d(x, y)$ and $\gamma_{h,x}(1) = y$. We set $y_j := \gamma^j_{h,x}(1)$ and consider J_0 and $C_K > 0$ as given by Theorem 3.1. For $j \geq J_3 := \max\{J_0, J_2(K, \operatorname{diam}_d K, (\varepsilon/C_K)^{s(K)})\}$ we have

$$|y_j - y| = |\gamma^j_{h,x}(1) - \gamma_{h,x}(1)| \leq \left(\frac{\varepsilon}{C_K}\right)^{s(K)}.$$

By Theorem 3.1 we deduce that $d_j(y_j, y) \leq \varepsilon$; in particular, for any $j \geq J_3$ one has

$$d_j(x, y) \leq d_j(x, y_j) + d_j(y_j, y) \leq d(x, y) + \varepsilon, \tag{10}$$

as claimed. Notice also that $\sup_{j \geq J_3} \operatorname{diam}_{d_j} K \leq \operatorname{diam}_d K + 1 =: L$ is finite.

We now prove that for any $x, y \in K$ and $\varepsilon \in (0,1)$ there exists $J_4 = J_4(K, x, y, \varepsilon) \in \mathbb{N}$ such that for every $j \geq J_4$

$$d(x, y) \leq d_j(x, y) + \varepsilon. \tag{11}$$

For every $j \geq J_3$ let $h^j \in L^\infty([0,1]; \mathbb{R}^m)$ be such that

$$\gamma^j_{h^j,x}(1) = y \qquad \text{and} \qquad \|h^j\|_{L^\infty} = d_j(x, y) \leq L.$$

The sequence $(h^j)_j$ is bounded in L^∞ and therefore there exists a subsequence $(h^{j_\ell})_\ell$ and $h \in L^\infty([0,1]; \mathbb{R}^m)$ such that

$$h^{j_\ell} \overset{*}{\rightharpoonup} h \text{ in } L^\infty \qquad \text{and} \qquad \lim_{\ell \to \infty} \|h^{j_\ell}\|_{L^\infty} = \liminf_{j \to \infty} \|h^j\|_{L^\infty} = \liminf_{j \to \infty} d_j(x, y).$$

Denoting $\gamma^{j_\ell} := \gamma^{j_\ell}_{h^{j_\ell},x}$ and considering $R = R(K, L) > 0$ as given by Lemma 3.2, one has $\gamma^{j_\ell}([0,1]) \subseteq B_e(0, R)$. Since X_i^j are converging to X_i uniformly in C^∞ ($i = 1, \dots, m$), such vector fields are equibounded on $B_e(0, R)$. By Ascoli-Arzelà Theorem, up to a further subsequence, there exists a curve $\gamma \in AC([0,1], \mathbb{R}^n)$ such that γ^{j_ℓ} uniformly converges to γ in $[0,1]$ as $\ell \to \infty$. For every $t \in [0,1]$ one has

$$\gamma^{j_\ell}(t) = x + \int_0^t \sum_{i=1}^m h_i^{j_\ell}(s) X_i^{j_\ell}(\gamma^{j_\ell}(s)) ds$$

and, taking into account that $X_i^{j_\ell} \circ \gamma^{j_\ell} \to X_i \circ \gamma$ uniformly in $[0,1]$ and that $h^j \overset{*}{\rightharpoonup} h$ in L^∞, by letting $\ell \to \infty$ one gets

$$\gamma(t) = x + \int_0^t \sum_{i=1}^m h_i(s) X_i(\gamma(s)) ds.$$

In particular $\gamma = \gamma_{h,x}$, $\gamma(1) = y$ and

$$d(x,y) \le \|h\|_{L^\infty} \le \liminf_{\ell\to\infty} \|h_{j_\ell}\|_{L^\infty} = \liminf_{j\to\infty} d_j(x,y),$$

which proves (11).

By the compactness of K we can find $x_1, \dots, x_k \in K$ such that $K \subseteq \bigcup_{\ell=1}^k B(x_\ell, \varepsilon)$. Using Theorem 3.1 and (11) we can find $\widetilde{C} = \widetilde{C}(K) > 0$ and $J_5 = J_5(K,\varepsilon) \in \mathbb{N}$ such that for $j \ge J_5$

$$\begin{aligned} &B(x_\ell, \varepsilon) \subseteq B^j(x_\ell, \widetilde{C}\varepsilon^{1/s(K)}) && \forall\, \ell = 1, \dots, k \\ &d(x_{\ell_1}, x_{\ell_2}) \le d_j(x_{\ell_1}, x_{\ell_2}) + \varepsilon && \forall\, \ell_1, \ell_2 = 1, \dots, k. \end{aligned}$$

For every $x, y \in K$ we can find $x_{\ell_1}, x_{\ell_2} \in K$ (with $1 \le \ell_1, \ell_2 \le k$) such that $x \in B(x_{\ell_1}, \varepsilon)$ and $y \in B(x_{\ell_2}, \varepsilon)$, hence for $j \ge J_5$ we have

$$\begin{aligned} d(x,y) &\le d(x, x_{\ell_1}) + d(x_{\ell_1}, x_{\ell_2}) + d(y, x_{\ell_2}) \\ &\le \varepsilon + d_j(x_{\ell_1}, x_{\ell_2}) + \varepsilon + \varepsilon \\ &\le d_j(x_{\ell_1}, x) + d_j(x,y) + d_j(y, x_{\ell_2}) + 3\varepsilon \\ &= d_j(x,y) + 3\varepsilon + 2\widetilde{C}\varepsilon^{1/s(K)}, \end{aligned}$$

which, combined with (10), concludes the proof. □

Let us recall that, given a CC space $(\mathbb{R}^n, X)$, a function $u \in L^1_{loc}(\Omega)$ is said to have *locally bounded X-variation* if the distributional derivatives $X_1 u, \dots, X_m u$ are represented by Radon measures. See e.g. [2, 4]. We denote by $BV_{X,loc}(\mathbb{R}^n)$ the set of functions of locally bounded X-variation in $\mathbb{R}^n$ and by $|D_X u|$ the total variation of the vector-valued measure $D_X u := (X_1 u, \dots, X_m u)$.

Sobolev- and Poincaré-type inequalities in CC spaces have been largely investigated; among the vast literature we mention only [9, 5, 7]. The following result is an easy consequence of [1, Theorem 7.2] or [11, Theorem 1.1]. Notice that the latter results are proved only when u is a smooth function on $\mathbb{R}^n$; in order to prove Theorem 3.5 as stated here one has to approximate functions in $BV_{X,loc}$ by smooth ones (see [4, 5]).

Theorem 3.5. *Let $X = (X_1, \dots, X_m)$ and $X^j = (X_1^j, \dots, X_m^j)$, $j \in \mathbb{N}$, be m-tuples of linearly independent smooth vector fields on $\mathbb{R}^n$ such that X satisfies the Chow-Hörmander condition and its CC balls are bounded in $\mathbb{R}^n$; assume that, for every $i = 1, \dots, m$, $X_i^j \to X_i$ in $C^\infty_{loc}(\mathbb{R}^n)$ as $j \to \infty$. Then, for every compact set $K \subseteq \mathbb{R}^n$ there exist $C_P > 1$, $\alpha \ge 1$, $R_P > 0$ and $J \in \mathbb{N}$ such that for every $j \ge J$, $u \in BV_{X^j,loc}(\mathbb{R}^n)$, $x \in K$ and $r \in (0, R_P)$ one has*

$$\int_{B^j} \left|u - u(B^j)\right| d\mathscr{L}^n \le C_P\, r\, |D_{X^j} u|(\alpha B^j), \tag{12}$$

where $B^j := B^j(x,r)$ and $u(B^j) = \fint_{B^j} u\, d\mathscr{L}^n$.

We can then state our main application. See [7, Section 8] for more references about compactness results for Sobolev or BV functions in CC spaces.

Theorem 3.6. *Let $X = (X_1, \dots, X_m)$ and $X^j = (X_1^j, \dots, X_m^j)$, $j \in \mathbb{N}$, be m-tuples of linearly independent smooth vector fields on $\mathbb{R}^n$ such that X satisfies the Chow-Hörmander condition and its CC balls are bounded in $\mathbb{R}^n$; assume that, for every $i = 1, \dots, m$, $X_i^j \to X_i$ in $C^\infty_{loc}(\mathbb{R}^n)$ as $j \to \infty$. Let $u_j \in BV_{X^j,loc}(\mathbb{R}^n)$ be a sequence of functions that is locally uniformly bounded in BV, i.e., such that for any compact set $K \subseteq \mathbb{R}^n$ there exists $M > 0$ such that*

$$\forall j \in \mathbb{N} \qquad \|u_j\|_{L^1(K)} + |D_{X^j} u_j|(K) \le M < \infty.$$

Then, there exist $u \in BV_{X,loc}(\mathbb{R}^n)$ and a subsequence $(u_{j_h})_h$ of $(u_j)_j$ such that $u_{j_h} \to u$ in $L^1_{loc}(\mathbb{R}^n)$ as $h \to \infty$. Moreover, for any bounded open set $\Omega \subseteq \mathbb{R}^n$ the semicontinuity of the total variation

$$|D_X u|(\Omega) \le \liminf_{j\to\infty} |D_{X^j} u_j|(\Omega)$$

holds.

Proof. We use Theorem 2.1 with $X = \mathbb{R}^n$, $\lambda = \mathscr{L}^n$, $\delta = q = 1$, $\mu_j := |D_{X^j} u|$ and d, d_j the CC distances associated with X, X^j respectively. Assumption (i) follows from Theorem 3.4, while the local doubling property (ii) of d is a well-known fact (see e.g. [13]). The validity of (iii) (with $\delta = q = 1$) follows from Theorem 3.5, while (iv) is satisfied by assumption.

Theorem 2.1 ensures that, up to subsequences, u_j converges to some u in $L^1_{loc}(\mathbb{R}^n)$; we need to show that $u \in BV_{X,loc}(\mathbb{R}^n)$. To this aim, for any $i = 1, \dots, m$ we denote by X_i^* the formal adjoint to X_i and write

$$X_i(x) = \sum_{k=1}^n a_{i,k}(x)\partial_k \qquad \text{and} \qquad X_i^j(x) = \sum_{k=1}^n a_{i,k}^j(x)\partial_k$$

for suitable smooth functions $a_{i,k}, a_{i,k}^j$. Then, for any bounded open set $\Omega \subseteq \mathbb{R}^n$ and any test function $\varphi \in C_c^1(\Omega)$ we have

$$\begin{aligned}\int_\Omega u\, X_i^*\varphi \, d\mathscr{L}^n = \int_\Omega u \sum_{k=1}^n \partial_k(a_{i,k}\varphi)\, d\mathscr{L}^n \;&=\; \lim_{j\to\infty} \int_\Omega u_j \sum_{k=1}^n \partial_k(a_{i,k}^j\varphi)\, d\mathscr{L}^n \\ &= -\lim_{j\to\infty} \int_\Omega \varphi\, dX_i^j u_j \;\le\; \|\varphi\|_{L^\infty(\Omega)} \liminf_{j\to\infty} |D_X^j u_j|(\Omega) < \infty.\end{aligned}$$

This proves that $u \in BV_{X,loc}(\mathbb{R}^n)$ as well as the semicontinuity of the total variation. The proof is accomplished. □

Remark 3.7. We conjecture that, when the CC space $(\mathbb{R}^n, X)$ is *equiregular*, the convergence $u_{j_h} \to u$ in Theorem 3.6 holds in L^q_{loc} for any $q \in [1, \frac{Q}{Q-1}[$, where Q is the Hausdorff dimension of $(\mathbb{R}^n, X)$. This would easily follow in case the Poincaré inequality (12) could be strengthened to

$$\|u - u(B^j)\|_{L^q(B^j)} \le C_P\, r^\delta\, |D_{X^j} u|(\alpha B^j)$$

for some $\delta > 0$ (arguably, $\delta = \frac{Q}{q} + 1 - Q$). The key point would be proving that the constant C_P can be chosen independent of j but, as far as we know, no investigation in this direction has been attempted in the literature.

Remark 3.8. Theorems 3.4, 3.5 and 3.6 hold also under a slightly weaker assumption: it is indeed enough that, for any compact set $K \subseteq \mathbb{R}^n$, the convergence $X_i^j \to X_i$ holds in $C^k(K)$ for a suitable $k = k(K)$ (actually, k depends only on $s(K)$) that one could explicitly compute. See [1, 11] for more details.

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