Geometric Variational Problems on Spaces of Multiple-Valued Functions

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ABSTRACT

In this thesis we deal with various problems arising in Calculus of Variations, Geometric Measure Theory and Geometric Analysis. Most of the variational problems treated here will be set in the context of multiple-valued functions, a tool introduced by F. Almgren in [Almoo] in order to study the regularity of generalized minimal surfaces (area minimizing integer rectifiable currents) in higher codimension. In recent years, C. De Lellis and E. Spadaro have revisited Almgren's work (cf. [DLS11, DS15, DLS14, DLS16a, DLS16b]), thus not only providing a shorter proof of his celebrated partial regularity result, but also renewing the interest towards multiple-valued functions and their strong interplay with the theory of minimal surfaces.

The topics studied in this thesis can be gathered in four groups, each corresponding to one of the parts of the thesis.

1. Multi-valued theory of the stability operator:
   One of the most relevant questions in Geometric Measure Theory is whether the class of tangent cones at each interior point of the support of an area minimizing integral current consists of a unique element or not. The answer to this question is known (and it is affirmative) only for few special classes of minimizing currents (e.g. two-dimensional currents, cf. [Whi83]). The problem in its full generality is instead widely open. In the cases when an isolated singularity of a minimizing current admits one tangent cone whose cross-section is a minimal submanifold of the sphere, uniqueness of the cone can be proved, provided the spectrum of the stability operator and the Jacobi fields of such a minimal submanifold satisfy certain properties (cf. [AA81, Sim94]). Motivated by the desire to extend the Allard-Almgren and Simon results to more general scenarios, we develop a multi-valued theory of the stability operator. Specifically, for every positive integer Q, we define Jacobi Q-fields as those Q-valued sections of the normal bundle of a minimal submanifold of a Riemannian manifold which minimize an energy (the Jacobi energy Jac) obtained from a multi-valued counterpart of the second variation formula. After studying sufficient conditions such that the minimum problem for the Jac functional admits a solution for any given boundary datum, we explore the regularity of the minimizers. In this direction, we first show that Jacobi Q-fields are locally Hölder continuous Q-valued functions. Then, we prove that every Jacobi Q-field can be written as the superposition of Q classical Jacobi fields in the neighborhood of every point *except* those belonging to a singular set of codimension at least 2 in the submanifold. These results extend analogous results of Almgren and De Lellis-Spadaro (cf. [Almoo, DLS11]) valid for Dirichlet minimizing Q-valued functions to a more general class of functionals on Sobolev spaces of multiple valued functions, and are contained in our paper [Stu17a].

2. Multiple-valued sections of vector bundles and applications:
   Following some ideas of W. Allard [All13], we define multiple-valued sections of

an abstract vector bundle over a Riemannian manifold, and we study some geometric properties which extend the concepts of continuity and Lipschitz continuity of multiple-valued functions to the vector-bundle setting. With these theoretical tools, we are able to provide a new geometric proof of the delicate reparametrization theorem for multiple-valued graphs contained in [DS15], which in turn is the key step for producing the normal Lipschitz Q-valued approximations of a minimizing current from the center manifold needed in the proof of Almgren's partial regularity theorem (cf. [DLS16a]). These results are contained in [Stu17b].

3. Regularity and singularities of multiple-valued harmonic maps:
Minimizing harmonic maps are minimizers of the Dirichlet energy with respect to boundary data under the constraint to take values in a given Riemannian manifold. Unlike their unconstrained counterpart, minimizing harmonic maps need not be everywhere smooth. Nonetheless, it is a well known result (cf. [SU82]) that the set where a minimizing harmonic map fails to be smooth is "small", in the sense that it has codimension at least 3 in the domain of the map. In [NV17], A. Naber and D. Valtorta prove that if the domain of the map is m-dimensional then the singular set is in fact $(m-3)$-rectifiable with uniformly finite $(m-3)$-dimensional Hausdorff measure. In [Hir16b], J. Hirsch initiated the analysis of multiple-valued Dirichlet minimizing harmonic maps, developing the basic continuity theory for these objects, analogous to [SU82]. Here, we extend the results of [NV17] to the multiple-valued framework, thus proving rectifiability and volume estimates of the singular set of multiple-valued minimizing harmonic maps. Moreover, we study the special case of non-positively curved target manifolds. Specifically, we show that if the target is non-positively curved and simply connected then any Q-valued minimizing harmonic map has empty singular set. If $Q = 1$, this result holds under the weaker assumption that the target is merely connected; we provide an example showing that this stronger theorem is false whenever $Q > 1$. These results, obtained in collaboration with J. Hirsch and D. Valtorta, are contained in [HSV17].

4. Results on real currents and currents with coefficients in groups:
In the paper [CDMS17], in collaboration with M. Colombo, A. De Rosa and A. Marchese, we study a general class of energies defined on real rectifiable currents via integration, over the rectifiable set supporting the current, of a function H of the multiplicity. These energies are known in the literature as H-masses, and they are usually considered, with specific choices for H, to represent the "costs" of transportation networks in branched transportation models. We prove that, under minimal assumptions on H, the H-mass coincides, on rectifiable currents, with the lower semi-continuous envelope of the H-mass functional defined on real polyhedral chains.

   In the paper [MS17], in collaboration with A. Marchese, we focus instead on the structure of currents with coefficients in the group $\mathbb{Z}_p$ of integers $\mathrm{mod}(p)$. We show that every equivalence class in the quotient group of integral 1-currents modulo p in Euclidean space contains an integral current, with quantitative estimates on its mass and the mass of its boundary. This affirmatively answers, at least in the simpler case of one-dimensional currents, a long-standing open question of F. Almgren (cf. [ope86, Prob-

lem 3.3]). Moreover, we show that the validity of this statement for m-dimensional integral currents modulo p implies that the family of $(m-1)$-dimensional flat chains of the form pT, with T a flat chain, is closed with respect to the flat norm. In particular, we deduce that such closedness property holds for 0-dimensional flat chains, and, using a proposition from [Whi79], also for flat chains of codimension 1.

ZUSAMMENFASSUNG

Diese Arbeit handelt von verschiedene Probleme der Variationsrechnung, geometrischer Masstheorie und geometrischer Analysis. Die meisten Probleme der Variationsrechnung hier behandelten sind im Zusammenhang mit mehrwertigen Funktionen beschrieben. Diese Technik wurde von F. Almgren in [Alm00] eingeführt, um die Regularität von verallgemeinerten Minimalflächen (flächenminimierende ganzzahlig rektifizierbare Ströme) in höherer Codimension zu untersuchen. In den vergangenen Jahren haben C. De Lellis und E. Spadaro die Arbeiten von F. Almgren wieder aufgegriffen (vgl. [DLS11, DS15, DLS14, DLS16a, DLS16b]). Sie haben nicht nur einen kürzeren Beweis von seinem berühmten Resultat über partielle Regularität erstellt, sondern auch die Wichtigkeit von mehrwertigen Funktionen und ihren starken Zusammenhang zu der Minimalflächen-Theorie aufgezeigt.

Die verschiedenen Themen dieser Arbeit sind auf die folgenden vier Kapitel aufgeteilt:

1. Mehrwertigkeits Theorie vom Stabilitätsoperator:
   Eine der bedeutendsten Fragen in der geometrischen Masstheorie ist, ob an jedem inneren Punkt des Trägers eines flächenminimierenden integralen Stroms die Klasse der Tangentialkegel aus einem eindeutigen Element besteht. Die dazugehörige Antwort ist unbekannt. Nur in wenigen Spezialfällen von Klassen von flächenminimierenden Strömen (z.B. zwei-dimensionale Ströme, siehe [Whi83]) ist die Antwort bejahend. Jedoch ist das Problem in seiner ganzen Allgemeinheit noch völlig offen. Die Eindeutigkeit des Kegels kann gezeigt werden, falls der flächenminimierender Strom eine isolierte Singularität besitzt, die Schnittfläche eines zugehörigen Tangentialkegels mit der Sphäre eine minimale Untermannigfaltigkeit der Sphäre ist und das Spektrum des Stabilitätsoperator und die Jacobi Felder der minimalen Untermannigfaltigkeit gewisse Eigenschaften erfüllen (siehe [AA81, Sim94]). Mit dem Ziel die Resultate von Allard-Almgren und Simon auf allgemeinere Situationen zu erweitern, entwickeln wir eine Mehrwertigkeits Theorie für den Stabilitätsoperator. Genauer definieren wir zu jeder positiven ganzen Zahl Q ein Jacobi Q-Feld als ein Q-wertiger Schnitt vom Normalenbündel einer minimalen Untermannigfaltigkeit von einer Riemannscher Mannigfaltigkeit, welcher eine Energie (die Jacobi Energie Jac) minimiert. Diese (Jacobi) Energie kommt von dem mehrwertigen Gegenstück der zweiten Variationsformel. Nachdem wir die hinreichenden Bedingungen dafür untersuchen, dass das Minimierungsproblem für das Jac Funktional eine Lösung für jeden beliebigen Anfangswert besitzt, beschäftigen wir uns mit der Regularität der minimierenden Schnitte. Dort zeigen wir zuerst, dass Jacobi Q-Felder Hölder-stetige mehrwertige Funktionen sind. Dann beweisen wir, dass ausser um Punkte, welche zu einer singulären Menge gehören, deren Codimension in der Untermannigfaltigkeit mindestens zwei beträgt, jedes Jacobi Q-Feld in einer offenen Umgebung als Superposition von klassischen Jacobi Q-Felder dargestellt werden kann. Diese Resultate sind Teile von unserem Artikel [Stu17a].

2. Mehrwertige Schnitte von Vektorbündel und Anwendungen:
   Mit ähnlichen Ideen wie diesen von W. Allard in [All13] definieren wir mehrwertige Schnitte von abstrakten Vektorbündel von Riemannschen Mannigfaltigkeiten. Wir untersuchen gewisse geometrische Eigenschaften, welche das Konzept von Stetigkeit und Lipschitz-Stetigkeit von mehrwertigen Funktionen auf Vektorbündel erweitern. Mit diesen theoretischen Mitteln erschaffen wir einen neuen geometrischen Beweis für das Umparametrisierungstheorem für Graphen von mehrwertigen Funktionen wie in [DS15]. Dieser Beweis ist der entscheidende Schritt, um die Q-wertigen Lipschitz-Funktionen zu konstruieren, welche normal zur zentralen Mannigfaltigkeit definiert sind und den minimierenden Strom approximieren. Dies ist im Beweis von F. Almgrens partiellem Regularitätstheorem notwendig (siehe [DLS16a]). Diese Resultate sind in [Stu17b] zu finden.

3. Regularität und Singularitäten von mehrwertigen harmonischen Abbildungen:
   Minimierende harmonische Abbildungen sind Abbildungen, deren Wertebereiche in einer gegebenen Riemannschen Mannigfaltigkeit liegen und welche die Dirichlet Energie bezüglich den Randwerten minimieren. Im Gegensatz zu harmonischen Funktionen sind die minimierenden harmonischen Abbildungen nicht zwingend überall glatt. Allerdings ist bekannt (siehe [SU82]), dass die Menge der Punkte, bei denen die minimierende harmonische Abbildung nicht glatt ist, "klein" ist in dem Sinne, dass die Codimension dieser Menge im Definitionsbereich der Abbildung mindestens drei beträgt. In [NV17] haben A. Naber und D. Valtorta gezeigt, dass falls der Definitionsbereich der Abbildung m-dimensional ist, dann ist die singuläre Menge $(m-3)$-rektifizierbar und ihr $(m-3)$-Hausdorfmass ist gleichmässig endlich. In [Hir16b] hat J. Hirsch die Analyse von mehrwertigen Dirichlet Energie minimierenden harmonischen Abbildungen eingeführt. Er hat analog zu [SU82] die grundlegende Stetigkeitstheorie für solche Abbildungen entwickelt. Hier erweitern wir die Resultate von [NV17] zum mehrwertigen Konstrukt. Das heisst, wir beweisen die Rektifizierbarkeit der singulären Menge und schätzen ihr Volumen ab. Des Weiteren untersuchen wir den Spezialfall, wenn die Mannigfaltigkeit, in welcher der Wertebereich der Abbildung liegt, nicht-positiv gekrümmt ist. Genauer zeigen wir, dass falls diese Mannigfaltigkeit nicht-positiv gekrümmt und einfach zusammenhängend ist, dann ist jede Q-wertige minimierende harmonische Abbildung überall stetig. Dies ist auch wahr, falls $Q = 1$ und die Mannigfaltigkeit nur zusammenhängend ist. Wir zeigen anhand eines Beispiels, dass diese stärkere Aussage jedoch nicht mehr gilt, falls $Q > 1$. Diese Resultate, welche in Zusammenarbeit mit J. Hirsch und D. Valtorta entstanden sind, sind in [HSV17] aufgeführt.

4. Resultate über reelle Ströme und Ströme mit Koeffizienten in Gruppen:
   In dem Artikel [CDMS17] untersuchen wir in Zusammenarbeit mit M. Colombo, A. De Rosa und A. Marchese eine allgemeine Klasse von Energien, welche auf reellen rektifizierbaren Strömen durch Integration einer Funktion H der Multiplizität über der rektifizierbaren Menge, welche der Träger des Stroms darstellt, definiert ist. Diese Energien sind in der Literatur mit H-Massen bezeichnet und werden gewöhnlich als "Kosten" von Transport-Netzwerken in verzweigten Transportmodellen angesehen.

Wir zeigen, dass falls H gewisse Annahmen erfüllt, dann ist die H-Masse auf rektifizierbaren Strömen die unterhalbstetige Hülle vom H-Mass-Funktional, welches auf polyedrischen Strömen definiert ist.

Im Artikel [MS17] beschäftigen wir uns in Zusammenarbeit mit A. Marchese mit der Struktur von Strömen mit Koeffizienten in der Gruppe $\mathbb{Z}_p$ von den ganzen Zahlen $\mathrm{mod}(p)$. Wir zeigen, dass jede Äquivalenzklasse in der Quotientengruppe von integralen 1-Strömen modulo p im Euklidischen Raum einen integralen Repräsentanten besitzt. Dies bejaht die lange offen gestandene Frage von F. Almgren (siehe [ope86, Problem 3.3]) im einfacheren Fall von 1-dimensionalen Strömen. Des Weiteren zeigen wir, dass die Gültigkeit dieser Aussage im Falle von m-dimensionalen integralen Ströme modulo p impliziert, dass die Familie der $(m-1)$-dimensionalen flat Chains der Form pT, wobei T ein flat Chain ist, abgeschlossen ist in der flat Norm. Insbesondere folgern wir, dass diese Abgeschlossenheit auch für 0-dimensionale flat Chains gilt. Zusammen mit einer Proposition von [Whi79] leiten wir dies auch für flat Chains mit Codimension 1 her.

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Wir müssen wissen,
wir werden wissen.

— David Hilbert (1862 - 1943)

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GENERAL CONVENTIONS ADOPTED IN THIS THESIS

Natural, integer, rational and real numbers are canonically denoted by $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ and $\mathbb{R}$ respectively. Complex numbers are denoted $\mathbb{C}$, and $\bar{z}$ denotes the complex conjugate of the complex number z. We will denote by $[a, b]$ the closed interval in $\mathbb{R}$, defined by $a \leqslant x \leqslant b$, and by (a, b) the open interval, defined by $a < x < b$. The Euclidean space of dimension d is canonically denoted $\mathbb{R}^d$. If $v, w \in \mathbb{R}^d$, then their Euclidean scalar product is denoted $\langle v, w \rangle := \sum_{i=1}^{d} v_i w_i$. The Euclidean norm of $v \in \mathbb{R}^d$ is $|v| := (\langle v, v \rangle)^{1/2} = \left(\sum_{i=1}^{d} v_i^2 \right)^{1/2}$. The $(d-1)$-dimensional *sphere* in $\mathbb{R}^d$, defined by the condition $|x| = 1$, is denoted S^{d-1}. The symbol $B_r(x)$ (or sometimes $B(x, r)$) will denote the open ball centered at $x \in \mathbb{R}^d$ and having radius $r > 0$. We will often simply write B_r if the center is the origin, whereas we will use the writing $B_r^d(x)$ to emphasize that the ambient space is $\mathbb{R}^d$. If $E \subset \mathbb{R}^d$, then $\overline{E}$, $\mathrm{int}E$, and ∂E denote its closure, its interior, and its boundary respectively. We will also set $E^c := \mathbb{R}^d \setminus E$.

If (X, d) is a metric space and $E \subset X$ then the *diameter* of E is the number

$$\mathrm{diam}(E) := \sup\{d(x, y) : x, y \in E\} .$$

If $\mathrm{diam}(E) < \infty$, then we say that E is *bounded*. If $E_1, E_2 \subset X$ then their *distance* is

$$\mathrm{dist}(E_1, E_2) := \inf\{d(x, y) : x \in E_1 , y \in E_2\} .$$

The Lebesgue measure of a measurable set $E \subset \mathbb{R}^d$ is denoted $\mathcal{L}^d(E)$ or simply $|E|$. If m is a non-negative real number, with the symbol $\mathcal{H}^m$ we denote the m-dimensional Hausdorff measure on $\mathbb{R}^d$, i.e.

$$\mathcal{H}^m(B) := \lim_{\delta \to 0} \inf \left\{ \sum_{h=1}^{\infty} \omega_m \left(\frac{\mathrm{diam}(E_h)}{2} \right)^m : B \subset \bigcup_{h=1}^{\infty} E_h , \mathrm{diam}(E_h) \leqslant \delta \right\},$$

with $\omega_m := \frac{\pi^{m/2}}{\Gamma\left(\frac{m}{2}+1\right)}$, Γ denoting the usual Gamma function. In particular, if m is integer then ω_m is the volume of the unit m-dimensional ball.

If μ is a positive Radon measure on $\mathbb{R}^d$, $m \geqslant 0$, and $x \in \mathbb{R}^d$, we will denote by $\Theta_*^m(\mu, x)$ and $\Theta^{m*}(\mu, x)$ the *lower* and *upper* m*-dimensional densities* of μ at x, respectively defined by

$$\Theta_*^m(\mu, x) := \liminf_{r \to 0} \frac{\mu(B_r(x))}{\omega_m r^m} , \quad \Theta^{m*}(\mu, x) := \limsup_{r \to 0} \frac{\mu(B_r(x))}{\omega_m r^m}.$$

In case the two limits above coincide, we will let $\Theta^m(\mu, x)$ denote their common value. We will then call $\Theta^m(\mu, x)$ the m-dimensional *density* of μ at x.

The *restriction* of a positive Radon measure μ to a Borel set E is denoted $\mu \llcorner E$.

With a slight abuse of notation, we will identify a linear map $L\colon \mathbb{R}^m \to \mathbb{R}^n$ with its representation matrix with respect to the standard bases of $\mathbb{R}^m$ and $\mathbb{R}^n$, and thus we will write $L \in \mathbb{R}^{n\times m}$. The action of $L \in \mathbb{R}^{n\times m}$ on a vector $v \in \mathbb{R}^m$ is denoted $L \cdot v$. If $L, M \in \mathbb{R}^{n\times m}$ then we will denote by $\langle L : M\rangle$ their Hilbert-Schmidt scalar product, given by $\langle L : M\rangle := \sum_{i=1}^{n}\sum_{j=1}^{m} L_{ij}M_{ij}$. The Hilbert-Schmidt norm of $L \in \mathbb{R}^{n\times m}$ is then simply $|L| := (\langle L : L\rangle)^{1/2}$.

We will use standard notations for the classical spaces of functions: Lebesgue spaces, Sobolev spaces, and Hölder spaces are denoted L^p, $W^{k,p}$, and $C^{k,\alpha}$ respectively. Spaces $C^{0,1}$ of Lipschitz functions are denoted Lip. If F is a space of functions, then the writing $f \in F(A, B)$ means that $f\colon A \to B$ and that $f \in F$. If X and Y are metric spaces and $f \in \mathrm{Lip}(X, Y)$ then $\mathrm{Lip}(f)$ denotes its Lipschitz constant. The support of a function f (or of a measure μ) is denoted $\mathrm{spt}(f)$ (or $\mathrm{spt}(\mu)$). The writing $f \in C_c^{k,\alpha}$ means that f is a function in $C^{k,\alpha}$ and that $\mathrm{spt}(f)$ is a compact subset of its domain.

All manifolds appearing in this thesis are assumed to be second countable and Hausdorff. The notation here used for classical objects in differential geometry is standard, and some notions will be recalled when the need arises. The class of differentiability of a manifold is either specified or assumed to be ∞. If Σ is a manifold then we will sometimes write Σ^m to underline that the dimension of Σ is m. If Σ^m is a manifold of class C^1 and $x \in \Sigma$ then $T_x\Sigma$ denotes the tangent space to Σ at x. If $f\colon \Sigma \to \mathbb{R}^q$ is a C^1 map and ξ is a tangent vector field to Σ, the symbol $D_\xi f$ will denote the directional derivative of f along ξ, that is

$$D_\xi f(x) := \frac{d}{dt}(f \circ \gamma)\Big|_{t=0}$$

whenever $\gamma = \gamma(t)$ is a C^1 curve on Σ with $\gamma(0) = x$ and $\dot\gamma(0) = \xi(x)$. The differential of f at $x \in \Sigma$ will be denoted $Df(x)$: we recall that this is the linear operator $Df(x)\colon T_x\Sigma \to \mathbb{R}^q$ such that $Df(x) \cdot \xi(x) = D_\xi f(x)$ for any tangent vector field ξ. The notation $Df|_x$ will sometimes be used in place of $Df(x)$. Moreover, the derivative along ξ of a scalar function $f\colon \Sigma \to \mathbb{R}$ will be sometimes simply denoted by $\xi(f)$.

We will assume that the reader is familiar with basic notions in multi-linear algebra (cf. [Sim83b, Section 25]). The vector spaces of m-vectors and m-covectors in $\mathbb{R}^d$ are denoted $\Lambda_m(\mathbb{R}^d)$ and $\Lambda^m(\mathbb{R}^d)$ respectively. The pairing between m-covectors and m-vectors is denoted $\langle\cdot,\cdot\rangle$. If $\vec v = v_1 \wedge \cdots \wedge v_m \in \Lambda_m(\mathbb{R}^d)$ is a simple m-vector, then its Euclidean norm is denoted $|\vec v|$. If $\omega \in \Lambda^m(\mathbb{R}^d)$ is an m-covector, we will let $\|\omega\|_c$ denote its *comass norm*, that is the quantity

$$\|\omega\|_c := \sup\{\langle \omega, \vec v\rangle : \vec v \text{ is a simple } m\text{-vector and } |\vec v| = 1\}.$$

Let $\Omega \subset \mathbb{R}^d$ be an open set. A smooth map

$$\omega\colon \Omega \to \Lambda^m(\mathbb{R}^d)$$

is called a smooth *differential* m*-form* in Ω. We will denote $\mathcal{D}^m(\Omega) := C_c^\infty(\Omega, \Lambda^m(\mathbb{R}^d))$ the space of smooth compactly supported differential m-forms in Ω, endowed with the usual

locally convex topology of uniform convergence on compact sets. If $\omega \in \mathcal{D}^m(\Omega)$ then $d\omega \in \mathcal{D}^{m+1}(\Omega)$ is its exterior differential.

Finally, constants will be usually denoted by C. The precise value of C may change from line to line throughout a computation. Moreover, we will write $C(a, b, \dots)$ or $C_{a,b,\dots}$ to specify that C depends on previously introduced quantities $a, b, \dots$.

1 INTRODUCTION

Understanding the structure of *minimal surfaces* has represented, and still represents, a great, long-standing challenge to mathematicians.

The starting point of our discussion is the following very simple question, historically attributed to the Belgian physicist J.A.F. Plateau (although it had been formulated much earlier by Lagrange, Meusnier and others), who originally raised it for two-dimensional surfaces in the three-dimensional Euclidean space.

Question 1.0.1 (Plateau's problem)**.** Given an $(m-1)$-dimensional "contour" Γ in $\mathbb{R}^{m+n}$ (or in an $(m+n)$-dimensional Riemannian manifold), is there an m-dimensional surface Σ having minimal m-dimensional area among all those spanning Γ?

The problem is deliberately stated in vague terms. The concepts "m-dimensional surface", "m-dimensional area" and "spanning" can be interpreted mathematically in many different ways, and different theories can be consequently produced. During the nineteenth century, Plateau's problem was solved for many special contours Γ, but a sufficiently general solution was only obtained in 1930 simultaneously by J. Douglas and T. Radó. Their solution considered only two-dimensional surfaces in Euclidean space $\mathbb{R}^3$, defined as images of mappings from the disc $\mathbb{D} := \{u^2 + v^2 \leqslant 1\} \subset \mathbb{R}^2$. As soon as one tries to tackle the problem in higher dimension (the number m in the statement of Plateau's problem) and codimension (the number n in the statement of Plateau's problem), it becomes clear that the "surface as a mapping" approach is not promising at all. The main reason being that the natural topology lacks the necessary compactness properties to solve the problem by direct methods. As it is often the case in the Calculus of Variations, in order to gain enough compactness to guarantee convergence of a minimizing sequence, one has to enlarge the class of competitors sacrificing some a priori regularity assumptions. Guided by this principle, many generalizations of the classical notion of surface have been introduced in the second half of the twentieth century: among others, we cite De Giorgi's *sets of finite perimeter*, Federer-Fleming's *integer rectifiable currents*, and Almgren-Allard's *integral varifolds*.

In this thesis, we will work in the framework of integer rectifiable currents. Born in the early 1960s after the foundational work [FF60] of H. Federer and W.H. Fleming, currents are a very broad generalization of surfaces. Nonetheless, they are powerful enough not only to produce a satisfactory analytical and topological formulation of "m-dimensional domains of integration in a d-dimensional ambient space", but also to provide a satisfactory and extremely general (affirmative) answer to Plateau's problem. Once that the solvability of Plateau's problem has been established in the framework of integer rectifiable currents, and consequently the notion of *area minimizing current* has been introduced, one is left with the regularity issue that was created when the class of competitors was enlarged. One could hope that every area minimizing current is, eventually, everywhere regular. Maybe surprisingly, it turns out that this is not the case, since there exist "minimal surfaces" with

singularities. In turn, this information tells us that some generalization of the classical notion of surface was, in fact, unavoidable.

Let us try to provide a glimpse to the (interior) regularity theory for area minimizing currents. In doing this, we will necessarily need to use some of the notation and terminology which is typical of the theory of currents. The reader who is not familiar with it can refer to Section 2.1 and, of course, to the references therein.

Let us assume that T is an area minimizing (locally) integer rectifiable m-dimensional current in $\mathbb{R}^d$. Let $n := d - m$ be the codimension of T. Also let $\mathrm{Sing}(T)$ denote the set of points $x \in \mathrm{spt}(T) \setminus \mathrm{spt}(\partial T)$ for which there exists a neighborhood $\mathcal{U}$ of x in $\mathbb{R}^d$ such that $\mathrm{spt}(T) \cap \mathcal{U}$ is a smooth embedded m-dimensional submanifold. Then, the problem is to understand whether $\mathrm{Sing}(T)$ is empty and, in case it is not, to estimate its Hausdorff dimension.

It turns out that the answer to the above question strongly depends on the codimension n. If $n = 1$, then it is known that the singular set $\mathrm{Sing}(T)$ of an m-dimensional area minimizing current T has Hausdorff dimension at most $m-7$ ([DG61, Fle62, DG65, Alm66, Sim68, Fed70]), and it is countably $(m-7)$-rectifiable ([Sim95b], later improved in [NV15]); furthermore, if $m = 7$ then $\mathrm{Sing}(T)$ consists of isolated points ([Fed70]). In particular, m-dimensional area minimizing currents in $\mathbb{R}^{m+1}$ are classical submanifolds for every $m \leqslant 6$. On the other hand, if $n \geqslant 2$ then the current might exhibit singularities already in dimension $m = 2$. Indeed, it is known that in this higher codimension case the singular set $\mathrm{Sing}(T)$ of an m-dimensional area minimizing current T has Hausdorff dimension at most $m-2$ ([Almoo]), and consists of isolated points if $m = 2$ ([Cha88]). In fact, both results are sharp (see [BDGG69] for the codimension one case and [Fed65] for the higher codimension case).

Now, not only the singularities already appear in low dimensions when $n \geqslant 2$, but in fact the degree of difficulty of the two problems is substantially different. It is fair to say that, by now, the techniques leading to the regularity result in codimension one have been well assimilated by the community of mathematicians working in Geometric Measure Theory and Geometric Analysis. On the other hand, the proof of the result in the case $n \geqslant 2$ has required the development of a whole range of new tools, in order to deal with the kind of singularities that can arise.

A fundamental notion that is needed when addressing the problem of the regularity of area minimizing currents is that of a *tangent cone*. A by now very standard *monotonicity formula* (see [All72]) states that if T is area minimizing and m-dimensional, then the function $r \mapsto \frac{\|T\|(B_r(x))}{\omega_m r^m}$ (where $\|T\|(B_r(x))$ denotes the "area" of T in the ball $B_r(x)$) is monotone non-decreasing at *every* point $x \in \mathrm{spt}(T) \setminus \mathrm{spt}(\partial T)$. There are two main consequences of this monotonicity result: first, that the m-density $\Theta^m(\|T\|, x)$ of the measure $\|T\|$ associated with T is well defined at every interior point x; second, that if we define, for any $r > 0$, the function $\iota_{x,r} \colon \mathbb{R}^d \to \mathbb{R}^d$ by $\iota_{x,r}(y) := r^{-1}(y - x)$, then for any sequence of radii $r_h \downarrow 0$ the rescalings $T_{x,r_h} := (\iota_{x,r_h})_\sharp T$ obtained by *push-forwarding* T via ι_{x,r_h} (see § 2.1.2) converge, up to subsequences, to a (locally) area minimizing current which is invariant with respect to homotheties centered at the origin: such a limit current is called a tangent cone to T at x.

Now, one key idea that can be exploited in the regularity theory when $n = 1$ is that if T is area minimizing and if at least one tangent cone to T at a point $x \in \mathrm{spt}(T) \setminus \mathrm{spt}(\partial T)$ is (the

current associated to) a flat plane π, then, at a sufficiently small scale ρ, the current is close to the graph of a harmonic function defined on π. In particular, every point that admits at least one flat tangent cone is regular. On the other hand, a careful analysis of minimizing hypercones, together with the classical *dimension reduction argument* by Federer implies that the Hausdorff dimension of the set of points which do not admit any flat tangent cone cannot exceed $m-7$.

Such an argument fails dramatically in the higher codimension case. As an example, consider the holomorphic curve

$$\Gamma := \{(z,w) \in \mathbb{C}^2 : z^2 = w^3\}.$$

The curve Γ is calibrated, and thus area-minimizing. Nonetheless, the origin is a singular point for Γ, even though the (unique!) tangent cone to Γ at $(0,0)$ is the plane $\{z = 0\}$ (counted with multiplicity two). In fact, Γ cannot be approximated with the graph of a (single-valued!) function in any neighborhood of the origin.

This phenomenon, typical of the higher codimension, is called *branching*. In [Almoo], Almgren introduces the notion of *multiple-valued functions* taking a fixed number Q of values in order to approximate area minimizing currents in a neighborhood of a singular point of branching type with multiplicity Q. Specifically, a Q-valued function is a function taking values in the space $\mathcal{A}_Q(\mathbb{R}^n)$ of *unordered* Q*-tuples* of points in $\mathbb{R}^n$ (Q-points): each Q-point $S = \sum_{\ell=1}^{Q} [\![p_\ell]\!]$ is naturally identified with the purely atomic measure of mass Q in $\mathbb{R}^n$ obtained by placing a Dirac delta at each p_ℓ, and $\mathcal{A}_Q(\mathbb{R}^n)$ is endowed with the structure of complete separable metric space when the distance between two Q-points is defined as the L^2-based Wasserstein distance between the associated measures. A brief overview of Almgren's theory of multiple valued functions is the content of Section 2.2.

Almgren's strategy to tackle the regularity problem in higher codimension can be very broadly summarized in three main steps:

(1) develop a regularity theory for multiple-valued functions minimizing a suitable generalization of the Dirichlet energy (Dir-minimizers, the Q-valued counterpart of harmonic functions), where suitable means that it appears as the first non-trivial term in the Taylor expansion of the mass of (the rectifiable current associated to) a multiple-valued graph;

(2) perform a delicate approximation of the rescalings T_{x,r_h} of an area minimizing current T at a singular point $x \in \operatorname{spt}(T) \setminus \operatorname{spt}(\partial T)$ with multiple-valued functions converging, in the limit, to a Dir-minimizer;

(3) argue by contradiction: if the singular set of T was too large, a large singular set would also be inherited by the Dir-minimizer in the limit, thus contradicting the "linear" regularity theory in (1).

This program was successfully implemented in the enormous Big Regularity Paper [Almoo]. In recent years, C. De Lellis and E. Spadaro have revisited Almgren's program, giving a much shorter version of it (cf. [DLS11, DS15, DLS14, DLS16a, DLS16b] and also [DL16a, DL16b]). The new techniques developed in the last two decades to perform analysis in

metric spaces allowed De Lellis and Spadaro to substantially reduce the complexity of several arguments contained in [Almoo], thus getting more transparent proofs and achieving stronger analytic estimates. Chang's theorem [Cha88], according to which the singularities of two-dimensional area minimizing currents are isolated, was also re-proved in the light of the new techniques by the same authors and L. Spolaor (cf. [DSS15a, DSS15b, DSS15c]).

The major role played by multiple-valued Dirichlet minimizers in the regularity theory for higher codimension minimal surfaces suggests that a finer analysis of their properties may be the starting point for solving some of the yet unanswered questions in the theory of rectifiable currents minimizing the area functional. In this thesis, after some unavoidable preliminaries aimed both at introducing the main objects of our study (Chapter 2), and at presenting some technical tools which will be used in the next chapters (Chapter 3), we will present the results of our investigations on multiple-valued functions, which go both in the direction of studying the minimization of more general functionals than the Dirichlet energy (coming from higher order Taylor expansions of the mass of graphs, cf. Part I, Chapters 4 to 7), and in the sense of more general target spaces than $\mathbb{R}^n$ (Part II, Chapters 8-9, and Part III, Chapters 10-11). Part IV, the last of the thesis, will contain some results on real currents (Chapter 12) and on currents with coefficients in the group $\mathbb{Z}_p$ of integers modulo p (Chapter 13).

1.1 PART I: MULTI-VALUED THEORY OF THE STABILITY OPERATOR

In the above discussion we have introduced the fundamental notion of a *tangent cone* C to an m-dimensional area minimizing integer rectifiable current T at a point $x \in \operatorname{spt}(T) \setminus \operatorname{spt}(\partial T)$. Now, if x is a regular point, and thus $\operatorname{spt}(T)$ is a classical m-dimensional minimal submanifold in a neighborhood of x, then the cone C is certainly unique, and in fact $C = Q\llbracket\pi\rrbracket$, where $\pi = T_x(\operatorname{spt}(T))$ is the tangent space to $\operatorname{spt}(T)$ at x, the double brackets roughly mean "the current associated to", and $Q = \Theta^m(\|T\|, x)$ is the m-dimensional density of the measure $\|T\|$ at x. If, on the other hand, x happens to be singular, then not only we have no information about the limit cone, but in fact it is still an open question whether in general such a limit cone is unique (that is, independent of the approximating sequence) or not. The problem of uniqueness of tangent cones at the singular points of area minimizing currents of general dimension and codimension stands still today as one of the most celebrated of the unsolved problems in Geometric Measure Theory (cf. [ope86, Problem 5.2]), and only a few partial answers corresponding to a limited number of particular cases are available in literature. In [Whi83], B. White showed such uniqueness for two-dimensional area minimizing currents in any codimension, building on a characterization of two-dimensional area minimizing cones proved earlier on by F. Morgan in [Mor82]. In general dimension, W. Allard and F. Almgren [AA81] were able to prove that uniqueness holds under some additional requirements on the limit cone. Specifically, they have the following theorem.

Theorem 1.1.1 ([AA81]). *Let T be an m-dimensional area minimizing integer rectifiable current in $\mathbb{R}^{m+n}$, and let $x \in \operatorname{spt}(T)$ be an isolated singular point. Assume that there exists a tangent cone C to T at x satisfying the following hypotheses:*

(H1) C *is the cone over an* $(m-1)$*-dimensional minimal submanifold* Σ *of* $\mathbb{S}^{m+n-1}$ [1]*, and thus* C *has an isolated singularity at* 0 *and* $\Theta^m(\|C\|, x) = 1$ *for every* $x \in \mathrm{spt}(C) \setminus \{0\}$*;*

(H2) *all normal* Jacobi fields N *of* Σ *in* $\mathbb{S}^{m+n-1}$ *are integrable, that is for every normal Jacobi field* N *there exists a one-parameter family of minimal submanifolds of* $\mathbb{S}^{m+n-1}$ *having velocity* N *at* Σ*.*

Then, C *is the* unique *tangent cone to* T *at* x*. Furthermore, the blow-up sequence* $T_{x,r} = (\iota_{x,r})_\sharp T$ *converges to* C *as* $r \downarrow 0$ *with rate* r^β *for some* $\beta > 0$*.*

The hypotheses (H1) and (H2) are however quite restrictive. Allard and Almgren were able to show that (H2) holds in case Σ is the product of two lower dimensional standard spheres (of appropriate radii to ensure minimality), since in this case all normal Jacobi fields of Σ in $\mathbb{S}^{m+n-1}$ arise from isometric motions of $\mathbb{S}^{m+n-1}$. It seems however rather unlikely that the condition can hold for any general Σ admitting normal Jacobi fields other than those generated by rigid motions of the sphere. In [Sim83a], L. Simon was able to prove Theorem 1.1.1 dropping the hypothesis (H2), with a quite different approach with respect to [AA81] and purely PDE-based techniques. Not much has been done, instead, in the direction of removing (or, at least, weakening) the hypothesis (H1): to our knowledge, indeed, the only result concerning the case when a tangent cone C has more than one isolated singularity at the origin is contained in L. Simon's work [Sim94], where the author proves uniqueness of tangent cones to any *codimension one* area minimizing m-current T whenever one limit cone C is of the form $C = C_0 \times \mathbb{R}$, with C_0 a *strictly stable*, *strictly minimizing* $(m-1)$-dimensional cone in $\mathbb{R}^m$ with an isolated singularity at the origin, and under additional assumptions on the Jacobi fields of C and on the spectrum of the Jacobi normal operator of C_0.

However, all the results discussed above do not cover the cases when a tangent cone has higher multiplicity: it is remarkable that uniqueness is still open even under the strong assumption that *all* tangent cones to an area minimizing m-current T ($m > 2$) at an interior singular point x are of the form $C = Q\llbracket\pi\rrbracket$, where $\llbracket\pi\rrbracket$ is the rectifiable current associated with an oriented m-dimensional linear subspace of $\mathbb{R}^{m+n}$ and $Q > 1$ (cf. [Almoo, Section I.11(2), p. 9]).

In Part I we will present our paper [Stu17a], where we develop a complete *multi-valued* theory of the Jacobi normal operator: we believe that such a theory may facilitate the understanding of the qualitative behaviour of the area functional near a minimal submanifold *with multiplicity*, and eventually lead to a generalization of Theorem 1.1.1 (and neighbouring results) to relevant cases when the condition that $\Theta^m(\|C\|, x) = 1$ for every $x \in \mathrm{spt}(C) \setminus \{0\}$ fails to hold.

As a byproduct, the theory of multiple-valued Jacobi fields will show that the regularity theory for Dir-minimizing Q-valued functions developed in step (1) of Almgren's program (of which we will give an account in § 2.2.4) is robust enough to allow one to produce analogous regularity results for minimizers of functionals defined on Sobolev spaces of Q-valued functions other than the Dirichlet energy (see also [Mat83, DLFS11] for a discussion about general integral functionals defined on spaces of multiple valued functions and their semi-continuity properties).

1 See Section 2.1.2 for the definition of a cone over a current.

1.1.1 Structure and main results

Let us first recall what is classically meant by Jacobi operator and Jacobi fields. Let Σ be an m-dimensional compact oriented submanifold (with or without boundary) of an $(m+k)$-dimensional Riemannian manifold $\mathcal{M} \subset \mathbb{R}^d$, and assume that Σ is stationary with respect to the m-dimensional area functional. Then, a one-parameter family of normal variations of Σ in $\mathcal{M}$ can be defined by setting $\Sigma_t := F_t(\Sigma)$, where F_t is the flow generated by a smooth cross-section N of the normal bundle $\mathcal{N}\Sigma$ of Σ in $\mathcal{M}$ which has compact support in Σ. It is known that the second variation formula corresponding to such a family of variations can be expressed in terms of an elliptic differential operator $\mathcal{L}$ defined on the space $\Gamma(\mathcal{N}\Sigma)$ of the cross-sections of the normal bundle. This operator, usually called the *Jacobi normal operator*, is given by $\mathcal{L} := -\Delta^{\perp}_{\Sigma} - \mathscr{A} - \mathscr{R}$, where $\Delta^{\perp}_{\Sigma}$ is the Laplacian on $\mathcal{N}\Sigma$, and $\mathscr{A}$ and $\mathscr{R}$ are linear transformations of $\mathcal{N}\Sigma$ defined in terms of the second fundamental form of the immersion $\iota\colon \Sigma \to \mathcal{M}$ and of a partial Ricci tensor of the ambient manifold $\mathcal{M}$, respectively. The notions of *Morse index*, *stability* and *Jacobi fields*, central in the analysis of the properties of the class of minimal submanifolds of a given Riemannian manifold, are all defined by means of the Jacobi normal operator and its spectral properties (see Section 4.2 for the precise definitions and for a discussion about the most relevant literature related to the topic). In particular, Jacobi fields are defined as those sections $N \in \Gamma(\mathcal{N}\Sigma)$ lying in the kernel of the operator $\mathcal{L}$, and thus solving the system of partial differential equations $\mathcal{L}(N) = 0$.

In Part I, we consider instead *multi-valued* normal variations in the following sense. Let Σ and $\mathcal{M}$ be as above, and consider, for a fixed integer $Q > 1$, a Lipschitz multiple valued vector field $N\colon \Sigma \to \mathcal{A}_Q(\mathbb{R}^d)$ vanishing at $\partial\Sigma$ and having the form $N = \sum_{\ell=1}^{Q} [\![N^\ell]\!]$, where $N^\ell(x)$ is tangent to $\mathcal{M}$ and orthogonal to Σ at every point $x \in \Sigma$ and for every $\ell = 1, \dots, Q$. The "flow" of such a multiple valued vector field generates a one-parameter family Σ_t of m-dimensional integer rectifiable currents in $\mathcal{M}$ such that $\Sigma_0 = Q[\![\Sigma]\!]$ and $\partial\Sigma_t = Q[\![\partial\Sigma]\!]$ for every t. The second variation

$$\delta^2[\![\Sigma]\!](N) := \frac{d^2}{dt^2}\mathbb{M}(\Sigma_t)\Big|_{t=0},$$

$\mathbb{M}(T)$ denoting the *mass* of a current T, is a well-defined functional on the space $\Gamma^{1,2}_Q(\mathcal{N}\Sigma)$ of Q-valued $W^{1,2}$ sections of the normal bundle $\mathcal{N}\Sigma$ of Σ in $\mathcal{M}$. We will denote such Jacobi functional by Jac. Explicitly, the Jac functional is given by

$$\mathrm{Jac}(N, \Sigma) := \int_\Sigma \sum_{\ell=1}^{Q} \left(|\nabla^\perp N^\ell|^2 - |A \cdot N^\ell|^2 - \mathcal{R}(N^\ell, N^\ell) \right) d\mathcal{H}^m, \tag{1.1}$$

where $\nabla^\perp$ is the projection of the Levi-Civita connection of $\mathcal{M}$ onto $\mathcal{N}\Sigma$, $|A \cdot N^\ell|$ is the Hilbert-Schmidt norm of the projection of the second fundamental form of the embedding $\Sigma \hookrightarrow \mathcal{M}$ onto N^ℓ and $\mathcal{R}(N^\ell, N^\ell)$ is a partial Ricci tensor of the ambient manifold $\mathcal{M}$ in the direction of N^ℓ (see Section 4.2 for the precise definition of the notation used in (1.1)).

Unlike the classical case, it is not possible to characterize the stationary maps of the Jac functional as the solutions of a certain Euler-Lagrange equation, and no PDE techniques seem available to study their regularity. Therefore, we develop a completely variational theory of multiple valued Jacobi fields. Hence, we give the following definition.

Definition 4.2.11 *Let $\Omega \subset \Sigma \hookrightarrow \mathcal{M}$ be a Lipschitz open set. A map $N \in \Gamma^{1,2}_Q(\mathcal{N}\Omega)$ is said to be a* Jac-minimizer, *or a* Jacobi Q-field *in Ω, if it minimizes the Jacobi functional among all Q-valued $W^{1,2}$ sections of the normal bundle of Ω in $\mathcal{M}$ having the same trace at the boundary, that is*

$$\mathrm{Jac}(N, \Omega) \leqslant \mathrm{Jac}(u, \Omega) \qquad \textit{for all } u \in \Gamma^{1,2}_Q(\mathcal{N}\Omega) \textit{ such that } u|_{\partial\Omega} = N|_{\partial\Omega}.$$

We are now ready to state the main theorems of this part. They develop the theory of Jacobi Q-fields along three main directions, concerning *existence*, *regularity*, and *estimate of the singular set*.

Theorem 5.0.1 (Conditional existence). *Let Ω be an open and connected subset of $\Sigma \hookrightarrow \mathcal{M}$ with C^2 boundary. Assume that the following* strict stability condition *is satisfied: the only Q-valued Jacobi field N in Ω such that $N|_{\partial\Omega} = Q\llbracket 0 \rrbracket$ is the null field $N_0 \equiv Q\llbracket 0 \rrbracket$. Then, for any $g \in \Gamma^{1,2}_Q(\mathcal{N}\Omega)$ such that $g|_{\partial\Omega} \in W^{1,2}(\partial\Omega, \mathcal{A}_Q(\mathbb{R}^d))$ there is a Jacobi Q-field $\overline{N}$ such that $\overline{N}|_{\partial\Omega} = g|_{\partial\Omega}$.*

Note that the above result strongly resembles the classical Fredholm alternative condition for solving linear elliptic boundary value problems: the solvability of the minimum problem for the Jac functional in $\Gamma^{1,2}_Q(\mathcal{N}\Omega)$ for any given boundary datum g as in the statement is guaranteed whenever Ω does not admit any non-trivial Jacobi Q-field vanishing at the boundary.

Theorem 6.0.1 (Regularity). *Let $\Omega \subset \Sigma$ be an open subset, with $\Sigma \hookrightarrow \mathcal{M}$ as above. There exists a universal constant $\alpha = \alpha(m, Q) \in (0, 1)$ such that if $N \in \Gamma^{1,2}_Q(\mathcal{N}\Omega)$ is* Jac*-minimizing then $N \in C^{0,\alpha}_{loc}(\Omega, \mathcal{A}_Q(\mathbb{R}^d))$.*

Theorem 6.0.3 (Estimate of the singular set). *Let N be a Q-valued Jacobi field in $\Omega \subset \Sigma^m$. Then, there exists a relatively closed set $\mathrm{sing}(N) \subset \Omega$ of Hausdorff dimension $\dim_{\mathcal{H}}(\mathrm{sing}(N)) \leqslant m-2$ (and at most countable if $m = 2$) such that on the open set $\Omega \setminus \mathrm{sing}(N)$ the vector field N can be locally written as the superposition of Q classical Jacobi fields. Furthermore, either two Jacobi fields of the local selection coincide, or they never cross.*

Note that Theorems 5.0.1, 6.0.1 and 6.0.3 all have a counterpart in (and can in fact be considered a generalization of) Almgren's theory of Dir-minimizing multiple valued functions (cf. Theorems 2.2.20, 2.2.21 and 2.2.23). The existence result for Jacobi Q-field is naturally more difficult than its Dir-minimizing counterpart, because in general the space of Q-valued $W^{1,2}$ sections of $\mathcal{N}\Sigma$ with bounded Jacobi energy is not weakly compact. Therefore, the proof of Theorem 5.0.1 requires a suitable extension result (cf. Corollary 5.1.3) for multiple valued Sobolev functions defined on the boundary of an open subset of Σ to a tubular neighborhood, which eventually allows one to exploit the strict stability condition in order to gain the desired compactness. In turn, such an extension theorem is obtained as a corollary of a multi-valued version of the celebrated Luckhaus' Lemma, cf. Proposition 5.1.1. The proof of Theorem 6.0.1 is obtained from the Hölder regularity of Dir-minimizing Q-valued functions by means of a perturbation argument. Finally, the estimate of the Hausdorff dimension of the singular set of a Jac-minimizer, Theorem 6.0.3, relies on its Dir-minimizing counterpart once we have shown that the *tangent maps* to a

Jacobi Q-field at a collapsed singularity p (obtained as uniform limits of suitable sequences of rescalings of N in a neighborhood of p) are non-trivial homogeneous Dir-minimizing functions, see Theorem 6.3.8. In turn, the proof of the Blow-up Theorem 6.3.8 is based on a delicate asymptotic analysis of an Almgren's type frequency function, which is shown to be almost monotone and bounded at every collapsed point. This is done by providing fairly general first variation integral identities satisfied by the Jac-minimizers.

Let us also remark that Theorem 6.3.8 does not guarantee that tangent maps to a Jacobi Q-field at a collapsed singularity are unique. Similarly to what happens for tangent cones to area minimizing currents (and for several other problems in Geometric Analysis), different blow-up sequences may converge to different limit profiles. Whether this phenomenon occurs or not is an open problem. On the other hand, if the dimension of the base manifold is $m = \dim \Sigma = 2$, then we are able to show that the limit profile must be a *unique* non-trivial Dirichlet minimizer. Indeed, we have the following theorem.

Theorem 7.0.1 (Uniqueness of tangent maps). *Let $m = \dim \Sigma = 2$, and let N be a Q-valued Jacobi field in $\Omega \subset \Sigma^2$. Let p be a collapsed singular point, that is, assume that $N(p) = Q\llbracket v \rrbracket$ for some $v \in T_p^{\perp}\Sigma \subset T_p\mathcal{M}$ but there exists no neighborhood U of p such that $N|_U = Q\llbracket \zeta \rrbracket$ for some single-valued section ζ. Then, there exists a* unique *tangent map $\mathscr{N}_p$ to N at p. $\mathscr{N}_p$ is a non-trivial homogeneous* Dir*-minimizer $\mathscr{N}_p : T_p\Sigma \to \mathcal{A}_Q(T_p^{\perp}\Sigma \subset T_p\mathcal{M})$.*

The key to prove Theorem 7.0.1 is to show that, in dimension $m = 2$, the rate of convergence of the frequency function at a collapsed singularity to its limit is a small power of the radius. In turn, this is achieved by exploiting one more time the variation formulae satisfied by N.

The part is organized as follows. After a quick review of some classical notions in Differential Geometry, Chapter 4 contains the derivation of the second variation formula generated by a Q-valued section of $\mathcal{N}\Sigma$, which leads to the definition of the Jac functional and to a first analysis of its properties: in particular, we show that the Jac functional is lower semicontinuous with respect to $W^{1,2}$ weak convergence (cf. Proposition 4.3.1) and we study the strict stability condition mentioned in the statement of Theorem 5.0.1 (cf. Lemma 4.3.4). Chapter 5 contains the proof of Theorem 5.0.1. The regularity theory in general dimension is instead developed in Chapter 6. In particular, the proof of Theorem 6.0.1 is contained in Section 6.1. In Section 6.2 we prove the properties of the frequency function which are needed to carry on the blow-up scheme, which is instead the content of Section 6.3. Theorem 6.0.3 is finally proved in Section 6.4. Last, Chapter 7 contains the uniqueness of tangent maps in dimension 2: the decay of the frequency function, Proposition 7.1.1, is proved in Section 7.1; Theorem 7.0.1 is finally proved in Section 7.2.

1.2 PART II: MULTIPLE-VALUED SECTIONS OF VECTOR BUNDLES AND APPLICATIONS

Many natural problems in the Calculus of Variations require to minimize a given functional among all functions in a certain functional space which not only attain a prescribed boundary datum, but also satisfy an assigned geometric constraint. The harmonic maps

problem, which is going to be the subject of Part III, can be naturally seen under this perspective: minimize the Dirichlet energy among all (single-valued or multi-valued) maps which are constrained to take values into a prescribed target manifold $\mathcal{N}$. The multi-valued theory of the stability operator developed in Part I provides another instance of this class of problems: Jacobi Q-fields are minimizers of the Jac functional among Q-valued maps satisfying the additional requirement of taking values in the normal bundle $\mathcal{N}\Sigma$ of the minimal submanifold Σ in $\mathcal{M}$. This last example in particular has driven our investigation towards the possibility of developing a theory of multiple-valued sections of an abstract vector bundle E over a Riemannian manifold Σ.

In part II we present our work [Stu17b], where we initiate this theory. First, in Chapter 8, building on some unpublished ideas of W. Allard [All13], we provide the elementary definition of Q-multisection of the bundle E over Σ, and we establish how these multisections relate to Almgren's multiple-valued functions and in which sense the former are a generalization of the latter.

Then, in Chapter 9, we apply the theory of Q-multisections to provide an elementary proof of a delicate "reparametrization statement" for multi-valued graphs, which in turn plays an important role in the regularity theory à la Almgren-De Lellis-Spadaro for minimizing currents in codimension higher than one. Specifically, the problem we are going to consider is the following: let $f\colon \Omega \subset \mathbb{R}^m \to \mathcal{A}_Q(\mathbb{R}^n)$ be a Lipschitz Q-valued function, and let Σ be a regular m-dimensional manifold which is the graph of a sufficiently smooth function $\boldsymbol{\varphi}\colon \Omega' \subset \Omega \to \mathbb{R}^n$. If the Lipschitz constant of f is small and Σ is sufficiently flat, then is it possible to represent the graph of f also as the image of a Lipschitz multiple-valued function F defined on Σ and taking values in its normal bundle? Furthermore, which control do we have on the Lipschitz constant of F in terms of the Lipschitz constant of f?

In order to motivate the interest in the above problem, we need to go back to the regularity theory for area-minimizing currents in high codimension, and in particular to steps (2) and (3) of the Almgren-De Lellis-Spadaro program. When performing the approximation procedure described in there, it is crucial, in order to close the contradiction argument, that the limiting Dir-minimizer "inherits" the singularities of the current. In order to guarantee that this happens, it is necessary to suitably construct a regular manifold (the *center manifold*) which is an approximate "average" of the sheets of the current itself, and to approximate with high degree of accuracy the current with Q-valued functions defined on the center manifold and taking values in its normal bundle. Using the center manifold as reference manifold from which the approximation is constructed is a way to prevent the sheets of the approximating Q-valued functions from collapsing, in the limit, onto a single sheet, that is a regular Dir-minimizer which would therefore fail in capturing the singular behavior of the current. The center manifold construction and the related normal approximation are performed in [DLS16a]. Solving the above reparametrization problem is a key step in the construction of the approximation.

The reparametrization problem has been tackled and successfully solved in [DS15]. On the other hand, the proof suggested by De Lellis and Spadaro makes use of the theory of currents in metric spaces developed by Ambrosio-Kirchheim (see [AK00]). It turns out, instead, that the theory of Q-multisections developed in Chapter 8 contains all the tools that are required to provide a completely elementary and purely geometric proof of the

reparametrization theorem for Lipschitz multiple-valued graphs needed in [DLS16a]. In turn, this new approach will also serve as an example of the fact that some a-priori elementary geometric concepts may turn out to be extremely powerful in proving deep analytical results.

1.2.1 Structure and main results

In Chapter 8 we consider an m-dimensional Riemannian manifold Σ of class C^1 and a vector bundle $\Pi\colon E \to \Sigma$ of rank n and class C^1 over Σ. For any fixed integer $Q \geqslant 1$, we give the following definition.

Definition 8.1.1 (Q-valued sections) *Given a vector bundle $\Pi\colon E \to \Sigma$ as above, and a subset $B \subset \Sigma$, a Q-multisection over B is a map*

$$M\colon E \to \mathbb{N} \cup \{0\}$$

with the property that

$$\sum_{\xi \in \Pi^{-1}(\{p\})} M(\xi) = Q \qquad \text{for every } p \in B.$$

It is immediate to observe that Q-multisections are a generalization of Almgren's Q-valued functions to vector bundle targets. In particular, if $E = \Sigma \times \mathbb{R}^n$ is the trivial bundle of rank n over Σ then every multisection M over $B \subset \Sigma$ defines a unique function $u_M\colon B \to \mathcal{A}_Q(\mathbb{R}^n)$, and, vice versa, every Q-valued function $u\colon B \to \mathcal{A}_Q(\mathbb{R}^n)$ induces a unique Q-multisection over B. These preliminary notions are the content of Section 8.1. In Section 8.2, instead, we turn our attention to two properties of Q-multisections suggested by Allard in [All13]: *coherence* and *vertical boundedness*. These properties are particularly relevant, as they "mimic" the classical notion of Lipschitz continuity in the vector bundle-valued case.

Roughly speaking, a Q-multisection M over Σ is coherent if the following holds. For every point $p \in \Sigma$ and for every disjoint family $\mathcal{V}$ of open sets $V \subset E$ such that each $V \in \mathcal{V}$ contains one and only one of the points $\xi \in \Pi^{-1}(\{p\})$ such that $M(\xi) > 0$, there exists a neighborhood U of p in Σ with the property that if $q \in U$ and V contains the point $\xi \in \Pi^{-1}(\{p\})$ then the sum of the multiplicities $M(\zeta)$ of the points $\zeta \in V$ with $\Pi(\zeta) = q$ is precisely equal to $M(\xi)$.

The vertical boundedness property can be easily described for multisections $M\colon \Omega \times \mathbb{R}^n \to \mathbb{N} \cup \{0\}$, $\Omega \subset \mathbb{R}^m$ open, and then extended to general multisections by means of charts and trivializations. A multisection M over the trivial bundle $\Omega \times \mathbb{R}^n$ is vertically bounded if there exists $\tau > 0$ such that for any $(x, \nu) \in \Omega \times \mathbb{R}^n$ with $M(x, \nu) > 0$ there exists a neighborhood $U \times V$ of (x, ν) in $\Omega \times \mathbb{R}^n$ such that the "graph" of M in $U \times V$ is contained in a τ-cone centered at (x, ν). We have the following result on coherent and vertically limited multisections.

Proposition 8.2.2 and **Proposition 8.2.4** *Let $\Omega \subset \mathbb{R}^m$ be open, and let $E = \Omega \times \mathbb{R}^n$ be the trivial bundle of rank n over Ω. A Q-multisection $M\colon E \to \mathbb{N} \cup \{0\}$ is coherent if and only if the associated multiple-valued function $u\colon \Omega \to \mathcal{A}_Q(\mathbb{R}^n)$ is continuous. If M is also τ-vertically bounded then u*

is Lipschitz with $\mathrm{Lip}(u) \leqslant \sqrt{Q}\tau$. *Vice versa, if* $u\colon \Omega \to \mathcal{A}_Q(\mathbb{R}^n)$ *is* L*-Lipschitz continuous then the associated multisection* M *is coherent and* L*-vertically bounded.*

The above theory is then applied in Chapter 9 to prove the following reparametrization result.

Theorem 9.1.4 *Let* Q, m *and* n *be positive integers, and* $0 < s < r < 1$. *Suppose that* Σ *is an open* m*-dimensional submanifold of* $\mathbb{R}^{m+n}$ *which is the graph of a* C^3 *function* $\boldsymbol{\varphi}\colon B_s(0) \subset \mathbb{R}^m \to \mathbb{R}^n$, *and that* $f\colon B_r(0) \subset \mathbb{R}^m \to \mathcal{A}_Q(\mathbb{R}^n)$ *is a Lipschitz* Q*-valued function. If the norms* $\|\boldsymbol{\varphi}\|_{C^2}$, $\|f\|_{C^0}$ *and* $\mathrm{Lip}(f)$ *are suitably small (depending on* Q, m, n, $r-s$ *and* $\frac{r}{s}$*), then there exist a normal tubular neighborhood* $\mathbf{U}$ *of* Σ *in* $\mathbb{R}^{m+n}$ *and a Lipschitz* Q*-valued section* N *of the normal bundle of* Σ *such that the graph of* f *in* $\mathbf{U}$ *coincides (as current) with the image of the map* $F(p) = \sum_{\ell=1}^{Q} [\![p + N^\ell(p)]\!]$. *Furthermore,* $\mathrm{Lip}(N) \leqslant C(\|D^2\boldsymbol{\varphi}\|_{C^0}, \|N\|_{C^0}, \mathrm{Lip}(f))$.

1.3 PART III: REGULARITY AND SINGULARITIES OF MULTIPLE-VALUED HARMONIC MAPS

The Dirichlet energy is one of the simplest functionals studied in the framework of Calculus of Variations. Unconstrained (that is, $\mathbb{R}^n$-valued) critical points of the Dirichlet energy are harmonic functions, and as such they enjoy strong regularity properties. Harmonic maps are maps taking values in a prescribed (compact) Riemannian manifold which are critical for the Dirichlet energy with respect to variations preserving the constraint on the target. The presence of the constraint gives rise to non-linear Euler-Lagrange equations for its critical points; in turn, these non-linearities make it a challenge to study the regularity of the solutions. In fact, it is not difficult to produce harmonic maps with singularities, and thus the best one can hope for is a *partial* regularity theory. Such a theory has been developed starting with the pioneering work [SU82] by Schoen and Uhlenbeck in the 1980s. An account for the theory of harmonic maps is the content of § 2.3.1.

Some of the methods and techniques developed for the analysis of harmonic maps are robust enough to be applicable in the study of even more general problems, such as the regularity theory for minimizers of the Dirichlet energy taking values in a metric space. Note, for instance, that Almgren's Dir-minimizers are precisely minimizers of the Dirichlet energy taking values in the very special metric space $\mathcal{A}_Q(\mathbb{R}^n)$. As a matter of fact, the "locally Euclidean" structure of $\mathcal{A}_Q(\mathbb{R}^n)$ makes Dir-minimizers better behaved even than classical energy minimizing harmonic maps into compact Riemannian manifolds (for instance, energy minimizing harmonic maps, in general, fail to be continuous everywhere).

In [Hir16b], Hirsch started analyzing Dirichlet-minimizing Q-valued maps into compact Riemannian manifolds, introducing the appropriate definitions and developing the basic continuity theory for such objects, which we will summarize in § 2.3.3. In particular, the partial regularity theory for multi-valued energy minimizing maps turns out to be completely equivalent to its classical single-valued counterpart. Motivated by this analogy, we decided to investigate the possibility of deducing finer properties (rectifiability, Minkowski estimates) of the singular set of Hirsch's multiple valued energy minimizing maps in the

spirit of the corresponding single-valued theory recently developed by Naber and Valtorta in [NV17] (see § 2.3.2 for a quick overview of the results of [NV17]).

As a byproduct of these investigations, we have also discovered that some of the standard results that are available in the single-valued situation fail to hold in the multiple-valued case. This shows once more that the properties of Q-valued maps can be very different from their single-valued counterparts.

Part III contains a presentation of our paper [HSV17], obtained in collaboration with J. Hirsch and D. Valtorta, in which the results of the aforementioned studies are recorded.

1.3.1 Structure and main results

Part III is divided into two chapters. Chapter 10 contains a proof of the following result.

Theorem 10.0.1 *Let m, n, Q be positive integers, let $\mathcal{N}^n \hookrightarrow \mathbb{R}^d$ be a compact Riemannian manifold, and set*

$$\mathcal{A}_Q(\mathcal{N}) := \left\{ T = \sum_{\ell=1}^{Q} \llbracket p_\ell \rrbracket : \textit{each } p_\ell \in \mathcal{N} \right\} .$$

Suppose that $u \colon B_2(0) \subset \mathbb{R}^m \to \mathcal{A}_Q(\mathcal{N})$ is energy minimizing with energy bounded by Λ. If

$$\operatorname{sing}_H(u) := \{ x \in B_2(0) : u \textit{ is not Hölder continuous in a neighborhood of } x \} ,$$

and

$$B_r(\operatorname{sing}_H(u)) := \bigcup_{x \in \operatorname{sing}_H(u)} B_r(x) ,$$

then we have the uniform Minkowski estimate

$$\mathcal{L}^m \left(B_r(\operatorname{sing}_H(u)) \cap B_1(0) \right) \leqslant C(m, \mathcal{N}, \Lambda) r^3 .$$

Furthermore, $\operatorname{sing}_H(u)$ is countably $(m-3)$-rectifiable.

The proof of Theorem 10.0.1 uses the techniques developed in [NV17] for the single-valued case, which roughly speaking rely on a quantitative version of the Federer-Almgren dimension reduction argument. In fact, as in [NV17] Theorem 10.0.1 will be obtained as a corollary of a more general statement on the *quantitative stratification* of $\operatorname{sing}_H(u)$, cf. Theorem 10.2.17. However, we present an alternative definition of the quantitative stratification used in [NV17] (which was originally introduced by Cheeger and Naber, see § 2.3.2 and the references therein). The new stratification turns out to be equivalent to the standard one in the case of minimizing maps, but easier to handle.

In Chapter 11 we consider instead the special case when the target $\mathcal{N}$ is connected and has non-positive sectional curvatures. For classical harmonic maps, this assumption implies full-blown continuity of the map u everywhere. On the other hand, in the case of Q-valued maps this is true only if $\mathcal{N}$ is assumed to be also *simply connected*. We will provide a counterexample to show that this assumption is needed. Specifically, we have the following result.

Theorem 11.0.1 and Proposition 11.4.1 *Let $\Omega \subset \mathbb{R}^m$ be an open set, and let $\mathcal{N}$ be a complete, simply connected Riemannian manifold all of whose sectional curvatures are non-positive. Then, every minimizing harmonic map $u \in W^{1,2}(\Omega, \mathcal{A}_Q(\mathcal{N}))$ satisfies*

$$\operatorname{sing}_H(u) = \emptyset.$$

If $Q > 1$, the result does not hold if $\mathcal{N}$ is merely connected. Indeed, there is a 2-valued Dirichlet minimizing map u from $B_1(0) \subset \mathbb{R}^3$ into the flat torus $\mathbb{T}^2 = \mathbb{C}/\mathbb{Z}^2$ with the property that $u|_{S^2}$ is Lipschitz continuous, $\operatorname{sing}_H(u) \Subset B_1(0)$ and $\operatorname{sing}_H(u) \neq \emptyset$.

1.4 PART IV: RESULTS ON REAL CURRENTS AND CURRENTS WITH COEFFICIENTS IN GROUPS

It is often the case in Mathematics that new concepts and ideas not only contribute to the solution of many long-standing problems, but also pose new questions and initiate new lines of research. Following this line of thinking, it is not surprising that the theory of currents is not limited solely to the solution of Plateau's problem. On the contrary, many generalizations of the classical theory of integer rectifiable currents have been studied, both in the direction of considering more general functionals than the mass, and in the direction of analyzing more general classes of currents. At the same time, currents have proven themselves to be a powerful and versatile tool for tackling problems coming from different areas of Analysis.

In particular, regarding the study of more general classes of currents, let us first mention the work of Ziemer [Zie62], who introduced the notion of *integral currents modulo 2* in order to develop a theory concerning the existence of solutions to the Plateau's problem among non-orientable surfaces spanning a given boundary. Further generalizations, such as *integral currents modulo* p and *flat chains modulo* p, were considered in order to treat a wider class of surfaces which can be realized, for instance, as soap films. An interesting property of such surfaces is that they can develop singularities in low codimension, unlike the classical solutions to Plateau's problem (see, for instance, [Mor86] and [Whi86]). The occurrence, in practical experiments, of two-dimensional soap films with a line of singularities (prohibited in the classical theory of area minimizing integer rectifiable currents) justifies the interest towards this kind of objects.

This line of research has in turn initiated the investigations on the more general classes of currents with coefficients in a normed abelian group G, of which currents modulo p represent the particular case when $G = \mathbb{Z}_p$. Introduced by Fleming in the seminal paper [Fle66], currents with coefficients in groups have been used in modelling immiscible fluids and soap bubble clusters (see [Whi96]), in proving that various surfaces are area minimizing [LM94], and in analyzing the properties of networks arising as solutions to classical problems such as the Steiner or the Gilbert-Steiner problem (cf. [MM16b, MM16a]). The interested reader can also see [AK11, DPH12, DPH14] for more on the topic.

1.4.1 Structure and main results

Part IV is divided into two chapters. In Chapter 12 we present our work [CDMS17], obtained in collaboration with M. Colombo, A. De Rosa and A. Marchese. We work in the framework of real currents, and we deal with a more general class of functionals than the mass. Specifically, consider any subadditive, lower semi-continuous and even function $H\colon \mathbb{R} \to [0,\infty)$ with $H(0) = 0$. Any such a function induces a functional Φ_H on the space of real polyhedral m-chains in $\mathbb{R}^d$ (roughly speaking, finite unions of non-overlapping oriented m-simplexes carrying real multiplicities) by setting, for any polyhedral m-chain $P = \sum_{i=1}^{N} \theta_i [\![\sigma_i]\!]$ associated to non-overlapping oriented m-simplexes σ_i with multiplicities $\theta_i \in (0,\infty)$,

$$\Phi_H(P) := \sum_{i=1}^{N} H(\theta_i)\mathcal{H}^m(\sigma_i).$$

Observe that the mass $\mathbb{M}(P)$ coincides with $\Phi_H(P)$ corresponding with the choice $H(\theta) := |\theta|$. The functional Φ_H extends to a functional $\mathbb{M}_H$, the so-called H-mass, defined on all real rectifiable m-currents (that is, currents supported on an m-rectifiable set E oriented by a tangent m-vector field $\vec{\tau}$ and carrying a real multiplicity function θ) by setting, for every $R = [\![E, \vec{\tau}, \theta]\!]$,

$$\mathbb{M}_H(R) := \int_E H(\theta(x))\, d\mathcal{H}^m(x)\,.$$

H-mass functionals naturally arise in the context of *branched transport problems*, variants of the optimal transport problem where the cost function not only depends on the spatial distribution of the masses that one wants to move, but also on the paths along which the flow takes place. These models have been proposed in order to describe a large variety of natural phenomena as well as engineering problems, and other very interesting mathematical problems can be related with it, such as the analysis of topological singularities for Ginzburg-Landau models (see the recent papers [MM16a, CDM17] and the references therein).

It is easy to see that the above assumptions on H are necessary for the functional $\mathbb{M}_H$ to be (well defined and) lower semi-continuous on rectifiable currents with respect to convergence in an appropriate topology - the so called *flat norm topology* -. In Chapter 12, we prove that they are also sufficient. Furthermore, we obtain the following theorem.

Theorem 12.2.4 *The lower semi-continuous envelope of Φ_H, namely the functional F_H defined on any* real flat chain T *by*

$$F_H(T) := \inf\left\{ \liminf_{j\to\infty} \Phi_H(P_j) :\ P_j \textit{ are polyhedral and converge to } T \textit{ in the flat topology} \right\},$$

coincides, on the class of rectifiable currents, with the functional $\mathbb{M}_H$.

The validity of the representation $F_H = \mathbb{M}_H$ on rectifiable currents has recently attracted some attention. For instance, it is clearly assumed in [Xia03] for the choice $H(\theta) = |\theta|^\alpha$, with $\alpha \in (0,1)$, in order to prove some regularity properties of minimizers of problems related to branched transportation (see also [PS06], [BCM09], [Peg16]), and in [CMF16] in

order to define suitable approximations of the Steiner problem, with the choice $H(\theta) = (1 + \beta|\theta|)\mathbf{1}_{\mathbb{R}\setminus\{0\}}$, where $\beta > 0$ and $\mathbf{1}_A$ denotes the indicator function of the Borel set A.

The proof that we suggest is based on the aforementioned lower semi-continuity result for $\mathbb{M}_H$ and on a polyhedral approximation of real rectifiable currents with real polyhedral chains having H-mass arbitrarily close to the H-mass of the current they approximate. We remark that in the last section of [Whi99a] the author sketches a strategy to prove an analogous version of Theorem 12.2.4 in the framework of flat chains with coefficients in a normed abelian group G. Motivated by the relevance of such result for real valued flat chains, the ultimate aim of Chapter 12 is to present a self-contained complete proof of it when $G = \mathbb{R}$.

In Chapter 13, instead, we present our paper [MS17], obtained in collaboration with A. Marchese, where we work in the framework of currents modulo p. It has to be noted that, despite the substantial interest in the subject, the very structure of flat chains and integral currents modulo p is yet to be completely understood. The initial idea of defining flat chains modulo p by identifying currents which differ by pT, where T is a "classical" flat chain, fails because of one major drawback: the closedness of the classes with respect to the flat norm is a-priori not guaranteed. Hence, it is more convenient to define the classes of flat chains modulo p as the *flat closure* of the equivalence classes mentioned above. The equivalence of the two definitions is still an open problem.

A second issue regards the structure of integral currents modulo p. They are defined as flat chains modulo p with finite p-mass and finite p-mass of the boundary. It is not known whether each equivalence class contains at least one classical integral current.

In Chapter 13, we specifically address these two problems. After introducing the basic terminology and properties concerning flat chains and integral currents modulo p, we formulate two questions related to the two above problems and collect some partial answers from the literature. Then, we throw light on the connection between the two questions, and we provide a positive answer to the second one in the case of 1-dimensional currents. Specifically, we have the following theorem.

Theorem 13.2.5 *Let $[T]_{\mathrm{mod}(p)}$ be an integral 1-current modulo p in $\mathbb{R}^d$. Then, there exists a 1-current T_0 in $\mathbb{R}^d$ such that $T_0 \equiv T \bmod(p)$ and T_0 is integral. In particular, every integer rectifiable 1-current without boundary modulo p admits an integer rectifiable representative without boundary in the classical sense.*

Finally, we conclude the chapter with an example illustrating how it is possible to produce situations in which the answer to the second question is negative in higher dimension.

2 PRELIMINARIES

This chapter is dedicated to give an overview of standard topics from the literature in Geometric Measure Theory and Geometric Analysis in which the research presented in this thesis has its roots. Nothing written in this chapter is original, and we will provide references to standard textbooks or research papers where the material here included is thoroughly developed. Clearly, our presentation is not supposed to be exhaustive: our main goal is rather to fix the notation and recall some important background results for further reference. The chapter is divided in three sections: Section 2.1 is dedicated to the theory of currents, Section 2.2 contains an introduction to the theory of multiple-valued functions, Section 2.3 deals with the theory of harmonic maps. Currents are ubiquitous in this thesis, and thus the contents of Section 2.1 are a prerequisite to all other chapters; multiple-valued functions do not appear in Part IV, so Section 2.2 is propaedeutic only to parts I, II and III; the material presented in Section 2.3 will only be used in Part III.

2.1 AN OVERVIEW OF THE THEORY OF CURRENTS

We start with a tutorial on the theory of currents. For a thorough discussion of the topic, the reader can refer to standard books in Geometric Measure Theory such as [Sim83b] and [KP08], to the monograph [GMS98] or to the treatise [Fed69].

As anticipated in the Introduction, currents are a generalization of the notion of surface, introduced in order to produce a general existence theory of solutions of Plateau's problem. The key idea motivating the definition of a current is the following. If Σ is an oriented m-dimensional submanifold (possibly with boundary) of an open set $\Omega \subset \mathbb{R}^d$ ($d \geqslant m$) with locally finite m-dimensional Hausdorff measure, and $\omega \in \mathcal{D}^m(\Omega)$ is a smooth differential m-form on Ω with compact support, then one can consider the quantity

$$[\![\Sigma]\!](\omega) := \int_\Sigma \omega\,.$$

It is then easy to check that the functional $\omega \in \mathcal{D}^m(\Omega) \mapsto [\![\Sigma]\!](\omega)$ is linear and continuous with respect to the standard locally convex topology of $\mathcal{D}^m(\Omega)$. Moreover, the map $\Sigma \mapsto [\![\Sigma]\!]$ is injective: thus, in a sense, the value of $[\![\Sigma]\!]$ on all possible forms $\omega \in \mathcal{D}^m(\Omega)$ determines the submanifold Σ itself.

The above discussion makes the following definition extremely natural.

Definition 2.1.1 (General currents)**.** Given an open set $\Omega \subset \mathbb{R}^d$, an m-dimensional *current* in Ω is a linear and continuous functional

$$T\colon \mathcal{D}^m(\Omega) \to \mathbb{R}.$$

The space of m-currents in Ω is therefore the topological dual space of $\mathcal{D}^m(\Omega)$, and will be denoted by $\mathcal{D}_m(\Omega)$. Rephrasing the observation above, if $\Sigma \subset \Omega$ is an oriented m-dimensional submanifold, then there is a corresponding m-current $[\![\Sigma]\!] \in \mathcal{D}_m(\Omega)$ defined by integration of m-forms on Σ in the usual sense of differential geometry:

$$[\![\Sigma]\!](\omega) := \int_\Sigma \omega \quad \forall\, \omega \in \mathcal{D}^m(\Omega).$$

The idea of "spanning" a given contour can be formalized by introducing the notion of *boundary* of a current. If $T \in \mathcal{D}_m(\Omega)$, then its boundary is the $(m-1)$-current ∂T whose action on any form $\omega \in \mathcal{D}^{m-1}(\Omega)$ is given by

$$\partial T(\omega) := T(d\omega).$$

Observe that the definition of boundary is obtained by enforcing Stokes' theorem: in particular, $\partial[\![\Sigma]\!] = [\![\partial\Sigma]\!]$ if Σ is a smooth oriented m-dimensional submanifold in Ω. Furthermore, from the fact that $d \circ d = 0$ immediately follows that $\partial(\partial T) = 0$ for every $T \in \mathcal{D}_m(\Omega)$.

The *mass* of $T \in \mathcal{D}_m(\Omega)$, denoted $\mathbb{M}(T)$, is the (possibly infinite) supremum of the values $T(\omega)$ among all forms $\omega \in \mathcal{D}^m(\Omega)$ with $\|\omega(x)\|_c \leqslant 1$ everywhere. Again, for a submanifold Σ, computing $\mathbb{M}([\![\Sigma]\!])$ produces the expected value $\mathcal{H}^m(\Sigma)$. Hence, $\mathbb{M}(T)$ can be thought of as the "m-dimensional area" of T. The definition of mass can be localized to any $W \Subset \Omega$ simply by restricting the class of competitors in the supremum only to those forms ω with $\mathrm{spt}(\omega) \subset W$. We will use the notation $\mathbb{M}_W(T)$ for the localized mass in W. Both the mass and the localized mass satisfy the triangle inequality $\mathbb{M}_W(T_1 + T_2) \leqslant \mathbb{M}_W(T_1) + \mathbb{M}_W(T_2)$.

The *support* $\mathrm{spt}(T)$ of the current T is the intersection of all closed subsets C such that $T(\omega) = 0$ whenever $\mathrm{spt}(\omega) \subset \mathbb{R}^d \setminus C$.

A suitable notion of convergence of currents can be defined by endowing $\mathcal{D}_m(\Omega)$ with the weak-* topology induced by the topology on $\mathcal{D}^m(\Omega)$. Hence, we will say that a sequence $\{T_h\}_{h=1}^\infty \subset \mathcal{D}_m(\Omega)$ *converges* to $T \in \mathcal{D}_m(\Omega)$ *in the sense of currents*, and we will write $T_h \rightharpoonup T$, if $T_h(\omega) \to T(\omega)$ for every $\omega \in \mathcal{D}^m(\Omega)$. It is clear that if $T_h \rightharpoonup T$ then also $\partial T_h \rightharpoonup \partial T$. Moreover, the mass is lower semi-continuous with respect to convergence in the sense of currents.

Since the mass is lower semi-continuous with respect to the weak-* convergence of currents, using standard functional analytic methods to gain compactness it is easy to provide an affirmative answer to the Plateau's problem in the framework of general currents with finite mass. Nonetheless, such a solution is highly unsatisfactory, because, for instance, the resulting minimizing current might be a surface carrying real multiplicities. This issue was overcome in [FF60], where the authors prove a highly non-trivial compactness result for a subclass of the class of general currents, which goes under the name of *integral currents* and is going to be defined in the next paragraph.

2.1.1 Integral currents and the solution of Plateau's problem

A subset $B \subset \Omega$ is (countably) m-*rectifiable* if B can be covered, up to a $\mathcal{H}^m$-null set, by countably many m-dimensional embedded submanifolds of $\mathbb{R}^d$ of class C^1. If B is

m-rectifiable, then to $\mathcal{H}^m$-a.e. point $x \in B$ can be suitably associated an m-dimensional *approximate tangent space*, denoted $\mathrm{Tan}(B, x)$, in such a way that $\mathrm{Tan}(B, x) = T_x\Sigma$ if B coincides with a C^1 submanifold Σ in a neighborhood of x (cf. [Sim83b, Theorem 11.6]).

Let B be m-rectifiable. An *orientation* of B is a $\mathcal{H}^m$-measurable function $\vec{\tau}\colon B \to \Lambda_m(\mathbb{R}^d)$ such that, for $\mathcal{H}^m$-a.e. $x \in B$, $\vec{\tau}(x)$ is a simple unit m-vector having the form $\vec{\tau}(x) = \tau_1(x) \wedge \cdots \wedge \tau_m(x)$, where $(\tau_1(x), \ldots, \tau_m(x))$ is an orthonormal basis of $\mathrm{Tan}(B, x)$.

A *multiplicity* on B is a real-valued function $\theta \in L^1_{loc}(\mathcal{H}^m \llcorner B)$.

To any triple $(B, \vec{\tau}, \theta)$ as above it is possible to associate a current T setting

$$T(\omega) := \int_B \langle \omega(x), \vec{\tau}(x) \rangle\, \theta(x)\, d\mathcal{H}^m(x) \qquad \forall\, \omega \in \mathcal{D}^m(\Omega).$$

If the action of T is given by the above expression, we will write $T = [\![B, \vec{\tau}, \theta]\!]$. Moreover, if $\theta(x) \in \mathbb{Z}$ for $\mathcal{H}^m$-a.e. $x \in B$ we will call T a *locally integer rectifiable current*. The set of locally integer rectifiable m-currents in Ω is denoted $\mathscr{R}^{loc}_m(\Omega)$. If $T = [\![B, \vec{\tau}, \theta]\!] \in \mathscr{R}^{loc}_m(\Omega)$, we denote by $\|T\|$ the Radon measure given by

$$\|T\|(A) := \int_{A \cap B} |\theta|\, d\mathcal{H}^m \qquad \text{for every } A \subset \mathbb{R}^d \text{ Borel.}$$

One can check that $\mathbb{M}_W(T) = \|T\|(W)$ for every $W \Subset \Omega$ and thus, in particular, locally integer rectifiable currents have locally finite mass. For $T \in \mathscr{R}^{loc}_m(\Omega)$ the m-dimensional density $\Theta^m(\|T\|, x)$, when it exists, will be sometimes simply denoted $\Theta(\|T\|, x)$.

Locally integer rectifiable currents with locally integer rectifiable boundary are called *locally integral currents*. We write $T \in \mathscr{I}^{loc}_m(\Omega)$ if T is a locally integral m-current in Ω. One of the cornerstones of the Federer-Fleming theory, known as the Boundary Rectifiability Theorem, shows that for $T \in \mathscr{R}^{loc}_m(\Omega)$ the condition $\mathbb{M}_W(\partial T) < \infty$ for every $W \Subset \Omega$ suffices to conclude that T is actually locally integral.

Theorem 2.1.2 (Boundary Rectifiability, cf. [Fed69, Theorem 4.2.16] and [Sim83b, Theorem 30.3])**.** *It holds:*

$$\mathscr{I}^{loc}_m(\Omega) = \{T \in \mathscr{R}^{loc}_m(\Omega) : \mathbb{M}_W(\partial T) < \infty \text{ for every } W \Subset \Omega\}. \tag{2.1}$$

More importantly for our purposes, the class of locally integral currents enjoys good compactness properties, as stated in the next, remarkable, *Compactness Theorem*.

Theorem 2.1.3 (Compactness, cf. [Sim83b, Theorem 27.3])**.** *Let $\{T_h\}_{h=1}^{\infty} \subset \mathscr{I}^{loc}_m(\Omega)$ be a sequence of locally integral currents such that*

$$\sup_{h \geqslant 1} \left(\mathbb{M}_W(T_h) + \mathbb{M}_W(\partial T_h)\right) < \infty \qquad \forall\, W \Subset \Omega. \tag{2.2}$$

Then, there exist $T \in \mathscr{I}^{loc}_m(\Omega)$ and a subsequence $\{T_{h_j}\}$ such that $T_{h_j} \rightharpoonup T$.

Theorem 2.1.3 allows to conclude that the Plateau's problem does admit a positive answer in the framework of locally integer rectifiable currents: in particular, for any locally integer rectifiable $(m-1)$-current S for which the class of locally integer rectifiable m-currents T

having $\partial T = S$ is non-empty it is possible to find a locally integer rectifiable current T_0 with $\partial T_0 = S$ which satisfies the following *minimality condition*:

$$\mathbb{M}_W(T_0) \leqslant \mathbb{M}_W(T_0 + \partial R) \quad \text{for every } W \Subset \Omega, \text{ for every } R \in \mathscr{I}^{loc}_{m+1}(\Omega) \text{ with } \mathrm{spt}(R) \Subset W. \tag{2.3}$$

Definition 2.1.4. A current $T \in \mathscr{R}^{loc}_m(\Omega)$ satisfying (2.3) is said to be *area minimizing* in Ω.

In § 2.1.5 we will record the main results available in literature concerning the regularity of area minimizing locally integer rectifiable currents.

Remark 2.1.5. A remark on the terminology we have used in this paragraph is now in order. The reader might indeed wonder why we have used the adverb "locally" when discussing the notions of integer rectifiable and integral currents. The reason being that what we have used is precisely the terminology adopted by Federer [Fed69], for which integer rectifiable and integral currents all have compact support and finite mass. In particular, the m-current $[\![\pi]\!]$ associated to an m-dimensional plane $\pi \subset \mathbb{R}^d$ with multiplicity one is *not* a rectifiable current in the sense of Federer, but rather a locally rectifiable current. On the other hand, when presenting the notion of minimality for a current it is rather advisable to include planes in the discussion. Anyway, as a matter of fact most of the integer rectifiable or integral currents appearing in this thesis are going to have compact support and finite mass. Hence, we have decided to stick to the notation of [Fed69] and to reserve the notations $\mathscr{R}_m(\Omega)$ and $\mathscr{I}_m(\Omega)$ for those (locally) integer rectifiable (resp. integral) currents with compact support and finite mass. In particular, if $K \subset \Omega$ is a compact set then we will denote $\mathscr{R}_{m,K}(\Omega)$ (resp. $\mathscr{I}_{m,K}(\Omega)$) the sets of currents T in $\mathscr{R}^{loc}_m(\Omega)$ (resp. $\mathscr{I}^{loc}_m(\Omega)$) with $\mathrm{spt}(T) \subset K$. Further, we will set

$$\mathscr{R}_m(\Omega) := \bigcup_K \mathscr{R}_{m,K}(\Omega),$$
$$\mathscr{I}_m(\Omega) := \bigcup_K \mathscr{I}_{m,K}(\Omega),$$

where the union is extended to all compact subsets $K \subset \Omega$. Currents in $\mathscr{R}_m(\Omega)$ and $\mathscr{I}_m(\Omega)$ will be called integer rectifiable and integral m-dimensional currents respectively.

2.1.2 Some relevant constructions with currents

Let $T \in \mathcal{D}_m(\Omega)$ and suppose $f\colon \Omega \to \mathbb{R}^n$ is a C^∞ map. If f is *proper* (i.e. $f^{-1}(K)$ is compact for any compact $K \subset \mathbb{R}^n$), then the *push-forward* of T through f is the current $f_\sharp T \in \mathcal{D}_m(\mathbb{R}^n)$ defined by

$$f_\sharp T(\omega) := T(f^\sharp \omega) \quad \forall\, \omega \in \mathcal{D}^m(\mathbb{R}^n),$$

where $f^\sharp\omega$ denotes the pull-back of the form ω through f. The push-forward operator $f_\sharp$ is linear, and moreover an elementary computation shows that it commutes with the boundary operator:

$$\partial(f_\sharp T) = f_\sharp(\partial T). \tag{2.4}$$

Next, we recall the important *homotopy formula* for currents. Let $f, g\colon \Omega \to \mathbb{R}^n$ be smooth, and let $\sigma\colon [0,1] \times \Omega \to \mathbb{R}^n$ be a smooth function such that $\sigma(0) \equiv f$ and $\sigma(1) \equiv g$. If $T \in \mathcal{D}_m(\Omega)$ and if σ is proper, then $\sigma_\sharp(\llbracket(0,1)\rrbracket \times T)$ [1] is a well defined current in $\mathcal{D}_{m+1}(\mathbb{R}^n)$, and moreover (cf. [Sim83b, (26.22)])

$$\partial\sigma_\sharp(\llbracket(0,1)\rrbracket \times T) = g_\sharp T - f_\sharp T - \sigma_\sharp(\llbracket(0,1)\rrbracket \times \partial T). \tag{2.5}$$

An important case of the above construction occurs when σ is the affine homotopy $\sigma(t,x) := (1-t)f(x) + tg(x)$. In this case, we have the following estimate on the mass of $\sigma_\sharp(\llbracket(0,1)\rrbracket \times T)$, which will be useful in the sequel (see [Sim83b, Section 26] for the proof):

$$\mathbb{M}(\sigma_\sharp(\llbracket(0,1)\rrbracket \times T)) \leqslant \left(\sup_{x\in \mathrm{spt}(T)} |f(x)-g(x)|\right)\left(\sup_{x\in \mathrm{spt}(T)} (|Df(x)|+|Dg(x)|)\right)\mathbb{M}(T). \tag{2.6}$$

Next, we define the *cone* $0 \ast\!\!\ast T$ over a current $T \in \mathcal{D}_m(\Omega)$. If Ω is star-shaped with respect to 0 and $\mathrm{spt}(T)$ is compact, we set

$$0 \ast\!\!\ast T := h_\sharp(\llbracket(0,1)\rrbracket \times T),$$

where $h\colon (0,1) \times \mathbb{R}^d \to \mathbb{R}^d$ is defined by $h(t,x) := tx$. Thus, $0 \ast\!\!\ast T \in \mathcal{D}_{m+1}(\Omega)$, and by the homotopy formula (2.5) one has

$$\partial(0 \ast\!\!\ast T) = T - 0 \ast\!\!\ast (\partial T).$$

Observe that if $T = \llbracket\Sigma\rrbracket$ is the current associated to an m-dimensional submanifold Σ of $\mathbb{S}^{d-1} \subset \mathbb{R}^d$, then $0 \ast\!\!\ast T = \llbracket C_\Sigma\rrbracket$, where $C_\Sigma = \{tx : x \in \Sigma, t \in (0,1)\}$.

The following theorem is very useful.

Theorem 2.1.6 (Constancy Theorem, cf. [Sim83b, Theorem 26.27]). *If Ω is open and connected in $\mathbb{R}^m$ (i.e. $d = m$), and if $T \in \mathcal{D}_m(\Omega)$ has $\partial T = 0$ then there is a constant $c \in \mathbb{R}$ such that $T = c\llbracket\Omega\rrbracket$.*

If $T = \llbracket B, \vec{\tau}, \theta\rrbracket$ is a (locally) rectifiable current, and $A \subset \mathbb{R}^d$ is a Borel set, we denote the *restriction* of T to A by setting $T \llcorner A := \llbracket B \cap A, \vec{\tau}, \theta\rrbracket$. The restriction operator analogously extends to all currents which can be represented by integration.

Now, we introduce the notion of *slicing* a locally integer rectifiable current by a Lipschitz function. Let $T = \llbracket B, \vec{\tau}, \theta\rrbracket \in \mathscr{R}_m^{\mathrm{loc}}(\Omega)$, and let $f\colon B \to \mathbb{R}$ be Lipschitz. Denote by $Df(x)$ the tangent map of f at x, which exists at $\mathcal{H}^m$-a.e. $x \in B$ since B is rectifiable and f is Lipschitz, and set $B_+ := \{x \in B : |Df(x)| > 0\}$. Then, for a.e. $t \in \mathbb{R}$ one has that the set $B_t := f^{-1}(\{t\}) \cap B_+$ is countably $(m-1)$-rectifiable (cf. [Sim83b, Lemma 28.1]). Moreover, at $\mathcal{H}^{m-1}$-a.e. $x \in B_t$ the approximate tangent spaces $\mathrm{Tan}(B,x)$ and $\mathrm{Tan}(B_t,x)$ both exist

1 Here, $\llbracket(0,1)\rrbracket \times T$ denotes the cartesian product of the currents $\llbracket(0,1)\rrbracket$ and T. Of course, when $T = \llbracket\Sigma\rrbracket$ is the current associated to a smooth submanifold Σ then $\llbracket(0,1)\rrbracket \times T$ coincides with the current which is naturally associated to the product manifold $(0,1) \times \Sigma$. For the general definition of the cartesian product of currents, the reader can refer to [Fed69, 4.1.8] or [Sim83b, Section 26].

and $\mathrm{Tan}(B_t, x) \subset \mathrm{Tan}(B, x)$. For any such x there is $\vec{\xi}(x) \in \Lambda_{m-1}(\mathrm{Tan}(B_t, x))$ such that $|\vec{\xi}(x)| = 1$, $\vec{\xi}(x)$ is simple and $\frac{Df(x)}{|Df(x)|} \wedge \vec{\xi}(x) = \vec{\tau}(x)$. For the ($\mathcal{L}^1$-almost all) $t \in \mathbb{R}$ such that B_t is rectifiable, one defines then the *slice* $\langle T, f, t\rangle$ of T by f at t by setting $\langle T, f, t\rangle := [\![B_t, \vec{\xi}, \theta_t]\!] \in \mathscr{R}^{loc}_{m-1}(\Omega)$, where $\theta_t := \theta|_{B_t}$. The properties of the slices are recorded in the following proposition.

Proposition 2.1.7 (Slicing, cf. [Sim83b, Lemma 28.5])**.**

(i) *For every $W \Subset \Omega$ it holds*

$$\int_{-\infty}^{+\infty} \mathbb{M}_W(\langle T, f, t\rangle)\, dt = \int_{W\cap B} |Df(x)||\theta(x)| d\mathcal{H}^m(x) \leqslant \|Df\|_{L^\infty(W\cap B)} \mathbb{M}_W(T)\,;$$

(ii) *if $\mathbb{M}_W(\partial T) < \infty$ for every $W \Subset \Omega$, then for $\mathcal{L}^1$-a.e. $t \in \mathbb{R}$ one has*

$$\partial\,(T \llcorner \{f < t\}) = (\partial T) \llcorner \{f < t\} + \langle T, f, t\rangle;$$

(iii) *if $T \in \mathscr{I}^{loc}_m(\Omega)$, then for $\mathcal{L}^1$-a.e. $t \in \mathbb{R}$ it holds*

$$\langle \partial T, f, t\rangle = -\partial\,(\langle T, f, t\rangle)\,.$$

The slicing theory can then be extended to slicing by $\mathbb{R}^n$-valued maps by iteratively slicing by the (real-valued) components of the map (see [Fed69, Section 4.3]).

2.1.3 Integral flat chains

Let $K \subset \Omega$ be a compact set. We set

$$\mathscr{F}_{m,K}(\Omega) := \{T = R + \partial S :\ R \in \mathscr{R}_{m,K}(\Omega) \text{ and } S \in \mathscr{R}_{m+1,K}(\Omega)\},$$

and we let $\mathscr{F}_m(\Omega)$ be the union of the sets $\mathscr{F}_{m,K}(\Omega)$ over all compact $K \subset \Omega$. Observe that $\mathscr{R}_m(\Omega) \subset \mathscr{F}_m(\Omega)$. Currents $T \in \mathscr{F}_m(\Omega)$ are called m-dimensional (integral) *flat chains* in Ω. On each set $\mathscr{F}_{m,K}(\Omega)$ one can define a metric as follows: for $T \in \mathscr{F}_{m,K}(\Omega)$, set

$$\mathbb{F}_K(T) := \inf\{\mathbb{M}(R) + \mathbb{M}(S) :\ R \in \mathscr{R}_{m,K}(\Omega), S \in \mathscr{R}_{m+1,K}(\Omega) \text{ such that } T = R + \partial S\},$$

and then let the distance between T_1 and T_2 (usually called *flat distance*) be given by

$$d_{\mathbb{F}_K}(T_1, T_2) := \mathbb{F}_K(T_1 - T_2).$$

It turns out that the resulting metric space $(\mathscr{F}_{m,K}(\Omega), d_{\mathbb{F}_K})$ is complete. Moreover, the mass functional is lower semi-continuous with respect to the *flat convergence*.

For every K, the class $\mathscr{I}_{m,K}(\Omega)$ is dense in $\mathscr{R}_{m,K}(\Omega)$ in mass, and consequently $\mathscr{I}_{m,K}(\Omega)$ is dense in $\mathscr{F}_{m,K}(\Omega)$ in flat norm. In particular, given $T \in \mathscr{F}_{m,K}(\Omega)$ there exist sequences $\{T_j\} \subset \mathscr{I}_{m,K}(\Omega)$, $\{R_j\} \subset \mathscr{R}_{m,K}(\Omega)$, $\{S_j\} \subset \mathscr{R}_{m+1,K}(\Omega)$ such that

$$T = T_j + R_j + \partial S_j$$

and

$$\mathbb{M}(R_j) + \mathbb{M}(S_j) \to 0.$$

If T has finite mass, then the ∂S_j's have finite mass too, and thus $S_j \in \mathscr{I}_{m+1,K}(\Omega)$ by Theorem 2.1.2. Therefore, the currents $T_j + \partial S_j \in \mathscr{R}_{m,K}(\Omega)$ approximate T in mass, and this suffices to conclude that $T \in \mathscr{R}_{m,K}(\Omega)$ (cf. [Sim83b, Lemma 27.5]). We have shown the following result:

Theorem 2.1.8 (Rectifiability of flat chains with finite mass, cf. [Fed69, Theorem 4.2.16])**.** *One has:*

$$\mathscr{R}_{m,K}(\Omega) = \{T \in \mathscr{F}_{m,K}(\Omega) : \mathbb{M}(T) < \infty\}. \tag{2.7}$$

It is immediate to show that if a sequence $\{T_h\}$ of flat chains converges to T with respect to the flat distance then it also weakly converges to T. The two notions of convergence are in fact equivalent if $\{T_h\}$ is a sequence of integral currents satisfying (2.2) (cf. [Sim83b, Theorem 31.2]). Hence, in Theorem 2.1.3 the conclusion $T_{h_j} \rightharpoonup T$ is in fact equivalent to $\mathbb{F}_K(T_{h_j} - T) \to 0$ for every $K \subset \Omega$ compact.

Finally, it is possible to show that the infimum in the definition of $\mathbb{F}_K(T)$ is, in fact, a minimum (see [Fed69, Corollary 4.2.18]).

Proposition 2.1.9. *If $T \in \mathscr{F}_{m,K}(\Omega)$, then there exists a current $S \in \mathscr{R}_{m+1,K}(\Omega)$ such that $T - \partial S \in \mathscr{R}_{m,K}(\Omega)$ and*

$$\mathbb{F}_K(T) = \mathbb{M}(T - \partial S) + \mathbb{M}(S). \tag{2.8}$$

2.1.4 Approximation theorems

When working with integral currents or flat chains, it is sometimes extremely useful to approximate such currents with more regular objects. Surprisingly enough, the "regular" objects we are referring to are not the currents associated with smooth submanifolds, but *polyhedral chains*.

Given an m-dimensional simplex σ in $\mathbb{R}^d$ with constant unit orientation $\vec{\tau}$, we denote by $[\![\sigma]\!]$ the rectifiable current $[\![\sigma, \vec{\tau}, 1]\!]$. Finite linear combinations of (the currents associated with) oriented m-simplexes with integer coefficients are called (integral) *polyhedral m-chains*. The set of polyhedral m-chains in $\mathbb{R}^d$ will be denoted $\mathscr{P}_m(\mathbb{R}^d)$.

The following *Deformation Theorem*, first proved by Federer and Fleming in [FF60], is a central result in the theory of currents.

Theorem 2.1.10 (Deformation, cf. [Fed69, Theorem 4.2.9])**.** *There exists a constant $\gamma = \gamma(m, d)$ with the following property. For any $T \in \mathscr{I}_m(\mathbb{R}^d)$ and $\varepsilon > 0$ there exist $P \in \mathscr{P}_m(\mathbb{R}^d)$, $R \in \mathscr{I}_m(\mathbb{R}^d)$ and $S \in \mathscr{I}_{m+1}(\mathbb{R}^d)$ such that the following holds:*

(i) $T = P + R + \partial S$;

(ii) $\mathbb{M}(P) \leqslant \gamma\left(\mathbb{M}(T) + \varepsilon\mathbb{M}(\partial T)\right)$, $\quad \mathbb{M}(\partial P) \leqslant \gamma\mathbb{M}(\partial T)$,
$\mathbb{M}(R) \leqslant \gamma\varepsilon\mathbb{M}(\partial T)$, $\quad \mathbb{M}(S) \leqslant \gamma\varepsilon\mathbb{M}(T)$;

(iii) $\operatorname{spt}(P) \cup \operatorname{spt}(S) \subset \{x : \operatorname{dist}(x, \operatorname{spt}(T)) \leqslant 2d\varepsilon\}$
$\operatorname{spt}(\partial P) \cup \operatorname{spt}(R) \subset \{x : \operatorname{dist}(x, \operatorname{spt}(\partial T)) \leqslant 2d\varepsilon\}$;

(iv) *if* ∂T *is an integral polyhedral chain, so is* R*;*

(v) *if* T *is an integral polyhedral chain, so is* S*.*

A great variety of results concerning the approximation of currents with polyhedral chains stem directly from the Deformation Theorem. In the sequel, we will mainly use the following two "flat norm" approximation theorems, stated in the next two propositions and concerning integral currents and flat chains respectively.

Proposition 2.1.11 (Polyhedral approximation of integral currents, cf. [Fed69, Corollary 4.2.21])**.** *If* $T \in \mathscr{I}_m(\Omega)$*,* $\rho > 0$ *and* $K \subset \Omega$ *is a compact subset such that* $\operatorname{spt}(T) \subset \operatorname{int} K$*, then there exists* $P \in \mathscr{P}_m(\mathbb{R}^d)$ *with* $\operatorname{spt}(P) \subset K$ *and*

$$\mathbb{F}_K(T - P) \leqslant \rho, \quad \mathbb{M}(P) \leqslant \mathbb{M}(T) + \rho, \quad \mathbb{M}(\partial P) \leqslant \mathbb{M}(\partial T) + \rho. \tag{2.9}$$

Proposition 2.1.12 (Polyhedral approximation of flat chains, cf. [Fed69, Theorem 4.2.22])**.** *If* $T \in \mathscr{F}_m(\Omega)$ *and* $K \subset \Omega$ *is a compact subset such that* $\operatorname{spt}(T) \subset \operatorname{int} K$*, then* $T \in \mathscr{F}_{m,K}(\Omega)$*, and for every* $\varepsilon > 0$ *there exists* $P \in \mathscr{P}_m(\mathbb{R}^d)$ *with* $\operatorname{spt}(P) \subset K$ *and*

$$\mathbb{F}_K(T - P) \leqslant \varepsilon, \qquad \mathbb{M}(P) \leqslant \mathbb{M}(T) + \varepsilon. \tag{2.10}$$

2.1.5 Area minimizing currents: interior regularity theory

Let $\Omega \subset \mathbb{R}^d$ be an open set, and let $T \in \mathscr{R}^{loc}_m(\Omega)$ be area minimizing in Ω. Define $\operatorname{Reg}(T)$ to be the set of points $x \in \operatorname{spt}(T) \setminus \operatorname{spt}(\partial T)$ for which there exists $r > 0$ such that $\operatorname{spt}(T) \cap B_r(x)$ is a smoothly embedded m-dimensional submanifold, and set

$$\operatorname{Sing}(T) := \operatorname{spt}(T) \setminus (\operatorname{Reg}(T) \cup \operatorname{spt}(\partial T)) .$$

As anticipated in the Introduction, in order to discuss the regularity of T we have to distinguish two cases.

Theorem 2.1.13 (Regularity in codimension one)**.** *Let* $\Omega \subset \mathbb{R}^d$ *be an open set, and let* $T \in \mathscr{R}^{loc}_m(\Omega)$ *be area minimizing in* Ω*. Suppose that* $n := d - m = 1$*. Then, the following holds:*

(i) *for* $m \leqslant 6$*,* $\operatorname{Sing}(T) \cap \Omega$ *is empty (Fleming and De Giorgi (*$m = 2$*), Almgren (*$m = 3$*), Simons (*$4 \leqslant m \leqslant 6$*), see [DG61, Fle62, DG65, Alm66, Sim68]);*

(ii) *for* $m = 7$*,* $\operatorname{Sing}(T) \cap \Omega$ *consists of isolated points (Federer [Fed70]);*

(iii) *for* $m \geqslant 8$*,* $\operatorname{Sing}(T) \cap \Omega$ *has Hausdorff dimension not larger than* $m - 7$ *(Federer [Fed70]), and it is countably* $(m-7)$*-rectifiable (Simon [Sim95b], Naber and Valtorta [NV15]);*

(iv) *the above results are optimal: for every* $m \geqslant 7$ *there are area minimizing* T *in* $\mathbb{R}^{m+1}$ *for which* $\operatorname{Sing}(T)$ *has positive* $\mathcal{H}^{m-7}$ *measure (Bombieri-De Giorgi-Giusti [BDGG69]).*

Theorem 2.1.14 (Regularity in higher codimension)**.** *Let* $\Omega \subset \mathbb{R}^d$ *be an open set, and let* $T \in \mathscr{R}^{loc}_m(\Omega)$ *be area minimizing in* Ω*. Suppose that* $n := d - m \geqslant 2$*. Then, the following holds:*

(i) *for* $m = 1$*,* $\operatorname{Sing}(T) \cap \Omega$ *is empty;*

(ii) *for* $m = 2$, $\mathrm{Sing}(T) \cap \Omega$ *consists of isolated points (Chang [Cha88]);*

(iii) *for* $m \geqslant 3$, $\mathrm{Sing}(T) \cap \Omega$ *has Hausdorff dimension not larger than* $m - 2$ *(Almgren [Almoo], De Lellis-Spadaro [DLS14, DLS16a, DLS16b]);*

(iv) *the above results are optimal: for every* $m \geqslant 2$ *there are area minimizing* T *in* $\mathbb{R}^{m+n}$ $(n \geqslant 2)$ *for which* $\mathrm{Sing}(T)$ *has positive* $\mathcal{H}^{m-2}$ *measure (Federer [Fed65]).*

The occurrence of branching-type singularities in the higher codimension case has made necessary the development of a whole range of new tools. Central among them is Almgren's theory of *multiple-valued* functions, which is the topic of our next section.

2.2 ALMGREN'S THEORY OF MULTIPLE-VALUED FUNCTIONS

Here we briefly recall the relevant definitions and properties concerning Q-valued functions. Our main reference is [DLS11].

2.2.1 The metric space of Q-points

From now on, let $Q \geqslant 1$ be a fixed positive integer. The set of Q-*points* in $\mathbb{R}^n$ is, roughly speaking, the set of *unordered* Q-tuples of vectors in $\mathbb{R}^n$. More precisely, we have the following definition.

Definition 2.2.1 (Q-points)**.** The set of Q-points in the Euclidean space $\mathbb{R}^n$ is denoted $\mathcal{A}_Q(\mathbb{R}^n)$ and defined as the quotient $(\mathbb{R}^n)^Q/\sim$ modulo the equivalence relation

$$(v_1, \dots, v_Q) \sim (v_{\sigma(1)}, \dots, v_{\sigma(Q)}) \qquad \forall\, \sigma \in \mathcal{P}_Q,$$

where $\mathcal{P}_Q$ is the group of permutations of $\{1, \dots, Q\}$. Equivalently, $\mathcal{A}_Q(\mathbb{R}^n)$ can be identified with the following set:

$$\mathcal{A}_Q(\mathbb{R}^n) = \left\{ T = \sum_{\ell=1}^{Q} \llbracket v_\ell \rrbracket : \, v_\ell \in \mathbb{R}^n \text{ for every } \ell = 1, \dots, Q \right\}, \tag{2.11}$$

where $\llbracket v \rrbracket$ is the Dirac mass δ_v centered at the point $v \in \mathbb{R}^n$. Hence, every Q-point T can in fact be identified with a purely atomic non-negative measure of mass Q in $\mathbb{R}^n$ which is the sum of Dirac deltas with integer multiplicities.

For the sake of notational simplicity, we will sometimes write $\mathcal{A}_Q$ instead of $\mathcal{A}_Q(\mathbb{R}^n)$ if there is no chance of ambiguity.

Remark 2.2.2. Observe that the notation $\llbracket v \rrbracket$ to denote the Dirac delta δ_v is consistent with that introduced in Section 2.1 for the current associated to a submanifold. Indeed, if $v \in \mathbb{R}^n$ then the action of the 0-dimensional current associated to v is precisely given by

$$\llbracket v \rrbracket(f) = f(v) \quad \text{for every } f \in \mathcal{D}^0(\mathbb{R}^n) = C_c^\infty(\mathbb{R}^n).$$

Hence, $\llbracket v \rrbracket$ is in fact the restriction of δ_v to $C_c^\infty(\mathbb{R}^n)$.

The identification of Q-points with measures plays a fundamental role in the development of calculus on $\mathcal{A}_Q(\mathbb{R}^n)$, as it allows one to define a distance between Q-points borrowing one of the distances defined for measures with finite mass. In particular, it is customary to use the Wasserstein distance of exponent two (cf. for instance [Vil03, Section 7.1]).

Definition 2.2.3. If $T_1 = \sum \llbracket v_\ell \rrbracket$ and $T_2 = \sum \llbracket w_\ell \rrbracket$, then the distance between T_1 and T_2 is denoted $\mathcal{G}(T_1, T_2)$ and given by

$$\mathcal{G}(T_1, T_2)^2 := \min_{\sigma \in \mathcal{P}_Q} \sum_{\ell=1}^{Q} |v_\ell - w_{\sigma(\ell)}|^2. \tag{2.12}$$

One can easily see that $(\mathcal{A}_Q(\mathbb{R}^n), \mathcal{G})$ is a complete, separable metric space.

If $T \in \mathcal{A}_Q(\mathbb{R}^n)$ can be written as $T = k\llbracket v \rrbracket + \sum_{i=1}^{Q-k} \llbracket v_i \rrbracket$ with each $v_i \neq v$, then we say that v has *multiplicity* k in T. Sometimes, when v has multiplicity k in T we will write $k = \Theta^0(T, v) =: \Theta_T(v)$, using a notation which is coherent with regarding T as a 0-dimensional integer rectifiable current in $\mathbb{R}^n$.

Also, to any point $T = \sum_\ell \llbracket v_\ell \rrbracket \in \mathcal{A}_Q(\mathbb{R}^n)$ one can naturally associate two objects, of which we will make use in the sequel: the *diameter* of T is the scalar

$$\operatorname{diam}(T) := \max_{i,j \in \{1,\dots,Q\}} |v_i - v_j|, \tag{2.13}$$

whereas the *center of mass* of T is the vector

$$\boldsymbol{\eta}(T) := \frac{1}{Q} \sum_{\ell=1}^{Q} v_\ell. \tag{2.14}$$

2.2.2 Q-valued functions

Let $\Sigma = \Sigma^m$ be an m-dimensional C^1 submanifold of $\mathbb{R}^d$. We will regard Σ as a Riemannian manifold with the metric induced by the flat metric of the ambient space $\mathbb{R}^d$. In particular, given two points $x, y \in \Sigma$ we will let $\mathbf{d}(x, y)$ denote their Riemannian geodesic distance. Furthermore, measures and integrals on Σ will always be computed with respect to the m-dimensional Hausdorff measure $\mathcal{H}^m$ defined in the ambient space (note that the Hausdorff measure can be defined also intrinsically in terms of the distance $\mathbf{d}$: however, since Σ is isometrically embedded in $\mathbb{R}^d$, the intrinsic $\mathcal{H}^m$ measure coincides with the restriction of the "Euclidean one").

Any map $u \colon \Sigma \to \mathcal{A}_Q(\mathbb{R}^n)$ will be called a Q-*valued function* on Σ.

Continuous, Lipschitz, Hölder and measurable functions $u \colon \Sigma \to \mathcal{A}_Q(\mathbb{R}^n)$ can be straightforwardly defined taking advantage of the metric space structure of both the domain and the target. As for the spaces $L^p(\Sigma, \mathcal{A}_Q)$, $1 \leqslant p \leqslant \infty$, they consist of those measurable maps $u \colon \Sigma \to \mathcal{A}_Q(\mathbb{R}^n)$ for which $\|u\|_{L^p} := \|\mathcal{G}(u, Q\llbracket 0 \rrbracket)\|_{L^p(\Sigma)}$ is finite. We will systematically use the notation $|u| := \mathcal{G}(u, Q\llbracket 0 \rrbracket)$, so that

$$\|u\|_{L^p}^p = \int_\Sigma |u|^p \, d\mathcal{H}^m \quad \text{when } 1 \leqslant p < \infty,$$
$$\|u\|_{L^\infty} = \operatorname*{ess\,sup}_\Sigma |u|.$$

In spite of this notation, we remark here that, when $Q > 1$, $\mathcal{A}_Q(\mathbb{R}^n)$ is not a linear space: thus, in particular, the map $T \mapsto |T|$ is not a norm.

Any measurable Q-valued function can be thought as coming together with a measurable selection, as specified in the following proposition.

Proposition 2.2.4 (Measurable selection, cf. [DLS11, Proposition 0.4]). *Let $B \subset \Sigma$ be a $\mathcal{H}^m$-measurable set and $u\colon B \to \mathcal{A}_Q(\mathbb{R}^n)$ be a measurable function. Then, there exist measurable functions $u_1, \dots, u_Q\colon B \to \mathbb{R}^n$ such that*

$$u(x) = \sum_{\ell=1}^{Q} [\![u_\ell(x)]\!] \quad \text{for } \mathcal{H}^m\text{-a.e. } x \in B. \tag{2.15}$$

It is possible to introduce a notion of differentiability for multiple-valued maps.

Definition 2.2.5 (Differentiable Q-valued functions). A map $u\colon \Sigma \to \mathcal{A}_Q(\mathbb{R}^n)$ is said to be *differentiable* at $x \in \Sigma$ if there exist Q linear maps $\lambda_\ell\colon T_x\Sigma \to \mathbb{R}^n$ satisfying:

(i) $\mathcal{G}\big(u(\exp_x(\xi)), T_x u(\xi)\big) = o(|\xi|)$ as $|\xi| \to 0$ for any $\xi \in T_x\Sigma$, where exp is the exponential map on Σ and

$$T_x u(\xi) := \sum_{\ell=1}^{Q} [\![u_\ell(x) + \lambda_\ell \cdot \xi]\!]; \tag{2.16}$$

(ii) $\lambda_\ell = \lambda_{\ell'}$ if $u_\ell(x) = u_{\ell'}(x)$.

We will use the notation $Du_\ell(x)$ for λ_ℓ, and formally set $Du(x) = \sum_\ell [\![Du_\ell(x)]\!]$: observe that one can regard $Du(x)$ as an element of $\mathcal{A}_Q(\mathbb{R}^{n\times m})$ as soon as a basis of $T_x\Sigma$ has been fixed. For any $\xi \in T_x\Sigma$, we define the directional derivative of u along ξ to be $D_\xi u(x) := \sum_\ell [\![Du_\ell(x) \cdot \xi]\!]$, and establish the notation $D_\xi u = \sum_\ell [\![D_\xi u_\ell]\!]$.

Differentiable functions enjoy a chain rule formula.

Proposition 2.2.6 (Chain rules, cf. [DLS11, Proposition 1.12]). *Let $u\colon \Sigma \to \mathcal{A}_Q(\mathbb{R}^n)$ be differentiable at x_0.*

(i) *Consider $\Phi\colon \tilde{\Sigma} \to \Sigma$ such that $\Phi(y_0) = x_0$, and assume that Φ is differentiable at y_0. Then, $u \circ \Phi$ is differentiable at y_0 and*

$$D(u \circ \Phi)(y_0) = \sum_{\ell=1}^{Q} [\![Du_\ell(x_0) \cdot D\Phi(y_0)]\!]. \tag{2.17}$$

(ii) *Consider $\Psi\colon \Sigma_x \times \mathbb{R}^n_y \to \mathbb{R}^q$ such that Ψ is differentiable at the point $(x_0, u_\ell(x_0))$ for every ℓ. Then, the map $\Psi(x,u)\colon x \in \Sigma \mapsto \sum_{\ell=1}^{Q} [\![\Psi(x, u_\ell(x))]\!] \in \mathcal{A}_Q(\mathbb{R}^q)$ fulfills* (i) *of Definition 2.2.5. Moreover, if also* (ii) *holds, then*

$$D\Psi(x,u)(x_0) = \sum_{\ell=1}^{Q} [\![D_x\Psi(x_0, u_\ell(x_0)) + D_y\Psi(x_0, u_\ell(x_0)) \cdot Du_\ell(x_0)]\!]. \tag{2.18}$$

(iii) *Consider a map* $F\colon (\mathbb{R}^n)^Q \to \mathbb{R}^q$ *with the property that, for any choice of* Q *points* $(y_1, \dots, y_Q) \in (\mathbb{R}^n)^Q$*, for any permutation* $\sigma \in \mathcal{P}_Q$

$$F(y_1, \dots, y_Q) = F(y_{\sigma(1)}, \dots, y_{\sigma(Q)}).$$

Then, if F *is differentiable at* $(u_1(x_0), \dots, u_Q(x_0))$ *the composition* $F \circ u$ [2] *is differentiable at* x_0 *and*

$$D(F \circ u)(x_0) = \sum_{\ell=1}^{Q} D_{v_\ell} F(u_1(x_0), \dots, u_Q(x_0)) \cdot Du_\ell(x_0). \tag{2.19}$$

Rademacher's theorem extends to the Q-valued setting, as shown in [DLS11, Theorem 1.13]: Lipschitz Q-valued functions are differentiable $\mathcal{H}^m$-almost everywhere in the sense of Definition 2.2.5. Moreover, for a Lipschitz Q-valued function the decomposition result stated in Proposition 2.2.4 can be improved as follows.

Proposition 2.2.7 (Lipschitz selection, cf. [DS15, Lemma 1.1])**.** *Let* $B \subset \Sigma$ *be measurable, and assume* $u\colon B \to \mathcal{A}_Q(\mathbb{R}^n)$ *is Lipschitz. Then, there are a countable partition of* B *in measurable subsets* B_i *(*$i \in \mathbb{N}$*) and Lipschitz functions* $u_i^\ell\colon B_i \to \mathbb{R}^n$ *(*$\ell \in \{1, \dots, Q\}$*) such that*

(a) $u|_{B_i} = \sum_{\ell=1}^{Q} [\![u_i^\ell]\!]$ *for every* $i \in \mathbb{N}$*, and* $\mathrm{Lip}(u_i^\ell) \leqslant \mathrm{Lip}(u)$ *for every* i, ℓ*;*

(b) *for every* $i \in \mathbb{N}$ *and* $\ell, \ell' \in \{1, \dots, Q\}$*, either* $u_i^\ell \equiv u_i^{\ell'}$ *or* $u_i^\ell(x) \neq u_i^{\ell'}(x)\ \forall x \in B_i$*;*

(c) *for every* i *one has* $Du(x) = \sum_{\ell=1}^{Q} [\![Du_i^\ell(x)]\!]$ *for* $\mathcal{H}^m$*-a.e.* $x \in B_i$*.*

We conclude this paragraph with the following useful *Lipschitz decomposition property*.

Proposition 2.2.8 (Lipschitz decomposition, cf. [DLS11, Proposition 1.6])**.** *Let* $u = \sum_{\ell=1}^{Q} [\![u_\ell]\!]$ *be a Lipschitz function,* $u\colon B \subset \Sigma \to \mathcal{A}_Q(\mathbb{R}^n)$*. Suppose that there exists* $x_0 \in B$ *and* $i, j \in \{1, \dots, Q\}$ *such that*

$$|u_i(x_0) - u_j(x_0)| > 3(Q-1)\mathrm{Lip}(u)\mathrm{diam}(B). \tag{2.20}$$

Then, there are integers $Q_1 < Q$ *and* $Q_2 < Q$ *with* $Q_1 + Q_2 = Q$ *and Lipschitz functions* $u_1\colon B \to \mathcal{A}_{Q_1}(\mathbb{R}^n)$ *and* $u_2\colon B \to \mathcal{A}_{Q_2}(\mathbb{R}^n)$ *such that* $u = [\![u_1]\!] + [\![u_2]\!]$*,* $\mathrm{Lip}(u_1), \mathrm{Lip}(u_2) \leqslant \mathrm{Lip}(u)$ *and* $\mathrm{spt}(u_1(x)) \cap \mathrm{spt}(u_2(x)) = \emptyset$ *for every* $x \in B$*.*

2.2.3 Q-valued Sobolev functions and their properties

Next, we study the Sobolev spaces $W^{1,p}(\Sigma, \mathcal{A}_Q)$, where Σ^m is compact (or it is an open subset of $\mathbb{R}^m$). The definition that we use here was proposed by C. De Lellis and E. Spadaro (cf. [DLS11, Definition 0.5 and Proposition 4.1]), and allowed the authors to develop an alternative, intrinsic approach to the study of Q-valued Sobolev mappings, which does not rely on Almgren's embedding of the space $\mathcal{A}_Q(\mathbb{R}^n)$ in a larger Euclidean space (cf. [Almoo] and [DLS11, Chapter 2]). Such an approach is close in spirit to the general theory of Sobolev maps taking values in abstract metric spaces and started in the works of Ambrosio [Amb90] and Reshetnyak [Res97, Reso4, Reso6].

2 Observe that $F \circ u$ is a well defined function $\Sigma \to \mathbb{R}^q$, because F is, by hypothesis, a well defined map on the quotient $\mathcal{A}_Q(\mathbb{R}^n) = (\mathbb{R}^n)^Q/\mathcal{P}_Q$.

Definition 2.2.9 (Sobolev Q-valued functions)**.** A measurable function $u\colon \Sigma \to \mathcal{A}_Q(\mathbb{R}^n)$ is in the Sobolev class $W^{1,p}$, $1 \leqslant p \leqslant \infty$ if and only if there exists a non-negative function $\psi \in L^p(\Sigma)$ such that, for every Lipschitz function $\phi\colon \mathcal{A}_Q(\mathbb{R}^n) \to \mathbb{R}$, the following two properties hold:

(i) $\phi \circ u \in W^{1,p}(\Sigma)$ [3];

(ii) $|D(\phi \circ u)(x)| \leqslant \mathrm{Lip}(\phi)\psi(x)$ for almost every $x \in \Sigma$.

We also recall (cf. [DLS11, Proposition 4.2]) that if $u \in W^{1,p}\left(\Sigma, \mathcal{A}_Q(\mathbb{R}^n)\right)$ and ξ is a tangent vector field defined on Σ, there exists a non-negative function $g_\xi \in L^p(\Sigma)$ with the following two properties:

(i) $|D_\xi(\mathcal{G}(u,T))| \leqslant g_\xi$ $\mathcal{H}^m$-a.e. in Σ for all $T \in \mathcal{A}_Q$;

(ii) if $h_\xi \in L^p(\Sigma)$ satisfies $|D_\xi(\mathcal{G}(u,T))| \leqslant h_\xi$ for all $T \in \mathcal{A}_Q$, then $g_\xi \leqslant h_\xi$ $\mathcal{H}^m$-a.e.

Such a function is clearly unique (up to sets of $\mathcal{H}^m$-measure zero), and will be denoted by $|D_\xi u|$. Moreover, chosen a countable dense subset $\{T_i\}_{i=0}^\infty \subset \mathcal{A}_Q$, it satisfies

$$|D_\xi u| = \sup_{i \in \mathbb{N}} |D_\xi \mathcal{G}(u, T_i)| \tag{2.21}$$

almost everywhere in Σ.

As in the classical theory, Sobolev Q-valued maps can be approximated by Lipschitz maps.

Proposition 2.2.10 (Lipschitz approximation, cf. [DLS11, Proposition 4.4])**.** *There exists a constant $C = C(m, \Sigma, Q)$ with the following property. Let u be a function in $W^{1,p}(\Sigma, \mathcal{A}_Q)$. For every $\lambda > 0$, there exists a Lipschitz Q-function u_λ such that $\mathrm{Lip}(u_\lambda) \leqslant C\lambda$ and*

$$\mathcal{H}^m\left(\{x \in \Sigma : u_\lambda(x) \neq u(x)\}\right) \leqslant \frac{C}{\lambda^p} \int_\Sigma |Du|^p \, d\mathcal{H}^m, \tag{2.22}$$

where

$$|Du| := \left(\sum_{i=1}^m |D_{\xi_i} u|^2 \right)^{1/2}$$

for any choice of an orthonormal frame $(\xi_i)_{i=1}^m$ of the tangent bundle $T\Sigma$.

As a corollary, Proposition 2.2.10 allows to prove that Sobolev Q-valued maps are approximately differentiable almost everywhere.

3 Here, the Sobolev space $W^{1,p}(\Sigma)$ is classically defined as the completion of $C^1(\Sigma)$ with respect to the $W^{1,p}$-norm

$$\|f\|^p_{W^{1,p}(\Sigma)} := \int_\Sigma \left(|f(x)|^p + |Df(x)|^p\right) d\mathcal{H}^m(x)$$

for $1 \leqslant p < \infty$ and

$$\|f\|_{W^{1,\infty}(\Sigma)} := \operatorname*{ess\,sup}_\Sigma \left(|f(x)| + |Df(x)|\right).$$

Corollary 2.2.11 (cf. [DLS11, Corollary 2.7]). *Let $u \in W^{1,p}(\Sigma, \mathcal{A}_Q)$. Then, u is* approximately differentiable $\mathcal{H}^m$*-a.e. in Σ: precisely, for $\mathcal{H}^m$-a.e. $x \in \Sigma$ there exists a measurable set $\Omega \subset \Sigma$ containing x such that Ω has density 1 at x and $u|_\Omega$ is differentiable at x.*

The next proposition explores the link between the metric derivative defined in (2.21) and the approximate differential of a Q-valued Sobolev function.

Proposition 2.2.12 (cf. [DLS11, Proposition 2.17]). *Let u be a map in $W^{1,2}\left(\Sigma, \mathcal{A}_Q(\mathbb{R}^n)\right)$. Then, for any vector field ξ defined on Σ and tangent to Σ the metric derivative $|D_\xi u|$ defined in (2.21) satisfies*

$$|D_\xi u|^2 = \sum_{\ell=1}^{Q} |D_\xi u^\ell|^2 \quad \mathcal{H}^m\text{- a.e. in } \Sigma, \tag{2.23}$$

where $\sum_\ell |D_\xi u^\ell|^2 = \mathcal{G}(D_\xi u, Q[\![0]\!])^2$ and $D_\xi u(x) \in \mathcal{A}_Q(\mathbb{R}^n)$ is the approximate directional derivative of u along ξ at the point $x \in \Sigma$. In particular, it holds

$$|Du(x)|^2 := \sum_{i=1}^{m} |D_{\xi_i} u(x)|^2 = \sum_{i=1}^{m} \sum_{\ell=1}^{Q} |D_{\xi_i} u^\ell(x)|^2, \tag{2.24}$$

with $(\xi_i)_{i=1}^m$ any orthonormal frame of $T\Sigma$, at all points x of approximate differentiability for u in Σ.

Remark 2.2.13. Observe that by the above formula the definition of $|Du(x)|$ is indeed independent of the choice of the frame (ξ_i), as in fact one has

$$|Du(x)|^2 = \sum_{\ell=1}^{Q} |Du^\ell(x)|^2,$$

where $|Du^\ell(x)|$ is the Hilbert-Schmidt norm of the linear map $Du^\ell(x)\colon T_x\Sigma \to \mathbb{R}^n$ at every point of approximate differentiability for u.

The main consequence of the above proposition is that essentially all the conclusions of the usual Sobolev space theory for single-valued functions can be recovered in the multi-valued setting modulo routine modifications of the usual arguments. Some of these conclusions will be useful in the coming chapters, thus we will list them here, again referring the interested reader to [DLS11] for their proofs and other useful considerations. In what follows, $\Omega \subset \Sigma$ is an open set with Lipschitz boundary.

Definition 2.2.14 (Trace of Sobolev Q-functions). Let $u \in W^{1,p}\left(\Omega, \mathcal{A}_Q(\mathbb{R}^n)\right)$. A function g belonging to $L^p\left(\partial\Omega, \mathcal{A}_Q(\mathbb{R}^n)\right)$ is said to be the *trace* of u at $\partial\Omega$ (and we write $g = u|_{\partial\Omega}$) if for any $T \in \mathcal{A}_Q$ the trace of the real-valued Sobolev function $\mathcal{G}(u, T)$ coincides with $\mathcal{G}(g, T)$.

Definition 2.2.15 (Weak convergence). Let $\{u_h\}_{h=1}^\infty$ be a sequence of maps in $W^{1,p}(\Omega, \mathcal{A}_Q)$. We say that u_h *converges weakly* to $u \in W^{1,p}(\Omega, \mathcal{A}_Q)$ for $h \to \infty$, and we write $u_h \rightharpoonup u$, if

(i) $\lim_{h\to\infty} \int_\Omega \mathcal{G}(u_h, u)^p \, d\mathcal{H}^m = 0$;

(ii) there exists a constant C such that $\sup_h \int_\Omega |Du_h|^p \, d\mathcal{H}^m \leqslant C$.

Proposition 2.2.16 (Weak sequential closure, cf. [DLS11, Proposition 2.10, Proposition 4.5]). *Let $u \in W^{1,p}(\Omega, \mathcal{A}_Q)$. Then, there is a unique function $g \in L^p(\partial\Omega, \mathcal{A}_Q)$ such that $g = u|_{\partial\Omega}$ in the sense of Definition 2.2.14. Moreover, the set*

$$W^{1,p}_g(\Omega, \mathcal{A}_Q) := \{u \in W^{1,p}(\Omega, \mathcal{A}_Q) : \ u|_{\partial\Omega} = g\}$$

is sequentially closed with respect to the notion of weak convergence introduced in Definition 2.2.15.

Proposition 2.2.17 (Sobolev embeddings, cf. [DLS11, Proposition 2.11, Proposition 4.6]). *The following embeddings hold:*

(i) *if $p < m$, then $W^{1,p}(\Omega, \mathcal{A}_Q) \subset L^q(\Omega, \mathcal{A}_Q)$ for every $q \in [1, p^*]$, $p^* := \frac{mp}{m-p}$, and the inclusion is compact when $q < p^*$;*

(ii) *if $p = m$, then $W^{1,p}(\Omega, \mathcal{A}_Q) \subset L^q(\Omega, \mathcal{A}_Q)$ for all $q \in [1, \infty)$, with compact inclusion;*

(iii) *if $p > m$, then $W^{1,p}(\Omega, \mathcal{A}_Q) \subset C^{0,\alpha}(\Omega, \mathcal{A}_Q)$ for all $\alpha \in \left[0, 1 - \frac{m}{p}\right]$, with compact inclusion if $\alpha < 1 - \frac{m}{p}$.*

Proposition 2.2.18 (Poincaré inequality, cf. [DLS11, Proposition 2.12, Proposition 4.9]). *Let Ω be a connected open subset of Σ with Lipschitz boundary, and let $p < m$. There exists a constant $C = C(p, m, n, Q, \Omega)$ with the following property: for every $u \in W^{1,p}\left(\Omega, \mathcal{A}_Q(\mathbb{R}^n)\right)$ there exists a point $\overline{u} \in \mathcal{A}_Q(\mathbb{R}^n)$ such that*

$$\left(\int_\Omega \mathcal{G}(u, \overline{u})^{p^*} \, d\mathcal{H}^m\right)^{1/p^*} \leqslant C \left(\int_\Omega |Du|^p \, d\mathcal{H}^m\right)^{1/p}. \tag{2.25}$$

Proposition 2.2.19 (Campanato-Morrey estimates, cf. [DLS11, Proposition 2.14]). *Let u be a $W^{1,2}(B_1, \mathcal{A}_Q)$ function, with $B_1 = B_1(0) \subset \mathbb{R}^m$, and assume $\alpha \in (0, 1]$ is such that*

$$\int_{B_r(y)} |Du|^2 \leqslant A r^{m-2+2\alpha} \qquad \text{for every } y \in B_1 \text{ and a.e. } r \in (0, 1 - |y|).$$

Then, for every $0 < \delta < 1$ there is a constant $C = C(m, n, Q, \delta)$ such that

$$[u]_{C^{0,\alpha}(\overline{B}_\delta)} := \sup_{x,y \in \overline{B}_\delta} \frac{\mathcal{G}(u(x), u(y))}{|x - y|^\alpha} \leqslant C\sqrt{A}. \tag{2.26}$$

2.2.4 The Dirichlet energy. Dir-minimizers

A simple corollary of Proposition 2.2.12 and Remark 2.2.13 is that the *Dirichlet energy* of a map $u \in W^{1,2}\left(\Omega, \mathcal{A}_Q(\mathbb{R}^n)\right)$ can be defined in a unique way by setting

$$\mathrm{Dir}(u, \Omega) := \int_\Omega \sum_{i=1}^m |D_{\xi_i} u|^2 \, d\mathcal{H}^m = \int_\Omega \sum_{i=1}^m \sum_{\ell=1}^Q |D_{\xi_i} u^\ell|^2 \, d\mathcal{H}^m, \tag{2.27}$$

for any choice of a (local) orthonormal frame $(\xi_1, \dots, \xi_m)$ of the tangent bundle of Σ.

As already mentioned before, the first step of Almgren's program towards the partial regularity for area minimizing currents in codimension higher than one is to develop a theory concerning existence and regularity properties of minimizers of the Dirichlet energy in $W^{1,2}$ (the so called Dir-minimizers). Such a theory, extensively studied by Almgren in [Almoo] and revisited by De Lellis and Spadaro in [DLS11], can be summarized in four main theorems.

Theorem 2.2.20 (Existence, cf. [DLS11, Theorem 0.8])**.** *Let $\Omega \subset \mathbb{R}^m$ be a bounded open subset with Lipschitz boundary. Let $g \in W^{1,2}(\Omega, \mathcal{A}_Q)$. Then, there exists a function $u \in W^{1,2}(\Omega, \mathcal{A}_Q)$ minimizing the Dirichlet energy (2.27) among all $W^{1,2}$ Q-valued functions v such that $v|_{\partial\Omega} = g|_{\partial\Omega}$.*

Theorem 2.2.21 (Hölder regularity, cf. [DLS11, Theorem 0.9])**.** *There exists a constant $\alpha = \alpha(m, Q) > 0$ with the following property. If $u \in W^{1,2}(\Omega, \mathcal{A}_Q)$ is* Dir*-minimizing, then $u \in C^{0,\alpha}(\Omega', \mathcal{A}_Q)$ for every $\Omega' \Subset \Omega$.*

The statement of the other two results requires the definition of regular and singular points of a Dir-minimizer u.

Definition 2.2.22 (Regular and singular points of a Dir-minimizing map)**.** A Q-valued Dir-minimizer u is regular at a point $x \in \Omega$ if there exist a neighborhood B of x in Ω and Q harmonic functions $u_\ell \colon B \to \mathbb{R}^n$ such that

$$u(y) = \sum_{\ell=1}^{Q} [\![u_\ell(y)]\!] \qquad \text{for almost every } y \in B$$

and either $u_\ell(y) \neq u_{\ell'}(y)$ for every $y \in B$ or $u_\ell \equiv u_{\ell'}$. We will write $x \in \mathrm{reg}(u)$ if x is a regular point. The complement of $\mathrm{reg}(u)$ in Ω is the *singular set*, and will be denoted $\mathrm{sing}(u)$.

Theorem 2.2.23 (Estimate of the singular set, cf. [DLS11, Theorem 0.11])**.** *Let u be a* Dir*-minimizer. Then, the Hausdorff dimension of $\mathrm{sing}(u)$ is at most $m-2$. If $m = 2$, then $\mathrm{sing}(u)$ is at most countable.*

Theorem 2.2.24 (Improved estimate of the singular set for $m = 2$, cf. [DLS11, Theorem 0.12])**.** *Let u be* Dir*-minimizing, and $m = 2$. Then, the singular set $\mathrm{sing}(u)$ consists of isolated points.*

Remark 2.2.25. It is worth observing that here we have only discussed those results in the theory of Dir-minimizing multiple-valued functions which will be useful for our purposes at a later stage of this thesis, and therefore our summary is far from being complete. Among the results that we have not included in the above presentation, we mention the paper [Hir16a], concerned with the problem of extending the Hölder regularity in Theorem 2.2.21 up to the boundary of Ω, and the recent result [DMSV16], where the authors prove that if u is Dir-minimizing then $\mathrm{sing}(u)$ is actually countably $(m-2)$-rectifiable (and hence $\mathcal{H}^{m-2}$ σ-finite), thus extending to general Q a previous result obtained for $Q = 2$ by Krummel and Wickramasekera in [KW13] and considerably improving Almgren's original theory.

2.3 THE THEORY OF HARMONIC MAPS

After having studied the regularity theory of "$\mathbb{R}^n$-valued" Sobolev functions minimizing the Dirichlet energy, a natural question is whether the same regularity properties are shared by functions minimizing the Dirichlet energy among those that are constrained to have target in a given (compact) Riemannian manifold. The resulting theory is known in the literature as *harmonic maps theory*. In this section we will first discuss the basic properties of single-valued harmonic maps (§ 2.3.1), then we will move to more recent advances in the theory (§ 2.3.2), and finally we will present the recent theory of multiple-valued harmonic maps (§ 2.3.3). References for the first part are the standard books by Simon [Sim96], Moser [Mos05] and Lin-Wang [LW08]; for the second part, we will mainly refer to the beautiful paper [NV17]; finally, for the third part our main reference is [Hir16b].

2.3.1 Harmonic maps: an overview

Suppose that Ω is an open subset of $\mathbb{R}^m$, where $m \geqslant 2$, and that $\mathcal{N}$ is a smooth compact Riemannian manifold of dimension $n \geqslant 2$ which is isometrically embedded in some Euclidean space $\mathbb{R}^d$. A map u of Ω into $\mathcal{N}$ will always be thought of as a map $u\colon \Omega \to \mathbb{R}^d$ with the additional property that $u(\Omega) \subset \mathcal{N}$. In particular, we set

$$W^{1,2}(\Omega, \mathcal{N}) := \left\{ u \in W^{1,2}(\Omega, \mathbb{R}^d) : \, u(x) \in \mathcal{N} \text{ for a.e. } x \in \Omega \right\},$$

and

$$W^{1,2}_{loc}(\Omega, \mathcal{N}) := \left\{ u \in W^{1,2}(\Omega', \mathcal{N}) \text{ for every } \Omega' \Subset \Omega \right\}.$$

If $u \in W^{1,2}_{loc}(\Omega, \mathcal{N})$ and $B_r(x) \Subset \Omega$, then the (rescaled) *Dirichlet energy* of u in $B_r(x)$ is the quantity

$$\mathscr{E}(u, B_r(x)) := r^{2-m} \int_{B_r(x)} |Du(y)|^2 \, dy,$$

where $Du(y) \in \mathbb{R}^{d \times m}$ is the classical differential of u and $|Du(y)|$ is its Hilbert-Schmidt norm. A map $u \in W^{1,2}_{loc}(\Omega, \mathcal{N})$ is a (local) minimizer of the Dirichlet energy if the following holds: for any ball $B_r(x) \Subset \Omega$ one has

$$\mathscr{E}(u, B_r(x)) \leqslant \mathscr{E}(v, B_r(x))$$

for every map $v \in W^{1,2}(B_r(x), \mathcal{N})$ such that $v \equiv u$ in a neighborhood of $\partial B_r(x)$. Observe that energy minimizing maps with values in $\mathbb{R}^n$ solve the *linear* system of equations $\Delta u = 0$ everywhere in Ω. As it will be clear in a few lines, an energy minimizing map which is constrained to take values in a manifold $\mathcal{N}$ solves a *nonlinear* version of the Laplace equation in Ω. It is therefore both extremely natural and very interesting to study the regularity for such an object.

The first observation is that if $u \in W^{1,2}_{loc}(\Omega, \mathcal{N})$ is energy minimizing and if $B_r(x) \Subset \Omega$ then one can test the minimality of u along suitably chosen families u_ε of competitors in order to infer that u satisfies some integral equations, known as *variational equations*, which turn out to be of fundamental importance for the regularity theory. Explicitly, if $\delta > 0$ and $\{u_s\}_{s \in (-\delta, \delta)}$ is a one-parameter family of maps $u_s \in W^{1,2}(B_r(x), \mathcal{N})$ having the properties

that $u_0 \equiv u$ in $B_r(x)$ and $u_s \equiv u$ in a neighborhood of $\partial B_r(x)$ for every $s \in (-\delta, \delta)$, then the minimizing property of u implies that

$$\frac{d}{ds}\mathscr{E}(u_s, B_r(x))\Big|_{s=0} = 0. \tag{2.28}$$

Equation (2.28) is the variational equation associated to the family $\{u_s\}$. There are two important kinds of variations of u:

(OV) *outer variations* are variations of the form

$$u_s := \Pi \circ (u + sY),$$

where $Y = \left(Y^1, \dots, Y^d\right) \in C^1_c(B_r(x), \mathbb{R}^d)$ is a vector field *in the target*, and where Π is the nearest point projection map from a tubular neighborhood of $\mathcal{N}$ in $\mathbb{R}^d$ onto $\mathcal{N}$. For such a kind of variations the variational equation (2.28) reads

$$\int_{B_r(x)} \sum_{i=1}^{m} \left(\langle D_i u, D_i Y\rangle - \langle A_u(D_i u, D_i u), Y\rangle\right) dy = 0\,, \tag{2.29}$$

where $D_i u \in \mathbb{R}^d$ is the derivative of u in the direction e_i, $\{e_i\}_{i=1}^m$ being the standard orthonormal basis of $\mathbb{R}^m$, and A is the second fundamental form of the embedding $\mathcal{N} \hookrightarrow \mathbb{R}^d$. Notice that if $u \in C^2$ then we can integrate by parts the left-hand side in the above equation, and, using that the map Y is arbitrary, conclude that u solves

$$\Delta u + \sum_{i=1}^{m} A_u(D_i u, D_i u) = 0 \quad \text{in } \Omega, \tag{2.30}$$

which is the nonlinear variant of the Laplace equation mentioned above. If $u \in C^2$ then equation (2.30) is in fact equivalent to

$$\mathbf{p}_{T_{u(y)}\mathcal{N}} \cdot \Delta u(y) = 0 \quad \text{for every } y \in \Omega, \tag{2.31}$$

where $\mathbf{p}_{T_{u(y)}\mathcal{N}}$ is the orthogonal projection of $\mathbb{R}^d$ onto the tangent space $T_{u(y)}\mathcal{N}$ at every y;

(IV) *inner variations* are instead variations of the form

$$u_s(y) := u(y + sX(y)),$$

where $X \in C^1_c(B_r(x), \mathbb{R}^m)$ is a vector field *in the domain*. For such a kind of variation the variational equation (2.28) reads

$$\int_{B_r(x)} \sum_{i,j=1}^{m} \left(|Du|^2 \delta_{ij} - 2\langle D_i u, D_j u\rangle\right) D_i X^j = 0. \tag{2.32}$$

Maps $u \in W^{1,2}_{loc}(\Omega, \mathcal{N})$ which satisfy both the variational equations (2.29) and (2.32) are called in the literature *stationary harmonic maps*. By the discussion above, every energy

minimizing map is stationary harmonic. Maps satisfying only the outer variation formula (2.29) are instead known as *weakly harmonic maps*. As a matter of fact, weakly harmonic maps admit far worse singularities than energy minimizing or stationary harmonic maps (see e.g. [Riv92]), except in the case $m = 2$, when there are no singularities at all (cf. [Hél91]). In this thesis we are never going to work with weakly harmonic maps.

A fundamental consequence of the variational equations is that if $u \in W^{1,2}_{loc}(\Omega, \mathcal{N})$ is stationary harmonic then for any $x \in \Omega$ the function $r \in (0, \operatorname{dist}(x, \partial\Omega)) \mapsto \mathscr{E}(u, B_r(x))$ is monotone non-decreasing. Therefore, the quantity

$$\Theta_u(x) := \lim_{r \downarrow 0} \mathscr{E}(u, B_r(x))$$

is well defined for every $x \in \Omega$. We will call $\Theta_u(x)$ the *density* of u at x. It is very easy to check that $\Theta_u \colon \Omega \to \mathbb{R}$ is an upper semi-continuous function. The monotonicity of the rescaled energy and the existence of the density at every point are key tools in the study of energy minimizing and stationary harmonic maps. The breakthrough result in the regularity theory is the following ε-regularity theorem, which is due to Schoen and Uhlenbeck [SU82] in the energy minimizing case.

Theorem 2.3.1 (ε-regularity). *There exist $\varepsilon_0 > 0$, $\alpha > 0$ and $C > 1$ depending on $m, \mathcal{N}$ with the property that if $u \in W^{1,2}_{loc}(\Omega, \mathcal{N})$ is energy minimizing in $B_{R_0}(x_0)$ with*

$$\mathscr{E}(u, B_{R_0}(x_0)) \leqslant \varepsilon_0 \,,$$

then the following energy decay estimate holds:

$$\mathscr{E}(u, B_r(x)) \leqslant C \left(\frac{r}{R}\right)^{2\alpha} \mathscr{E}(u, B_R(x)) \quad \forall\, x \in B_{\frac{R_0}{2}}(x_0)\,, \forall\, 0 < r \leqslant R \leqslant \frac{R_0}{2}.$$

In particular, $u \in C^{0,\alpha}(B_{\frac{R_0}{2}}(x_0), \mathcal{N})$.

Observe that by standard elliptic regularity theory the Hölder continuity of u in $B_{\frac{R_0}{2}}(x_0)$ upgrades to C^∞ regularity in a smaller ball. Also note that it is now a simple consequence of Theorem 2.3.1 that minimizing harmonic maps u in dimension $m = 2$ are smooth. On the other hand, concrete examples indicate that minimizing harmonic maps in dimension $m \geqslant 3$ may exhibit singularities. Indeed, it is known that for $m \geqslant 3$ the map $u_0 \colon B_1 \subset \mathbb{R}^m \to S^{m-1}$ defined by $u_0(x) := \frac{x}{|x|}$ is a minimizing harmonic map. This was proved first by Jäger-Kaul [JK83] for $m \geqslant 7$, later by Brezis-Coron-Lieb [BCL86] for $m = 3$ and finally by Lin [Lin87] and Coron-Gulliver [CG89] independently for all $m \geqslant 3$. Needless to say, u_0 is singular at the origin. Furthermore, starting from u_0 it is easy to produce, for any $m \geqslant 3$, minimizing harmonic maps $u \colon \mathbb{R}^m \to S^2$ which are singular along an $(m-3)$-dimensional linear subspace of $\mathbb{R}^m$. By the following celebrated partial regularity theorem by Schoen and Uhlenbeck [SU82], these are, in a sense, the "worst" singularities for minimizing harmonic maps.

Theorem 2.3.2 (Partial regularity for minimizing maps). *For $m \geqslant 3$, let $u \in W^{1,2}_{loc}(\Omega, \mathcal{N})$ be a minimizing harmonic map, and set*

$$\operatorname{sing}(u) := \{x \in \Omega : \, u \text{ is discontinuous at } x\}.$$

Then, $\operatorname{sing}(u)$ *is a closed set which is discrete for* $m = 3$ *and has Hausdorff dimension* $\dim_{\mathcal{H}}(\operatorname{sing}(u)) \leqslant m-3$ *for* $m \geqslant 4$. *Furthermore,* $u \in C^\infty(\Omega \setminus \operatorname{sing}(u), \mathcal{N})$.

Let us now briefly comment on the strategy to prove Theorem 2.3.2. From the ε-regularity Theorem 2.3.1 it follows that

$$\operatorname{sing}(u) = \{x \in \Omega : \Theta_u(x) \geqslant \varepsilon_0\} = \{x \in \Omega : \Theta_u(x) > 0\}.$$

Using this information, it is not difficult to prove that $\mathcal{H}^{m-2}(\operatorname{sing}(u)) = 0$. The refined dimension estimate on $\operatorname{sing}(u)$ is based on a compactness theorem for minimizing harmonic maps and on a variant of the Federer-Almgren dimension reduction argument. Let us first state the compactness theorem.

Theorem 2.3.3 (Compactness). *Let* $\{u_h\}_{h=1}^\infty \subset W^{1,2}(\Omega, \mathcal{N})$ *be a sequence of minimizing harmonic maps with* $\sup_{h \geqslant 1} \mathcal{E}(u_h, B_r(x)) < \infty$ *for each ball* $B_r(x) \Subset \Omega$. *Then, there is a subsequence* u_{h_j} *and a minimizing harmonic map* $u \in W^{1,2}(\Omega, \mathcal{N})$ *such that* $u_{h_j} \to u$ strongly *in* $W^{1,2}$ *in every* $B_r(x) \Subset \Omega$.

Theorem 2.3.3 allows to prove the following proposition.

Proposition 2.3.4. *Let* $u \in W^{1,2}_{loc}(\Omega, \mathcal{N})$ *be energy minimizing, and let* $x_0 \in \Omega$.

(1) *For any sequence* r_h *of radii with* $r_h \downarrow 0$ *there exists a subsequence* r_{h_j} *such that the maps* $T^u_{x_0, r_{h_j}}(y) := u(x_0 + r_{h_j} y)$ *converge strongly in* $W^{1,2}_{loc}(\mathbb{R}^m, \mathcal{N})$ *to a minimizing harmonic map* ϕ;

(2) ϕ *is homogeneous of degree zero, and thus*

$$\Theta_u(x_0) = \Theta_\phi(0) = \mathcal{E}(\phi, B_\rho(0)) \, \forall \, \rho > 0;$$

(3)
$$\Theta_\phi(0) = \max_{y \in \mathbb{R}^m} \Theta_\phi(y);$$

(4) *The set*
$$S(\phi) := \{y \in \mathbb{R}^m : \Theta_\phi(y) = \Theta_\phi(0)\}$$
is a linear subspace of $\mathbb{R}^m$. *Moreover,* ϕ *is invariant under composition with translations by elements in* $S(\phi)$, *i.e.* $\phi(x+y) = \phi(x)$ *for every* $x \in \mathbb{R}^m$, *for every* $y \in S(\phi)$.

Any map ϕ arising as in (1) as a limit of the maps $T^u_{x_0, r_j}$ for some sequence $r_j \downarrow 0$ is called a *tangent map* to u at x_0. Note that tangent maps to u at a given point x_0 may not be unique. Nonetheless, every tangent map ϕ to u at x_0 is a zero-homogeneous minimizer for which (2), (3), and (4) above hold. If ϕ is a tangent map, then $S(\phi)$ is called its *spine*. The dimension of $S(\phi)$ is the number of independent directions along which ϕ is invariant. Now, if $x_0 \in \operatorname{reg}(u) := \Omega \setminus \operatorname{sing}(u)$, then $\Theta_u(x_0) = 0$, and thus, by (2) and (3), u has a *constant* tangent map at x_0. If instead $x_0 \in \operatorname{sing}(u)$ then for every tangent map ϕ to u at x_0 it holds $\Theta_\phi(0) = \Theta_u(x_0) > 0$, and thus $0 \in \operatorname{sing}(\phi) \neq \emptyset$. Furthermore, the spine $S(\phi)$ is a subset of the singular set $\operatorname{sing}(\phi)$. Since $S(\phi)$ is a linear subspace, and since

$\mathcal{H}^{m-2}(\mathrm{sing}(\phi)) = 0$ because ϕ is minimizing, it immediately follows that $\dim S(\phi) \leqslant m-3$. It is now possible to *stratify* the singular set $\mathrm{sing}(u)$ of the minimizing harmonic map u we started with according to the number of *symmetries* a tangent map has at each point. Specifically, for every $0 \leqslant k \leqslant m-3$ we set

$$\begin{aligned} \mathcal{S}^k(u) :&= \{x \in \mathrm{sing}(u) : \ \dim S(\phi) \leqslant k \text{ for all tangent maps } \phi \text{ to } u \text{ at } x\} \\ &= \{x \in \mathrm{sing}(u) : \ \text{no tangent map to } u \text{ at } x \text{ is invariant along } k+1 \text{ directions}\}. \end{aligned} \tag{2.33}$$

Then, one has

$$\mathcal{S}^0(u) \subset \mathcal{S}^1(u) \subset \ldots \subset \mathcal{S}^{m-3}(u) = \mathrm{sing}(u),$$

and the estimate on the Hausdorff dimension of $\mathrm{sing}(u)$ follows from the Federer-Almgren dimension reduction argument:

Lemma 2.3.5 (Federer-Almgren dimension reduction argument)**.** *For $0 \leqslant k \leqslant m-3$, one has $\dim_{\mathcal{H}}(\mathcal{S}^k(u)) \leqslant k$.*

As for stationary harmonic maps, the situation is more involved, and we are not going further into the details. Let us only mention that the ε-regularity Theorem 2.3.1 was extended to stationary harmonic maps by Bethuel [Bet93] and later improved by Rivière and Struwe in [RS08]. As a consequence, the following theorem holds.

Theorem 2.3.6 (Partial regularity for stationary harmonic maps)**.** *For $m \geqslant 3$, if $u \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathcal{N})$ is stationary harmonic then $\mathcal{H}^{m-2}(\mathrm{sing}(u)) = 0$ and $u \in C^\infty(\Omega \setminus \mathrm{sing}(u), \mathcal{N})$.*

2.3.2 Fine properties of the singular set

Beyond the dimension estimate in Lemma 2.3.5, little else was known about the structure of the singular strata $\mathcal{S}^k(u)$ until very recently. In [Sim95a], Simon proved that if the target $\mathcal{N}$ is analytic then the singular set $\mathrm{sing}(u) = \mathcal{S}^{m-3}(u)$ is countably $(m-3)$-rectifiable. The breakthrough in this direction was made by Naber and Valtorta in the pioneering work [NV17], where they were able to prove that $\mathcal{S}^k(u)$ is countably k-rectifiable for any k whenever u is stationary harmonic and under minimal assumptions on the regularity of the target manifold $\mathcal{N}$.

Theorem 2.3.7 (Stratification for stationary harmonic maps, cf. [NV17, Theorem 1.5])**.** *Let $u\colon B_2 \subset \mathbb{R}^m \to \mathcal{N}$ be a stationary harmonic map with $\mathscr{E}(u, B_2) \leqslant \Lambda$. Then for every k the singular stratum $\mathcal{S}^k(u)$ is k-rectifiable. Furthermore, for $\mathcal{H}^k$-a.e. $x \in \mathcal{S}^k(u)$ there exists a unique k-dimensional linear subspace $V^k \subset \mathbb{R}^m$ such that every tangent map to u at x is invariant with respect to compositions with translations by vectors in V^k.*

If u is energy minimizing, then the result can be improved: indeed, not only one has that $\mathrm{sing}(u) = \mathcal{S}^{m-3}(u)$ is countably $(m-3)$-rectifiable, but also that it has uniformly finite $(m-3)$-dimensional Hausdorff measure. More precisely, the following theorem holds.

Theorem 2.3.8 (Fine structure of the singular set, cf. [NV17, Theorem 1.6]). *Let* $u\colon B_2 \subset \mathbb{R}^m \to \mathcal{N}$ *be a minimizing harmonic map with* $\mathscr{E}(u, B_2) \leqslant \Lambda$. *Then,* $\mathrm{sing}(u)$ *is countably* $(m-3)$*-rectifiable and there exists* $C = C(m, \mathcal{N}, \Lambda)$ *such that, denoting* $B_r(\mathrm{sing}(u)) := \bigcup_{x\in\mathrm{sing}(u)} B_r(x)$ *the* r*-tubular neighborhood of* $\mathrm{sing}(u)$ *in* $\mathbb{R}^m$*, one has*

$$\mathcal{L}^m\left(B_r(\mathrm{sing}(u)) \cap B_1\right) \leqslant Cr^3\,.$$

In particular, $\mathcal{H}^{m-3}(\mathrm{sing}(u) \cap B_1) \leqslant C$.

Theorems 2.3.7 and 2.3.8 were in fact obtained as corollaries of more general statements on the *quantitative stratification* for stationary harmonic maps, and not on the standard stratification itself. A first version of the quantitative stratification can be found in [Almoo, §2.25]; the concept was later developed by Cheeger and Naber first in [CN13a] in order to prove new estimates on the singular set of Gromov-Hausdorff limits of non-collapsed manifolds with Ricci curvature bounded below, and then in [CN13b] to obtain new L^p estimates on the second derivatives of minimizing harmonic maps and on the second fundamental form of area minimizing integral currents in codimension one. Since then, the quantitative stratification has appeared in several works to obtain similar results in different contexts of Geometric Analysis, including mean curvature flow, critical sets of elliptic equations, harmonic map flow among others.

Before explaining what the quantitative stratification is, we need to discuss the notion of symmetry associated to a harmonic map.

Notation 2.3.9. We will use the notation ${}^{[CN]}\mathcal{S}^k_{\varepsilon,r}(u)$ to denote the standard quantitative stratification à la Cheeger-Naber, in order to distinguish it from the new notion of quantitative stratification $\mathcal{S}^k_{\varepsilon,r}(u)$ that we will use in our discussion on multiple-valued harmonic maps in Chapter 10. The notion of quantitative stratification we will propose, although naturally inspired by the original Cheeger-Naber one, will allow us to obtain a slightly better control on the different strata. A careful comparison between ${}^{[CN]}\mathcal{S}^k_{\varepsilon,r}(u)$ and $\mathcal{S}^k_{\varepsilon,r}$ is carried on in § 10.2.

Notation 2.3.10. For any $x \in \mathbb{R}^m$, we shall denote by r_x the radial unit vector field with respect to x, defined by

$$r_x(y) := \frac{y-x}{|y-x|} \quad \text{for every } y \in \mathbb{R}^m \setminus \{x\}.$$

Definition 2.3.11 (k-symmetric maps, cf. [NV17, Definition 1.1]). A map $h \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^m, \mathcal{N})$ is said to be:

- *homogeneous* with respect to $x \in \mathbb{R}^m$ if

$$h(x+\lambda v) = h(x+v) \quad \text{for all } \lambda > 0, \text{ for every } v \in \mathbb{R}^m,$$

  or equivalently if

$$D_{r_x} h = 0 \quad \text{a.e. in } \mathbb{R}^m.$$

- k-*symmetric* if it is homogeneous with respect to the origin and there exists a linear subspace $L \subset \mathbb{R}^m$ with $\dim(L) = k$ along which h is invariant, that is

$$h(x+v) = h(x) \quad \text{for every } x \in \mathbb{R}^m, \text{ for all } v \in L,$$

or, equivalently, such that

$$D_v h(x) = 0 \quad \text{for a.e. } x \in \mathbb{R}^m, \text{ for all } v \in L.$$

Definition 2.3.12 ((k,ε)-symmetric balls, cf. [NV17, Definition 1.1])**.** Let $u \in W^{1,2}_{loc}(\Omega, \mathcal{N})$, and fix $k \in \{0,\dots,m\}$ and $\varepsilon > 0$. A ball $B_r(x) \subset \Omega$ is said to be (k,ε)-symmetric for u *in the sense of Cheeger-Naber*, or briefly [CN] (k,ε)-symmetric, if there exists some k-symmetric map $h \in W^{1,2}_{loc}(\mathbb{R}^m, \mathcal{N})$ such that

$$\fint_{B_r(x)} |u(y) - h(y-x)|^2 \, dy \leqslant \varepsilon. \tag{2.34}$$

Definition 2.3.13 ([CN] Quantitative stratification)**.** Let $u \in W^{1,2}_{loc}(\Omega, \mathcal{N})$ be stationary harmonic, and let $\varepsilon, r > 0$ and $k \in \{0,\dots,m\}$. We will set

$${}^{[CN]}\mathcal{S}^k_{\varepsilon,r}(u) := \{x \in \Omega : \text{ for no } r \leqslant s < 1 \text{ the ball } B_s(x) \text{ is [CN] } (k+1,\varepsilon)\text{-symmetric w.r.t. } u\}.$$

It is an immediate consequence of the definition that if $k' \leqslant k$, $\varepsilon' \geqslant \varepsilon$ and $r' \leqslant r$ then

$${}^{[CN]}\mathcal{S}^{k'}_{\varepsilon',r'}(u) \subseteq {}^{[CN]}\mathcal{S}^k_{\varepsilon,r}(u).$$

Hence, we can set:

$${}^{[CN]}\mathcal{S}^k_\varepsilon(u) := \bigcap_{r>0} {}^{[CN]}\mathcal{S}^k_{\varepsilon,r}(u), \qquad {}^{[CN]}\mathcal{S}^k(u) := \bigcup_{\varepsilon>0} {}^{[CN]}\mathcal{S}^k_\varepsilon(u).$$

Remark 2.3.14. It is of great importance to observe that the set ${}^{[CN]}\mathcal{S}^k(u)$ coincides with the standard singular stratum $\mathcal{S}^k(u)$ as defined in (2.33), cf. [NV17, Section 9.3].

We are now ready to state the results of [NV17] concerning the quantitative stratification.

Theorem 2.3.15 (Quantitative stratification for stationary harmonic maps, cf. [NV17, Theorems 1.3 and 1.4])**.** *Let $u\colon B_2 \subset \mathbb{R}^m \to \mathcal{N}$ be a stationary harmonic map with $\mathscr{E}(u, B_2) \leqslant \Lambda$. Then, for any $\varepsilon > 0$ there exists $C = C(m, \mathcal{N}, \Lambda, \varepsilon)$ such that the following k-dimensional Minkowski content estimates hold:*

$$\mathcal{L}^m\left(B_r\left({}^{[CN]}\mathcal{S}^k_{\varepsilon,r}(u)\right) \cap B_1\right) \leqslant C_\varepsilon r^{m-k} \tag{2.35}$$

and

$$\mathcal{L}^m\left(B_r\left({}^{[CN]}\mathcal{S}^k_{\varepsilon}(u)\right) \cap B_1\right) \leqslant C_\varepsilon r^{m-k}. \tag{2.36}$$

In particular, $\mathcal{H}^k\left({}^{[CN]}\mathcal{S}^k_\varepsilon(u) \cap B_1\right) \leqslant C_\varepsilon$. Moreover, ${}^{[CN]}\mathcal{S}^k_\varepsilon(u)$ is countably k-rectifiable, and for $\mathcal{H}^k$-a.e. $x \in {}^{[CN]}\mathcal{S}^k_\varepsilon(u)$ there exists a unique k-dimensional linear subspace $V^k \subset \mathbb{R}^m$ such that every tangent map to u at x is invariant with respect to compositions with translations by vectors in V^k.

The core of Theorem 2.3.15 is represented by the Minkowski bounds (2.35) and (2.36). For the moment, let us just mention that the two new main ingredients exploited in [NV17] in order to get the estimates above are new Reifenberg-type results and a new L^2-subspace approximation theorem for stationary harmonic maps. We will not enter further into the details, since we will present these tools "directly in action" in the context of multiple-valued minimizing harmonic maps when we need them. Let us also point out that the tools introduced in [NV17] are so robust that they have been already fruitfully applied in several different areas: we mention here the recent preprints [NV15], dealing with the quantitative stratification of integral varifolds with bounded mean curvature, [NV16], which extends (and simplifies) the theory of [NV17] to *approximate* harmonic maps, and [DMSV16], where the rectifiability of the singular set of multiple-valued Dir-minimizers is discussed. A Reifenberg-type result for general non-negative Borel measures can be instead found in [ENV16].

2.3.3 Multiple-valued harmonic maps

As anticipated, in this paragraph, the last containing preparatory material, we are going to provide a brief overview of the notion and properties of *multiple-valued* harmonic maps. The main reference is [Hir16b], where Hirsch introduces the notion of multiple-valued harmonic maps and develops a parallel theory to the one presented in § 2.3.1.

For $\Omega \subset \mathbb{R}^m$ open, $\mathcal{N}^n \hookrightarrow \mathbb{R}^d$ compact Riemannian manifold and $Q \geqslant 1$ integer, we define

$$W^{1,2}(\Omega, \mathcal{A}_Q(\mathcal{N})) := \left\{ u \in W^{1,2}(\Omega, \mathcal{A}_Q(\mathbb{R}^d)) : \operatorname{spt}(u(x)) \subset \mathcal{N} \text{ for a.e. } x \in \Omega \right\},$$

and $W^{1,2}_{loc}(\Omega, \mathcal{A}_Q(\mathcal{N}))$ accordingly. If $u \in W^{1,2}_{loc}(\Omega, \mathcal{A}_Q(\mathcal{N}))$ and $B_r(x) \Subset \Omega$, we set

$$\mathscr{E}(u, B_r(x)) := r^{2-m} \mathrm{Dir}(u, B_r(x)) = r^{2-m} \int_{B_r(x)} |Du(y)|^2 \, dy .$$

Definition 2.3.16 (Q-valued energy minimizers, cf. [Hir16b, Definition 1.1])**.** A map $u \in W^{1,2}_{loc}(\Omega, \mathcal{A}_Q(\mathcal{N}))$ is a *local minimizer*, or simply *minimizer*, of the Dirichlet energy if for any $B_r(x) \Subset \Omega$ it holds

$$\mathscr{E}(u, B_r(x)) \leqslant \mathscr{E}(v, B_r(x)) \tag{2.37}$$

for every $v \in W^{1,2}_{loc}(\Omega, \mathcal{A}_Q(\mathcal{N}))$ such that $v \equiv u$ in a neighborhood of $\partial B_r(x)$.

As a consequence of the minimality condition, Q-valued minimizers satisfy inner variation and outer variation formulae, which we record in the following proposition.

Proposition 2.3.17 (Variational equations, cf. [Hir16b, Equations (2.2) and (2.5)])**.** *Let* $u = \sum_\ell \llbracket u_\ell \rrbracket \in W^{1,2}_{loc}(\Omega, \mathcal{A}_Q(\mathcal{N}))$ *be energy minimizing, and assume* $B_r(x) \Subset \Omega$*. Then, for every vector field* $X = (X^1, \dots, X^m) \in C^1_c(B_r(x), \mathbb{R}^m)$ *the following* inner variation formula *holds:*

$$\int_{B_r(x)} \sum_{i,j=1}^{m} \left(|Du|^2 \delta_{ij} - 2 \sum_{\ell=1}^{Q} \langle D_i u_\ell, D_j u_\ell \rangle \right) D_i X^j \, dy = 0. \tag{2.38}$$

Moreover, for any vector field $Y \in C^1(B_r(x) \times \mathbb{R}^d, \mathbb{R}^d)$ *such that* $Y(y,p) = 0$ *for* y *in a neighborhood of* $\partial B_r(x)$ *we have the following* outer variation formula*:*

$$\int_{B_r(x)} \sum_{i=1}^{m} \sum_{\ell=1}^{Q} \left(\langle D_i u_\ell, D_i(Y(y,u_\ell)) \rangle - \langle A_{u_\ell}(D_i u_\ell, D_i u_\ell), Y(y,u_\ell) \rangle \right) dy = 0. \tag{2.39}$$

In analogy with the classical case, we will call *stationary* Q*-harmonic* any map u in $W^{1,2}_{loc}(\Omega, \mathcal{A}_Q(\mathcal{N}))$ for which both equations (2.38) and (2.39) hold. As a consequence of the inner variation formula (2.38), one classically recovers the monotonicity of the function $r \in (0, \operatorname{dist}(x, \partial\Omega)) \mapsto \mathscr{E}(u, B_r(x))$ and, therefore, the existence of the density

$$\Theta_u(x) := \lim_{r \downarrow 0} \mathscr{E}(u, B_r(x))$$

at every $x \in \Omega$ whenever u is a stationary Q-harmonic map. Furthermore, Q-valued minimizers enjoy the following compactness theorem, which extends Theorem 2.3.3 to the multiple-valued context.

Theorem 2.3.18 (Compactness, cf. [Hir16b, Lemma 4.1])**.** *Let* $\{u_h\}_{h=1}^{\infty} \subset W^{1,2}(\Omega, \mathcal{A}_Q(\mathcal{N}))$ *be a sequence of* Q*-valued minimizing harmonic maps with* $\sup_{h \geqslant 1} \mathscr{E}(u_h, B_r(x)) < \infty$ *for each ball* $B_r(x) \Subset \Omega$*. Then, there is a subsequence* u_{h_j} *and a minimizing harmonic map* $u \in W^{1,2}(\Omega, \mathcal{A}_Q(\mathcal{N}))$ *such that*

(i) $\lim_{j\to\infty} \int_\Omega \mathcal{G}(u_{h_j}, u)^2 \, dy = 0$;

(ii) $\lim_{j\to\infty} \mathscr{E}(u_{h_j}, B_r(x)) = \mathscr{E}(u, B_r(x))$ *for every ball* $B_r(x) \Subset \Omega$.

As a consequence, Proposition 2.3.4 holds modulo replacing the target $\mathcal{N}$ with $\mathcal{A}_Q(\mathcal{N})$ whenever it occurs. If $x_0 \in \Omega$ then every tangent map to u at x_0 is a Q-valued map $\phi \in W^{1,2}_{loc}(\mathbb{R}^m, \mathcal{A}_Q(\mathcal{N}))$ which is homogeneous of degree zero with respect to $0 \in \mathbb{R}^m$. Further, $\Theta_\phi(0) = \Theta_u(x_0)$, and the (upper semi-continuous) map $y \in \mathbb{R}^m \mapsto \Theta_\phi(y)$ attains its maximum at $y = 0$. The spine $S(\phi)$ is a linear subspace of $\mathbb{R}^m$ with respect to which ϕ is invariant. Extending the terminology introduced in § 2.3.2, if $\dim S(\phi) = k$ then ϕ is a k-symmetric map with invariance space given by $S(\phi)$.

Before proceeding with the regularity theory for Q-valued minimizing harmonic maps, we need to discuss the different notions of singularities that can be taken into consideration in this context.

Definition 2.3.19 (Regular and singular sets, cf. [Hir16b, Definitions 1.2 and 1.4])**.** An energy minimizing map $u \in W^{1,2}_{loc}(\Omega, \mathcal{A}_Q(\mathcal{N}))$ is *regular* at a point $x \in \Omega$ if there exist a neighborhood $\mathcal{U}$ of x in Ω and Q *smooth* minimizing harmonic maps $u_\ell : \mathcal{U} \to \mathcal{N}$ such that

$$u(y) = \sum_{\ell=1}^{Q} [\![u_\ell(y)]\!] \quad \text{for a.e. } y \in \mathcal{U}.$$

If u is regular at x then we write $x \in \operatorname{reg}(u)$. The singular set of u is the set $\operatorname{sing}(u) := \Omega \setminus \operatorname{reg}(u)$.

If $x \in \mathrm{reg}(u)$, then a fortiori u is Hölder continuous in a neighborhood of x. Hence, one can regard $\mathrm{reg}(u)$ as a subset of a larger set, called the *Hölder regular set* of u and defined by

$$\mathrm{reg}_H(u) := \{x \in \Omega : \ u \text{ is Hölder continuous in a neighborhood of } y\} \supset \mathrm{reg}(u)\,.$$

The *Hölder singular set* of u is then defined by $\mathrm{sing}_H(u) := \Omega \setminus \mathrm{reg}_H(u)$. Observe that $\mathrm{sing}_H(u) \subset \mathrm{sing}(u)$.

Remark 2.3.20. Note that trivially one has $\mathrm{reg}_H(u) = \mathrm{reg}(u)$ and $\mathrm{sing}_H(u) = \mathrm{sing}(u)$ if $Q = 1$. Also observe that for Dir-minimizers, i.e. for Q-valued energy minimizers with target $\mathbb{R}^n$, one has $\mathrm{sing}_H(u) = \emptyset$ and $\dim_{\mathcal{H}}(\mathrm{sing}(u)) \leqslant m-2$ by Theorems 2.2.21 and 2.2.23 respectively. Of course, if the target is a manifold then in general $\mathrm{sing}_H(u)$ is not empty, as it is not empty even in the classical $Q = 1$ case.

The results about tangent maps described above allow to apply the standard stratification to the set $\mathrm{sing}_H(u)$, and classically define, for $0 \leqslant k \leqslant m$,

$$\mathcal{S}^k(u) := \left\{x \in \mathrm{sing}_H(u) : \ \dim S(\phi) \leqslant k \text{ for all tangent maps } \phi \text{ to } u \text{ at } x\right\}\,.$$

The last ingredient to complete the regularity theory for Q-valued minimizers is then clearly only the Schoen-Uhlenbeck ε-regularity result, which is indeed the core of [Hir16b].

Theorem 2.3.21 (Q-valued ε-regularity, cf. [Hir16b, Lemma 5.2])**.** *There exist $\varepsilon_0 > 0$, $\alpha > 0$ and $C > 1$ depending on $m, \mathcal{N}, Q$ with the property that if $u \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathcal{A}_Q(\mathcal{N}))$ is energy minimizing in $B_{R_0}(x_0)$ with*

$$\mathcal{E}(u, B_{R_0}(x_0)) \leqslant \varepsilon_0\,,$$

then the following energy decay estimate holds:

$$\mathcal{E}(u, B_r(x)) \leqslant C\left(\frac{r}{R}\right)^{2\alpha} \mathcal{E}(u, B_R(x)) \quad \forall\, x \in B_{\frac{R_0}{2}}(x_0)\,, \forall\, 0 < r \leqslant R \leqslant \frac{R_0}{2}.$$

In particular, $u \in C^{0,\alpha}(B_{\frac{R_0}{2}}(x_0), \mathcal{A}_Q(\mathcal{N}))$.

Theorem 2.3.21 implies that $\mathrm{sing}_H(u) = \{\Theta_u(x) > 0\}$, and thus that $\mathcal{H}^{m-2}(\mathrm{sing}_H(u)) = 0$. The condition $x \in \mathrm{reg}_H(u)$ is equivalent to u having a constant tangent map at x. On the other hand, if $x \in \mathrm{sing}_H(u)$ then every tangent map ϕ to u at x has a non-empty Hölder singular set $\mathrm{sing}_H(\phi) \supset S(\phi)$. Since $S(\phi)$ is a linear subspace and $\mathcal{H}^{m-2}(\mathrm{sing}_H(\phi)) = 0$ it is necessarily $\dim S(\phi) \leqslant m-3$, and thus $\mathrm{sing}_H(u) = \mathcal{S}^{m-3}(u)$. The Federer-Almgren dimension reduction argument then allows to conclude the following theorem.

Theorem 2.3.22 (Partial regularity for Q-valued minimizers, cf. [Hir16b, Theorem 0.1])**.** *If $u \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathcal{A}_Q(\mathcal{N}))$ is energy minimizing then $\mathrm{sing}_H(u)$ is a closed set having Hausdorff dimension $\dim_{\mathcal{H}}(\mathrm{sing}_H(u)) \leqslant m-3$.*

3 A TECHNICAL TOOL: MULTIPLE-VALUED PUSH-FORWARDS

In this chapter we study an important technical tool, of which we will often make use in the coming chapters: the extension of the push-forward operator introduced in § 2.1.2 to multiple-valued functions. Multiple-valued push-forwards were already considered by Almgren in his monumental Big Regularity Paper [Almoo], and later revisited by De Lellis and Spadaro in [DS15], and most of the results here presented have a counterpart in there. Our contribution is mainly to present a homogeneous treatment of the subject and to simplify the arguments. In particular, in Section 3.1 we quote some elementary definitions and results from [DS15] in order to fix notation and terminology related to the subject. In Section 3.2 we provide a slightly simplified proof of the fact that the multi-valued push-forward operator acting on Lipschitz manifolds commutes with the boundary operator. Finally, in Section 3.3 we extend the multi-valued push-forward operator to integral flat chains. This is the most original part of the chapter: indeed, in order to extend the push-forward operator to the class $\mathscr{F}_m(\mathbb{R}^d)$ Almgren relies on the intersection theory of flat chains. Our approach, instead, makes use only of the polyhedral approximation results discussed in § 2.1.4. The material covered in this chapter is taken from our paper [Stu17b].

3.1 THE PUSH-FORWARD OF RECTIFIABLE CURRENTS. GRAPHS

Let $\Omega \subset \mathbb{R}^d$ be an open set, and let $f\colon \Omega \to \mathbb{R}^n$ be smooth and proper. We have seen in § 2.1.2 that if $T \in \mathcal{D}_m(\Omega)$ then the push-forward of T through f is the current $f_\sharp T \in \mathcal{D}_m(\mathbb{R}^n)$ defined by

$$f_\sharp T(\omega) := T(f^\sharp \omega) \quad \text{for every } \omega \in \mathcal{D}^m(\mathbb{R}^n).$$

Now, if $T = [\![B, \vec{\tau}, \theta]\!]$ is rectifiable then it is straightforward to verify that the push-forward $f_\sharp T$ is given explicitly by

$$f_\sharp T(\omega) = \int_B \langle \omega(f(x)), Df(x)_\sharp \vec{\tau}(x)\rangle\, \theta(x)\, d\mathcal{H}^m(x) \qquad \forall\, \omega \in \mathcal{D}^m(\mathbb{R}^n),$$

where

$$Df(x)_\sharp \vec{\tau}(x) := (Df(x)\cdot \tau_1(x)) \wedge \cdots \wedge (Df(x)\cdot \tau_m(x)).$$

The hypotheses on f can in fact be relaxed, as the above formula makes sense whenever $f\colon B \to \mathbb{R}^n$ is Lipschitz and proper. In this case, $Df(x)$ has to be regarded as the tangent map of f at x, which exists at $\mathcal{H}^m$-a.e. $x \in B$ since B is rectifiable and f is Lipschitz. Furthermore, since $|Df(x)_\sharp \vec{\tau}(x)|$ coincides with the Jacobian determinant

$$\mathbf{J}f(x) := \sqrt{\det\left((Df(x))^T \cdot Df(x)\right)},$$

from the area formula it follows that

$$f_\sharp T(\omega) = \int_{f(B)} \left\langle \omega(y), \sum_{x\in B_+ :\, f(x)=y} \theta(x) \frac{Df(x)_\sharp \vec{\tau}(x)}{|Df(x)_\sharp \vec{\tau}(x)|} \right\rangle d\mathcal{H}^m(y),$$

with $B_+ := \{x \in B : \mathbf{J}f(x) > 0\}$. Moreover, $f(B)$ is an m-rectifiable subset of $\mathbb{R}^n$, and for $\mathcal{H}^m$-a.e. $y \in f(B)$ one has

$$\frac{Df(x)_\sharp \vec{\tau}(x)}{|Df(x)_\sharp \vec{\tau}(x)|} = \pm \vec{\eta}(y)$$

for all $x \in B_+$ such that $f(x) = y$, where $\vec{\eta}(y) = \eta_1(y) \wedge \cdots \wedge \eta_m(y)$ is a simple unit m-vector orienting $\mathrm{Tan}(f(B), y)$. It follows that $f_\sharp T$ is a rectifiable m-current in $\mathbb{R}^n$, and in fact $f_\sharp T = [\![f(B), \vec{\eta}, \Theta]\!]$, with

$$\Theta(y) := \sum_{x\in B_+ :\, f(x)=y} \theta(x) \left\langle \vec{\eta}(y), \frac{Df(x)_\sharp \vec{\tau}(x)}{|Df(x)_\sharp \vec{\tau}(x)|} \right\rangle.$$

We discuss now how to extend the above results to the context of multiple-valued functions. The Lipschitz selection property, already recalled in Proposition 2.2.7, plays a fundamental role in achieving the goal.

The first step is to define the push-forward of C^1 submanifolds. Hence, in what follows we will assume that Σ is an m-dimensional C^1 submanifold of $\mathbb{R}^d$, and $B \subset \Sigma$ is $\mathcal{H}^m$-measurable. We will also assume that Σ is oriented with orientation $\vec{\tau}$.

Definition 3.1.1 (Proper Q-valued functions, cf. [DS15, Definition 1.2])**.** A measurable function $u\colon B \subset \Sigma \to \mathcal{A}_Q(\mathbb{R}^n)$ is *proper* if there exists a measurable selection $u = \sum_{\ell=1}^Q [\![u_\ell]\!]$ such that the set $\bigcup_{\ell=1}^Q \overline{u_\ell^{-1}(K)}$ is compact for any compact $K \subset \mathbb{R}^n$. If such a selection exists, then clearly the same property is indeed satisfied by every measurable selection.

Definition 3.1.2 (Q-valued push-forward, cf. [DS15, Definition 1.3])**.** Let $B \subset \Sigma$ be as above, and let $u\colon B \to \mathcal{A}_Q(\mathbb{R}^n)$ be Lipschitz and proper. Then, the *push-forward* of B through u is the current $\mathbf{T}_u := \sum_{i\in\mathbb{N}} \sum_{\ell=1}^Q (u_i^\ell)_\sharp [\![B_i]\!]$, where B_i and u_i^ℓ are as in Proposition 2.2.7: that is,

$$\mathbf{T}_u(\omega) := \sum_{i\in\mathbb{N}} \sum_{\ell=1}^Q \int_{B_i} \left\langle \omega(u_i^\ell(x)), Du_i^\ell(x)_\sharp \vec{\tau}(x) \right\rangle d\mathcal{H}^m(x) \quad \forall\, \omega \in \mathcal{D}^m(\mathbb{R}^n). \tag{3.1}$$

Using the classical results concerning the push-forward of integer rectifiable currents through (single valued) proper Lipschitz functions recalled above and the properties of Lipschitz selections, it is not difficult to conclude the validity of the following proposition.

Proposition 3.1.3 (Representation of the push-forward, cf. [DS15, Proposition 1.4])**.** *The definition of the action of* $\mathbf{T}_u$ *in* (3.1) *does not depend on the chosen partition* B_i*, nor on the chosen decomposition* $\{u_i^\ell\}$*. If* $u = \sum_\ell [\![u_\ell]\!]$*, we are allowed to write*

$$\mathbf{T}_u(\omega) = \int_B \sum_{\ell=1}^Q \left\langle \omega(u_\ell(x)), Du_\ell(x)_\sharp \vec{\tau}(x) \right\rangle d\mathcal{H}^m(x) \quad \forall\, \omega \in \mathcal{D}^m(\mathbb{R}^n). \tag{3.2}$$

Thus, $\mathbf{T}_u$ *is a (well-defined) locally integer rectifiable* m*-current in* $\mathbb{R}^n$ *given by* $\mathbf{T}_u = [\![\mathrm{Im}(u), \vec{\eta}, \Theta]\!]$*, where:*

(R1) $\mathrm{Im}(u) = \bigcup_{x\in B} \mathrm{spt}(u(x)) = \bigcup_{i\in\mathbb{N}} \bigcup_{\ell=1}^{Q} u_i^\ell(B_i)$ *is an* m*-rectifiable set in* $\mathbb{R}^n$*;*

(R2) $\vec{\eta}$ *is a Borel unit* m*-vector field orienting* $\mathrm{Im}(u)$*; moreover, for* $\mathcal{H}^m$*-a.e.* $y \in \mathrm{Im}(u)$*, we have* $Du_i^\ell(x)_\sharp \vec{\tau}(x) \neq 0$ *for every* i, ℓ, x *such that* $u_i^\ell(x) = y$ *and*

$$\vec{\eta}(y) = \pm \frac{Du_i^\ell(x)_\sharp \vec{\tau}(x)}{|Du_i^\ell(x)_\sharp \vec{\tau}(x)|}; \tag{3.3}$$

(R3) *for* $\mathcal{H}^m$*-a.e.* $y \in \mathrm{Im}(u)$*, the (Borel) multiplicity function* Θ *equals*

$$\Theta(y) = \sum_{i,\ell,x\,:\, u_i^\ell(x)=y} \left\langle \vec{\eta}(y), \frac{Du_i^\ell(x)_\sharp \vec{\tau}(x)}{|Du_i^\ell(x)_\sharp \vec{\tau}(x)|} \right\rangle. \tag{3.4}$$

Remark 3.1.4. The definition of push-forward can be easily extended to the case when the domain Σ is a *Lipschitz* oriented m-dimensional submanifold. In this case, indeed, there are countably many submanifolds Σ_j of class C^1 which cover $\mathcal{H}^m$-a.a. Σ, and such that the orientations of Σ_j and Σ coincide on their intersection (see [Sim83b, Theorem 5.3]). Hence, if $B \subset \Sigma$ is a measurable subset and $u\colon B \to \mathcal{A}_Q(\mathbb{R}^n)$ is Lipschitz and proper, then the push-forward of $[\![B]\!]$ through u can be defined to be the integer rectifiable current $\mathbf{T}_u := \sum_{j=1}^\infty \mathbf{T}_{u_j}$, where $u_j := u|_{B\cap\Sigma_j}$. All the conclusions of Proposition 3.1.3 remain valid in this context (cf. [DS15, Lemma 1.7]). Furthermore, the push-forward is invariant with respect to bi-Lipschitz homeomorphisms: if $u\colon \Sigma \to \mathcal{A}_Q(\mathbb{R}^n)$ is Lipschitz and proper, $\phi\colon \tilde{\Sigma} \to \Sigma$ is bi-Lipschitz and $\tilde{u} := u \circ \phi$, then $\mathbf{T}_{\tilde{u}} = \mathbf{T}_u$.

The following Q-valued area formula is a fundamental tool to compute the mass of $\mathbf{T}_u$. If $u = \sum_\ell [\![u_\ell]\!]$ is Lipschitz, we will denote by $\mathbf{J}u_\ell(x)$ the Jacobian determinant of Du_ℓ, i.e. the number

$$\mathbf{J}u_\ell(x) := |Du_\ell(x)_\sharp \vec{\tau}(x)| = \sqrt{\det\left((Du_\ell(x))^T \cdot Du_\ell(x)\right)}. \tag{3.5}$$

Proposition 3.1.5 (Q-valued area formula, cf. [DS15, Lemma 1.9])**.** *Let* B *be a measurable subset of a Lipschitz oriented* m*-dimensional submanifold* $\Sigma \subset \mathbb{R}^d$*, and let* $u = \sum_\ell [\![u_\ell]\!]\colon B \to \mathcal{A}_Q(\mathbb{R}^n)$ *be a Lipschitz and proper* Q*-valued function. Then, for every Borel function* $h\colon \mathbb{R}^n \to [0,\infty)$*, we have*

$$\int h(p)\, d\|\mathbf{T}_u\|(p) \leqslant \int_B \sum_{\ell=1}^{Q} h(u_\ell(x))\mathbf{J}u_\ell(x)\, d\mathcal{H}^m(x). \tag{3.6}$$

Equality holds in (3.6) *if there is a set* $B' \subset B$ *of full* $\mathcal{H}^m$*-measure for which*

$$\langle Du_\ell(x)_\sharp \vec{\tau}(x), Du_{\ell'}(y)_\sharp \vec{\tau}(y)\rangle \geqslant 0 \quad \forall\, x, y \in B' \text{ and } \ell, \ell' \text{ with } u_\ell(x) = u_{\ell'}(y). \tag{3.7}$$

The notion of push-forward allows one to associate a rectifiable current to the *graph* of a multiple-valued function. Here and in the sequel, if $\Sigma \subset \mathbb{R}^d$ is an m-dimensional Lipschitz submanifold and $u\colon B \subset \Sigma \to \mathcal{A}_Q(\mathbb{R}^n)$ is a Q-valued map we will denote by $\mathrm{Gr}(u)$ the *set-theoretical graph* of u, given by

$$\mathrm{Gr}(u) := \{(x, v) \in \mathbb{R}^d \times \mathbb{R}^n :\, x \in B,\, v \in \mathrm{spt}(u(x))\}.$$

Definition 3.1.6. Let $u = \sum_\ell \llbracket u_\ell \rrbracket \colon B \subset \Sigma \to \mathcal{A}_Q(\mathbb{R}^n)$ be a proper Lipschitz Q-valued map, and define the map

$$\mathrm{Id} \times u \colon x \in B \mapsto \sum_{\ell=1}^{Q} \llbracket (x, u_\ell(x)) \rrbracket \in \mathcal{A}_Q(\mathbb{R}^d \times \mathbb{R}^n).$$

Then, the push-forward $\mathbf{T}_{\mathrm{Id}\times u}$ is the locally integer rectifiable current associated to $\mathrm{Gr}(u)$, and will be denoted by $\mathbf{G}_u$.

Using similar arguments to those carried in Remark 3.1.4, it is not difficult to extend the above results to multi-valued push-forwards of general integer rectifiable currents. This was already observed by De Lellis and Spadaro in [DS15], without going further into the details. Indeed, if $\Omega \subset \mathbb{R}^d$ is open and if $T \in \mathscr{R}_m(\Omega)$ then there exist a sequence of C^1 oriented m-dimensional submanifolds $\Sigma_j \subset \mathbb{R}^d$, a sequence of pairwise disjoint closed subsets $K_j \subset \Sigma_j$, and a sequence of positive integers k_j such that $\sum_{j=1}^\infty k_j \mathcal{H}^m(K_j) < \infty$ and

$$T = \sum_{j=1}^{\infty} k_j \llbracket K_j \rrbracket. \tag{3.8}$$

Now, if $u \colon \Omega \to \mathcal{A}_Q(\mathbb{R}^n)$ is Lipschitz and proper, we define the push-forward of T through u by setting

$$u_\sharp T := \sum_{j=1}^{\infty} k_j \mathbf{T}_{u_j}, \tag{3.9}$$

where $u_j := u|_{K_j}$. We record the properties of $u_\sharp T$ in the following proposition.

Proposition 3.1.7 (Q-valued push-forward of rectifiable currents.)**.** *The integer rectifiable current $u_\sharp T \in \mathscr{R}_m(\mathbb{R}^n)$ defined in (3.9) is independent of the particular representation (3.8) of T. If $T = \llbracket B, \vec{\tau}, \theta \rrbracket$, then $u_\sharp T$ acts on forms $\omega \in \mathcal{D}^m(\mathbb{R}^n)$ as follows:*

$$(u_\sharp T)(\omega) = \int_B \sum_{\ell=1}^{Q} \langle \omega(u_\ell(x)), Du_\ell(x)_\sharp \vec{\tau}(x) \rangle \, \theta(x) \, d\mathcal{H}^m(x). \tag{3.10}$$

Moreover, $u_\sharp T$ can be represented by $u_\sharp T = \llbracket \mathrm{Im}(u|_B), \vec{\eta}, \Theta \rrbracket$, where

(R1)′ *$\mathrm{Im}(u|_B) = \bigcup_{j=1}^\infty \mathrm{Im}(u_j)$ is an m-rectifiable set in $\mathbb{R}^n$;*

(R2)′ *$\vec{\eta}$ is a Borel unit m-vector field orienting $\mathrm{Im}(u|_B)$; moreover, if $K_j = \bigcup_{i\in\mathbb{N}} K_j^i$ is a countable partition of $K_j \subset \Sigma_j$ in measurable subsets associated to a Lipschitz selection $u|_{K_j^i} = \sum_\ell \llbracket (u_j^i)^\ell \rrbracket$ of u as in Proposition 2.2.7, then for $\mathcal{H}^m$-a.e. $y \in \mathrm{Im}(u|_B)$ one has that*

$$\frac{D(u_j^i)^\ell(x)_\sharp \vec{\tau}(x)}{|D(u_j^i)^\ell(x)_\sharp \vec{\tau}(x)|} = \pm \vec{\eta}(y) \tag{3.11}$$

for all j, i, ℓ, x such that $(u_j^i)^\ell(x) = y$;

(R3)′ *for $\mathcal{H}^m$-a.e. $y \in \mathrm{Im}(u|_B)$, the (Borel) multiplicity function Θ equals*

$$\Theta(y) = \sum_{j,i,\ell,x\,:\,(u_j^i)^\ell(x)=y} \theta(x) \left\langle \vec{\eta}(y), \frac{D(u_j^i)^\ell(x)_\sharp \vec{\tau}(x)}{|D(u_j^i)^\ell(x)_\sharp \vec{\tau}(x)|} \right\rangle . \tag{3.12}$$

Notation 3.1.8. In the sequel, we will use the symbol $u_\sharp T$ to denote the push-forward of a current $T \in \mathcal{D}_m(\Omega)$ through a multiple-valued function $u\colon \Omega \to \mathcal{A}_Q(\mathbb{R}^n)$ whenever such a push-forward is defined. The symbol $\mathbf{T}_u$ will be still used when it is understood that the push-forward operator is acting on the whole domain of u. In particular, if $\Sigma \subset \mathbb{R}^d$ is an m-dimensional Lipschitz submanifold and $u\colon \Sigma \to \mathcal{A}_Q(\mathbb{R}^n)$ then the writings $\mathbf{T}_u$ and $u_\sharp[\![\Sigma]\!]$ are equivalent.

As already anticipated, in Section 3.3, we will take advantage of the polyhedral approximation of flat chains, Theorem 2.1.12, to give a meaning to $u_\sharp T$ when $T \in \mathscr{F}_m(\Omega)$. Before doing that, we have to investigate the behaviour of the multi-valued push forward with respect to the boundary operator.

3.2 PUSH-FORWARD AND BOUNDARY

An important feature of the notion of push-forward of Lipschitz manifolds through multiple-valued functions is that, exactly as in the single valued context (cf. (2.4)), it behaves nicely with respect to the boundary operator. The first instance of such a result appears already in [Almoo, Section 1.6], where Almgren relies on the intersection theory of flat chains to define a multi-valued push-forward operator acting on flat chains and study its properties. A more elementary proof was then suggested by De Lellis and Spadaro in [DS15, Theorem 2.1]. Here we provide a slightly simplified version of their proof, relying on a double inductive process, both on the number Q of values that the function takes and on the dimension m of the domain.

Theorem 3.2.1 (Boundary of the push-forward)**.** *Let $\Sigma \subset \mathbb{R}^d$ be an m-dimensional Lipschitz manifold with Lipschitz boundary, and let $u\colon \Sigma \to \mathcal{A}_Q(\mathbb{R}^n)$ be a proper Lipschitz map. Then, $\partial \mathbf{T}_u = \mathbf{T}_{u|_{\partial\Sigma}}$.*

Proof. First observe that since every Lipschitz manifold can be triangulated, and since the statement is invariant under bi-Lipschitz homeomorphisms, it is enough to prove the theorem with $\Sigma = [0,1]^m$. Furthermore, it suffices to show that the theorem holds in the case of the currents associated to graphs. Indeed, suppose to know that $\partial \mathbf{G}_u = \mathbf{G}_{u|_{\partial\Sigma}}$, and let $\mathbf{p}\colon \mathbb{R}^d \times \mathbb{R}^n \to \mathbb{R}^n$ be the orthogonal projection onto the second components. Then, it is immediate to see that

$$\mathbf{p}_\sharp \mathbf{G}_u = \mathbf{p}_\sharp \mathbf{T}_{\mathrm{Id}\times u} = \mathbf{T}_{\mathbf{p}\circ(\mathrm{Id}\times u)} = \mathbf{T}_u\,,$$

where, for given Lipschitz $F\colon \mathbb{R}^d \to \mathcal{A}_Q(\mathbb{R}^n)$ and $\phi\colon \mathbb{R}^n \to \mathbb{R}^k$, we have used the notation $\phi \circ F$ for the Q-valued function $\phi \circ F(x) := \sum_{\ell=1}^Q [\![\phi(F_\ell(x))]\!] \in \mathcal{A}_Q(\mathbb{R}^k)$. Then, using that push-forward and boundary do commute in the case of single valued Lipschitz functions, one readily concludes

$$\partial \mathbf{T}_u = \partial \mathbf{p}_\sharp \mathbf{G}_u = \mathbf{p}_\sharp \partial \mathbf{G}_u = \mathbf{p}_\sharp \mathbf{G}_{u|_{\partial\Sigma}} = \mathbf{T}_{u|_{\partial\Sigma}}.$$

Hence, we show that $\partial \mathbf{G}_u = \mathbf{G}_{u|_{\partial\Sigma}}$. The proof is by induction on both m and Q. If $Q = 1$, the result is classical. On the other hand, the case $m = 1$ is a consequence of [DLS11, Proposition 1.2]: if $u\colon [0,1] \to \mathcal{A}_Q(\mathbb{R}^n)$ is Lipschitz, then there exist Lipschitz functions $u_1, \dots, u_Q\colon [0,1] \to \mathbb{R}^n$ such that $u = \sum_{\ell=1}^{Q} \llbracket u_\ell \rrbracket$. Therefore, $\mathbf{T}_u = \sum_\ell (u_\ell)_\sharp \llbracket (0,1) \rrbracket$, and thus

$$\partial \mathbf{T}_u = \sum_\ell \partial (u_\ell)_\sharp \llbracket (0,1) \rrbracket = \sum_\ell (u_\ell)_\sharp \left(\llbracket 1 \rrbracket - \llbracket 0 \rrbracket \right) = \sum_\ell \left(\llbracket u_\ell(1) \rrbracket - \llbracket u_\ell(0) \rrbracket \right) = \mathbf{T}_{u|_{\partial\Sigma}}.$$

Then, we make the following inductive hypotheses:

(H1) the theorem is true when $\dim(\Sigma) \leqslant m-1$,

(H2) the theorem is true for $\dim(\Sigma) = m$ when the function u takes Q^* values for every $Q^* < Q$,

and we show that the theorem is true for (m, Q). In order to do this, we consider a dyadic decomposition of $\Sigma = [0,1]^m$ in m-cubes of side length 2^{-h} with $h \in \mathbb{N}$, and for any integer vector $k \in \{0, 1, \dots, 2^h - 1\}^m$ we let $C_{h,k}$ be the cube $C_{h,k} := 2^{-h}\left(k + [0,1]^m\right)$.

Now, for fixed h, let $\mathscr{B}_h$ be the set of all $k \in \{0, 1, \dots, 2^h - 1\}^m$ such that on the corresponding cube $C_{h,k}$ one has

$$\max_{x \in C_{h,k}} \operatorname{diam}(u(x)) > 3(Q-1)\operatorname{Lip}(u) 2^{-h} \sqrt{m}. \tag{3.13}$$

By Proposition 2.2.8, if $k \in \mathscr{B}_h$ then on the cube $C_{h,k}$ the function u is well separated into the sum

$$u|_{C_{h,k}} = \llbracket u_{k,Q_1} \rrbracket + \llbracket u_{k,Q_2} \rrbracket, \tag{3.14}$$

where $u_{k,Q_1} \in \operatorname{Lip}(C_{h,k}, \mathcal{A}_{Q_1}(\mathbb{R}^n))$, $u_{k,Q_2} \in \operatorname{Lip}(C_{h,k}, \mathcal{A}_{Q_2}(\mathbb{R}^n))$ and $Q_1, Q_2 < Q$. Therefore, by the inductive hypothesis (H2) we can conclude that

$$\partial \mathbf{G}_{u|_{C_{h,k}}} = \mathbf{G}_{u|_{\partial C_{h,k}}} \tag{3.15}$$

for every $k \in \mathscr{B}_h$.

If on the other hand $k \notin \mathscr{B}_h$, consider the affine homotopy $\sigma\colon [0,1] \times C_{h,k} \times \mathbb{R}^n \to \mathbb{R}^d \times \mathbb{R}^n$ defined by

$$\sigma(t, x, v) := \left(x, (1-t)\boldsymbol{\eta} \circ u(x) + tv\right), \tag{3.16}$$

and define the current

$$R_k := Q\mathbf{G}_{(\boldsymbol{\eta}\circ u)|_{C_{h,k}}} + \sigma_\sharp(\llbracket (0,1) \rrbracket \times \mathbf{G}_{u|_{\partial C_{h,k}}}). \tag{3.17}$$

Here, $\boldsymbol{\eta} \circ u$ denotes the (single valued) Lipschitz function $\boldsymbol{\eta} \circ u\colon \Sigma \to \mathbb{R}^n$ given by

$$\boldsymbol{\eta} \circ u(x) := \boldsymbol{\eta}(u(x)) = \frac{1}{Q} \sum_{\ell=1}^{Q} u_\ell(x).$$

Since $\boldsymbol{\eta} \circ u$ is a classical Lipschitz function, the classical commutation rule of push-forward and boundary gives

$$\partial(Q\mathbf{G}_{(\boldsymbol{\eta}\circ u)|_{C_{h,k}}}) = Q\mathbf{G}_{(\boldsymbol{\eta}\circ u)|_{\partial C_{h,k}}}. \tag{3.18}$$

On the other hand, the homotopy formula (2.5) yields

$$\partial\sigma_\sharp([\![(0,1)]\!]\times \mathbf{G}_{u|_{\partial C_{h,k}}}) = \mathbf{G}_{u|_{\partial C_{h,k}}} - Q\mathbf{G}_{(\eta\circ u)|_{\partial C_{h,k}}} - \sigma_\sharp([\![(0,1)]\!]\times \partial\mathbf{G}_{u|_{\partial C_{h,k}}}). \tag{3.19}$$

Since $\partial C_{h,k}$ is the union of $(m-1)$-dimensional cubes, the inductive hypothesis (H1) ensures that in fact $\partial\mathbf{G}_{u|_{\partial C_{h,k}}} = 0$, and thus the last addendum in the r.h.s. of equation (3.19) vanishes. Combining (3.18) and (3.19) therefore yields

$$\partial R_k = \mathbf{G}_{u|_{\partial C_{h,k}}}. \tag{3.20}$$

For every $h \in \mathbb{N}$, define the current

$$T_h := \sum_{k\in\mathscr{B}_h} \mathbf{G}_{u|_{C_{h,k}}} + \sum_{k\notin\mathscr{B}_h} R_k, \tag{3.21}$$

and notice that by (3.15) and (3.20) one has

$$\partial T_h = \sum_k \mathbf{G}_{u|_{\partial C_{h,k}}} = \mathbf{G}_{u|_{\partial\Sigma}} \tag{3.22}$$

because the common faces to adjacent cubes have opposite orientations. Furthermore, it is easy to see that for every $h \in \mathbb{N}$ and for every $k \in \mathscr{B}_h$ one has

$$\mathbb{M}(\mathbf{G}_{u|_{C_{h,k}}}) \leqslant C(1+\mathrm{Lip}(u))^m \mathcal{H}^m(C_{h,k}) \leqslant C(2^{-h})^m, \tag{3.23}$$

whereas

$$\begin{aligned}
\mathbb{M}(R_k) &\overset{(2.6)}{\leqslant} C(2^{-h})^m + C\mathbb{M}(\mathbf{G}_{u|_{\partial C_{h,k}}}) \sup_{(x,\nu)\in \mathrm{Gr}(u|_{\partial C_{h,k}})} |(x,\nu)-(x,\eta\circ u(x))| \\
&\leqslant C(2^{-h})^m + C(2^{-h})^{m-1} \sup_{x\in\partial C_{h,k}} \max_{\ell\in\{1,\dots,Q\}} |u_\ell(x) - \eta\circ u(x)| \\
&\leqslant C(2^{-h})^m + C(2^{-h})^{m-1} \sup_{x\in\partial C_{h,k}} \mathrm{diam}(u(x)) \\
&\leqslant C(2^{-h})^m
\end{aligned} \tag{3.24}$$

if $k \notin \mathscr{B}_h$, for a constant $C = C(m, Q, \mathrm{Lip}(u))$.

By equations (3.21), (3.22), (3.23) and (3.24) we immediately conclude that

$$\mathbb{M}(T_h) + \mathbb{M}(\partial T_h) \leqslant C, \tag{3.25}$$

where $C = C(m, Q, \mathrm{Lip}(u))$ is a constant independent of h. It then follows from the Compactness Theorem 2.1.3 that when $h \uparrow \infty$ a subsequence of the T_h's converges to an integral current T such that $\partial T = \mathbf{G}_{u|_{\partial\Sigma}}$.

We are only left to prove that in fact $T = \mathbf{G}_u$. Since clearly $\mathrm{spt}(T) \subset \mathrm{Gr}(u)$ and T is integral, we have that $T = [\![\mathrm{Gr}(u), \vec{\eta}, \Theta_T]\!]$ and $\mathbf{G}_u = [\![\mathrm{Gr}(u), \vec{\eta}, \Theta_{\mathbf{G}_u}]\!]$. We only need to show that $\Theta_T(x,\nu) = \Theta_{\mathbf{G}_u}(x,\nu)$ at $\mathcal{H}^m$-a.e. $(x,\nu) \in \mathrm{Gr}(u)$. Let $x \in \Sigma$, and denote by $D_Q(u)$ the closed set

$$D_Q(u) := \{x \in \Sigma : \ u(x) = Q[\![\nu]\!] \text{ for some } \nu \in \mathbb{R}^n\}$$

of multiplicity Q points of the function u. If $x \notin D_Q(u)$, then there exists a suitably large $\bar{h}$ such that for every $h \geqslant \bar{h}$ one has $x \in C_{h,k}$ for some $k \in \mathscr{B}_h$, and thus it follows naturally that $\Theta_T(x, u_\ell(x)) = \Theta_{\mathbf{G}_u}(x, u_\ell(x))$ for every ℓ. Hence, if $\mathcal{H}^m(D_Q(u)) = 0$ then we are done. Otherwise, consider the 1-Lipschitz orthogonal projection on the first components $\bar{\mathbf{p}} \colon \mathbb{R}^d \times \mathbb{R}^n \to \mathbb{R}^d$. One has that $\bar{\mathbf{p}}_\sharp T = \bar{\Theta}_T \llbracket \Sigma \rrbracket$ and $\bar{\mathbf{p}}_\sharp \mathbf{G}_u = \bar{\Theta}_{\mathbf{G}_u} \llbracket \Sigma \rrbracket$, with

$$\bar{\Theta}_T(x) = \sum_{(x,\nu)\in \mathrm{Gr}(u)} \Theta_T(x,\nu) \quad \text{and} \quad \bar{\Theta}_{\mathbf{G}_u}(x) = \sum_{(x,\nu)\in \mathrm{Gr}(u)} \Theta_{\mathbf{G}_u}(x,\nu) \quad \text{for } \mathcal{H}^m\text{-a.e. } x \in \Sigma.$$

In particular, for $\mathcal{H}^m$-a.e. $x \in D_Q(u)$, if $u(x) = Q\llbracket \nu(x) \rrbracket$ then $\bar{\Theta}_T(x) = \Theta_T(x, \nu(x))$ and $\bar{\Theta}_{\mathbf{G}_u}(x) = \Theta_{\mathbf{G}_u}(x, \nu(x))$. On the other hand, by the definitions of u and T_h it also holds $\bar{\mathbf{p}}_\sharp \mathbf{G}_u = Q\llbracket \Sigma \rrbracket = \bar{\mathbf{p}}_\sharp T_h$ for every h. Since T is the limit of (a subsequence of) the T_h, then necessarily $\bar{\Theta}_{\mathbf{G}_u}(x) = Q = \bar{\Theta}_T(x)$ $\mathcal{H}^m$-a.e. on Σ, and thus finally $\Theta_{\mathbf{G}_u}(x, \nu(x)) = Q = \Theta_T(x, \nu(x))$ for $\mathcal{H}^m$-a.e. $x \in D_Q(u)$. This completes the proof. □

3.3 THE PUSH-FORWARD OF FLAT CHAINS

The goal of this section is to extend the definition of multiple-valued push-forward to the class of integral flat chains. As mentioned before, the existence of a multi-valued push-forward operator acting on flat chains has already been investigated by Almgren in [Almoo, Section 1.6]. In what follows, we deduce it as a rather immediate consequence of Theorem 3.2.1 and of the polyhedral approximation of flat chains, Proposition 2.1.12.

We fix the following hypotheses.

Assumption 3.3.1. We will consider:

- a Lipschitz Q-valued function $u \colon \Omega \to \mathcal{A}_Q(\mathbb{R}^n)$ defined in an open subset $\Omega \subset \mathbb{R}^d$;
- a compact subset $K \subset \Omega$;
- an integral flat m-chain $T \in \mathscr{F}_m(\mathbb{R}^d)$ with $\mathrm{spt}(T) \subset \mathrm{int} K$.

Given K and T as in Assumptions 3.3.1, by Proposition 2.1.12 there exists a sequence $\{P_j\}_{j=1}^\infty$ of integral polyhedral m-chains supported in K such that

$$\mathbb{F}_K(T - P_j) \leqslant \frac{1}{j} \quad \text{and} \quad \mathbb{M}(P_j) \leqslant \mathbb{M}(T) + \frac{1}{j}. \tag{3.26}$$

Now, integral polyhedral chains are a subclass of the class of integer rectifiable currents, as any P_j can be written as the linear combination $P_j = \sum_{i=1}^{k_j} \beta_{ji} \llbracket \sigma_{ji} \rrbracket$ of a finite number of oriented simplexes $\llbracket \sigma_{ji} \rrbracket$ with coefficients $\beta_{ji} \in \mathbb{Z}$. Since we have a well defined notion of multi-valued push-forward of an integer rectifiable current, we can consider the currents

$$u_\sharp P_j = \sum_{i=1}^{k_j} \beta_{ji}\, u_\sharp \llbracket \sigma_{ji} \rrbracket. \tag{3.27}$$

We also know that the mass of $u_\sharp P_j$ can be estimated by

$$\mathbb{M}(u_\sharp P_j) \leqslant C\mathbb{M}(P_j), \tag{3.28}$$

where C is a constant depending on $\mathrm{Lip}(u)$, and Theorem 3.2.1 guarantees that

$$\partial(u_\sharp P_j) = u_\sharp(\partial P_j). \tag{3.29}$$

Clearly, $\{P_j\}$ is a Cauchy sequence with respect to the flat distance $\mathbb{F}_K$. Indeed, for any $j, h \in \mathbb{N}$ one can explicitly estimate

$$\mathbb{F}_K(P_j - P_h) \leqslant \mathbb{F}_K(P_j - T) + \mathbb{F}_K(T - P_h) \leqslant \frac{1}{j} + \frac{1}{h}. \tag{3.30}$$

Now, we have the following

Theorem 3.3.2 (Push-forward of a flat chain)**.** *Let u, K and T be as in Assumptions 3.3.1. Then, for any open subset $W \Subset \Omega$ with $K \subset W$, for any compact $K' \subset \mathbb{R}^n$ containing $\mathrm{Im}(u|_W) = \bigcup_{x\in W} \mathrm{spt}(u(x))$, and for any sequence $\{P_j\}_{j=1}^\infty$ of integral polyhedral m-chains converging to T with respect to $d_{\mathbb{F}_K}$, the sequence $\{u_\sharp P_j\}_{j=1}^\infty$ is Cauchy with respect to $d_{\mathbb{F}_{K'}}$. Therefore, there exists an integral flat m-chain $Z \in \mathscr{F}_{m,K'}(\mathbb{R}^n)$ such that $\mathbb{F}_{K'}(Z - u_\sharp P_j) \to 0$ as $j \uparrow \infty$. Such a Z does not depend on the approximating sequence P_j converging to T.*

Definition 3.3.3. The current $Z \in \mathscr{F}_m(\mathbb{R}^n)$ given by Theorem 3.3.2 is the push-forward of T through u. Coherently with Notation 3.1.8, we will set $Z = u_\sharp T$.

The proof of Theorem 3.3.2 is a simple consequence of the following lemma, which is proved for real polyhedral chains in [Fed69, Lemma 4.2.23]. For the reader's convenience, we provide here also the proof.

Lemma 3.3.4. *If $K \subset W \subset \mathbb{R}^d$ with K compact, W open, and $P \in \mathscr{P}_m(\mathbb{R}^d)$ with $\mathrm{spt}(P) \subset K$, then the quantity*

$$G(P) := \inf\left\{\mathbb{M}(P - \partial S) + \mathbb{M}(S) :\ S \in \mathscr{P}_{m+1}(\mathbb{R}^d) \text{ with } \mathrm{spt}(S) \subset W\right\} \tag{3.31}$$

does not exceed $\mathbb{F}_K(P)$.

Proof. Preliminarly, we show that

$$G(P) \leqslant \tilde{\gamma}\mathbb{F}_K(P) \tag{3.32}$$

for some constant $\tilde{\gamma} = \tilde{\gamma}(m, d)$. In order to do this, first use Proposition 2.1.9 to determine a current $N \in \mathscr{I}_{m+1,K}(\mathbb{R}^d)$ such that

$$\mathbb{F}_K(P) = \mathbb{M}(P - \partial N) + \mathbb{M}(N). \tag{3.33}$$

Observe that $\partial(P - \partial N) = \partial P \in \mathscr{P}_{m-1}(\mathbb{R}^d)$. Therefore, we can apply the Deformation Theorem 2.1.10 with $T = P - \partial N$ and small ε, to conclude the existence of $R_1 \in \mathscr{P}_m(\mathbb{R}^d)$ and $S_1 \in \mathscr{I}_{m+1}(\mathbb{R}^d)$ with $\mathrm{spt}(R_1) \cup \mathrm{spt}(S_1) \subset W$ such that

$$P - \partial N = R_1 + \partial S_1, \tag{3.34}$$

and furthermore satisfying the estimates

$$\begin{aligned} \mathbb{M}(R_1) &\leqslant \gamma\left(\mathbb{M}(P-\partial N)+\varepsilon\mathbb{M}(\partial P)\right), \\ \mathbb{M}(S_1) &\leqslant \gamma\varepsilon\mathbb{M}(P-\partial N), \end{aligned} \tag{3.35}$$

for a constant $\gamma=\gamma(m,d)$. Again, since $\partial(N+S_1)=P-R_1\in\mathscr{P}_m(\mathbb{R}^d)$ from (3.34), another application of the Deformation Theorem with $T=N+S_1$ and ε suitably small implies that there exist $R_2\in\mathscr{P}_{m+1}(\mathbb{R}^d)$ and $S_2\in\mathscr{I}_{m+2}(\mathbb{R}^d)$ with $\operatorname{spt}(R_2)\cup\operatorname{spt}(S_2)\subset W$ such that

$$N+S_1=R_2+\partial S_2 \tag{3.36}$$

and furthermore satisfying

$$\mathbb{M}(R_2)\leqslant\gamma\left(\mathbb{M}(N+S_1)+\varepsilon\mathbb{M}(P-R_1)\right). \tag{3.37}$$

Combining (3.34) and (3.36), we see that

$$P=R_1+\partial(N+S_1)=R_1+\partial R_2, \tag{3.38}$$

with $R_1\in\mathscr{P}_m(\mathbb{R}^d)$, $R_2\in\mathscr{P}_{m+1}(\mathbb{R}^d)$, $\operatorname{spt}(R_1)\cup\operatorname{spt}(R_2)\subset W$ satisfying

$$G(P)\leqslant\mathbb{M}(R_1)+\mathbb{M}(R_2)\overset{(3.35),(3.37)}{\leqslant}\gamma\left(1+2\varepsilon\gamma\right)\left(\mathbb{F}_K(P)+\varepsilon\left(\mathbb{M}(P)+\mathbb{M}(\partial P)\right)\right). \tag{3.39}$$

The preliminary estimate (3.32), then, follows from (3.39) by letting $\varepsilon\to 0$.

Next, in order to complete the proof of the lemma, fix $\rho>0$, let N be as above and select a compact $K_1\subset W$ such that $K\subset\operatorname{int}K_1$. Apply Proposition 2.1.11 twice, first with $T=P-\partial N$ and then with $T=N$ to conclude the existence of $P_1\in\mathscr{P}_m(\mathbb{R}^d)$ and $P_2\in\mathscr{P}_{m+1}(\mathbb{R}^d)$ with $\operatorname{spt}(P_1)\cup\operatorname{spt}(P_2)\subset K_1$ such that

$$\mathbb{F}_{K_1}(P-\partial N-P_1)\leqslant\rho\quad\text{and}\quad\mathbb{F}_{K_1}(N-P_2)\leqslant\rho \tag{3.40}$$

and satisfying

$$\mathbb{M}(P_1)\leqslant\mathbb{M}(P-\partial N)+\rho\quad\text{and}\quad\mathbb{M}(P_2)\leqslant\mathbb{M}(N)+\rho. \tag{3.41}$$

Observe now that the current $P-P_1-\partial P_2\in\mathscr{P}_m(\mathbb{R}^d)$ satisfies

$$\begin{aligned} \mathbb{F}_{K_1}(P-P_1-\partial P_2) &\leqslant \mathbb{F}_{K_1}(P-\partial N-P_1)+\mathbb{F}_{K_1}(\partial N-\partial P_2) \\ &\leqslant \mathbb{F}_{K_1}(P-\partial N-P_1)+\mathbb{F}_{K_1}(N-P_2) \\ &\overset{(3.40)}{\leqslant} 2\rho \end{aligned} \tag{3.42}$$

because $\mathbb{F}_K(\partial T)\leqslant\mathbb{F}_K(T)$ for any $T\in\mathscr{F}_{m,K}(\mathbb{R}^d)$. Applying the estimate (3.32) with P and K replaced by $P-P_1-\partial P_2$ and K_1 respectively, we finally conclude

$$\begin{aligned} G(P) &\leqslant G(P_1+\partial P_2)+G(P-P_1-\partial P_2) \\ &\leqslant \mathbb{M}(P_1)+\mathbb{M}(P_2)+\tilde{\gamma}\mathbb{F}_{K_1}(P-P_1-\partial P_2) \\ &\overset{(3.41),(3.42)}{\leqslant} \mathbb{M}(P-\partial N)+\mathbb{M}(N)+2\rho(1+\tilde{\gamma}) \\ &= \mathbb{F}_K(P)+2\rho(1+\tilde{\gamma}). \end{aligned} \tag{3.43}$$

The conclusion follows by letting $\rho\searrow 0$. □

Proof of Theorem 3.3.2. Fix any open set $W \Subset \Omega$ with $K \subset W$, let $K' \subset \mathbb{R}^n$ be any compact set containing $\mathrm{Im}(u|_W)$, and let $\{P_j\}_{j=1}^\infty$ be any sequence of integral polyhedral m-chains supported in K and satisfying (3.26). For any $j, h \in \mathbb{N}$, consider the current $P_j - P_h \in \mathscr{P}_m(\mathbb{R}^d)$, and notice that $\mathrm{spt}(P_j - P_h) \subset K$. For any choice of polyhedral currents $R \in \mathscr{P}_m(\mathbb{R}^d)$, $S \in \mathscr{P}_{m+1}(\mathbb{R}^d)$ with $\mathrm{spt}(R) \cup \mathrm{spt}(S) \subset W$ such that

$$P_j - P_h = R + \partial S, \tag{3.44}$$

Theorem 3.2.1 guarantees that

$$u_\sharp P_j - u_\sharp P_h = u_\sharp R + \partial(u_\sharp S). \tag{3.45}$$

Since $u_\sharp R$ and $u_\sharp S$ are rectifiable currents supported in K', one has

$$\begin{aligned} \mathbb{F}_{K'}(u_\sharp P_j - u_\sharp P_h) &\leqslant \mathbb{M}(u_\sharp R) + \mathbb{M}(u_\sharp S) \\ &\leqslant C\left(\mathbb{M}(R) + \mathbb{M}(S)\right), \end{aligned} \tag{3.46}$$

for some constant C depending on $\mathrm{Lip}(u)$. Taking the infimum among all integral polyhedral currents R and S supported in W such that (3.44) holds, we immediately conclude from Lemma 3.3.4 that

$$\mathbb{F}_{K'}(u_\sharp P_j - u_\sharp P_h) \leqslant C G(P_j - P_h) \leqslant C \mathbb{F}_K(P_j - P_h) \leqslant \frac{C}{j} + \frac{C}{h}. \tag{3.47}$$

This proves that the sequence $\{u_\sharp P_j\}_{j=1}^\infty$ is Cauchy with respect to $d_{\mathbb{F}_{K'}}$, and, thus, has a limit $Z \in \mathscr{F}_{m,K'}(\mathbb{R}^n)$. In order to see that the limit does not depend on the approximating sequence $\{P_j\}$, consider two sequences of integral polyhedral m-currents $\{P_j\}$ and $\{\tilde{P}_j\}$ both approximating T in the $\mathbb{F}_K$ distance, and assume that $u_\sharp P_j$ and $u_\sharp \tilde{P}_j$ flat converge to Z and $\tilde{Z}$ respectively. For any $\varepsilon > 0$, let $j_0 \in \mathbb{N}$ be such that both $\mathbb{F}_K(T - P_{j_0}) + \mathbb{F}_K(T - \tilde{P}_{j_0}) < \varepsilon$ and $\mathbb{F}_{K'}(Z - u_\sharp P_{j_0}) + \mathbb{F}_{K'}(\tilde{Z} - u_\sharp \tilde{P}_{j_0}) < \varepsilon$. Then, we can estimate:

$$\begin{aligned} \mathbb{F}_{K'}(Z - \tilde{Z}) &\leqslant \mathbb{F}_{K'}(Z - u_\sharp P_{j_0}) + \mathbb{F}_{K'}(u_\sharp P_{j_0} - u_\sharp \tilde{P}_{j_0}) + \mathbb{F}_{K'}(u_\sharp \tilde{P}_{j_0} - \tilde{Z}) \\ &\leqslant \varepsilon + \mathbb{F}_{K'}(u_\sharp P_{j_0} - u_\sharp \tilde{P}_{j_0}) \end{aligned} \tag{3.48}$$

On the other hand, applying the same argument that we have used above to prove (3.47) to $P_{j_0} - \tilde{P}_{j_0} \in \mathscr{P}_m(\mathbb{R}^d)$ shows that

$$\mathbb{F}_{K'}(u_\sharp P_{j_0} - u_\sharp \tilde{P}_{j_0}) \leqslant C \mathbb{F}_K(P_{j_0} - \tilde{P}_{j_0}) \leqslant C\varepsilon. \tag{3.49}$$

Combining (3.48) and (3.49), and letting $\varepsilon \downarrow 0$ yields that $Z = \tilde{Z}$. □

Corollary 3.3.5. *Let u, K and T be as in Assumption 3.3.1. If $Z = u_\sharp T$, then it also holds $\partial Z = u_\sharp(\partial T)$.*

Proof. Let $W \Subset \Omega$ and $K' \subset \mathbb{R}^n$ be as in Theorem 3.3.2, and let $\{P_j\}_{j=1}^\infty$ be any sequence of integral polyhedral m-chains $\mathbb{F}_K$-converging to T. Then, by Theorem 3.3.2 Z is the $\mathbb{F}_{K'}$-limit of the currents $u_\sharp P_j$. Hence, since in general $\mathbb{F}_K(\partial T) \leqslant \mathbb{F}_K(T)$, we also have that ∂Z is the $\mathbb{F}_{K'}$-limit of the currents $\partial(u_\sharp P_j) = u_\sharp(\partial P_j)$ by Theorem 3.2.1. On the other hand, since the ∂P_j's are a sequence of integral polyhedral $(m-1)$-chains which $\mathbb{F}_K$-approximates ∂T, the sequence $u_\sharp(\partial P_j)$ necessarily $\mathbb{F}_{K'}$-converges to $u_\sharp(\partial T)$. The claim follows by uniqueness of the limit. □

Part I

Multi-valued theory of the stability operator

4 THE JACOBI FUNCTIONAL

In this chapter we initiate the study of the multi-valued theory for the stability operator. After a preliminary section in which we introduce the main geometric objects and the terminology that we are going to use throughout the whole Part I, we will turn to Section 4.2, where we compute the second variation formula corresponding to Q-valued normal deformations of a minimal submanifold Σ of an ambient Riemannian manifold $\mathcal{M}$ (cf. Theorem 4.2.4 below). This will naturally lead us to the definition of the Jac functional on the space of Q-valued $W^{1,2}$ sections of the normal bundle of Σ in $\mathcal{M}$. Multiple-valued Jacobi fields are then defined as the minimizers of Jac with respect to boundary data. We conclude the chapter with the analysis of the first elementary properties of Jacobi Q-fields and with the proof of Proposition 4.3.1, which establishes the weak lower semi-continuity of Jac in the aforementioned space of sections. This is the first step towards the developments of the existence theory to be carried on in Chapter 5.

4.1 GEOMETRIC PRELIMINARIES

Throughout the whole Part I, we will work under the following assumptions.

Assumption 4.1.1. We will consider:

(M) a closed (i.e. compact with empty boundary) Riemannian manifold $\mathcal{M}$ of dimension $m+k$ and class $C^{3,\beta}$ for some $\beta \in (0,1)$;

(S) a compact oriented *minimal* submanifold Σ of the ambient manifold $\mathcal{M}$ of dimension $\dim(\Sigma) = m$ and class $C^{3,\beta}$.

Without loss of generality, we will regard $\mathcal{M}$ as an isometrically embedded submanifold of some Euclidean space $\mathbb{R}^d$. We will let $n := d - m$ and $K := d - (m+k)$ be the codimensions of Σ and $\mathcal{M}$ in $\mathbb{R}^d$ respectively.

Let $\Sigma^m \hookrightarrow \mathcal{M}^{m+k} \subset \mathbb{R}^d$ be as in Assumption 4.1.1. Since the metric on $\mathcal{M}$ and Σ is induced by the flat metric in $\mathbb{R}^d$, the symbol $\langle \cdot, \cdot \rangle$ adopted for the Euclidean scalar product in $\mathbb{R}^d$ will also denote the scalar product between tangent vectors to $\mathcal{M}$ or to Σ.

If $z \in \mathcal{M}$, then the maps $\mathbf{p}^{\mathcal{M}}_z \colon \mathbb{R}^d \to T_z\mathcal{M}$ and $\mathbf{p}^{\mathcal{M}\perp}_z \colon \mathbb{R}^d \to T^\perp_z\mathcal{M}$ denote orthogonal projections of $\mathbb{R}^d$ onto the tangent space to $\mathcal{M}$ at z and its orthogonal complement in $\mathbb{R}^d$ respectively. If $x \in \Sigma$, the tangent space $T_x\mathcal{M}$ can be decomposed into the direct sum

$$T_x\mathcal{M} = T_x\Sigma \oplus T^\perp_x\Sigma,$$

where $T^\perp_x\Sigma$ is the orthogonal complement of $T_x\Sigma$ in $T_x\mathcal{M}$. At each point $x \in \Sigma$, we define orthogonal projections $\mathbf{p}_x \colon T_x\mathcal{M} \to T_x\Sigma$ and $\mathbf{p}^\perp_x \colon T_x\mathcal{M} \to T^\perp_x\Sigma$.

This decomposition at the level of the tangent spaces induces an orthogonal decomposition at the level of the tangent bundle, namely

$$T\mathcal{M} = T\Sigma \oplus N\Sigma,$$

where $N\Sigma$ denotes the normal bundle of Σ in $\mathcal{M}$.

We will use D to denote the standard flat connection in $\mathbb{R}^d$. The symbol ∇ will instead identify the Levi-Civita connection on $\mathcal{M}$. If ξ and X are tangent vector fields to Σ, then for every $x \in \Sigma$ we have

$$\nabla_\xi X(x) = \mathbf{p}_x \cdot \nabla_\xi X(x) + \mathbf{p}_x^\perp \cdot \nabla_\xi X(x) =: \nabla^\Sigma_\xi X(x) + A_x\left(\xi(x), X(x)\right),$$

where ∇^Σ is the Levi-Civita connection on Σ and A is the 2-covariant tensor with values in $N\Sigma$ defined by $A_x(X,Y) := \mathbf{p}_x^\perp \cdot \nabla_X Y$ for any $x \in \Sigma$, for any $X, Y \in T_x\Sigma$. A is called the *second fundamental form* of the embedding $\Sigma \hookrightarrow \mathcal{M}$ by some authors (cf. [Sim83b, Section 7], where the tensor is denoted B, or [Lee97, Chapter 8], where the author uses the notation II) and we will use the same terminology, although in the literature in differential geometry (above all when working with embedded hypersurfaces, that is in case the codimension of the submanifold is $k = 1$) it is sometimes more customary to call A "shape operator" and to use "second fundamental form" for scalar products $h(X,Y) = \langle A(X,Y), \eta\rangle$ with a fixed normal vector field η (cf. [dC92, Chapter 6, Section 2]).

Observe that, since we have assumed Σ to be minimal in $\mathcal{M}$, the mean curvature $H := \mathrm{tr}(A)$ is everywhere vanishing on Σ.

The curvature endomorphism of the ambient manifold $\mathcal{M}$ is denoted by R: we recall that this is a tensor field on $\mathcal{M}$ of type $(3,1)$, whose action on vector fields is defined by

$$R(X,Y)Z := \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z,$$

where $[X,Y]$ is the Lie bracket of the vector fields X and Y.

Recall also that the Riemann tensor can be defined by setting

$$\mathrm{Rm}(X,Y,Z,W) := \langle R(X,Y)Z, W\rangle$$

for any choice of the vector fields X, Y, Z, W, and that the Ricci tensor is the trace of the curvature endomorphism with respect to its first and last indices, that is $\mathrm{Ric}(X,Y)$ is the trace of the linear map

$$Z \mapsto R(Z,X)Y.$$

For any pair of points $x, y \in \Sigma$, $\mathbf{d}(x,y)$ will be their Riemannian geodesic distance, while measures and integrals will be computed with respect to the m-dimensional Hausdorff measure $\mathcal{H}^m$ defined in the ambient space $\mathbb{R}^d$. Boldface characters will be used to denote quantities which are related to the Riemannian geodesic distance: for instance, if $x \in \Sigma$ and r is a positive number, $\mathbf{B}_r(x)$ is the geodesic ball with center x and radius r, namely the set of points $y \in \Sigma$ such that $\mathbf{d}(y,x) < r$. In the same fashion, if U and V are two subsets of Σ we will set

$$\mathbf{dist}(U,V) := \inf\{\mathbf{d}(x,y) : x \in U, y \in V\}.$$

In this part we will work with multi-valued functions $u\colon \Sigma \to \mathcal{A}_Q(\mathbb{R}^d)$, where $Q \geqslant 1$ is a fixed integer. Together with the notions introduced in Section 2.2, we will need the following definition of *Dirichlet energy of a tangent vector field* to the manifold $\mathcal{M}$.

Definition 4.1.2 (Dirichlet energy of a tangent Q-field)**.** Let $\Sigma \hookrightarrow \mathcal{M}$ be as in Assumption 4.1.1, and let $\Omega \subset \Sigma$ be an open set. Let $u = \sum_\ell \llbracket u^\ell \rrbracket \in W^{1,2}\left(\Omega, \mathcal{A}_Q(\mathbb{R}^d)\right)$ be a Sobolev Q-valued tangent vector field to $\mathcal{M}$: that is, assume that $\mathrm{spt}(u(x)) \subset T_x\mathcal{M}$ for $\mathcal{H}^m$-a.e. $x \in \Omega$. Then, for any point x of approximate differentiability for u in Ω, and for any tangent vector field ξ, we set

$$\nabla_\xi u(x) := \sum_{\ell=1}^{Q} \llbracket \mathbf{p}_x^{\mathcal{M}} \cdot D_\xi u^\ell(x) \rrbracket. \tag{4.1}$$

The Dirichlet energy of the vector field u in Ω is thus given by

$$\mathrm{Dir}^{T\mathcal{M}}(u, \Omega) := \int_\Omega \sum_{i=1}^{m} |\nabla_{\xi_i} u|^2 \, d\mathcal{H}^m \tag{4.2}$$

for any (local) orthonormal frame $(\xi_1, \dots, \xi_m)$ of $T\Sigma$.

Remark 4.1.3. Observe that, when u is Lipschitz continuous and $u|_{B_i} = \sum_{\ell=1}^{Q} \llbracket u_i^\ell \rrbracket$ is a local Lipschitz selection of u as in Proposition 2.2.7, one has

$$|\nabla_\xi u(x)|^2 = \sum_{\ell=1}^{Q} |\nabla_\xi u_i^\ell(x)|^2 \qquad \text{for } \mathcal{H}^m - \text{a.e. } x \in B_i, \text{ for all vector fields } \xi,$$

where the ∇ on the right-hand side has to be intended as the classical covariant derivative (which can be extended to Lipschitz maps by means of Rademacher's theorem).

The functional $\mathrm{Dir}^{T\mathcal{M}}$ defined in (4.2) is the "right" geometric quantity to consider when dealing with tangent vector fields, since it does not involve any geometric structure which is external to the manifold $\mathcal{M}$. In particular, it does not depend on the isometric embedding of the Riemannian manifold $\mathcal{M}$ in the Euclidean space $\mathbb{R}^d$.

4.2 Q-VALUED SECOND VARIATION OF THE AREA FUNCTIONAL

Let $\mathcal{M}$ and Σ be as in Assumption 4.1.1. The goal of this section is to define the admissible Q-valued normal variations of Σ in $\mathcal{M}$ and to compute the associated second variation functional. In what follows, we will denote by $\mathcal{A}_Q(\mathcal{M})$ the space of Q-points $T = \sum_\ell \llbracket p_\ell \rrbracket \in \mathcal{A}_Q(\mathbb{R}^d)$ with each p_ℓ in $\mathcal{M}$.

Definition 4.2.1. An *admissible* variational Q-field of Σ in $\mathcal{M}$ is a Lipschitz map

$$N := \sum_{\ell=1}^{Q} \llbracket N^\ell \rrbracket : \Sigma \to \mathcal{A}_Q(\mathbb{R}^d)$$

satisfying the following assumptions:

(H1) $N^\ell(x) \in T_x^\perp \Sigma \subset T_x\mathcal{M}$ for every $\ell \in \{1, \dots, Q\}$, for every $x \in \Sigma$;

(H2) N^ℓ vanishes in a neighborhood of $\partial\Sigma$ for every $\ell \in \{1, \dots, Q\}$.

Definition 4.2.2. Given an admissible variational Q-field N, the one-parameter family of Q-valued deformations of Σ in $\mathcal{M}$ induced by N is the map

$$F\colon \Sigma \times (-\delta, \delta) \to \mathcal{A}_Q(\mathcal{M})$$

defined by

$$F(x,t) := \sum_{\ell=1}^{Q} [\![\exp_x(tN^\ell(x))]\!], \tag{4.3}$$

where exp denotes the exponential map on $\mathcal{M}$.

Observe that, for any given N as in Definition 4.2.1, the induced one-parameter family of Q-valued deformations is always well defined for a positive δ which depends on the L^∞ norm of N and on the injectivity radius of $\mathcal{M}$. Note, furthermore, that $F(x,0) = Q[\![x]\!]$ for every $x \in \Sigma$, and that $F(x,t) = Q[\![x]\!]$ for all t if $x \in \partial\Sigma$.

If F is an admissible one-parameter family of Q-valued deformations, we will often write $F_t(x)$ instead of $F(x,t)$. Moreover, we will set $F^\ell_t(x) := \exp_x(tN^\ell(x))$.

In what follows, we will always assume to have fixed an orthonormal frame $(\xi_1, \dots, \xi_m)$ of the tangent bundle $T\Sigma$, so that $\vec{\xi} = \xi_1 \wedge \cdots \wedge \xi_m$ is a continuous simple unit m-vector field orienting Σ. Given any admissible variational Q-field N, we can now apply the results of Chapter 3, and consider the push-forward of Σ through the family F_t induced by N. An immediate consequence of Proposition 3.1.3 is that the resulting object is a one-parameter family of integer rectifiable m-currents, denoted $\Sigma_t := \mathbf{T}_{F_t} = (F_t)_\sharp[\![\Sigma]\!]$ with $\operatorname{spt}(\Sigma_t) \subset \mathcal{M}$. From (3.2), we have also the explicit representation formula

$$\Sigma_t(\omega) = \int_\Sigma \sum_{\ell=1}^{Q} \left\langle \omega\left(F^\ell_t(x)\right), DF^\ell_t(x)_\sharp \vec{\xi}(x) \right\rangle d\mathcal{H}^m(x) \quad \forall\, \omega \in \mathcal{D}^m(\mathbb{R}^d). \tag{4.4}$$

We will denote $\mu(t)$ the *mass* $\mathbb{M}(\Sigma_t)$ of the current Σ_t.

Definition 4.2.3. Let $\Sigma \subset \mathcal{M}$, and let N be an admissible variational Q-field. For any integer $j \geqslant 1$, the j^{th} order *variation* of Σ generated by N is the quantity

$$\delta^j[\![\Sigma]\!](N) := \left.\frac{d^j \mu}{dt^j}\right|_{t=0}. \tag{4.5}$$

$\delta^1[\![\Sigma]\!]$ is usually denoted $\delta[\![\Sigma]\!]$, and called first variation. $\delta^2[\![\Sigma]\!]$ is called second variation.

For every j, $\delta^j[\![\Sigma]\!]$ is a functional defined on the space of admissible variational Q-fields. In the following theorem we show that the first variation functional $\delta[\![\Sigma]\!]$ is identically zero under the assumption that Σ is minimal in $\mathcal{M}$. Furthermore, and more importantly for our purposes, we provide an explicit representation formula for $\delta^2[\![\Sigma]\!]$.

Theorem 4.2.4. *Let $\Sigma \hookrightarrow \mathcal{M}$ be as in Assumption 4.1.1. If* N *is an admissible variational* Q*-field of Σ in $\mathcal{M}$, then*

$$\delta[\![\Sigma]\!](N) = 0, \tag{4.6}$$

and

$$\delta^2[\![\Sigma]\!](N) = \mathrm{Dir}^{\mathcal{TM}}(N,\Sigma) - 2\int_\Sigma \sum_{\ell=1}^{Q} |A\cdot N^\ell|^2\, d\mathcal{H}^m - \int_\Sigma \sum_{\ell=1}^{Q} \mathcal{R}(N^\ell, N^\ell)\, d\mathcal{H}^m, \tag{4.7}$$

where

$$|A\cdot N^\ell|^2 := \sum_{i,j=1}^{m} |\langle A(\xi_i,\xi_j), N^\ell\rangle|^2 \tag{4.8}$$

and

$$\mathcal{R}(N^\ell, N^\ell) := \sum_{i=1}^{m} \langle R(N^\ell,\xi_i)\xi_i, N^\ell\rangle. \tag{4.9}$$

Remark 4.2.5. Observe that formula (4.7) makes sense because the quantity on the right-hand side does not depend on the particular selection chosen for N, nor on the orthonormal frame chosen for the tangent bundle $\mathcal{T}\Sigma$.

The first addendum in the sum is the Dirichlet energy of the multi-valued vector field N on the manifold $\mathcal{M}$ as defined in (4.2).

The second term in the sum can as well be given an intrinsic formulation, once we observe that $|A\cdot N^\ell|$ is the Hilbert-Schmidt norm of the symmetric bilinear form $A\cdot N^\ell\colon \mathcal{T}\Sigma\times\mathcal{T}\Sigma\to\mathbb{R}$ defined by $A\cdot N^\ell(\xi,\eta) := \langle A(\xi,\eta), N^\ell\rangle$.

Regarding the third term, the symmetry properties of the Riemann tensor allow to write

$$\langle R(N^\ell,\xi_i)\xi_i, N^\ell\rangle = \langle R(\xi_i, N^\ell)N^\ell, \xi_i\rangle = \langle \mathbf{p}\cdot R(\xi_i, N^\ell)N^\ell, \xi_i\rangle,$$

which in turn implies that $\mathcal{R}(N^\ell, N^\ell)$ coincides with the trace of the endomorphism

$$\xi \mapsto \mathbf{p}\cdot R(\xi, N^\ell)N^\ell$$

of the tangent bundle $\mathcal{T}\Sigma$. In other words, this term is a partial Ricci curvature in the direction of the vector field N^ℓ.

Proof of Theorem 4.2.4. Let N be an admissible variational Q-field of Σ in $\mathcal{M}$, and let $F = F(x,t)$ denote the induced one-parameter family of Q-valued deformations. The proof of the representation formulae (4.6) and (4.7) will be obtained by direct computation.

The starting point is the Q-valued area formula, Proposition 3.1.5, which yields an explicit formula for the function $\mu(t)$. Indeed, we may write

$$\mu(t) = \int_\Sigma \sum_{\ell=1}^{Q} \mathbf{J}F_t^\ell(x)\, d\mathcal{H}^m(x), \tag{4.10}$$

provided condition (3.7) is satisfied: that is, provided there is a set $B\subset\Sigma$ of full measure for which

$$\langle DF_t^\ell(x)_\sharp\vec{\xi}(x), DF_t^{\ell'}(y)_\sharp\vec{\xi}(y)\rangle \geqslant 0 \quad \forall x,y\in B \text{ and } \ell,\ell' \text{ with } F_t^\ell(x) = F_t^{\ell'}(y). \tag{4.11}$$

Now, it is not difficult to show that in fact condition (4.11) holds with $B=\Sigma$: to see this, first observe that since Σ is compact there exists a number $\varepsilon>0$ such that $\langle\vec{\xi}(x),\vec{\xi}(y)\rangle \geqslant \frac{1}{2}$

for all points $x, y \in \Sigma$ such that $\mathbf{d}(x,y) \leqslant \varepsilon$. On the other hand, the very definition of F implies that for any $x \in \Sigma$ one may write $F^\ell_t(x) = x + tN^\ell(x) + o(t)$ for $t \to 0$. Therefore, if $|t|$ is chosen small enough, depending on Σ, ε and on the L^∞ norm of N in Σ, the condition $F^\ell_t(x) = F^{\ell'}_t(y)$ implies $\mathbf{d}(x,y) \leqslant \varepsilon$ and consequently the condition $\langle \vec{\xi}(x), \vec{\xi}(y)\rangle \geqslant \frac{1}{2}$. But now, since $DF^\ell_t(x) = \mathrm{Id} + tDN^\ell(x) + o(t)$, we easily infer that $\langle DF^\ell_t(x)_\sharp \vec{\xi}(x), DF^{\ell'}_t(y)_\sharp \vec{\xi}(y)\rangle \geqslant \frac{1}{4}$ for all $x, y \in \Sigma$ and ℓ, ℓ' with $F^\ell_t(x) = F^{\ell'}_t(y)$ provided $|t| \leqslant \delta_0$ for some $\delta_0 = \delta_0\big(\Sigma, \varepsilon, \|N\|_{L^\infty(\Sigma)}, \mathrm{Lip}(N)\big)$.

Thus, we can work on each component F^ℓ of the decomposition of F separately: in the end, we will just apply (4.10) to obtain the desired variation formulae. Moreover, since the coming arguments are local, we will assume in what follows that the frame $\{\xi_i\}_{i=1}^m$ is C^2 and that the selection $N = \sum_\ell [\![N^\ell]\!]$ is Lipschitz in a neighborhood of any given point x.

With that being said, let us now consider a fixed value of $\ell \in \{1, \dots, Q\}$ and introduce the following quantities. For any x point of differentiability for N in Σ, let $Z^\ell(x) := \partial_{tt} F^\ell(x, 0)$ denote the initial acceleration of the ℓ^{th} sheet at the point x, so that the second order Taylor expansion of $F^\ell(x, \cdot)$ around $t = 0$ is

$$F^\ell(x,t) = x + tN^\ell(x) + \frac{1}{2}t^2 Z^\ell(x) + o(t^2)$$

in a suitable δ-neighborhood of $t = 0$. Then, for any $i \in \{1, \dots, m\}$, define

$$e^\ell_i = e^\ell_i(x,t) := D_{\xi_i} F^\ell_t(x) = \xi_i(x) + tD_{\xi_i} N^\ell(x) + \frac{1}{2}t^2 D_{\xi_i} Z^\ell(x) + o(t^2) \tag{4.12}$$

and

$$V^\ell = V^\ell(x,t) := \partial_t F^\ell(x,t). \tag{4.13}$$

Observe that e^ℓ_i and V^ℓ are tangent vector fields to $\mathcal{M}$.

Next, for $i, j \in \{1, \dots, m\}$ denote

$$\mathbf{g}^\ell_{ij} = \mathbf{g}^\ell_{ij}(x,t) := \langle e^\ell_i(x,t), e^\ell_j(x,t)\rangle \tag{4.14}$$

and

$$\mathbf{g}^\ell = \mathbf{g}^\ell(x,t) := \det(\mathbf{g}^\ell_{ij}(x,t)). \tag{4.15}$$

Using the above notation, we readily see that the Jacobian determinant $\mathbf{J}F^\ell_t$ can be written as follows:

$$J^\ell = J^\ell(x,t) := \mathbf{J}F^\ell_t(x) = \sqrt{\mathbf{g}^\ell(x,t)}, \tag{4.16}$$

so that, finally, the mass of the push-forwarded current is given by

$$\mu(t) = \sum_{\ell=1}^{Q} \mu^\ell(t), \tag{4.17}$$

where

$$\mu^\ell(t) := \int_\Sigma J^\ell(x,t)\, d\mathcal{H}^m(x). \tag{4.18}$$

Thus, we conclude that the first and second variation of Σ under the deformation generated by N can be represented in the following way:

$$\delta[\![\Sigma]\!](N) = \sum_{\ell=1}^{Q} \frac{d\mu^\ell}{dt}\Big|_{t=0} \tag{4.19}$$

and

$$\delta^2[\![\Sigma]\!](N) = \sum_{\ell=1}^{Q} \frac{d^2\mu^\ell}{dt^2}\Big|_{t=0}. \tag{4.20}$$

In what follows, in order to simplify the notation, we will drop the superscript ℓ when carrying on the computation.

One has:

$$\frac{d\mu}{dt}\Big|_{t=0} = \int_\Sigma \partial_t J(x,0)\, d\mathcal{H}^m(x). \tag{4.21}$$

Now, since

$$\partial_t J = \frac{1}{2J}\partial_t \mathbf{g},$$

and since $\mathbf{g}_{ij} = \delta_{ij}$ at time $t = 0$, easy computations show that

$$\partial_t J|_{t=0} = \frac{1}{2}\sum_{i=1}^{m} \partial_t \mathbf{g}_{ii}|_{t=0} = \sum_{i=1}^{m} \langle e_i, \partial_t e_i\rangle|_{t=0}, \tag{4.22}$$

and thus

$$\delta[\![\Sigma]\!](N) = \int_\Sigma \sum_{\ell=1}^{Q}\sum_{i=1}^{m} \langle \xi_i, D_{\xi_i} N^\ell\rangle\, d\mathcal{H}^m = \int_\Sigma \sum_{\ell=1}^{Q}\sum_{i=1}^{m} \langle \xi_i, \nabla_{\xi_i} N^\ell\rangle\, d\mathcal{H}^m = \int_\Sigma \sum_{\ell=1}^{Q} \mathrm{div}_\Sigma(N^\ell)\, d\mathcal{H}^m.$$

In particular, recalling the definition of the map $\boldsymbol{\eta}$ in (2.14), we deduce from the linearity of the divergence operator that

$$\delta[\![\Sigma]\!](N) = Q\int_\Sigma \mathrm{div}_\Sigma(\boldsymbol{\eta}\circ N)\, d\mathcal{H}^m, \tag{4.23}$$

where $\boldsymbol{\eta}\circ N\colon \Sigma \to \mathbb{R}^d$, the "average" of the sheets of the vector field N, is a classical single-valued Lipschitz map. Note that if N is single-valued then $\boldsymbol{\eta}\circ N \equiv N$, and we recover the usual formulation of the first variation formula in terms of the divergence of the variational vector field. Observe now that the average $\boldsymbol{\eta}\circ N$ vanishes in a neighborhood of $\partial\Sigma$ and satisfies $\boldsymbol{\eta}\circ N(x) \in T_x^\perp\Sigma \subset T_x\mathcal{M}$ for every $x \in \Sigma$. Hence, for every $i \in \{1,\dots,m\}$ the scalar product $\langle \xi_i, \boldsymbol{\eta}\circ N\rangle$ is everywhere vanishing, and we have that $\langle \xi_i, \nabla_{\xi_i}(\boldsymbol{\eta}\circ N)\rangle = -\langle \nabla_{\xi_i}\xi_i, \boldsymbol{\eta}\circ N\rangle = -\langle A(\xi_i,\xi_i), \boldsymbol{\eta}\circ N\rangle$. Therefore, recalling the definition of the mean curvature vector H as the trace of the second fundamental form, one can also write

$$\delta[\![\Sigma]\!](N) = Q\int_\Sigma \sum_{i=1}^{m} \langle \xi_i, \nabla_{\xi_i}(\boldsymbol{\eta}\circ N)\rangle\, d\mathcal{H}^m = -Q\int_\Sigma \langle H, \boldsymbol{\eta}\circ N\rangle\, d\mathcal{H}^m = 0 \tag{4.24}$$

because Σ is minimal in $\mathcal{M}$. This proves (4.6).

Next, we go further and we compute the second variation of the mass. We first write, for every t and for every $x \in \Sigma$ of differentiability for the variational field:

$$\partial_t J = \frac{1}{2\sqrt{\mathbf{g}}}\partial_t \mathbf{g} = \frac{1}{2} J \frac{1}{\mathbf{g}} \partial_t \mathbf{g} = \frac{1}{2} J \partial_t \left(\log(\mathbf{g})\right) = \frac{1}{2} J \mathbf{g}^{ij} \partial_t \mathbf{g}_{ji},$$

where in the last identity we have used Jacobi's formula

$$\partial_t \log \det A(t) = \operatorname{tr}\left(A(t)^{-1} \cdot \partial_t A(t)\right)$$

for any invertible matrix $A(t)$ with positive determinant. Moreover, $\left(\mathbf{g}^{ij}\right)$ is the inverse matrix of $\left(\mathbf{g}_{ij}\right)$, and Einstein's convention on the summation of repeated indices has been used. Now, since

$$\partial_t \mathbf{g}_{ji} = \partial_t \left(\langle e_j, e_i \rangle\right) = \langle \partial_t e_j, e_i \rangle + \langle e_j, \partial_t e_i \rangle,$$

and using the fact that the matrix $\left(\mathbf{g}^{ij}\right)$ is symmetric, we can conclude the following identity:

$$\partial_t J = J \mathbf{g}^{ij} \langle e_i, \partial_t e_j \rangle.$$

In turn, this produces:

$$\partial_{tt} J = \underbrace{\left(\partial_t J\right) \mathbf{g}^{ij} \langle e_i, \partial_t e_j \rangle}_{=:I} + \underbrace{J \left(\partial_t \mathbf{g}^{ij}\right) \langle e_i, \partial_t e_j \rangle}_{=:II} + \underbrace{J \mathbf{g}^{ij} \partial_t \left(\langle e_i, \partial_t e_j \rangle\right)}_{=:III}. \tag{4.25}$$

Now, we evaluate equation (4.25) at time $t = 0$. Regarding the first term in the sum, we use (4.22), the orthonormality condition $\mathbf{g}^{ij}\big|_{t=0} = \delta^{ij}$ and the fact that $e_i\big|_{t=0} = \xi_i$, $\partial_t e_i\big|_{t=0} = D_{\xi_i} N$ (here, of course, we are writing N instead of N^ℓ) to conclude

$$I\big|_{t=0} = \left(\sum_{i=1}^{m} \langle \xi_i, \nabla_{\xi_i} N \rangle\right)^2. \tag{4.26}$$

Since $N = N^\ell$ is Lipschitz, and since $\langle \xi_i, N \rangle \equiv 0$, we have $\langle \xi_i, \nabla_{\xi_i} N \rangle = -\langle A(\xi_i, \xi_i), N \rangle$, and thus

$$I\big|_{t=0} = \left(\langle H, N \rangle\right)^2 = 0 \tag{4.27}$$

due to the minimality of Σ.

In order to derive a formula for $II\big|_{t=0}$, we first differentiate the identity

$$\mathbf{g}^{ij} \mathbf{g}_{jh} = \delta^i_h$$

to obtain that

$$\partial_t \mathbf{g}^{ij} = -\mathbf{g}^{ik} \left(\partial_t \mathbf{g}_{kh}\right) \mathbf{g}^{hj},$$

whence

$$\partial_t \mathbf{g}^{ij}\big|_{t=0} = -\partial_t \mathbf{g}_{ij}\big|_{t=0} = -\left(\langle \nabla_{\xi_i} N, \xi_j \rangle + \langle \xi_i, \nabla_{\xi_j} N \rangle\right). \tag{4.28}$$

Since $\langle \nabla_{\xi_i} N, \xi_j \rangle = -\langle A\left(\xi_i, \xi_j\right), N \rangle$, the symmetry of the second fundamental form implies

$$\partial_t \mathbf{g}^{ij}\big|_{t=0} = 2 \langle A\left(\xi_i, \xi_j\right), N \rangle. \tag{4.29}$$

Again, since

$$\langle e_i, \partial_t e_j\rangle\big|_{t=0} = \langle \xi_i, \nabla_{\xi_j} N\rangle = -\langle A\left(\xi_i, \xi_j\right), N\rangle,$$

we can finally obtain

$$\mathrm{II}\big|_{t=0} = -2\sum_{i,j=1}^{m} |\langle A(\xi_i, \xi_j), N\rangle|^2. \tag{4.30}$$

Finally, we compute $\mathrm{III}\big|_{t=0}$. The simplest way to do it is to regard the operator ∂_t as the covariant derivative along the vector field $V = V^\ell$. One therefore has:

$$\begin{aligned}\partial_t\left(\langle e_i, \partial_t e_j\rangle\right) &= V\langle e_i, \nabla_V e_j\rangle\\ &= \langle \nabla_V e_i, \nabla_V e_j\rangle + \langle e_i, \nabla_V \nabla_V e_j\rangle\\ &= \langle \nabla_{e_i} V, \nabla_{e_j} V\rangle + \langle e_i, \nabla_V \nabla_{e_j} V\rangle,\end{aligned}$$

where in the last identity we have used the fact that the vector fields e_i and V commute, and, of course, that the Riemannian connection on $\mathcal{M}$ is torsion-free. Now, using again that $[V, e_j] = 0$ and the definition of the curvature tensor R, we may write

$$\nabla_V \nabla_{e_j} V = \nabla_{e_j} \nabla_V V + R(V, e_j)V,$$

so that, finally, the evaluation of $\partial_t\left(\langle e_i, \partial_t e_j\rangle\right)$ at time $t = 0$ yields

$$\partial_t\left(\langle e_i, \partial_t e_j\rangle\right)\big|_{t=0} = \langle \nabla_{\xi_i} N, \nabla_{\xi_j} N\rangle + \langle \xi_i, \nabla_{\xi_j} Z\rangle + \langle \xi_i, R(N, \xi_j)N\rangle,$$

with $Z = Z^\ell$. Since $J\big|_{t=0} = 1$ and $\mathbf{g}^{ij}\big|_{t=0} = \delta^{ij}$, we conclude the following identity:

$$\mathrm{III}\big|_{t=0} = \sum_{i=1}^{m} |\nabla_{\xi_i} N|^2 + \mathrm{div}_\Sigma Z - \sum_{i=1}^{m} \langle R(N, \xi_i)\xi_i, N\rangle. \tag{4.31}$$

Observe that, in deriving formula (4.31), we have used that $\langle R(X, Y)U, W\rangle = -\langle R(X, Y)W, U\rangle$ for any choice of X, Y, U, W vector fields on $\mathcal{M}$.

We have now all the tools to conclude: from the Q-valued area formula (4.10) it follows that

$$\left.\frac{d^2\mu}{dt^2}\right|_{t=0} = \int_\Sigma \sum_{\ell=1}^{Q} \partial_{tt} J^\ell(x, 0)\, d\mathcal{H}^m(x),$$

thus it suffices to plug equations (4.27), (4.30), (4.31) in (4.25) to get

$$\begin{aligned}\delta^2[\![\Sigma]\!](N) = &\int_\Sigma \sum_{\ell=1}^{Q}\left(\sum_{i=1}^{m} |\nabla_{\xi_i} N^\ell|^2 - 2\sum_{i,j=1}^{m} |\langle A(\xi_i, \xi_j), N^\ell\rangle|^2 - \sum_{i=1}^{m}\langle R(N^\ell, \xi_i)\xi_i, N^\ell\rangle\right) d\mathcal{H}^m\\ &+ Q\int_\Sigma \mathrm{div}_\Sigma(\boldsymbol{\eta}\circ Z)\, d\mathcal{H}^m,\end{aligned} \tag{4.32}$$

where $Z := \sum_\ell [\![Z^\ell]\!]$. Now, we decompose

$$\boldsymbol{\eta}\circ Z = \mathbf{p}\cdot(\boldsymbol{\eta}\circ Z) + \mathbf{p}^\perp\cdot(\boldsymbol{\eta}\circ Z) = (\boldsymbol{\eta}\circ Z)^\top + (\boldsymbol{\eta}\circ Z)^\perp, \tag{4.33}$$

and we see that, since $\langle \xi_i, (\eta \circ Z)^\perp \rangle = 0$ for all i,

$$\operatorname{div}_\Sigma((\eta \circ Z)^\perp) = \sum_{i=1}^m \langle \xi_i, \nabla_{\xi_i}(\eta \circ Z)^\perp \rangle = -\sum_{i=1}^m \langle A(\xi_i, \xi_i), \eta \circ Z \rangle = -\langle H, \eta \circ Z \rangle = 0. \quad (4.34)$$

On the other hand, Stokes' theorem and the fact that N is vanishing in a neighborhood of $\partial\Sigma$ readily imply that

$$\int_\Sigma \operatorname{div}_\Sigma((\eta \circ Z)^\top)\, d\mathcal{H}^m = 0, \quad (4.35)$$

and thus the last addendum on the right-hand side of (4.32) vanishes. This completes the proof of formula (4.7). □

We note now that the quantity appearing on the right-hand side of formula (4.32) is in fact well defined for any Q-valued vector field tangent to $\mathcal{M}$ and belonging to the class $W^{1,2}\left(\Sigma, \mathcal{A}_Q(\mathbb{R}^d)\right)$. This motivates the following definitions.

Definition 4.2.6 ($W^{1,2}$ sections of the normal bundle)**.** Let $\Sigma \hookrightarrow \mathcal{M}$ be as above, and let $\Omega \subset \Sigma$ be open. We define the class of $W^{1,2}$ *sections* of the normal bundle of Ω in $\mathcal{M}$, denoted $\Gamma^{1,2}_Q(\mathcal{N}\Omega)$, as follows:

$$\Gamma^{1,2}_Q(\mathcal{N}\Omega) := \left\{ N \in W^{1,2}\left(\Omega, \mathcal{A}_Q(\mathbb{R}^d)\right) \,:\, \operatorname{spt}(N(x)) \subset T_x^\perp\Sigma \subset T_x\mathcal{M} \text{ for } \mathcal{H}^m\text{-a.e. } x \in \Omega \right\}. \quad (4.36)$$

Definition 4.2.7 (Jacobi functional)**.** For a section $N \in \Gamma^{1,2}_Q(\mathcal{N}\Omega)$, the *Jacobi functional*, or *stability functional*, is defined by:

$$\operatorname{Jac}(N, \Omega) := \operatorname{Dir}^{\mathcal{TM}}(N, \Omega) - 2\int_\Omega \sum_{\ell=1}^Q |A \cdot N^\ell|^2\, d\mathcal{H}^m - \int_\Omega \sum_{\ell=1}^Q \mathcal{R}(N^\ell, N^\ell)\, d\mathcal{H}^m. \quad (4.37)$$

Our first observation is that the classical theory of the Jacobi normal operator can be recovered within the above framework by simply setting $Q = 1$.

Remark 4.2.8. Consider the classical single-valued setting, corresponding to $Q = 1$, let $\Omega = \Sigma$ and recall that

$$\operatorname{Dir}^{\mathcal{TM}}(N, \Sigma) = \int_\Sigma \sum_{i=1}^m |\nabla_{\xi_i} N|^2\, d\mathcal{H}^m$$

for any orthonormal frame $(\xi_1, \dots, \xi_m)$ of $\mathcal{T}\Sigma$. Assume also that N is Lipschitz continuous for convenience. Let $(\nu_1, \dots, \nu_k)$ be local sections of the normal bundle $\mathcal{N}\Sigma$ of Σ in $\mathcal{M}$ such that, at each point $x \in \Sigma$, the system $\left((\xi_j(x))_{j=1}^m, (\nu_\alpha(x))_{\alpha=1}^k\right)$ is an orthonormal basis of $T_x\mathcal{M}$. Then, for every point of differentiability for N and for every $i = 1, \dots, m$ we have:

$$|\nabla_{\xi_i} N|^2 = \sum_{j=1}^m |\langle \nabla_{\xi_i} N, \xi_j \rangle|^2 + \sum_{\alpha=1}^k |\langle \nabla_{\xi_i} N, \nu_\alpha \rangle|^2.$$

Now, the usual considerations about the orthogonality of N and ξ_j imply that $\langle \nabla_{\xi_i} N, \xi_j \rangle = -\langle A(\xi_i, \xi_j), N \rangle$. We therefore obtain that

$$\int_\Sigma \left(\sum_{i=1}^m |\nabla_{\xi_i} N|^2 - \sum_{i,j=1}^m |\langle A(\xi_i, \xi_j), N \rangle|^2 \right) d\mathcal{H}^m = \int_\Sigma \sum_{i=1}^m \sum_{\alpha=1}^k |\langle \nabla_{\xi_i} N, \nu_\alpha \rangle|^2 \, d\mathcal{H}^m,$$

and finally conclude the identity

$$\mathrm{Jac}(N, \Sigma) = \int_\Sigma \left(\sum_{i=1}^m \sum_{\alpha=1}^k |\langle \nabla_{\xi_i} N, \nu_\alpha \rangle|^2 - \sum_{i,j=1}^m |\langle A(\xi_i, \xi_j), N \rangle|^2 - \sum_{i=1}^m \langle R(N, \xi_i)\xi_i, N \rangle \right) d\mathcal{H}^m. \tag{4.38}$$

It is immediately seen that the Euler-Lagrange operator associated to the second variation functional (4.38) is the linear elliptic operator $\mathcal{L}$ defined on the space of sections of $\mathcal{N}\Sigma$ and given by

$$\mathcal{L} := -\Delta_\Sigma^\perp - \mathscr{A} - \mathscr{R}, \tag{4.39}$$

where $\Delta_\Sigma^\perp$ is the Laplacian on the normal bundle of Σ, $\mathscr{A}$ is Simons' operator, defined by

$$\mathscr{A}(N) := \sum_{i,j=1}^m \langle A(\xi_i, \xi_j), N \rangle A(\xi_i, \xi_j), \tag{4.40}$$

and $\mathscr{R}$ is given by

$$\mathscr{R}(N) := \sum_{i=1}^m \mathbf{p}^\perp \cdot R(N, \xi_i)\xi_i. \tag{4.41}$$

As already anticipated in the Introduction, the operator $\mathcal{L}$ is classically called *Jacobi normal operator*, and the solutions of the differential equation $\mathcal{L}(N) = 0$ (that is, the normal vector fields that are in its kernel) are known in the literature as *Jacobi fields*. The importance of studying the second variation operator of minimal submanifolds into Riemannian manifolds is well justified by the arguments given earlier on in this section: in the single valued case $Q = 1$, the Jacobi operator $\mathcal{L}$ carries the information about the stability properties of the submanifold itself, when it is thought of as a stationary point for the m-dimensional volume. In particular, non-trivial Jacobi fields vanishing on $\partial\Sigma$ are, when they exist, the infinitesimal normal deformations of Σ which preserve the volume up to second order. From a functional analytic point of view, $\mathcal{L}$ is a second-order strongly elliptic operator. When diagonalized on the space of sections of $\mathcal{N}\Sigma$ vanishing on $\partial\Sigma$ with respect to the standard inner product, it exhibits distinct, real eigenvalues $\{\lambda_h\}_{h=1}^\infty$ (counted with multiplicities) such that

$$\lambda_1 < \lambda_2 < \cdots < \lambda_h < \cdots \to +\infty.$$

Moreover, the dimension of each eigenspace is finite. The sum of the dimensions of the eigenspaces corresponding to negative eigenvalues is called the *Morse index* of Σ: it accounts for the number of independent infinitesimal normal deformations of Σ which do decrease the volume at second order. If $\lambda = 0$ is an eigenvalue, then the dimension of $\ker(\mathcal{L})$ is called *nullity*. We recall that Σ is called *stable* if its Morse index is 0, and *strictly stable* if there exist no non-trivial Jacobi fields vanishing at the boundary, i.e. if $\mathrm{index}(\Sigma) + \mathrm{nullity}(\Sigma) = 0$.

A systematic study of the Jacobi normal operator was initiated by J. Simons in [Sim68]. One of Simons' main results was to prove that if $\mathcal{M} = \mathbb{S}^{m+1}$ and Σ^m is a closed minimal hypersurface immersed in $\mathbb{S}^{m+1}$ which is not totally geodesic then the first eigenvalue of the operator $\mathcal{L}$ satisfies the upper bound $\lambda_1 \leqslant -m$. As a consequence of this, he was able to show that no non-trivial stable minimal hypercones exist in $\mathbb{R}^{m+1}$ for $m \leqslant 6$. In turn, this led to the proof of the Bernstein conjecture, stating that the only entire solutions $u\colon \mathbb{R}^m \to \mathbb{R}$ of the minimal surface equation are linear, for every $m \leqslant 7$. The result is sharp, as the Bernstein conjecture was proved to be false for $m > 7$ by E. Bombieri, E. De Giorgi and E. Giusti in [BDGG69].

The considerations leading to formula (4.38) can be repeated in the Q-valued setting, thus showing that the Definition 4.2.7 of the Jacobi functional agrees with the one given in formula (1.1). This equivalence is recorded in Lemma 4.2.10 below. We first need a definition.

Definition 4.2.9 (Normal Dirichlet energy of a section)**.** Let $N \in \Gamma^{1,2}_Q(\mathcal{N}\Omega)$. For any point $x \in \Omega$ where N is approximately differentiable, and for any tangent vector field ξ, set

$$\nabla^\perp_\xi N(x) := \sum_{\ell=1}^{Q} [\![\mathbf{p}^{\Sigma\perp}_x \cdot D_\xi N^\ell(x)]\!], \tag{4.42}$$

where $\mathbf{p}^{\Sigma\perp}_x = \mathbf{p}^\perp_x \circ \mathbf{p}^{\mathcal{M}}_x$ is the orthogonal projection of $\mathbb{R}^d$ onto $T^\perp_x\Sigma$. Then, the normal Dirichlet energy of N in Ω is the quantity

$$\mathrm{Dir}^{\mathcal{N}\Sigma}(N,\Omega) := \int_\Omega \sum_{i=1}^{m} |\nabla^\perp_{\xi_i} N|^2 \, d\mathcal{H}^m, \tag{4.43}$$

for any choice of a (local) orthonormal frame $\{\xi_i\}_{i=1}^m$ of $\mathcal{T}\Sigma$.

Lemma 4.2.10 (Equivalence of the definitions of the Jac functional)**.** *For any* $N = \sum_\ell [\![N^\ell]\!] \in \Gamma^{1,2}_Q(\mathcal{N}\Omega)$ *it holds*

$$\begin{aligned} \mathrm{Jac}(N,\Omega) &= \mathrm{Dir}^{\mathcal{N}\Sigma}(N,\Omega) - \int_\Omega \sum_{\ell=1}^{Q} |A\cdot N^\ell|^2 \, d\mathcal{H}^m - \int_\Omega \sum_{\ell=1}^{Q} \mathcal{R}(N^\ell, N^\ell)\, d\mathcal{H}^m \\ &= \int_\Omega \sum_{\ell=1}^{Q} \left(\sum_{i=1}^{m}\sum_{\alpha=1}^{k} |\langle D_{\xi_i} N^\ell, \nu_\alpha\rangle|^2 - \sum_{i,j=1}^{m} |\langle A(\xi_i,\xi_j), N^\ell\rangle|^2 - \sum_{i=1}^{m} \langle R(\xi_i, N^\ell)N^\ell, \xi_i\rangle \right) d\mathcal{H}^m, \end{aligned} \tag{4.44}$$

where $\{\xi_i\}_{i=1}^m$ *and* $\{\nu_\alpha\}_{\alpha=1}^k$ *are (local) orthonormal frames of* $\mathcal{T}\Sigma$ *and* $\mathcal{N}\Sigma$ *respectively.*

Proof. It is a straightforward consequence of the arguments contained in Remark 4.2.8 combined with the Lipschitz approximation theorem, Proposition 2.2.10 (cf. also [DLS14, Lemma 4.5]) and the Lipschitz selection property in Proposition 2.2.7. □

On the other hand, unlike the single-valued case, the lack of linear structure of $\Gamma^{1,2}_Q(\mathcal{N}\Omega)$ in the multi-valued case $Q > 1$ does not allow one to associate an operator to the Jacobi

functional, nor to characterize multiple valued Jacobi fields as the solutions of a certain (Euler-Lagrange) PDE. Nonetheless, the variational structure of the problem suggests that the minimizers of the Jacobi functional for a given boundary datum have the right to be considered the multi-valued counterpart of the classical Jacobi fields. This justifies the following definition.

Definition 4.2.11. Let $\Sigma \hookrightarrow \mathcal{M}$ be as in Assumption 4.1.1, and let $\Omega \subset \Sigma$ be a Lipschitz open set. A map $N \in \Gamma^{1,2}_Q(\mathcal{N}\Omega)$ is said to be a *Jac-minimizer*, or a *Jacobi Q-field* in Ω, if it minimizes the Jacobi functional among all Q-valued $W^{1,2}$ sections of the normal bundle of Ω in $\mathcal{M}$ having the same trace at the boundary, that is

$$\mathrm{Jac}(N, \Omega) \leqslant \mathrm{Jac}(u, \Omega) \qquad \text{for all } u \in \Gamma^{1,2}_Q(\mathcal{N}\Omega) \text{ such that } u|_{\partial\Omega} = N|_{\partial\Omega}. \tag{4.45}$$

4.3 JACOBI Q-FIELDS

The goal of this section is to provide the two fundamental tools which will be used in Chapter 5 to address the question of the existence of Jacobi Q-fields N in Ω with prescribed boundary value $N|_{\partial\Omega} = g|_{\partial\Omega}$ for some fixed $g \in \Gamma^{1,2}_Q(\mathcal{N}\Omega)$, and ultimately to prove Theorem 5.0.1. These tools are encoded in Proposition 4.3.1 and Lemma 4.3.4 below. The proof of Theorem 5.0.1 will be obtained by direct methods in the Calculus of Variations, and therefore it is natural to analyze the properties of *lower semi-continuity* and *compactness* of the stability functional. The proof of the weak lower semi-continuity is rather simple, and it is the content of the following proposition.

Proposition 4.3.1. *The Jacobi functional is lower semi-continuous with respect to the weak topology of* $\Gamma^{1,2}_Q(\mathcal{N}\Omega)$.

Before coming to the proof, it will be useful to make some further considerations about the structure of the Jacobi functional, in order to simplify the notation and to express it as a perturbation of the Dirichlet functional $\mathrm{Dir}(u, \Omega)$.

Remark 4.3.2. Given any Q-valued Lipschitz map $u = \sum_\ell \llbracket u^\ell \rrbracket$ satisfying $u^\ell(x) \in T^\perp_x\Sigma \subset T_x\mathcal{M}$ for all $x \in \Omega$, the orthogonal decomposition

$$D_\xi u^\ell(x) = \mathbf{p}^{\mathcal{M}}_x \cdot D_\xi u^\ell(x) + \mathbf{p}^{\mathcal{M}\perp}_x \cdot D_\xi u^\ell(x) = \nabla_\xi u^\ell(x) + \overline{A}_x\left(\xi(x), u^\ell(x)\right)$$

holds for any tangent vector field ξ at any point x of differentiability for u, hence $\mathcal{H}^m$-a.e. in Ω. Here, $\overline{A}$ denotes the second fundamental form of the embedding $\mathcal{M} \hookrightarrow \mathbb{R}^d$. Hence, at any such point we may write

$$|\nabla_\xi u^\ell|^2 = |D_\xi u^\ell|^2 - |\overline{A}(\xi, u^\ell)|^2.$$

These considerations are extended in the obvious way to all sections $u \in \Gamma^{1,2}_Q(\mathcal{N}\Omega)$ at all points of approximate differentiability. Ultimately, we will write

$$\mathrm{Jac}(u, \Omega) = \mathrm{Dir}(u, \Omega) - \mathcal{B}_\Omega(u), \tag{4.46}$$

where $\mathcal{B}_\Omega$ is the functional on $\Gamma^{1,2}_Q(\mathcal{N}\Omega)$ defined by

$$\mathcal{B}_\Omega(u) := \int_\Omega \sum_{\ell=1}^{Q} \left(\sum_{i=1}^{m} |\overline{A}(\xi_i, u^\ell)|^2 + 2 \sum_{i,j=1}^{m} |\langle A(\xi_i, \xi_j), u^\ell \rangle|^2 + \sum_{i=1}^{m} \langle R(u^\ell, \xi_i)\xi_i, u^\ell \rangle \right) d\mathcal{H}^m. \tag{4.47}$$

Observe that $\mathcal{B}_\Omega$ satisfies an estimate of the form

$$|\mathcal{B}_\Omega(u)| \leqslant C_0 \|u\|^2_{L^2}, \tag{4.48}$$

where C_0 is a *geometric* constant, depending on $\mathbf{A}, \overline{\mathbf{A}}, \mathbf{R}$, where

$$\mathbf{A} = \|A\|_{L^\infty} := \max_{x\in\Sigma} \max \left\{ |A_x(X,Y)| : X, Y \in \mathbb{S}^{m-1} \subset T_x\Sigma \right\}, \tag{4.49}$$

$$\overline{\mathbf{A}} = \|\overline{A}\|_{L^\infty} := \max_{x\in\Sigma} \max \left\{ |\overline{A}_x(X,Y)| : X, Y \in \mathbb{S}^{m+k-1} \subset T_x\mathcal{M} \right\}, \tag{4.50}$$

$$\mathbf{R} = \|R\|_{L^\infty} := \max_{x\in\Sigma} \max \left\{ |\mathbf{p}^{\Sigma\perp}_x \cdot R_x(X,Y)Z| : X \in T^\perp_x\Sigma, \quad Y, Z \in T_x\Sigma, \quad |X| = |Y| = |Z| = 1 \right\}. \tag{4.51}$$

Proof of Proposition 4.3.1. Consider Q-valued sections $u_h, u \in \Gamma^{1,2}_Q(\mathcal{N}\Omega)$ and assume that $u_h \rightharpoonup u$ weakly in $W^{1,2}(\Omega; \mathcal{A}_Q)$ as in Definition 2.2.15. Now, use (4.46) in order to write

$$\mathrm{Jac}(u_h, \Omega) = \mathrm{Dir}(u_h, \Omega) - \mathcal{B}_\Omega(u_h).$$

The weak lower semi-continuity of the Dirichlet energy was proved by De Lellis and Spadaro in [DLS11, Proof of Theorem 0.8, p.30]. On the other hand, the condition

$$\lim_{h\to\infty} \int_\Omega \mathcal{G}(u_h, u)^2 \, d\mathcal{H}^m = 0$$

is enough to show that in fact

$$\lim_{h\to\infty} \mathcal{B}_\Omega(u_h) = \mathcal{B}_\Omega(u). \tag{4.52}$$

To see this, just write (4.52) as

$$\lim_h \int_\Omega b_h \, d\mathcal{H}^m = \int_\Omega b \, d\mathcal{H}^m, \tag{4.53}$$

with

$$b_h(x) = \sum_{\ell=1}^{Q} \left(\sum_{i=1}^{m} |\overline{A}(\xi_i, u^\ell_h)|^2 + 2 \sum_{i,j=1}^{m} |\langle A(\xi_i, \xi_j), u^\ell_h \rangle|^2 + \sum_{i=1}^{m} \langle R(u^\ell_h, \xi_i)\xi_i, u^\ell_h \rangle \right)$$

and

$$b(x) = \sum_{\ell=1}^{Q} \left(\sum_{i=1}^{m} |\overline{A}(\xi_i, u^\ell)|^2 + 2 \sum_{i,j=1}^{m} |\langle A(\xi_i, \xi_j), u^\ell \rangle|^2 + \sum_{i=1}^{m} \langle R(u^\ell, \xi_i)\xi_i, u^\ell \rangle \right),$$

and observe that the strong convergence $u_h \to u$ in $L^2(\Omega, \mathcal{A}_Q)$ suffices to prove that along a subsequence (not relabeled) $b_h(x) \to b(x)$ pointwise $\mathcal{H}^m$-a.e. in Ω. Equation (4.53) then follows by standard techniques in integration theory. □

As for compactness, the situation is much more involved. Indeed, as already anticipated, the existence of a solution of the minimum problem for the Jacobi functional with any boundary datum g is strictly related with showing that $N_0 \equiv Q[\![0]\!]$ is in fact the *only* minimizer under $Q[\![0]\!]$ boundary conditions.

Remark 4.3.3. If $N \in \Gamma^{1,2}_Q(\mathcal{N}\Omega)$ is a Jacobi Q-field with $N|_{\partial\Omega} = Q[\![0]\!]$, then $\mathrm{Jac}(N,\Omega) = 0$. This is a consequence of the homogeneity properties of the functional: in this case, indeed, for any $t \in \mathbb{R}$ the Q-field $tN := \sum_\ell [\![tN^\ell]\!]$ is a suitable competitor, and

$$\mathrm{Jac}(tN,\Omega) = t^2 \mathrm{Jac}(N,\Omega).$$

Hence, were $\mathrm{Jac}(N,\Omega) < 0$ [1], one would obtain that $\lim_{t\to\infty} \mathrm{Jac}(tN,\Omega) = -\infty$, thus contradicting the definition of N.

We are then able to conclude that if the minimum problem for the Jacobi functional with $Q[\![0]\!]$ boundary data does admit a solution, then for any minimizer N one has $\mathrm{Jac}(N,\Omega) = 0$. In particular, $N_0 \equiv Q[\![0]\!]$ is a minimizer.

The condition that $N_0 \equiv Q[\![0]\!]$ is the only minimizer for its boundary value will be referred to as *strict stability condition*, as it characterizes the strictly stable submanifolds in the $Q = 1$ case. In the following lemma we provide an equivalent condition to the strict stability.

Lemma 4.3.4. *There exists a unique solution $N_0 \equiv Q[\![0]\!]$ of the problem*

$$\min\left\{\mathrm{Jac}(u,\Omega) :\ u \in \Gamma^{1,2}_Q(\mathcal{N}\Omega) \text{ such that } u|_{\partial\Omega} = Q[\![0]\!]\right\}$$

if and only if there exists a positive constant $c = c(\Omega)$ such that

$$\mathrm{Jac}(u,\Omega) \geqslant c \int_\Omega |u|^2 d\mathcal{H}^m, \tag{4.54}$$

for all $u \in \Gamma^{1,2}_Q(\mathcal{N}\Omega)$ with $u|_{\partial\Omega} = Q[\![0]\!]$.

Remark 4.3.5. If $Q = 1$ and Σ is strictly stable, then the largest positive constant $c(\Omega)$ such that (4.54) holds for every $W^{1,2}$ section u of $\mathcal{N}\Omega$ with $u|_{\partial\Omega} = 0$ is the first eigenvalue λ_1 of the Jacobi normal operator $\mathcal{L}$.

Proof. Assume first that (4.54) holds. Then, the Jacobi functional is non-negative on the space of $W^{1,2}$ sections of $\mathcal{N}\Omega$ with vanishing trace at the boundary. It is then clear that $N_0 \equiv Q[\![0]\!]$ is a minimizer. Moreover, it is the only one. Indeed, if u is a minimizer, then $\mathrm{Jac}(u,\Omega) = 0$, and therefore (4.54) forces

$$\int_\Omega \mathcal{G}(u, Q[\![0]\!])^2 d\mathcal{H}^m = 0.$$

For the converse, assume that the minimum problem for the Jacobi functional with vanishing boundary condition admits $N_0 \equiv Q[\![0]\!]$ as the only solution. In particular, this implies

1 Observe that if $N|_{\partial\Omega} = Q[\![0]\!]$, then the null Q-field $N_0 \equiv Q[\![0]\!]$ is a competitor, whence $\mathrm{Jac}(N,\Omega) \leqslant \mathrm{Jac}(N_0,\Omega) = 0$ if N is a minimizer.

that $\mathrm{Jac}(u,\Omega) \geqslant 0$ for all sections $u \in \Gamma^{1,2}_Q(\mathcal{N}\Omega)$ such that $u|_{\partial\Omega} = Q[\![0]\!]$, and in fact that $\mathrm{Jac}(u,\Omega) > 0$ for all such sections except the trivial one N_0. Now, assume by contradiction that (4.54) does not hold. Then, for any $h \in \mathbb{N}$ there is a competitor $u_h \in \Gamma^{1,2}_Q(\mathcal{N}\Omega)$ such that $u_h|_{\partial\Omega} = Q[\![0]\!]$, $\|u_h\|_{L^2} = 1$ and

$$\mathrm{Jac}(u_h,\Omega) \leqslant \frac{1}{h}.$$

In particular, as a consequence of (4.48), we conclude that

$$\int_\Omega |Du_h|^2 \, d\mathcal{H}^m \leqslant C. \tag{4.55}$$

By the compact embedding theorem for Q-valued maps, Proposition 2.2.17, and by Proposition 2.2.16, we infer that there exist a subsequence $\{u_{h'}\}$ of $\{u_h\}$ and a section $u_\infty \in \Gamma^{1,2}_Q(\mathcal{N}\Omega)$, $u_\infty|_{\partial\Omega} = Q[\![0]\!]$, such that

$$\lim_{h'\to\infty} \int_\Omega \mathcal{G}(u_{h'}, u_\infty)^2 \, d\mathcal{H}^m = 0,$$

that is $u_{h'} \rightharpoonup u_\infty$ weakly in $W^{1,2}$. Then, from the semi-continuity of the Jacobi functional follows:

$$0 \leqslant \mathrm{Jac}(u_\infty,\Omega) \leqslant \liminf_{h'\to\infty} \mathrm{Jac}(u_{h'},\Omega) = 0.$$

Hence, $\mathrm{Jac}(u_\infty,\Omega) = 0$, and thus u_∞ is a minimizer. By hypothesis, $u_\infty \equiv Q[\![0]\!]$, which contradicts $\|u_\infty\|_{L^2} = 1$. □

5 EXISTENCE THEORY

In this chapter we provide a proof of the following (conditional) existence theorem for Jacobi Q-fields.

Theorem 5.0.1 (Conditional existence). *Let $\Sigma \hookrightarrow \mathcal{M}$ be as in Assumption 4.1.1, and let $\Omega \subset \Sigma$ be an open and connected subset with C^2 boundary. Assume that the* strict stability condition *is satisfied: the only Q-valued Jacobi field N in Ω such that $N|_{\partial\Omega} = Q[\![0]\!]$ is the null field $N_0 \equiv Q[\![0]\!]$. Then, for any $g \in \Gamma^{1,2}_Q(\mathcal{N}\Omega)$ such that $g|_{\partial\Omega} \in W^{1,2}(\partial\Omega; \mathcal{A}_Q(\mathbb{R}^d))$ there is a Jacobi Q-field $\overline{N}$ such that $\overline{N}|_{\partial\Omega} = g|_{\partial\Omega}$.*

The proof of Theorem 5.0.1 will be obtained by direct methods. Since we know that the Jac functional is weakly lower semi-continuous in $\Gamma^{1,2}_Q(\mathcal{N}\Omega)$, there are only two other pieces of information that we need in order to achieve the result. First, we need to know that for any fixed boundary datum g the Jacobi functional is bounded below in the space of sections $u \in \Gamma^{1,2}_Q(\mathcal{N}\Omega)$ with $u|_{\partial\Omega} = g|_{\partial\Omega}$. Second, we need to prove a compactness theorem for sections in $\Gamma^{1,2}_Q(\mathcal{N}\Omega)$ with Jac bounded above. Both these two results will follow from the strict stability condition, but in order to obtain them we first need to derive an extension theorem for Q-valued $W^{1,2}$ maps. In turn, the extension theorem will follow as a corollary of a Luckhaus type result for multiple-valued maps.

5.1 Q-VALUED LUCKHAUS LEMMA AND THE EXTENSION THEOREM

Proposition 5.1.1. *Let $\mathcal{N}$ be a d-dimensional closed Riemannian manifold of class C^2. Assume $0 < \lambda < 1$ and $f^1, f^2 \colon \mathcal{N} \to \mathcal{A}_Q(\mathbb{R}^q)$ are two maps in $W^{1,2}$. Then, there exist a constant $C = C(\mathcal{N}, d, q, Q)$ and a map $h \in W^{1,2}\left(\mathcal{N} \times [0,\lambda], \mathcal{A}_Q(\mathbb{R}^q)\right)$ such that*

$$h(\cdot, 0) \equiv f^1 \quad \text{and} \quad h(\cdot, \lambda) \equiv f^2 \quad \text{in } \mathcal{N}, \tag{5.1}$$

satisfying

$$\int_{\mathcal{N}\times[0,\lambda]} |h|^2 \leqslant C\lambda \int_{\mathcal{N}} \left(|f^1|^2 + |f^2|^2\right), \tag{5.2}$$

$$\int_{\mathcal{N}\times[0,\lambda]} |Dh|^2 \leqslant C\lambda \int_{\mathcal{N}} \left(|Df^1|^2 + |Df^2|^2\right) + \frac{C}{\lambda} \int_{\mathcal{N}} \mathcal{G}(f^1, f^2)^2. \tag{5.3}$$

Such a result adapts to the Q-valued setting a classical result by S. Luckhaus, concerning the extension of a Sobolev map defined on the boundary of an annulus $\partial(B_1 \setminus B_{1-\lambda})$ in its interior with control on the L^2 norm of the gradient of the extension map (for the precise statement and the proof, see the original paper [Luc88, Lemma 1], or the nice presentations given by L. Simon in [Sim96, Section 2.12.2] or by R. Moser in [Mos05, Lemma 4.4]).

A version of this result in the Q-valued setting was given by C. De Lellis in [DL13], where the author interpolates between two functions defined on flat cubes, and by J. Hirsch in [Hir16b, Lemma C.1] in the original Luckhaus setting of functions defined on the boundary of an annulus. The proof of the interpolation between maps defined over a closed Riemannian manifold presented here is based on De Lellis' result and on a decomposition of the manifold that is bi-Lipschitz to a d-dimensional cubic complex, following ideas already contained in [Whi88] and [Han05]. We will make an extensive use of the following Lemma, which provides the elementary step in the construction of the interpolation.

Lemma 5.1.2. *There is a constant C depending only on j and Q with the following property. Assume that $0 < \lambda \leqslant 1$, $L = [0,\lambda]^j + z$ is a j-dimensional cube of side length λ, and $\varphi \in W^{1,2}\left(\partial L, \mathcal{A}_Q(\mathbb{R}^q)\right)$. Then, there is an extension ψ of φ to L such that*

$$\int_L |\psi|^2 \, d\mathcal{H}^j \leqslant C\lambda \int_{\partial L} |\varphi|^2 \, d\mathcal{H}^{j-1} \tag{5.4}$$

and

$$\mathrm{Dir}(\psi, L) \leqslant C\lambda \mathrm{Dir}(\varphi, \partial L). \tag{5.5}$$

Proof. First observe that, since the inequalities (5.4) and (5.5) are invariant with respect to translations and dilations, it suffices to prove the lemma when $L = [0,1]^j$. Moreover, since L is bi-Lipschitz equivalent to the unit ball, it is enough to show the claim for $L = \overline{B}_1 \subset \mathbb{R}^j$.

For reasons that will soon become clear, the proof when working in dimension $j = 2$ is different with respect to the one in the higher dimensional case ($j \geqslant 3$): for this reason, we will distinguish between these two scenarios.

The higher dimensional case ($j \geqslant 3$). This is the easiest situation: indeed, it suffices to define ψ as the zero-degree homogeneous extension of φ. That is, if $\varphi = \sum_\ell \llbracket \varphi^\ell \rrbracket$ on ∂B_1, then one just sets

$$\psi(x) := \sum_{\ell=1}^{Q} \left\llbracket \varphi^\ell \left(\frac{x}{|x|} \right) \right\rrbracket \qquad \text{for } x \in B_1 \setminus \{0\}. \tag{5.6}$$

A simple computation in radial coordinates shows that both estimates (5.4) and (5.5) hold with $C = \max\{j^{-1}, (j-2)^{-1}\} = (j-2)^{-1}$. Observe that this proof breaks down when $j = 2$, because the zero-degree homogeneous extension of φ has infinite Dirichlet energy in two dimensions.

The planar case ($j = 2$). For this proof, it will be useful to introduce a suitable notation. We identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$, and the unit ball $B_1 \subset \mathbb{R}^2$ with the disk $\mathbb{D}$, as usual defined as

$$\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\} = \{re^{i\theta} : 0 \leqslant r < 1, \theta \in \mathbb{R}\}.$$

The boundary of $\mathbb{D}$ is the unit circle $\mathbb{S}^1$, described by

$$\mathbb{S}^1 := \{z \in \mathbb{C} : |z| = 1\} = \{e^{i\theta} : \theta \in \mathbb{R}\}.$$

Consider now a function $\varphi \in W^{1,2}(\mathbb{S}^1, \mathcal{A}_Q)$. By [DLS11, Proposition 1.5], there exists a decomposition $\varphi = \sum_{\ell=1}^{P} \llbracket \varphi_\ell \rrbracket$, where each φ_ℓ is an *irreducible* map in $W^{1,2}(\mathbb{S}^1, \mathcal{A}_{k_\ell})$. This

means that $\sum_{\ell=1}^{P} k_\ell = Q$, and furthermore that for every $\ell = 1, \dots, P$ there exists a $W^{1,2}$ map $\gamma_\ell \colon S^1 \to \mathbb{R}^q$ such that

$$\varphi_\ell(x) = \sum_{z^{k_\ell}=x} [\![\gamma_\ell(z)]\!], \tag{5.7}$$

and with $\gamma_\ell(z_1) \neq \gamma_\ell(z_2)$ if $z_1 \neq z_2$ are two distinct $k_\ell{}^{\text{th}}$ roots of x. In other words, if we identify the point $x = e^{i\theta} \in S^1$ with the phase $\theta \in [0, 2\pi)$ we have that

$$\varphi_\ell(\theta) = \sum_{m=0}^{k_\ell - 1} \left[\!\!\left[\gamma_\ell \left(\frac{\theta + 2\pi m}{k_\ell} \right) \right]\!\!\right], \tag{5.8}$$

with

$$\gamma_\ell \left(\frac{\theta + 2\pi m}{k_\ell} \right) \neq \gamma_\ell \left(\frac{\theta + 2\pi \tilde{m}}{k_\ell} \right) \quad \text{for } m \neq \tilde{m}.$$

The idea, now, is to consider the *harmonic extension* to the disk $\mathbb{D}$ of the function γ_ℓ: if

$$\gamma_\ell(\theta) = \frac{a_{\ell,0}}{2} + \sum_{n=1}^{\infty} \left(a_{\ell,n} \cos(n\theta) + b_{\ell,n} \sin(n\theta) \right) \tag{5.9}$$

is the Fourier series of γ_ℓ, we let

$$\zeta_\ell(r, \theta) = \frac{a_{\ell,0}}{2} + \sum_{n=1}^{\infty} r^n \left(a_{\ell,n} \cos(n\theta) + b_{\ell,n} \sin(n\theta) \right) \tag{5.10}$$

denote its harmonic extension to the whole disk. Then, for each $\ell = 1, \dots, P$, we consider the k_ℓ-valued function ψ_ℓ obtained "rolling" back the ζ_ℓ, that is

$$\psi_\ell(x) := \sum_{z^{k_\ell}=x} [\![\zeta_\ell(z)]\!] \tag{5.11}$$

for $x \in \mathbb{D}$, and, finally, we set $\psi := \sum_{\ell=1}^{P} [\![\psi_\ell]\!]$. We claim now that ψ is a $W^{1,2}(\mathbb{D}, \mathcal{A}_Q)$ extension of φ satisfying the estimates (5.4) and (5.5). To see this, fix $\ell \in \{1, \dots, P\}$ and define the following subsets of the unit disk,

$$\mathcal{D}_m := \left\{ re^{i\theta} : 0 < r < 1, \frac{2\pi m}{k_\ell} < \theta < \frac{2\pi(m+1)}{k_\ell} \right\}$$

for $m = 0, \dots, k_\ell - 1$, and

$$\mathcal{C} := \{ re^{i\theta} : 0 < r < 1, \theta \neq 0 \}.$$

One immediately sees that $\psi_\ell|_{\mathcal{C}} = \sum_{m=0}^{k_\ell - 1} [\![\zeta_\ell \circ \sigma_m]\!]$, where $\sigma_m \colon \mathcal{C} \to \mathcal{D}_m$ are the k_ℓ determinations of the $k_\ell{}^{\text{th}}$ root, that is

$$\sigma_m(re^{i\theta}) = r^{\frac{1}{k_\ell}} e^{i\left(\frac{\theta + 2\pi m}{k_\ell}\right)}.$$

Similarly, if the arcs $\mathcal{S}_m$ are defined by

$$\mathcal{S}_m := \left\{ e^{i\theta} : \frac{2\pi m}{k_\ell} < \theta < \frac{2\pi(m+1)}{k_\ell} \right\},$$

we have that $\varphi_\ell|_{S^1\setminus\{1\}} = \sum_{m=0}^{k_\ell-1} [\![\gamma_\ell \circ \tau_m]\!]$, where $\tau_m \colon S^1 \setminus \{1\} \to \mathcal{S}_m$ is given by $\tau_m := \sigma_m|_{S^1}$. Thus, we can immediately compute

$$\begin{aligned}\int_{S^1} |\varphi_\ell|^2 &= \sum_{m=0}^{k_\ell-1} \int_{S^1} |\gamma_\ell \circ \tau_m|^2 \\ &= k_\ell \sum_{m=0}^{k_\ell-1} \int_{\mathcal{S}_m} |\gamma_\ell|^2 \\ &= k_\ell \int_{S^1} |\gamma_\ell|^2 = k_\ell \pi \left(\frac{|a_{\ell,0}|^2}{2} + \sum_{n=1}^{\infty} (|a_{\ell,n}|^2 + |b_{\ell,n}|^2) \right)\end{aligned} \tag{5.12}$$

by Plancherel's theorem. On the other hand, we can use polar coordinates to compute the integral of the extension ψ_ℓ to the disk and see that

$$\begin{aligned}\int_{\mathbb{D}} |\psi_\ell|^2 &= \sum_{m=0}^{k_\ell-1} \int_{\mathcal{C}} |\zeta_\ell \circ \sigma_m|^2 \\ &= \sum_{m=0}^{k_\ell-1} \int_0^1 \left(\int_0^{2\pi} \left| \zeta_\ell \left(\rho^{\frac{1}{k_\ell}}, \frac{\alpha + 2\pi m}{k_\ell} \right) \right|^2 \, d\alpha \right) \rho \, d\rho \\ &= k_\ell^2 \sum_{m=0}^{k_\ell-1} \int_0^1 \left(\int_{\frac{2\pi m}{k_\ell}}^{\frac{2\pi(m+1)}{k_\ell}} |\zeta_\ell(r,\theta)|^2 \, d\theta \right) r^{2k_\ell-1} \, dr \\ &= k_\ell^2 \int_0^1 \left(\int_0^{2\pi} |\zeta_\ell(r,\theta)|^2 \, d\theta \right) r^{2k_\ell-1} \, dr \\ &= k_\ell^2 \pi \int_0^1 \left(\frac{|a_{\ell,0}|^2}{2} + \sum_{n=1}^{\infty} r^{2n} (|a_{\ell,n}|^2 + |b_{\ell,n}|^2) \right) r^{2k_\ell-1} \, dr \\ &\overset{(5.12)}{\leqslant} k_\ell \left(\int_{S^1} |\varphi_\ell|^2 \right) \left(\int_0^1 r^{2k_\ell-1} \, dr \right) \\ &= \frac{1}{2} \int_{S^1} |\varphi_\ell|^2.\end{aligned} \tag{5.13}$$

Summing over $\ell \in \{1, \dots, P\}$, we finally conclude that

$$\int_{\mathbb{D}} |\psi|^2 \leqslant \frac{1}{2} \int_{S^1} |\varphi|^2, \tag{5.14}$$

that is, (5.4) holds with $C = \frac{1}{2}$. Concerning (5.5), we exploit the invariance of the Dirichlet energy under conformal mappings in order to infer that, for any $\ell = 1, \dots, P$,

$$\mathrm{Dir}(\psi_\ell, \mathcal{C}) = \sum_{m=0}^{k_\ell-1} \mathrm{Dir}(\zeta_\ell \circ \sigma_m, \mathcal{C}) = \sum_{m=0}^{k_\ell-1} \mathrm{Dir}(\zeta_\ell, \mathcal{D}_m) = \int_{\mathbb{D}} |D\zeta_\ell|^2. \tag{5.15}$$

Now, by a simple computation on planar harmonic functions, it is easy to see that

$$\int_{\mathbb{D}} |D\zeta_\ell|^2 \leqslant \int_{S^1} |\partial_\theta \gamma_\ell|^2, \tag{5.16}$$

where ∂_θ is the tangential derivative along the circle. On the other hand, for every $\ell = 1, \dots, P$,

$$\begin{aligned} \mathrm{Dir}(\varphi_\ell, \mathbb{S}^1) &= \sum_{m=0}^{k_\ell - 1} \int_{\mathbb{S}^1} |\partial_\theta (\gamma_\ell \circ \tau_m)|^2 \\ &= \sum_{m=0}^{k_\ell - 1} \int_{\mathbb{S}^1} \frac{1}{k_\ell^2} |\partial_\theta \gamma_\ell \circ \tau_m|^2 \\ &= \sum_{m=0}^{k_\ell - 1} \int_{\mathbb{S}_m} \frac{1}{k_\ell} |\partial_\theta \gamma_\ell|^2 \\ &= \frac{1}{k_\ell} \int_{\mathbb{S}^1} |\partial_\theta \gamma_\ell|. \end{aligned} \tag{5.17}$$

Finally, summing on ℓ, the above arguments produce

$$\mathrm{Dir}(\psi, \mathbb{D}) \overset{(5.15)}{=} \sum_{\ell=1}^{P} \int_{\mathbb{D}} |D\zeta_\ell|^2 \overset{(5.16)}{\leqslant} \sum_{\ell=1}^{P} \int_{\mathbb{S}^1} |\partial_\theta \gamma_\ell|^2 \overset{(5.17)}{=} \sum_{\ell=1}^{P} k_\ell \mathrm{Dir}(\varphi_\ell, \mathbb{S}^1) \leqslant Q \mathrm{Dir}(\varphi, \mathbb{S}^1), \tag{5.18}$$

whence (5.5) holds with $C = Q$. □

Proof of Proposition 5.1.1. Without loss of generality, we assume that $\mathcal{N}$ is an embedded submanifold of some Euclidean space $\mathbb{R}^N$. We shall divide the proof into steps.

Step 1. We first consider a Lipschitz cubic decomposition of the manifold $\mathcal{N}$, that is a pair $(\mathcal{K}, \sigma)$, where $\mathcal{K}$ is a d-dimensional cubic complex, and $\sigma \colon |\mathcal{K}| \to \mathcal{N}$ is a bi-Lipschitz map, $|\mathcal{K}|$ denoting the union of all cells of $\mathcal{K}$. Without loss of generality, we may assume that each cell in $\mathcal{K}$ has unit d-dimensional volume. Set $m := \lfloor \frac{1}{\lambda} \rfloor + 1$. Using that $[0,1] = \bigcup_{i=1}^{m} \left[\frac{i-1}{m}, \frac{i}{m}\right]$, we can divide each cell in $\mathcal{K}$ into m^d smaller d-dimensional cubes, whose side length is at most λ. We will denote the resulting cubic complex by $\mathcal{K}_m$, and regard σ as a bi-Lipschitz map $\sigma \colon |\mathcal{K}_m| \to \mathcal{N}$: observe that if L is any cell in $\mathcal{K}_m$ then the image $\sigma(L)$ is a domain in $\mathcal{N}$ with diameter (computed with respect to the geodesic distance on $\mathcal{N}$) $\mathbf{diam}(\sigma(L)) \leqslant \sqrt{d}\, \mathrm{Lip}(\sigma)\lambda$.

For each $j \in \{0, 1, \dots, d\}$, $\mathcal{K}_m^j$ will denote the j-skeleton of the complex $\mathcal{K}_m$, that is the family of all j-dimensional faces, and $|\mathcal{K}_m^j|$ will be their union.

Step 2. Let now $\eta = \eta(\mathcal{N}) > 0$ be so small that the set $\mathbf{U} = \mathbf{U}_{2\eta}(\mathcal{N}) := \{x \in \mathbb{R}^N : \mathrm{dist}(x, \mathcal{N}) < 2\eta\}$ is a tubular neighborhood of $\mathcal{N}$, with (unique) differentiable nearest point projection $\Pi \colon \mathbf{U}_{2\eta}(\mathcal{N}) \to \mathcal{N}$. For $i = 1, 2$, we extend f^i to a map $F^i \colon \mathbf{U} \to \mathcal{A}_Q(\mathbb{R}^q)$ by setting $F^i := f^i \circ \Pi$. One has that

$$\int_{\mathbf{U}} |F^i|^2 \leqslant c_1 \int_{\mathcal{N}} |f^i|^2, \tag{5.19}$$

$$\int_{\mathbf{U}} |DF^i|^2 \leqslant c_1 \int_{\mathcal{N}} |Df^i|^2, \tag{5.20}$$

and

$$\int_{\mathbf{U}} \mathcal{G}(F^1, F^2)^2 \leqslant c_1 \int_{\mathcal{N}} \mathcal{G}(f^1, f^2)^2, \tag{5.21}$$

where the constant c_1 depends only on the retraction Π (and, thus, on the dimensions d and N and on the width η of the tubular neighborhood).

Furthermore, for every $z \in |\mathcal{K}_m|$ and for every vector $\nu \in B^N_\eta$, we define $\sigma_\nu(z) := \Pi(\sigma(z)+\nu)$. Assume that η is so small that all σ_ν's are bi-Lipschitz maps $|\mathcal{K}_m| \to \mathcal{N}$, and set $f^i_\nu := f^i \circ \sigma_\nu$, that is $f^i_\nu(z) = F^i(\sigma(z)+\nu)$. By Fubini's theorem, for all $j = 1, \dots, d$ and for a.e. $\nu \in B^N_\eta$ one has that $f^i_\nu \in W^{1,2}\left(F, \mathcal{A}_Q(\mathbb{R}^q)\right)$ for all faces $F \in \mathcal{K}^j_m$.

Consider now any non-negative function $\alpha \in L^1(\mathbf{U})$. It is easily seen that there exists a constant $c_2 = c_2(\mathcal{N}, d, N, \eta)$ such that for any $j = 0, \dots, d$ and for every $\theta \in (0,1)$

$$\int_{|\mathcal{K}^j_m|} \alpha(\sigma(z)+\nu)\, d\mathcal{H}^j(z) \leqslant c_2 \theta^{-1} \lambda^{j-d} \int_{\mathbf{U}} \alpha \tag{5.22}$$

for all $\nu \in B^N_\eta$ with the exception of a set E of $\mathcal{L}^N$-measure $|E| \leqslant \theta |B^N_\eta|$. To prove this, first note that

$$\int_{|\mathcal{K}^j_m|} \alpha(\sigma(z)+\nu)\, d\mathcal{H}^j(z) \leqslant \frac{1}{\theta |B^N_\eta|} \int_{B^N_\eta} \left(\int_{|\mathcal{K}^j_m|} \alpha(\sigma(z)+\nu)\, d\mathcal{H}^j(z) \right) d\nu \tag{5.23}$$

for all $\nu \in B^N_\eta \setminus E$, $|E| \leqslant \theta |B^N_\eta|$. Then, conclude by estimating:

$$\begin{aligned}
\int_{B^N_\eta} \left(\int_{|\mathcal{K}^j_m|} \alpha(\sigma(z)+\nu)\, d\mathcal{H}^j(z) \right) d\nu &= \int_{|\mathcal{K}^j_m|} \left(\int_{B^N_\eta} \alpha(\sigma(z)+\nu)\, d\nu \right) d\mathcal{H}^j(z) \\
&= \int_{|\mathcal{K}^j_m|} \left(\int_{B^N_\eta(\sigma(z))} \alpha(w)\, dw \right) d\mathcal{H}^j(z) \\
&\leqslant \mathcal{H}^j(|K^j_m|) \int_{\mathbf{U}} \alpha \\
&\leqslant C m^d \lambda^j \int_{\mathbf{U}} \alpha \\
&\leqslant C \lambda^{j-d} \int_{\mathbf{U}} \alpha,
\end{aligned} \tag{5.24}$$

where the constant C appearing in the last line depends only on the number of cells in the original cubic complex $\mathcal{K}$ and on the dimension d.

Now, it suffices to apply (5.22) with $\alpha = |F^i|^2$, $\alpha = |DF^i|^2$ and $\alpha = \mathcal{G}(F^1, F^2)^2$, and, say, $\theta = \frac{1}{2}$, and to plug in equations (5.19), (5.20) and (5.21) to finally show the following: there exists $\nu \in B^N_\eta$ such that for all $j \in \{1, \dots, d\}$ the following inequalities

$$\int_{|\mathcal{K}^j_m|} \left(|f^1_\nu|^2 + |f^2_\nu|^2 \right) \leqslant C \lambda^{j-d} \int_{\mathcal{N}} \left(|f^1|^2 + |f^2|^2 \right), \tag{5.25}$$

and

$$\int_{|\mathcal{K}^j_m|} \left(|Df^1_\nu|^2 + |Df^2_\nu|^2 + \mathcal{G}(f^1_\nu, f^2_\nu)^2 \right) \leqslant C \lambda^{j-d} \int_{\mathcal{N}} \left(|Df^1|^2 + |Df^2|^2 + \mathcal{G}(f^1, f^2)^2 \right) \tag{5.26}$$

hold true with a constant $C = C(c_1, c_2, \mathrm{Lip}(\sigma))$. Furthermore, for $j = 0$:

$$\sum_{z \in |\mathcal{K}^0_m|} \left(|f^1_\nu|^2(z) + |f^2_\nu|^2(z) \right) \leqslant C \lambda^{-d} \int_{\mathcal{N}} \left(|f^1|^2 + |f^2|^2 \right), \tag{5.27}$$

$$\sum_{z\in|\mathcal{K}^0_m|} \mathcal{G}\left(f^1_\nu(z), f^2_\nu(z)\right)^2 \leqslant C\lambda^{-d} \int_{\mathcal{N}} \mathcal{G}\left(f^1, f^2\right)^2. \tag{5.28}$$

From now on, we will then assume to have fixed a $\nu \in B^N_\eta$ such that the corresponding maps $f^i_\nu \colon |\mathcal{K}_m| \to \mathcal{A}_Q(\mathbb{R}^q)$ satisfy equations (5.25), (5.26), (5.27), (5.28) and the following condition: for every $j \geqslant 1$, for each $\tau \in \mathcal{K}^j_m$ and for all $\gamma \in \mathcal{K}^{j-1}_m$ with $\gamma \subset \tau$, the restrictions $f^i_\nu|_\tau$ and $f^i_\nu|_\gamma$ are all $W^{1,2}$, and moreover the trace of $f^i_\nu|_\tau$ at γ is precisely $f^i_\nu|_\gamma$.

Step 3. Consider now the $(d+1)$-dimensional cubic complex $\overline{\mathcal{K}} := \mathcal{K}_m \times [0,\lambda]$ whose $(d+1)$-dimensional cells are cubes of the form $L \times [0,\lambda]$ for some $L \in \mathcal{K}^d_m$. A face $\tau \in \overline{\mathcal{K}}^j$, $j < d+1$, is said to be *horizontal* if it is contained in $\mathcal{K}_m \times \{0\}$ (lower horizontal) or $\mathcal{K}_m \times \{\lambda\}$ (upper horizontal), *vertical* otherwise. The collection of j-dimensional faces of $\overline{\mathcal{K}}$ is hence given by

$$\overline{\mathcal{K}}^j = \mathscr{L}^j \cup \mathscr{U}^j \cup \mathscr{V}^j, \tag{5.29}$$

where $\mathscr{L}^j$, $\mathscr{U}^j$ and $\mathscr{V}^j$ are the lower horizontal, upper horizontal and vertical j-dimensional faces respectively. Observe that $\mathscr{V}^0 = \emptyset$, $\mathscr{L}^0$ consists of points $(z,0)$, while $\mathscr{U}^0$ consists of points (z,λ) with $z \in \mathcal{K}^0_m$; note, furthermore, that all $(d+1)$-dimensional cells are vertical.

We are now in the position to define a map $\overline{h} \colon |\overline{\mathcal{K}}| \to \mathcal{A}_Q(\mathbb{R}^q)$. First of all, we set $\overline{h}|_\beta \equiv f^1_\nu|_\beta$ if β is a lower horizontal face, and $\overline{h}|_\tau \equiv f^2_\nu|_\tau$ if τ is an upper horizontal face. Consider next any vertical segment $\gamma \in \mathscr{V}^1$. Its two endpoints are given by $(z,0)$ and (z,λ) for some $z \in \mathcal{K}^0_m$. Now, if $f^1_\nu(z) = \sum_\ell [\![(f^1_\nu)_\ell(z)]\!]$ and $f^2_\nu(z) = \sum_\ell [\![(f^2_\nu)_\ell(z)]\!]$ are ordered in such a way that $\mathcal{G}\left(f^1_\nu(z), f^2_\nu(z)\right)^2 = \sum_\ell |(f^1_\nu)_\ell(z) - (f^2_\nu)_\ell(z)|^2$, then a natural extension is obtained by setting

$$\overline{h}(z,\theta) := \sum_{\ell=1}^{Q} \left[\!\!\left[(f^1_\nu)_\ell(z) + \frac{\theta}{\lambda}\left((f^2_\nu)_\ell(z) - (f^1_\nu)_\ell(z)\right)\right]\!\!\right], \tag{5.30}$$

for all $\theta \in [0,\lambda]$. In this way, we obtain the bounds

$$\int_\gamma |\overline{h}|^2 \leqslant 2\lambda\left(|f^1_\nu|^2(z) + |f^2_\nu|^2(z)\right) \tag{5.31}$$

and

$$\int_\gamma |D\overline{h}|^2 \leqslant \lambda^{-1}\mathcal{G}\left(f^1_\nu(z), f^2_\nu(z)\right)^2. \tag{5.32}$$

If we carry on this procedure for all vertical segments, we obtain a well defined Q-valued map $\overline{h}$ on all the vertical 1-skeleton $\mathscr{V}^1$, which, thanks to (5.27) and (5.28), satisfies

$$\int_{|\mathscr{V}^1|} |\overline{h}|^2 \leqslant C\lambda^{1-d} \int_{\mathcal{N}} \left(|f^1|^2 + |f^2|^2\right) \tag{5.33}$$

and

$$\int_{|\mathscr{V}^1|} |D\overline{h}|^2 \leqslant C\lambda^{-1-d} \int_{\mathcal{N}} \mathcal{G}\left(f^1, f^2\right)^2. \tag{5.34}$$

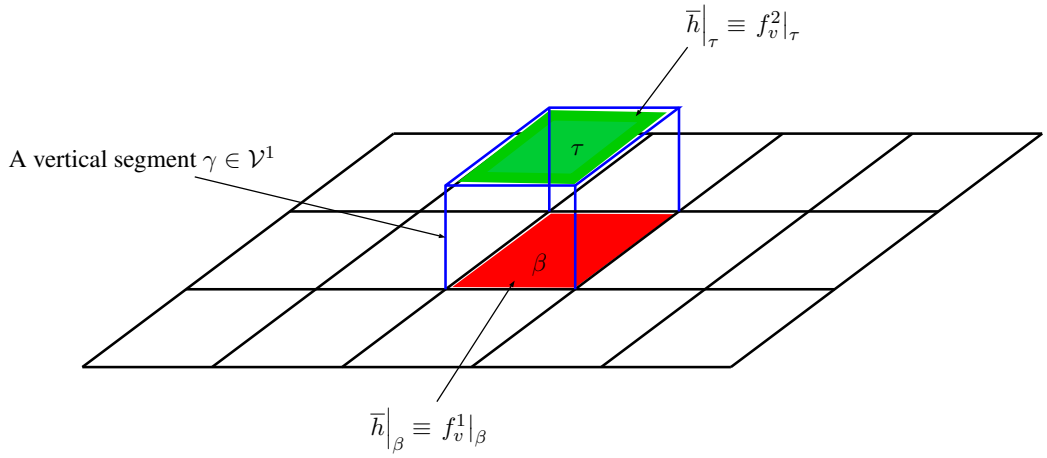


Figure 1: The cubic complex $\overline{\mathcal{K}}$ and the first step in the construction of $\overline{h}$.

Pick next a vertical 2-dimensional face τ. Its boundary consists of two horizontal segments $\beta \in \mathscr{L}^1$ and $\delta \in \mathscr{U}^1$, and two vertical segments joining the points $(z,0)$, $(w,0)$ to the points (z,λ), (w,λ) respectively. Using our assumptions on v, we can conclude that $\overline{h}|_{\partial\tau}$ is in $W^{1,2}$, whence Lemma 5.1.2 yields an extension of $\overline{h}$ to τ with estimates

$$\int_\tau |\overline{h}|^2 \leqslant C\lambda \left(\int_\beta |f_v^1|^2 + \int_\delta |f_v^2|^2 \right) + C\lambda^2 \left((|f_v^1|^2 + |f_v^2|^2)(z) + (|f_v^1|^2 + |f_v^2|^2)(w) \right) \tag{5.35}$$

and

$$\int_\tau |D\overline{h}|^2 \leqslant C\lambda \left(\int_\beta |Df_v^1|^2 + \int_\delta |Df_v^2|^2 \right) + C \left(\mathcal{G}\left(f_v^1(z), f_v^2(z)\right)^2 + \mathcal{G}\left(f_v^1(w), f_v^2(w)\right)^2 \right). \tag{5.36}$$

Summing over the 2-skeleton $\mathscr{V}^2$, from the estimates (5.25) and (5.27) we deduce

$$\int_{|\mathscr{V}^2|} |\overline{h}|^2 \leqslant C\lambda^{2-d} \int_{\mathcal{N}} \left(|f^1|^2 + |f^2|^2 \right), \tag{5.37}$$

whereas (5.26) and (5.28) imply

$$\int_{|\mathscr{V}^2|} |D\overline{h}|^2 \leqslant C\lambda^{2-d} \int_{\mathcal{N}} \left(|Df^1|^2 + |Df^2|^2 \right) + C\lambda^{-d} \int_{\mathcal{N}} \mathcal{G}\left(f^1, f^2\right)^2. \tag{5.38}$$

We then proceed inductively over $\mathscr{V}^j$, iteratively applying Lemma 5.1.2 and using the inequalities (5.25) to (5.28) at each step. At the final iteration, namely for $j = d+1$, we construct a map $\overline{h}$ which is $W^{1,2}$ on each $(d+1)$-dimensional cell $L \times [0,\lambda]$, coinciding with f_v^1 on $L \times \{0\}$ and with f_v^2 on $L \times \{\lambda\}$. Furthermore, if two cells $H = L_1 \times [0,\lambda]$ and $K = L_2 \times [0,\lambda]$ have a common face $S \in \mathscr{V}^d$, the traces of $\overline{h}|_H$ and $\overline{h}|_K$ at S coincide. Thus, we can regard $\overline{h}$ as a $W^{1,2}$ map defined on the whole cubic complex $\overline{\mathcal{K}}$. Moreover, since $|\overline{\mathcal{K}}| = \bigcup \mathscr{V}^{d+1} = |\mathscr{V}^{d+1}|$, the inductive step provides the following estimates:

$$\int_{|\overline{\mathcal{K}}|} |\overline{h}|^2 \leqslant C\lambda \int_{\mathcal{N}} \left(|f^1|^2 + |f^2|^2 \right), \tag{5.39}$$

$$\int_{|\overline{\mathcal{K}}|} |D\overline{h}|^2 \leqslant C\lambda \int_{\mathcal{N}} \left(|Df^1|^2 + |Df^2|^2\right) + \frac{C}{\lambda} \int_{\mathcal{N}} \mathcal{G}\left(f^1, f^2\right)^2. \tag{5.40}$$

Step 4. Finally, we simply define a map $h \in W^{1,2}\left(\mathcal{N} \times [0,\lambda], \mathcal{A}_Q(\mathbb{R}^q)\right)$ by setting

$$h(x,\theta) := \overline{h}\left(\sigma_\nu^{-1}(x), \theta\right). \tag{5.41}$$

It is immediate to check that such a map indeed satisfies (5.1), (5.2) and (5.3) in the statement. □

Corollary 5.1.3. *Let $\Sigma^m \hookrightarrow \mathbb{R}^d$ be a regular compact submanifold, and let $\lambda_0 := \mathrm{inj}(\Sigma) > 0$ be the injectivity radius of Σ. Then, for any $0 < \lambda < \lambda_0$, for any $\mathcal{V} \subsetneq \Sigma$ connected, open subset with C^2 boundary and such that*

$$\mathbf{dist}(x, \partial\Sigma) \geqslant \lambda \quad \textit{for every } x \in \partial\mathcal{V},$$

and for any $\tilde{g}_0 \in W^{1,2}(\partial\mathcal{V}, \mathcal{A}_Q(\mathbb{R}^d))$ there exist an open set $\mathcal{V}_\lambda \subset \Sigma$ with $\mathcal{V} \Subset \mathcal{V}_\lambda$, $\mathbf{dist}(\mathcal{V}, \partial\mathcal{V}_\lambda) \leqslant \lambda$, and a map $\overline{g}_\lambda \in W^{1,2}\left(\mathcal{V}_\lambda \setminus \overline{\mathcal{V}}, \mathcal{A}_Q(\mathbb{R}^d)\right)$ satisfying:

$$\overline{g}_\lambda|_{\partial\mathcal{V}} = \tilde{g}_0 \quad \textit{and} \quad \overline{g}_\lambda|_{\partial\mathcal{V}_\lambda} = Q[\![0]\!], \tag{5.42}$$

$$\int_{\mathcal{V}_\lambda \setminus \overline{\mathcal{V}}} |\overline{g}_\lambda|^2 \, d\mathcal{H}^m \leqslant C\lambda \int_{\partial\mathcal{V}} |\tilde{g}_0|^2 \, d\mathcal{H}^{m-1}, \tag{5.43}$$

$$\mathrm{Dir}(\overline{g}_\lambda, \mathcal{V}_\lambda \setminus \overline{\mathcal{V}}) \leqslant C\lambda \mathrm{Dir}(\tilde{g}_0, \partial\mathcal{V}) + \frac{C}{\lambda} \int_{\partial\mathcal{V}} |\tilde{g}_0|^2 \, d\mathcal{H}^{m-1}, \tag{5.44}$$

for a constant $C = C(\mathcal{V}, m, d, Q)$.

Proof. Let $\mathcal{V}$ and $\tilde{g}_0$ be as in the statement. Then, by the very definition of injectivity radius, for any $0 < \lambda < \lambda_0$ the exponential map, restricted to $\partial\mathcal{V}$, is injective in a ball of radius λ around the zero section of the normal bundle of $\partial\mathcal{V}$ in Σ. In turn, this allows one to define, for any such λ, a λ-tubular neighborhood $\mathbf{U}_\lambda$ of $\partial\mathcal{V}$ in Σ by setting

$$\mathbf{U}_\lambda := \{\exp_\pi(\theta\eta(\pi)) \,:\, \pi \in \partial\mathcal{V}, |\theta| < \lambda\}, \tag{5.45}$$

where for every point $\pi \in \partial\mathcal{V}$ we have denoted $\eta(\pi) \in T_\pi\Sigma$ the unit outer co-normal vector to $\partial\mathcal{V}$ at π.

Note that it is well defined a differentiable parametrization $x \in \mathbf{U}_\lambda \mapsto (\pi(x), \theta(x)) \in \partial\mathcal{V} \times (-\lambda, \lambda)$ such that $\exp_{\pi(x)}(\theta(x)\eta(\pi(x))) = x$ for all $x \in \mathbf{U}_\lambda$.

Next, the *positive* and *negative* λ-tubular neighborhoods of $\partial\mathcal{V}$ in Σ are respectively defined by

$$\mathbf{U}_\lambda^+ := \{\exp_\pi(\theta\eta(\pi)) \,:\, \pi \in \partial\mathcal{V}, 0 < \theta < \lambda\}, \tag{5.46}$$

$$\mathbf{U}_\lambda^- := \{\exp_\pi(\theta\eta(\pi)) \,:\, \pi \in \partial\mathcal{V}, -\lambda < \theta < 0\}. \tag{5.47}$$

We set $\mathcal{V}_\lambda := \mathcal{V} \cup \mathbf{U}_\lambda$. The claimed result is then simply obtained by applying Proposition 5.1.1 with $\mathcal{N} = \partial\mathcal{V}$, $f^1 = \tilde{g}_0$, $f^2 = Q[\![0]\!]$ and setting $\overline{g}_\lambda(x) := h(\pi(x), \theta(x))$ for $x \in \mathcal{V}_\lambda \setminus \overline{\mathcal{V}} = \mathbf{U}_\lambda^+$. □

5.2 THE COMPACTNESS THEOREM

Here we go back to our original setting, in order to finally prove Theorem 5.0.1. Let Ω be an open and connected subset of $\Sigma \hookrightarrow \mathcal{M}$ in which we wish to solve the minimum problem for the Jac functional. We will assume C^2 regularity for $\partial\Omega$. Let $\lambda_0 := \mathrm{inj}(\Sigma)$. For $0 < \lambda < \lambda_0$, set $\mathcal{V} := \{x \in \Omega : \mathbf{dist}(x, \partial\Omega) > \lambda\}$, so that Ω coincides with the set $\mathcal{V}_\lambda = \mathcal{V} \cup \mathbf{U}_\lambda$ which was obtained in the proof of Corollary 5.1.3. Using the same notations introduced in the proof of Corollary 5.1.3, we parametrize $\mathbf{U}_\lambda$ with coordinates $(\pi, \theta) \in \partial\mathcal{V} \times (-\lambda, \lambda)$.

Let us now define $\Phi_\lambda \colon \mathcal{V} \to \Omega$ to be the diffeomorphism given by:

$$\Phi_\lambda(x) := \begin{cases} \exp_{\pi(x)}\left(\varphi_\lambda(\theta(x))\eta(\pi(x))\right) & \text{if } x \in \mathbf{U}_\lambda^- \\ x & \text{otherwise}, \end{cases} \tag{5.48}$$

where φ_λ is any monotone increasing diffeomorphism $\varphi_\lambda \colon (-\lambda, 0) \to (-\lambda, \lambda)$ such that $\varphi_\lambda(\theta) = \theta$ for $\theta \in \left(-\lambda, -\frac{\lambda}{2}\right)$. From this moment on, we will assume that such a family of diffeomorphisms φ_λ has been fixed, and satisfies a bound of the form

$$c^{-1} \leqslant |\varphi_\lambda'| \leqslant c \tag{5.49}$$

for a positive constant c which does not depend on λ.

Furthermore, if $u = \sum_\ell [\![u_\ell]\!]$ is any map in $W^{1,2}\left(\Omega, \mathcal{A}_Q(\mathbb{R}^d)\right)$, we set:

$$u^\perp(x) := \sum_{\ell=1}^{Q} [\![\mathbf{p}_x^{\Sigma\perp} \cdot u_\ell(x)]\!], \tag{5.50}$$

where $\mathbf{p}^{\Sigma\perp}$ is the normal bundle projection defined in Definition 4.2.9. Observe that $u^\perp \in \Gamma_Q^{1,2}(\mathcal{N}\Omega)$. The following Lemma yields a useful formula to relate the Dirchlet energy of u with the Dirichlet energy of $u^\perp$.

Lemma 5.2.1. *For every $\varepsilon > 0$ there exists a positive constant C_ε such that the following estimate holds true:*

$$\mathrm{Dir}(u^\perp, \Omega) \leqslant (1+\varepsilon)\mathrm{Dir}(u, \Omega) + C_\varepsilon \int_\Omega |u|^2 \, d\mathcal{H}^m. \tag{5.51}$$

Proof. Write $\mathbf{p}^{\Sigma\perp}(x, \nu) := \mathbf{p}_x^{\Sigma\perp} \cdot \nu$ for $x \in \Omega$, $\nu \in \mathbb{R}^d$. If $\nu = \nu(x)$ is a (single valued) Lipschitz map defined in Ω, then for any tangent vector field ξ one has

$$D_\xi \mathbf{p}^{\Sigma\perp}(x, \nu(x)) = \partial_x \mathbf{p}^{\Sigma\perp}(x, \nu(x)) \cdot \xi(x) + \partial_\nu \mathbf{p}^{\Sigma\perp}(x, \nu(x)) \cdot D_\xi \nu(x).$$

Since, for fixed x, the map $\nu \in \mathbb{R}^d \mapsto \mathbf{p}^{\Sigma\perp}(x, \nu) \in T_x^\perp \Sigma$ is linear with Lipschitz constant not larger than 1, we conclude that for any $\nu \colon \Omega \to \mathbb{R}^d$ Lipschitz one has

$$\mathrm{Dir}(\nu^\perp, \Omega) \leqslant \mathrm{Dir}(\nu, \Omega) + C \int_\Omega |\nu|^2 \, d\mathcal{H}^m + C \int_\Omega |\nu||D\nu| \, d\mathcal{H}^m,$$

where C is a constant depending on $\max_{\overline{\Omega} \times S^{d-1}} |\partial_x \mathbf{p}^{\Sigma\perp}|$.

Formula (5.51) is then a consequence of Young's inequality. The formula is then extended to Lipschitz Q-valued maps via decomposition into Q Lipschitz functions (Proposition 2.2.7), and finally to Sobolev Q-maps via approximation (Proposition 2.2.10). □

We are now ready to state and prove the proposition that will provide the key towards Theorem 5.0.1.

Proposition 5.2.2. *Let $\Omega \subset \Sigma$ be open, connected with C^2 boundary. Assume the strict stability condition (4.54) holds for every $u \in \Gamma^{1,2}_Q(\mathcal{N}\Omega)$ such that $u|_{\partial\Omega} = Q[\![0]\!]$. Then, if $g \in \Gamma^{1,2}_Q(\mathcal{N}\Omega)$ has boundary trace $g_0 := g|_{\partial\Omega} \in W^{1,2}(\partial\Omega, \mathcal{A}_Q(\mathbb{R}^d))$, the following estimate*

$$\mathrm{Jac}(N,\Omega) \geqslant c(\Omega) \int_\Omega |N|^2 \, d\mathcal{H}^m - C(\Omega, g_0) \tag{5.52}$$

holds true for any $N \in \Gamma^{1,2}_Q(\mathcal{N}\Omega)$ such that $N|_{\partial\Omega} = g_0$.

Proof. Fix $\lambda < \lambda_0$ to be chosen, and let $\mathcal{V} \Subset \Omega$ be such that $\Omega = \mathcal{V}_\lambda$ as above. For any $N \in \Gamma^{1,2}_Q(\mathcal{N}\Omega)$ such that $N|_{\partial\Omega} = g_0$, consider the map $\tilde{N} := N \circ \Phi_\lambda \in W^{1,2}(\mathcal{V}, \mathcal{A}_Q(\mathbb{R}^d))$, and observe that $\tilde{N}|_{\partial\mathcal{V}} = \tilde{g}_0$, where $\tilde{g}_0(\pi) = g_0(\exp_\pi(\lambda\eta(\pi)))$ for $\pi \in \partial\mathcal{V}$.

Now, apply Corollary 5.1.3 with this choice of $\mathcal{V}$, $\tilde{g}_0$ and λ in order to extend $\tilde{N}$ to the map $u \in W^{1,2}(\Omega, \mathcal{A}_Q(\mathbb{R}^d))$ given by

$$u(x) := \begin{cases} \tilde{N}(x) & \text{if } x \in \mathcal{V} \\ \overline{g}_\lambda(x) & \text{if } x \in \Omega \setminus \mathcal{V} = \mathbf{U}^+_\lambda, \end{cases} \tag{5.53}$$

Observe that the normal bundle projection $u^\perp$ satisfies $u^\perp \in \Gamma^{1,2}_Q(\mathcal{N}\Omega)$ and the boundary condition $u^\perp|_{\partial\Omega} = Q[\![0]\!]$. From the hypothesis, we are therefore able to conclude that

$$\mathrm{Jac}(u^\perp, \Omega) \geqslant c(\Omega) \int_\Omega |u^\perp|^2 \, d\mathcal{H}^m. \tag{5.54}$$

Now, note that $u^\perp \equiv N$ in $\Omega \setminus \mathbf{U}_\lambda$. Combining this observation with (5.54), we trivially deduce

$$\mathrm{Jac}\left(N, \Omega \setminus \mathbf{U}_\lambda\right) + \mathrm{Jac}\left(u^\perp, \mathbf{U}_\lambda\right) \geqslant c(\Omega) \left(\int_{\Omega\setminus\mathbf{U}_\lambda} |N|^2 \, d\mathcal{H}^m + \int_{\mathbf{U}_\lambda} |u^\perp|^2 \, d\mathcal{H}^m \right). \tag{5.55}$$

In order to prove our result, we then clearly have to provide suitable estimates for $\mathrm{Jac}\left(u^\perp, \mathbf{U}_\lambda\right)$ and $\int_{\mathbf{U}_\lambda} |u^\perp|^2$.

We observe first that

$$\int_{\mathbf{U}_\lambda} |u^\perp|^2 = \int_{\mathbf{U}^-_\lambda} |u^\perp|^2 + \int_{\mathbf{U}^+_\lambda} |u^\perp|^2 \geqslant \int_{\mathbf{U}^-_\lambda} |u^\perp|^2. \tag{5.56}$$

Recall that

$$u^\perp|_{\mathbf{U}^-_\lambda} = \sum_{\ell=1}^Q [\![\mathbf{p}^{\Sigma\perp} \cdot (N_\ell \circ \Phi_\lambda)]\!] = \sum_{\ell=1}^Q \left[\!\!\left[\sum_{\beta=1}^k \langle N_\ell \circ \Phi_\lambda, \nu_\beta \rangle \nu_\beta \right]\!\!\right],$$

whence

$$\int_{\mathbf{U}^-_\lambda} |u^\perp|^2 = \int_{\mathbf{U}^-_\lambda} \sum_{\ell=1}^Q \sum_{\beta=1}^k |\langle N_\ell\left(\Phi_\lambda(x)\right), \nu_\beta(x)\rangle|^2 \, d\mathcal{H}^m(x). \tag{5.57}$$

Now, changing variable $y = \Phi_\lambda(x)$, integrating along geodesics and using (5.49) one easily proves that from this follows

$$\int_{\mathbf{U}_\lambda^-} |u^\perp|^2 \geqslant C \left(\int_{\mathbf{U}_\lambda} |N|^2 \, d\mathcal{H}^m - \mathcal{E}_\lambda^{(1)} \right), \tag{5.58}$$

where the error term $\mathcal{E}_\lambda^{(1)}$ satisfies the estimate

$$|\mathcal{E}_\lambda^{(1)}| \leqslant \frac{1}{2} \int_{\mathbf{U}_\lambda} |N|^2 \, d\mathcal{H}^m, \tag{5.59}$$

provided λ satisfies some suitable smallness conditions which are *not* depending on N. Combining (5.55), (5.56), (5.58) and (5.59), we conclude that for suitably small λ

$$\mathrm{Jac}\,(N, \Omega \setminus \mathbf{U}_\lambda) + \mathrm{Jac}\left(u^\perp, \mathbf{U}_\lambda\right) \geqslant c(\Omega) \int_\Omega |N|^2 \, d\mathcal{H}^m, \tag{5.60}$$

up to possibly changing the value of $c(\Omega)$.

Now, we work on $\mathrm{Jac}\left(u^\perp, \mathbf{U}_\lambda\right)$. As before, decompose

$$\mathrm{Jac}\left(u^\perp, \mathbf{U}_\lambda\right) = \mathrm{Jac}\left(u^\perp, \mathbf{U}_\lambda^-\right) + \mathrm{Jac}\left(u^\perp, \mathbf{U}_\lambda^+\right). \tag{5.61}$$

Concerning the first addendum, one shows that

$$\mathrm{Jac}\left(u^\perp, \mathbf{U}_\lambda^-\right) \leqslant C\mathrm{Jac}\,(N, \mathbf{U}_\lambda) + \mathcal{E}_\lambda^{(2)}, \tag{5.62}$$

where the error $\mathcal{E}_\lambda^{(2)}$ satisfies

$$|\mathcal{E}_\lambda^{(2)}| \leqslant \varepsilon \left(\mathrm{Dir}\,(N, \mathbf{U}_\lambda) + \int_{\mathbf{U}_\lambda} |N|^2 \, d\mathcal{H}^m \right) \tag{5.63}$$

for any choice of $\varepsilon > 0$, provided λ is smaller than some λ_* depending on ε and on the geometry of the problem, but, again, not on the map N. In particular, this allows to absorb the error term and conclude, under the previously considered smallness assumptions on λ, that

$$\mathrm{Jac}\,(N, \Omega) \geqslant c(\Omega) \left(\int_\Omega |N|^2 \, d\mathcal{H}^m - \mathrm{Jac}\left(u^\perp, \mathbf{U}_\lambda^+\right) \right) \tag{5.64}$$

after possibly having redefined $c(\Omega)$.

Now we are able to conclude: following the same strategy as before, it is not difficult to estimate

$$|\mathrm{Jac}\left(u^\perp, \mathbf{U}_\lambda^+\right)| \leqslant C \left(\mathrm{Dir}\left(\overline{g}_\lambda, \mathbf{U}_\lambda^+\right) + \int_{\mathbf{U}_\lambda^+} |\overline{g}_\lambda|^2 \, d\mathcal{H}^m \right), \tag{5.65}$$

where λ is small, but fixed, and does not depend on N. Our result, equation (5.52), is then an immediate consequence of Corollary 5.1.3 and of the definition of $\tilde{g}_0$. □

We are now ready to prove the Conditional Existence Theorem 5.0.1.

Proof of Theorem 5.0.1. The proof is an application of the direct methods in the Calculus of Variations. Fix any $g \in \Gamma^{1,2}_Q(\mathcal{N}\Omega)$ with boundary trace $g_0 := g|_{\partial\Omega} \in W^{1,2}(\partial\Omega; \mathcal{A}_Q(\mathbb{R}^d))$. Then, the inequality (5.52) implies that for any $N \in \Gamma^{1,2}_Q(\mathcal{N}\Omega)$ with $N|_{\partial\Omega} = g_0$ one has

$$\mathrm{Jac}(N, \Omega) \geqslant -C(\Omega, g_0),$$

thus the Jacobi functional is bounded from below in the class of competitors.

Set

$$\Lambda := \inf\{\mathrm{Jac}(N, \Omega) : \ N \in \Gamma^{1,2}_Q(\mathcal{N}\Omega), N|_{\partial\Omega} = g_0\} > -\infty,$$

and consider a minimizing sequence $\{N_h\}_{h=1}^{\infty} \subset \Gamma^{1,2}_Q(\mathcal{N}\Omega)$, $N_h|_{\partial\Omega} = g_0$, $\lim_{h\to\infty} \mathrm{Jac}(N_h, \Omega) = \Lambda$. Then, for $h \geqslant h_0$ sufficiently large, one has

$$\mathrm{Jac}(N_h, \Omega) \leqslant \Lambda + 1,$$

from which we deduce

$$\mathrm{Dir}(N_h, \Omega) \leqslant C \int_\Omega |N_h|^2 \, d\mathcal{H}^m + |\Lambda| + 1.$$

On the other hand, (5.52) immediately implies that

$$\int_\Omega |N_h|^2 \, d\mathcal{H}^m \leqslant C(|\Lambda|, \Omega, g_0).$$

Putting all together, we conclude that

$$\mathrm{Dir}(N_h, \Omega) + \int_\Omega |N_h|^2 \, d\mathcal{H}^m \leqslant C, \tag{5.66}$$

where C is a constant depending only on $|\Lambda|$, Ω, g_0 and the geometry of the embeddings $\Sigma \hookrightarrow \mathcal{M} \hookrightarrow \mathbb{R}^d$. Hence, up to extracting a subsequence, N_h converges weakly in $W^{1,2}$, strongly in L^2, to a map $\overline{N} \in \Gamma^{1,2}_Q(\mathcal{N}\Omega)$ with $\overline{N}|_{\partial\Omega} = g_0$. The lower semi-continuity of the Jacobi functional with respect to weak convergence, Proposition 4.3.1, allows us to conclude that $\overline{N}$ is the desired minimizer. □

6 REGULARITY THEORY

In this chapter we develop the regularity theory for Jacobi Q-fields. First, we show that any Jac-minimizing map $N \in \Gamma^{1,2}_Q(\mathcal{N}\Omega)$ is locally Hölder continuous in Ω.

Theorem 6.0.1 (Hölder regularity of Jacobi multi-fields)**.** *Let Ω be an open subset of $\Sigma \hookrightarrow \mathcal{M}$ as in Assumption 4.1.1. There exist universal constants $\alpha = \alpha(m, Q) \in (0,1)$ and $\Lambda = \Lambda(m, Q) > 0$ and a radius $0 < r_0 = r_0(m, Q) < \mathrm{inj}(\Sigma)$ with the following property. If $N \in \Gamma^{1,2}_Q(\mathcal{N}\Omega)$ is* Jac*-minimizing, then for every $0 < \theta < 1$ there exists a constant $C = C(m, d, Q, \Sigma, \theta)$ such that*

$$\begin{aligned}[N]_{C^{0,\alpha}(\overline{\mathbf{B}}_{\theta r}(p))} &:= \sup_{x_1,x_2 \in \overline{\mathbf{B}}_{\theta r}(p)} \frac{\mathcal{G}(N(x_1), N(x_2))}{\mathbf{d}(x_1, x_2)^\alpha} \\ &\leqslant C\left(r^{2-m-2\alpha}\left(\mathrm{Dir}(N, \mathbf{B}_r(p)) + \Lambda \int_{\mathbf{B}_r(p)} |N|^2 \, d\mathcal{H}^m\right)\right)^{1/2}\end{aligned} \tag{6.1}$$

for every $p \in \Omega$ and for every $r \leqslant \min\{r_0, \mathbf{dist}(p, \partial\Omega)\}$. In particular, $N \in C^{0,\alpha}_{loc}(\Omega, \mathcal{A}_Q(\mathbb{R}^d))$.

Then, we turn our attention to finer regularity properties. Let us give the following definition of regular and singular points.

Definition 6.0.2 (Regular and singular set)**.** Let $N \in \Gamma^{1,2}_Q(\mathcal{N}\Omega)$ be Jac-minimizing. A point $p \in \Omega$ is *regular* for N (and we write $p \in \mathrm{reg}(N)$) if there exists a neighborhood B of p in Ω and Q classical Jacobi fields $N^\ell \colon B \to \mathbb{R}^d$ such that

$$N(x) = \sum_{\ell=1}^{Q} [\![N^\ell(x)]\!] \qquad \forall\, x \in B$$

and either $N^\ell \equiv N^{\ell'}$ or $N^\ell(x) \neq N^{\ell'}(x)$ for all $x \in B$, for any $\ell, \ell' \in \{1, \dots, Q\}$. The *singular set* of N is defined by

$$\mathrm{sing}(N) := \Omega \setminus \mathrm{reg}(N).$$

We have the following theorem.

Theorem 6.0.3 (Estimate of the singular set)**.** *Let N be a Q-valued Jacobi field in $\Omega \subset \Sigma^m$. Then, the singular set $\mathrm{sing}(N)$ is relatively closed in Ω. Furthermore, if $m = 2$, then $\mathrm{sing}(N)$ is at most countable; if $m \geqslant 3$, then the Hausdorff dimension $\dim_{\mathcal{H}} \mathrm{sing}(N)$ does not exceed $m - 2$.*

Starting from Section 6.2, the chapter will be devoted to the proof of Theorem 6.0.3, which will eventually be obtained in Section 6.4 as a consequence of the analogous theorem valid for Dir-minimizing Q-valued functions, after we have shown the key fact that at any multiplicity Q point for N every tangent map is a *non trivial* homogeneous Dirichlet minimizer. This is the content of Theorem 6.3.8. A careful analysis of Almgren's frequency function will be indispensable to prove Theorem 6.3.8.

6.1 HÖLDER REGULARITY OF JACOBI Q-FIELDS

The proof of Theorem 6.0.1 is a fairly easy consequence of Proposition 6.1.1 below. Before stating it, we need to introduce some further notation.

Let us fix a point $p \in \Omega$, and a radius $r < \min\{\mathrm{inj}(\Sigma), \mathbf{dist}(p, \partial\Omega)\}$, in such a way that the exponential map $\exp_p$ defines a diffeomorphism

$$\exp_p : B_r(0) \subset T_p\Sigma \to \mathbf{B}_r(p) \subset \Omega.$$

Denote by $y = (y^1, \dots, y^m)$ coordinates in $T_p\Sigma$ corresponding to the choice of an orthonormal basis $(e_1, \dots, e_m)$, and set $u := N \circ \exp_p$. Observe that for any $y \in B_r$ the differential $d(\exp_p)|_y$ realizes a linear isomorphism between $T_p\Sigma$ and $T_{\exp_p(y)}\Sigma$. Fix an orthonormal frame $(\xi_1, \dots, \xi_m)$ of the tangent bundle $T\Sigma|_{\mathbf{B}_r(p)}$ extending $(e_1, \dots, e_m)$ (i.e. such that $\xi_i|_p = e_i$ for $i = 1, \dots, m$), and define, for $y \in B_r$,

$$\varepsilon_i(y) := \left(d(\exp_p)|_y\right)^{-1} \cdot \xi_i(\exp_p(y)). \tag{6.2}$$

Then, an elementary computation shows that

$$\int_{\mathbf{B}_r(p)} |N(x)|^2 \, d\mathcal{H}^m(x) = \int_{B_r} |u(y)|^2 \mathbf{J}\exp_p(y) \, dy \tag{6.3}$$

and

$$\mathrm{Dir}(N, \mathbf{B}_r(p)) = \int_{B_r} \sum_{i=1}^{m} |D_{\varepsilon_i} u(y)|^2 \mathbf{J}\exp_p(y) \, dy, \tag{6.4}$$

where $\mathbf{J}\exp_p$ is the Jacobian determinant of the exponential map. From this it is immediate to deduce that the following asymptotic behaviors are satisfied for $r \to 0$ uniformly in p:

$$\int_{\mathbf{B}_r(p)} |N(x)|^2 \, d\mathcal{H}^m(x) = (1 + O(r)) \int_{B_r} |u(y)|^2 \, dy, \tag{6.5}$$

$$\mathrm{Dir}(N, \mathbf{B}_r(p)) = (1 + O(r)) \int_{B_r} \sum_{i=1}^{m} |D_{e_i} u(y)|^2 \, dy = (1 + O(r))\mathrm{Dir}(u, B_r). \tag{6.6}$$

We can now state the key result from which we will conclude the Hölder regularity of Jacobi Q-fields.

Proposition 6.1.1. *There exist a universal positive constant $\Lambda = \Lambda(m, Q)$ and a radius $0 < r_0 = r_0(m, Q) < \mathrm{inj}(\Sigma)$ with the following property. Let $N \in \Gamma_Q^{1,2}(\mathcal{N}\Omega)$ be* Jac*-minimizing and $p \in \Omega$. Then, for a.e. radius $r \leqslant \min\{r_0, \mathbf{dist}(p, \partial\Omega)\}$ one has*

$$\mathrm{Dir}(u, B_r) + \Lambda \int_{B_r} |u|^2 \, dy \leqslant C(m) r \left(\mathrm{Dir}(u, \partial B_r) + \Lambda \int_{\partial B_r} |u|^2 \, d\mathcal{H}^{m-1} \right), \tag{6.7}$$

where $u := N \circ \exp_p |_{B_r} \in W^{1,2}\left(B_r, \mathcal{A}_Q(\mathbb{R}^d)\right)$ and $C(m) < (m-2)^{-1}$ when $m \geqslant 3$.

In order to prove Proposition 6.1.1, we will need the following simple result on classical Sobolev functions in the Euclidean space.

Lemma 6.1.2. *For every $\varepsilon > 0$ there exists a constant $C = C_\varepsilon > 0$ such that the inequality*

$$\int_{B_r} |g|^2 \, dy \leqslant \left(\frac{1}{m} + \varepsilon\right) r \int_{\partial B_r} |g|^2 \, d\mathcal{H}^{m-1} + C_\varepsilon r^2 \int_{B_r} |Dg|^2 \, dy \tag{6.8}$$

holds for any function $g \in W^{1,2}(B_r^m)$.

Proof. First observe that, by a simple scaling argument, it is enough to prove the lemma for $r = 1$. Assume the lemma is false: suppose, by contradiction, that there exists $\varepsilon_0 > 0$ such that for any $h \in \mathbb{N}$ there is $g_h \in W^{1,2}(B_1^m)$, with $\|g_h\|_{L^2} = 1$, such that

$$1 > \left(\frac{1}{m} + \varepsilon_0\right) \int_{\partial B_1} |g_h|^2 \, d\mathcal{H}^{m-1} + h \int_{B_1} |Dg_h|^2 \, dy. \tag{6.9}$$

The inequality (6.9) readily implies that

$$\lim_{h\to\infty} \int_{B_1} |Dg_h|^2 \, dy = 0, \tag{6.10}$$

whence, by Rellich's compactness theorem, the sequence g_h converges up to a subsequence (not relabeled) weakly in $W^{1,2}$, strongly in L^2, to a constant function $g \equiv c$. The condition $\|g\|_{L^2} = 1$ forces the constant to satisfy $|c|^2 = \omega_m^{-1}$, where ω_m is the volume of the unit ball in $\mathbb{R}^m$. Hence, it suffices to pass to the limit the inequality

$$1 > \left(\frac{1}{m} + \varepsilon_0\right) \int_{\partial B_1} |g_h|^2 \, d\mathcal{H}^{m-1} \tag{6.11}$$

to obtain the desired contradiction:

$$1 > \left(\frac{1}{m} + \varepsilon_0\right) m. \tag{6.12}$$

□

Corollary 6.1.3. *For every $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ such that for any function $v \in W^{1,2}\left(B_r, \mathcal{A}_Q(\mathbb{R}^d)\right)$ one has:*

$$\int_{B_r} |v|^2 \, dy \leqslant \left(\frac{1}{m} + \varepsilon\right) r \int_{\partial B_r} |v|^2 \, d\mathcal{H}^{m-1} + C_\varepsilon r^2 \mathbf{Dir}(v, B_r). \tag{6.13}$$

Proof. Fix $\varepsilon > 0$ and $v \in W^{1,2}(B_r, \mathcal{A}_Q(\mathbb{R}^d))$, and apply Lemma 6.1.2 to the function $g = |v| = \mathcal{G}(v, Q[\![0]\!]) \in W^{1,2}(B_r)$. The inequality (6.13) then follows immediately, because $g|_{\partial B_r} = |v|_{\partial B_r}|$ and $|\partial_j g| \leqslant |\partial_j v|$ for every $j = 1, \ldots, m$. □

Proof of Proposition 6.1.1. Let $N \in \Gamma_Q^{1,2}(\mathcal{N}\Omega)$ be Jac-minimizing, and fix any point $p \in \Omega$. For every radius $r < \min\{\mathrm{inj}(\Sigma), \mathbf{dist}(p, \partial\Omega)\}$ the exponential map $\exp_p$ maps the Euclidean ball $B_r(0) \subset T_p\Sigma$ diffeomorphically onto the geodesic ball $\mathbf{B}_r(p) \subset \Sigma$, and the composition $u := N \circ \exp_p$ is a $W^{1,2}$ Q-valued map defined in B_r.

Let now $f \in W^{1,2}\left(B_r, \mathcal{A}_Q(\mathbb{R}^d)\right)$ be Dir-minimizing in B_r such that $f|_{\partial B_r} = u|_{\partial B_r}$ [1], and set $h := f \circ \exp_p^{-1}$. Then, the normal bundle projection $h^\perp \in \Gamma_Q^{1,2}(\mathcal{N}\mathbf{B}_r(p))$ satisfies $h^\perp|_{\partial \mathbf{B}_r(p)} = N|_{\partial \mathbf{B}_r(p)}$ and is therefore a competitor for the Jacobi functional. Hence, using minimality, the definition of the Jacobi functional and (4.48), we deduce:

$$\mathrm{Jac}(N, \mathbf{B}_r(p)) \leqslant \mathrm{Jac}(h^\perp, \mathbf{B}_r(p)) \leqslant \mathrm{Dir}(h^\perp, \mathbf{B}_r(p)) + C_0 \int_{\mathbf{B}_r(p)} |h|^2 \, d\mathcal{H}^m, \tag{6.14}$$

which in turn produces

$$\mathrm{Dir}(N, \mathbf{B}_r(p)) \leqslant \mathrm{Dir}(h^\perp, \mathbf{B}_r(p)) + C_0 \left(\int_{\mathbf{B}_r(p)} |h|^2 \, d\mathcal{H}^m + \int_{\mathbf{B}_r(p)} |N|^2 \, d\mathcal{H}^m \right). \tag{6.15}$$

Hence, combining Lemma 5.2.1 with (6.5) and (6.6), we can conclude that for any $\varepsilon_1 \in (0,1)$ there exists a radius $0 < r_{\varepsilon_1} < \mathrm{inj}(\Sigma)$ such that the estimate

$$\mathrm{Dir}(u, B_r) \leqslant (1+\varepsilon_1)\,\mathrm{Dir}(f, B_r) + C_{\varepsilon_1} \left(\int_{B_r} |f|^2 \, dy + \int_{B_r} |u|^2 \, dy \right), \tag{6.16}$$

holds true whenever $r \leqslant r_{\varepsilon_1}$.

Now we apply [DLS11, Proposition 3.10]: since f is Dir-minimizing in B_r, the estimate

$$\mathrm{Dir}(f, B_r) \leqslant C(m) r \mathrm{Dir}(u, \partial B_r) \tag{6.17}$$

holds with constants $C(2) = Q$ and $C(m) < (m-2)^{-1}$ for $m \geqslant 3$ whenever $\mathrm{Dir}(u, \partial B_r)$ is finite, and thus for a.e. r. Combining (6.16) with (6.17), we deduce that we can choose $\varepsilon_1 = \varepsilon_1(m, Q)$ so small that the inequality

$$\mathrm{Dir}(u, B_r) \leqslant C(m) r \mathrm{Dir}(u, \partial B_r) + C \left(\int_{B_r} |f|^2 \, dy + \int_{B_r} |u|^2 \, dy \right) \tag{6.18}$$

holds with, say, $C(2) = 2Q$ and again $C(m) < (m-2)^{-1}$ when $m \geqslant 3$ for a.e. $r \leqslant r_{m,Q}$.

Now, fix $\varepsilon > 0$ and apply the result of Corollary 6.1.3 first with $v = f$ and then with $v = u$, and plug the resulting inequalities in (6.18). Using the fact that f and u have the same boundary value and that $\mathrm{Dir}(f, B_r) \leqslant \mathrm{Dir}(u, B_r)$, we obtain the following key inequality:

$$\mathrm{Dir}(u, B_r) \leqslant C(m) r \mathrm{Dir}(u, \partial B_r) + C \left(\frac{1}{m} + \varepsilon \right) r \int_{\partial B_r} |u|^2 d\mathcal{H}^{m-1} + C_\varepsilon r^2 \mathrm{Dir}(u, B_r). \tag{6.19}$$

This implies the following: for every $\Lambda > 0$ one has

$$\begin{aligned} \mathrm{Dir}(u, B_r) + \Lambda \int_{B_r} |u|^2 \, dy \leqslant\ & C(m) r \mathrm{Dir}(u, \partial B_r) \\ & + (C + \Lambda) \left(\frac{1}{m} + \varepsilon \right) r \int_{\partial B_r} |u|^2 \, d\mathcal{H}^{m-1} \\ & + C_{\varepsilon, \Lambda} r^2 \mathrm{Dir}(u, B_r). \end{aligned} \tag{6.20}$$

1 Recall that the existence of such a map f is guaranteed by Theorem 2.2.20.

For a suitable choice of $\Lambda = \Lambda_{m,\varepsilon} \gg 1$ this yields:

$$(1 - C_{m,\varepsilon} r^2)\,\mathrm{Dir}(u, B_r) + \Lambda \int_{B_r} |u|^2\,dy \leqslant C(m) r \mathrm{Dir}(u, \partial B_r) + \Lambda \left(\frac{1}{m} + 2\varepsilon\right) r \int_{\partial B_r} |u|^2\,d\mathcal{H}^{m-1}. \tag{6.21}$$

Finally, we divide the whole inequality by $1 - C_{m,\varepsilon} r^2$ and conclude that if r is sufficiently small, say $r \leqslant r_{m,\varepsilon,Q}$ then the inequality

$$\mathrm{Dir}(u, B_r) + \Lambda \int_{B_r} |u|^2\,dy \leqslant C(m) r \mathrm{Dir}(u, \partial B_r) + \Lambda \left(\frac{1}{m} + 4\varepsilon\right) r \int_{\partial B_r} |u|^2\,d\mathcal{H}^{m-1} \tag{6.22}$$

holds with a possible new choice of $C(m)$, say $C(2) = 4Q$ and still $C(m) < (m-2)^{-1}$ for $m \geqslant 3$. The conclusion immediately follows, by choosing $\varepsilon = \varepsilon(m, Q)$ in such a way that $\frac{1}{m} + 4\varepsilon < 4Q$ when $m = 2$ and $\frac{1}{m} + 4\varepsilon < \frac{1}{m-2}$ when $m \geqslant 3$. □

We have now all the ingredients that are needed to prove Theorem 6.0.1: as announced at the beginning of the section, the proof can be easily achieved by combining our Proposition 6.1.1 with the Campanato-Morrey estimates 2.2.19.

Proof of Theorem 6.0.1. Let r_0 be the radius given in Proposition 6.1.1. Fix any point $p \in \Omega$, and assume without loss of generality that $\mathbf{B}_{r_0}(p) \Subset \Omega$. Consider the corresponding exponential map $\exp_p : B_{r_0}(0) \subset T_p\Sigma \to \mathbf{B}_{r_0}(p) \subset \Sigma$, and set $u := N \circ \exp_p$. By Proposition 6.1.1, for a.e. radius $r \leqslant r_0$ the inequality (6.7) is satisfied with universal constants Λ and $C(m)$, with $C(m) < (m-2)^{-1}$ when $m \geqslant 3$. We set:

$$\gamma(m) := \begin{cases} C(m)^{-1} & \text{if } m = 2 \\ C(m)^{-1} - m + 2 & \text{if } m \geqslant 3, \end{cases} \tag{6.23}$$

and we denote by $\phi = \phi(r)$ the absolutely continuous function

$$\phi(r) := \mathrm{Dir}(u, B_r) + \Lambda \int_{B_r} |u|^2\,dy \tag{6.24}$$

for $r \in (0, r_0]$. By (6.7), ϕ satisfies the differential inequality

$$(m - 2 + \gamma)\phi \leqslant r\phi' \tag{6.25}$$

almost everywhere in the interval $(0, r_0]$. Integrating (6.25) we obtain:

$$\mathrm{Dir}(u, B_r) \leqslant \phi(r) \leqslant \frac{\phi(r_0)}{r_0^{m-2+\gamma}} r^{m-2+\gamma} =: A r^{m-2+\gamma}. \tag{6.26}$$

As a consequence of the Campanato - Morrey estimates, Proposition 2.2.19, we conclude that u is Hölder continuous with exponent $\alpha := \frac{\gamma}{2}$, with

$$[u]_{C^{0,\alpha}(\overline{B}_{\theta r_0})} := \sup_{y_1, y_2 \in \overline{B}_{\theta r_0}} \frac{\mathcal{G}(u(y_1), u(y_2))}{|y_1 - y_2|^\alpha} \leqslant C\sqrt{A}, \tag{6.27}$$

for any $0 < \theta < 1$ and for a constant $C = C(m, d, Q, \theta)$.

The estimate (6.1) is an immediate consequence of (6.27) and the properties of the exponential map. □

6.2 FIRST VARIATION FORMULAE AND THE ANALYSIS OF THE FREQUENCY FUNCTION

In this section we start the machinery that will eventually lead us, in Section 6.4, to the proof of Theorem 6.0.3.

The first step towards this result consists of deriving some Euler-Lagrange conditions for Jac-minimizing multi-valued maps. Throughout the whole section, we will assume, as usual, that N is a Q-valued section of the normal bundle of Σ in $\mathcal{M}$ defined in an open set Ω, where it minimizes the Jacobi functional as specified in Definition 4.2.11.

6.2.1 First variations

Suppose that for some $\delta > 0$ we have a 1-parameter family $\{N_s\}_{s\in(-\delta,\delta)} \subset \Gamma^{1,2}_Q(\mathcal{N}\Omega)$ such that $N_0 = N$ and $N_s \equiv N$ in a neighborhood of $\partial\Omega$ for all s. Then, the minimization property of N implies that the map $s \mapsto \mathrm{Jac}(N_s, \Omega)$ takes its minimum at $s = 0$, and thus

$$\frac{d}{ds}\mathrm{Jac}(N_s, \Omega)\Big|_{s=0} = 0 \tag{6.28}$$

whenever the derivative on the left exists. The family $\{N_s\}$ is called an (admissible) 1-parameter family of variations of N in Ω, and formula (6.28) is the first variation formula corresponding to the given variation.

We will consider two natural types of variations in order to perturb the map N. The *inner variations* are generated by right compositions with diffeomorphisms of the domain and by a suitable "orthogonalization procedure"; the *outer variations* correspond instead to "left compositions". The relevant definition is the following.

Definition 6.2.1. Let $N = \sum_\ell \llbracket N^\ell \rrbracket \in \Gamma^{1,2}_Q(\mathcal{N}\Omega)$ be Jac-minimizing in Ω.

(OV) Given $\psi \in C^1(\Omega \times \mathbb{R}^d, \mathbb{R}^d)$ such that $\mathrm{spt}(\psi) \subset \Omega' \times \mathbb{R}^d$ for some $\Omega' \Subset \Omega$ and $\psi(x,u) \in T^\perp_x\Sigma \subset T_x\mathcal{M}$ for all $(x,u) \in \Omega \times T^\perp_x\Sigma$, an admissible variation of N in Ω can be defined by $N_s(x) := \sum_{\ell=1}^Q \llbracket N^\ell(x) + s\psi(x, N^\ell(x)) \rrbracket$. Such a family is called outer variation (OV);

(IV) Given a C^1 vector field X of $T\Sigma$ compactly supported in Ω, for s sufficiently small the map $x \mapsto \Phi_s(x) := \exp_x(sX(x))$ is a diffeomorphism of Ω which leaves $\partial\Omega$ fixed. As a consequence, the family $\{N_s\}$ defined by $N_s := (N \circ \Phi_s)^\perp$ is an admissible variation of N in Ω, which we call inner variation (IV).

In the next proposition, we obtain an explicit formulation of (6.28) in the case of outer variations induced by maps ψ as above. Consistently with the notation introduced for multi-fields in Definition 4.2.9, given $(x,u) \in \Omega \times \mathbb{R}^d$ we will denote by $\nabla^\perp\psi(x,u)$ the linear operator $T_x\Sigma \to T^\perp_x\Sigma$ obtained by projecting $D_x\psi(x,u)$ onto $T^\perp_x\Sigma$ at every x. Also, recall the definitions of $A \cdot u$, u being a (single-valued) section of $\mathcal{N}\Omega$, and of the quadratic form $\mathcal{R}$. Finally, recall that the symbol $\langle L : M \rangle$ denotes the usual Hilbert-Schmidt scalar product of two matrices L and M.

Proposition 6.2.2 (Outer variation formula)**.** *Let ψ be as in (OV) and such that*

$$|D_u\psi| \leqslant C < \infty \quad \textit{and} \quad |\psi| + |D_x\psi| \leqslant C(1+|u|). \tag{6.29}$$

Then, the first variation formula corresponding to the outer variation N_s defined by ψ is

$$\int_\Omega \sum_{\ell=1}^{Q} \langle \nabla^\perp N^\ell(x) : \left(\nabla^\perp \psi(x, N^\ell(x)) + D_u\psi(x, N^\ell(x)) \cdot DN^\ell(x)\right)\rangle \, d\mathcal{H}^m(x) = \mathcal{E}_{OV}(\psi), \tag{6.30}$$

where

$$\mathcal{E}_{OV}(\psi) := \int_\Omega \sum_{\ell=1}^{Q} \left(\langle A \cdot N^\ell(x) : A \cdot \psi(x, N^\ell(x))\rangle + \mathcal{R}(N^\ell(x), \psi(x, N^\ell(x)))\right) d\mathcal{H}^m(x). \tag{6.31}$$

Proof. The proof is straightforward: using (4.44) with N_s in place of u, it suffices to differentiate in s and recall that $\mathcal{R}$ is a symmetric quadratic form on the normal bundle of Σ in $\mathcal{M}$ (the hypotheses in (6.29) ensure the summability of the various integrands involved in the computation). □

An explicit formula for (6.28) in the case of inner variations induced by vector fields X as in Definition 6.2.1 is the content of the following proposition. Recall that $\overline{A}$ denotes the second fundamental form of the embedding $\mathcal{M} \hookrightarrow \mathbb{R}^d$.

Proposition 6.2.3 (Inner variation formula)**.** *Let X be as in (IV). Then, the first variation formula corresponding to the inner variation N_s defined by the family Φ_s of diffeomorphisms induced by X is*

$$-\int_\Omega |\nabla^\perp N|^2 \mathrm{div}_\Sigma(X)\, d\mathcal{H}^m + 2\int_\Omega \sum_{\ell=1}^{Q} \langle \nabla^\perp N^\ell : \nabla^\perp N^\ell \cdot \nabla^\Sigma X\rangle \, d\mathcal{H}^m = \mathcal{E}_{IV}(X), \tag{6.32}$$

where

$$\mathcal{E}_{IV}(X) = \mathcal{E}_{IV}^{(1)}(X) + \mathcal{E}_{IV}^{(2)}(X) + \mathcal{E}_{IV}^{(3)}(X)$$

is defined by

$$\mathcal{E}_{IV}^{(1)}(X) := 2\int_\Omega \sum_{\ell=1}^{Q} \left(\mathrm{tr}_\Sigma(\langle \overline{A}(\cdot, N^\ell), \overline{A}(X, \nabla^\perp_{(\cdot)} N^\ell)\rangle) - \mathrm{tr}_\Sigma(\langle \overline{A}(X, N^\ell), \overline{A}(\cdot, \nabla^\perp_{(\cdot)} N^\ell)\rangle)\right) d\mathcal{H}^m, \tag{6.33}$$

$$\mathcal{E}_{IV}^{(2)}(X) := 2\int_\Omega \sum_{\ell=1}^{Q} \langle A \cdot N^\ell : A \cdot \nabla^\perp_X N^\ell\rangle \, d\mathcal{H}^m, \tag{6.34}$$

and

$$\mathcal{E}_{IV}^{(3)}(X) := 2\int_\Omega \sum_{\ell=1}^{Q} \mathcal{R}(N^\ell, \nabla^\perp_X N^\ell)\, d\mathcal{H}^m. \tag{6.35}$$

Proof. Fix the vector field X, and consider the associated variation $\{N_s\}$, with $N_s = \sum_\ell [\![N_s^\ell]\!]$ defined by

$$N_s^\ell(x) = (N^\ell \circ \Phi_s)^\perp(x) = \sum_{\beta=1}^{k} \langle N^\ell(\Phi_s(x)), \nu_\beta(x)\rangle \nu_\beta(x), \tag{6.36}$$

$(\nu_\beta)_{\beta=1}^k$ being a (local) orthonormal frame of $\mathcal{N}\Omega$. Recall that $\Phi_0 = \mathrm{Id}_\Omega$ and that $\frac{\partial}{\partial s}\Phi_s(x)\big|_{s=0} = X(x)$. Now, using (4.44), we write the first variation formula as

$$\begin{aligned} 0 &= \frac{d}{ds}\mathrm{Jac}(N_s,\Omega)\Big|_{s=0} \\ &= \underbrace{\frac{d}{ds}\Big|_{s=0}\mathrm{Dir}^{\mathcal{N}\Sigma}(N_s,\Omega)}_{=:I_1} + \underbrace{\left(-\frac{d}{ds}\Big|_{s=0}\int_\Omega \sum_{\ell=1}^{Q} |A\cdot N_s^\ell|^2\, d\mathcal{H}^m\right)}_{=:I_2} + \underbrace{\left(-\frac{d}{ds}\Big|_{s=0}\int_\Omega \sum_{\ell=1}^{Q} \mathcal{R}(N_s^\ell, N_s^\ell)\, d\mathcal{H}^m\right)}_{=:I_3}, \end{aligned} \tag{6.37}$$

and we will work on the three terms separately.

Step 1: computing I_1. Write $I_1 = \sum_\ell I_1^\ell$, where

$$I_1^\ell = \frac{d}{ds}\Big|_{s=0}\int_\Omega \sum_{i=1}^{m}\sum_{\alpha=1}^{k} |\langle D_{\xi_i} N_s^\ell, \nu_\alpha\rangle|^2\, d\mathcal{H}^m. \tag{6.38}$$

Using the representation formula (6.36), one immediately computes

$$\begin{aligned} \langle D_{\xi_i} N_s^\ell, \nu_\alpha\rangle(x) &= \langle DN^\ell|_{\Phi_s(x)}\cdot D\Phi_s|_x\cdot \xi_i(x), \nu_\alpha(x)\rangle \\ &\quad + \langle N^\ell(\Phi_s(x)), D_{\xi_i}\nu_\alpha(x)\rangle \\ &\quad + \sum_{\beta=1}^{k}\langle N^\ell(\Phi_s(x)),\nu_\beta(x)\rangle\langle D_{\xi_i}\nu_\beta(x),\nu_\alpha(x)\rangle. \end{aligned} \tag{6.39}$$

Now, since $\langle \nu_\alpha, \nu_\beta\rangle = \delta_{\alpha\beta}$, we have that $\langle D_{\xi_i}\nu_\beta, \nu_\alpha\rangle = -\langle \nu_\beta, D_{\xi_i}\nu_\alpha\rangle$, so that the last term in formula (6.39) becomes

$$-\sum_{\beta=1}^{k}\langle N^\ell(\Phi_s(x)),\nu_\beta(x)\rangle\langle D_{\xi_i}\nu_\alpha(x),\nu_\beta(x)\rangle = -\langle N^\ell(\Phi_s(x)), \nabla^\perp_{\xi_i}\nu_\alpha(x)\rangle, \tag{6.40}$$

and we can write

$$\begin{aligned} \langle D_{\xi_i} N_s^\ell, \nu_\alpha\rangle(x) &= \langle DN^\ell|_{\Phi_s(x)}\cdot D\Phi_s|_x\cdot \xi_i(x), \nu_\alpha(x)\rangle \\ &\quad + \langle N^\ell(\Phi_s(x)), (D_{\xi_i}\nu_\alpha - \nabla^\perp_{\xi_i}\nu_\alpha)(x)\rangle. \end{aligned} \tag{6.41}$$

For small values of the parameter s, the map Φ_s is a diffeomorphism of Ω, and we will denote by Φ_s^{-1} its inverse. Then, we can change variable $x = \Phi_s^{-1}(y)$ in the integral, and finally write

$$I_1^\ell = \frac{d}{ds}\Big|_{s=0}\int_\Omega \sum_{i=1}^{m}\sum_{\alpha=1}^{k} |g_{i\alpha}^\ell(s,y)|^2 \mathbf{J}\Phi_s^{-1}(y)\, d\mathcal{H}^m(y), \tag{6.42}$$

where $\mathbf{J}\Phi_s^{-1}$ is the Jacobian determinant of $D\Phi_s^{-1}$ and

$$g^\ell_{i\alpha}(s,y) = \langle DN^\ell|_y \cdot \zeta_i(s,y), \nu_\alpha(\Phi_s^{-1}(y))\rangle + \langle N^\ell(y), (D_{\xi_i}\nu_\alpha - \nabla^\perp_{\xi_i}\nu_\alpha)(\Phi_s^{-1}(y))\rangle, \tag{6.43}$$

with $\zeta_i(s,y) := D\Phi_s|_{\Phi_s^{-1}(y)} \cdot \xi_i(\Phi_s^{-1}(y))$. Hence, we have:

$$\begin{aligned} I_1^\ell &= -\int_\Omega \sum_{i=1}^m \sum_{\alpha=1}^k |g^\ell_{i\alpha}(0,y)|^2 \mathrm{div}_\Sigma(X)\, d\mathcal{H}^m + 2\int_\Omega \sum_{i=1}^m \sum_{\alpha=1}^k g^\ell_{i\alpha}(0,y)\partial_s g^\ell_{i\alpha}(0,y)\, d\mathcal{H}^m \\ &= -\int_\Omega \sum_{i=1}^m \sum_{\alpha=1}^k |g^\ell_{i\alpha}(0,y)|^2 \mathrm{div}_\Sigma(X)\, d\mathcal{H}^m + \int_\Omega \frac{\partial}{\partial s}\left(\sum_{i=1}^m \sum_{\alpha=1}^k |g^\ell_{i\alpha}(s,y)|^2\right)\Big|_{s=0} d\mathcal{H}^m(y). \end{aligned} \tag{6.44}$$

Now, since

$$\sum_{i=1}^m \sum_{\alpha=1}^k |g^\ell_{i\alpha}(s,y)|^2 = |\nabla^\perp N^\ell_s|^2(\Phi_s^{-1}(y)),$$

its value is independent of the orthonormal frame chosen: thus, having fixed a point $y \in \Omega$, we can impose $\nabla\xi_i = \nabla\nu_\alpha = 0$ at y.

We can now proceed computing explicitly (6.44). Clearly, $g^\ell_{i\alpha}(0,y) = \langle D_{\xi_i}N^\ell, \nu_\alpha\rangle(y)$, so we are only left with the computation of $\partial_s g^\ell_{i\alpha}(0,y)$. We start observing that

$$\partial_s \zeta_i(0,y) = (D_{\xi_i}X - D_X\xi_i)(y) = -[X,\xi_i](y), \tag{6.45}$$

from which we easily deduce

$$\begin{aligned} \partial_s|_{s=0}\left(\langle DN^\ell|_y \cdot \zeta_i(s,y), \nu_\alpha(\Phi_s^{-1}(y))\rangle\right) &= -\langle D_{[X,\xi_i]}N^\ell, \nu_\alpha\rangle - \langle D_{\xi_i}N^\ell, D_X\nu_\alpha\rangle \\ &= \langle D_{\nabla^\Sigma_{\xi_i}X}N^\ell, \nu_\alpha\rangle - \langle \overline{A}(\xi_i, N^\ell), \overline{A}(X,\nu_\alpha)\rangle, \end{aligned} \tag{6.46}$$

where we have used that $\nabla_X\xi_i = \nabla_X\nu_\alpha = 0$ at y (and, therefore, $[X,\xi_i](y) = -\nabla^\Sigma_{\xi_i}X(y)$ and $D_X\nu_\alpha = \overline{A}(X,\nu_\alpha)$).

On the other hand,

$$\begin{aligned} \partial_s|_{s=0}\left(\langle N^\ell(y), (D_{\xi_i}\nu_\alpha - \nabla^\perp_{\xi_i}\nu_\alpha)(\Phi_s^{-1}(y))\rangle\right) &= -\langle N^\ell, D_X(D_{\xi_i}\nu_\alpha - \nabla^\perp_{\xi_i}\nu_\alpha)\rangle \\ &= \langle D_X N^\ell, D_{\xi_i}\nu_\alpha - \nabla^\perp_{\xi_i}\nu_\alpha\rangle \\ &= \langle \overline{A}(X,N^\ell), \overline{A}(\xi_i,\nu_\alpha)\rangle \end{aligned} \tag{6.47}$$

because the fields N^ℓ and $D_{\xi_i}\nu_\alpha - \nabla^\perp_{\xi_i}\nu_\alpha$ are mutually orthogonal and, again, because $\nabla\nu_\alpha = 0$ at y.

This allows to conclude:

$$\begin{aligned} I_1 &= -\int_\Omega |\nabla^\perp N|^2 \mathrm{div}_\Sigma(X)\, d\mathcal{H}^m + 2\int_\Omega \sum_{\ell=1}^Q \langle \nabla^\perp N^\ell : \nabla^\perp N^\ell \cdot \nabla^\Sigma X\rangle\, d\mathcal{H}^m \\ &\quad + 2\int_\Omega \sum_{\ell=1}^Q \left(\mathrm{tr}_\Sigma(\langle \overline{A}(X,N^\ell), \overline{A}(\cdot, \nabla^\perp_{(\cdot)}N^\ell)\rangle) - \mathrm{tr}_\Sigma(\langle \overline{A}(\cdot,N^\ell), \overline{A}(X, \nabla^\perp_{(\cdot)}N^\ell)\rangle)\right) d\mathcal{H}^m \\ &= -\int_\Omega |\nabla^\perp N|^2 \mathrm{div}_\Sigma(X)\, d\mathcal{H}^m + 2\int_\Omega \sum_{\ell=1}^Q \langle \nabla^\perp N^\ell : \nabla^\perp N^\ell \cdot \nabla^\Sigma X\rangle\, d\mathcal{H}^m - \mathcal{E}^{(1)}_{IV}(X). \end{aligned} \tag{6.48}$$

Step 2: computing I_2. Write $I_2 = \sum_\ell I_2^\ell$, where

$$I_2^\ell = -\frac{d}{ds}\Big|_{s=0} \int_\Omega |A \cdot N_s^\ell|^2 \, d\mathcal{H}^m. \tag{6.49}$$

Since the tensor A takes values in the normal bundle of Σ, clearly $A \cdot N_s^\ell = A \cdot (N^\ell \circ \Phi_s)$, whence

$$\int_\Omega |A \cdot N_s^\ell|^2 \, d\mathcal{H}^m = \int_\Omega \sum_{i,j=1}^m |\langle A_x(\xi_i(x), \xi_j(x)), N^\ell(\Phi_s(x))|^2 \, d\mathcal{H}^m(x). \tag{6.50}$$

We can now differentiate in s and evaluate for $s = 0$ in formula (6.50) to obtain:

$$I_2^\ell = -2 \int_\Omega \sum_{i,j=1}^m \langle A_x(\xi_i(x), \xi_j(x)), N^\ell(x)\rangle \langle A_x(\xi_i(x), \xi_j(x)), D_X N^\ell(x)\rangle, \tag{6.51}$$

which readily yields

$$I_2 = -2 \int_\Omega \sum_{\ell=1}^Q \langle A \cdot N^\ell : A \cdot \nabla_X^\perp N^\ell \rangle \, d\mathcal{H}^m = -\mathcal{E}_{IV}^{(2)}(X). \tag{6.52}$$

Step 3: computing I_3. As before, write $I_3 = \sum_\ell I_3^\ell$, where

$$I_3^\ell = -\frac{d}{ds}\Big|_{s=0} \int_\Omega \mathcal{R}(N_s^\ell, N_s^\ell) \, d\mathcal{H}^m. \tag{6.53}$$

Now, it suffices to differentiate in s and evaluate at $s = 0$ inside the integral keeping in mind that $\mathcal{R}$ is a symmetric 2-tensor to get

$$I_3 = -2 \int_\Omega \sum_{\ell=1}^Q \mathcal{R}(N^\ell, \nabla_X^\perp N^\ell) \, d\mathcal{H}^m = -\mathcal{E}_{IV}^{(3)}(X). \tag{6.54}$$

Conclusion. The statement, formula (6.32), is immediately obtained by plugging equations (6.48), (6.52) and (6.54) into (6.37).

□

The first variation formulae (6.30) and (6.32) will play a fundamental role in the next section to discuss the almost monotonicity properties of the frequency function. Before proceeding, we apply the outer variation formula to show that minimizers of the Jac functional enjoy a Caccioppoli type inequality.

Proposition 6.2.4 (Caccioppoli inequality)**.** *There exists a geometric constant* $C > 0$ *such that for any* Jac*-minimizing* Q*-valued map* $N = \sum_\ell [\![N^\ell]\!] \in \Gamma_Q^{1,2}(\mathcal{N}\Omega)$ *the inequality*

$$\int_\Omega \eta(x)^2 |\nabla^\perp N(x)|^2 \, d\mathcal{H}^m(x) \leqslant 4 \int_\Omega |D\eta(x)|^2 |N(x)|^2 \, d\mathcal{H}^m(x) + C \int_\Omega \eta(x)^2 |N(x)|^2 \, d\mathcal{H}^m(x) \tag{6.55}$$

holds for any choice of $\eta \in C_c^1(\Omega)$. *In particular, for every* $p \in \Omega$ *and for every* $r < \min\{\mathrm{inj}(\Sigma), \mathbf{dist}(p, \partial\Omega)\}$ *one has*

$$\int_{\mathbf{B}_{\frac{r}{2}}(p)} |\nabla^\perp N|^2 \, d\mathcal{H}^m \leqslant \frac{C}{r^2} \int_{\mathbf{B}_r(p)} |N|^2 \, d\mathcal{H}^m. \tag{6.56}$$

Proof. Fix N and η as in the statement, and apply the outer variation formula (6.30) with $\psi(x,u) := \eta(x)^2 u$. Since $D_x\psi(x,u) = 2\eta(x)u \otimes D\eta(x)$ and $D_u\psi(x,u) = \eta(x)^2 \mathrm{Id}$, for this choice of ψ the outer variation formula reads

$$\int_\Omega \eta^2 |\nabla^\perp N|^2 + 2\sum_{\ell=1}^{Q} \langle \eta \nabla^\perp N^\ell : N^\ell \otimes D\eta \rangle \, d\mathcal{H}^m = \int_\Omega \eta^2 \sum_{\ell=1}^{Q} \left(|A \cdot N^\ell|^2 + \mathcal{R}(N^\ell, N^\ell) \right) d\mathcal{H}^m. \tag{6.57}$$

Applying Young's inequality we immediately deduce that for any $\delta > 0$ one has

$$\int_\Omega \eta^2 |\nabla^\perp N|^2 \, d\mathcal{H}^m \leqslant \delta \int_\Omega \eta^2 |\nabla^\perp N|^2 \, d\mathcal{H}^m + \frac{1}{\delta} \int_\Omega |D\eta|^2 |N|^2 \, d\mathcal{H}^m + C \int_\Omega \eta^2 |N|^2 \, d\mathcal{H}^m, \tag{6.58}$$

for a constant $C = C(\mathbf{A}, \mathbf{R})$, where, we recall, $\mathbf{A} = \|A\|_{L^\infty}$ and $\mathbf{R} = \|R\|_{L^\infty}$ are defined in (4.49) and (4.51). Choose $\delta = \frac{1}{2}$ to obtain (6.55). In order to deduce (6.56), apply (6.55) with $\eta(x) := \phi\left(\frac{d(x)}{r}\right)$, where $d(x) := \mathbf{d}(x,p)$ and ϕ is a cut-off function $0 \leqslant \phi \leqslant 1$ such that $\phi(t) = 1$ for $0 \leqslant t \leqslant \frac{1}{2}$, $\phi(t) = 0$ for $t \geqslant 1$ and $|\phi'| \leqslant 2$. □

6.2.2 Almost monotonicity of the frequency function and its consequences

The next step towards the proof of Theorem 6.0.3 consists of a careful asymptotic analysis of the celebrated *frequency function*.

Definition 6.2.5 (Frequency function)**.** Fix any point $p \in \Omega$. For any radius $0 < r < \min\{\mathrm{inj}(\Sigma), \mathbf{dist}(p, \partial\Omega)\}$, define the *energy function*

$$\mathbf{D}_{N,p}(r) := \int_{\mathbf{B}_r(p)} |\nabla^\perp N|^2(x) \, d\mathcal{H}^m(x) \tag{6.59}$$

and the *height function*

$$\mathbf{H}_{N,p}(r) := \int_{\partial\mathbf{B}_r(p)} |N|^2(x) \, d\mathcal{H}^{m-1}(x). \tag{6.60}$$

The *frequency function* is then defined by

$$\mathbf{I}_{N,p}(r) := \frac{r\mathbf{D}_{N,p}(r)}{\mathbf{H}_{N,p}(r)} \tag{6.61}$$

for all r such that $\mathbf{H}_{N,p}(r) > 0$. When the Q-field N and the point p are fixed and there is no ambiguity, we will drop the subscripts and simply write $\mathbf{D}(r)$, $\mathbf{H}(r)$ and $\mathbf{I}(r)$.

Remark 6.2.6. Observe that $\mathbf{D}(r) = \mathrm{Dir}^{\mathcal{N}\Sigma}(N, \mathbf{B}_r(p))$; $\mathbf{D}$ is an absolutely continuous function with derivative

$$\mathbf{D}'(r) = \int_{\partial\mathbf{B}_r(p)} |\nabla^\perp N|^2 \, d\mathcal{H}^{m-1}$$

almost everywhere. As for $\mathbf{H}(r)$, note that $|N|$ is the composition of N with the Lipschitz function $\mathcal{G}(\cdot, Q[\![0]\!])$, thus it belongs to $W^{1,2}$. Hence, $|N|^2$ is a $W^{1,1}$ function, and also $\mathbf{H} \in W^{1,1}$.

Remark 6.2.7. It is an easy consequence of the Hölder regularity of N that the frequency function $\mathbf{I}(r)$ is well defined and bounded for suitably small radii at any point $p \in \Omega$ such that $N(p) \neq Q[\![0]\!]$. Indeed, if such assumption is satisfied then

$$\lim_{r\to 0^+} \frac{1}{\mathcal{H}^{m-1}(\partial \mathbf{B}_r(p))}\mathbf{H}(r) = |N|^2(p) = \mathcal{G}(N(p), Q[\![0]\!])^2 > 0,$$

which in turn implies that $\mathbf{H}(r) > 0$ for small values of r. Furthermore, from the proof of Theorem 6.0.1 (cf. in particular formula (6.26)) we can also infer that if r is sufficiently small then

$$\mathbf{D}(r) \leqslant C r^{m-2+2\alpha},$$

where α is the Hölder exponent of N. In particular, from this one immediately concludes that there exists the limit

$$\lim_{r\to 0} \mathbf{I}(r) = 0$$

at every point p such that $N(p) \neq Q[\![0]\!]$.
As we shall see, we will obtain as a byproduct of the improved regularity theory developed in this section that the frequency function is well defined and bounded also in a suitable neighborhood of $r = 0$ at every point p such that $N(p) = Q[\![0]\!]$, and that also at such points the limit $\lim_{r\to 0^+} \mathbf{I}(r)$ exists, but it is strictly positive.

The main analytic feature of the frequency function is the following *almost monotonicity* property.

Theorem 6.2.8 (Almost monotonicity of the frequency)**.** *There exist a geometric constant C_0 and a radius $0 < r_0 < \min\{\mathrm{inj}(\Sigma), \mathbf{dist}(p, \partial\Omega)\}$ such that for all $0 < s < t \leqslant r_0$ with $\mathbf{H}\big|_{[s,t]} > 0$ one has*

$$\mathbf{I}(s) \leqslant C_0\,(1 + \mathbf{I}(t))\,. \tag{6.62}$$

Propositions 6.2.9 and 6.2.10 below contain the most relevant consequences of Theorem 6.2.8. Both these results will be derived under the additional assumption that $p \in \Omega$ has been fixed in such a way that $N(p) = Q[\![0]\!]$. As already observed in Remark 6.2.7 above, these are exactly the points where we lack a precise description of the behavior of the frequency function. The arguments contained in the next sections will illustrate the reason why an analysis of the Jacobi multi-field N in a neighborhood of such a point is indeed crucial in order to obtain the proof of Theorem 6.0.3.

The first result we are interested in is the following dichotomy: if $N(p) = Q[\![0]\!]$, then either there exists a neighborhood of p where the map N is identically vanishing, and thus where the frequency function is not defined at all, or, conversely, there is a neighborhood of p where the frequency function is well defined everywhere and bounded.

Proposition 6.2.9. *Let $N \in \Gamma^{1,2}_Q(\mathcal{N}\Omega)$ be* Jac*-minimizing. Assume $p \in \Omega$ is such that $N(p) = Q[\![0]\!]$. Then, the following dichotomy holds:*

(i) *either $N \equiv Q[\![0]\!]$ in a neighborhood of p;*

(ii) *or there exists a radius $r_0 > 0$ such that*

$$\mathbf{H}(r) > 0 \text{ for all } r \in (0, r_0] \qquad \text{and} \qquad \limsup_{r\to 0} \mathbf{I}(r) < \infty.$$

As it is natural, the most interesting situation is when condition (ii) in the above Proposition 6.2.9 is observed. As a first remark, we observe that the fact that the frequency function is bounded in a neighborhood of a point p allows to improve the almost monotonicity property itself.

Proposition 6.2.10 (Improved almost monotonicity of the frequency). *Let $N \in \Gamma^{1,2}_Q(\mathcal{N}\Omega)$ be* Jac*-minimizing. Assume $p \in \Omega$ is such that $N(p) = Q[\![0]\!]$ but N does not vanish in a neighborhood of p. Then, there exist $r_0 > 0$ and a constant $\lambda = \lambda(\mathbf{I}(r_0)) > 0$ such that the function*

$$r \in (0, r_0] \mapsto e^{\lambda r}\mathbf{I}(r) \tag{6.63}$$

is monotone non-decreasing. The limit

$$\lim_{r\to 0} \mathbf{I}(r) =: I_0(p) \tag{6.64}$$

exists and is strictly positive.

The rest of the section will be devoted to the proofs of Theorem 6.2.8, Proposition 6.2.9 and Proposition 6.2.10.

6.2.3 First variation estimates and the proof of Theorem 6.2.8

The proof of Theorem 6.2.8 is a consequence of some estimates involving the functions $\mathbf{D}$ and $\mathbf{H}$ and their derivatives, which in turn can be obtained by testing the first variations formulae (6.30) and (6.32) with a suitable choice of the maps ψ and X. The derivation of these estimates is the content of Lemma 6.2.13 below. We need to define the following auxiliary functions.

Definition 6.2.11. We denote by $\frac{\partial}{\partial \hat{r}}$ the vector field which is tangent to geodesic arcs parametrized by arc length and emanating from p. We will set $\nabla_{\hat{r}} := \nabla_{\frac{\partial}{\partial \hat{r}}}$, the directional derivative along $\frac{\partial}{\partial \hat{r}}$, and we will let $\nabla^{\perp}_{\hat{r}}$ be its projection onto the normal bundle of Σ in $\mathcal{M}$. We set:

$$\mathbf{E}(r) = \mathbf{E}_{N,p}(r) := \int_{\partial \mathbf{B}_r(p)} \sum_{\ell=1}^{Q} \langle N^\ell(x), \nabla^{\perp}_{\hat{r}} N^\ell(x)\rangle \, d\mathcal{H}^{m-1}(x), \tag{6.65}$$

$$\mathbf{G}(r) = \mathbf{G}_{N,p}(r) := \int_{\partial \mathbf{B}_r(p)} |\nabla^{\perp}_{\hat{r}} N|^2(x) \, d\mathcal{H}^{m-1}(x), \tag{6.66}$$

and

$$\mathbf{F}(r) = \mathbf{F}_{N,p}(r) := \int_{\mathbf{B}_r(p)} |N|^2(x) \, d\mathcal{H}^{m}(x). \tag{6.67}$$

Remark 6.2.12. Note that $\mathbf{F}(r) = \|N\|^2_{L^2(\mathbf{B}_r(p))}$ is an absolutely continuous function, and for a.e. r

$$\mathbf{F}'(r) = \int_{\partial \mathbf{B}_r(p)} |N|^2 \, d\mathcal{H}^{m-1} = \mathbf{H}(r). \tag{6.68}$$

Lemma 6.2.13 (First variation estimates). *There exist a geometric constant $C_0 > 0$ and a radius $0 < r_0 < \min\{\mathrm{inj}(\Sigma), \mathbf{dist}(p, \partial\Omega)\}$ such that the following inequalities hold true for a.e. $0 < r \leqslant r_0$:*

$$|\mathbf{D}(r) - \mathbf{E}(r)| \leqslant C_0 \mathbf{F}(r), \tag{6.69}$$

$$|\mathbf{D}'(r) - 2\mathbf{G}(r) - \frac{m-2}{r}\mathbf{D}(r)| \leqslant C_0 r \mathbf{D}(r) + C_0 (\mathbf{D}(r)\mathbf{F}(r))^{1/2}, \tag{6.70}$$

$$|\mathbf{H}'(r) - \frac{m-1}{r}\mathbf{H}(r) - 2\mathbf{E}(r)| \leqslant C_0 r \mathbf{H}(r). \tag{6.71}$$

Furthermore, if $\mathbf{I}(r) \geqslant 1$ then

$$|\mathbf{D}(r) - \mathbf{E}(r)| \leqslant C_0 r^2 \mathbf{D}(r), \tag{6.72}$$

and

$$|\mathbf{D}'(r) - 2\mathbf{G}(r) - \frac{m-2}{r}\mathbf{D}(r)| \leqslant C_0 r \mathbf{D}(r). \tag{6.73}$$

Proof. Step 1: proof of (6.69). We test the outer variation formula (6.30) with the map ψ given by

$$\psi(x, u) := \phi\left(\frac{d(x)}{r}\right) u, \tag{6.74}$$

where $d(\cdot) := \mathbf{d}(\cdot, p)$, and $\phi = \phi(t) \in C^\infty([0, \infty))$ is a cut-off function such that:

$$0 \leqslant \phi \leqslant 1, \quad \phi \equiv 1 \text{ in a neighborhood of } t = 0, \quad \phi \equiv 0 \text{ for } t \geqslant 1. \tag{6.75}$$

Observe first that this choice of ψ induces an admissible family of outer variations: indeed, one clearly sees that $\mathrm{spt}(\psi) \subset \mathbf{B}_r(p)$, the geodesic ball centered at p and of radius r, which is compactly supported in Ω, and that the orthogonality conditions and the assumptions in (6.29) are satisfied. We compute:

$$D_x \psi(x, u) = r^{-1} \phi'\left(\frac{d(x)}{r}\right) u \otimes \nabla d,$$

which yields

$$\langle \nabla^\perp N^\ell(x) : \nabla^\perp \psi(x, N^\ell(x)) \rangle = r^{-1} \phi'\left(\frac{d(x)}{r}\right) \langle \nabla^\perp_{\hat{r}} N^\ell(x), N^\ell(x) \rangle. \tag{6.76}$$

On the other hand, $D_u \psi(x, u) = \phi\left(\frac{d(x)}{r}\right) \mathrm{Id}$, whence

$$\langle \nabla^\perp N^\ell(x) : D_u \psi(x, N^\ell(x)) \cdot DN^\ell(x) \rangle = \phi\left(\frac{d(x)}{r}\right) |\nabla^\perp N^\ell|^2(x). \tag{6.77}$$

Analogously, we can compute explicitly the right-hand side of (6.30) corresponding to our choice of ψ and get:

$$\mathcal{E}_{\mathrm{OV}}(\psi) = \int_\Sigma \phi\left(\frac{d(x)}{r}\right) \sum_{\ell=1}^{Q} \left(|A \cdot N^\ell|^2(x) + \mathcal{R}(N^\ell(x), N^\ell(x))\right) d\mathcal{H}^m(x). \tag{6.78}$$

By a standard approximation procedure, the details of which are left to the reader, it is easy to see that we can test with

$$\phi(t) = \phi_h(t) = \begin{cases} 1 & \text{for } 0 \leqslant t \leqslant 1 - \frac{1}{h} \\ h(1-t) & \text{for } 1 - \frac{1}{h} \leqslant t \leqslant 1 \\ 0 & \text{for } t \geqslant 1. \end{cases} \tag{6.79}$$

Inserting into (6.76), (6.77) and (6.78), the outer variation formula (6.30) becomes

$$\begin{aligned} -\frac{h}{r} \int_{\mathbf{B}_r(p)\setminus\mathbf{B}_{r-\frac{r}{h}}(p)} \sum_{\ell=1}^{Q} \langle \nabla^{\perp}_{\hat{r}} N^{\ell}(x), N^{\ell}(x)\rangle \, d\mathcal{H}^m(x) + \int_{\Sigma} \phi_h\left(\frac{d(x)}{r}\right) |\nabla^{\perp} N(x)|^2 \, d\mathcal{H}^m(x) \\ = \int_{\Sigma} \phi_h\left(\frac{d(x)}{r}\right) \sum_{\ell=1}^{Q} \left(|A\cdot N^{\ell}|^2(x) + \mathcal{R}(N^{\ell}(x), N^{\ell}(x))\right) \, d\mathcal{H}^m(x). \end{aligned} \tag{6.80}$$

Now, let $h \uparrow \infty$. The left-hand side of (6.80) converges to $\mathbf{D}(r) - \mathbf{E}(r)$, whereas the right-hand side converges to

$$\int_{\mathbf{B}_r(p)} \sum_{\ell=1}^{Q} \left(|A\cdot N^{\ell}|^2 + \mathcal{R}(N^{\ell}, N^{\ell})\right) \, d\mathcal{H}^m.$$

In particular, the inequality (6.69) readily follows with a constant C_0 depending on $\mathbf{A} = \|A\|_{L^\infty}$ and $\mathbf{R} = \|R\|_{L^\infty}$.

Step 2: proof of (6.70). We test now the inner variation formula (6.32) with the vector field X defined by

$$\begin{aligned} X(x) &:= \frac{d(x)}{r}\phi\left(\frac{d(x)}{r}\right)\frac{\partial}{\partial \hat{r}} \\ &= \phi\left(\frac{d(x)}{r}\right)\frac{1}{2r}\nabla(d(x)^2), \end{aligned} \tag{6.81}$$

with ϕ as in (6.75).

Standard computations lead to

$$\begin{aligned} \nabla^{\Sigma} X(x) &= \phi'\left(\frac{d(x)}{r}\right)\frac{d(x)}{r^2}\frac{\partial}{\partial\hat{r}}\otimes\frac{\partial}{\partial\hat{r}} + \phi\left(\frac{d(x)}{r}\right)\frac{1}{2r}\mathrm{Hess}^{\Sigma}(d(x)^2) \\ &= \phi'\left(\frac{d(x)}{r}\right)\frac{d(x)}{r^2}\frac{\partial}{\partial\hat{r}}\otimes\frac{\partial}{\partial\hat{r}} + \phi\left(\frac{d(x)}{r}\right)\left(\frac{\mathrm{Id}}{r} + O(r)\right) \end{aligned}$$

for $r \to 0$, and consequently

$$\begin{aligned} \mathrm{div}_{\Sigma} X(x) &= \phi'\left(\frac{d(x)}{r}\right)\frac{d(x)}{r^2} + \phi\left(\frac{d(x)}{r}\right)\frac{1}{2r}\Delta_{\Sigma}(d(x)^2) \\ &= \phi'\left(\frac{d(x)}{r}\right)\frac{d(x)}{r^2} + \phi\left(\frac{d(x)}{r}\right)\left(\frac{m}{r} + O(r)\right). \end{aligned}$$

Choosing again tests of the form $\phi = \phi_h$ as in (6.79), plugging into (6.32) and taking the limit $h \uparrow \infty$, we see that the left-hand side of the inner variation formula reads

$$\mathbf{D}'(r) - 2\mathbf{G}(r) - \frac{m-2}{r}\mathbf{D}(r) + O(r)\mathbf{D}(r) \tag{6.82}$$

for $r \to 0$.

We proceed with the analysis of the error term $\mathcal{E}_{IV}(X)$. Straightforward computations imply the following estimates:

$$|\mathcal{E}_{IV}^{(1)}| \leqslant C_1 \int_{\mathbf{B}_r(p)} |N(x)||\nabla^{\perp} N(x)|\, d\mathcal{H}^m(x),$$

$$|\mathcal{E}_{IV}^{(2)}| + |\mathcal{E}_{IV}^{(3)}| \leqslant C_{2,3} \int_{\mathbf{B}_r(p)} |N(x)||\nabla^{\perp}_{\hat{r}} N(x)|\, d\mathcal{H}^m(x),$$

where C_1 is a geometric constant depending on $\overline{\mathbf{A}} = \|\overline{A}\|_{L^\infty}$, and $C_{2,3}$ depends on $\mathbf{A}$ and $\mathbf{R}$. Applying the Cauchy-Schwarz inequality we conclude

$$|\mathcal{E}_{IV}(X)| \leqslant C_0 \left(\mathbf{D}(r)\mathbf{F}(r)\right)^{1/2}. \tag{6.83}$$

Combining (6.82) and (6.83), we deduce the inequality (6.70) whenever r is small enough.

Step 3: proof of (6.71). Let $\exp_p : \mathcal{V} \subset T_p\Sigma \to \Sigma$ be the exponential map with pole p. Since $B_r(0) \Subset \mathcal{V}$ for every $r < \mathrm{inj}(\Sigma)$, we can use the change of coordinates $x = \exp_p(y)$ to write:

$$\begin{aligned}\mathbf{H}(r) &= \int_{\partial B_r} |N|^2(\exp_p(y))\, \mathbf{J}\exp_p(y)\, d\mathcal{H}^{m-1}(y)\\ &= r^{m-1}\int_{\partial B_1} |N|^2(\exp_p(rz))\, \mathbf{J}\exp_p(rz)\, d\mathcal{H}^{m-1}(z).\end{aligned}$$

Thus, we differentiate under the integral sign and compute

$$\begin{aligned}\mathbf{H}'(r) &= (m-1)r^{m-2}\int_{\partial B_1} |N|^2(\exp_p(rz))\, \mathbf{J}\exp_p(rz)\, d\mathcal{H}^{m-1}(z)\\ &+ 2r^{m-1}\int_{\partial B_1} \sum_{\ell=1}^{Q} \langle N^\ell(\exp_p(rz)), \nabla^{\perp}_{\hat{r}} N^\ell(\exp_p(rz))\rangle\, \mathbf{J}\exp_p(rz)\, d\mathcal{H}^{m-1}(z)\\ &+ r^{m-1}\int_{\partial B_1} |N|^2(\exp_p(rz))\frac{d}{dr}\left(\mathbf{J}\exp_p(rz)\right)\, d\mathcal{H}^{m-1}(z).\end{aligned}$$

Since $\frac{d}{dr}\left(\mathbf{J}\exp_p(rz)\right) = O(r)$ for $r \to 0$, we are able to conclude

$$\mathbf{H}'(r) = \frac{m-1}{r}\mathbf{H}(r) + 2\mathbf{E}(r) + O(r)\mathbf{H}(r), \tag{6.84}$$

from which (6.71) readily follows.

Step 4: proof of (6.72) *and* (6.73). It suffices to exploit the inequality

$$\mathbf{F}(r) \leqslant C_0 r\mathbf{H}(r) + C_0 r^2\mathbf{D}(r), \tag{6.85}$$

which can be easily deduced from the Poincaré inequality (note also that the same inequality has been already proved in the Euclidean setting earlier on, cf. Corollary 6.13). In the regime $\mathbf{I}(r) \geqslant 1$, that is $\mathbf{H}(r) \leqslant r\mathbf{D}(r)$, (6.85) simply reads

$$\mathbf{F}(r) \leqslant C_0 r^2\mathbf{D}(r). \tag{6.86}$$

Then, (6.72) and (6.73) are an immediate consequence of (6.69) and (6.70) respectively. □

We can now proceed with the proof of the almost monotonicity property of the frequency.

Proof of Theorem 6.2.8. Set $\mathbf{\Omega}(r) := \log(\max\{\mathbf{I}(r), 1\})$. In order to prove the theorem, it suffices to show that

$$\mathbf{\Omega}(s) \leqslant C + \mathbf{\Omega}(t) \tag{6.87}$$

for some positive geometric constant C. If $\mathbf{\Omega}(s) = 0$ there is nothing to prove. Thus, we assume that $\mathbf{\Omega}(s) > 0$. Define

$$\tau := \sup\{r \in (s,t] \,:\, \mathbf{\Omega}(r) > 0 \text{ on } (s,r)\}.$$

If $\tau < t$, then by continuity it must be $\mathbf{\Omega}(\tau) = 0$: hence, in this case we would have $\mathbf{\Omega}(\tau) = 0 \leqslant \mathbf{\Omega}(t)$, and therefore proving that $\mathbf{\Omega}(s) \leqslant C + \mathbf{\Omega}(\tau)$ would imply (6.87). Thus, we can assume without loss of generality that $\mathbf{\Omega}(r) > 0$ in (s,t): in other words, $\mathbf{I}(r) > 1$, and $\mathbf{\Omega}(r) = \log(\mathbf{I}(r))$. Then, as a consequence of (6.72), if r_0 is taken small enough one has

$$\frac{\mathbf{D}(r)}{2} \leqslant \mathbf{E}(r) \leqslant 2\mathbf{D}(r), \tag{6.88}$$

that is the quantity $\mathbf{E}(r)$ is positive and comparable to $\mathbf{D}(r)$ at small scales.

Guided by this principle, we compute:

$$\begin{aligned} -\frac{d}{dr}(\log \mathbf{I}(r)) &= \frac{\mathbf{H}'(r)}{\mathbf{H}(r)} - \frac{\mathbf{D}'(r)}{\mathbf{D}(r)} - \frac{1}{r} \\ &= \frac{\mathbf{H}'(r)}{\mathbf{H}(r)} - \frac{\mathbf{D}'(r)}{\mathbf{E}(r)} - \mathbf{D}'(r)\mathbf{Z}(r) - \frac{1}{r}, \end{aligned} \tag{6.89}$$

where $\mathbf{Z}(r) := \dfrac{1}{\mathbf{D}(r)} - \dfrac{1}{\mathbf{E}(r)}$ satisfies

$$|\mathbf{Z}(r)| = \frac{|\mathbf{D}(r) - \mathbf{E}(r)|}{\mathbf{D}(r)\mathbf{E}(r)} \overset{(6.88)}{\leqslant} 2\frac{|\mathbf{D}(r) - \mathbf{E}(r)|}{\mathbf{D}(r)^2} \overset{(6.69)}{\leqslant} C_0\frac{\mathbf{F}(r)}{\mathbf{D}(r)^2}. \tag{6.90}$$

Now, by (6.71) one has that

$$\frac{\mathbf{H}'(r)}{\mathbf{H}(r)} \leqslant Cr + \frac{m-1}{r} + 2\frac{\mathbf{E}(r)}{\mathbf{H}(r)}, \tag{6.91}$$

whereas the inner variation formula (6.73) yields

$$\begin{aligned} -\frac{\mathbf{D}'(r)}{\mathbf{E}(r)} &\leqslant Cr\frac{\mathbf{D}(r)}{\mathbf{E}(r)} - 2\frac{\mathbf{G}(r)}{\mathbf{E}(r)} - \frac{m-2}{r}\frac{\mathbf{D}(r)}{\mathbf{E}(r)} \\ &\overset{(6.88)}{\leqslant} Cr - 2\frac{\mathbf{G}(r)}{\mathbf{E}(r)} - \frac{m-2}{r}\left(1 - \mathbf{D}(r)\mathbf{Z}(r)\right) \\ &\overset{(6.90)}{\leqslant} Cr - 2\frac{\mathbf{G}(r)}{\mathbf{E}(r)} - \frac{m-2}{r} + Cr^{-1}\frac{\mathbf{F}(r)}{\mathbf{D}(r)} \\ &\leqslant Cr - 2\frac{\mathbf{G}(r)}{\mathbf{E}(r)} - \frac{m-2}{r} \end{aligned} \tag{6.92}$$

because of (6.86).

Plugging (6.91) and (6.92) into (6.89), and using the estimate on the error term $\mathbf{Z}(r)$ given by (6.90), we obtain the following:

$$-\frac{d}{dr}(\log \mathbf{I}(r)) \leqslant Cr + 2\left(\frac{\mathbf{E}(r)}{\mathbf{H}(r)} - \frac{\mathbf{G}(r)}{\mathbf{E}(r)}\right) + C\frac{\mathbf{D}'(r)}{\mathbf{D}(r)^2}\mathbf{F}(r). \tag{6.93}$$

Now, by the Cauchy-Schwarz inequality one has

$$\mathbf{E}(r)^2 \leqslant \mathbf{G}(r)\mathbf{H}(r),$$

whence the term $\dfrac{\mathbf{E}(r)}{\mathbf{H}(r)} - \dfrac{\mathbf{G}(r)}{\mathbf{E}(r)}$ is non-positive and (6.93) yields

$$-\frac{d}{dr}(\log \mathbf{I}(r)) \leqslant Cr + C\frac{\mathbf{D}'(r)}{\mathbf{D}(r)^2}\mathbf{F}(r). \tag{6.94}$$

Integrating for $r \in (s,t)$, we obtain

$$\mathbf{\Omega}(s) - \mathbf{\Omega}(t) \leqslant C + C\left(\frac{\mathbf{F}(s)}{\mathbf{D}(s)} - \frac{\mathbf{F}(t)}{\mathbf{D}(t)}\right) + C\int_s^t \frac{\mathbf{F}'(r)}{\mathbf{D}(r)}\,dr \leqslant C, \tag{6.95}$$

where the last inequality follows from the above observation that, in the regime $\mathbf{I} \geqslant 1$, the inequalities

$$\mathbf{F}(r) \leqslant C_0 r^2 \mathbf{D}(r), \qquad \mathbf{F}'(r) = \mathbf{H}(r) \leqslant r\mathbf{D}(r)$$

hold almost everywhere. This completes the proof. □

6.2.4 Proof of Propositions 6.2.9 and 6.2.10

We will need the following version of the Poincaré inequality.

Lemma 6.2.14. *There exist a radius $0 < r_0 = r_0(m, Q) < \mathrm{inj}(\Sigma)$ and a geometric constant $C > 0$ with the following property. Let $N \in \Gamma_Q^{1,2}(\mathcal{N}\Omega)$ be a multiple valued section of $\mathcal{N}\Sigma$ Jac-minimizing in Ω. Assume $p \in \Omega$ is such that $N(p) = Q[\![0]\!]$. Then, the inequality*

$$\|N\|^2_{L^2(\mathbf{B}_r(p))} \leqslant Cr^2 \mathrm{Dir}^{\mathcal{N}\Sigma}(N, \mathbf{B}_r(p)) \tag{6.96}$$

holds true for every $0 < r \leqslant \min\{r_0, \mathbf{dist}(p, \partial\Omega)\}$.

Proof. Let $r_0 = r_0(m, Q)$ be the radius given by Theorem 6.0.1, and let $r \leqslant \min\{r_0, \mathbf{dist}(p, \partial\Omega)\}$ be arbitrary. Let $\rho \in \left(0, \frac{r}{2}\right]$ be a radius to be chosen later and split $\|N\|^2_{L^2(\mathbf{B}_r(p))}$ into the sum

$$\int_{\mathbf{B}_r(p)} |N|^2 \, d\mathcal{H}^m = \int_{\mathbf{B}_\rho(p)} |N|^2 \, d\mathcal{H}^m + \int_{\mathbf{B}_r(p)\setminus\mathbf{B}_\rho(p)} |N|^2 \, d\mathcal{H}^m. \tag{6.97}$$

In order to estimate the first term in the sum, we recall that $|N|^2(x) = \mathcal{G}(N(x), Q[\![0]\!])^2 = \mathcal{G}(N(x), N(p))^2$ and exploit the α-Hölder continuity of N to conclude

$$\begin{aligned}
\int_{\mathbf{B}_\rho(p)} |N|^2 \, d\mathcal{H}^m &\leqslant \rho^{2\alpha} \, [N]^2_{C^{0,\alpha}(\overline{\mathbf{B}}_\rho(p))} \, \mathcal{H}^m(\mathbf{B}_\rho(p)) \\
&\overset{(6.1)}{\leqslant} C\rho^2 \left(\mathrm{Dir}(N, \mathbf{B}_{2\rho}(p)) + \Lambda \int_{\mathbf{B}_{2\rho}(p)} |N|^2 \, d\mathcal{H}^m \right) \\
&\leqslant C\rho^2 \mathrm{Dir}(N, \mathbf{B}_r(p)) + C\Lambda\rho^2 \int_{\mathbf{B}_r(p)} |N|^2 \, d\mathcal{H}^m \\
&\leqslant \underbrace{Cr^2 \mathrm{Dir}^{\mathcal{N}\Sigma}(N, \mathbf{B}_r(p))}_{=:I_1} + \underbrace{C(\Lambda + C_0)\rho^2 \int_{\mathbf{B}_r(p)} |N|^2 \, d\mathcal{H}^m}_{=:I_2},
\end{aligned} \tag{6.98}$$

where C_0 depends on $\mathbf{A}$ and $\overline{\mathbf{A}}$.

As for the second addendum in (6.97), we integrate in normal polar coordinates with pole p to write

$$\int_{\mathbf{B}_r(p)\setminus\mathbf{B}_\rho(p)} |N|^2 \, d\mathcal{H}^m = \int_\rho^r \left(\int_{\partial\mathbf{B}_\tau(p)} |N|^2 \, d\mathcal{H}^{m-1} \right) d\tau. \tag{6.99}$$

Now, fix any $\tau \in (\rho, r)$, and for every $x \in \partial\mathbf{B}_\tau(p)$ let $\gamma_x = \gamma_x(s)$, $s \in [0, \tau]$, be the unique geodesic parametrized by arclength joining p to x. Also denote by $\overline{x}$ the point where γ_x intersects $\partial\mathbf{B}_\rho(p)$. Then, the fundamental theorem of calculus immediately yields

$$|N|^2(x) \leqslant |N|^2(\overline{x}) + 2\int_\rho^\tau (|N||\nabla^\perp N|)(\gamma_x(s)) \, ds. \tag{6.100}$$

Integrate the above inequality in $x \in \partial\mathbf{B}_\tau(p)$ to get

$$\int_{\partial\mathbf{B}_\tau(p)} |N|^2 \, d\mathcal{H}^{m-1} \leqslant C \left(\frac{\tau}{\rho}\right)^{m-1} \left(\int_{\partial\mathbf{B}_\rho(p)} |N|^2 \, d\mathcal{H}^{m-1} + 2\int_{\mathbf{B}_\tau(p)\setminus\mathbf{B}_\rho(p)} |N||\nabla^\perp N| \, d\mathcal{H}^m \right). \tag{6.101}$$

Using once again the Hölder estimate (6.1) and recalling that $\rho \leqslant \frac{r}{2}$, we are able to control

$$\begin{aligned}
\left(\frac{\tau}{\rho}\right)^{m-1} \int_{\partial\mathbf{B}_\rho(p)} |N|^2 \, d\mathcal{H}^{m-1} &\leqslant \left(\frac{\tau}{\rho}\right)^{m-1} \rho^{2\alpha} \, [N]^2_{C^{0,\alpha}(\overline{\mathbf{B}}_{\frac{r}{2}}(p))} \, \mathcal{H}^{m-1}(\partial\mathbf{B}_\rho(p)) \\
&\leqslant C\tau^{m-1}\rho^{2\alpha} r^{2-m-2\alpha} \left(\mathrm{Dir}(N, \mathbf{B}_r(p)) + \Lambda \int_{\mathbf{B}_r(p)} |N|^2 \, d\mathcal{H}^m \right).
\end{aligned} \tag{6.102}$$

We can now integrate in $\tau \in (\rho, r)$, so that using the estimates in (6.101) and (6.102) we can easily deduce from (6.99) the following inequality:

$$\begin{aligned}\int_{\mathbf{B}_r(p)\setminus\mathbf{B}_\rho(p)} |N|^2 \, d\mathcal{H}^m \leqslant & \underbrace{C\left(\frac{\rho}{r}\right)^{2\alpha} r^2 \mathrm{Dir}^{\mathcal{N}\Sigma}(N, \mathbf{B}_r(p))}_{=:J_1} \\ & \underbrace{+ C(\Lambda + C_0)\rho^{2\alpha} \int_{\mathbf{B}_r(p)} |N|^2 \, d\mathcal{H}^m}_{=:J_2} \\ & \underbrace{+ C\left(\frac{r}{\rho}\right)^{m-1} r \int_{\mathbf{B}_r(p)} |N||\nabla^\perp N| \, d\mathcal{H}^m}_{=:J_3},\end{aligned} \tag{6.103}$$

where C and C_0 are geometric constants. Now we can sum up the contributions coming from the ball $\mathbf{B}_\rho(p)$ and from the annulus $\mathbf{B}_r(p) \setminus \mathbf{B}_\rho(p)$ and choose $\rho = \rho(\Lambda, C, C_0)$ so small that the terms I_2 and J_2 are absorbed in the left-hand side of the equation, thus ultimately providing

$$\int_{\mathbf{B}_r(p)} |N|^2 \, d\mathcal{H}^m \leqslant Cr^2 \mathrm{Dir}^{\mathcal{N}\Sigma}(N, \mathbf{B}_r(p)) + Cr \int_{\mathbf{B}_r(p)} |N||\nabla^\perp N| \, d\mathcal{H}^m. \tag{6.104}$$

Finally, use Young's inequality: for any choice of the parameter $\eta > 0$, (6.104) implies that

$$\int_{\mathbf{B}_r(p)} |N|^2 \, d\mathcal{H}^m \leqslant Cr^2 \mathrm{Dir}^{\mathcal{N}\Sigma}(N, \mathbf{B}_r(p)) + Cr\left(\eta \int_{\mathbf{B}_r(p)} |N|^2 \, d\mathcal{H}^m + \frac{1}{\eta}\mathrm{Dir}^{\mathcal{N}\Sigma}(N, \mathbf{B}_r(p))\right). \tag{6.105}$$

The conclusion immediately follows by choosing η such that $Cr\eta = \frac{1}{2}$. □

Proof of Proposition 6.2.9. First observe that if N does not vanish identically in a neighborhood of p, then there exists $r_0 > 0$ such that $\mathbf{H}(r_0) > 0$. Clearly, without loss of generality we can suppose that (6.96) holds for every $0 < r \leqslant r_0$, and also that (6.62) holds in any interval $[s, t] \subset (0, r_0]$ such that $\mathbf{H}|_{[s,t]} > 0$. We claim that in fact $\mathbf{H}(r) > 0$ for all $0 < r \leqslant r_0$. Indeed, if this is not true, let $\rho > 0$ be given by $\rho := \sup\{r \in (0, r_0] : \mathbf{H}(r) = 0\}$. By definition $\mathbf{H}(r) > 0$ for $\rho < r \leqslant r_0$, whence for such r's we can take advantage of Theorem 6.2.8 and write

$$\mathbf{I}(r) \leqslant C_0(1 + \mathbf{I}(r_0)).$$

By letting $r \downarrow \rho$ we conclude

$$\rho \mathbf{D}(\rho) \leqslant C_0(1 + \mathbf{I}(r_0))\mathbf{H}(\rho) = 0,$$

which in turn produces $\mathrm{Dir}^{\mathcal{N}\Sigma}(N, \mathbf{B}_\rho(p)) = 0$. Then, by Lemma 6.2.14, N vanishes identically in $\mathbf{B}_\rho(p)$, contradiction.

It is now a simple consequence of Theorem 6.2.8 that

$$\limsup_{r\to 0} \mathbf{I}(r) \leqslant C_0(1 + \mathbf{I}(r_0)),$$

which completes the proof. □

Proof of Proposition 6.2.10. Under the assumptions in the statement, case (ii) in Proposition 6.2.9 must hold, and thus the frequency function is well defined and bounded in an interval $(0, r_0]$. Moreover, the Poincaré inequality (6.96) implies that, modulo possibly taking a smaller value of r_0, the first variation estimates of Lemma 6.2.13 can be again re-written as in (6.72) and (6.73), and that (6.88) holds. Thus, we can compute:

$$\begin{aligned}\mathbf{I}'(r) &= \frac{\mathbf{D}(r)}{\mathbf{H}(r)} + \frac{r\mathbf{D}'(r)}{\mathbf{H}(r)} - \frac{r\mathbf{D}(r)\mathbf{H}'(r)}{\mathbf{H}(r)^2}\\ &= \frac{\mathbf{D}(r)}{\mathbf{H}(r)} + \frac{r}{\mathbf{H}(r)}\left(2\mathbf{G}(r) + \frac{m-2}{r}\mathbf{D}(r) + \mathcal{E}_1(r)\right) - \frac{r\mathbf{D}(r)}{\mathbf{H}(r)^2}\left(\frac{m-1}{r}\mathbf{H}(r) + 2\mathbf{E}(r) + \mathcal{E}_2(r)\right)\\ &= \frac{2r}{\mathbf{H}(r)^2}\left(\mathbf{G}(r)\mathbf{H}(r) - \mathbf{E}(r)^2\right) + \frac{r}{\mathbf{H}(r)}\,\mathcal{E}_1(r) - \frac{r\mathbf{D}(r)}{\mathbf{H}(r)^2}\,\mathcal{E}_2(r) + \mathcal{E}_3(r),\end{aligned} \tag{6.106}$$

where

$$|\mathcal{E}_1(r)| \overset{(6.73)}{\leqslant} C_0 r\mathbf{D}(r), \tag{6.107}$$

$$|\mathcal{E}_2(r)| \overset{(6.71)}{\leqslant} C_0 r\mathbf{H}(r), \tag{6.108}$$

and

$$|\mathcal{E}_3(r)| = \frac{2r\mathbf{E}(r)}{\mathbf{H}(r)^2}|\mathbf{E}(r) - \mathbf{D}(r)| \overset{(6.72),(6.88)}{\leqslant} C_0\frac{r^3\mathbf{D}(r)^2}{\mathbf{H}(r)^2}, \tag{6.109}$$

if r_0 is chosen small enough. Since $\mathbf{G}(r)\mathbf{H}(r) - \mathbf{E}(r)^2 \geqslant 0$ by the Cauchy-Schwartz inequality, the above arguments show the existence of a radius $r_0 > 0$ and a geometric constant $C_0 > 0$ such that

$$\mathbf{I}'(r) \geqslant -C_0 r\mathbf{I}(r) - C_0 r\mathbf{I}(r)^2 \tag{6.110}$$

for all $r \in (0, r_0]$. On the other hand, for such r's one has $\mathbf{I}(r) \leqslant C_0(1 + \mathbf{I}(r_0))$ by Theorem 6.2.8. Thus, this allows to conclude that

$$\mathbf{I}'(r) \geqslant -\lambda\mathbf{I}(r), \tag{6.111}$$

for some positive λ depending only on r_0 and $\mathbf{I}(r_0)$. The monotonicity of the function $r \mapsto e^{\lambda r}\mathbf{I}(r)$ is now a simple consequence of (6.111).

Next, we conclude the proof showing that the limit

$$I_0 := \lim_{r\to 0} e^{\lambda r}\mathbf{I}(r) = \lim_{r\to 0}\mathbf{I}(r) \tag{6.112}$$

is positive. To see this, we show that the Poincaré inequality (6.96) allows to bound the frequency function from below with a positive constant. Indeed, arguing as in the proof of

Lemma 6.2.14 (cf. in particular the equations (6.101) and (6.102)), it is easily seen that one can estimate

$$\begin{aligned}\mathbf{H}(r) &= \int_{\partial \mathbf{B}_r(p)} |N|^2 \, d\mathcal{H}^{m-1} \leqslant C \left(\int_{\partial \mathbf{B}_{\frac{r}{2}}(p)} |N|^2 \, d\mathcal{H}^{m-1} + \int_{\mathbf{B}_r(p) \setminus \mathbf{B}_{\frac{r}{2}}(p)} |N| |\nabla^\perp N| \, d\mathcal{H}^m \right) \\ &\leqslant Cr \left(\mathrm{Dir}(N, \mathbf{B}_r(p)) + \Lambda \int_{\mathbf{B}_r(p)} |N|^2 \, d\mathcal{H}^m \right) + C \int_{\mathbf{B}_r(p)} |N| |\nabla^\perp N| \, d\mathcal{H}^m \\ &\leqslant Cr\mathbf{D}(r) + Cr\mathbf{F}(r) + C \int_{\mathbf{B}_r(p)} |N| |\nabla^\perp N| \, d\mathcal{H}^m. \end{aligned} \tag{6.113}$$

In turn, applying Young's inequality to the last addendum in the right-hand side of (6.113) yields

$$\int_{\mathbf{B}_r(p)} |N| |\nabla^\perp N| \, d\mathcal{H}^m \leqslant \frac{r}{2} \mathbf{D}(r) + \frac{1}{2r} \mathbf{F}(r). \tag{6.114}$$

Plugging (6.114) in (6.113) and using the Poincaré inequality (6.96) finally gives

$$\mathbf{H}(r) \leqslant C(1 + r^2) r \mathbf{D}(r) \leqslant C(1 + r_0^2) r \mathbf{D}(r), \tag{6.115}$$

thus completing the proof. □

6.3 REVERSE POINCARÉ AND ANALYSIS OF BLOW-UPS FOR THE TOP STRATUM

The final goal of this section is to perform the key step in the proof of Theorem 6.0.3, namely the blow-up procedure, see Theorem 6.3.8 below. In doing this, we will clarify the importance of the results obtained in the previous paragraph.

6.3.1 Reverse Poincaré inequalities

The proof of the blow-up theorem will heavily rely on an important technical tool, a reverse Poincaré inequality for Jac-minimizers. In Proposition 6.2.4, we have already shown that Jac-minimizers enjoy a Caccioppoli type inequality: the L^2-norm of a Jacobi Q-field N in a ball $\mathbf{B}_r(p)$ controls the Dirichlet energy in the ball with half the radius. As an immediate consequence of the boundedness of the frequency function, one can actually show that the Dirichlet energy in $\mathbf{B}_{\frac{r}{2}}(p)$ can be controlled with the L^2-norm of N in the annulus $\mathbf{B}_r(p) \setminus \mathbf{B}_{\frac{r}{2}}(p)$, provided that we allow the constant to depend on the value of the frequency at a given scale.

Proposition 6.3.1. *Let $N \in \Gamma^{1,2}_Q(\mathcal{N}\Omega)$ be* Jac*-minimizing. Then, there exists $r_0 > 0$ such that for any $r \in (0, r_0]$ one has*

$$\mathrm{Dir}^{\mathcal{N}\Sigma}(N, \mathbf{B}_{\frac{r}{2}}(p)) \leqslant \frac{C}{r^2} \int_{\mathbf{B}_r(p) \setminus \mathbf{B}_{\frac{r}{2}}(p)} |N|^2 \, d\mathcal{H}^m \tag{6.116}$$

for some positive $C = C(\mathbf{I}(r_0))$.

Proof. If N is vanishing identically in a neighborhood of p there is nothing to prove. Therefore, we can assume that either $N(p) \neq Q[\![0]\!]$ or $N(p) = Q[\![0]\!]$ but N does not vanish identically in any neighborhood of p. In any of the two cases, either by the arguments contained in Remark 6.2.7 or by Proposition 6.2.9, there exists a positive radius r_0 such that the frequency function is well defined and bounded for all $r \in (0, r_0]$. Thus, there exists a positive constant $C = C(\mathbf{I}(r_0))$ such that, for fixed $r \leqslant r_0$, $\tau \mathbf{D}(\tau) \leqslant C\mathbf{H}(\tau)$ for τ in the interval $\left[\frac{r}{2}, r\right]$. Integrate with respect to τ to get (6.116):

$$\begin{aligned}\frac{3}{8}r^2 \mathrm{Dir}^{\mathcal{N}\Sigma}(N, \mathbf{B}_{\frac{r}{2}}(p)) = \frac{3}{8}r^2 \mathbf{D}\left(\frac{r}{2}\right) &\leqslant \int_{\frac{r}{2}}^{r} \tau \mathbf{D}(\tau)\, d\tau \\ &\leqslant C \int_{\frac{r}{2}}^{r} \mathbf{H}(\tau)\, d\tau = C \int_{\mathbf{B}_r(p)\setminus \mathbf{B}_{\frac{r}{2}}(p)} |N|^2\, d\mathcal{H}^m.\end{aligned}$$

□

The Caccioppoli inequality can in fact be improved further under the assumption that $N(p) = Q[\![0]\!]$: indeed, at small scales the inequality (6.56) holds without having to increase the support of the ball on the right-hand side. Again, for this to be true we need to allow the constant to depend on the value of the frequency at scale r_0.

Proposition 6.3.2 (Reverse Poincaré Inequality)**.** *Let* $N \in \Gamma^{1,2}_Q(\mathcal{N}\Omega)$ *be* Jac*-minimizing. Assume* $N(p) = Q[\![0]\!]$. *Then, there exists* $r_0 > 0$ *such that for any* $r \in (0, r_0]$ *the following inequality*

$$\mathrm{Dir}^{\mathcal{N}\Sigma}(N, \mathbf{B}_r(p)) \leqslant \frac{C}{r^2} \int_{\mathbf{B}_r(p)} |N|^2\, d\mathcal{H}^m \tag{6.117}$$

holds for some positive $C = C(\mathbf{I}(r_0))$.

Proof. Once again, we observe that (6.117) is trivial when $N \equiv Q[\![0]\!]$ in a neighborhood of p. We assume then that case (ii) in Proposition 6.2.9 holds, and we let r_0 be the radius given in there. Since the frequency function is well defined and bounded in $(0, r_0]$, there exists $C = C(\mathbf{I}(r_0)) > 0$ such that

$$\int_{\mathbf{B}_r(p)} |\nabla^\perp N|^2\, d\mathcal{H}^m \leqslant \frac{C}{r} \int_{\partial \mathbf{B}_r(p)} |N|^2\, d\mathcal{H}^{m-1}, \tag{6.118}$$

for all r's in the above interval. Arguing once again as in the proof of Lemma 6.2.14, we have that for every $\rho \in \left(0, \frac{r}{2}\right]$ it holds

$$\int_{\partial \mathbf{B}_r(p)} |N|^2\, d\mathcal{H}^{m-1} \leqslant C \left(\frac{r}{\rho}\right)^{m-1} \left(\int_{\partial \mathbf{B}_\rho(p)} |N|^2\, d\mathcal{H}^{m-1} + 2 \int_{\mathbf{B}_r(p)\setminus \mathbf{B}_\rho(p)} |N||\nabla^\perp N|\, d\mathcal{H}^m \right). \tag{6.119}$$

Furthermore, by the Hölder continuity of N and since $\rho \leqslant \frac{r}{2}$ we also have

$$\left(\frac{r}{\rho}\right)^{m-1} \int_{\partial \mathbf{B}_\rho(p)} |N|^2\, d\mathcal{H}^{m-1} \overset{(6.96)}{\leqslant} C \left(\frac{\rho}{r}\right)^{2\alpha} r \left(\int_{\mathbf{B}_r(p)} |\nabla^\perp N|^2\, d\mathcal{H}^m + (C_0 + \Lambda) \int_{\mathbf{B}_r(p)} |N|^2\, d\mathcal{H}^m \right). \tag{6.120}$$

Combining (6.118), (6.119) and (6.120) gives

$$\int_{\mathbf{B}_r(p)} |\nabla^\perp N|^2 \, d\mathcal{H}^m \leqslant C \left(\frac{\rho}{r}\right)^{2\alpha} \left(\int_{\mathbf{B}_r(p)} |\nabla^\perp N|^2 \, d\mathcal{H}^m + \int_{\mathbf{B}_r(p)} |N|^2 \, d\mathcal{H}^m \right) + \frac{C}{r} \left(\frac{r}{\rho}\right)^{m-1} \int_{\mathbf{B}_r(p)} |N||\nabla^\perp N| \, d\mathcal{H}^m. \tag{6.121}$$

Now, if we choose ρ so small that $C\left(\frac{\rho}{r}\right)^{2\alpha} \leqslant \frac{1}{2}$ then from (6.121) follows

$$\begin{aligned} \int_{\mathbf{B}_r(p)} |\nabla^\perp N|^2 \, d\mathcal{H}^m &\leqslant \int_{\mathbf{B}_r(p)} |N|^2 \, d\mathcal{H}^m + \frac{C}{r} \int_{\mathbf{B}_r(p)} |N||\nabla^\perp N| \, d\mathcal{H}^m \\ &\leqslant \int_{\mathbf{B}_r(p)} |N|^2 \, d\mathcal{H}^m + \frac{C}{2r}\eta \int_{\mathbf{B}_r(p)} |\nabla^\perp N|^2 \, d\mathcal{H}^m + \frac{C}{2r\eta} \int_{\mathbf{B}_r(p)} |N|^2 \, d\mathcal{H}^m, \end{aligned} \tag{6.122}$$

by the Young's inequality. Choose $\eta = \frac{r}{C}$ to obtain

$$\int_{\mathbf{B}_r(p)} |\nabla^\perp N|^2 \, d\mathcal{H}^m \leqslant \left(\frac{C}{r^2} + 2\right) \int_{\mathbf{B}_r(p)} |N|^2, \tag{6.123}$$

which immediately implies (6.117). □

Now that we have the Reverse Poincaré inequality at our disposal, we can enter the core of the blow-up scheme.

6.3.2 The top-multiplicity singular stratum. Blow-up

The main difficulty in the proof of Theorem 6.0.3 consists of estimating the Hausdorff dimension of the set of singular points with multiplicity exactly equal to Q. The proof of the general result then follows in a relatively easy way by an induction argument on Q. Therefore, it is fundamental to study the structure of the *top-multiplicity singular stratum* of N, denoted $\operatorname{sing}_Q(N)$ and defined as follows.

Definition 6.3.3 (Top-multiplicity points)**.** Let $N \in \Gamma^{1,2}_Q(\mathcal{N}\Omega)$ be Jac-minimizing. A point $p \in \Omega$ has multiplicity Q, or simply is a Q-point for N, and we will write $p \in D_Q(N)$, if there exists $\nu \in T^\perp_p\Sigma$ such that $N(p) = Q[\![\nu]\!]$. We will define the top-multiplicity regular and singular strata of N by

$$\operatorname{reg}_Q(N) := \operatorname{reg}(N) \cap D_Q(N), \qquad \operatorname{sing}_Q(N) := \operatorname{sing}(N) \cap D_Q(N),$$

respectively.

From this point onward, we will assume to have fixed a point $p \in D_Q(N)$. The first step is to show that without loss of generality we can always assume that $N(p) = Q[\![0]\!]$. Recall the definition of the map $\boldsymbol{\eta}$ given in (2.14).

Lemma 6.3.4. *Let* $N \in \Gamma^{1,2}_Q(\mathcal{N}\Omega)$ *be* Jac*-minimizing. Then:*

(i) *the map* $\boldsymbol{\eta} \circ N \colon \Omega \to \mathbb{R}^d$ *is a classical Jacobi field;*

(ii) *if* $\zeta \colon \Omega \to \mathbb{R}^d$ *is a classical Jacobi field, then the* $\mathcal{Q}$*-valued map* $u := \sum_\ell \llbracket N^\ell + \zeta \rrbracket$ *is* Jac*-minimizing.*

Proof. Recall (cf. Remark 4.2.8 and the notation therein) that a normal vector field $\zeta \in \Gamma^{1,2}(\mathcal{N}\Omega) := \Gamma_1^{1,2}(\mathcal{N}\Sigma)$ is a Jacobi field if it is a weak solution of the linear elliptic PDE on the normal bundle $\mathcal{N}\Sigma$

$$\left(-\Delta_\Sigma^\perp - \mathscr{A} - \mathscr{R}\right) \zeta = 0,$$

that is, if the identity

$$\int_\Omega \left(\langle \nabla^\perp \zeta : \nabla^\perp \phi \rangle - \langle A \cdot \zeta : A \cdot \phi \rangle - \mathcal{R}(\zeta, \phi) \right) d\mathcal{H}^m = 0 \tag{6.124}$$

holds for all test functions $\phi \in C^1\left(\Omega, \mathbb{R}^d\right)$ with $\operatorname{spt}(\phi) \subset \Omega' \Subset \Omega$ and $\phi(x) \in T_x^\perp \Sigma \subset T_x \mathcal{M}$ for every $x \in \Omega$.

In order to prove (i), first observe that the map $\boldsymbol{\eta}$ preserves the fibers of the normal bundle, so that $\boldsymbol{\eta} \circ N(x) \in T_x^\perp \Sigma$ for a.e. $x \in \Omega$ and thus $\boldsymbol{\eta} \circ N \in \Gamma^{1,2}(\mathcal{N}\Omega) = \Gamma_1^{1,2}(\mathcal{N}\Omega)$. Now, fix any vector field ϕ as above. It is immediate to see that we can test the outer variation formula (6.30) with $\psi(x, u) := \phi(x)$, and that the resulting equation is precisely

$$\int_\Omega \left(\langle \nabla^\perp (\boldsymbol{\eta} \circ N) : \nabla^\perp \phi \rangle - \langle A \cdot (\boldsymbol{\eta} \circ N) : A \cdot \phi \rangle - \mathcal{R}(\boldsymbol{\eta} \circ N, \phi) \right) d\mathcal{H}^m = 0,$$

that is $\boldsymbol{\eta} \circ N$ solves (6.124) and the proof of (i) is complete.

In order to prove (ii), we take any $h \in \Gamma_Q^{1,2}(\mathcal{N}\Omega)$ such that $h|_{\partial\Omega} = N|_{\partial\Omega}$ and we show that

$$\operatorname{Jac}(u, \Omega) \leqslant \operatorname{Jac}(\tilde{h}, \Omega),$$

with $\tilde{h} = \sum_\ell \llbracket h^\ell + \zeta \rrbracket$. We compute:

$$\begin{aligned}
\operatorname{Jac}(u, \Omega) &= \int_\Omega \sum_{\ell=1}^{Q} \left(|\nabla^\perp (N^\ell + \zeta)|^2 - |A \cdot (N^\ell + \zeta)|^2 - \mathcal{R}(N^\ell + \zeta, N^\ell + \zeta) \right) d\mathcal{H}^m \\
&= \operatorname{Jac}(N, \Omega) + Q \left(\int_\Omega \left(|\nabla^\perp \zeta|^2 - |A \cdot \zeta|^2 - \mathcal{R}(\zeta, \zeta) \right) d\mathcal{H}^m \right) \\
&\quad + 2Q \left(\int_\Omega \left(\langle \nabla^\perp (\boldsymbol{\eta} \circ N) : \nabla^\perp \zeta \rangle - \langle A \cdot (\boldsymbol{\eta} \circ N) : A \cdot \zeta \rangle - \mathcal{R}(\boldsymbol{\eta} \circ N, \zeta) \right) d\mathcal{H}^m \right).
\end{aligned}$$

Using that $\operatorname{Jac}(N, \Omega) \leqslant \operatorname{Jac}(h, \Omega)$ and recalling the definition of $\tilde{h}$, we see that

$$\begin{aligned}
\operatorname{Jac}(u, \Omega) - \operatorname{Jac}(\tilde{h}, \Omega) \leqslant\ & 2Q \int_\Omega \langle \nabla^\perp (\boldsymbol{\eta} \circ N - \boldsymbol{\eta} \circ h) : \nabla^\perp \zeta \rangle \\
& - 2Q \int_\Omega \langle A \cdot (\boldsymbol{\eta} \circ N - \boldsymbol{\eta} \circ h) : A \cdot \zeta \rangle \\
& - 2Q \int_\Omega \mathcal{R}(\boldsymbol{\eta} \circ N - \boldsymbol{\eta} \circ h, \zeta) = 0,
\end{aligned}$$

because ζ is a Jacobi field and the function $\phi = \boldsymbol{\eta} \circ N - \boldsymbol{\eta} \circ h$ is a $W^{1,2}$ section of the normal bundle vanishing at $\partial\Omega$. This completes the proof. □

Remark 6.3.5. As a simple corollary of Lemma 6.3.4, if N is Jac-minimizing then the Q-valued map $\tilde{N} = \sum_\ell [\![N^\ell - \boldsymbol{\eta} \circ N]\!]$ is a Jac-minimizer with $\boldsymbol{\eta} \circ \tilde{N} \equiv 0$ and the same singular set as N. Therefore, there is no loss of generality in assuming that $\boldsymbol{\eta} \circ N \equiv 0$, and, thus, that every Q-point p satisfies $N(p) = Q[\![0]\!]$. In particular, when $p \in D_Q(N)$ we can apply all the results of the previous section that were proved under the above assumption. Furthermore, the content of Proposition 6.2.9 becomes more apparent in this context. Indeed, the dichotomy stated in there discriminates perfectly between regular and singular top-multiplicity points: $p \in \mathrm{reg}_Q(N)$ if and only if the condition (i) is observed; on the other hand, $p \in \mathrm{sing}_Q(N)$ if and only if the frequency function is well defined and bounded in a neighborhood of p.

In view of the above remark, we assume from this point onwards that N is Jac-minimizing and such that $\boldsymbol{\eta} \circ N = 0$. We fix a point $p \in \mathrm{sing}_Q(N)$, and an orthonormal basis

$$(e_1, \dots, e_m, e_{m+1}, \dots, e_{m+k}, e_{m+k+1}, \dots, e_d)$$

of the euclidean space $\mathbb{R}^d$ with the property that $T_p\Sigma = \mathrm{span}(e_1, \dots, e_m)$ and $T_p^\perp\Sigma = \mathrm{span}(e_{m+1}, \dots, e_{m+k})$. Choose local orthonormal frames $(\xi_i)_{i=1}^m$ and $(\nu_\alpha)_{\alpha=1}^k$ of $\mathcal{T}\Sigma$ and $\mathcal{N}\Sigma$ respectively which extend the basis at p, that is, such that $\xi_i(p) = e_i$ for $i = 1, \dots, m$ and $\nu_\alpha(p) = e_{m+\alpha}$ for $\alpha = 1, \dots, k$. With a slight abuse of notation, we will sometimes denote the linear subspace $\mathbb{R}^m \times \{0\} \times \{0\}$ by $\mathbb{R}^m$ and $\{0\} \times \mathbb{R}^k \times \{0\}$ by $\mathbb{R}^k$.

Let $r_0 > 0$ be such that all the conclusions from the previous paragraphs hold. For every $r \in (0, r_0]$, translate and rescale the manifolds $\mathcal{M}$ and Σ, setting

$$\mathcal{M}_r := \frac{\mathcal{M} - p}{r}, \qquad \Sigma_r := \frac{\Sigma - p}{r},$$

that is $\mathcal{M}_r = \iota_{p,r}(\mathcal{M})$ and $\Sigma_r = \iota_{p,r}(\Sigma)$, where

$$\iota_{p,r}(x) := \frac{x - p}{r}.$$

The manifolds $\mathcal{M}_r$ and Σ_r will be regarded as Riemannian submanifolds of $\mathbb{R}^d$ with the induced euclidean metric. We will let

$$\mathbf{ex}_r \colon B_1 \subset T_0\Sigma_r \simeq \mathbb{R}^m \to \Sigma_r$$

be the exponential map, and we will use the symbol $\psi_{p,r}$ to denote the map

$$\psi_{p,r} := \iota_{p,r}^{-1} \circ \mathbf{ex}_r.$$

Observe that $\psi_{p,r}$ maps the euclidean ball $B_1(0) \subset \mathbb{R}^m$ diffeomorphically onto the geodesic ball $\mathbf{B}_r(p) \subset \Sigma$.

Remark 6.3.6. Observe that, since $T_0\mathcal{M}_r = T_p\mathcal{M} = \mathrm{span}(e_1, \dots, e_{m+k})$ for every r, the ambient manifolds $\mathcal{M}_r$ converge, as $r \downarrow 0$, to $\mathbb{R}^{m+k} \times \{0\}$ in $C^{3,\beta}$. For the same reason, the Σ_r's converge to $\mathbb{R}^m \times \{0\} \times \{0\}$ in $C^{3,\beta}$ and the exponential maps $\mathbf{ex}_r$ converge in $C^{2,\beta}$ to the identity map of the ball $B_1 \subset \mathbb{R}^m$ (cf. [DLS16b, Proposition A.4]).

Definition 6.3.7. We define the *blow-ups* of N at p as the one-parameter family of maps $N_{p,r}\colon B_1 \subset T_0\Sigma_r \to \mathcal{A}_Q(\mathbb{R}^d)$ indexed by $r \in (0, r_0]$ and given by

$$N_{p,r}(y) := \frac{r^{\frac{m}{2}} N(\psi_{p,r}(y))}{\|N\|_{L^2(\mathbf{B}_r(p))}} = \frac{r^{\frac{m}{2}} N(p + r\mathbf{ex}_r(y))}{\|N\|_{L^2(\mathbf{B}_r(p))}}. \tag{6.125}$$

Observe that the maps $N_{p,r}$ are well defined because N is not vanishing in any ball $\mathbf{B}_r(p)$ with $0 < r \leqslant r_0$.

The next theorem is the anticipated convergence result for the blow-up maps.

Theorem 6.3.8. *Let $N \in \Gamma^{1,2}_Q(\mathcal{N}\Omega)$ be* Jac*-minimizing with $\boldsymbol{\eta} \circ N = 0$. Assume $p \in \mathrm{sing}_Q(N)$. Then, for any sequence N_{p,r_j} with $r_j \downarrow 0$, a subsequence, not relabeled, converges weakly in $W^{1,2}$, strongly in L^2 and locally uniformly to a continuous Q-valued function $\mathscr{N}_p\colon B_1 \subset \mathbb{R}^m \to \mathcal{A}_Q(\mathbb{R}^k)$ such that:*

(a) *$\mathscr{N}_p(0) = Q[\![0]\!]$ and $\boldsymbol{\eta} \circ \mathscr{N}_p \equiv 0$, but $\|\mathscr{N}_p\|_{L^2(B_1)} = 1$, and thus, in particular, $\mathscr{N}_p$ is non-trivial;*

(b) *$\mathscr{N}_p$ is locally* Dir*-minimizing in B_1;*

(c) *$\mathscr{N}_p$ is μ-homogeneous, with $\mu = I_0(p)$, the frequency of N at p defined in (6.64).*

Remark 6.3.9. Any map $\mathscr{N}_p$ which is the limit of a blow-up sequence N_{p,r_j} in the sense specified above will be called a *tangent map* to N at p.

Proof. Let N and p be as in the statement. For any sequence $r_j \downarrow 0$, let us denote $\mathcal{M}_j := \mathcal{M}_{r_j}$, $\Sigma_j := \Sigma_{r_j}$, $\mathbf{ex}_j := \mathbf{ex}_{r_j}$, $\psi_j := \psi_{p,r_j}$ and $N_j := N_{p,r_j}$ in order to simplify the notation. We will divide the proof into steps.

Step 1: boundedness in $W^{1,2}$. Assume for the moment that $j \in \mathbb{N}$ is fixed. We start estimating $\|N_j\|_{L^2(B_1)}$. Changing coordinates $x = \psi_j(y)$ in the integral, we compute explicitly

$$\begin{aligned} \|N\|^2_{L^2(\mathbf{B}_{r_j}(p))} &= \int_{\mathbf{B}_{r_j}(p)} |N(x)|^2 \, d\mathcal{H}^m(x) \\ &= \|N\|^2_{L^2(\mathbf{B}_{r_j}(p))} \int_{B_1} |N_j(y)|^2 \, \mathbf{Jex}_j(y) \, dy, \end{aligned} \tag{6.126}$$

and thus

$$\int_{B_1} |N_j(y)|^2 \, \mathbf{Jex}_j(y) \, dy = 1 \tag{6.127}$$

for every j. By the considerations in Remark 6.3.6, we can deduce that necessarily

$$\frac{1}{2} \leqslant \|N_j\|^2_{L^2(B_1)} \leqslant 2 \tag{6.128}$$

when j is large enough.

Next, we bound the Dirichlet energy of the blow-up maps in B_1. For any $y \in B_1 \subset T_0\Sigma_j$, and for all $i = 1, \dots, m$, let $\varepsilon_i = \varepsilon_i(y)$ be the vector in $T_0\Sigma_j$ defined by

$$d(\mathbf{ex}_j)|_y \cdot \varepsilon_i(y) = \xi_i(p + r_j \mathbf{ex}_j(y)).$$

We note that, when $j \uparrow \infty$, ε_i converges to $e_i = \xi_i(p)$ uniformly in B_1.

Again by changing variable $x = \psi_j(y)$ in the integral, we compute:

$$\begin{aligned} \mathrm{Dir}(N, \mathbf{B}_{r_j}(p)) &= \int_{\mathbf{B}_{r_j}(p)} \sum_{i=1}^{m} |D_{\xi_i} N(x)|^2 \, d\mathcal{H}^m(x) \\ &= \frac{\|N\|^2_{L^2(\mathbf{B}_{r_j}(p))}}{r_j^2} \int_{B_1} \sum_{i=1}^{m} |D_{\varepsilon_i} N_j(y)|^2 \, \mathbf{Jex}_j(y) \, dy. \end{aligned} \tag{6.129}$$

On the other hand, using that N takes values in the normal bundle, we have the usual estimate

$$\mathrm{Dir}(N, \mathbf{B}_{r_j}(p)) \leqslant \mathrm{Dir}^{\mathcal{N}\Sigma}(N, \mathbf{B}_{r_j}(p)) + C_0 \|N\|^2_{L^2(\mathbf{B}_{r_j}(p))}, \tag{6.130}$$

for some positive geometric constant $C_0 = C_0(\mathbf{A}, \overline{\mathbf{A}})$. From (6.129) and (6.130) we conclude that for any j

$$\int_{B_1} \sum_{i=1}^{m} |D_{\varepsilon_i} N_j(y)|^2 \, \mathbf{Jex}_j(y) \, dy \leqslant \frac{r_j^2 \, \mathrm{Dir}^{\mathcal{N}\Sigma}(N, \mathbf{B}_{r_j}(p))}{\|N\|^2_{L^2(\mathbf{B}_{r_j}(p))}} + C_0 r_j^2 \leqslant C(1 + r_j^2), \tag{6.131}$$

because of the reverse Poincaré inequality (6.117). Thus, we conclude that the Dirichlet energy of the blow-up maps in B_1 is bounded:

$$\mathrm{Dir}(N_j, B_1) := \int_{B_1} \sum_{i=1}^{m} |D_{e_i} N_j|^2 \, dy \leqslant C. \tag{6.132}$$

Step 2: convergence. The $W^{1,2}$ bounds given by estimates (6.128) and (6.132), together with Proposition 2.2.17, clearly imply the $W^{1,2}$-weak and L^2-strong convergence of a subsequence in B_1. We claim now that the N_j's are locally Hölder equi-continuous. This is an easy consequence of the Hölder estimate in (6.1) and of the reverse Poincaré inequality. Indeed, for any $0 < \theta < 1$ and for any points $y_1, y_2 \in \overline{B}_\theta$ one has the following:

$$\begin{aligned} \mathcal{G}(N_j(y_1), N_j(y_2)) &= \frac{r_j^{\frac{m}{2}}}{\|N\|_{L^2(\mathbf{B}_{r_j}(p))}} \mathcal{G}(N(p + r_j \mathbf{ex}_j(y_1)), N(p + r_j \mathbf{ex}_j(y_2))) \\ &\leqslant \frac{r_j^{\frac{m}{2}}}{\|N\|_{L^2(\mathbf{B}_{r_j}(p))}} [N]_{C^{0,\alpha}(\overline{\mathbf{B}}_{\theta r_j}(p))} \, \mathbf{d}(p + r_j \mathbf{ex}_j(y_1), p + r_j \mathbf{ex}_j(y_2))^\alpha \\ &\overset{(6.1)}{\leqslant} C \frac{r_j}{\|N\|_{L^2(\mathbf{B}_{r_j}(p))}} \left(\mathrm{Dir}^{\mathcal{N}\Sigma}(N, \mathbf{B}_{r_j}(p)) + (\Lambda + C_0) |N\|^2_{L^2(\mathbf{B}_{r_j}(p))} \right)^{1/2} |y_1 - y_2|^\alpha \\ &\overset{(6.117)}{\leqslant} C(1 + r_j) |y_1 - y_2|^\alpha. \end{aligned}$$

Hence, for every $0 < \theta < 1$ there exists $C = C(\theta) > 0$ such that

$$[N_j]_{C^{0,\alpha}(\overline{B}_\theta)} := \sup_{y_1, y_2 \in \overline{B}_\theta} \frac{\mathcal{G}(N_j(y_1), N_j(y_2))}{|y_1 - y_2|^\alpha} \leqslant C \tag{6.133}$$

for all j. Since $N_j(0) = Q[\![0]\!]$, the N_j's are also locally uniformly bounded, hence the Ascoli-Arzelà theorem implies that, up to extracting another subsequence if necessary, the convergence is locally uniform, and the limit is a continuous Q-valued function $\mathscr{N}_p \colon B_1 \subset \mathbb{R}^m \to \mathcal{A}_Q(\mathbb{R}^d)$.

Step 3: properties of the limit: proof of (a). It is immediate to see that $\boldsymbol{\eta} \circ \mathscr{N}_p \equiv 0$. Indeed, from the assumption that $\boldsymbol{\eta} \circ N \equiv 0$ and the definition of the blow-up maps, we deduce that $\boldsymbol{\eta} \circ N_j \equiv 0$ for every j. Now, the N_j's converge to $\mathscr{N}_p$ locally uniformly, and thus

$$\boldsymbol{\eta} \circ \mathscr{N}_p(y) = \lim_{j\to\infty} \boldsymbol{\eta} \circ N_j(y) = 0$$

for all $y \in B_1$. With the same argument, using the pointwise convergence of N_j to $\mathscr{N}_p$ and the fact that $N_j(0) = Q[\![0]\!]$ for every j we conclude that $\mathscr{N}_p(0) = Q[\![0]\!]$.

Nonetheless, the map $\mathscr{N}_p$ is non-trivial. Indeed, since $N_j \to \mathscr{N}_p$ strongly in L^2, estimate (6.127) guarantees that:

$$\|\mathscr{N}_p\|^2_{L^2(B_1)} = \lim_{j\to\infty} \int_{B_1} |N_j(y)|^2 \, \mathbf{Jex}_j(y)\, dy = 1. \tag{6.134}$$

Next, we see that $\mathscr{N}_p(y) \in \mathcal{A}_Q(\mathbb{R}^k)$ for every $y \in B_1$. Indeed, considering the projection $\mathscr{N}_p^{(1)}$ of $\mathscr{N}_p$ onto the subspace $\mathbb{R}^m \times \{0\} \times \{0\}$ we easily infer that

$$\begin{aligned}\int_{B_1} |\mathscr{N}_p^{(1)}(y)|^2\, dy &= \int_{B_1} \sum_{\ell=1}^{Q} \sum_{i=1}^{m} |\langle \mathscr{N}_p^{\ell}(y), e_i\rangle|^2\, dy \\ &= \lim_{j\to\infty} \int_{B_1} \sum_{\ell=1}^{Q} \sum_{i=1}^{m} |\langle N_j^{\ell}(y), \xi_i(p + r_j \mathbf{ex}_j(y))\rangle|^2\, dy = 0,\end{aligned}$$

because of the definition of N_j. Analogously, the projection $\mathscr{N}_p^{(3)}$ onto $\{0\} \times \{0\} \times \mathbb{R}^K$ satisfies

$$\begin{aligned}\int_{B_1} |\mathscr{N}_p^{(3)}(y)|^2\, dy &= \int_{B_1} \sum_{\ell=1}^{Q} \sum_{\beta=1}^{K} |\langle \mathscr{N}_p^{\ell}(y), e_{m+k+\beta}\rangle|^2\, dy \\ &= \lim_{j\to\infty} \int_{B_1} \sum_{\ell=1}^{Q} \sum_{\beta=1}^{K} |\langle N_j^{\ell}(y), \eta_\beta(p + r_j \mathbf{ex}_j(y))\rangle|^2\, dy = 0,\end{aligned}$$

where the η_β's are a local orthonormal frame of the normal bundle of $\mathcal{M}$ in $\mathbb{R}^d$ extending the $e_{m+k+\beta}$'s in a neighborhood of p.

Step 4: harmonicity of the limit: proof of (b). We show now that $\mathscr{N}_p$ is locally Dir-minimizing in B_1 and, moreover, that for every $0 < \rho < 1$ the following identity holds true:

$$\mathrm{Dir}(\mathscr{N}_p, B_\rho) = \liminf_{j\to\infty} \mathrm{Dir}(N_j, B_\rho). \tag{6.135}$$

In order to obtain the proof of the above claim, we need to exploit the minimizing property of the Jacobi Q-valued field N in order to deduce some crucial information on the

blow-up sequence N_j. Fix $j \in \mathbb{N}$, and let $u\colon B_1 \subset T_0\Sigma_j \simeq \mathbb{R}^m \to \mathcal{A}_Q(\mathbb{R}^d)$ be any $W^{1,2}$ map such that $u|_{\partial B_1} = N_j|_{\partial B_1}$. Then, the map $\tilde{u} \in W^{1,2}\left(\mathbf{B}_{r_j}(p), \mathcal{A}_Q(\mathbb{R}^d)\right)$ defined by

$$\begin{aligned}\tilde{u}(x) &:= r_j^{-\frac{m}{2}} \|N\|_{L^2(\mathbf{B}_{r_j}(p))} \left(u \circ \psi_j^{-1}\right)^{\perp}(x) \\ &= r_j^{-\frac{m}{2}} \|N\|_{L^2(\mathbf{B}_{r_j}(p))} \sum_{\ell=1}^{Q} \left[\!\!\left[\sum_{\beta=1}^{k} \langle u^\ell \circ \psi_j^{-1}(x), \nu_\beta(x)\rangle \nu_\beta(x) \right]\!\!\right]\end{aligned}$$

is a section of $\mathcal{N}\Sigma$ in $\mathbf{B}_{r_j}(p)$ such that $\tilde{u}|_{\partial \mathbf{B}_{r_j}(p)} = N|_{\partial \mathbf{B}_{r_j}(p)}$. By minimality, it follows then that

$$\mathrm{Jac}(N, \mathbf{B}_{r_j}(p)) \leqslant \mathrm{Jac}(\tilde{u}, \mathbf{B}_{r_j}(p)). \tag{6.136}$$

Standard computations show that (6.136) is equivalent to the condition

$$\mathscr{F}_j(N_j) \leqslant \mathscr{F}_j(u), \tag{6.137}$$

where $\mathscr{F}_j(u)$ is the functional defined by

$$\begin{aligned}\mathscr{F}_j(u) :=& \int_{B_1} \sum_{\ell=1}^{Q} \sum_{i=1}^{m} \sum_{\alpha=1}^{k} \left| \langle D_{\xi_i} u^\ell(y), \nu_\alpha \circ \psi_j(y)\rangle + r_j \langle u^\ell(y), (D_{\xi_i}\nu_\alpha - \nabla^{\perp}_{\xi_i}\nu_\alpha) \circ \psi_j(y)\right|^2 \mathbf{Jex}_j(y)\, dy \\ &- r_j^2 \int_{B_1} \sum_{\ell=1}^{Q} \left(\left| A \circ \psi_j(y) \cdot (u^\ell)^{\perp_{\psi_j(y)}}\right|^2 + \mathcal{R} \circ \psi_j((u^\ell)^{\perp_{\psi_j(y)}}, (u^\ell)^{\perp_{\psi_j(y)}}) \right) \mathbf{Jex}_j(y)\, dy \\ &= \mathscr{F}_j^{(1)}(u) + \mathscr{F}_j^{(2)}(u)\end{aligned} \tag{6.138}$$

on the space of $u \in W^{1,2}\left(B_1, \mathcal{A}_Q(\mathbb{R}^d)\right)$ such that $u|_{\partial B_1} = N_j|_{\partial B_1}$. Note that the following notation has been adopted in formula (6.128): $(u^\ell)^{\perp_{\psi_j(y)}}$ is the orthogonal projection of the vector $u^\ell(y)$ onto $T^{\perp}_{\psi_j(y)}\Sigma$, given by

$$(u^\ell)^{\perp_{\psi_j(y)}} = \sum_{\beta=1}^{k} \langle u^\ell(y), \nu_\beta(\psi_j(y))\rangle \nu_\beta(\psi_j(y)).$$

Hence, one has

$$\left| A \circ \psi_j(y) \cdot (u^\ell)^{\perp_{\psi_j(y)}}\right|^2 = \sum_{i,h=1}^{m} \left| \sum_{\beta=1}^{k} A^\beta_{ih}(\psi_j(y)) \langle u^\ell(y), \nu_\beta(\psi_j(y))\rangle \right|^2,$$

with $A^\beta_{ih} := \langle A(\xi_i, \xi_h), \nu_\beta\rangle$, and

$$\mathcal{R} \circ \psi_j((u^\ell)^{\perp_{\psi_j(y)}}, (u^\ell)^{\perp_{\psi_j(y)}}) = \sum_{i=1}^{m} \sum_{\beta,\gamma=1}^{k} R^i_{i\beta\gamma}(\psi_j(x)) \langle u^\ell(y), \nu_\beta(\psi_j(y))\rangle \langle u^\ell(y), \nu_\gamma(\psi_j(y))\rangle,$$

with $R^i_{i\beta\gamma} := \langle R(\xi_i, \nu_\beta)\nu_\gamma, \xi_i\rangle$.

Now, for every $0 < \rho < 1$, set

$$D_\rho := \liminf_{j\to\infty} \mathrm{Dir}(N_j, B_\rho) = \liminf_{j\to\infty} \int_{B_\rho} \sum_{i=1}^{m} |D_{e_i} N_j|^2 \, dy,$$

and suppose by contradiction that either $\mathscr{N}_p$ is not Dir-minimizing in B_ρ or $\mathrm{Dir}(\mathscr{N}_p, B_\rho) < D_\rho$ [2] for some ρ. In any of the two situations, there exists a $\rho_0 > 0$ such that for any $\rho \geqslant \rho_0$ there exists a multiple valued map $g \in W^{1,2}(B_\rho, \mathcal{A}_Q(\mathbb{R}^k))$ with

$$g|_{\partial B_\rho} = \mathscr{N}_p|_{\partial B_\rho} \quad \text{and} \quad \gamma_\rho := D_\rho - \mathrm{Dir}(g, B_\rho) > 0. \tag{6.139}$$

A simple application of Fatou's lemma shows that for almost every $\rho \in (0,1)$ both the quantities $\liminf_j \mathrm{Dir}(N_j, \partial B_\rho)$ and $\liminf_j \|N_j\|^2_{L^2(\partial B_\rho)}$ are finite:

$$\int_0^1 \liminf_{j\to\infty} \mathrm{Dir}(N_j, \partial B_\rho)\, d\rho \leqslant \liminf_{j\to\infty} \int_0^1 \mathrm{Dir}(N_j, \partial B_\rho)\, d\rho = \liminf_{j\to\infty} \mathrm{Dir}(N_j, B_1) \leqslant M < \infty,$$

$$\int_0^1 \liminf_{j\to\infty} \|N_j\|^2_{L^2(\partial B_\rho)}\, d\rho \leqslant \liminf_{j\to\infty} \int_0^1 \|N_j\|^2_{L^2(\partial B_\rho)}\, d\rho = \lim_{j\to\infty} \|N_j\|^2_{L^2(B_\rho)} = \|\mathscr{N}_p\|^2_{L^2(B_\rho)} \leqslant 1.$$

Therefore, passing if necessary to a subsequence, we can fix a radius $\rho \geqslant \rho_0$ such that

$$\mathrm{Dir}(\mathscr{N}_p, \partial B_\rho) \leqslant \lim_{j\to\infty} \mathrm{Dir}(N_j, \partial B_\rho) \leqslant M < \infty \tag{6.140}$$

and

$$\|\mathscr{N}_p\|^2_{L^2(\partial B_\rho)} \leqslant \lim_{j\to\infty} \|N_j\|^2_{L^2(\partial B_\rho)} \leqslant 1. \tag{6.141}$$

This allows us also to fix the corresponding map g satisfying the conditions in (6.139). The strategy to complete the proof is now the following: we will use the map g to construct, for every j, a competitor u_j for the functional $\mathscr{F}_j$, that is a map $u_j \in W^{1,2}(B_1, \mathcal{A}_Q(\mathbb{R}^d))$ with $u_j|_{\partial B_1} = N_j|_{\partial B_1}$. Then, we will show that if j is chosen sufficiently large then $\mathscr{F}_j(u_j) < \mathscr{F}_j(N_j)$, thus contradicting (6.137) and, in turn, the minimality of N in $\mathbf{B}_{r_j}(p)$.

The construction of the maps u_j is analogous to the one presented in [DLS11, Proposition 3.20]: we fix a number $0 < \delta < \frac{\rho}{2}$ to be suitably chosen later, and for every $j \in \mathbb{N}$ we define u_j on B_1 as follows:

$$u_j(y) := \begin{cases} g\left(\frac{\rho y}{\rho - \delta}\right) & \text{for } y \in B_{\rho-\delta}, \\ h_j(y) & \text{for } y \in B_\rho \setminus B_{\rho-\delta}, \\ N_j(y) & \text{for } y \in B_1 \setminus B_\rho, \end{cases}$$

where the maps h_j interpolate between $g\left(\frac{\rho y}{\rho-\delta}\right) = \mathscr{N}_p\left(\frac{\rho y}{\rho-\delta}\right) \in W^{1,2}(\partial B_{\rho-\delta}, \mathcal{A}_Q)$ and $N_j \in W^{1,2}(\partial B_\rho, \mathcal{A}_Q)$. Observe that the existence of the h_j's is guaranteed by Proposition 5.1.1 (also cf. [DLS11, Lemma 2.15]).

2 Observe that the inequality

$$\mathrm{Dir}(\mathscr{N}_p, B_\rho) \leqslant D_\rho$$

is guaranteed for every ρ because the Dirichlet functional is lower semi-continuous with respect to weak convergence in $W^{1,2}$.

As anticipated, the goal is now to show that this map u_j has less $\mathscr{F}_j$ energy than N_j when j is big enough (and thus r_j is suitably close to 0). We first note that u_j differs from N_j only on B_ρ, therefore our analysis will be carried on this smaller ball only. Then, fix a small number $\theta > 0$, and recall that, in the limit as $j \uparrow \infty$, the exponential maps $\mathbf{ex}_j$ converge uniformly to the identity map of the unit ball in $\mathbb{R}^m$, whereas the maps ψ_j converge uniformly to the constant map identically equal to p. Hence, the first line in the definition of $\mathscr{F}_j(u_j)$ can be estimated by

$$\mathscr{F}_j^{(1)}(u_j)\big|_{B_\rho} \leqslant (1+\theta)\mathrm{Dir}(u_j, B_\rho) + \theta\|u_j\|^2_{L^2(B_\rho)} \tag{6.142}$$

for all $j \geqslant j_0(\theta)$. On the other hand, the definition of u_j together with the estimate (5.3) imply that

$$\begin{aligned}\mathrm{Dir}(u_j, B_\rho) &\leqslant \mathrm{Dir}(u_j, B_{\rho-\delta}) + C\delta\big(\mathrm{Dir}(u_j, \partial B_{\rho-\delta}) + \mathrm{Dir}(N_j, \partial B_\rho)\big) + \frac{C}{\delta}\int_{\partial B_\rho} \mathcal{G}(u_j, N_j)^2 \\ &\leqslant \mathrm{Dir}(g, B_\rho) + C\delta\mathrm{Dir}(\mathscr{N}_p, \partial B_\rho) + C\delta\mathrm{Dir}(N_j, \partial B_\rho) + \frac{C}{\delta}\int_{\partial B_\rho} \mathcal{G}(\mathscr{N}_p, N_j)^2\end{aligned} \tag{6.143}$$

whereas

$$\begin{aligned}\|u_j\|^2_{L^2(B_\rho)} &= \|u_j\|^2_{L^2(B_{\rho-\delta})} + \|u_j\|^2_{L^2(B_\rho \setminus B_{\rho-\delta})} \\ &\leqslant \|g\|^2_{L^2(B_\rho)} + \|h_j\|^2_{L^2(B_\rho\setminus B_{\rho-\delta})} \\ &\overset{(5.2)}{\leqslant} \|g\|^2_{L^2(B_\rho)} + C\delta\left(\|\mathscr{N}_p\|^2_{L^2(\partial B_\rho)} + \|N_j\|^2_{L^2(\partial B_\rho)}\right) \\ &\overset{(6.141)}{\leqslant} \|g\|^2_{L^2(B_\rho)} + 3C\delta.\end{aligned} \tag{6.144}$$

Concerning the second term in the functional $\mathscr{F}_j$, it is easy to compute that

$$\begin{aligned}\mathscr{F}_j^{(2)}(u_j)\big|_{B_\rho} &\leqslant \mathscr{F}_j^{(2)}(N_j)\big|_{B_\rho} + Cr_j^2\left(\int_{B_\rho}|N_j|^2\,dy\right)^{\frac12}\left(\int_{B_\rho}\mathcal{G}(u_j,N_j)^2\,dy\right)^{\frac12} + Cr_j^2\int_{B_\rho}\mathcal{G}(u_j,N_j)^2\,dy \\ &\leqslant \mathscr{F}_j^{(2)}(N_j)\big|_{B_\rho} + Cr_j^2,\end{aligned} \tag{6.145}$$

because the L^2 norms of both the maps u_j and N_j are uniformly bounded in j.

We can finally close the argument. Choose δ such that $4C\delta(M+1) \leqslant \gamma_\rho$, where M and γ_ρ are the constants in (6.140) and (6.139) respectively. Invoking (6.140) and using the uniform convergence of N_j to $\mathscr{N}_p$, from (6.143) follows

$$\begin{aligned}\mathrm{Dir}(u_j, B_\rho) &\leqslant D_\rho - \gamma_\rho + C\delta M + C\delta(M+1) + \frac{C}{\delta}\int_{\partial B_\rho}\mathcal{G}(\mathscr{N}_p, N_j)^2 \\ &\leqslant D_\rho - \frac{\gamma_\rho}{2} + \frac{C}{\delta}\int_{\partial B_\rho}\mathcal{G}(\mathscr{N}_p, N_j)^2 \leqslant D_\rho - \frac{\gamma_\rho}{4}\end{aligned} \tag{6.146}$$

whenever j is sufficiently large. A suitable choice of the parameter θ in (6.142) depending on γ_ρ, D_ρ and $\|g\|_{L^2(B_\rho)}$ allows to conclude that for j big enough (depending on the same quantities):

$$\begin{aligned}\mathscr{F}_j^{(1)}(u_j)\big|_{B_\rho} &\leqslant D_\rho - \frac{\gamma_\rho}{8}\\ &\leqslant \mathrm{Dir}(N_j, B_\rho) - \frac{\gamma_\rho}{16}\\ &\leqslant \mathscr{F}_j^{(1)}(N_j)\big|_{B_\rho} - \frac{\gamma_\rho}{32}.\end{aligned} \tag{6.147}$$

Observe that in the last inequality we have used again that the manifolds Σ_j are becoming more and more flat in the limit $j \uparrow \infty$, and also that the projection of the N_j's on the orthogonal complement to $\mathbb{R}^k$ is vanishing in an L^2 sense in the same limit. Now, summing (6.145) and (6.147), we conclude:

$$\mathscr{F}_{r_j}(u_j) \leqslant \mathscr{F}_{r_j}(N_j) - \frac{\gamma_\rho}{32} + Cr_j^2. \tag{6.148}$$

The desired contradiction is immediately obtained by choosing j so big that $Cr_j^2 \leqslant \frac{\gamma_\rho}{64}$.

Step 5: homogeneity of the limit: proof of (c). We conclude the proof of the theorem showing that the limit map $\mathscr{N}_p$ admits a homogeneous extension to the whole $\mathbb{R}^m$. In other words, the goal is to show that

$$\mathscr{N}_p\left(\frac{\rho y}{|y|}\right) = \left(\frac{\rho}{|y|}\right)^\mu \mathscr{N}_p(y)$$

for all $y \in B_1 \setminus \{0\}$, for all $0 < \rho < 1$, and with $\mu = I_0(p)$.

The strategy is to take advantage of [DLS11, Corollary 3.16]: since $\mathscr{N}_p$ is Dir-minimizing, in order to prove that it is homogeneous it suffices to show that its frequency function at the origin $y = 0$ is constant. Hence, we set for $0 < \rho < 1$:

$$\mathscr{I}(\rho) := \frac{\rho\mathscr{D}(\rho)}{\mathscr{H}(\rho)}, \tag{6.149}$$

where

$$\mathscr{D}(\rho) := \mathrm{Dir}(\mathscr{N}_p, B_\rho) = \int_{B_\rho} |D\mathscr{N}_p|^2\,dy = \int_{B_\rho} \sum_{\ell=1}^{Q}\sum_{i=1}^{m}\sum_{\alpha=1}^{k} |\langle D_{e_i}\mathscr{N}_p^\ell, e_{m+\alpha}\rangle|^2\,dy, \tag{6.150}$$

and

$$\mathscr{H}(\rho) := \int_{\partial B_\rho} |\mathscr{N}_p|^2\,dy. \tag{6.151}$$

We first observe that $\mathscr{I}(\rho)$ is well defined for all $\rho \in (0,1)$. Indeed, if there is ρ_0 such that $\mathscr{H}(\rho_0) = 0$, then by minimality it must be $\mathscr{N}_p \equiv Q[\![0]\!]$ in B_{ρ_0}. On the other hand, the unique continuation property of Dir-minimizers (cf. [DLS16a, Lemma 7.1]) would then imply that $\mathscr{N}_p \equiv Q[\![0]\!]$ in the whole B_1, which in turn contradicts the fact that $\|\mathscr{N}_p\|_{L^2(B_1)} = 1$. In other words, this shows that the origin is singular for $\mathscr{N}_p$.

Extract, if necessary, a subsequence such that the lim inf in (6.135) can be replaced by a lim, and compute:

$$\begin{aligned}\mathscr{I}(\rho) &= \frac{\rho\int_{B_\rho}\sum_{\ell=1}^{Q}\sum_{i=1}^{m}\sum_{\alpha=1}^{k}|\langle D_{e_i}\mathscr{N}_p^\ell, e_{m+\alpha}\rangle|^2\,dy}{\int_{\partial B_\rho}|\mathscr{N}_p|^2\,dy}\\ &= \lim_{j\to\infty}\frac{\rho\int_{B_\rho}\sum_{\ell=1}^{Q}\sum_{i=1}^{m}\sum_{\alpha=1}^{k}|\langle D_{\varepsilon_i}N_j^\ell, \nu_\alpha\circ\psi_j\rangle|^2\,\mathbf{Jex}_j(y)\,dy}{\int_{\partial B_\rho}|N_j|^2\,\mathbf{Jex}_j(y)\,dy}\\ &= \lim_{j\to\infty}\frac{\rho r_j \mathrm{Dir}^{\mathcal{N}\Sigma}(N,\mathbf{B}_{\rho r_j}(p))}{\int_{\partial\mathbf{B}_{\rho r_j}(p)}|N|^2\,d\mathcal{H}^{m-1}}\\ &= \lim_{j\to\infty}\frac{\rho r_j \mathbf{D}(\rho r_j)}{\mathbf{H}(\rho r_j)} = \lim_{j\to\infty}\mathbf{I}(\rho r_j) = I_0,\end{aligned} \tag{6.152}$$

where we have used (modifications of) formulae (6.126), (6.129) and finally (6.112).

As already anticipated, [DLS11, Corollary 3.16] implies now that $\mathscr{N}_p$ is a μ-homogeneous Q-valued function, with $\mu = I_0(p) > 0$. □

Remark 6.3.10. Note that from the proof of Theorem 6.3.8 it follows that the convergence of (a subsequence of) the N_{p,r_j} to $\mathscr{N}_p$ is actually *strong* in $W^{1,2}$ in any ball $B_\rho \subset B_1$ (cf. formula (6.135)). This stronger convergence has been in fact tacitly used in deriving (6.152).

6.4 THE CLOSING ARGUMENT: PROOF OF THEOREM 6.0.3

Proposition 6.4.1. *Let $N \in \Gamma^{1,2}_Q(\mathcal{N}\Omega)$ be* Jac*-minimizing. Assume $\Omega \subset \Sigma^m$ is connected. Then:*

(i) *either $N = Q[\![\zeta]\!]$ with $\zeta\colon \Omega \to \mathbb{R}^d$ a classical Jacobi field,*

(ii) *or the set $D_Q(N)$ of multiplicity Q points is a relatively closed proper subset of Ω consisting of isolated points if $m = 2$ and with $\dim_{\mathcal{H}}(D_Q(N)) \leqslant m-2$ if $m \geqslant 3$.*

Proof. Assume without loss of generality that $\boldsymbol{\eta}\circ N = 0$, so that $p \in D_Q(N)$ if and only if $N(p) = Q[\![0]\!]$. We first observe the following fact: the set $D_Q(N)$ is relatively closed in Ω. This can be rapidly seen writing $D_Q(N) = \sigma^{-1}(\{Q\})$, where $\sigma\colon \Omega \to \mathbb{N}$ is the function given by

$$\sigma(x) := \mathrm{card}(\mathrm{spt}(N(x))), \tag{6.153}$$

and noticing that, since N is continuous, σ is lower semi-continuous.

We will now treat the two cases $m = 2$ and $m \geqslant 3$ separately.

Case 1: dimension $m = 2$. In this case, we claim that the points $p \in \mathrm{sing}_Q(N)$ are isolated in $D_Q(N)$. Assume by contradiction that this is not the case, and let $p \in \mathrm{sing}_Q(N)$ be the limit of a sequence $\{x_j\}_{j=1}^\infty$ of points in $D_Q(N)$. Set $r_j := \mathbf{d}(x_j, p)$. Since $r_j \downarrow 0$, by Theorem 6.3.8 the corresponding blow-up family N_{p,r_j} converges uniformly, up to a subsequence, to a Dir-minimizing, μ-homogeneous tangent map $\mathscr{N}_p\colon B_1 \subset \mathbb{R}^2 \to \mathcal{A}_Q(\mathbb{R}^k)$ with $\|\mathscr{N}_p\|_{L^2(B_1)} = 1$ and $\boldsymbol{\eta}\circ\mathscr{N}_p \equiv 0$. Moreover, since each $x_j \in D_Q(N)$, the points $y_j := \psi^{-1}_{p,r_j}(x_j)$ are a sequence of multiplicity Q points for the corresponding N_{p,r_j} in $S^1 = \partial B_1$: from this, we conclude

that there exists $w \in \mathbb{S}^1$ such that $\mathscr{N}_p(w) = Q[\![0]\!]$. Up to rotations, we can assume that $w = e_1$. Denote $z := \frac{1}{2}e_1$, and observe that, since $\mathscr{N}_p$ is homogeneous, necessarily $\mathscr{N}_p(z) = Q[\![0]\!]$. Consider now the blow-up of $\mathscr{N}_p$ at z: by [DLS11, Lemma 3.24], any tangent map h to $\mathscr{N}_p$ at z is a non-trivial β-homogeneous Dir-minimizer, with β equal to the frequency of $\mathscr{N}_p$ at z, and such that $h(x_1, x_2) = \hat{h}(x_2)$, for some function $\hat{h}\colon \mathbb{R} \to \mathcal{A}_Q(\mathbb{R}^k)$ which is Dir-minimizing on every interval. Moreover, since $\mathrm{Dir}(h, B_1) > 0$, it must also be $\mathrm{Dir}(\hat{h}, I) > 0$, where $I := [-1, 1]$. On the other hand, every 1-dimensional Dir-minimizer $\hat{h}$ is affine, that is it has the form $\hat{h}(x) = \sum_{i=1}^{Q} [\![L_i(x)]\!]$, where the L_i's are affine functions such that either $L_i \equiv L_j$ or $L_i(x) \neq L_j(x)$ for every $x \in \mathbb{R}$, for any i, j. Now, since $\hat{h}(0) = Q[\![0]\!]$, we deduce that $\hat{h} = Q[\![L]\!]$; on the other hand, $\boldsymbol{\eta} \circ h \equiv 0$, and thus necessarily $L = 0$. This contradicts $\mathrm{Dir}(\hat{h}, I) > 0$.

Hence, if $p \in D_Q(N)$ then either p is isolated or, in case p is a regular multiplicity Q point, there exists an open neighborhood V of p such that $V \subset D_Q(N)$. From this we deduce that $\mathrm{reg}_Q(N)$ is both open and closed in Ω. Since Ω is connected, then either $\mathrm{reg}_Q(N) = \Omega$, and $N \equiv Q[\![0]\!]$, or $\mathrm{reg}_Q(N) = \emptyset$, and $D_Q(N) = \mathrm{sing}_Q(N)$ consists of isolated points. This completes the proof in the dimension $m = 2$ case.

Case 2: dimension $m \geqslant 3$. In this case, the goal is to show that $\mathcal{H}^s(D_Q(N)) = 0$ for every $s > m-2$, unless $N \equiv Q[\![0]\!]$. Consider the set $\mathrm{sing}_Q(N)$. We first claim that $\mathcal{H}^s(\mathrm{sing}_Q(N)) = 0$ for every $s > m-2$. Suppose by contradiction that this is not the case. Then, by [Sim83b, Theorem 3.6 (2)], there exist $s > m-2$ and a subset $F \subset \mathrm{sing}_Q(N)$ with $\mathcal{H}^s(F) > 0$ such that every point $p \in F$ is a point of positive upper s-density for the measure $\mathcal{H}^s_\infty$, that is

$$\limsup_{r \to 0} \frac{\mathcal{H}^s_\infty(\mathrm{sing}_Q(N) \cap \mathbf{B}_r(p))}{r^s} > 0 \qquad \text{for every } p \in F. \tag{6.154}$$

Here, the symbol $\mathcal{H}^s_\infty$ denotes as usual the s-dimensional Hausdorff pre-measure, defined by

$$\mathcal{H}^s_\infty(A) := \inf\left\{ \sum_{h=1}^{\infty} \omega_s \left(\frac{\mathrm{diam}(E_h)}{2}\right)^s : A \subset \bigcup_{h=1}^{\infty} E_h \right\},$$

with $\omega_s := \frac{\pi^{\frac{s}{2}}}{\Gamma(\frac{s}{2}+1)}$, where $\Gamma(s)$ is the usual Gamma function. Among the properties of $\mathcal{H}^s_\infty$, it is worth recalling now that it is upper semi-continuous with respect to Hausdorff convergence of compact sets: in other words, if K_j is a sequence of compact sets converging to K in the sense of Hausdorff, then

$$\limsup_{j \to \infty} \mathcal{H}^s_\infty(K_j) \leqslant \mathcal{H}^s_\infty(K). \tag{6.155}$$

Now, (6.154) together with Theorem 6.3.8 imply the existence of a point $p \in \mathrm{sing}_Q(N)$ and a sequence of radii $r_j \downarrow 0$ such that the blow-up maps $N_j = N_{p, 2r_j}$ converge uniformly to a Dir-minimizing homogeneous Q-valued function $\mathscr{N}_p\colon B_1 \subset \mathbb{R}^m \to \mathcal{A}_Q(\mathbb{R}^k)$ with $\boldsymbol{\eta} \circ \mathscr{N}_p \equiv 0$ and $\|\mathscr{N}_p\|_{L^2(B_1)} = 1$, and furthermore such that

$$\limsup_{j \to \infty} \frac{\mathcal{H}^s_\infty(\mathrm{sing}_Q(N) \cap \mathbf{B}_{r_j}(p))}{r_j^s} > 0, \tag{6.156}$$

or, equivalently,

$$\limsup_{j\to\infty} \mathcal{H}^s_\infty(\mathrm{sing}_Q(N_j) \cap B_{\frac{1}{2}}) > 0, \tag{6.157}$$

where $B_{\frac{1}{2}} \subset T_0\Sigma_j \simeq \mathbb{R}^m$. Set $K_j := \mathrm{sing}_Q(N_j) \cap \overline{B}_{\frac{1}{2}}$, and observe that, since N_j converge to $\mathscr{N}_p$ uniformly, the compact sets K_j converge in the sense of Hausdorff to a compact set $K \subset D_Q(\mathscr{N}_p)$. From the semi-continuity property (6.155), we can therefore deduce that:

$$\begin{aligned} \mathcal{H}^s(D_Q(\mathscr{N}_p)) &\geqslant \mathcal{H}^s_\infty(D_Q(\mathscr{N}_p)) \geqslant \mathcal{H}^s_\infty(K) \\ &\geqslant \limsup_{j\to\infty} \mathcal{H}^s_\infty(K_j) \geqslant \limsup_{j\to\infty} \mathcal{H}^s_\infty(\mathrm{sing}_Q(N_j) \cap B_{\frac{1}{2}}) > 0. \end{aligned} \tag{6.158}$$

Since $s > m-2$, [DLS11, Proposition 3.22] implies that this can happen only if $\mathscr{N}_p \equiv Q\llbracket\zeta\rrbracket$, where $\zeta\colon B_1 \to \mathbb{R}^k$ is a harmonic function. Since $\boldsymbol{\eta} \circ \mathscr{N}_p \equiv 0$, then it must be $\mathscr{N}_p \equiv Q\llbracket 0\rrbracket$, which in turns contradicts the fact that $\|\mathscr{N}_p\|_{L^2(B_1)} = 1$.

We can therefore conclude that necessarily $\mathcal{H}^s(\mathrm{sing}_Q(N)) = 0$ for every $s > m-2$. Since $\mathrm{sing}_Q(N) = \partial D_Q(N) \cap \Omega$, either $D_Q(N) = \Omega$ and $N \equiv Q\llbracket 0\rrbracket$, or $D_Q(N) = \mathrm{sing}_Q(N)$. The proof is complete. □

Remark 6.4.2. As a corollary of Proposition 6.4.1, one easily deduces the following: if N is a Jac-minimizing Q-valued vector field in the open and connected subset $\Omega \subset \Sigma$ which is not of the form $N = Q\llbracket\zeta\rrbracket$ for some classical Jacobi field ζ, then $D_Q(N) = \mathrm{sing}_Q(N)$, that is all multiplicity Q points are singular.

We have now all the tools that are needed to prove Theorem 6.0.3.

Proof of Theorem 6.0.3. Since our manifolds are always assumed to be second-countable spaces, Ω can have at most countably many connected components, and these connected components are open. Hence, there is no loss of generality in assuming that Ω itself is connected: in the general case, we would just work on each connected component separately.

The fact that $\mathrm{sing}(N)$ is a relatively closed set in Ω (whereas $\mathrm{reg}(N)$ is open) is an immediate consequence of Definition 6.0.2. Let σ be the function defined in (6.153). If $x \in \Omega$ is a regular point, then it is clear that σ is continuous at x. On the other hand, assume x is a point of continuity for σ, and write $N(x) = \sum_{j=1}^J k_j\llbracket P^j\rrbracket$, where the k_j's are integers such that $\sum_{j=1}^J k_j = Q$, each $P^j \in T^\perp_x\Sigma$ and $P^i \neq P^j$ if $i \neq j$. Since the target of σ is discrete, the fact that σ is continuous at x implies that in fact $\sigma(z) = J$ for all z in a neighborhood U of x. Hence, since N is continuous, there exists a neighborhood $x \in V \subset U$ such that the map $N|_V$ admits a continuous decomposition $N(z) = \sum_{j=1}^J k_j\llbracket N^j(z)\rrbracket$, where each map $N^j\colon V \to \mathcal{N}\Sigma$ is a classical Jacobi field. Therefore, $x \in \mathrm{reg}(N)$.

The above argument implies that $\mathrm{sing}(N)$ coincides with the set of points where σ is discontinuous. The proof that its Hausdorff dimension cannot exceed $m-2$ will be obtained via induction on Q. If $Q = 1$, there is nothing to prove, since single-valued Jac-minimizers are classical Jacobi fields. Assume now that the theorem holds for every Q^*-valued Jac-minimizer with $Q^* < Q$ and we prove it for N. If $N \equiv Q\llbracket\zeta\rrbracket$ with ζ a classical Jacobi field, then $\mathrm{sing}(N)$ is empty, and the theorem follows. Assume, therefore, this is not the case. By Proposition 6.4.1, the set $D_Q(N) = \mathrm{sing}_Q(N)$ is a closed subset of Ω which is at most countable if $m = 2$ and has Hausdorff dimension at most $m-2$ if $m \geqslant 3$. Consider now

the open set $\Omega' := \Omega \setminus D_Q(N)$. Since N is continuous, we can find countably many open geodesic balls $\mathbf{B}_j$ such that $\Omega' = \bigcup_j \mathbf{B}_j$ and $N|_{\mathbf{B}_j}$ can be written as the superposition of two multiple-valued functions:

$$N|_{\mathbf{B}_j} = [\![N_{j,Q_1}]\!] + [\![N_{j,Q_2}]\!] \qquad \text{with } Q_1 < Q, Q_2 < Q \tag{6.159}$$

and

$$\operatorname{spt}(N_{j,Q_1}(x)) \cap \operatorname{spt}(N_{j,Q_2}(x)) = \emptyset \qquad \text{for every } x \in \mathbf{B}_j. \tag{6.160}$$

From this last condition, it follows that

$$\operatorname{sing}(N) \cap \mathbf{B}_j = \operatorname{sing}(N_{j,Q_1}) \cup \operatorname{sing}(N_{j,Q_2}). \tag{6.161}$$

The maps N_{j,Q_1} and N_{j,Q_2} are both Jac-minimizing, and thus, by inductive hypothesis, their singular set has Hausdorff dimension at most $m-2$, and is at most countable if $m = 2$. Finally:

$$\operatorname{sing}(N) = \operatorname{sing}_Q(N) \cup \bigcup_{j=1}^{\infty} \left(\operatorname{sing}(N_{j,Q_1}) \cup \operatorname{sing}(N_{j,Q_2})\right) \tag{6.162}$$

also has Hausdorff dimension at most $m-2$ and is at most countable if $m = 2$. □

7 UNIQUENESS OF TANGENT MAPS IN DIMENSION 2

This last chapter dedicated to the theory of multiple-valued Jacobi fields is devoted to the proof of the following result.

Theorem 7.0.1 (Uniqueness of the tangent map at collapsed singularities)**.** *Let* $\Sigma \hookrightarrow \mathcal{M}$ *be as in Assumption 4.1.1, with* $m = \dim \Sigma = 2$*. Let* $N \in \Gamma^{1,2}_Q(\mathcal{N}\Omega)$ *be* Jac*-minimizing in the open set* $\Omega \subset \Sigma^2$*, and assume, without loss of generality, that* $\boldsymbol{\eta} \circ N \equiv 0$*. Let* $p \in \Omega$ *be such that* $N(p) = Q[\![0]\!]$ *but* N *does not vanish in a neighborhood of* p*. Then, there is a* unique *tangent map* $\mathscr{N}_p$ *to* N *at* p *(that is, the blow-up family* $N_{p,r}$ *defined in* (6.125) *converges locally uniformly to* $\mathscr{N}_p$*).*

Theorem 7.0.1 has the following natural corollary.

Corollary 7.0.2. *Let* Ω *be an open subset of the two-dimensional manifold* $\Sigma^2 \hookrightarrow \mathcal{M}$ *as in Assumption 4.1.1, and let* $N \in \Gamma^{1,2}_Q(\mathcal{N}\Omega)$ *be* Jac*-minimizing. Then, for every* $p \in \Omega$ *there exists a unique tangent map* $\mathscr{N}_p$ *to* N *at* p*.*

Proof. The proof is by induction on $Q \geqslant 1$. If $Q = 1$ then the result is trivial, since N is a classical Jacobi field. Let us then assume that the claim holds true for every $Q' < Q$, and we prove that it holds true for Q as well. Let $N \in \Gamma^{1,2}_Q(\mathcal{N}\Omega)$ be Jac-minimizing, and let $p \in \Omega$. Without loss of generality, assume that $\boldsymbol{\eta} \circ N \equiv 0$. If $\operatorname{diam}(N(p)) > 0$, then, since N is continuous, there exists a neighborhood U of p in Ω such that $N|_U = [\![N^1]\!] + [\![N^2]\!]$, where each $N^i \in \Gamma^{1,2}_{Q_i}(\mathcal{N}U)$ is Jac-minimizing, $Q_i < Q$ for $i = 1, 2$ and $\operatorname{spt}(N^1(x)) \cap \operatorname{spt}(N^2(x)) = \emptyset$ for every $x \in U$. By induction hypothesis, N^1 and N^2 have unique tangent maps $\mathscr{N}^1_p$ and $\mathscr{N}^2_p$ at p respectively. Hence, $\mathscr{N}_p := [\![\mathscr{N}^1_p]\!] + [\![\mathscr{N}^2_p]\!]$ is the unique tangent map to N at p.

If, on the other hand, $\operatorname{diam}(N(p)) = 0$, $N(p) = Q[\![0]\!]$ because of the hypotheses on N. If $N \equiv Q[\![0]\!]$ in a neighborhood of p, then the unique tangent map to N at p is $\mathscr{N}_p \equiv Q[\![0]\!]$. If N does not vanish identically in any neighborhood of p, then the tangent map $\mathscr{N}_p$ is unique by Theorem 7.0.1. In either case, this completes the proof of the corollary. □

It only remains to prove Theorem 7.0.1. The plan is the following: first, in Section 7.1 we show that under the assumptions of Theorem 7.0.1 the frequency function $\mathbf{I}_r = \mathbf{I}_{N,p}(r)$ converges, as $r \downarrow 0$, to its limit $I_0 = I_0(p) > 0$ with rate r^β for some $\beta > 0$ (cf. Proposition 7.1.1 below). Then, we will use this key fact to deduce Theorem 7.0.1 in Section 7.2.

7.1 DECAY OF THE FREQUENCY FUNCTION

The main result of this section is the following proposition. Recall the definitions of the energy function $\mathbf{D}(r)$, the height function $\mathbf{H}(r)$ and the frequency function $\mathbf{I}(r)$.

Proposition 7.1.1. *Let* $N \in \Gamma^{1,2}_Q(\mathcal{N}\Omega)$ *be* Jac*-minimizing in* $\Omega \subset \Sigma^2$*, and let* p *be such that* $N(p) = Q[\![0]\!]$ *but* N *does not vanish in a neighborhood of* p*. Let* $I_0 := I_0(p) > 0$ *be the frequency of* N *at* p *(which exists and is strictly positive by Proposition 6.2.10). Then, there are* $\beta, C, D_0, H_0 > 0$ *such that for every* r *sufficiently small one has*

$$|\mathbf{I}(r) - I_0| + \left|\frac{\mathbf{H}(r)}{r^{2I_0+1}} - H_0\right| + \left|\frac{\mathbf{D}(r)}{r^{2I_0}} - D_0\right| \leqslant Cr^\beta\,. \tag{7.1}$$

We will need a preliminary lemma.

Lemma 7.1.2. *Let* N *and* p *be as in Proposition 7.1.1. For every* $\mu > 0$ *there exists* $\beta_0 = \beta_0(\mu)$ *and* $C = C(\mu)$ *such that for every* $0 < \beta < \beta_0$ *the inequality*

$$\mathbf{D}(r) \leqslant \frac{r}{2(2\mu+\beta)}\mathbf{D}'(r) + \frac{\mu(\mu+\beta)}{r(2\mu+\beta)}\mathbf{H}(r) + C_\mu r\mathbf{D}(r) \tag{7.2}$$

holds true for every r *small enough.*

Proof. Let $r_0 < \mathrm{inj}(\Sigma)$ be a radius such that $\mathbf{I}(r) = \mathbf{I}_{N,p}(r)$ is well defined and bounded in $\mathbf{B}_r(p)$ for every $0 < r < \min\{r_0, \mathrm{dist}(p, \partial\Omega)\}$. Recall that for every $0 < r < \min\{r_0, \mathrm{dist}(p, \partial\Omega)\}$ the exponential map $\exp_p$ defines a diffeomorphism $\exp_p : \mathbb{D}_r \to \mathbf{B}_r(p)$, where $\mathbb{D}_r$ is the disk of radius r in $\mathbb{R}^2 \simeq \mathbb{C}$. Let $g := N \circ \exp_p : \mathbb{D}_r \to \mathcal{A}_Q(\mathbb{R}^d)$, and let $f \in W^{1,2}(\mathbb{D}_r, \mathcal{A}_Q(\mathbb{R}^d))$ be the "harmonic extension" of $g|_{r\mathbb{S}^1}$ already considered in Section 5.1. In particular, let $\varphi(\theta) := g(re^{i\theta})$, and let $\varphi = \sum_{\ell=1}^P [\![\varphi_\ell]\!]$ be an irreducible decomposition of φ in maps $\varphi_\ell \in W^{1,2}(\mathbb{S}^1, \mathcal{A}_{Q_\ell}(\mathbb{R}^d))$ such that for some $\gamma_\ell \in W^{1,2}(\mathbb{S}^1, \mathbb{R}^d)$

$$\varphi(\theta) = \sum_{\ell=1}^P \sum_{m=0}^{Q_\ell - 1} \left[\!\!\left[\gamma_\ell\left(\frac{\theta + 2\pi m}{Q_\ell}\right)\right]\!\!\right]\,.$$

Such an irreducible decomposition exists by [DLS11, Proposition 1.5]. Then, if the Fourier expansions of the γ_ℓ's are given by

$$\gamma_\ell(\theta) = \frac{a_{\ell,0}}{2} + \sum_{n=1}^\infty r^n\left(a_{\ell,n}\cos(n\theta) + b_{\ell,n}\sin(n\theta)\right)\,,$$

we consider their harmonic extensions to $\mathbb{D}_r$, namely the functions defined by

$$\zeta_\ell(\rho,\theta) := \frac{a_{\ell,0}}{2} + \sum_{n=1}^\infty \rho^n\left(a_{\ell,n}\cos(n\theta) + b_{\ell,n}\sin(n\theta)\right) \quad \text{for every } 0 < \rho \leqslant r\,,$$

and finally we let

$$f(\rho e^{i\theta}) := \sum_{\ell=1}^P \sum_{m=0}^{Q_\ell} \left[\!\!\left[\zeta_\ell\left(\rho^{1/Q_\ell}, \frac{\theta + 2\pi m}{Q_\ell}\right)\right]\!\!\right] \quad \text{for } \rho e^{i\theta} \in \mathbb{D}_r\,.$$

Recalling [DLS11, Lemma 3.12], one can explicitly compute the following quantities:

$$\int_{\mathbb{D}_r} |Df|^2 = \sum_{\ell=1}^{P} \mathrm{Dir}(\zeta_\ell, \mathbb{D}_r) = \pi \sum_{\ell=1}^{P} \sum_{n=1}^{\infty} r^{2n} n \left(|a_{\ell,n}|^2 + |b_{\ell,n}|^2\right), \tag{7.3}$$

$$\int_{r\mathbb{S}^1} |\partial_\tau f|^2 = \sum_{\ell=1}^{P} \mathrm{Dir}(\varphi_\ell, r\mathbb{S}^1) = \frac{1}{r} \sum_{\ell=1}^{P} \frac{1}{Q_\ell} \int_0^{2\pi} |\gamma_\ell'(\alpha)|^2 \, d\alpha = \pi \sum_{\ell=1}^{P} \sum_{n=1}^{\infty} \frac{r^{2n-1} n^2}{Q_\ell} \left(|a_{\ell,n}|^2 + |b_{\ell,n}|^2\right), \tag{7.4}$$

$$\int_{r\mathbb{S}^1} |f|^2 = r \sum_{\ell=1}^{P} Q_\ell \int_0^{2\pi} |\gamma_\ell(\alpha)|^2 \, d\alpha = \pi \sum_{\ell=1}^{P} Q_\ell \left(\frac{r|a_{\ell,0}|^2}{2} + \sum_{n=1}^{\infty} r^{2n+1} \left(|a_{\ell,n}|^2 + |b_{\ell,n}|^2\right) \right), \tag{7.5}$$

where ∂_τ denotes the tangential derivative along $r\mathbb{S}^1$.

Now, it is an elementary fact (cf. [DLS11, proof of Proposition 5.2]) that for any $\mu > 0$ there exists $\beta_0 = \beta_0(\mu) > 0$ such that for every $0 < \beta < \beta_0$ it holds

$$(2\mu + \beta) n \leqslant \frac{n^2}{Q_\ell} + \mu(\mu + \beta) Q_\ell \quad \text{for every } n \in \mathbb{N} \text{ and for every } Q_\ell \,. \tag{7.6}$$

Multiplying (7.6) by $\pi r^{2n} \left(|a_{\ell,n}|^2 + |b_{\ell,n}|^2\right)$ and summing over n and ℓ, we obtain from (7.3), (7.4), and (7.5) that for every $\mu > 0$ there exists $\beta_0 > 0$ such that for every $0 < \beta < \beta_0$

$$(2\mu + \beta) \int_{\mathbb{D}_r} |Df|^2 \leqslant r \int_{r\mathbb{S}^1} |\partial_\tau g|^2 + \frac{\mu(\mu + \beta)}{r} \int_{r\mathbb{S}^1} |g|^2 \,. \tag{7.7}$$

Now, let $u := f \circ \exp_p^{-1} : \mathbf{B}_r(p) \to \mathcal{A}_Q(\mathbb{R}^d)$, so that the orthogonal projection $u^\perp$ is a map in $\Gamma^{1,2}_Q(\mathcal{N}\mathbf{B}_r(p))$ with $u^\perp\big|_{r\mathbb{S}^1} = N|_{r\mathbb{S}^1}$. The minimality of N then implies that

$$\begin{aligned} \mathrm{Jac}(N, \mathbf{B}_r(p)) &\leqslant \mathrm{Jac}(u^\perp, \mathbf{B}_r(p)) \\ &\leqslant \mathrm{Dir}(u^\perp, \mathbf{B}_r(p)) + C_0 \int_{\mathbf{B}_r(p)} |u|^2 \\ &\overset{(5.51)}{\leqslant} (1 + r)\mathrm{Dir}(u, \mathbf{B}_r(p)) + \frac{C}{r} \int_{\mathbf{B}_r(p)} |u|^2 \,, \end{aligned}$$

from which in turn follows

$$\mathbf{D}(r) \leqslant (1 + r)\mathrm{Dir}(u, \mathbf{B}_r(p)) + \frac{C}{r} \int_{\mathbf{B}_r(p)} |u|^2 + C_0 \int_{\mathbf{B}_r(p)} |N|^2 \,.$$

Using that the metric of Σ in $\mathbf{B}_r(p)$ is almost euclidean when $r \to 0$, we conclude that for small r's

$$\mathbf{D}(r) \leqslant (1 + Cr) \left[(1 + r)\mathrm{Dir}(f, \mathbb{D}_r) + \frac{C}{r} \int_{\mathbb{D}_r} |f|^2 \right] + C_0 \int_{\mathbf{B}_r(p)} |N|^2 \,.$$

Now, by definition of f one has

$$\int_{\mathbb{D}_r} |f|^2 \leqslant Cr \int_{r\mathbb{S}^1} |g|^2 \leqslant Cr(1 + Cr) \int_{\partial\mathbf{B}_r(p)} |N|^2 \,. \tag{7.8}$$

Combining (7.8) with (7.7), we deduce that for every $\mu > 0$ there exists $\beta_0 = \beta_0(\mu)$ such that for every $0 < \beta < \beta_0$

$$\mathbf{D}(r) \leqslant (1 + Cr)\left[\frac{r}{2\mu+\beta}\int_{\partial \mathbf{B}_r(p)} |\nabla^{\perp}_{\tau} N|^2 + \frac{\mu(\mu+\beta)}{r(2\mu+\beta)}\mathbf{H}(r)\right] + C_\mu \mathbf{H}(r) + C_0 \int_{\mathbf{B}_r(p)} |N|^2 . \quad (7.9)$$

Next, observe that the inner variation formula (6.70) together with the Poincaré inequality (6.96) imply that in dimension $m = 2$

$$\left| \int_{\partial B_r(p)} |\nabla^{\perp}_{\tau} N|^2 - \frac{\mathbf{D}'(r)}{2} \right| \leqslant Cr\mathbf{D}(r) ,$$

and thus, since, again by the Poincaré inequality

$$\int_{\mathbf{B}_r(p)} |N|^2 \leqslant Cr^2 \mathbf{D}(r) ,$$

equation (7.9) reads

$$\mathbf{D}(r) \leqslant (1 + Cr)\left[\frac{r}{2(2\mu+\beta)}\mathbf{D}'(r) + \frac{\mu(\mu+\beta)}{r(2\mu+\beta)}\mathbf{H}(r)\right] + C_\mu \mathbf{H}(r) + C_\mu r^2 \mathbf{D}(r) . \quad (7.10)$$

Finally, divide by $1 + Cr$ and use that $\mathbf{H}(r) \leqslant \frac{2}{I_0} r\mathbf{D}(r)$ for small r's to finally conclude the validity of (7.2). □

Proof of Proposition 7.1.1. Let N and p be as in the statement, and fix a suitably small radius $r_0 > 0$. In particular, take $r_0 < \min\{\mathrm{inj}(\Sigma), \mathrm{dist}(p, \partial\Omega)\}$ such that the conclusions of Proposition 6.2.10 hold. Recall from the aforementioned proposition that there exists $\lambda > 0$ such that the function $r \in (0, r_0) \mapsto e^{\lambda r}\mathbf{I}(r)$ is monotone non-decreasing. As an immediate corollary we deduce that when r is small enough

$$\mathbf{I}(r) - I_0 \geqslant -Cr . \quad (7.11)$$

The goal now is to get the upper bound. In order to do this, first we exploit the variation estimates deduced in Lemma 6.2.13 to compute again the derivative

$$\begin{aligned}
\mathbf{I}'(r) &= \frac{\mathbf{D}(r)}{\mathbf{H}(r)} + \frac{r\mathbf{D}'(r)}{\mathbf{H}(r)} - \frac{r\mathbf{D}(r)\mathbf{H}'(r)}{\mathbf{H}(r)^2} \\
&= \frac{r\mathbf{D}'(r)}{\mathbf{H}(r)} - 2\frac{r\mathbf{D}(r)\mathbf{E}(r)}{\mathbf{H}(r)^2} - \frac{r\mathbf{D}(r)}{\mathbf{H}(r)^2}\mathcal{E}_{(6.71)}(r) \\
&= \frac{r\mathbf{D}'(r)}{\mathbf{H}(r)} - 2\frac{r\mathbf{D}(r)^2}{\mathbf{H}(r)^2} - \frac{r\mathbf{D}(r)}{\mathbf{H}(r)^2}\left(\mathcal{E}_{(6.71)}(r) + 2\mathcal{E}_{(6.69)}(r)\right) ,
\end{aligned}$$

where

$$|\mathcal{E}_{(6.71)}(r)| \overset{(6.71)}{\leqslant} C_0 r\mathbf{H}(r) , \quad (7.12)$$

$$|\mathcal{E}_{(6.69)}(r)| = |\mathbf{D}(r) - \mathbf{E}(r)| \overset{(6.69),(6.96)}{\leqslant} C_0 r^2 \mathbf{D}(r) . \quad (7.13)$$

Now, apply the estimate (7.2) from the previous lemma with $\mu = I_0$ to deduce that for every $0 < \beta < \beta_0(I_0)$ one has

$$\frac{r\mathbf{D}'(r)}{\mathbf{H}(r)} - 2\frac{r\mathbf{D}(r)^2}{\mathbf{H}(r)^2} \geqslant \frac{2}{r}\left(I_0 + \beta - \mathbf{I}(r)\right)\left(\mathbf{I}(r) - I_0\right) - 2C_{I_0}(2I_0 + \beta)\mathbf{I}(r)\,,$$

so that, recalling that $\mathbf{I}(r) \leqslant C$, we can finally estimate

$$\mathbf{I}'(r) \geqslant \frac{2}{r}\left(I_0 + \beta - \mathbf{I}(r)\right)\left(\mathbf{I}(r) - I_0\right) - C\mathbf{I}(r)\,. \tag{7.14}$$

Hence, if we fix $0 < \beta < \beta_0(I_0)$ we easily conclude

$$\begin{aligned}\frac{d}{dr}\left[\frac{\mathbf{I}(r) - I_0}{r^\beta}\right] &= \frac{\mathbf{I}'(r)}{r^\beta} - \beta\frac{\mathbf{I}(r) - I_0}{r^{\beta+1}} \\ &\geqslant \frac{1}{r^{\beta+1}}\left(2I_0 + \beta - 2\mathbf{I}(r)\right)\left(\mathbf{I}(r) - I_0\right) - \frac{C}{r^\beta} \geqslant -\frac{C}{r^\beta}\,,\end{aligned} \tag{7.15}$$

for all radii $0 < r < r_0(\beta)$.

Integrating in $[r, r_0]$ we conclude

$$\frac{\mathbf{I}(r_0) - I_0}{r_0^\beta} - \frac{\mathbf{I}(r) - I_0}{r^\beta} \geqslant -Cr_0^{1-\beta}\,, \tag{7.16}$$

that is

$$\mathbf{I}(r) - I_0 \leqslant Cr^\beta\,. \tag{7.17}$$

This concludes the proof of

$$|\mathbf{I}(r) - I_0| \leqslant Cr^\beta \tag{7.18}$$

To get the other estimates, compute

$$\begin{aligned}\frac{d}{dr}\left[\log\left(\frac{\mathbf{H}(r)}{r^{2I_0+1}}\right)\right] &= \frac{\mathbf{H}'(r)}{\mathbf{H}(r)} - \frac{2I_0+1}{r} = \frac{2\mathbf{E}(r)}{\mathbf{H}(r)} + \frac{1}{\mathbf{H}(r)}\mathcal{E}_{(6.71)}(r) - \frac{2I_0}{r} \\ &= \frac{2}{r}\left(\mathbf{I}(r) - I_0\right) + \frac{1}{\mathbf{H}(r)}\left(\mathcal{E}_{(6.71)}(r) + 2\mathcal{E}_{(6.69)}(r)\right)\,,\end{aligned}$$

with $\mathcal{E}_{(6.71)}(r)$ and $\mathcal{E}_{(6.69)}(r)$ satisfying the same bounds as in (7.12), (7.13). Using that $\mathbf{I}(r) \leqslant C$, this allows to conclude that

$$\frac{2}{r}\left(\mathbf{I}(r) - I_0\right) - Cr \leqslant \frac{d}{dr}\left[\log\left(\frac{\mathbf{H}(r)}{r^{2I_0+1}}\right)\right] \leqslant \frac{2}{r}\left(\mathbf{I}(r) - I_0\right) + Cr\,. \tag{7.19}$$

After applying (7.18), integrating on $0 < s < r < r_0$ and taking exponentials, we therefore obtain the estimate

$$e^{-C_\beta(r^\beta - s^\beta)} \leqslant \frac{\mathbf{H}(r)}{r^{2I_0+1}}\frac{s^{2I_0+1}}{\mathbf{H}(s)} \leqslant e^{C_\beta(r^\beta - s^\beta)}\,. \tag{7.20}$$

In particular, (7.20) implies that the map

$$r \in (0, r_0) \mapsto \frac{\mathbf{H}(r)e^{-C_\beta r^\beta}}{r^{2I_0+1}}$$

is monotone non-increasing. In turn, from this immediately follows the existence of the limit

$$H_0 := \lim_{r\to 0} \frac{\mathbf{H}(r)}{r^{2I_0+1}}\,.$$

The rate of convergence

$$\left|\frac{\mathbf{H}(r)}{r^{2I_0+1}} - H_0\right| \leqslant Cr^\beta \quad \text{for } r \text{ small enough}$$

is also a standard consequence of (7.20).

Finally, we set $D_0 := I_0 \cdot H_0$ and immediately obtain

$$\left|\frac{\mathbf{D}(r)}{r^{2I_0}} - D_0\right| = \left|\mathbf{I}(r)\frac{\mathbf{H}(r)}{r^{2I_0+1}} - I_0\cdot H_0\right| \leqslant |\mathbf{I}(r)-I_0|\,\frac{\mathbf{H}(r)}{r^{2I_0+1}} + I_0\left|\frac{\mathbf{H}(r)}{r^{2I_0+1}} - H_0\right| \overset{(7.18),(7.20)}{\leqslant} Cr^\beta\,. \tag{7.21}$$

□

7.2 UNIQUENESS OF THE TANGENT MAP AT COLLAPSED SINGULARITIES

We are now ready to prove Theorem 7.0.1.

Proof of Theorem 7.0.1. Let N and p be as in the statement, and recall the definition of the blow-up maps $N_r = N_{p,r}$ given in (6.125) (together with the definitions of the maps $\mathbf{ex}_r$ and $\psi_r = \psi_{p,r}$ used in there). We first remark that by the Poincaré inequality (6.96) and the reverse Poincaré inequality (6.117) any convergence result for the maps N_r as $r \downarrow 0$ is equivalent to the same result obtained for the maps

$$\tilde N_r(y) := \frac{r^{\frac{m-2}{2}} N(\psi_r(y))}{\sqrt{\mathbf{D}(r)}}\,.$$

Let us assume without loss of generality that $D_0 = 1$. Then, in dimension $m = 2$ the decay estimate (7.21) implies that for $r \downarrow 0$

$$\tilde N_r(z) = r^{-I_0} N(\psi_r(z))\left(1 + o(r^{\beta/2})\right)\,, \tag{7.22}$$

and therefore in order to show the existence of a uniform limit for the maps $\tilde N_r$ in $\mathbb{D}_1$ it suffices to show the existence of a uniform limit for the maps $\tilde u_r(z) := r^{-I_0} N(\psi_r(z))$. Furthermore, if we write $z = \rho e^{i\theta} \in \mathbb{D}_1$ we see immediately that

$$\tilde u_r(\rho e^{i\theta}) = r^{-I_0} N(\psi_r(\rho e^{i\theta})) = \rho^{I_0}(\rho r)^{-I_0} N(\psi_{\rho r}(e^{i\theta})) = \rho^{I_0}\tilde u_{\rho r}(e^{i\theta})\,,$$

and thus our goal will be achieved if we show uniform convergence of the maps $\tilde u_r|_{S^1}$. For the sake of notational simplicity we will then remove the tilde, call $w = e^{i\theta}$ the variable on S^1 and consider the one-parameter family of maps $u_r \colon S^1 \to \mathcal{A}_Q(\mathbb{R}^d)$ given by

$$u_r(w) = r^{-I_0} N(\psi_r(w))\,.$$

We then fix $\frac{r}{2} \leqslant s \leqslant r$ and compute

$$\begin{aligned}\int_{S^1} \mathcal{G}(u_r, u_s)^2 \, d\mathcal{H}^1 &= \int_{S^1} \mathcal{G}\left(\frac{N(\psi_r(w))}{r^{I_0}}, \frac{N(\psi_s(w))}{s^{I_0}}\right)^2 d\mathcal{H}^1(w) \\ &\leqslant \int_{S^1} \sum_{\ell=1}^{Q} \left(\int_s^r \left|\frac{d}{dt}\left(\frac{N^\ell(\psi_t(w))}{t^{I_0}}\right)\right| dt\right)^2 d\mathcal{H}^1(w) \\ &\leqslant (r-s)\int_{S^1}\int_s^r \sum_{\ell=1}^{Q} \left|\frac{d}{dt}\left(\frac{N^\ell(\psi_t(w))}{t^{I_0}}\right)\right|^2 dt\, d\mathcal{H}^1(w).\end{aligned} \tag{7.23}$$

Note that in the above computation we have used [DLS11, Proposition 1.2] and the fact that the map $t \in (s,r) \mapsto \frac{N(\psi_t(w))}{t^{I_0}}$ is in $W^{1,2}$ for a.e. $w \in S^1$.

Now, we have

$$\frac{d}{dt}\left(\frac{N^\ell(\psi_t(w))}{t^{I_0}}\right) = \frac{DN^\ell(\psi_t(w)) \cdot d\exp_p\big|_{tw}(w)}{t^{I_0}} - I_0 \frac{N^\ell(\psi_t(w))}{t^{I_0+1}},$$

and thus

$$\begin{aligned}\left|\frac{d}{dt}\left(\frac{N^\ell(\psi_t(w))}{t^{I_0}}\right)\right|^2 \leqslant &\frac{|\nabla^\perp_{\hat r} N^\ell(\psi_t(w))|^2}{t^{2I_0}} + I_0^2 \frac{|N^\ell(\psi_t(w))|^2}{t^{2I_0+2}} - 2I_0 \frac{\langle \nabla^\perp_{\hat r} N^\ell(\psi_t(w)), N^\ell(\psi_t(w))\rangle}{t^{2I_0+1}} \\ &+ Err,\end{aligned}$$

where

$$Err \leqslant Ct^{1-2I_0}|\nabla^\perp_{\hat r} N^\ell(\psi_t(w))|^2 + Ct^{-2I_0}|N^\ell(\psi_t(w))|^2$$

for small t.

Inserting in (7.23) and changing variable $x = \psi_t(w)$ we easily obtain from the variation estimates in Lemma 6.2.13:

$$\begin{aligned}\int_{S^1} \mathcal{G}(u_r, u_s)^2 d\mathcal{H}^1 \leqslant &(r-s)(1+Cr)\int_s^r \frac{\mathbf{G}(t)}{t^{2I_0+1}} + I_0^2\frac{\mathbf{H}(t)}{t^{2I_0+3}} - 2I_0\frac{\mathbf{E}(t)}{t^{2I_0+2}}\, dt \\ &+ C(r-s)\int_s^r \frac{\mathbf{G}(t)}{t^{2I_0}} + \frac{\mathbf{H}(t)}{t^{2I_0+1}}\, dt \\ = &\underbrace{(r-s)(1+Cr)\int_s^r \frac{\mathbf{D}'(t)}{2t^{2I_0+1}} + I_0^2\frac{\mathbf{H}(t)}{t^{2I_0+3}} - 2I_0\frac{\mathbf{D}(t)}{t^{2I_0+2}}\, dt}_{=:A} \\ &\underbrace{+ C(r-s)\int_s^r \frac{\mathbf{G}(t)}{t^{2I_0}} + \frac{\mathbf{H}(t)}{t^{2I_0+1}}\, dt}_{=:E_1} \\ &\underbrace{+ (r-s)(1+Cr)\int_s^r \frac{\mathcal{E}_{(6.70)}(t)}{t^{2I_0+1}} + \frac{\mathcal{E}_{(6.69)}(t)}{t^{2I_0+2}}\, dt}_{=:E_2}.\end{aligned} \tag{7.24}$$

Now, we have

$$\begin{aligned}A &= (r-s)(1+Cr)\int_s^r \frac{1}{2t}\left(\frac{\mathbf{D}(t)}{t^{2I_0}}\right)' + I_0^2\frac{\mathbf{H}(t)}{t^{2I_0+3}} - I_0\frac{\mathbf{D}(t)}{t^{2I_0+2}}\, dt \\ &= (r-s)(1+Cr)\int_s^r \frac{1}{2t}\left(\frac{\mathbf{D}(t)}{t^{2I_0}}\right)' + I_0\frac{\mathbf{H}(t)}{t^{2I_0+3}}\,(I_0 - \mathbf{I}(t))\, dt,\end{aligned} \tag{7.25}$$

so that, for $s = \frac{r}{2}$

$$A \leqslant C \left| \frac{\mathbf{D}(r)}{r^{2I_0}} - \frac{\mathbf{D}(r/2)}{(r/2)^{2I_0}} \right| + C \int_{r/2}^{r} \frac{I_0 - \mathbf{I}(t)}{t} \, dt \overset{(7.21),(7.18)}{\leqslant} Cr^{\beta} . \tag{7.26}$$

For what concerns the error terms, we can use the variation estimates (6.69) and (6.70) together with the Poincaré inequality (6.96) to control

$$|E_2| \leqslant C \frac{r}{2} \int_{r/2}^{r} \frac{\mathbf{D}(t)}{t^{2I_0}} \, dt \overset{(7.21)}{\leqslant} Cr^2 , \tag{7.27}$$

and

$$\begin{aligned} |E_1| &\leqslant C \frac{r}{2} \int_{r/2}^{r} \frac{\mathbf{D}'(t)}{t^{2I_0}} + \frac{|\mathcal{E}_{(6.70)}(t)|}{t^{2I_0}} + \frac{\mathbf{H}(t)}{t^{2I_0+1}} \, dt \\ &\leqslant Cr^{1-2I_0} |\mathbf{D}(r) - \mathbf{D}(r/2)| + Cr^2 \int_{r/2}^{r} \frac{\mathbf{D}(t)}{t^{2I_0}} \, dt + Cr \int_{r/2}^{r} \frac{\mathbf{H}(t)}{t^{2I_0+1}} \, dt \\ &\overset{(7.21),(7.20)}{\leqslant} Cr^{1+\beta} . \end{aligned} \tag{7.28}$$

Plugging (7.26), (7.27) and (7.28) in (7.24) we conclude that

$$\int_{S^1} \mathcal{G}(u_r, u_{\frac{r}{2}})^2 \, d\mathcal{H}^1 \leqslant Cr^{\beta} . \tag{7.29}$$

With an elementary dyadic argument analogous to [DLS11, proof of Theorem 5.3], we conclude that the family u_r is L^2-Cauchy. Since the u_r's are equi-Hölder (cf. (6.133)), this suffices to conclude uniform convergence to a unique limit.

□

Part II

Multiple-valued sections of vector bundles and applications

8 MULTIPLE-VALUED SECTIONS OF VECTOR BUNDLES

In this chapter we introduce the notion of Q-multisection of an abstract vector bundle over a Riemannian manifold, developing some ideas contained in the unpublished note [All13] by W. Allard. The techniques contained in this chapter will be then applied in Chapter 9 to provide a new proof of the reparametrization theorem for Lipschitz Q-valued functions.

8.1 PRELIMINARY DEFINITIONS

In what follows, $\Sigma = \Sigma^m$ denotes an m-dimensional Riemannian manifold of class C^1, and E is an $(m+n)$-dimensional manifold which is the total space of a vector bundle $\Pi\colon E \to \Sigma$ of rank n and class C^1 over the base manifold Σ. Following standard notations, we will denote by $E_p = \Pi^{-1}(\{p\})$ the *fiber* over the base point $p \in \Sigma$. We will let $\{(U_\alpha, \Psi_\alpha)\}_{\alpha\in I}$ be a locally finite family of *local trivializations* of the bundle: thus, $\{U_\alpha\}$ is a locally finite open covering of the manifold Σ, and

$$\Psi_\alpha\colon \Pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^n$$

are differentiable maps satisfying:

(i) $\mathbf{p}_1 \circ \Psi_\alpha = \Pi|_{\Pi^{-1}(U_\alpha)}$, where $\mathbf{p}_1\colon U_\alpha \times \mathbb{R}^n \to U_\alpha$ is the projection on the first factor;

(ii) for any $\alpha, \beta \in I$ with $U_\alpha \cap U_\beta \neq \emptyset$, there exists a differentiable map

$$\tau_{\alpha\beta}\colon U_\alpha \cap U_\beta \to \mathrm{GL}(n, \mathbb{R})$$

with the property that

$$\Psi_\alpha \circ \Psi_\beta^{-1}(p, \nu) = (p, \tau_{\alpha\beta}(p)\cdot\nu) \qquad \forall\, p \in U_\alpha \cap U_\beta,\ \forall \nu \in \mathbb{R}^n.$$

Without loss of generality, we can assume that each open set U_α is also the domain of a local chart $\psi_\alpha\colon U_\alpha \to \mathbb{R}^m$ on Σ.

Let now Q be an integer, $Q \geqslant 1$. We adopt the convention that the set $\mathbb{N}$ of natural numbers contains zero.

Definition 8.1.1 (Q-valued sections, Allard [All13])**.** Given a vector bundle $\Pi\colon E \to \Sigma$ as above, and a subset $B \subset \Sigma$, a Q-multisection over B is a map

$$M\colon E \to \mathbb{N} \tag{8.1}$$

with the property that

$$\sum_{\xi\in E_p} M(\xi) = Q \qquad \text{for every } p \in B. \tag{8.2}$$

Remark 8.1.2. If $s\colon B \to E$ is a classical local section, then the map $M\colon E \to \mathbb{N}$ defined by

$$M(\xi) := \begin{cases} 1, & \text{if there exists } p \in B \text{ such that } \xi = s(p), \\ 0, & \text{otherwise} \end{cases} \tag{8.3}$$

is evidently a 1-multisection over B, according to Definition 8.1.1. On the other hand, given a 1-multisection M, condition (8.2) ensures that for every $p \in B$ there exists a unique $\xi \in E_p$ such that $M(\xi) > 0$. If such an element ξ is denoted $s(p)$, then the map $p \mapsto s(p)$ defines a classical section of the bundle E over B. Hence, 1-multisections over a subset B are just (possibly rough) sections over B in the classical sense.

The above Remark justifies the name that was adopted for the objects introduced in Definition 8.1.1: Q-multisections are simply the Q-valued counterpart of classical sections of a vector bundle. From a different point of view, we may say that Q-multisections generalize Almgren's Q-valued functions to vector bundle targets. Indeed, Q-valued functions defined on a manifold Σ might be seen as Q-multisections of a trivial bundle over Σ, as specified in the following remark.

Remark 8.1.3. Assume E is the trivial bundle of rank n over Σ, that is $E = \Sigma \times \mathbb{R}^n$ and Π is the projection on the first factor. Then, to any Q-multisection M over Σ it is possible to associate the multiple-valued function $u_M\colon \Sigma \to \mathcal{A}_Q(\mathbb{R}^n)$ defined by

$$u_M(p) := \sum_{\nu \in \mathbb{R}^n} M(p,\nu)[\![\nu]\!]. \tag{8.4}$$

Conversely, if $u\colon \Sigma \to \mathcal{A}_Q(\mathbb{R}^n)$ is a multiple-valued function then one can define the Q-multisection M_u induced by u simply setting

$$M_u(p,\nu) := \Theta_{u(p)}(\nu), \tag{8.5}$$

where $\Theta_{u(p)}(\nu)$ is the multiplicity of the vector ν in $u(p)$.

8.2 COHERENT AND VERTICALLY LIMITED MULTISECTIONS

Definition 8.2.1 (Coherence, Allard [All13])**.** A Q-multisection M of the vector bundle $\Pi\colon E \to \Sigma$ over Σ is said to be *coherent* if the following holds. For every $p \in \Sigma$ and for every disjoint family $\mathcal{V}$ of open sets $V \subset E$ such that each member $V \in \mathcal{V}$ contains exactly one element of $M_p := \{\xi \in E_p : M(\xi) > 0\}$, there is an open neighborhood U of p in Σ such that for any $q \in U$

$$\sum_{\zeta \in M_q \cap V} M(\zeta) = M(\xi) \qquad \text{if } \xi \in M_p \cap V. \tag{8.6}$$

The following proposition motivates the necessity of introducing the notion of coherence: it is a way of generalizing the continuity of Q-valued functions in the vector bundle-valued context.

Proposition 8.2.2. *Let $E = \Sigma \times \mathbb{R}^n$ be the trivial bundle of rank n over Σ. Then, a Q-multisection M is coherent if and only if the associated multiple-valued function $u_M \colon \Sigma \to \mathcal{A}_Q(\mathbb{R}^n)$ is continuous.*

Proof. Let $u \colon \Sigma \to \mathcal{A}_Q(\mathbb{R}^n)$ be a continuous Q-valued function, and let $M \colon \Sigma \times \mathbb{R}^n \to \mathbb{N}$ be the induced multisection defined by (8.5). In order to show that M is coherent, fix a point p in the base manifold Σ, and decompose $u(p) = \sum_{j=1}^{J} m_j [\![v_j]\!]$ so that $v_j \neq v_{j'}$ when $j \neq j'$ and $m_j := M(p, v_j)$. Now, let $\mathcal{V} = \{V_1, \dots, V_J\}$ be a disjoint family of open sets $V_j \subset \mathbb{R}^n$ with the property that if $M_p := \{v \in \mathbb{R}^n : M(p, v) > 0\}$ then $M_p \cap V_j = \{v_j\}$. Let $\varepsilon > 0$ be a radius such that $B_\varepsilon(v_j) \subset V_j$ for every $j = 1, \dots, J$. Then, since u is continuous, there exists a neighborhood U of p in Σ such that

$$u(q) \in \mathcal{B}_{\frac{\varepsilon}{2}}(u(p)) := \left\{ T \in \mathcal{A}_Q(\mathbb{R}^n) : \mathcal{G}(T, u(p)) < \frac{\varepsilon}{2} \right\},$$

for every $q \in U$. From the definition of the metric $\mathcal{G}(\cdot, \cdot)$ in $\mathcal{A}_Q(\mathbb{R}^n)$, it follows naturally that for every $q \in U$ it has to be

$$\sum_{w \in V_j} M(q, w) = m_j \qquad \text{for every } j \in \{1, \dots, J\},$$

and thus M is coherent.

Conversely, suppose M is a coherent Q-multisection of the trivial bundle $\Sigma \times \mathbb{R}^n$, and let u be the associated multiple-valued function as defined in (8.4). The goal is to prove that u is continuous. Fix any point $p \in \Sigma$, and let $\{p_h\}_{h=1}^{\infty} \subset \Sigma$ be any sequence such that $p_h \to p$. Since M is coherent, for any ball $B_R \subset \mathbb{R}^n$ such that $\operatorname{spt}(u(p)) \subset B_R$ there exists $h_0 \in \mathbb{N}$ such that $\operatorname{spt}(u(p_h)) \subset B_R$ for every $h \geqslant h_0$. In particular, $|u(p_h)|^2 := \mathcal{G}(u(p_h), Q[\![0]\!])^2 \leqslant QR^2$ for every $h \geqslant h_0$, and thus the measures $\{u(p_h)\}$ have uniformly finite second moment. Therefore, since the metric $\mathcal{G}$ on $\mathcal{A}_Q(\mathbb{R}^n)$ coincides with the L^2-based Wasserstein distance on the space of positive measures with finite second moment, from [AGS08, Proposition 7.1.5] immediately follows that $\mathcal{G}(u(p_h), u(p)) \to 0$ if and only if the sequence $u(p_h)$ *narrowly converges* to $u(p)$, that is if and only if

$$\lim_{h \to \infty} \langle u(p_h), f \rangle = \langle u(p), f \rangle \qquad \forall\, f \in C_b(\mathbb{R}^n), \tag{8.7}$$

that is, explicitly,

$$\lim_{h \to \infty} \sum_{v \in \mathbb{R}^n} M(p_h, v) f(v) = \sum_{v \in \mathbb{R}^n} M(p, v) f(v) \qquad \forall\, f \in C_b(\mathbb{R}^n), \tag{8.8}$$

where $C_b(\mathbb{R}^n)$ denotes the space of bounded continuous functions on $\mathbb{R}^n$. So, in order to prove this, fix $f \in C_b(\mathbb{R}^n)$ and $\varepsilon > 0$. Let $v_1, \dots, v_J$ be distinct points in M_p, and let $\eta = \eta(\varepsilon) > 0$ be a number chosen in such a way that

$$|v - v_j| < \eta \implies |f(v) - f(v_j)| < \varepsilon \qquad \text{for every } j = 1, \dots, J. \tag{8.9}$$

Choose now radii $r_1, \dots, r_J$ such that $r_j < \frac{\eta}{2}$, the balls $B_j := B_{r_j}(v_j)$ are pairwise disjoint and $M(p, v) = 0$ for any $v \in B_j \setminus \{v_j\}$. Since M is coherent, in correspondence with the choice of the family $\{B_j\}$ there is an open neighborhood U of p in Σ with the property that

$$\sum_{v \in B_j} M(q, v) = M(p, v_j) \qquad \text{for every } q \in U. \tag{8.10}$$

Since $\sum_{j=1}^{J} M(p, v_j) = Q$, equation (8.10) implies that

$$\sum_{j=1}^{J} \sum_{v \in B_j} M(q, v) = Q \qquad \text{for every } q \in U, \tag{8.11}$$

and thus, whenever $q \in U$, $M(q, v) = 0$ if $v \notin \bigcup_{j=1}^{J} B_j$. Therefore, only the balls B_j are relevant, namely

$$\sum_{v \in \mathbb{R}^n} M(q, v) = \sum_{j=1}^{J} \sum_{v \in B_j} M(q, v) \qquad \text{for every } q \in U. \tag{8.12}$$

We can now finally conclude the validity of (8.8): Let $N \in \mathbb{N}$ be such that $p_h \in U$ for every $h \geqslant N$ and estimate, for such h's:

$$\begin{aligned}
\left| \sum_{v \in \mathbb{R}^n} M(p_h, v) f(v) - \sum_{v \in \mathbb{R}^n} M(p, v) f(v) \right| &\overset{(8.12)}{=} \left| \sum_{j=1}^{J} \sum_{v \in B_j} M(p_h, v) f(v) - \sum_{j=1}^{J} M(p, v_j) f(v_j) \right| \\
&\leqslant \sum_{j=1}^{J} \left| \sum_{v \in B_j} M(p_h, v) f(v) - M(p, v_j) f(v_j) \right| \\
&\overset{(8.10)}{\leqslant} \sum_{j=1}^{J} \sum_{v \in B_j} M(p_h, v) |f(v) - f(v_j)| \\
&\overset{(8.11)}{\leqslant} Q\varepsilon,
\end{aligned}$$

which completes the proof. □

The next step will be to define a suitable property of Q-multisections that is equivalent to Lipschitz continuity of the associated multiple-valued function whenever such an association is possible. We start from a definition in the easy case when the vector bundle E coincides with the trivial bundle $\Omega \times \mathbb{R}^n$ over an open subset $\Omega \subset \mathbb{R}^m$.

Definition 8.2.3 (τ-cone condition, Allard [All13])**.** Let $\tau > 0$ be a real number. We say that a Q-multisection $M \colon \Omega \times \mathbb{R}^n \to \mathbb{N}$ satisfies the τ-*cone condition* if the following holds. For any $x \in \Omega$, for any $v \in M_x = \{v \in \mathbb{R}^n : M(x, v) > 0\}$, there exist neighborhoods U of x in Ω and V of v in $\mathbb{R}^n$ such that

$$\{(y, w) \in U \times V : M(y, w) > 0\} \subset \mathcal{K}^{\tau}_{x,v}, \tag{8.13}$$

where

$$\mathcal{K}^{\tau}_{x,v} := \{(y, w) \in \mathbb{R}^m \times \mathbb{R}^n : |w - v| \leqslant \tau |y - x|\} \tag{8.14}$$

is the τ-cone centered at (x, v) in $\mathbb{R}^m \times \mathbb{R}^n$.

Proposition 8.2.4. *Let $\Omega \subset \mathbb{R}^m$ be open and convex. If u is an L-Lipschitz Q-valued function, then the induced multisection $M_u \colon \Omega \times \mathbb{R}^n \to \mathbb{N}$ is coherent and satisfies a τ-cone condition with $\tau = L$. Conversely, if a Q-multisection of the bundle $\Omega \times \mathbb{R}^n$ is coherent and satisfies the τ-cone condition, then the associated Q-valued function $u_M \colon \Omega \to \mathcal{A}_Q(\mathbb{R}^n)$ is Lipschitz with $\operatorname{Lip}(u_M) \leqslant \sqrt{Q}\tau$.*

Proof. The first part of the statement is immediate. Indeed, first observe that the continuity of u implies that $M = M_u$ is coherent by Proposition 8.2.2. Then, fix $x \in \Omega$, and suppose that $u(x) = \sum_{j=1}^{J} m_j [\![\tilde{v}_j]\!]$, with the $\tilde{v}_j$'s all distinct and $m_j := M(x, \tilde{v}_j)$. Let $\varepsilon > 0$ be such that the balls $B_\varepsilon(\tilde{v}_j) \subset \mathbb{R}^n$ are a disjoint family of open sets such that $M_x \cap B_\varepsilon(\tilde{v}_j) = \{\tilde{v}_j\}$. Since M is coherent, there exists an open neighborhood U of x in Ω such that the following two properties are satisfied for any $y \in U$:

(i) $\sum_{w \in B_\varepsilon(\tilde{v}_j)} M(y, w) = m_j$;

(ii) if $u(x) = \sum_{\ell=1}^{Q} [\![v_\ell]\!]$ with the first m_1 of the v_ℓ's all identically equal to $\tilde{v}_1$, the next m_2 all identically equal to $\tilde{v}_2$ and so on, and if $u(y) = \sum_{\ell=1}^{Q} [\![w_\ell]\!]$ with the w_ℓ (not necessarily all distinct) ordered in such a way that $\mathcal{G}(u(x), u(y)) = \left(\sum_{\ell=1}^{Q} |v_\ell - w_\ell|^2 \right)^{1/2}$, then $w_\ell \in B_\varepsilon(v_\ell)$ for every $\ell \in \{1, \dots, Q\}$.

Thus, for such y's it is evident that the Lipschitz condition $\mathcal{G}(u(y), u(x)) \leqslant L|y - x|$ forces $|w_\ell - v_\ell| \leqslant L|y - x|$ for every $\ell = 1, \dots, Q$, which is to say that for every $j = 1, \dots, J$

$$\{(y, w) \in U \times B_\varepsilon(\tilde{v}_j) : M(y, w) > 0\} \subset \mathcal{K}^L_{x, \tilde{v}_j},$$

as we wanted.

For the converse, consider a Q-multisection M, and assume it is coherent and satisfies the τ-cone condition. Define $u \colon \Omega \to \mathcal{A}_Q(\mathbb{R}^n)$ as in (8.4). We will first prove the following claim, from which the Lipschitz continuity of u will easily follow:

Claim. For every $x \in \Omega$ there exists an open neighborhood U_x of x in Ω such that

$$\mathcal{G}(u(y), u(x)) \leqslant \sqrt{Q}\tau |y - x| \qquad \text{for every } y \in U_x. \tag{8.15}$$

In order to show this, fix a point $x \in \Omega$, and let $\{v_1, \dots, v_J\}$ be distinct vectors in M_x. Since M satisfies the τ-cone condition, there exist open neighborhoods U of x in Ω and V_j of v_j in $\mathbb{R}^n$ for every $j = 1, \dots, J$ such that

$$\{(y, w) \in U \times V_j : M(y, w) > 0\} \subset \mathcal{K}^\tau_{x, v_j} \qquad \forall\, j = 1, \dots, J. \tag{8.16}$$

In particular, condition (8.16) implies that $M_x \cap V_j = \{v_j\}$ for every j. Up to shrinking the V_j's if necessary, we can also assume that they are pairwise disjoint. Hence, since M is also coherent, we can conclude the existence of a (possibly smaller) neighborhood of x, which we will still denote U, with the property that not only (8.16) is satisfied but also

$$\sum_{w \in V_j} M(y, w) = M(x, v_j) \qquad \forall\, j = 1, \dots, J, \ \forall\, y \in U. \tag{8.17}$$

Therefore, if $y \in U$ we can write

$$u(y) = \sum_{j=1}^{J} \sum_{w \in V_j} M(y, w) [\![w]\!], \tag{8.18}$$

whereas

$$u(x) = \sum_{j=1}^{J} M(x, v_j) [\![v_j]\!]. \tag{8.19}$$

Using (8.17), (8.18), (8.19) and the fact that if $(y,w) \in U \times V_j$ then $M(y,w) > 0 \implies |w - v_j| \leqslant \tau|y-x|$, we immediately conclude that

$$\mathcal{G}(u(y),u(x))^2 \leqslant \left(\sum_{j=1}^{J} M(x,v_j)\right) \tau^2 |y-x|^2 \qquad \text{for every } y \in U, \tag{8.20}$$

which proves our claim.

Next, we prove that u is Lipschitz continuous with $\mathrm{Lip}(u) \leqslant \sqrt{Q}\tau$. To achieve this, fix two distinct points $p, q \in \Omega$. Since Ω is convex, the segment $[p,q]$ is contained in Ω, and let e denote the unit vector orienting the segment $[p,q]$ in the direction from p to q. By the claim, for every $x \in [p,q]$ there exists a radius $r_x > 0$ such that

$$\mathcal{G}(u(y),u(x)) \leqslant \sqrt{Q}\tau|y-x| \qquad \text{for every } y \in I_x := (x - r_x e, x + r_x e)\,. \tag{8.21}$$

The open intervals I_x are clearly an open covering of $[p,q]$. Since the segment is compact, it admits a finite subcovering, which will be denoted $\{I_{x_i}\}_{i=0}^{N}$. We may assume, refining the subcovering if necessary, that an interval I_{x_i} is not completely contained in an interval I_{x_j} if $i \neq j$. If we relabel the indices of the points x_i in a non-decreasing order along the segment, we can now choose an auxiliary point $y_{i,i+1}$ in $I_{x_i} \cap I_{x_{i+1}} \cap (x_i, x_{i+1})$ for each $i = 0, \dots, N-1$. We can finally conclude:

$$\begin{aligned}
\mathcal{G}(u(p),u(q)) &\leqslant \mathcal{G}(u(p),u(x_0)) \\
&\quad + \sum_{i=0}^{N-1} (\mathcal{G}(u(x_i),u(y_{i,i+1})) + \mathcal{G}(u(y_{i,i+1}),u(x_{i+1}))) + \mathcal{G}(u(x_N),u(q)) \\
&\overset{(8.21)}{\leqslant} \sqrt{Q}\tau \left(|x_0 - p| + \sum_{i=0}^{N-1} (|y_{i,i+1} - x_i| + |x_{i+1} - y_{i,i+1}|) + |q - x_N| \right) \\
&= \sqrt{Q}\tau |q-p|,
\end{aligned} \tag{8.22}$$

which completes the proof. □

Definition 8.2.5 (Allard, [All13])**.** Let $\Pi\colon E \to \Sigma$ be a vector bundle, M a Q-multisection over Σ and $\tau > 0$. We say that M is τ-*vertically limited* if for any coordinate domain U_α on Σ with associated chart $\psi_\alpha\colon U_\alpha \to \mathbb{R}^m$ and trivialization $\Psi_\alpha\colon \Pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^n$ the multisection

$$M_\alpha := M \circ \Psi_\alpha^{-1} \circ \left(\psi_\alpha^{-1} \times \mathrm{id}_{\mathbb{R}^n}\right) : \psi_\alpha(U_\alpha) \times \mathbb{R}^n \to \mathbb{N}$$

satisfies the τ-cone condition.

9 REPARAMETRIZATION OF MULTIPLE-VALUED GRAPHS

Here we revisit the proof of the reparametrization theorem for multiple-valued Lipschitz graphs [DS15, Theorem 5.1], a fundamental tool for the construction of the normal approximation of an area minimizing current from the center manifold performed in [DLS16a]. In the proof that we are going to suggest, we deduce the Lipschitz continuity of the reparametrization from a simple geometric argument which is completely given in terms of the Q-multisections introduced in Chapter 8, rather than from an application of the Ambrosio-Kirchheim theory of currents in metric spaces [AK00] as in [DS15].

9.1 THE REPARAMETRIZATION THEOREM

Before stating the precise result we are aiming at, we need to introduce some notation and terminology, which will be used throughout the whole chapter.

Assumption 9.1.1. Let m, n and Q denote fixed positive integers. Let also $0 < s < r < 1$. We will consider the following:

(A1) an open m-dimensional submanifold Σ of the Euclidean space $\mathbb{R}^{m+n}$ with $\mathcal{H}^m(\Sigma) < \infty$ which is the graph of a function $\boldsymbol{\varphi}\colon B_s \subset \mathbb{R}^m \to \mathbb{R}^n$ with $\|\boldsymbol{\varphi}\|_{C^3} \leqslant \bar{c}$;

(A2) a regular tubular neighborhood $\mathbf{U}$ of Σ, that is the set of points

$$\mathbf{U} := \{\xi = p + v : p \in \Sigma,\, v \in T_p^{\perp}\Sigma,\, |v| < c_0\} \subset \mathbb{R}^{m+n}, \tag{9.1}$$

where the thickness c_0 is small enough to guarantee that the nearest point projection $\Pi\colon \mathbf{U} \to \Sigma$ is well defined and C^2;

(A3) a proper Lipschitz Q-valued function $f\colon B_r \subset \mathbb{R}^m \to \mathcal{A}_Q(\mathbb{R}^n)$.

Some comments about the objects introduced in Assumptions 9.1.1 are now in order. First observe that the map $\boldsymbol{\varphi}$ induces a parametrization of the manifold Σ, which we denote by

$$\boldsymbol{\Phi}\colon x \in B_s \subset \mathbb{R}^m \mapsto \boldsymbol{\Phi}(x) := (x, \boldsymbol{\varphi}(x)) \in \mathbb{R}^{m+n}. \tag{9.2}$$

The inverse of $\boldsymbol{\Phi}$ can be used as a global chart on Σ. If $p \in \Sigma$, then π_p and $\varkappa_p$ will denote the tangent space $T_p\Sigma$ and its orthogonal complement in $\mathbb{R}^{m+n}$ respectively. The symbols π_0 and $\pi_0^{\perp}$, instead, will be reserved for the planes $\mathbb{R}^m \times \{0\} \simeq \mathbb{R}^m$ and $\{0\} \times \mathbb{R}^n \simeq \mathbb{R}^n$ respectively. In general, if π is a linear subspace of $\mathbb{R}^{m+n}$, the symbol $\mathbf{p}_\pi$ will denote orthogonal projection onto it.

Concerning the tubular neighborhood $\mathbf{U}$, we will denote by $\{\nu_1, \dots, \nu_n\}$ the standard orthonormal frame of the normal bundle of Σ described in [DS15, Appendix A]. Such a

frame is simply obtained by applying, at every point $p \in \Sigma$, the Gram-Schmidt orthogonalization algorithm to the vectors $\mathbf{p}_{\varkappa_p}(e_{m+1}), \dots, \mathbf{p}_{\varkappa_p}(e_{m+n})$, where $\{e_{m+1}, \dots, e_{m+n}\}$ is the standard orthonormal basis of $\{0\} \times \mathbb{R}^n \subset \mathbb{R}^{m+n}$. The analytic properties of the frame $\nu_1, \dots \nu_n$ are recorded in the following lemma.

Lemma 9.1.2 (cf. [DS15, Lemma A.1]). *If $\|D\boldsymbol{\varphi}\|_{C^0}$ is smaller than a geometric constant, then $\nu_1, \dots \nu_n$ is an orthonormal frame spanning $\varkappa_p$ at every $p \in \Sigma$. Consider ν_i as functions of $x \in B_s$ using the inverse of $\boldsymbol{\Phi}$ as a chart. For every $\gamma + k \geqslant 0$, there is a constant $C = C(m, n, \gamma, k)$ such that if $\|\boldsymbol{\varphi}\|_{C^{k+1,\gamma}} \leqslant 1$, then $\|D\nu_i\|_{C^{k,\gamma}} \leqslant C\|D\boldsymbol{\varphi}\|_{C^{k+1,\gamma}}$.*

Recall that, for any Q-valued function f as in assumption (A3), $\mathrm{Gr}(f)$ and $\mathbf{G}_f$ denote the set-theoretical graph of f and the integral m-current associated to it respectively. The concept of *reparametrization* of f is introduced next.

Definition 9.1.3. Given Σ, $\mathbf{U}$ and f as in Assumptions 9.1.1, we call a *Lipschitz normal reparametrization* of the Q-function f in the tubular neighborhood $\mathbf{U}$ any Q-valued function $F\colon \Sigma \to \mathcal{A}_Q(\mathbb{R}^{m+n})$ such that the following conditions are satisfied:

(i) for every $p \in \Sigma$, $F(p) = \sum_{\ell=1}^{Q} [\![p + N_\ell(p)]\!]$, with $N\colon \Sigma \to \mathcal{A}_Q(\mathbb{R}^{m+n})$ a Lipschitz continuous Q-valued function;

(ii) $p + N_\ell(p) \in \mathbf{U}$ and $N_\ell(p) \in \varkappa_p = T_p^\perp \Sigma$ for every $\ell \in \{1, \dots, Q\}$, for every $p \in \Sigma$;

(iii) $\mathbf{T}_F = \mathbf{G}_f \llcorner \mathbf{U}$.

We are now ready to state the main theorem we are aiming at.

Theorem 9.1.4 (Existence of the reparametrization). *Let Q, m and n be positive integers, and $0 < s < r < 1$. Then, there are constants $c_0, C > 0$ (depending on m, n, Q, $r - s$ and $\frac{r}{s}$) with the following property. For any $\boldsymbol{\varphi}$, Σ, $\mathbf{U}$ and f as in Assumptions 9.1.1 such that*

$$\|\boldsymbol{\varphi}\|_{C^2} + \mathrm{Lip}(f) \leqslant c_0, \qquad \|\boldsymbol{\varphi}\|_{C^0} + \|f\|_{C^0} \leqslant c_0 s, \tag{9.3}$$

there exists a Lipschitz normal reparametrization F of the Q-valued function f in $\mathbf{U}$. Furthermore, the associated normal multi-valued vector field N satisfies:

$$\mathrm{Lip}(N) \leqslant C\left(\|N\|_{C^0}\|D^2\boldsymbol{\varphi}\|_{C^0} + \|D\boldsymbol{\varphi}\|_{C^0} + \mathrm{Lip}(f)\right), \tag{9.4}$$

$$\frac{1}{2\sqrt{Q}}|N(\boldsymbol{\Phi}(x))| \leqslant \mathcal{G}(f(x), Q[\![\boldsymbol{\varphi}(x)]\!]) \leqslant 2\sqrt{Q}|N(\boldsymbol{\Phi}(x))| \ \ \forall x \in B_s, \tag{9.5}$$

$$|\boldsymbol{\eta} \circ N(\boldsymbol{\Phi}(x))| \leqslant C|\boldsymbol{\eta} \circ f(x) - \boldsymbol{\varphi}(x)| + C\mathrm{Lip}(f)|D\boldsymbol{\varphi}(x)||N(\boldsymbol{\Phi}(x))| \ \ \forall x \in B_s. \tag{9.6}$$

Finally, assume $x \in B_s$ and $(x, \boldsymbol{\eta} \circ f(x)) = p + v$ for some $p \in \Sigma$ and $v \in T_p^\perp \Sigma$. Then,

$$\mathcal{G}(N(p), Q[\![v]\!]) \leqslant 2\sqrt{Q}\mathcal{G}(f(x), Q[\![\boldsymbol{\eta} \circ f(x)]\!]). \tag{9.7}$$

9.2 THE PROOF OF THEOREM 9.1.4

The argument will be divided into two parts: in the first part, we will suppose to be given Σ, $\mathbf{U}$ and f as in Assumptions 9.1.1, and we will associate in an extremely natural way to the Q-valued function f a Q-multisection M of the tubular neighborhood $\mathbf{U}$, regarded as (the diffeomorphic image of) an open subset of a rank n vector bundle of class C^2 over Σ. Under suitable smallness assumptions on the universal constant c_0 which controls the relevant norms of the functions $\boldsymbol{\varphi}$ and f as in (9.3), we will be able to show that the multisection M so defined enjoys good properties of coherence and vertical boundedness. In the second part of the argument, we will produce the reparametrization F using the multisection M previously analyzed, and we will prove that the aforementioned geometric properties of M do suffice to conclude the proof of Theorem 9.1.4, using techniques that have been already introduced in the proofs of Propositions 8.2.2 and 8.2.4.

We start with the first part of our program. Assume, therefore, that the manifold Σ, the tubular neighborhood $\mathbf{U}$ and the Q-valued function f are given, and that the functions $\boldsymbol{\varphi}$ and f satisfy the bounds in (9.3). Suitable restrictions on the size of the constant c_0 will appear throughout the argument. Let

$$M_f \colon \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{N} \tag{9.8}$$

be the Q-multisection over B_r associated to f. Observe that, setting $L := \mathrm{Lip}(f)$, Proposition 8.2.4 guarantees that M_f is coherent and satisfies an L-cone condition.

Now, we define a Q-multisection M of the tubular neighborhood $\mathbf{U}$ as follows: for any $\xi \in \mathbf{U}$, $M(\xi)$ coincides with the multiplicity of the "vertical coordinate" $\mathbf{p}_{\pi_0^\perp}(\xi)$ in $f(\mathbf{p}_{\pi_0}(\xi))$. In symbols, we set:

$$M(\xi) := \Theta_{f(\mathbf{p}_{\pi_0}(\xi))}(\mathbf{p}_{\pi_0^\perp}(\xi)) = M_f(\mathbf{p}_{\pi_0}(\xi), \mathbf{p}_{\pi_0^\perp}(\xi)), \qquad \text{for every } \xi \in \mathbf{U}. \tag{9.9}$$

The following Proposition shows that, under suitable smallness conditions on c_0, M is indeed a coherent Q-multisection over the base manifold Σ.

Proposition 9.2.1. *If c_0 is small enough, depending on m, n, $r - s$ and $\frac{r}{s}$, then the identity*

$$\sum_{\xi \in \Pi^{-1}(\{p\})} M(\xi) = Q \tag{9.10}$$

holds for every $p \in \Sigma$, and thus M is a Q-multisection over Σ. Moreover, M is coherent.

Proof. First, we claim the following: the current $T := \mathbf{G}_f \llcorner (\Pi^{-1}(\Sigma))$ satisfies $\Pi_\sharp T = Q[\![\Sigma]\!]$. In order to see this, fix a point $\xi \in \mathrm{spt}(T)$. By definition, $\xi = (y, f_\ell(y))$ for some $y \in B_r$ and for some $\ell \in \{1, \dots, Q\}$; furthermore, there exist a point $p = (x, \boldsymbol{\varphi}(x)) \in \Sigma$ and a vector $v \in T_p^\perp \Sigma$ with $|v| < c_0$ such that $\xi = p + v$. Hence, we can easily estimate

$$|y| = |\mathbf{p}_{\pi_0}(p + v)| \leqslant |x| + |v| < s + c_0.$$

This implies that if we choose c_0 suitably small, say

$$c_0 \leqslant \frac{1}{2}(r - s), \tag{9.11}$$

then the current $(\mathbf{p}_{\pi_0})_\sharp T$ is compactly supported in B_r, and thus $(\partial \mathbf{G}_f) \llcorner \Pi^{-1}(\Sigma) = (\mathbf{G}_{f|_{\partial B_r}}) \llcorner \Pi^{-1}(\Sigma) = 0$. Now, we estimate more carefully the quantity $|v| = |\xi - p| = \operatorname{dist}(\xi, \Sigma)$. Decompose

$$|v|^2 = |\mathbf{p}_{\pi_0}(v)|^2 + |\mathbf{p}_{\pi_0^\perp}(v)|^2, \tag{9.12}$$

and observe that the hypothesis (9.3) readily implies that

$$|\mathbf{p}_{\pi_0^\perp}(v)|^2 = |f_\ell(y) - \boldsymbol{\varphi}(x)|^2 \leqslant c_0^2 s^2. \tag{9.13}$$

As for the "horizontal" component of the vector v, write

$$v = \sum_{i=1}^{n} v^i \nu_i(x), \tag{9.14}$$

where $\nu = (v^1, \dots, v^n) \in \mathbb{R}^n$, $\{\nu_1, \dots, \nu_n\}$ is the standard orthonormal frame on the normal bundle of Σ previously introduced, and where, with a slight abuse of notation, we are writing $\nu_i(x)$ instead of $\nu_i(\boldsymbol{\Phi}(x))$. In this way,

$$|\mathbf{p}_{\pi_0}(v)|^2 \leqslant \left(\sum_{i=1}^{n} |v^i| |\mathbf{p}_{\pi_0}(\nu_i(x))| \right)^2. \tag{9.15}$$

Clearly, in doing this we are tacitly assuming that c_0 is chosen so small that all the conclusions of Lemma 9.1.2 hold (in particular, we will always assume $c_0 \leqslant 1$). Now, the quantity $|\mathbf{p}_{\pi_0}(\nu_i(x))|$ can be estimated by

$$|\mathbf{p}_{\pi_0}(\nu_i(x))| \leqslant \left| \cos\left(\frac{\pi}{2} - \theta_i(x) \right) \right|, \tag{9.16}$$

where $\theta_i(x)$ is the angle between $\nu_i(x)$ and $\mathbf{p}_{\pi_0^\perp}(\nu_i(x))$. In turn, this angle is controlled by $C|D\boldsymbol{\varphi}(x)|$, with C a geometric constant, because $\nu_i(x)$ is orthogonal to $T_{\boldsymbol{\Phi}(x)}\Sigma$. Thus, one has

$$|\mathbf{p}_{\pi_0}(\nu_i(x))| \leqslant |\sin(\theta_i(x))| \leqslant C|D\boldsymbol{\varphi}(x)|. \tag{9.17}$$

Further estimating $|D\boldsymbol{\varphi}(x)| \leqslant \|D\boldsymbol{\varphi}\|_{C^0} \leqslant c_0$ by (9.3) and inserting into (9.15) yields:

$$|\mathbf{p}_{\pi_0}(v)|^2 \leqslant C c_0^2 \left(\sum_{i=1}^{n} |v^i| \right)^2 \leqslant C c_0^2 \sum_{i=1}^{n} |v^i|^2 = C c_0^2 |v|^2, \tag{9.18}$$

with $C = C(m, n)$. Combining (9.12), (9.13) and (9.18) produces

$$|v|^2 \leqslant C c_0^2 |v|^2 + c_0^2 s^2. \tag{9.19}$$

If

$$c_0^2 \leqslant C^{-1} \left(1 - \left(\frac{s}{r} \right)^2 \right), \tag{9.20}$$

then the term $C c_0^2 |v|^2$ on the right-hand side can be absorbed on the left-hand side, and in turn (9.19) leads to

$$\operatorname{dist}(\xi, \Sigma)^2 \leqslant c_0^2 r^2, \tag{9.21}$$

which shows that the current T is in fact compactly supported in $\mathbf{U}$. Together with the fact that $\mathbf{G}_f$ has no boundary in $\Pi^{-1}(\Sigma)$, such a result implies that the boundary of T is actually supported in $\Pi^{-1}(\partial\Sigma)$ as soon as the constant c_0 is chosen in agreement with (9.11) and (9.20). Hence, under these conditions we can deduce that $\partial\Pi_\sharp T$ is supported in $\partial\Sigma$. Thus, we are allowed to apply the Constancy Theorem 2.1.6, and consequently conclude that $\Pi_\sharp T = k[\![\Sigma]\!]$ for some $k \in \mathbb{Z}$. In order to show that $k = Q$, we consider the functions $\boldsymbol{\varphi}_t := t\boldsymbol{\varphi}$ for $t \in [0,1]$, the corresponding manifolds $\Sigma_t := \mathrm{Gr}(\boldsymbol{\varphi}_t)$ with the associated projections $\Pi_t \colon \mathbf{U}_t \to \Sigma_t$. Also in this case, the constancy theorem produces $(\Pi_t)_\sharp(\mathbf{G}_f \llcorner (\Pi_t^{-1}(\Sigma_t))) = k(t)[\![\Sigma_t]\!]$. On the other hand, since the map

$$t \in [0,1] \mapsto (\Pi_t)_\sharp(\mathbf{G}_f \llcorner (\Pi_t^{-1}(\Sigma_t)))$$

is continuous in the space of currents, one infers that $t \mapsto k(t)$ is a continuous integer-valued function, and thus is constant. Since $k(0) = Q$, then necessarily also $k = k(1) = Q$, and the claim is proved.

Now, the fact that $\Pi_\sharp T = Q[\![\Sigma]\!]$ does not immediately imply that $\sum_{\xi\in M_p} M(\xi) = Q$, since there could in principle be cancellations and the total mass on the fiber could in principle be larger than Q. To see that this is not the case, consider, for every $p \in \Sigma$, the 0-dimensional current $T_p := \langle \mathbf{G}_f, \Pi, p\rangle$ supported on the intersection $\mathrm{Gr}(f) \cap \Pi^{-1}(\{p\})$. By the slicing theory (cf. § 2.1.2 or [Fed69, Section 4.3]), one has that there exists a set $Z \subset \Sigma$ with $\mathcal{H}^m(Z) = 0$ such that the following holds for every $p \in \Sigma \setminus Z$:

(i) T_p consists of a finite sum of Dirac masses $\sum_{j=1}^{J_p} m_j[\![\xi_j]\!]$ with coefficients $m_j \in \mathbb{Z}$;

(ii) for every $j \in \{1, \dots J_p\}$, $\xi_j \in \mathrm{Gr}(f) \cap \Pi^{-1}(\{p\})$ and $|m_j| = M_f(\mathbf{p}_{\pi_0}(\xi_j), \mathbf{p}_{\pi_0^\perp}(\xi_j)) = M(\xi_j)$;

(iii) if $\vec{\nu}$ is the continuous unit n-vector orienting $\Pi^{-1}(\{p\})$ compatibly with the orientation of Σ, then the sign of m_j is $\mathrm{sgn}\left(\langle \vec{T}(\xi_j) \wedge \vec{\nu}(\xi_j), \vec{e}\rangle\right)$, where $\vec{e} := e_1 \wedge \cdots \wedge e_m \wedge \cdots \wedge e_{m+n}$, with $\{e_1, \dots, e_m\}$ the standard orthonormal basis of $\mathbb{R}^m \times \{0\}$ and $\{e_{m+1}, \dots, e_{m+n}\}$ the standard orthonormal basis of $\{0\} \times \mathbb{R}^n$.

Since $\|\boldsymbol{\varphi}\|_{C^1} + \mathrm{Lip}(f) \leqslant c_0$, if c_0 is suitably small then every $\vec{T}(\xi_j)$ is close to $\vec{e}_m := e_1 \wedge \cdots \wedge e_m$, whereas every $\vec{\nu}(\xi_j)$ is close to $\vec{e}_n := e_{m+1} \wedge \cdots \wedge e_{m+n}$, and therefore every m_j is positive. Since $\sum_{j=1}^{J_p} m_j = Q$ because $\Pi_\sharp T = Q[\![\Sigma]\!]$, we conclude that (9.10) holds for every $p \in \Sigma \setminus Z$.

Therefore, if $\tilde{Z}$ denotes the set of points $p \in \Sigma$ such that (9.10) does not hold (and hence $\sum_{\xi\in M_p} M(\xi) > Q$) then one has $\tilde{Z} \subset Z$. Now, we claim that in fact $\tilde{Z} = \emptyset$. This will be an easy consequence of the fact that M is coherent. Indeed, the coherence of M would immediately imply that $\tilde{Z}$ is open in Σ, and thus empty, since $\mathcal{H}^m(\tilde{Z}) = 0$. Hence, we only have to prove that M is coherent. Fix $p \in \Sigma$, and assume that $M_p = \{\xi_1, \dots, \xi_J\}$, with $m_j := M(\xi_j)$. Let $\mathcal{V} = \{V_1, \dots V_J\}$ denote a collection of disjoint bounded open sets in $\mathbf{U}$ such that $M_p \cap V_j = \{\xi_j\}$ for every $j = 1, \dots, J$. We will show that for every j there is an open neighborhood $\mathcal{U}_j$ of p in Σ such that

$$\sum_{\zeta\in M_q\cap V_j} M(\zeta) = m_j \quad \text{for every } q \in \mathcal{U}_j, \tag{9.22}$$

so that the coherence condition will hold in $\mathcal{U} := \bigcap_j \mathcal{U}_j$. Consider the current $T_j := \Pi_\sharp(\mathbf{G}_f \llcorner V_j)$. We claim the following: there exists $\mathcal{U}_j \subset \Sigma$ open neighborhood of p such that

$$\operatorname{spt}\left(\partial(T_j \llcorner \mathcal{U}_j)\right) \subset \partial \mathcal{U}_j. \tag{9.23}$$

If (9.23) holds, the proof is finished. Indeed, the constancy theorem would imply the existence of a constant $k_j \in \mathbb{Z}$ such that $T_j \llcorner \mathcal{U}_j = k_j[\![\mathcal{U}_j]\!]$. On the other hand, it would necessarily be $k_j = m_j$, because $\langle \mathbf{G}_f \llcorner V_j, \Pi, p\rangle = m_j[\![\xi_j]\!]$. Then, since no cancellations are allowed, if $q \in \mathcal{U}_j$ the slice $\langle \mathbf{G}_f \llcorner V_j, \Pi, q\rangle$ must be necessarily supported in a set of points $\{\zeta_1, \dots, \zeta_{J_q}\} \subset \operatorname{Gr}(f) \cap V_j$ with $\sum_{j=1}^{J_q} M(\zeta_j) = m_j$, which concludes the proof of (9.22).

Therefore, we just have to prove (9.23). By contradiction, assume that there exists a sequence $\{p_h\}_{h=1}^\infty \subset \Sigma$ with $p_h \to p$ and such that $p_h \in \operatorname{spt}(\partial T_j)$ for every h. Since the push-forward and boundary operators commute, and since $\mathbf{G}_f$ has no boundary in V_j, this would imply the existence of a sequence of points $\zeta_h \in \operatorname{Gr}(f) \cap \partial V_j$ such that $\Pi(\zeta_h) = p_h$. By the compactness of ∂V_j and the continuity of the projection, a subsequence of the ζ_h's would converge to a point $\bar\zeta \in \partial V_j$ such that $\Pi(\bar\zeta) = p$. Furthermore, since f is continuous $\operatorname{Gr}(f)$ is closed, and thus $\bar\zeta \in \operatorname{Gr}(f)$. But this is an evident contradiction, since by assumption $\mathbf{G}_f$ is supported outside of $\Pi^{-1}(\{p\}) \cap \partial V_j$. This shows the validity of (9.23), and concludes the proof of the Proposition. □

As an immediate consequence, the above result allows us to define the required reparametrization F: if Σ, $\mathbf{U}$ and f are such that (9.3) holds with the constant c_0 given by Proposition 9.2.1, we set

$$F(p) := \sum_{\xi \in \Pi^{-1}(\{p\})} M(\xi)[\![\xi]\!] \quad \text{for every } p \in \Sigma. \tag{9.24}$$

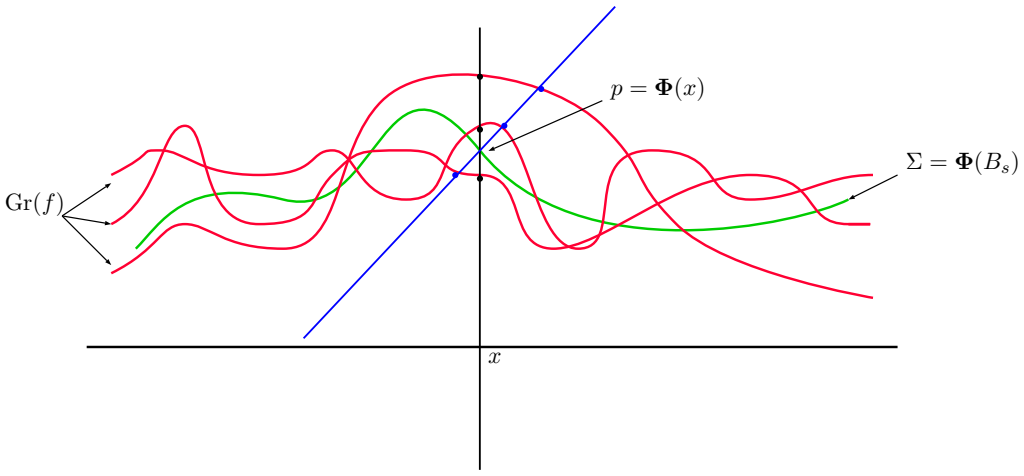


Figure 2: The reparametrization F. The black points are the support of $\mathrm{id} \times f(x)$; the blue points are the support of $F(\mathbf{\Phi}(x))$.

By Proposition 9.2.1, F is a well defined Q-valued function on Σ. By construction, the associated map $N\colon \Sigma \to \mathcal{A}_Q(\mathbb{R}^{m+n})$ given by

$$N(p) := \sum_{\xi \in \Pi^{-1}(\{p\})} M(\xi)[\![\xi - p]\!] \tag{9.25}$$

is a well defined Q-valued vector field with values in the normal bundle, and hence it satisfies property (ii) in Definition 9.1.3. Furthermore, it is evident from the very definition of M that property (iii) in Definition 9.1.3 is satisfied as well.

Hence, we are only left with proving that N is Lipschitz continuous and that properties (9.4)-(9.7) are satisfied.

Proposition 9.2.2. *If c_0 is small enough, depending on m, n, $r - s$ and $\frac{r}{s}$, then there exists $\tilde{\tau} > 0$ such that the multisection M is $\tilde{\tau}$-vertically limited. Furthermore,*

$$\tilde{\tau} \leqslant C\left(\|N\|_{C^0}\|D^2\boldsymbol{\varphi}\|_{C^0} + \|D\boldsymbol{\varphi}\|_{C^0} + \mathrm{Lip}(f)\right), \tag{9.26}$$

where $C = C(m, n)$ and $\|N\|_{C^0} := \sup_{p\in\Sigma}|N(p)| = \sup_{p\in\Sigma}\mathcal{G}(N(p), Q[\![0]\!])$.

Proof. First, let us exploit again the orthonormal frame $\{\nu_1, \dots, \nu_n\}$ in order to introduce coordinates on $\mathbf{U}$. Precisely, we let Ψ denote the map $\xi \in \mathbf{U} \mapsto (\Pi(\xi), \nu(\xi)) \in \Sigma \times \mathbb{R}^n$, where $p := \Pi(\xi)$ is the base point of ξ on Σ, and $\nu(\xi) = \left(\nu^1(\xi), \dots, \nu^n(\xi)\right)$ is the set of coordinates of the vector $v := \xi - p \in \varkappa_p$ with respect to the basis $\nu_1(p), \dots, \nu_n(p)$, explicitly given by $\nu^i(\xi) = \langle \xi - p, \nu_i(p)\rangle$ for $i = 1, \dots, n$. The map Ψ is a global trivialization of the bundle $\mathbf{U}$; moreover, since $\boldsymbol{\Phi}^{-1}$ is a global chart on Σ, then, in order to show that M is $\tilde{\tau}$-vertically limited, it suffices to prove that the Q-multisection

$$\widetilde{M} := M \circ \Psi^{-1} \circ (\boldsymbol{\Phi} \times \mathrm{id}_{\mathbb{R}^n}) : B_s \times \mathbb{R}^n \to \mathbb{N}$$

satisfies the $\tilde{\tau}$-cone condition. In order to see this, fix $(x, \nu) \in B_s \times \mathbb{R}^n$, and denote by $\xi = \xi(x, \nu)$ the corresponding point in $\mathbf{U}$, given by

$$\xi(x, \nu) := \boldsymbol{\Phi}(x) + \sum_{i=1}^{n} \nu^i \nu_i(x). \tag{9.27}$$

Assume that $\widetilde{M}(x, \nu) = M(\xi) > 0$: the goal is then to prove that there exists a positive number ε such that if $(y, w) \in B^m_\varepsilon(x) \times B^n_\varepsilon(\nu)$ satisfies $\widetilde{M}(y, w) > 0$, then necessarily

$$|w - \nu| \leqslant \tilde{\tau}|y - x|. \tag{9.28}$$

Let (x', ν') denote the coordinates of ξ in the standard reference frame on $\mathbb{R}^{m+n}$, that is $x' := \mathbf{p}_{\pi_0}(\xi)$ and $\nu' := \mathbf{p}_{\pi_0^\perp}(\xi)$. Observe that the condition $M(\xi) > 0$ is equivalent to say that $\nu' \in \mathrm{spt}(f(x'))$, and in fact $M(\xi) = M_f(x', \nu')$. Now, since the Q-valued function f is L-Lipschitz continuous, M_f satisfies the L-cone condition, and thus there exists $\delta > 0$ such that if $(y', w') \in B^m_\delta(x') \times B^n_\delta(\nu')$ is such that $M_f(y', w') > 0$ then

$$|w' - \nu'| \leqslant L|y' - x'|. \tag{9.29}$$

We first claim the following: there exists $0 < \varepsilon = \varepsilon(\delta, m, n)$ with the property that if

$$\zeta = \mathbf{\Phi}(y) + \sum_{i=1}^{n} w^i \nu_i(y) \qquad \text{with } (y, w) \in B^m_\varepsilon(x) \times B^n_\varepsilon(\nu),$$

then

$$|\mathbf{p}_{\pi_0}(\zeta) - x'| < \delta, \qquad |\mathbf{p}_{\pi_0^\perp}(\zeta) - \nu'| < \delta.$$

This can be immediately seen by estimating:

$$\begin{aligned} |\zeta - \xi| &\leqslant |\mathbf{\Phi}(y) - \mathbf{\Phi}(x)| + \sum_{i=1}^{n} |w^i \nu_i(y) - \nu^i \nu_i(x)| \\ &\leqslant (1 + \|D\boldsymbol{\varphi}\|_{C^0})\, |y - x| + \sum_{i=1}^{n} \left(|w^i| |\nu_i(y) - \nu_i(x)| + |\nu^i - w^i| \right) \\ &\leqslant C\,(1 + \|D\boldsymbol{\varphi}\|_{C^1})\, |y - x| + C|w - \nu|, \end{aligned} \tag{9.30}$$

where $C = C(m, n)$ is a geometric constant. The conclusion immediately follows, since $|\mathbf{p}_{\pi_0}(\zeta) - x'| = |\mathbf{p}_{\pi_0}(\zeta - \xi)| \leqslant |\zeta - \xi|$ and $|\mathbf{p}_{\pi_0^\perp}(\zeta) - \nu'| = |\mathbf{p}_{\pi_0^\perp}(\zeta - \xi)| \leqslant |\zeta - \xi|$.

Now, let (y, w) be any point in $B^m_\varepsilon(x) \times B^n_\varepsilon(\nu)$ such that for the corresponding $\zeta \in \mathbf{U}$ one has $M(\zeta) > 0$. By the above claim, if we set $y' := \mathbf{p}_{\pi_0}(\zeta)$ and $w' := \mathbf{p}_{\pi_0^\perp}(\zeta)$, then $(y', w') \in B^m_\delta(x') \times B^n_\delta(\nu')$, and thus the condition $M_f(y', w') > 0$ implies that (9.29) holds. Hence, we proceed with the proof of (9.28). For any $i = 1, \dots, n$, one has:

$$\begin{aligned} |w^i - \nu^i| &= |\langle \zeta - \mathbf{\Phi}(y), \nu_i(y) \rangle - \langle \xi - \mathbf{\Phi}(x), \nu_i(x) \rangle| \\ &\leqslant |\langle \xi - \mathbf{\Phi}(x), \nu_i(x) - \nu_i(y) \rangle| + |\langle \xi - \zeta, \nu_i(y) \rangle| + |\langle \mathbf{\Phi}(x) - \mathbf{\Phi}(y), \nu_i(y) \rangle|. \end{aligned} \tag{9.31}$$

Now, since $\xi \in M_{\mathbf{\Phi}(x)}$, the vector $\xi - \mathbf{\Phi}(x)$ is in the support of $N(\mathbf{\Phi}(x))$, and thus

$$|\xi - \mathbf{\Phi}(x)| \leqslant |N(\mathbf{\Phi}(x))|.$$

Therefore, if we apply Lemma 9.1.2 we easily estimate

$$|\langle \xi - \mathbf{\Phi}(x), \nu_i(x) - \nu_i(y) \rangle| \leqslant C \|N\|_{C^0(\Sigma)} \|D^2 \boldsymbol{\varphi}\|_{C^0} |y - x|. \tag{9.32}$$

In order to estimate the second and third term of (9.31), instead, we first decompose both $\xi - \zeta$ and $\mathbf{\Phi}(x) - \mathbf{\Phi}(y)$ by projecting them onto the planes π_0 and $\pi_0^\perp$. Then, we use (9.17) to conclude that

$$\begin{aligned} |\langle \xi - \zeta, \nu_i(y) \rangle| &\leqslant |\langle y' - x', \mathbf{p}_{\pi_0}(\nu_i(y)) \rangle| + |\langle w' - \nu', \mathbf{p}_{\pi_0^\perp}(\nu_i(y)) \rangle| \\ &\leqslant C |D\boldsymbol{\varphi}(y)| |y' - x'| + |w' - \nu'| \\ &\overset{(9.29)}{\leqslant} (C \|D\boldsymbol{\varphi}\|_{C^0} + L)\, |y' - x'|, \end{aligned} \tag{9.33}$$

and analogously

$$\begin{aligned} |\langle \mathbf{\Phi}(x) - \mathbf{\Phi}(y), \nu_i(y) \rangle| &\leqslant C |D\boldsymbol{\varphi}(y)| |y - x| + |\boldsymbol{\varphi}(y) - \boldsymbol{\varphi}(x)| \\ &\leqslant C \|D\boldsymbol{\varphi}\|_{C^0} |y - x|. \end{aligned} \tag{9.34}$$

Inserting (9.32), (9.33) and (9.34) into (9.31), we then conclude the following estimate:

$$|w^i - \nu^i| \leqslant C\left(\|N\|_{C^0}\|D^2\boldsymbol{\varphi}\|_{C^0} + \|D\boldsymbol{\varphi}\|_{C^0}\right)|y - x| + \left(C\|D\boldsymbol{\varphi}\|_{C^0} + L\right)|y' - x'|. \tag{9.35}$$

Therefore, in order to conclude, we need to bound:

$$\begin{aligned} |y' - x'| &= |\mathbf{p}_{\pi_0}(\zeta) - \mathbf{p}_{\pi_0}(\xi)| \\ &= \left| y + \sum_{i=1}^{n} w^i \mathbf{p}_{\pi_0}(\nu_i(y)) - x - \sum_{i=1}^{n} \nu^i \mathbf{p}_{\pi_0}(\nu_i(x)) \right| \\ &\leqslant |y - x| + \sum_{i=1}^{n} \left(|w^i||\nu_i(y) - \nu_i(x)| + |w^i - \nu^i||\mathbf{p}_{\pi_0}(\nu_i(x))|\right) \\ &\leqslant \left(1 + C\|D^2\boldsymbol{\varphi}\|_{C^0}\right)|y - x| + C\|D\boldsymbol{\varphi}\|_{C^0}|w - \nu|. \end{aligned} \tag{9.36}$$

If we combine (9.35) and (9.36), after standard algebraic computations we obtain:

$$|w - \nu| \leqslant C\left(\|N\|_{C^0}\|D^2\boldsymbol{\varphi}\|_{C^0} + \|D\boldsymbol{\varphi}\|_{C^0} + \mathrm{Lip}(f)\right)|y - x| + Cc_0^2|w - \nu|, \tag{9.37}$$

where the constant C appearing on the right-hand side of the inequality is purely geometric, and, in particular, does not depend on c_0. This allows us to conclude that if c_0 is such that

$$Cc_0^2 \leqslant \frac{1}{2}, \tag{9.38}$$

then a cone condition for $\widetilde{M}$ holds in the form

$$|w - \nu| \leqslant \tilde{\tau}|y - x| \tag{9.39}$$

with $\tilde{\tau}$ as in (9.26) for any (y, w) in a suitable neighborhood of (x, ν) such that $\widetilde{M}(y, w) > 0$. Since the choice of the point (x, ν) was arbitrary, the proof is complete. □

Proof of Theorem 9.1.4. We start proving that N is Lipschitz continuous. Let c_0 be such that Proposition 9.2.1 and Proposition 9.2.2 both hold. We make the following

Claim. For every $p \in \Sigma$ there exists an open neighborhood $\mathcal{U}_p$ of p in Σ such that

$$\mathcal{G}(N(q), N(p)) \leqslant \sqrt{Q}\tilde{\tau}'\mathbf{d}(q, p) \quad \text{for every } q \in \mathcal{U}_p, \tag{9.40}$$

where $\tilde{\tau}'$ satisfies the same estimate as in equation (9.26) and $\mathbf{d}(\cdot, \cdot)$ is the geodesic distance function on Σ. In order to see this, fix a point $p \in \Sigma$ and let M_p denote, as usual, the set of points $\xi \in \mathbf{U}$ such that $\Pi(\xi) = p$ and $M(\xi) > 0$. Assume that $M_p = \{\xi_1, \dots \xi_J\}$. If $p = \boldsymbol{\Phi}(x)$, then for any $j = 1, \dots, J$ one has

$$\xi_j = \boldsymbol{\Phi}(x) + \sum_{i=1}^{n} \nu_j^i \nu_i(x). \tag{9.41}$$

By Proposition 9.2.2, there exist neighborhoods U_j of x in B_s and V_j of $\nu_j := \left(\nu_j^1, \dots, \nu_j^n\right)$ in $\mathbb{R}^n$ such that if

$$\zeta = \zeta(y, w) := \boldsymbol{\Phi}(y) + \sum_{i=1}^{n} w^i \nu_i(y) \quad \text{with } (y, w) \in U_j \times V_j \tag{9.42}$$

is such that $M(\zeta) > 0$ then necessarily

$$|w - v_j| \leqslant \tilde{\tau}|y - x|. \tag{9.43}$$

Let $(x(\zeta), v(\zeta))$ denote the inverse mapping of $\zeta(x, v)$, given by

$$x(\zeta) := \mathbf{p}_{\pi_0} \circ \Pi(\zeta), \qquad v^i(\zeta) := \langle \zeta - \Pi(\zeta), v_i(\Pi(\zeta)) \rangle, \tag{9.44}$$

and let

$$\mathcal{V}_j := \{\zeta \in \mathbf{U} : \ (x(\zeta), v(\zeta)) \in U_j \times V_j\}. \tag{9.45}$$

Each $\mathcal{V}_j$ is an open neighborhood of ξ_j, and moreover the cone condition (9.43) forces $\mathcal{V}_j \cap M_p = \{\xi_j\}$. We can also assume without loss of generality that the $\mathcal{V}_j$'s are pairwise disjoint. By Proposition 9.2.1, since M is coherent there exists a neighborhood $\mathcal{U}_p$ of p in Σ such that

$$\sum_{\zeta \in \Pi^{-1}(\{q\}) \cap \mathcal{V}_j} M(\zeta) = M(\xi_j) \quad \text{for every } q \in \mathcal{U}_p. \tag{9.46}$$

Since

$$\sum_{j=1}^{J} M(\xi_j) = Q, \tag{9.47}$$

it is evident that when q is chosen in $\mathcal{U}_p$ then any $\zeta \in \Pi^{-1}(\{q\})$ with $M(\zeta) > 0$ must be an element of one and only one $\mathcal{V}_j$, and thus we can write

$$N(q) = \sum_{j=1}^{J} \sum_{\zeta \in \Pi^{-1}(\{q\}) \cap \mathcal{V}_j} M(\zeta) [\![\zeta - q]\!], \tag{9.48}$$

whereas

$$N(p) = \sum_{j=1}^{J} M(\xi_j) [\![\xi_j - p]\!]. \tag{9.49}$$

We can now estimate, for any $\zeta \in \Pi^{-1}(\{q\}) \cap \mathcal{V}_j$ with $M(\zeta) > 0$ and $q = \mathbf{\Phi}(y) \in \mathcal{U}_p$:

$$\begin{aligned} \big|(\zeta - q) - (\xi_j - p)\big| &= \left| \sum_{i=1}^{n} w^i v_i(y) - \sum_{i=1}^{n} v_j^i v_i(x) \right| \\ &\leqslant \sum_{i=1}^{n} \left(|w^i| |v_i(y) - v_i(x)| + |w^i - v_j^i| \right) \\ &\leqslant C \|N\|_{C^0} \|D^2 \boldsymbol{\varphi}\|_{C^0} |y - x| + C|w - v_j| \\ &\overset{(9.43)}{\leqslant} C \left(\|N\|_{C^0} \|D^2 \boldsymbol{\varphi}\|_{C^0} + \tilde{\tau} \right) |y - x| = \tilde{\tau}' |y - x|. \end{aligned} \tag{9.50}$$

Observe that the constant C appearing in (9.50) is purely geometric, and that $\tilde{\tau}'$ also satisfies the bound in (9.26). It is now evident that

$$\mathcal{G}(N(q), N(p))^2 \leqslant Q \tilde{\tau}'^2 |y - x|^2, \tag{9.51}$$

from which the claim follows because $|y - x| \leqslant |q - p| \leqslant \mathbf{d}(q, p)$.

Now, we show how from the claim one can easily conclude the Lipschitz continuity of N with the required estimates. Fix two distinct points $p, q \in \Sigma$, and let $\gamma\colon [a,b] \to \Sigma$ be any (piecewise) smooth curve such that $\gamma(a) = p$ and $\gamma(b) = q$. By the claim, for every $t \in [a,b]$ there exists a neighborhood $\mathcal{U}_{\gamma(t)}$ such that

$$\mathcal{G}(N(z), N(\gamma(t))) \leqslant \sqrt{Q}\tilde{\tau}'\mathbf{d}(z, \gamma(t)) \quad \text{for every } z \in \mathcal{U}_{\gamma(t)}. \tag{9.52}$$

Since γ is continuous, there exist numbers δ_t such that

$$I_t := (t - \delta_t, t + \delta_t) \subset \gamma^{-1}(\mathcal{U}_{\gamma(t)}). \tag{9.53}$$

The family $\{I_t\}$ is an open covering of the interval $[a,b]$, and thus by compactness we can extract a finite subcovering $\{I_{t_i}\}_{i=0}^{K}$. We may assume, refining the subcovering if necessary, that an interval I_{t_i} is not completely contained in an interval I_{t_j} if $i \neq j$. If we relabel the indices of the points t_i in a non-decreasing order, and thus in such a way that $\gamma(t_i)$ precedes $\gamma(t_{i+1})$, we can now choose an auxiliary point $s_{i,i+1}$ in $I_{t_i} \cap I_{t_{i+1}} \cap (t_i, t_{i+1})$ for each $i = 0, \dots, K-1$. We can finally conclude:

$$\begin{aligned}\mathcal{G}(N(q),N(p)) &\leqslant \mathcal{G}(N(p), N(\gamma(t_0))) \\ &\quad + \sum_{i=0}^{K-1} \left(\mathcal{G}(N(\gamma(t_i)), N(\gamma(s_{i,i+1}))) + \mathcal{G}(N(\gamma(s_{i,i+1})), N(\gamma(t_{i+1})))\right) + \mathcal{G}(N(\gamma(t_K)), N(q)) \\ &\overset{(9.53)}{\leqslant} \sqrt{Q}\tilde{\tau}'\mathscr{L}(\gamma),\end{aligned} \tag{9.54}$$

where $\mathscr{L}(\gamma)$ is the length of the curve γ. Minimizing among all the piecewise smooth curves γ joining p to q, one finally obtains

$$\mathcal{G}(N(q), N(p)) \leqslant \sqrt{Q}\tilde{\tau}'\mathbf{d}(q,p), \tag{9.55}$$

that is

$$\mathrm{Lip}(N) \leqslant \sqrt{Q}\tilde{\tau}'. \tag{9.56}$$

The estimate (9.4) is now just a consequence of (9.26).

In order to complete the proof, we are left with showing the validity of (9.5), (9.6) and (9.7). This can be done by reproducing verbatim the proof suggested by De Lellis and Spadaro in [DS15]; the arguments will be presented here only for completeness.

We start with the proof of (9.5) and (9.6). Fix a point $x \in B_s$, and let $p := \boldsymbol{\Phi}(x) \in \Sigma$. Observe that, by (9.9) and (9.25), the definition of the value of $N(p)$ does not change if we replace $\boldsymbol{\varphi}$ with its first order Taylor expansion at x, since this operation preserves the fiber $\Pi^{-1}(\{p\})$. Furthermore, we can assume without loss of generality that $x = 0$ and $\boldsymbol{\varphi}(0) = 0$. We will still use the symbols π_0 and $\pi_0^\perp$ to denote the planes $\mathbb{R}^m \times \{0\} \simeq \mathbb{R}^m$ and $\{0\} \times \mathbb{R}^n \simeq \mathbb{R}^n$ respectively, whereas the tangent space $T_0\Sigma$ and its orthogonal complement $T_0^\perp\Sigma$ will be denoted π and $\varkappa$. Now, concerning the estimate (9.5), assume that $f(0) = \sum_{\ell=1}^{Q} \llbracket \nu_\ell \rrbracket$, set $\xi_\ell := (0, \nu_\ell) \in \pi_0 \times \pi_0^\perp$ and $q_\ell := \mathbf{p}_\pi(\xi_\ell)$. If $N(q_\ell) = \sum_{j=1}^{Q} \llbracket \zeta_{\ell,j} \rrbracket$, then there is an index $j(\ell)$ such that $\zeta_{\ell,j(\ell)} = \xi_\ell$. If the point $\zeta_{\ell,j(\ell)}$ has coordinates (q_ℓ, ν'_ℓ) in the frame $\pi \times \varkappa$, we get

$$\begin{aligned}|\nu_\ell| &\leqslant |q_\ell| + |\nu'_\ell| \leqslant |q_\ell| + |N(0)| + \mathcal{G}(N(0), N(q_\ell)) \\ &\leqslant |N(0)| + (1 + \mathrm{Lip}(N))\,|q_\ell| \leqslant |N(0)| + C\,(1 + \mathrm{Lip}(N))\,\|D\boldsymbol{\varphi}\|_{C^0}|\nu_\ell|,\end{aligned} \tag{9.57}$$

where we have used that $q_\ell = |\mathbf{p}_\pi(\xi_\ell)| \leqslant C|D\boldsymbol{\varphi}(0)||\xi_\ell| = C\|D\boldsymbol{\varphi}\|_{C^0}|v_\ell|$. Now, we use (9.4) with $\boldsymbol{\varphi}$ linear to estimate

$$\mathrm{Lip}(N) \leqslant C\left(\|D\boldsymbol{\varphi}\|_{C^0} + \mathrm{Lip}(f)\right) \leqslant Cc_0. \tag{9.58}$$

Thus, we conclude

$$|v_\ell| \leqslant |N(0)| + C(1 + Cc_0)c_0|v_\ell|. \tag{9.59}$$

Since the constant C is purely geometric and does not depend on c_0, we deduce that if c_0 is sufficiently small then $|v_\ell| \leqslant 2|N(0)|$. Summing over $\ell \in \{1, \dots Q\}$ we obtain $|f(0)| \leqslant 2\sqrt{Q}|N(0)|$. The proof of the other inequality, namely $|N(0)| \leqslant 2\sqrt{Q}|f(0)|$, is analogous, reversing the roles of the systems of coordinates $\pi_0 \times \pi_0^\perp$ and $\pi \times \varkappa$. This concludes the proof of (9.5).

We proceed with the proof of (9.6). Assume once again that $f(0) = \sum_{\ell=1}^Q \llbracket v_\ell \rrbracket$, and write $N(0) = \sum_{\ell=1}^Q \llbracket \xi_\ell \rrbracket$. For every $\ell \in \{1, \dots, Q\}$, we set $x_\ell := \mathbf{p}_{\pi_0}(\xi_\ell)$, $w_\ell := \mathbf{p}_{\pi_0^\perp}(\xi_\ell)$ and $w'_\ell := \mathbf{p}_\varkappa(\xi_\ell)$, so that the point ξ_ℓ is represented by coordinates (x_ℓ, w_ℓ) in the standard reference frame $\pi_0 \times \pi_0^\perp$ and by coordinates $(0, w'_\ell)$ in the frame $\pi \times \varkappa$. As usual, we have:

$$|x_\ell| = |\mathbf{p}_{\pi_0}(\xi_\ell)| \leqslant C|D\boldsymbol{\varphi}(0)||\xi_\ell| \leqslant C|D\boldsymbol{\varphi}(0)||N(0)| =: \rho. \tag{9.60}$$

Using these notations, one has $|\boldsymbol{\eta} \circ N(0)| = Q^{-1}\left|\sum_\ell w'_\ell\right|$. On the other hand, under our usual smallness assumptions on the size of c_0, we can also assume that the operator norm of the linear and invertible transformation $L\colon \pi_0^\perp \to \varkappa$ is bounded by 2. Thus, we can further estimate $|\boldsymbol{\eta} \circ N(0)| \leqslant 2Q^{-1}\left|\sum_{\ell=1}^Q w_\ell\right|$, so that in order to get (9.6) it would suffice to prove the following:

$$\left|\sum_{\ell=1}^Q w_\ell\right| \leqslant \left|\sum_{\ell=1}^Q v_\ell\right| + C\mathrm{Lip}(f)\rho. \tag{9.61}$$

In order to show the validity of (9.61), we notice that if we set $h := \mathrm{Lip}(f)\rho$, then we can decompose $f(0) = \sum_{j=1}^J \llbracket T_j \rrbracket$, where each $T_j \in \mathcal{A}_{Q_j}(\mathbb{R}^n)$, $\sum_{j=1}^J Q_j = Q$ and with the property that:

(i) $\mathrm{diam}(T_j) \leqslant 4Qh$;

(ii) $|y - z| > 4h$ for all $y \in \mathrm{spt}(T_i)$ and $z \in \mathrm{spt}(T_j)$ when $i \neq j$.

This claim can be justified with the following simple argument. First, we order the vectors v_ℓ, and then we partition them in subcollections T_j according to the following algorithm: T_1 contains v_1 and any other vector v_ℓ for which there exists a chain $v_{\ell(1)}, \dots v_{\ell(k)}$ with $\ell(1) = 1$, $\ell(k) = \ell$ and $|v_{\ell(i+1)} - v_{\ell(i)}| \leqslant 4h$ for every $i = 0, \dots, k-1$. By construction, $\mathrm{diam}(T_1) \leqslant 4Qh$, and if $\mathrm{spt}(T_1) = \mathrm{spt}(f(0))$ then we are finished. Otherwise, we construct T_2 applying the same algorithm to the vectors in $\mathrm{spt}(f(0)) \setminus \mathrm{spt}(T_1)$. The construction of the algorithm guarantees that also property (ii) is satisfied.

Given the above decomposition of $f(0)$, we observe that from the choice of the constants it follows that in the ball B_ρ the function f decomposes into the sum $f = \sum_{j=1}^J \llbracket f^j \rrbracket$ of J Lipschitz functions $f^j\colon B_\rho \to \mathcal{A}_{Q_j}(\mathbb{R}^n)$ with $\mathrm{Lip}(f^j) \leqslant \mathrm{Lip}(f)$ for every j. In agreement with this decomposition, also the graph $\mathrm{Gr}(f|_{B_\rho})$ separates into the union $\bigcup_{j=1}^J \mathrm{Gr}(f^j)$. By the definition of the vector field N (cf. again (9.9) and (9.25)), the support of $N(0)$ contains points from

each of these sets; furthermore, if $\xi \in \operatorname{spt}(N(0)) \cap \operatorname{Gr}(f^j)$ then $M(\xi) = M_{f^j}(\mathbf{p}_{\pi_0}(\xi), \mathbf{p}_{\pi_0^\perp}(\xi))$. It follows that also $N(0)$ can be decomposed into $N(0) = \sum_{j=1}^{J} \sum_{i=1}^{Q_j} \llbracket \xi_i^j \rrbracket$ with the property that $\xi_i^j \in \operatorname{Gr}(f^j)$ for every $i = 1, \dots, Q_j$.

Now, by the definition of the distance $\mathcal{G}$, for each $\xi_i^j \in \operatorname{spt}(N(0))$ there exists a point $v_{k(j,i)} \in \operatorname{spt}(f^j(0))$ such that $|w_i^j - v_{k(j,i)}| \leqslant \mathcal{G}(f^j(x_i^j), f^j(0)) \leqslant \operatorname{Lip}(f)|x_i^j| \overset{(9.60)}{\leqslant} \operatorname{Lip}(f)\rho = h$. Hence, we conclude:

$$\begin{aligned}\left|\sum_{\ell=1}^{Q} w_\ell\right| &= \left|\sum_{j=1}^{J}\sum_{i=1}^{Q_j} w_i^j\right| \leqslant \left|\sum_{j=1}^{J}\sum_{i=1}^{Q_j} v_i^j\right| + \sum_{j=1}^{J}\sum_{i=1}^{Q_j} |w_i^j - v_i^j| \\ &\leqslant \left|\sum_{\ell=1}^{Q} v_\ell\right| + \sum_{j=1}^{J}\sum_{i=1}^{Q_j} \left(|w_i^j - v_{k(j,i)}| + |v_{k(j,i)} - v_i^j|\right) \leqslant \left|\sum_{\ell=1}^{Q} v_\ell\right| + Ch,\end{aligned}$$

where we used that $\operatorname{diam}(f^j(0)) \leqslant 4Qh$. This proves (9.61) and concludes the proof of (9.6).

Finally, we show that (9.7) holds. Let $x \in B_s$, and assume that $(x, \boldsymbol{\eta} \circ f(x)) = p + v$ for some $p \in \Sigma$ and $v \in T_p^\perp \Sigma$. Now, if $v = 0$ then the above assumption implies that $\boldsymbol{\eta} \circ f(x) = \boldsymbol{\varphi}(x)$, and thus (9.7) reduces to the first inequality in (9.5). On the other hand, if $v \neq 0$ then we shift Σ to $\tilde{\Sigma} := v + \Sigma$. Then, if we apply Theorem 9.1.4 with $\tilde{\Sigma}$ in place of Σ we obtain a vector field $\tilde{N}$ which satisfies $\tilde{N}(p + v) = \sum_\ell \llbracket N_\ell(p) - v \rrbracket$. Hence, $\mathcal{G}(N(p), Q\llbracket v \rrbracket) = \mathcal{G}(\tilde{N}(p + v), Q\llbracket 0 \rrbracket)$, which reduces the problem again to the case $v = 0$. This completes the proof of Theorem 9.1.4. □

Part III

Regularity and singularities of multiple-valued harmonic maps

10 RECTIFIABILITY OF THE HÖLDER SINGULAR STRATA

Let $\Omega \subset \mathbb{R}^m$ be an open set, and let $\mathcal{N}^n$ be a smooth compact Riemannian manifold, smoothly embedded in $\mathbb{R}^d$. In this chapter we study the fine properties of the Hölder singular set $\mathrm{sing}_H(u)$ of Dirichlet-minimizing Q-valued maps $u\colon \Omega \to \mathcal{A}_Q(\mathcal{N})$. In particular, we prove the following theorem.

Theorem 10.0.1. *Given a Dirichlet-minimizing* Q*-valued map* $u\colon B_2(0) \subset \mathbb{R}^m \to \mathcal{A}_Q(\mathcal{N})$ *with energy* $\mathscr{E}(u, B_2(0)) \leqslant \Lambda$*, if we denote* $B_r\left(\mathrm{sing}_H(u)\right) := \bigcup_{x\in \mathrm{sing}_H(u)} B_r(x)$ *then we have*

$$\mathcal{L}^m\left(B_r\left(\mathrm{sing}_H(u)\right) \cap B_1(0)\right) \leqslant Cr^3$$

for some constant $C = C(m, \mathcal{N}, Q, \Lambda)$*. Moreover,* $\mathrm{sing}_H(u)$ *is countably* $(m-3)$*-rectifiable.*

The plan of the chapter is the following. In Section 10.1 we consider a slightly modified version of the rescaled Dirichlet energy $\mathscr{E}(u, B_r(x))$ in a ball $B_r(x) \Subset \Omega$, and we prove its monotonicity with respect to the radius r of the ball. With section 10.2 we enter the core of the chapter. First, we provide an alternative definition of the quantitative singular strata $\mathcal{S}^k_{\varepsilon,r}(u)$: as in the classical Cheeger-Naber quantitative stratification (cf. Definitions 2.3.12 and 2.3.13), $\mathcal{S}^k_{\varepsilon,r}$ is, roughly speaking, the set of points $x \in \Omega$ for which u on $B_r(x)$ is ε-far away from being homogeneous and invariant with respect to a k-dimensional subspace. While the notion of closeness employed by Cheeger and Naber relies on the L^2 distance of the map u from some k-symmetric model map h, we propose a notion that focuses on the L^2 norm of the *gradient* of u along arbitrary k-subspaces. Once we have the notion of quantitative stratification at our disposal, we can state the main theorem of this chapter, that is Theorem 10.2.17, concerning Minkowski estimates and rectifiability of the quantitative strata: Theorem 10.0.1, in fact, is essentially a simple corollary of Theorem 10.2.17 and (a slightly refined version of) the ε-regularity theorem. After discussing some quantitative versions of the ε-regularity theorem in Section 10.3, we start the machinery which will eventually lead us to the proof of Theorem 10.2.17. Sections 10.4 and 10.5 contain the most important technical tools needed for the proof, which is instead completed in Section 10.6 with a double inductive covering argument in the spirit of Naber-Valtorta [NV17].

10.1 THE MOLLIFIED DIRICHLET ENERGY AND ITS MONOTONICITY

From now on, we fix an open subset $\Omega \subset \mathbb{R}^m$, a smooth compact Riemannian manifold $\mathcal{N}^n \hookrightarrow \mathbb{R}^d$ and an integer $Q \geqslant 1$. Recall from § 2.3.3 the definition of the space $W^{1,2}_{loc}(\Omega, \mathcal{A}_Q(\mathcal{N}))$ and the notions of stationary and minimizing Q-harmonic maps $u \in W^{1,2}_{loc}(\Omega, \mathcal{A}_Q(\mathcal{N}))$.

Definition 10.1.1 (Mollified energy)**.** Let $\varphi = \varphi(t)$ be any non-negative function in $C^1_c([0,1))$ which is constant in a neighborhood of $t = 0$.

Then, for any $u \in W^{1,2}_{loc}(\Omega, \mathcal{A}_Q(\mathcal{N}))$ and for any $B_r(x) \subset \Omega$ we define the quantity

$$\theta_u(x,r) := r^{2-m} \int \varphi\left(\frac{|x-y|}{r}\right) |Du(y)|^2 \, dy. \tag{10.1}$$

When the map u is fixed, we will simply write $\theta(x,r)$ for the sake of notational simplicity. In what follows, we show that, under suitable assumptions on φ, the function $r \mapsto \theta(x,r)$ is monotone non-decreasing for fixed x, and we explicitly compute its derivative. Recall that for any $x \in \mathbb{R}^m$ the radial unit vector field with respect to x is denoted $r_x(y) := \frac{y-x}{|y-x|}$.

Lemma 10.1.2. *Let $u \in W^{1,2}_{loc}(\Omega, \mathcal{A}_Q(\mathcal{N}))$ be a stationary Q-harmonic map, and let $x \in \Omega$. For any φ as in Definition 10.1.1, the following identity holds true for all r such that $B_r(x) \subset \Omega$:*

$$\frac{d}{dr}\theta(x,r) = -2r^{2-m} \int \varphi'\left(\frac{|x-y|}{r}\right) \frac{|x-y|}{r^2} |D_{r_x} u(y)|^2 \, dy. \tag{10.2}$$

In particular, if we let $\psi = \psi(t)$ denote a primitive function of $\varphi'(t)t^{m-2}$, then for $0 < s < r < \operatorname{dist}(x, \partial\Omega)$ we have:

$$\theta(x,r) - \theta(x,s) = \int \left(\psi\left(\frac{|x-y|}{r}\right) - \psi\left(\frac{|x-y|}{s}\right)\right) |x-y|^{2-m} |D_{r_x} u(y)|^2 \, dy. \tag{10.3}$$

In case we choose φ to be non-increasing, we have that $r \mapsto \theta(x,r)$ is non-decreasing; furthermore, if $-\varphi'(t) \geqslant (1-t)^+$ then it holds

$$\theta(x,r) - \theta(x,r/2) \geqslant C \int_{B_{\frac{r}{2}}(x)} \frac{|x-y|}{r^{m-1}} |D_{r_x} u(y)|^2 \, dy \tag{10.4}$$

for some positive constant $C = C(m)$.

Proof. The identity (10.2) follows from the inner variation formula, equation (2.38). Indeed, for any fixed $x \in \Omega$ and $0 < r < \operatorname{dist}(x, \partial\Omega)$ define the vector field $X(y) := \varphi\left(\frac{|x-y|}{r}\right)(y-x)$. If we plug this choice of X in (2.38), we easily deduce the identity

$$(m-2) \int \varphi\left(\frac{|x-y|}{r}\right) |Du(y)|^2 \, dy + \int \varphi'\left(\frac{|x-y|}{r}\right) \frac{|x-y|}{r} \left(|Du(y)|^2 - 2|D_{r_x} u(y)|^2\right) dy = 0.$$

To conclude, we can differentiate the quantity $\theta(x,r)$ in r and obtain the differential identity (10.2).

Now, let ψ be a primitive function of $\varphi'(t)t^{m-2}$. We have

$$\frac{d}{dr}\psi\left(\frac{|x-y|}{r}\right) = -\frac{1}{r}\varphi'\left(\frac{|x-y|}{r}\right)\left(\frac{|x-y|}{r}\right)^{m-1},$$

and thus we can rewrite (10.2) as

$$\frac{d}{dr}\theta(x,r) = 2\frac{d}{dr}\int \psi\left(\frac{|x-y|}{r}\right) |x-y|^{2-m} |D_{r_x} u(y)|^2 \, dy.$$

Integrating immediately leads to (10.3).

If we choose $\varphi'(t) \leqslant 0$, then (10.2) implies that $r \mapsto \theta(x,r)$ is non-decreasing. In case $-\varphi'(t) \geqslant (1-t)^+$, we have for $0 < a \leqslant \frac{1}{2}$

$$\psi(a) - \psi(2a) = -\int_a^{2a} \varphi'(t) t^{m-2} \geqslant a^{m-1} \left(\frac{2^{m-1}-1}{m-1} - a\frac{2^m-1}{m} \right) \geqslant C_m a^{m-1}. \quad (10.5)$$

Hence, the estimate (10.4) can be deduced from (10.3) by using the fact that ψ is non-increasing to estimate

$$\theta(x,r) - \theta(x,r/2) \geqslant \int_{B_{\frac{r}{2}}(x)} \left(\psi\left(\frac{|x-y|}{r}\right) - \psi\left(\frac{2|x-y|}{r}\right) \right) |x-y|^{2-m} |D_{r_x} u(y)|^2 \, dy, \quad (10.6)$$

and then using the inequality in (10.5) with $a = \frac{|x-y|}{r}$ for $y \in B_{\frac{r}{2}}(x)$. □

Assumption 10.1.3. For the rest of the chapter, we will assume that φ has been fixed, and that it satisfies the condition $-\varphi'(t) \geqslant (1-t)^+$, so that the inequality (10.4) holds.

10.2 QUANTITATIVE STRATIFICATION

The first step towards the definition of the quantitative singular strata is to introduce the notion of "model maps" having a given number of symmetries. This definition is analogous to Definition 2.3.11.

Definition 10.2.1 (k-symmetric maps)**.** A map $h \in W^{1,2}_{loc}(\mathbb{R}^m, \mathcal{A}_Q(\mathcal{N}))$ is said to be:

- *homogeneous* with respect to $x \in \mathbb{R}^m$ if

$$h(x+\lambda v) = h(x+v) \quad \text{for all } \lambda > 0, \text{ for every } v \in \mathbb{R}^m,$$

  or equivalently if

$$D_{r_x} h = Q[\![0]\!] \quad \text{a.e. in } \mathbb{R}^m.$$

- k-*symmetric* if it is homogeneous with respect to the origin and there exists a linear subspace $L \subset \mathbb{R}^m$ with $\dim(L) = k$ along which h is invariant, that is

$$h(x+v) = h(x) \quad \text{for every } x \in \mathbb{R}^m, \text{ for all } v \in L,$$

  or, equivalently, such that

$$D_v h(x) = Q[\![0]\!], \quad \text{for a.e. } x \in \mathbb{R}^m, \text{ for all } v \in L.$$

Observe that if $h \in W^{1,2}_{loc}(\mathbb{R}^m, \mathcal{A}_Q(\mathcal{N}))$ is stationary and homogeneous with respect to x then $\theta_h(x,s) = \theta_h(x,r)$ for every $0 < s < r$ by (10.3). Also, if h is k-symmetric with invariance subspace L then the energy of h in the direction of any $v \in L$ vanishes. Hence, it is very natural to give the following definition, which is the starting point for introducing the quantitative stratification.

Definition 10.2.2. Given a stationary Q-harmonic map $u \in W^{1,2}_{loc}(\Omega, \mathcal{A}_Q(\mathcal{N}))$, we say that a ball $B_r(x)$ with $B_{2r}(x) \subset \Omega$ is (k, ε)-symmetric for u if and only if the following conditions hold:

(a) $\theta_u(x, 2r) - \theta_u(x, r) < \varepsilon$;

(b) there exists a linear subspace $L \subset \mathbb{R}^m$ with $\dim(L) = k$ such that

$$r^{2-m} \int_{B_r(x)} |D_L u(y)|^2 \, dy \leqslant \varepsilon,$$

where

$$\int_{B_r(x)} |D_L u(y)|^2 \, dy := \int_{B_r(x)} \sum_{i=1}^{k} |D_{e_i} u(y)|^2 \, dy,$$

for any orthonormal basis $\{e_i\}_{i=1}^k$ of L.

Remark 10.2.3. Observe that the conditions (a) and (b) above are scale-invariant in the following sense. For $x \in \Omega$ and $r > 0$ such that $B_{2r}(x) \subset \Omega$, consider the blow-up map $T^u_{x,r}$ given by

$$T^u_{x,r}(y) := u(x + ry).$$

Then, $B_r(x)$ is (k, ε)-symmetric with respect to u if and only if $B_1(0)$ is (k, ε)-symmetric with respect to $T^u_{x,r}$.

Definition 10.2.4 (Quantitative stratification)**.** Let $u \in W^{1,2}_{loc}(\Omega, \mathcal{A}_Q(\mathcal{N}))$ be stationary Q-harmonic, and let $\varepsilon, r > 0$ and $k \in \{0, \dots, m\}$. We will set

$$\mathcal{S}^k_{\varepsilon,r}(u) := \{x \in \Omega : \text{ for no } r \leqslant s < 1 \text{ the ball } B_s(x) \text{ is } (k+1, \varepsilon)\text{-symmetric with respect to } u\}.$$

It is an immediate consequence of the definition that if $k' \leqslant k$, $\varepsilon' \geqslant \varepsilon$ and $r' \leqslant r$ then

$$\mathcal{S}^{k'}_{\varepsilon',r'}(u) \subseteq \mathcal{S}^k_{\varepsilon,r}(u).$$

Hence, we can set:

$$\mathcal{S}^k_\varepsilon(u) := \bigcap_{r>0} \mathcal{S}^k_{\varepsilon,r}(u), \qquad \mathcal{S}^k(u) := \bigcup_{\varepsilon>0} \mathcal{S}^k_\varepsilon(u).$$

Remark 10.2.5. Note that from Theorem 2.3.21 one easily deduces that if $u \in W^{1,2}_{loc}(\Omega, \mathcal{A}_Q(\mathcal{N}))$ is energy minimizing and a ball $B_r(x)$ is (m, ε_0)-symmetric for u, with the ε_0 given in there, then u is Hölder continuous in $B_{\frac{r}{2}}(x)$, and thus in particular $\mathcal{S}^k(u) \cap B_{\frac{r}{2}}(x) = \emptyset$ for every $k \leqslant m-1$. In fact, we can also conclude that $\mathcal{S}^m(u) \setminus \mathcal{S}^{m-1}(u)$ coincides with the set $\mathrm{reg}_H(u) := \Omega \setminus \mathrm{sing}_H(u)$ of points of Hölder continuity for u, and $\mathrm{sing}_H(u) = \mathcal{S}^{m-1}(u)$.

The first important property of the quantitative stratification is that the sets $\mathcal{S}^k(u)$ coincide with the classical singular k-strata defined by means of the number of symmetries of the tangent maps (cf. § 2.3.3).

Proposition 10.2.6. *Let $u \in W^{1,2}(\Omega, \mathcal{A}_Q(\mathcal{N}))$ be energy minimizing. Then*

$$\mathcal{S}^k(u) = \{x : \textit{no tangent map to } u \textit{ at } x \textit{ is } (k+1)\textit{-symmetric}\}.$$

Proof. First recall that for any $x \in \Omega$ there exists at least one tangent map $\phi \in W^{1,2}_{loc}(\mathbb{R}^m, \mathcal{A}_Q(\mathcal{N}))$ to u at x, and that all tangent maps are energy minimizing and 0-symmetric.

Now, let x be a point such that there exists a tangent map ϕ to u at x which is $(k+1)$-symmetric. Then, there is a sequence $r_j \searrow 0$ of radii such that the corresponding sequence of blow-up maps $u_j := T^u_{x,r_j}$ satisfies $\mathcal{G}(u_j, \phi) \to 0$ in $L^2_{loc}(\mathbb{R}^m)$ as $j \to \infty$ and furthermore

$$\theta_\phi(0,\rho) = \lim_{j\to\infty} \theta_{u_j}(0,\rho) \quad \forall\, \rho > 0.$$

In particular, since ϕ is homogeneous with respect to the origin, and thus $\theta_\phi(0,2) - \theta_\phi(0,1) = 0$ by (10.3), for any $\varepsilon > 0$ there exists $j_0 = j_0(\varepsilon)$ such that

$$\theta_{u_j}(0,2) - \theta_{u_j}(0,1) < \varepsilon \qquad \forall\, j \geqslant j_0. \tag{10.7}$$

Moreover, since ϕ is $(k+1)$-symmetric there exists a linear subspace $L \subset \mathbb{R}^m$ with $\dim(L) = k+1$ such that $D_L\phi = Q[\![0]\!]$ a.e. in $\mathbb{R}^m$. Hence, from the convergence of energy for minimizers we deduce that if j_0 is chosen suitably large then also

$$\int_{B_1(0)} |D_L u_j|^2 \, dy \leqslant \varepsilon \qquad \forall\, j \geqslant j_0. \tag{10.8}$$

Together, equations (10.7) and (10.8) imply that $B_{r_j}(x)$ is $(k+1,\varepsilon)$-symmetric for u if $j \geqslant j_0(\varepsilon)$, and thus $x \notin \mathcal{S}^k(u)$. This proves the first inclusion, namely

$$\mathcal{S}^k(u) \subseteq \{x \in \Omega : \text{ no tangent map to } u \text{ at } x \text{ is } (k+1)\text{-symmetric}\}.$$

In order to prove the other inclusion, assume that $x \notin \mathcal{S}^k(u)$. Then, for every $j \in \mathbb{N}$ there exist a radius $r_j > 0$ and a $(k+1)$-dimensional linear subspace $L_j \subset \mathbb{R}^m$ such that if we set $u_j := T^u_{x,r_j}$ then

$$\theta_{u_j}(0,2) - \theta_{u_j}(0,1) < \frac{1}{j} \tag{10.9}$$

and

$$\int_{B_1(0)} |D_{L_j} u_j|^2 \, dy \leqslant \frac{1}{j}. \tag{10.10}$$

Modulo a simple right composition of each u_j with a rotation, we can assume that the invariant subspace is a fixed $(k+1)$-dimensional subspace $L \subset \mathbb{R}^m$. By the compactness theorem for Q-valued energy minimizing maps, a subsequence (not relabeled) of the u_j's converges in L^2_{loc} and in energy to an energy minimizing map ϕ. From (10.9) together with (10.4) we deduce that the limit map ϕ is homogeneous with respect to the origin. Furthermore, (10.10) implies that ϕ is invariant along the subspace L, and thus ϕ is $(k+1)$-symmetric. Now, if a subsequence of the r_j's converges to 0 then ϕ is by definition a tangent map to u at x. If, on the other hand, the r_j's are bounded away from 0 then $u = \phi$ on a ball of positive radius centered at x, and thus, in particular, all tangent maps to u at x coincide with ϕ. In either case, this completes the proof. □

Corollary 10.2.7. *Let $u \in W^{1,2}(\Omega, \mathcal{A}_Q(\mathcal{N}))$ be energy minimizing. Then*

$$\mathcal{S}^{m-1}(u) \setminus \mathcal{S}^{m-3}(u) = \emptyset.$$

Proof. This is a direct consequence of the Proposition 10.2.6, since the identity $\mathcal{S}^{m-1}(u) = \mathcal{S}^{m-3}(u)$ holds for the standard stratification. □

The definition of quantitative stratification that we have proposed differs from the original one introduced by Cheeger and Naber in [CN13a, CN13b] and then used by Naber and Valtorta in [NV17]. Of course, the Cheeger-Naber quantitative stratification can be without any difficulties extended to the Q-valued context. We recall the definition here, in order to compare it with Definition 10.2.4.

Definition 10.2.8. Let $u \in W^{1,2}_{loc}(\Omega, \mathcal{A}_Q(\mathcal{N}))$, and fix $k \in \{0, \dots, m\}$ and $\varepsilon > 0$. A ball $B_r(x)$ with $B_{2r}(x) \subset \Omega$ is said to be (k, ε)-symmetric for u *in the sense of Cheeger-Naber*, or briefly [CN] (k, ε)-symmetric, if there exists some k-symmetric map $h \in W^{1,2}_{loc}(\mathbb{R}^m, \mathcal{A}_Q(\mathcal{N}))$ such that

$$\fint_{B_r(x)} \mathcal{G}(u(y), h(y-x))^2 \, dy \leqslant \varepsilon. \tag{10.11}$$

The definitions of [CN] (ε, r)-singular strata ${}^{[CN]}\mathcal{S}^k_{\varepsilon,r}(u)$ and [CN] ε-singular strata ${}^{[CN]}\mathcal{S}^k_{\varepsilon}(u)$ can be then straightforwardly obtained according to the definition of [CN] (k, ε)-symmetry precisely as in Definition 2.3.13. In particular, ${}^{[CN]}\mathcal{S}^k(u)$ classically consists of all points $x \in \Omega$ having the property that there exists $\varepsilon > 0$ such that no ball $B_r(x)$ is [CN] $(k+1, \varepsilon)$-symmetric with respect to u.

The following simple proposition shows that if u is a minimizing Q-valued map then Definition 10.2.2 and Definition 10.2.8 are equivalent, in the sense that they generate the same stratification. In order to fix the ideas, for the vast majority of the following results we will work under the following assumption.

Assumption 10.2.9. Assume that $u \in W^{1,2}(B_{10}(0), \mathcal{A}_Q(\mathcal{N}))$ is a Q-valued energy minimizing map, and that $\mathscr{E}(u, B_{10}(0)) \leqslant \Lambda$.

Proposition 10.2.10. *For every $\varepsilon > 0$ there exists $\delta = \delta(m, \mathcal{N}, Q, \Lambda, \varepsilon) > 0$ such that for any u satisfying Assumption 10.2.9:*

(i) *if $B_r(x)$ is (k, δ)-symmetric for u, then it is* [CN] *(k, ε)-symmetric for u;*

(ii) *if $B_r(x)$ is* [CN] *(k, δ)-symmetric for u, then it is (k, ε)-symmetric for u.*

Proof. Since both the definitions of symmetry are scale-invariant, modulo translations and dilations it suffices to show the validity of the proposition for $x = 0$ and $r = 1$. We start proving the first claim. Assume by contradiction that there exist $\varepsilon_0 > 0$ and a sequence $\{u_j\}_{j \in \mathbb{N}}$ of maps as in Assumption 10.2.9 for which the ball B_1 is (k, j^{-1})-symmetric but such that

$$\fint_{B_1} \mathcal{G}(u_j(y), h(y))^2 \, dy > \varepsilon_0 \quad \text{for every } k\text{-symmetric function } h, \text{ for every } j \in \mathbb{N}. \tag{10.12}$$

Modulo rotations, we can assume that the k-dimensional linear subspace L such that condition (b) in Definition 10.2.2 is satisfied is fixed along the sequence: namely, we can assume without loss of generality that

$$\theta_{u_j}(0, 2) - \theta_{u_j}(0, 1) < j^{-1}$$

and that

$$\int_{B_1} |D_L u_j(y)|^2 \, dy \leqslant j^{-1}$$

for some fixed k-dimensional plane $L \subset \mathbb{R}^m$. Now, the compactness theorem for Q-valued energy minimizing maps implies that a subsequence of the u_j's (not relabeled) converges in $L^2(B_{10}(0), \mathcal{A}_Q(\mathbb{R}^N))$ and in energy to a Q-valued energy minimizing map h for which

$$\theta_h(0,2) - \theta_h(0,1) = 0$$

and

$$D_L h = Q[\![0]\!] \quad \text{a.e. in } B_1.$$

Hence, by (10.4) the map $h|_{B_1}$ can be extended to a k-symmetric map (which we still denote by h), and the fact that $\mathcal{G}(u_j, h) \to 0$ in $L^2(B_1)$ contradicts (10.12).

For the converse, assume again by contradiction that there exist ε_0, a sequence $\{u_j\}_{j\in\mathbb{N}}$ of maps as in Assumption 10.2.9 and a sequence $\{h_j\}_{j\in\mathbb{N}} \subset W^{1,2}_{loc}(\mathbb{R}^m, \mathcal{A}_Q(\mathcal{N}))$ of k-symmetric maps such that

$$\fint_{B_1} \mathcal{G}(u_j(y), h_j(y))^2 \, dy \leqslant j^{-1} \tag{10.13}$$

and such that the ball B_1 is not (k, ε_0)-symmetric. Again, after applying suitable rotations we can assume that the invariant subspace for the maps h_j is a fixed k-dimensional plane $L \subset \mathbb{R}^m$. By compactness, the maps u_j converge, up to subsequences, to an energy minimizing $u \in W^{1,2}(B_{10}(0), \mathcal{A}_Q(\mathcal{N}))$. By (10.13), also $h_j \to u$ strongly in $L^2(B_1, \mathcal{A}_Q(\mathcal{N}))$. Since the space of k-symmetric maps is L^2-closed, we deduce that u is k-symmetric. Since the u_j's converge to u also in energy, the ball B_1 must be (k, ε_0)-symmetric for u_j if j is sufficiently large, which is the required contradiction. □

Corollary 10.2.11. *Let u satisfy Assumption 10.2.9. Then, for every $k \in \{0, \dots, m\}$ one has*

$$\mathcal{S}^k(u) = {}^{[CN]}\mathcal{S}^k(u). \tag{10.14}$$

Using more quantitative estimates, the comparison between the two notions of quantitative symmetry can be carried to the case of stationary Q-harmonic maps.

Proposition 10.2.12. *There exists a constant $C = C(m, \mathcal{N}, Q) > 0$ with the following property. Let $u \in W^{1,2}_{loc}(\Omega, \mathcal{A}_Q(\mathcal{N}))$ be a stationary Q-harmonic map. If a ball $B_r(x) \Subset \Omega$ is (k, ε)-symmetric for u, then $B_{\frac{r}{4}}(x)$ is $(k, C|\varepsilon \ln(\varepsilon)|)$-symmetric for u in the sense of Cheeger-Naber.*

Proof. Without loss of generality, we prove the claim for $x = 0$ and $r = 1$. The idea of the proof is to explicitly construct from u a k-symmetric map in $B_{\frac{1}{4}}$. Modulo a rotation, we can assume that the k-dimensional plane L of ε-almost symmetry is $L = \{x_i = 0 : i > k\} = \mathbb{R}^k \times \{0\}$. For convenience, we will denote the variables of $\mathbb{R}^k$ with y, y' and the variables of $\mathbb{R}^{m-k}$ with z, z'. The point $x \in \mathbb{R}^m$ will be therefore given coordinates $x = (y, z) \in \mathbb{R}^k \times \mathbb{R}^{m-k}$. With a slight abuse of notation, we will also sometimes regard y and z as vectors in $\mathbb{R}^m$, thus avoiding the cumbersome, although more correct, writings $(y, 0)$ and $(0, z)$. Finally, when we integrate a function with respect to the variable y over a ball $B^k_r \subset \mathbb{R}^k$ we will use the notation B^y_r as domain of integration (and analogously for the variables y', z, z').

In order to construct the k-symmetric map, we need to prove two simple inequalities for multiple-valued functions.

Claim 1: there exists a constant $C = C(k, d, Q)$ with the following property. For any function $f = f(y, z)$ in $W^{1,2}\left(B_1^k \times B_1^{m-k}, \mathcal{A}_Q(\mathbb{R}^d)\right)$, one has

$$\int_{B_1^{y'}} \int_{B_1^y \times B_1^z} \mathcal{G}(f(y,z), f(y',z))^2 \leqslant C \int_{B_1^y \times B_1^z} |D_L f|^2. \tag{10.15}$$

Claim 2: Let $0 < s_0 < a < 1$. There exists a constant $C = C(j, a, Q)$ such that for any $f \in W^{1,2}(B_1^j \subset \mathbb{R}^j, \mathcal{A}_Q(\mathbb{R}^d))$ and every $a < t \leqslant 1$ such that $f|_{\partial B_t^j} \in W^{1,2}(\partial B_t^j, \mathcal{A}_Q(\mathbb{R}^d))$ the following holds:

$$\int_{B_1^j \setminus B_{s_0}^j} \mathcal{G}\left(f(x), f\left(t\frac{x}{|x|}\right)\right)^2 \leqslant C|\ln(s_0)| \int_{B_1^j \setminus B_{s_0}^j} |Df(x) \cdot x|^2. \tag{10.16}$$

Proof of Claim 1: The proof is a consequence of the Poincaré inequality for multiple valued functions, Proposition 2.2.18. Indeed, first observe that for a.e. $z \in B_1^{m-k}$ the map $y \mapsto f(y, z)$ is in $W^{1,2}(B_1^k, \mathcal{A}_Q(\mathbb{R}^d))$. Hence, by the aforementioned Poincaré inequality, for any such a z there exists a point $\bar{f}(z) \in \mathcal{A}_Q(\mathbb{R}^d)$ such that

$$\int_{B_1^y} \mathcal{G}(f(y,z), \bar{f}(z))^2 \leqslant C \int_{B_1^y} |D_L f(y,z)|^2,$$

where $C = C(k, d, Q)$. Hence, by triangle inequality we infer that

$$\begin{aligned}\int_{B_1^y} \int_{B_1^{y'}} \mathcal{G}(f(y,z), f(y',z))^2 &\leqslant 2\mathcal{H}^k(B_1^k)\left(\int_{B_1^y} \mathcal{G}(f(y,z), \bar{f}(z))^2 + \int_{B_1^{y'}} \mathcal{G}(f(y',z), \bar{f}(z))^2\right) \\ &\leqslant C \int_{B_1^y} |D_L f(y,z)|^2.\end{aligned}$$

Integrating now this inequality in $z \in B_1^{m-k}$ gives (10.15).

Proof of Claim 2: First note that for $\mathcal{H}^{j-1}$-a.e. $w \in \partial B_1^j$ the map $r \mapsto g^w(r) := f(rw)$ is in $W^{1,2}((0,1), \mathcal{A}_Q(\mathbb{R}^d))$. By the $W^{1,2}$-selection theorem for multiple-valued functions of one variable (cf. [DLS11, Proposition 1.2]), there exist $W^{1,2}$ functions $g_\ell^w \colon (0,1) \to \mathbb{R}^d$ for $\ell = 1, \dots, Q$ such that $\left|\frac{d}{dr} g_\ell^w(r)\right| \leqslant |D_w f(rw)|$ for a.e. $r \in (0,1)$. Now, fix $t \in (a, 1)$. Then, by one-dimensional calculus, we have for $s_0 < s \leqslant t$ and for every $\ell \in \{1, \dots, Q\}$ that

$$\begin{aligned}|g_\ell^w(s) - g_\ell^w(t)|^2 &\leqslant \left(\int_s^t r^{-j-1} dr\right)\left(\int_s^t \left|\frac{d}{dr} g_\ell^w(r)\right|^2 r^{j+1} dr\right) \\ &\leqslant \frac{s^{-j}}{j} \int_{s_0}^1 |Df(rw) \cdot w|^2 \, r^{j+1} dr \\ &= \frac{s^{-j}}{j} \int_{s_0}^1 |Df(rw) \cdot rw|^2 \, r^{j-1} dr.\end{aligned}$$

For $t \leqslant s \leqslant 1$ the same computation holds true interchanging t and s: in this case, we estimate $\frac{t^{-j}}{j} \leqslant \frac{a^{-j}}{j} s^{-j}$. Hence in both cases we have

$$|g_\ell^w(s) - g_\ell^w(t)|^2 \leqslant C s^{-j} \int_{s_0}^1 |Df(rw) \cdot rw|^2 \, r^{j-1} dr.$$

Summing over $\ell \in \{1, \dots, Q\}$ and recalling the definition of the metric $\mathcal{G}$ this produces

$$\mathcal{G}(f(sw), f(tw))^2 \leqslant C s^{-j} \int_{s_0}^1 |Df(rw) \cdot rw|^2 \, r^{j-1} dr \quad \text{for every } s \in (s_0, 1),$$

where $C = C(j, a, Q)$. Multiply by s^{j-1} and integrate in s between s_0 and 1 to obtain

$$\int_{s_0}^1 \mathcal{G}(f(sw), f(tw))^2 s^{j-1} ds \leqslant C|\ln(s_0)| \int_{s_0}^1 |Df(rw) \cdot rw|^2 \, r^{j-1} dr.$$

Integrating now in $w \in \partial B_1$ gives inequality (10.16).

We are now ready to prove the proposition. Let $u \in W^{1,2}(\Omega, \mathcal{A}_Q(\mathcal{N}))$ be a stationary Q-harmonic map, and assume that B_1 is (k, ε)-symmetric for u. By (b) in Definition 10.2.2, we can fix $\frac{1}{4} \leqslant t \leqslant \frac{1}{\sqrt{2}}$ such that $x \in \partial B_t \mapsto u(x)$ is in $W^{1,2}(\partial B_t, \mathcal{A}_Q(\mathcal{N}))$ and satisfies $\int_{\partial B_t} |D_L u|^2 \leqslant C \int_{B_1} |D_L u|^2$.
For a.e. $y' \in B_t^k$ we have that the map $z \mapsto v_{y'}(z) := u(y', z)$ is in $W^{1,2}(B_t^{m-k}, \mathcal{A}_Q(\mathcal{N}))$. Hence, by the scaled version of (10.16) with $j = m - k$ we have for any $0 < s_0 < \frac{1}{4}$ that

$$\int_{B_t^z \setminus B_{s_0}^z} \mathcal{G}\left(v_{y'}(z), v_{y'}\left(t\frac{z}{|z|}\right)\right)^2 \leqslant C|\ln(s_0)| \int_{B_t^z} \left|Du(y', z) \cdot z\right|^2,$$

where $C = C(m, Q)$. Integrating this now in $y' \in B_t^k$ we obtain

$$\int_{B_t^y} \int_{B_t^z \setminus B_{s_0}^z} \mathcal{G}\left(v_{y'}(z), v_{y'}\left(t\frac{z}{|z|}\right)\right)^2 \leqslant C|\ln s_0| \int_{B_t^k \times B_t^{m-k}} |Du(y, z) \cdot z|^2 \, dy dz.$$

Adding the scaled version of (10.15), since $B_t^k \times B_t^{m-k} \subset B_1$ we obtain

$$\int_{B_t^y} \left(\int_{B_t^y \times B_t^z} \mathcal{G}(u(y, z), v_{y'}(z))^2 + \int_{B_t^z \setminus B_{s_0}^z} \mathcal{G}\left(v_{y'}(z), v_{y'}\left(t\frac{z}{|z|}\right)\right)^2 \right) \leqslant C \left(\int_{B_1} |D_L u|^2 + |\ln(s_0)| \int_{B_1} |Du(x) \cdot x|^2 \right).$$

Hence there exists $y_0' \in B_t^k$ such that

$$\begin{aligned} \int_{B_t^y \times B_t^z} \mathcal{G}(u(y, z), v_{y_0'}(z))^2 + \int_{B_t^z \setminus B_{s_0}^z} \mathcal{G}\left(v_{y_0'}(z), v_{y_0'}\left(t\frac{z}{|z|}\right)\right)^2 \\ \leqslant \frac{C}{\mathcal{H}^k(B_t^k)} \left(\int_{B_1} |D_L u|^2 + |\ln(s_0)| \int_{B_1} |Du(x) \cdot x|^2 \right) \\ \leqslant C(1 + |\ln(s_0)|)\varepsilon, \end{aligned}$$

where in the last inequality we have used that the ball B_1 is, by assumption, (k,ε)-symmetric for u together with (10.4).

Set $h(x) = h(y,z) := v_{y_0'}\left(t\frac{z}{|z|}\right) \in W^{1,2}(B_t, \mathcal{A}_Q(\mathcal{N}))$. Note that, by definition, h is homogeneous with respect to 0. Furthermore, h is k-symmetric. An application of the triangle inequality gives

$$\int_{B_t^y\times B_t^z} \mathcal{G}(u(y,z),h(x))^2 \leqslant 2\int_{B_t^y\times B_t^z} \mathcal{G}(u(y,z),v_{y_0'}(z))^2 + 2\int_{B_t^y\times (B_t^z\setminus B_{s_0}^z)} \mathcal{G}(v_{y_0'}(z),h(x))^2 + 2\int_{B_t^y\times B_{s_0}^z} \mathcal{G}(v_{y_0'}(z),h(x))^2.$$

As we have shown above, the first two integrals can be bounded by $C(1+|\ln(s_0)|)\varepsilon$. As for the last integral, we estimate it brutally by fixing a point $p\in\mathcal{N}$ and computing

$$\int_{B_t^y\times B_{s_0}^z} \mathcal{G}(v_{y_0'}(z),h(x))^2 \leqslant 2\sup_{x\in B_1} \mathcal{G}(u(x),Q[\![p]\!])^2\mathcal{H}^k(B_1^k)\mathcal{H}^{m-k}(B_{s_0}^{m-k}) \leqslant CQ\mathrm{diam}(\mathcal{N})^2 s_0^{m-k}$$

Hence choosing $s_0=\varepsilon$ proves the proposition, since we get

$$\int_{B_t} \mathcal{G}(u(x),h(x))^2 \leqslant C|\varepsilon\ln(\varepsilon)|.$$

□

Corollary 10.2.13. *Let $u\in W^{1,2}_{loc}(\Omega,\mathcal{A}_Q(\mathcal{N}))$ be a stationary Q-harmonic map. Then*

$$^{[CN]}\mathcal{S}^k(u)\subset \mathcal{S}^k(u).$$

We conclude the section with two propositions about the characterization of the singular set for minimizing and stationary maps. The first one is the following effective version of Corollary 10.2.7.

Proposition 10.2.14. *There exists $\varepsilon=\varepsilon(m,\mathcal{N},Q,\Lambda)$ such that for any map u satisfying Assumption 10.2.9 the following holds:*

$$B_1\cap\left(\mathcal{S}^{m-1}(u)\setminus\mathcal{S}^{m-3}_{\varepsilon}(u)\right)=\emptyset.$$

Proof. The proof is by contradiction. Assume, therefore, that for every $j\in\mathbb{N}$ there exists u_j as in Assumption 10.2.9 with a point $x_j\in B_1\cap\left(\mathcal{S}^{m-1}(u_j)\setminus\mathcal{S}^{m-3}_{j^{-1}}(u_j)\right)$. Since $x_j\notin\mathcal{S}^{m-3}_{j^{-1}}(u_j)$, there exists $0<r_j<1$ and a linear subspace $L_j\subset\mathbb{R}^m$ with $\dim(L_j)=m-2$ such that

$$\theta_{u_j}(x_j,2r_j)-\theta_{u_j}(x_j,r_j)\leqslant j^{-1}, \tag{10.17}$$

$$r_j^{2-m}\int_{B_{r_j}(x_j)}|D_{L_j}u_j|^2\leqslant j^{-1}. \tag{10.18}$$

As usual, without loss of generality we assume that the $(m-2)$-planes of j^{-1}-almost symmetry are a fixed subspace L along the sequence. Set $v_j(y) := u_j(x_j + r_j y)$, and re-write the equations (10.17) and (10.18) in terms of v_j:

$$\theta_{v_j}(0,2) - \theta_{v_j}(0,1) \leqslant j^{-1}\,, \tag{10.19}$$

$$\int_{B_1} |D_L v_j|^2 \leqslant j^{-1}\,. \tag{10.20}$$

Now, by an elementary computation it is immediate to see that for every $\rho \in (0,8)$ one has

$$\rho^{2-m} \int_{B_\rho} |Dv_j|^2 = (\rho r_j)^{2-m} \int_{B_{\rho r_j}(x_j)} |Du_j|^2 \leqslant C_m \Lambda.$$

Hence, by the Compactness Theorem 2.3.18, the sequence $\{v_j\}_{j\in\mathbb{N}}$ converges up to subsequences in $L^2(B_8, \mathcal{A}_Q(\mathbb{R}^N))$ and in energy to a Q-valued energy minimizing map v for which

$$\theta_v(0,2) - \theta_v(0,1) = 0\,, \tag{10.21}$$

$$\int_{B_1} |D_L v|^2 = 0\,. \tag{10.22}$$

In particular, $v|_{B_1}$ can be extended to an $(m-2)$-symmetric energy minimizer. This implies that a fortiori $0 \in \mathrm{reg}_H(v)$. Thus, $0 \notin \mathcal{S}^{m-1}(v_j)$ for j large, which contradicts the fact that $x_j \in \mathcal{S}^{m-1}(u_j)$. □

In the single-valued case $Q = 1$, we have the following result on the quantitative stratification for stationary harmonic maps.

Proposition 10.2.15. *There exists $\varepsilon = \varepsilon(m, \mathcal{N})$ such that for any* single-valued *stationary harmonic map $u \in W^{1,2}_{loc}(\Omega, \mathcal{N})$ the following holds:*

$$\mathcal{S}^{m-1}(u) \setminus \mathcal{S}^{m-2}_\varepsilon(u) = \emptyset.$$

Proof. Proposition 10.2.15 is a consequence of the inner variation formula. First we derive a general estimate and show afterwards how it implies the proposition.

Let us consider a single-valued harmonic map u in B_1 that satisfies the inner variation formula (2.32). We fix two non-negative, non-increasing functions $\psi, \varphi \in C^1_c\left(\left[0, \frac{1}{\sqrt{2}}\right)\right)$ and a k-dimensional subspace $L \subset \mathbb{R}^m$. After a rotation, we may assume that $L = \{x_i = 0 \colon i = k+1, \dots, m\}$. To make the notation a bit simpler we will write $x = (y, z) \in L \times L^\perp$, and by a slight abuse of notation we shall again consider $z = (0, z)$ as a vector in $\mathbb{R}^m$. Consider the vector field $X(y,z) := \psi(|y|)\varphi(|z|)z = \psi\varphi z$. We have $DX = \psi\varphi P^\perp + \frac{\psi\varphi'}{|z|} z \otimes z + \frac{\psi'\varphi}{|y|} z \otimes y$, where $P^\perp$ denotes the orthogonal projection onto $L^\perp$. We use this vector field in the inner variation formula (2.32) and obtain

$$0 = \int |Du|^2 \left((m-k)\psi\varphi + \psi\varphi'|z|\right) - 2\left(\psi\varphi |D_{L^\perp} u|^2 + \psi\varphi' \frac{1}{|z|} |Du \cdot z|^2 + \psi'\varphi \left\langle Du \cdot z, Du \cdot \frac{y}{|y|} \right\rangle\right).$$

Observe that

$$(m-k)\psi\varphi|Du|^2 - 2\psi\varphi|D_{L^\perp} u|^2 = (m-k-2)\varphi\psi|Du|^2 + 2\psi\varphi|D_L u|^2.$$

Furthermore, we can write $z = x - y$ to estimate

$$|Du \cdot z|^2 \leqslant 2|Du \cdot x|^2 + 2|Du \cdot y|^2$$
$$\left\langle Du \cdot z, Du \cdot \frac{y}{|y|} \right\rangle \leqslant \frac{1}{2}|Du \cdot x|^2 + \frac{1}{2}\left|Du \cdot \frac{y}{|y|}\right|^2.$$

Combining all together, and recalling that $\varphi', \psi' \leqslant 0$, we obtain the inequality

$$\begin{aligned}&\int -\left((m-k-2)\psi\varphi + \psi\varphi'|z|\right)|Du|^2 \\ &\leqslant \int 2\psi\varphi\,|D_L u|^2 + \int \frac{-4\psi\varphi'}{|z|}\left(|Du \cdot x|^2 + |Du \cdot y|^2\right) - \psi'\varphi\left(|Du \cdot x|^2 + \left|Du \cdot \frac{y}{|y|}\right|^2\right).\end{aligned} \tag{10.23}$$

We are ready to prove the proposition. Fix $\varepsilon > 0$ to be determined later, and suppose by contradiction that there is a point $x \in \mathcal{S}^{m-1}(u) \setminus \mathcal{S}^{m-2}_\varepsilon(u)$. Since $x \notin \mathcal{S}^{m-2}_\varepsilon$, there exists $r = r(\varepsilon) > 0$ and an $(m-1)$-dimensional subspace $L = L(\varepsilon)$ such that $r^{2-m}\int_{B_r(x)}|D_L u|^2 < \varepsilon$ and $\theta(x, 2r) - \theta(x, r) < \varepsilon$. By translation and scaling, i.e. passing to $T^u_{x,r}$, we may assume that $x = 0$ and $r = 1$. However, for notational convenience, we will still write u for $T^u_{x,r}$. After a further rotation we may assume that $L = \{x_m = 0\}$. Now, we have $B_{\frac{1}{2}} \subset B^{m-1}_{\frac{1}{\sqrt{2}}} \times \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \subset B_1$. Fix a function $\eta \in C^1$ with $\eta' \leqslant 0$ and $\eta(t) = 1$ for $t \leqslant \frac{1}{2}$, $\eta(t) = 0$ for $t \geqslant \frac{1}{\sqrt{2}}$. Set $\varphi = \psi := \eta$ in (10.23). Recall that in our situation $k = m-1$, and thus $-(m-k-2) = 1$. Furthermore, we have $\psi\varphi \geqslant \mathbf{1}_{B_{\frac{1}{2}}}$, and $\frac{|4\psi\varphi'|}{|z|}, |\psi'\varphi|$ are bounded and supported in B_1. Hence (10.23) reads in our case

$$\int_{B_{\frac{1}{2}}} |Du|^2 \leqslant C\int_{B_1}\left(|x||D_r u|^2 + |D_L u|^2\right),$$

where $r(x) = r_0(x) = \frac{x}{|x|}$. By (10.4), we deduce that $\int_{B_{\frac{1}{2}}}|Du|^2 \leqslant C\varepsilon$. If $\varepsilon > 0$ is chosen sufficient small, i.e. $C\varepsilon < \varepsilon_0$ where $\varepsilon_0 = \varepsilon_0(m, \mathcal{N})$ is the threshold in the ε-regularity theorem for stationary harmonic maps (cf. [Bet93, RS08]), this allows to infer that u is Hölder continuous in $B_{\frac{1}{4}}$, and hence 0 is a regular point. This contradicts the assumption that $0 \in \mathcal{S}^{m-1}$. □

Remark 10.2.16. Note that the above proposition could be extended (with exactly the same proof) to the case of stationary Q-harmonic maps *if* an ε-regularity theorem was available in that case.

10.2.1 Main theorem on the quantitative strata

Since the relevant terminology has been introduced now, we can finally state the main estimates that we are going to prove on the singular strata.

Theorem 10.2.17. *Given a Dirichlet-minimizing* Q*-valued map* $u : B_2(0) \subseteq \mathbb{R}^m \to \mathcal{A}_Q(\mathcal{N})$ *with* $\mathscr{E}(u, B_2(0)) \leqslant \Lambda$*, let* $\mathcal{S}^k_{\varepsilon,r}(u)$ *be its quantitative singular strata. Then, if* $B_r\left(\mathcal{S}^k_{\varepsilon,r}(u)\right) := \bigcup_{x\in\mathcal{S}^k_{\varepsilon,r}(u)} B_r(x)$*, we have*

$$\mathcal{L}^m\left(B_r\left(\mathcal{S}^k_{\varepsilon,r}(u)\cap B_1(0)\right)\right) \leqslant C(m,\mathcal{N},Q,\Lambda,\varepsilon)r^{m-k}. \tag{10.24}$$

Moreover, $\mathcal{S}^k_\varepsilon(u)$ *is* k*-rectifiable for all* $\varepsilon \geqslant 0$*.*

Remark 10.2.18. This theorem is similar in spirit to [NV17, Theorem 1.3].

Note that this and the ε-regularity theorem immediately imply Theorem 10.0.1 as a corollary.

Proof of Theorem 10.0.1. By remark 10.2.5 and proposition 10.2.14, there exists an ε such that

$$\mathcal{S}^{m-3}_\varepsilon(u)\cap B_1(0) = \operatorname{sing}_H(u)\cap B_1(0)\,. \tag{10.25}$$

Thus Theorem 10.2.17 immediately proves the volume estimates and rectifiability for $\operatorname{sing}_H(u)$. □

We postpone the proof of Theorem 10.2.17 to Section 10.6, after having discussed a few technical tools needed to complete it.

10.3 QUANTITATIVE ε–REGULARITY THEOREMS

In this section we are going to present the proof of a quantitative version of the ε-regularity theorem for Q-valued minimizers, cf. Theorem 10.3.3 below, which in turn implies Corollary 10.3.4, providing sufficient conditions under which the singular set $\operatorname{sing}_H(u)$ is constrained to live in the tubular neighborhood of an affine subspace of $\mathbb{R}^m$ of appropriate dimension. We start with the following definition, analogous to [NV17, Definition 4.5].

Definition 10.3.1. Let $y_0, y_1, \dots, y_k$ be $(k+1)$ points in $B_1(0) \subset \mathbb{R}^m$, and let $\rho > 0$. We say that these points ρ-effectively span a k-dimensional affine subspace if

$$\operatorname{dist}(y_i, y_0 + \operatorname{span}[y_1 - y_0, \dots, y_{i-1} - y_0]) \geqslant 2\rho \quad \text{for every } i = 1, \dots, k. \tag{10.26}$$

A set $F \subset B_1(0)$ ρ-effectively spans a k-dimensional subspace if there exist points $\{y_i\}_{i=0}^k \subset F$ which ρ-effectively span a k-dimensional subspace.

Remark 10.3.2. It is easy to see that if the points $\{y_i\}_{i=0}^k$ ρ-effectively span a k-dimensional affine subspace then for every point

$$x \in y_0 + \operatorname{span}[y_1 - y_0, \dots, y_k - y_0]$$

there exists a unique set of numbers $\{\alpha_i\}_{i=1}^k$ such that

$$x = y_0 + \sum_{i=1}^k \alpha_i(y_i - y_0), \quad |\alpha_i| \leqslant C(m,\rho)|x - y_0|.$$

Furthermore, the notion of ρ-effectively spanning a k-dimensional affine subspace passes to the limit: if for every $j \in \mathbb{N}$ the points $\{y_i^j\}_{i=0}^k$ ρ-effectively span a k-dimensional subspace and there exist the limits $y_i := \lim_{j\to\infty} y_i^j$, then also the points $\{y_i\}_{i=0}^k$ ρ-effectively span a k-dimensional subspace.

We can now state the main theorem of this section.

Theorem 10.3.3. *Let $\varepsilon, \rho > 0$ be fixed. There exist $\delta, \bar{r} > 0$, depending on $m, \rho, \Lambda, \varepsilon$, with the following property. Let $u \in W^{1,2}(B_{10}(0), \mathcal{A}_Q(\mathcal{N}))$ be a stationary Q-harmonic map with energy bounded by Λ, let $r \leqslant 1$, and let*

$$F := \{y \in B_r(0) : \ \theta(y, 4r) - \theta(y, 2r) < \delta\}.$$

If F $(\rho \cdot r)$-effectively spans a k-dimensional subspace L, then

$$\left(\mathcal{S}^k_{\varepsilon, r\bar{r}}(u) \cap B_{\frac{r}{2}}(0)\right) \setminus B_{r\rho}(L) = \emptyset. \tag{10.27}$$

Corollary 10.3.4. *For every $\rho > 0$, there exists $\delta = \delta(m, \mathcal{N}, Q, \Lambda, \rho) > 0$ with the following property. Let $u\colon B_{10}(0) \subset \mathbb{R}^m \to \mathcal{A}_Q(\mathcal{N})$ be a $W^{1,2}$ map with energy bounded by Λ, and let $r \leqslant 1$.*

(i) *In case u is energy minimizing, if there exist $m-2$ points $\{y_i\}_{i=0}^{m-3} \subset B_r(0)$ which $(\rho \cdot r)$-effectively span an $(m-3)$-dimensional affine subspace $L \subset \mathbb{R}^m$ and such that*

$$\theta(y_i, 4r) - \theta(y_i, 2r) < \delta \quad \text{for every } i = 0, \dots, m-3,$$

then

$$\left(\operatorname{sing}_H(u) \cap B_{\frac{r}{2}}(0)\right) \setminus B_{\rho r}(L) = \emptyset;$$

(ii) *in case u is single-valued and stationary harmonic, if there exist $m-1$ points $\{y_i\}_{i=0}^{m-2} \subset B_r(0)$ which $(\rho \cdot r)$-effectively span an $(m-2)$-dimensional affine subspace $L \subset \mathbb{R}^m$ and such that*

$$\theta(y_i, 4r) - \theta(y_i, 2r) < \delta \quad \text{for every } i = 0, \dots, m-2,$$

then

$$\left(\operatorname{sing}_H(u) \cap B_{\frac{r}{2}}(0)\right) \setminus B_{\rho r}(L) = \emptyset.$$

In particular, if $m = 3$ and u is a Q-valued energy minimizer, and if $\theta(y_0, 4r) - \theta(y_0, 2r) < \delta$ then

$$B_{\frac{r}{2}}(y_0) \setminus B_{\rho r}(y_0) \subset \operatorname{reg}_H(u).$$

The same holds if $m = 2$ and u is single-valued and stationary.

Proof of Corollary 10.3.4. It follows immediately from Theorem 10.3.3 and Propositions 10.2.14 for the minimizing case and 10.2.15 for the stationary harmonic case. □

For the proof of Theorem 10.3.3 we will need the following lemma.

Lemma 10.3.5. *Let $u \in W^{1,2}(B_{10}(0), \mathcal{A}_Q(\mathcal{N}))$ be a stationary Q-harmonic map, and let $r \leqslant 1$. If $\{y_i\}_{i=0}^k \subset B_r(0)$ $(\rho \cdot r)$-effectively span a k-dimensional affine subspace $L \subset \mathbb{R}^m$, then*

$$r^{-m} \int_{B_r(0)} \left(r^2 |D_{\hat{L}} u(z)|^2 + |D_v u(z)|^2 \right) dz \leqslant C(m,\rho) \sum_{i=0}^{k} \left(\theta(y_i, 4r) - \theta(y_i, 2r) \right), \tag{10.28}$$

where $\hat{L}$ is the linear part of L and v is the vector field $v(z) := D\left(\frac{1}{2}\mathrm{dist}^2(z,L)\right)$.

Proof. It is an immediate consequence of (10.4) that there exists a constant $C = C(m)$ such that

$$r^{-m} \int_{B_r(x)} |Du(z)\cdot(z-x)|^2 \, dz \leqslant \int_{B_r(x)} \frac{|z-x|}{r^{m-1}} \left| Du(z) \cdot \frac{z-x}{|z-x|} \right|^2 dz \leqslant C(m) \left(\theta(x,2r) - \theta(x,r) \right) \tag{10.29}$$

whenever $B_{2r}(x) \subset B_{10}(0)$. Now, assume that $y_0, y_1, \ldots, y_k$ are as in the statement, and observe that for every unit vector e in the linear part $\hat{L}$ of L there exists a unique set of numbers $\{\alpha_i\}_{i=1}^k$ such that

$$e = r^{-1} \sum_{i=1}^{k} \alpha_i (y_i - y_0), \quad |\alpha_i| \leqslant C(m,\rho).$$

Hence, we get

$$\begin{aligned}
r^{2-m} \int_{B_r(0)} |D_e u(z)|^2 \, dz &\leqslant C(m,\rho) r^{-m} \sum_{i=1}^{k} \int_{B_r(0)} |Du(z)\cdot(y_i - y_0)|^2 \, dz \\
&\leqslant C(m,\rho) r^{-m} \sum_{i=0}^{k} \int_{B_r(0)} |Du(z)\cdot(z - y_i)|^2 \, dz \\
&\leqslant C(m,\rho) r^{-m} \sum_{i=0}^{k} \int_{B_{2r}(y_i)} |Du(z)\cdot(z - y_i)|^2 \, dz \\
&\overset{(10.29)}{\leqslant} C(m,\rho) \sum_{i=0}^{k} \left(\theta(y_i, 4r) - \theta(y_i, 2r) \right).
\end{aligned}$$

Summing over an orthonormal basis $e_1, \ldots, e_k$ of $\hat{L}$ produces

$$r^{2-m} \int_{B_r(0)} \left| D_{\hat{L}} u(z) \right|^2 dz \leqslant C(m,\rho) \sum_{i=0}^{k} \left(\theta(y_i, 4r) - \theta(y_i, 2r) \right). \tag{10.30}$$

As for the second term, let $z \in B_r(0)$, and let $\pi := \pi_L(z)$ be the orthogonal projection of z onto L. Of course,

$$v(z) := D\left(\frac{1}{2}\mathrm{dist}^2(z,L)\right) = z - \pi.$$

On the other hand, we have as usual that

$$\pi = y_0 + \sum_{i=1}^{k} \alpha_i (y_i - y_0), \quad |\alpha_i| \leqslant C(m,\rho)|\pi - y_0| \leqslant C(m,\rho) r,$$

and thus

$$v(z) = z - \left(y_0 + \sum_{i=1}^{k} \alpha_i (y_i - y_0) \right).$$

Arguing as above, one concludes that also

$$r^{-m} \int_{B_r(0)} |D_v u(z)|^2 \, dz \leqslant C(m, \rho) \sum_{i=0}^{k} \left(\theta(y_i, 4r) - \theta(y_i, 2r) \right), \tag{10.31}$$

which together with (10.30) completes the proof of (10.29).

□

For $k = m - 2$, the conclusions of the previous lemma can be improved using again the inner variation formula.

Lemma 10.3.6. *Let $u \in W^{1,2}(B_{10}(0), \mathcal{A}_Q(\mathcal{N}))$ be a stationary Q-harmonic map, and let $r \leqslant 1$. If $\{y_i\}_{i=0}^{m-2} \subset B_r(0)$ $(\rho \cdot r)$-effectively span an $(m-2)$-dimensional affine subspace $L \subset \mathbb{R}^m$, then*

$$r^{-m} \int_{B_r(0)} \left(r^2 |D_{\hat{L}} u|^2 + |D_{\hat{L}^\perp} u|^2 |v|^2 \right) \leqslant C(m, \rho) \sum_{i=0}^{m-2} \left(\theta(y_i, 8r) - \theta(y_i, 4r) \right), \tag{10.32}$$

where $\hat{L}$ is the linear part of L, $\hat{L}^\perp$ is its orthogonal complement in $\mathbb{R}^m$ and v is the vector field $v(x) := D\left(\frac{1}{2}\mathrm{dist}^2(x, L)\right)$.

Proof. The proof is very similar to the one of Proposition 10.2.15: also in this case, we will make use of the stationary equation with a suitable choice of the vector field X. Without loss of generality, we can assume that $r = 1$. Furthermore, modulo translations and rotations we can assume that $L = \{x_i = 0 : i = m-1, m\}$. As usual, coordinates on L and $L^\perp$ will be denoted by y and z respectively, and in order to simplify our notation the vectors $(y, 0)$ and $(0, z)$ in $L \times L^\perp$ will be simply denoted by y and z. Observe that under these assumptions one has $v(x) = z$ for every $x = (y, z) \in B_1$. Now, let $\psi = \psi(y)$ be a cut-off function of the variable $y \in L$, with $\psi \equiv 1$ in B_1^{m-2}, $\mathrm{spt}(\psi) \subset B_2^{m-2}$ and $|D\psi| \leqslant 1$. Let also $\varphi(t) := \max\{1 - t, 0\}$, and consider the vector field $X(y, z) := \psi(y)\varphi(|z|^2)z = \psi\varphi z$. We can immediately compute $DX = \psi\varphi P^\perp + \varphi z \otimes D\psi + 2\psi\varphi' z \otimes z$. With this choice of X, the inner variation formula (2.38) reads

$$\begin{aligned} 0 &= \int |Du|^2 \left(2\psi\varphi + 2\psi\varphi' |z|^2 \right) - 2 \left(\psi\varphi |D_{L^\perp} u|^2 + \varphi \langle Du \cdot z, Du \cdot D\psi \rangle + 2\psi\varphi' |Du \cdot z|^2 \right) \\ &= 2 \int \psi\varphi |D_L u|^2 + \psi\varphi' |Du|^2 |z|^2 - \left(\varphi \langle Du \cdot z, Du \cdot D\psi \rangle + 2\psi\varphi' |Du \cdot z|^2 \right). \end{aligned}$$

In particular, since $\varphi'(|z|^2) = -\chi_{\{|z| \leqslant 1\}}$, $\psi|_{\{|y| \leqslant 1\}} \equiv 1$ and $B_1 \subset B_1^{m-2} \times B_1^2 \subset B_2$, we immediately deduce

$$\int_{B_1} |Du|^2 |z|^2 \leqslant C \int_{B_2} \left(|D_L u|^2 + |Du \cdot z|^2 \right).$$

The estimate (10.32) then follows from Lemma 10.3.5. □

We are now ready to prove Theorem 10.3.3.

Proof of Theorem 10.3.3. Since the statement is scale-invariant, there is no loss of generality in proving it only in the case $r = 1$. Let $\{y_i\}_{i=0}^k \subset F$ ρ-effectively span the k-dimensional subspace L, and let x be any point in $B_{\frac{1}{2}}(0) \setminus B_\rho(L)$. The goal is to prove that $x \notin \mathcal{S}^k_{\varepsilon,\overline{r}}(u)$ for some $\overline{r} > 0$, and thus that there exists $\overline{r} > 0$ and a radius $r_x \in [\overline{r}, 1)$ such that the ball $B_{r_x}(x)$ is $(k+1,\varepsilon)$-symmetric for u. Let $0 < \delta \ll 1$ to be chosen later. Since $x \in B_{\frac{1}{2}}(0)$, $B_\sigma(x) \subset B_1(0)$ for every $0 < \sigma < \frac{1}{2}$. Hence, we deduce from Lemma 10.3.5 that

$$\int_{B_\sigma(x)} |D_{\hat{L}} u|^2 \leqslant C(m,\rho)\delta$$

for any such σ. In order to gain another direction along which the energy is small, we let $v(z) := D\left(\frac{1}{2}\mathrm{dist}^2(z,L)\right)$, and we set $e := \frac{v(x)}{|v(x)|}$. Note that $|v(x)| = \mathrm{dist}(x,L) \geqslant \rho$. Again by Lemma 10.3.5 and by the monotonicity of the function $r \mapsto \mathscr{E}(u, B_r(x))$, we have

$$\begin{aligned}
\int_{B_\sigma(x)} |D_e u|^2 &\leqslant \rho^{-2} \int_{B_\sigma(x)} |Du(z)\cdot v(x)|^2 \\
&\leqslant 2\rho^{-2}\left(\int_{B_\sigma(x)} |Du(z)\cdot v(z)|^2 + \int_{B_\sigma(x)} |Du(z)\cdot(v(z)-v(x))|^2\right) \\
&\leqslant C\int_{B_1(0)} |D_v u|^2 + C\sigma^2 \int_{B_\sigma(x)} |Du|^2 \\
&\leqslant C\delta + C\Lambda\sigma^m,
\end{aligned}$$

where $C = C(m,\rho)$. Hence, if $V := \hat{L} \oplus \mathrm{span}(e)$ then

$$\int_{B_\sigma(x)} |D_V u|^2 \leqslant C\delta + C\Lambda\sigma^m \tag{10.33}$$

for every $0 < \sigma < \frac{1}{2}$. Note that $\dim(V) = k+1$.

Fix now $\varepsilon > 0$, and let $\overline{\sigma} = \overline{\sigma}(m,\rho,\Lambda,\varepsilon) < \frac{1}{2}$ be such that $C\Lambda\overline{\sigma}^2 \leqslant \frac{\varepsilon}{2}$. We claim that for any $0 < \tau \ll 1$ there exists $\tau\overline{\sigma} \leqslant r_x < \overline{\sigma}$ such that

$$\theta(x,2r_x) - \theta(x,r_x) \leqslant \frac{2c_1(m)\Lambda}{-\log_2(2\tau)}\,. \tag{10.34}$$

Indeed, otherwise for any integer $M \in \left(\frac{3}{4}\log_2\left(\frac{1}{2\tau}\right), \log_2\left(\frac{1}{2\tau}\right)\right)$ we would get

$$c_1(m)\Lambda \geqslant \theta(x,\overline{\sigma}) \geqslant \sum_{i=0}^{M} \theta(x,2^{-i}\overline{\sigma}) - \theta(x,2^{-(i+1)}\overline{\sigma}) \geqslant M\frac{2c_1(m)\Lambda}{-\log_2(2\tau)} \geqslant \frac{3}{2}c_1(m)\Lambda,$$

which is impossible. Hence, if we fix $\tau = \tau(m,\Lambda,\varepsilon)$ so small that $\frac{2c_1(m)\Lambda}{-\log_2(2\tau)} \leqslant \varepsilon$, the above argument allows to conclude that if we set $\overline{r} := \tau\overline{\sigma}$ then there is a radius $r_x \in (\overline{r},\overline{\sigma})$ such that

$$\theta(x,2r_x) - \theta(x,r_x) \leqslant \varepsilon. \tag{10.35}$$

Furthermore, formula (10.33) with r_x in place of σ implies that

$$r_x^{2-m} \int_{B_{r_x}(x)} |D_V u|^2 \leqslant C\delta \bar{r}^{2-m} + C\Lambda \bar{\sigma}^2. \tag{10.36}$$

We can finally chose $\delta = \delta(m, \rho, \Lambda, \varepsilon)$ such that $C\delta\bar{r}^{2-m} \leqslant \frac{\varepsilon}{2}$. From equations (10.35) and (10.36) we infer that $B_{r_x}(x)$ is $(k+1, \varepsilon)$-symmetric for u.

□

We conclude the section with the following proposition, according to which if the mollified energy is pinched enough at k points spanning a k-plane L, then it is almost constant along this L.

Proposition 10.3.7. *Let u satisfy Assumption 10.2.9. Let $0 < \rho < 1$ and $\eta > 0$ be fixed, and assume that $\theta(y, 8) \leqslant E$ for every $y \in B_1(0)$. There exists $\delta_0 = \delta_0(m, \mathcal{N}, Q, \Lambda, \rho, \eta) > 0$ such that if the set $F := \{y \in B_1(0) : \theta(y, \rho) > E - \delta_0\}$ (2ρ)-effectively spans a k-dimensional affine subspace $L \subset \mathbb{R}^m$ then*

$$|\theta(x, \rho) - E| < \eta \quad \textit{for every } x \in L \cap B_1(0).$$

Proof. The proof is by contradiction. Assume that there are $0 < \rho_0 < 1$, $\eta_0 > 0$ and a sequence u_i of maps satisfying Assumptions 10.2.9 and the condition $\theta_{u_i}(y, 8) \leqslant E$ everywhere in B_1, and with the property that for every $i \in \mathbb{N}$ there are points $\{y_j^i\}_{j=0}^k \subset B_1(0)$ with $\theta_{u_i}(y_j^i, \rho_0) > E - i^{-1}$ $(2\rho_0)$-effectively spanning a k-dimensional affine subspace $L_i \subset \mathbb{R}^m$ but with $\theta(x_i, \rho_0) \leqslant E - \eta_0$ for some $x_i \in L \cap B_1(0)$. As usual, without loss of generality we can assume that the subspace $L = L_i$ is fixed along the sequence. By the usual compactness for energy minimizers, modulo passing to a subsequence (not relabeled) the u_i's converge in L^2 and in energy to a minimizer u. Up to further extracting another subsequence, we can also assume that $y_j^i \to y_j$ and $x_i \to x$. By Remark 10.3.2, also the y_j's $(2\rho_0)$-effectively span L. Moreover, $\theta_u(y_j, \rho_0) \geqslant E$, and thus $\theta_u(y_j, 8) - \theta_u(y_j, \rho_0) \leqslant 0$. By monotonicity, then it has to be

$$\theta_u(y_j, 8) - \theta_u(y_j, \rho_0) = 0,$$

and hence, by Lemma 10.3.5, u is invariant along L in $B_2(0)$. Since $\theta_u(y_j, \rho_0) = E$, it has to be $\theta_u(y, \rho_0) = E$ everywhere on $L \cap B_1(0)$, which contradicts the existence of x. □

10.4 REIFENBERG THEOREM

This section is dedicated to Reifenberg-type results needed for the proof of the main theorem. The results will only be quoted without proof, and they are in some sense a quantitative generalization of Reifenberg's topological disk theorem (see [Rei60]). Many generalizations of this landmark theorem are available in literature, we limit ourselves to citing [Tor95, DT12] among the various present. Here we will need two versions of this theorem originally proved in [NV17].

Before quoting the theorems, we need the following definition of the the so-called Jones' β_2 numbers.

Definition 10.4.1. Given a positive Borel measure μ defined in $\mathbb{R}^m$, for all positive radii $r > 0$ and dimensions $k \in \mathbb{N}$, we define

$$D^k_\mu(x,r) := \min\left\{ \int_{B_r(x)} \frac{\mathrm{dist}^2(y,V)}{r^2}\,\frac{d\mu(y)}{r^k} : V \subset \mathbb{R}^m \text{ is an affine subspace with } \dim(V) = k \right\}. \tag{10.37}$$

Usually in literature this quantity is referred to as Jones' β-2 number $\beta^k_{2,\mu}(x,r)^2$.

D captures in a scale invariant way the distance between the support of μ and some k-dimensional subspace V. Indeed, the factor r^{-2} in the distance term makes the integrand scale-invariant, while $r^{-k}\mu$ is scale invariant if we assume that μ is Ahlfors upper k-regular, in the sense $\mu(B_r(x)) \leqslant Cr^k$ for some constant C. For example, this is the case if μ is the k-dimensional Hausdorff measure on a k-dimensional subspace $V \subset \mathbb{R}^m$.

Here we mention two easy and crucial properties of D.

Lemma 10.4.2 (Bounds on D). *Given two measures μ, μ' such that $\mu' \leqslant \mu$, for all x, r and $k \in \mathbb{N}$ we can bound*

$$D^k_{\mu'}(x,r) \leqslant D^k_\mu(x,r). \tag{10.38}$$

Also, for all x, y, r such that $|x-y| \leqslant r$:

$$D^k_\mu(x,r) \leqslant 2^{k+2} D^k_\mu(y,2r). \tag{10.39}$$

Proof. The proof follows immediately from the definition. □

10.4.1 Quantitative Reifenberg Theorems

Assuming a sort of integral Carleson-type condition on the D numbers, we can obtain uniform scale invariant properties on the measure μ. For the reader's convenience, here we quote two key theorems that we are going to use in order to get the final estimates on the singular set of Q-valued minimizers. The first one is about upper Ahlfors bounds for discrete measures, and is quoted from [NV17, Theorem 3.4]. This theorem is enough for our purposes, but we mention that some generalizations have been obtained in [ENV16]. The second important theorem is about rectifiability properties for general μ, and is quoted from [AT15, Theorem 1.1].

Theorem 10.4.3. *[NV17, Theorem 3.4] For some constants $\delta_R(m)$ and $C_R(m)$ depending only on the dimension m, the following holds. Let $\{B_{r_x/10}(x)\}_{x\in\mathcal{D}} \subseteq B_3(0) \subset \mathbb{R}^m$ be a collection of pairwise disjoint balls with their centers $x \in B_1(0)$, and let $\mu \equiv \sum_{x\in\mathcal{D}} r_x^k \delta_x$ be the associated measure. Assume that for each $B_r(x) \subseteq B_2$*

$$\int_{B_r(x)} \left(\int_0^r D^k_\mu(y,s)\,\frac{ds}{s} \right) d\mu(y) < \delta_R^2 r^k. \tag{10.40}$$

Then, we have the uniform estimate

$$\sum_{x\in\mathcal{D}} r_x^k < C_R(m). \tag{10.41}$$

Condition (10.40) prescribes some integral Carleson-type control over the quantity $D(x,r)$. If the measure μ is the k-dimensional Hausdorff measure restricted to some S, this bound is enough to guarantee also the rectifiability of S, as seen in the following theorem. Note that in [AT15] the theorem is presented in a more general form.

Theorem 10.4.4 ([AT15, Corollary 1.3]). *Given a Borel measurable subset S of $\mathbb{R}^m$, let $\mu := \mathcal{H}^k \llcorner S$ be the k-dimensional Hausdorff measure restricted to S. The set S is countably k-rectifiable if and only if*

$$\int_0^1 D_\mu^k(x,s)\frac{ds}{s} < \infty \quad \text{for } \mu\text{-a.e. } x\,. \tag{10.42}$$

10.5 BEST APPROXIMATING PLANE

In this section, we record the main technical lemma needed for the final proof of Theorem 10.2.17. Although several technical points need to be addressed, this lemma contains most of the important estimates in the paper and provides an estimate on the D numbers using the normalized energy $\theta(x,r)$.

The basic ideas behind the estimates in this section are similar to the ones in [NV17, Theorem 7.1], however the new definition of (k,ε)-symmetries allows for more quantitative and easier proofs.

For any $f \in W^{1,2}(\Omega, \mathcal{A}_Q(\mathcal{N}))$, and for all $B_r(x) \subseteq \Omega$, we introduce the following quantity

$$\mathcal{P}_f(x,r) := r^{-m}\int_{B_r(x)} |Df(y)\cdot(y-x)|^2\,dy\,. \tag{10.43}$$

Note that in the case u is a Dirichlet minimizing Q-valued harmonic map we have by (10.29):

$$\mathcal{P}_u(x,r) \leqslant C(m)\,[\theta(x,2r)-\theta(x,r)]\,. \tag{10.44}$$

However, here we carry out the estimates in a very general setting, and we will exploit this bound only at the very last step in our main proof.

Theorem 10.5.1. *Let $u \in W^{1,2}(B_2(0), \mathcal{A}_Q(\mathcal{N}))$, and fix $\varepsilon > 0$, $0 < r \leqslant 1$ and some $x \in B_1(0)$. Let also μ be any positive Radon measure supported on $B_1(0)$. Assuming that*

$$\inf\left\{ r^{2-m}\int_{B_r(x)} |D_V u|^2 : V \subset \mathbb{R}^m \text{ linear with } \dim(V) = k+1 \right\} \geqslant \varepsilon\,, \tag{10.45}$$

we conclude

$$D_\mu^k(x,r) \leqslant \frac{(m-k)(k+1)2^m}{\varepsilon r^k}\int_{B_r(x)} \mathcal{P}_u(y,2r)\,d\mu(y)\,. \tag{10.46}$$

Remark 10.5.2. We remark that (10.46) does not change if μ is multiplied by a positive constant, thus for convenience for the rest of this section we are going to assume without loss of generality that μ is a probability measure. Moreover, we can also assume without loss of generality that $x = 0$ and $r = 1$.

Note that for this theorem we will not exploit any property specific to Dirichlet-minimizers. For future convenience, we record a simple corollary that rephrases the previous theorem with the language of Dirichlet-minimizers and quantitative stratification.

Corollary 10.5.3. *Under Assumption 10.2.9, fix $\varepsilon > 0$, $0 < r \leqslant 1$ and some $x \in B_1(0)$. Let also μ be any positive Radon measure supported on $B_1(0)$. Assuming that $B_r(x)$ is (k, ε)-symmetric but NOT $(k+1, \varepsilon)$-symmetric, we conclude*

$$D^k_\mu(x, r) \leqslant \frac{C(m)}{\varepsilon r^k} \int_{B_r(x)} [\theta(y, 4r) - \theta(y, 2r)] \, d\mu(y) \,. \tag{10.47}$$

Proof. The proof follows immediately from the definition of $(k+1, \varepsilon)$-symmetric and the bound in (10.44). □

10.5.1 Properties of the best approximating plane

For fixed k, and given any probability measure μ, for all (x, r) we set $V(x, r)$ to be the k-dimensional affine subspace minimizing

$$\int_{B_r(x)} \mathrm{dist}^2(y, V) \, d\mu(y) \,, \tag{10.48}$$

so that, in particular,

$$D^k_\mu(x, r) = r^{-(k+2)} \int_{B_r(x)} \mathrm{dist}^2(y, V(x, r)) \, d\mu(y) \,. \tag{10.49}$$

Since in this section we focus on $x = 0$ and $r = 1$, we will in fact mostly consider only the k-dimensional subspace $V(0, 1)$.

First of all, note that necessarily $V(x, r)$ will pass through the center of mass of μ in $B_r(x)$, defined as

$$x_m(\mu, x, r) = x_m := \int_{B_r(x)} x \, d\mu(x) \,. \tag{10.50}$$

It will be convenient to phrase some of the estimates needed for theorem 10.5.1 in terms of a suitable quadratic form on $\mathbb{R}^m$, defined as

$$R(w) := \int_{B_1(0)} |\langle x - x_m, w \rangle|^2 \, d\mu(x) \,. \tag{10.51}$$

By standard linear algebra, there exists an orthonormal basis $\{e_1, \cdots, e_m\}$ of eigenvectors for R with non-negative eigenvalues $\lambda_1, \cdots, \lambda_m$, which we will take for convenience in decreasing order. Note that by the variational characterization of λ_k we have that

$$e_k \in \mathrm{argmax} \left\{ \int_{B_1(0)} |\langle x - x_m, e \rangle|^2 \, d\mu(x) \;\; \text{s.t.} \;\; |e|^2 = 1 \;\; \text{and} \;\; \langle e, e_i \rangle = 0 \;\; \forall i \leqslant k \right\} , \tag{10.52}$$

$$\lambda_k = \int_{B_1(0)} |\langle x - x_m, e_k \rangle|^2 \, d\mu(x) \,, \tag{10.53}$$

and so

$$D^k_\mu(0,1) = \int_{B_1(0)} \operatorname{dist}^2(x, V(0,1))\, d\mu(x) = \sum_{i=k+1}^{m} \lambda_i\,. \tag{10.54}$$

Indeed, by minimality of V, $V(0,1) = x_m + \operatorname{span}[e_1, \cdots, e_k]$, and thus

$$\int_{B_1(0)} \operatorname{dist}^2(x, V(0,1))\, d\mu(x) = \sum_{i=k+1}^{m} \int_{B_1(0)} |\langle x - x_m, e_i\rangle|^2\, d\mu(x) = \sum_{i=k+1}^{m} \lambda_i\,. \tag{10.55}$$

Using simple geometry, it is possible to prove that for *any* map $f \in W^{1,2}$ we have the following estimate involving λ_k and $\mathcal{P}_f$.

Lemma 10.5.4. *Let* $f = \sum_{\ell=1}^{Q} [\![f_\ell]\!] \in W^{1,2}(B_{3r}(x), \mathcal{A}_Q(\mathcal{N}))$, *and let* μ *be a probability measure on* $B_r(x)$. *Then*

$$\lambda_k \int_{B_r(x)} |Df(z)\cdot e_k|^2\, dz \leqslant 2^m \int_{B_r(x)} \mathcal{P}_f(y, 2r)\, d\mu(y) \quad \textit{for every } k = 1, \dots, m\,. \tag{10.56}$$

Proof. For simplicity, we assume $x = 0$ and $r = 1$. Moreover, note that evidently we can assume $\lambda_k > 0$, otherwise there is nothing to prove. Fix some $z \in B_1(0)$. By definition of eigenvectors e_k, we have for every $\ell \in \{1, \dots, Q\}$ that

$$\int_{B_1(0)} \langle x - x_m, e_k\rangle\, (Df_\ell(z)\cdot(x - x_m))\, d\mu(x) = \lambda_k Df_\ell(z)\cdot e_k\,. \tag{10.57}$$

By definition of center of mass, we can write

$$\int_{B_1(0)} \langle x - x_m, e_k\rangle\, (z - x_m)\, d\mu(x) = 0\,, \tag{10.58}$$

and so

$$\lambda_k Df_\ell(z)\cdot e_k = \int_{B_1(0)} \langle x - x_m, e_k\rangle\, (Df_\ell(z)\cdot(x - z))\, d\mu(x)\,. \tag{10.59}$$

By Cauchy-Schwartz and by (10.53), we have

$$\lambda_k^2 |Df_\ell(z)\cdot e_k|^2 \leqslant \lambda_k \int |Df_\ell(z)\cdot(x-z)|^2\, d\mu(x)\,, \tag{10.60}$$

and thus, summing over ℓ,

$$\lambda_k |Df(z)\cdot e_k|^2 \leqslant \int |Df(z)\cdot(x-z)|^2\, d\mu(x)\,. \tag{10.61}$$

Taking the integral of this inequality in $B_1(0)$ with respect to the volume measure in z, we obtain the estimate

$$\begin{aligned} \lambda_k \int_{B_1(0)} |Df(z)\cdot e_k|^2\, dz &\leqslant \iint_{B_1(0)\times B_1(0)} |Df(z)\cdot(x-z)|^2\, dz\, d\mu(x) \\ &\leqslant \int_{B_1(0)} \int_{B_2(x)} |Df(z)\cdot(x-z)|^2\, dz\, d\mu(x) \leqslant 2^m \int_{B_1(0)} \mathcal{P}_f(x, 2)\, d\mu(x)\,. \end{aligned} \tag{10.62}$$

□

From this proposition, the proof of Theorem 10.5.1 follows as a simple corollary.

Proof of theorem 10.5.1. As before, we assume without loss of generality that $x = 0$ and $r = 1$. Moreover, by (10.54) it is sufficient to prove that

$$\lambda_{k+1} \leqslant \frac{C(m)}{\varepsilon} \int_{B_1(0)} \mathcal{P}_u(y, 2)\, d\mu(y)\,. \tag{10.63}$$

By the previous lemma, we have

$$\lambda_{k+1} \sum_{j=1}^{k+1} \int_{B_1(0)} |Du \cdot e_j|^2 \leqslant \sum_{j=1}^{k+1} \lambda_j \int_{B_1(0)} |Du \cdot e_j|^2 \leqslant C(m) \int_{B_1(0)} \mathcal{P}_u(x, 2)\, d\mu(x)\,. \tag{10.64}$$

By the lower bound in (10.45), we must have

$$\sum_{j=1}^{k+1} \int_{B_1(0)} |Du \cdot e_j|^2 \geqslant \varepsilon\,, \tag{10.65}$$

and this concludes the proof. □

10.6 PROOF OF THE MAIN THEOREM VIA COVERING ARGUMENTS

This section is dedicated to the proof of the Theorem 10.2.17. We split it into two pieces, one containing the uniform Minkowski bounds and one with the rectifiability part. Once the Minkowski bounds are obtained, the rectifiability is almost an immediate corollary.

The Minkowski bounds will be obtained with a covering argument similar to the one in [NV16].

Proposition 10.6.1. *There exist a small constant $\delta = \delta(m, \mathcal{N}, Q, \Lambda, \varepsilon) > 0$ and $C_{III}(m)$ such that the following holds. Let u satisfy assumption 10.2.9, let $\varepsilon > 0$, $p \in B_1(0)$, and $0 < r \leqslant R$, $0 < R \leqslant 1$ be chosen in an arbitrary fashion. For any subset $S \subseteq S^k_{\varepsilon,\delta r}(u)$, setting $E = \sup_{x \in B_{2R}(p) \cap S} \theta(x, 3R)$, there exists a covering*

$$S \cap B_R(p) \subseteq \bigcup_{x \in \mathcal{D}} B_{r_x}(x)\,, \quad \textit{with} \quad r_x \geqslant r \quad \textit{and} \quad \sum_{x \in \mathcal{D}} r_x^k \leqslant 2C_{III}(m) R^k\,. \tag{10.66}$$

Moreover, for all $x \in \mathcal{D}$, either $r_x = r$, or for all $y \in B_{2r_x}(x)$:

$$\theta(y, 3r_x) \leqslant E - \delta\,. \tag{10.67}$$

10.6.1 Proof of Theorem 10.2.17

Before we move to the proof of the proposition, we use it to prove the main theorem. This proof is basically a corollary of the covering proposition 10.6.1. We will use this proposition

inductively to produce a family of coverings of $S = S^k_{\varepsilon,\delta r}(u) \cap B_1(0)$ indexed by a parameter $i \in \mathbb{N}$ of the form

$$S \subseteq \bigcup_{x \in \mathcal{D}^i} B_{r_x}(x)\,, \qquad \sum_{x \in \mathcal{D}^i} r_x^k \leqslant (c(m)C_F(m))^i\,. \tag{10.68}$$

Moreover, if $E = \sup_{x \in S^k_{\varepsilon,\delta r}(u) \cap B_2(0)} \theta(x, 3)$, we have for all i

$$r_x \leqslant r \quad \text{or} \quad \forall y \in S \cap B_{2r_x}(x)\,,\ \ \theta(y, 3r_x) \leqslant E - i\delta\,. \tag{10.69}$$

Evidently, for $i \geqslant \lfloor E/\delta \rfloor + 1$, the second condition cannot be verified, and so all the radii in the covering are going to be equal to r. As a consequence, we have the Minkowski bound

$$\mathcal{L}^m\left(B_r\left(S^k_{\varepsilon,\delta r}(u)\right) \cap B_1(0)\right) \leqslant (c(m)C_F(m))^{\lfloor \delta^{-1}E \rfloor + 1} r^{m-k}\,. \tag{10.70}$$

Since $\delta = \delta(m, \Lambda)$, it is clear that, up to enlarging the constant in the estimate, the same bound holds also for $S^k_{\varepsilon,r}(u)$ in the place of $S^k_{\varepsilon,\delta r}(u)$, and this concludes the proof of the Minkowski bounds in (10.24).

In order to produce the covering in (10.68), we will apply inductively the covering proposition 10.6.1. For $i = 1$, we can apply this proposition to $B_1(0)$ and obtain the desired covering. Inductively, consider all the balls $\{B_{r_x}(x)\}_{x \in \mathcal{D}^i}$ and apply proposition 10.6.1 to these balls. For each $x \in \mathcal{D}^i$, we obtain a covering of the form

$$S \cap B_{r_x}(x) \subseteq \bigcup_{y \in \mathcal{D}_x} B_{r_y}(y)\,, \qquad \sum_{y \in \mathcal{D}_x} r_y^k \leqslant 2C_{III}(m) r_x^k\,, \tag{10.71}$$

$$r_y \leqslant r \quad \text{or} \quad \forall z \in S \cap B_{2r_y}(y)\,,\ \ \theta(z, 3r_y) \leqslant E - (i+1)\delta\,. \tag{10.72}$$

Set

$$\mathcal{D}^{i+1} = \bigcup_{x \in \mathcal{D}^i} \mathcal{D}_x\,, \tag{10.73}$$

and the induction step is completed.

Proof of the rectifiability of S^k_ε

As for the rectifiability, this is going to be a corollary of Theorem 10.4.4, the uniform Minkowski bound (10.24) and the approximation theorem 10.5.1.

In particular, let $\mu = \mathcal{H}^k \llcorner \left\{S^k_\varepsilon(u) \cap B_1(0)\right\}$. From (10.24) we deduce that this measure is finite, as

$$\mu(B_1(0)) \leqslant C(m, \Lambda, \varepsilon).$$

In turn, by scaling this implies that for all $x \in B_1(0)$ and $r > 0$

$$\mu\left(B_r(x)\right) \leqslant C(m, \Lambda, \varepsilon) r^k\,, \tag{10.74}$$

and thus μ is Ahlfors upper k-regular.

Now by the best approximation theorem 10.5.1 and a simple change of variables we can write

$$\begin{aligned}\int_{B_1(0)}\int_0^1 D_\mu^k(x,r)\frac{dr}{r}d\mu(x) &\leqslant C(m,\varepsilon)\int_0^1\int_{B_1(0)} r^{-k}\int_{B_r(x)}\mathcal{P}_u(y,2r)\,d\mu(y)d\mu(x)\frac{dr}{r}\\ &\leqslant C(m,\varepsilon)\int_0^1\int_{B_1(0)}\mathcal{P}_u(y,2r)\left(r^{-k}\int_{B_r(y)}d\mu(x)\right)d\mu(y)\frac{dr}{r}\\ &\leqslant C(m,\varepsilon,\Lambda)\int_{B_1(0)}\int_0^1\mathcal{P}_u(y,2r)\frac{dr}{r}d\mu(x)\\ &\leqslant C(m,\varepsilon,\Lambda)\Lambda\,,\end{aligned} \tag{10.75}$$

where the last inequality follows from

$$\begin{aligned}&\int_0^1\mathcal{P}_u(y,2r)\frac{dr}{r}\overset{(10.44)}{\leqslant}\int_0^1[\theta(y,4r)-\theta(y,2r)]\frac{dr}{r}=\lim_{t\to0}\int_t^1[\theta(y,4r)-\theta(y,2r)]\frac{dr}{r} \qquad (10.76)\\ &=\int_{1/2}^1\theta(y,4r)\frac{dr}{r}+\underbrace{\lim_{t\to0}\int_t^{1/2}\theta(y,4r)\frac{dr}{r}-\int_{2t}^1\theta(y,2r)\frac{dr}{r}}_{=0}-\lim_{t\to0}\int_t^{2t}\theta(y,2r)\frac{dr}{r}\leqslant C(m)\Lambda\,.\end{aligned}$$

The rectifiability of $\mathcal{S}^k_\varepsilon(u)$ is now a consequence of theorem 10.4.4.

By countable additivity, the rectifiability of $\mathcal{S}^k(u)$ is a corollary of the rectifiability of $\mathcal{S}^k_\varepsilon(u)$ for all $\varepsilon>0$.

It is worth remarking that the uniform Ahlfors upper estimates obtained a priori for the measure $\mu=\mathcal{H}^k\llcorner\{\mathcal{S}^k_\varepsilon(u)\cap B_1(0)\}$ are *essential* to carry out this computation, and actually they are the most difficult part of the estimate. This is why the proof of the rectifiability property is so easy.

□

10.6.2 Proof of Proposition 10.6.1

Now we turn to the proof of the covering proposition. We split this proof in two pieces by introducing a secondary covering proposition.

Proposition 10.6.2. *Under the assumptions of proposition 10.6.1, for all $0<\rho<1/100$, there exist $\delta=\delta(m,N,Q,\Lambda,\varepsilon,\rho)>0$ and $C_{II}(m)$ such that the following is true.*

There exists a finite covering of $\mathcal{S}=\mathcal{S}^k_{\varepsilon,\delta r}(u)\cap B_R(p)$ of the form

$$\mathcal{S}\subseteq\bigcup_{x\in\mathcal{D}}B_{r_x}(x)\,,\quad \textit{with}\quad r_x\geqslant r\quad\textit{and}\quad\sum_{x\in\mathcal{D}}r_x^k\leqslant C_{II}(m)R^k\,. \tag{10.77}$$

Moreover, for each $x\in\mathcal{D}$, either there exists a $(k-1)$-dimensional space W_x such that

$$F_{x,r_x}\equiv\{y\in\mathcal{S}\cap B_{2r_x}(x)\quad\textit{with}\quad\theta(y,\rho r_x/20)\geqslant E-\delta\}\subseteq B_{\rho r_x/10}(W_x)\,, \tag{10.78}$$

or $r_x=r$.

Assuming this proposition, we prove proposition 10.6.1. The idea is simple: we consider this second covering, and refine it inductively on each ball with $r_x \geqslant r$ and no uniform energy drop.

Proof of proposition 10.6.1. Let $0 < \rho < 1/100$ to be fixed later, and let $A \in \mathbb{N}$ be the first integer such that $\rho^A < r$. Also assume without loss of generality $p = 0$ and $R = 1$.

For all $i = 1, \cdots, A$, we construct a covering of S of the form

$$S \cap B_1(0) \subseteq \bigcup_{x \in \mathcal{R}_i} B_r(x) \cup \bigcup_{x \in \mathcal{F}_i} B_{r_x}(x) \cup \bigcup_{x \in \mathcal{B}_i} B_{r_x}(x), \tag{10.79}$$

where $\mathcal{R}_i$ are the balls of radius r in the covering, $\mathcal{F}_i$ are the balls where the uniform energy drop condition (10.67) is satisfied, and $\mathcal{B}_i$ are the bad balls, where none of the two conditions is verified. We want to obtain uniform packing bounds on $\mathcal{R}_i$ and $\mathcal{F}_i$, and exponentially small packing bounds on $\mathcal{B}_i$. We will refine our covering only on bad balls by re-applying the second covering lemma on those, and this is why we need smallness on their packing bounds. In detail, we want

$$\sum_{x \in \mathcal{R}_i \cup \mathcal{F}_i} r_x^k \leqslant C_F(m) \left(\sum_{j=0}^{i} 7^{-j} \right), \quad \sum_{x \in \mathcal{B}_i} r_x^k \leqslant 7^{-i}. \tag{10.80}$$

Induction step

Pick a generic ball $B_R(p)$, and apply the second covering in Proposition 10.6.2 to it. We obtain a covering of the form

$$S \cap B_R(p) \subseteq \bigcup_{x \in \mathcal{D}} B_{r_x}(x), \quad \text{with} \quad r_x \geqslant r \quad \text{and} \quad \sum_{x \in \mathcal{D}} r_x^k \leqslant C_{II}(m) R^k. \tag{10.81}$$

We split $\mathcal{D}$ into two disjoint sets: $\mathcal{D} = \mathcal{D}_r \cup \mathcal{D}_+$, where the first set is the one with $r_x \leqslant 60\rho^{-1} r$. Observe that if x is in the second set then (10.78) is valid. For all $x \in \mathcal{D}_r$, consider a simple covering of $B_{r_x}(x)$ by balls of radius r with number bounded by $c(m)\rho^{-m}$, and let $\mathcal{R}_p$ be the union of all centers in these coverings. Note that if $r_x = r$, we can keep this ball unchanged.

For all $x \in \mathcal{D}_+$, consider a covering of $B_{r_x}(x)$ made of balls of radius $\rho r_x/60 > r$ centered inside this ball and such that the family of balls with half the radius are pairwise disjoint. In particular, let

$$B_{r_x}(x) \subseteq \bigcup_{y \in \mathcal{B}_x} B_{\rho r_x/60}(y) \cup \bigcup_{y \in \mathcal{F}_x} B_{\rho r_x/60}(y), \tag{10.82}$$

where

$$F_{x,r_x} \cap \bigcup_{y \in \mathcal{F}_x} B_{2\cdot(\rho r_x/60)}(y) = \emptyset, \quad \mathcal{B}_x \subseteq B_{\rho r_x}(W_x). \tag{10.83}$$

Thus, the balls in $\mathcal{F}_x$ will have a uniform energy drop, in particular we have that for all $y \in \mathcal{F}_x$ and $z \in B_{2\cdot(\rho r_x/60)}(y) = B_{2r_y}(y)$,

$$\theta(z, 3r_y) < E - \delta. \tag{10.84}$$

Moreover, the number of balls in $\mathcal{B}_x$ is well controlled. Indeed, since $\mathcal{B}_x \subseteq B_{\rho r_x}(W_x)$, $\{B_{\rho r_x/40}(y)\}_{y\in\mathcal{B}_x}$ are pairwise disjoint and W_x is a k-dimensional subspace, then

$$\#\{\mathcal{F}_x\} \leqslant c(m)\rho^{-m}, \quad \#\{\mathcal{B}_x\} \leqslant c(m)\rho^{1-k}. \tag{10.85}$$

Set $\mathcal{B}_p = \cup_{x\in\mathcal{D}}\mathcal{B}_x$ and $\mathcal{F}_p = \cup_{x\in\mathcal{D}}\mathcal{F}_x$. We have

$$\sum_{z\in\mathcal{R}_p\cup\mathcal{F}_p} r_z^k \leqslant c(m)\rho^{-m+k}\sum_{x\in\mathcal{D}} r_x^k \leqslant c(m)\rho^{-m+k}C_{II}(m)R^k, \tag{10.86}$$

$$\sum_{z\in\mathcal{B}_p} r_z^k \leqslant c(m)\rho^{1}\sum_{x\in\mathcal{D}} r_x^k \leqslant c(m)\rho C_{II}(m)R^k. \tag{10.87}$$

We choose $\rho = \rho(m) \leqslant 1/100$ sufficiently small so that

$$c(m)\rho C_{II}(m) \leqslant 1/7. \tag{10.88}$$

In this way, we have the estimates

$$\sum_{z\in\mathcal{R}_p\cup\mathcal{F}_p} r_z^k \leqslant C_{III}(m)R^k \qquad \sum_{z\in\mathcal{B}_p} r_z^k \leqslant 7^{-1}R^k. \tag{10.89}$$

Finishing the proof

With the induction step, the proof follows easily. For $i = 1$, apply the induction step to $B_1(0)$ and we obtain (10.79) for $i = 1$ with (10.80).

For generic i, we have by induction

$$\mathcal{S}\cap B_1(0) \subseteq \bigcup_{x\in\mathcal{R}_i} B_r(x) \cup \bigcup_{x\in\mathcal{F}_i} B_{r_x}(x) \cup \bigcup_{x\in\mathcal{B}_i} B_{r_x}(x). \tag{10.90}$$

Apply the induction step on all the balls $\{B_{r_x}(x)\}_{x\in\mathcal{B}_i}$ separately, and define

$$\mathcal{R}_{i+1} = \mathcal{R}_i \cup \bigcup_{x\in\mathcal{B}_i}\mathcal{R}_x, \quad \mathcal{F}_{i+1} = \mathcal{F}_i \cup \bigcup_{x\in\mathcal{B}_i}\mathcal{F}_x, \quad \mathcal{B}_{i+1} = \bigcup_{x\in\mathcal{B}_i}\mathcal{B}_x. \tag{10.91}$$

By construction, we have the estimates

$$\sum_{z\in\mathcal{R}_{i+1}\cup\mathcal{F}_{i+1}} r_z^k \leqslant C_{III}(m)\sum_{s=0}^{i} 7^{-s} \qquad \sum_{z\in\mathcal{B}_{i+1}} r_z^k \leqslant 7^{-i-1}. \tag{10.92}$$

Note that at the step $i = A$ all the balls in our covering will either have energy drop (if they are in $\mathcal{F}_A$) or have radius $= r$ (if they are in $\mathcal{R}_A$). Equation (10.92) for $i = A$ gives the desired bound on the final covering.

□

Now we turn our attention to the proof of proposition 10.6.2, which is the last one needed to complete the main theorem.

10.6.3 Proof of Proposition 10.6.2

For convenience, we assume $p = 0$ and $R = 1$. Fix $\varepsilon, \rho > 0$, and let A be such that $\rho^A \leqslant r < \rho^{A-1}$. The proof is based on an inductive covering by balls, where the discrete Reifenberg is applied in order to control the number of these balls.

Construction of the covering

We split the inductive covering in two parts: at first we simply construct the covering inductively, and then we prove the packing bounds using Reifenberg's theorem. Specifically, we start by looking for an inductive (for $i = 0, 1, \cdots, A$) covering of the form

$$S \subseteq \bigcup_{x \in \mathcal{B}_i} B_{r_x}(x) \cup \bigcup_{x \in \mathcal{G}_i} B_{r_x}(x)\,, \tag{10.93}$$

where the elements of $\mathcal{B}_i$ are the centers of the *bad balls* in our covering, and $\mathcal{G}_i$ are the centers of the *good balls*. In particular, we want :

1. for all i and $x \in \mathcal{B}_i$, $r_x \geqslant \rho^i$ and there exists a $(k-1)$-dimensional subspace W_x such that

$$F_{x,r_x} \equiv \{y \in S \cap B_{2r_x}(x) \quad \text{such that} \quad \theta(y, \rho r_x/20) \geqslant E - \delta\} \subseteq B_{\rho r_x/10}(W_x)\,, \tag{10.94}$$

   where $\delta > 0$ is fixed, to be determined later;

2. for all $i = 1, \cdots, A$ and $x \in \mathcal{G}_i$, $r_x = \rho^i$ and the set F_{x,r_x} defined above $(\rho r_x/20)$-effectively spans some k-dimensional affine subspace V_x;

3. for $i = A$, we have the bound

$$\sum_{x \in \mathcal{B}_A \cup \mathcal{G}_A} r_x^k \leqslant C_{II}(m)\,. \tag{10.95}$$

Moreover, we request some extra properties of the centers of the covering in order to apply the discrete-Reifenberg theorem:

4. for all i, the balls in the collection $\left\{B_{r_x/10}(x)\right\}_{x \in \mathcal{G}_i \cup \mathcal{B}_i}$ are pairwise disjoint;

5. for all $i \geqslant 1$ and $x \in \mathcal{G}_i$, we have the energy bound

$$\theta(x, r_x) \geqslant E - \eta \quad \text{for some } \eta > 0\,; \tag{10.96}$$

6. there exists a constant $c(m)$ such that for all i, the balls in the collection $\{B_s(x)\}_{x \in \mathcal{G}_i,\ s \in [r_x, 1]}$ are not $(k+1, \varepsilon/c(m))$-symmetric.

At each induction step, we will refine our covering on the good balls, while leaving the bad balls untouched.

For $i = 0$, consider the set $F_{0,1}$. If this set does NOT $\rho/20$-effectively span something k-dimensional, then we call $B_1(0)$ a bad ball, set $\mathcal{G}_i = \emptyset$ for all i and $\{0\} = \mathcal{B}_0 = \mathcal{B}_A$ with $r_0 = 1$. This covering immediately satisfies all the properties of proposition 10.6.2.

In the other case, set $\mathcal{G}_0 = \{0\}$ with $r_0 = 1$.

INDUCTION STEP Assuming by induction that all the properties listed above are valid up to the index i, we want to produce the covering for $i+1$. In order to do so, we want to refine our covering on good balls, and leave the previous bad balls intact.

Fix an arbitrary $x \in \mathcal{G}_i$, and consider the set F_{x,r_x}. Since $B_{\rho^i}(x)$ is a good ball, by definition this set $[\rho^{i+1}/20]$-effectively spans a k-dimensional affine subspace V_x. By applying theorem 10.3.3 to the ball $B_{4\rho^i}(x)$, we find that there exists a $\delta(m, \Lambda, \varepsilon, \rho)$ sufficiently small so that

$$S^k_{\varepsilon,\delta r}(u) \cap B_{2\rho^i}(x) \subset B_{\rho^{i+1}/10}(V_x) \tag{10.97}$$

Consider the set

$$K = \bigcup_{x\in\mathcal{G}_i} \left(B_{\rho^i}(x) \cap V_x\right) \setminus \bigcup_{x\in\mathcal{B}_i} B_{r_x}(x)\,. \tag{10.98}$$

Given the inclusion (10.97), and since we have chosen $\rho \leqslant 1/100$, we have

$$\mathcal{S} \setminus \bigcup_{x\in\mathcal{B}_i} B_{r_x}(x) \subseteq B_{\rho^{i+1}/5}(K)\,. \tag{10.99}$$

Let $\mathcal{D}_K \subseteq K$ be a maximal subset of points at least $\rho^{i+1}/5$ apart, so that the balls $\left\{B_{\rho^{i+1}/10}(x)\right\}_{x\in\mathcal{D}_K}$ are pairwise disjoint. Note that these balls are also disjoint from $\left\{B_{r_x/3}(x)\right\}_{x\in\mathcal{B}_i}$ by construction. Moreover, by maximality of the subset

$$\mathcal{S} \setminus \bigcup_{x\in\mathcal{B}_i} B_{r_x}(x) \subseteq \bigcup_{x\in\mathcal{D}_K} B_{2\rho^{i+1}/5}(x) \tag{10.100}$$

We can discard from this collection all the balls $B_{2\rho^{i+1}/5}(x)$ that have empty intersection with $\mathcal{S}$. Now consider the collection

$$\left\{B_{\rho^{i+1}}(x)\right\}_{x\in\mathcal{D}_K}\,, \tag{10.101}$$

and classify these points into good and bad balls according to whether or not (10.94) is satisfied. In particular, if $F_{x,\rho^{i+1}}$ $\rho^{i+2}/20$-effectively spans a k-dimensional subspace V_x, then we say that $x \in \tilde{\mathcal{G}}_{i+1}$, and $x \in \tilde{\mathcal{B}}_{i+1}$ otherwise. We set

$$\mathcal{B}_{i+1} = \mathcal{B}_i \cup \tilde{\mathcal{B}}_{i+1}\,, \quad \mathcal{G}_{i+1} = \tilde{\mathcal{G}}_{i+1}\,. \tag{10.102}$$

This takes care of properties 1 and 2 in the induction.

Now fix any $x \in \mathcal{D}_K$. By construction, there exists an $x' \in \mathcal{G}_i$ such that $x \in V_{x'} \cap B_{r_{x'}}(x')$. Hence, we can apply proposition 10.3.7, and prove that for all $\eta > 0$ there exists a $\delta(m, \mathcal{N}, Q, \Lambda, \rho, \eta)$ sufficiently small so that

$$\theta\left(x, \rho^{i+1}/40\right) \geqslant E - \eta\,. \tag{10.103}$$

Moreover, there also exists some $x' \in \mathcal{S} \cap B_{2\rho^{i+1}/5}(x)$. By definition of $\mathcal{S}$, this implies that for every $(k+1)$-dimensional subspace $T = T_{x'}$:

$$c(m)\rho^{(2-m)(i+1)} \int_{B_{\rho^{i+1}}(x)} |D_T u|^2 \geqslant \left(2\rho^{i+1}/5\right)^{2-m} \int_{B_{2\rho^{i+1}/5}(x')} |D_T u|^2 \geqslant \varepsilon\,. \tag{10.104}$$

In other words, $B_{\rho^{i+1}}(x)$ is not $(k+1, \varepsilon/c(m))$-symmetric. Thus all the properties of our inductive covering are satisfied.

I=A For $i = A$, one can use the same construction as above, but with radius r instead of radius ρ^A. At this stage, one also does not need to make any distinction between good and bad balls.

At this stage, we also set

$$\mathcal{D} = \mathcal{B}_A \cup \mathcal{G}_A \,. \tag{10.105}$$

We are left to prove the packing estimates (10.77).

Volume estimates

We will apply the discrete Reifenberg theorem to the measure

$$\mu_{\mathcal{D}} = \sum_{x \in \mathcal{D}} r_x^k \delta_x \,. \tag{10.106}$$

In order to do so, we need to check that (10.40) is satisfied for this μ, and we exploit the best approximation theorem 10.5.1.

However, as it will be evident later on, we cannot apply this theorem directly. Instead, we will prove the volume estimate with an upwards induction.

Inductive statement

For convenience, we define the one-parameter family of measures μ_t by setting

$$\mathcal{D}_t = \mathcal{D} \cap \{r_x \leqslant t\} \,, \quad \mu_t = \mu \llcorner \mathcal{D}_t \leqslant \mu \,. \tag{10.107}$$

Let T be such that $2^{T-1} r < 1/70 \leqslant 2^T r$. We will prove by induction on $j = 0, 1, \cdots, T$ that there exists a constant $C_I(m)$ such that for all $x \in B_1(0)$ and $s = 2^j r$:

$$\mu_s(B_s(x)) = \sum_{y \in \mathcal{D} \cap B_s(x) \text{ s.t. } r_y \leqslant s} r_y^k \leqslant C_I(m) s^k \,. \tag{10.108}$$

Once this has been proved, with a simple covering argument we can turn the estimates for $j = T$ into the estimates (10.77), replacing $C_I(m)$ with $C_{II}(m) = c(m) C_I(m)$ if necessary.

BASE STEP IN THE INDUCTION, $j = 0$. The first step of the induction is easy. Since by construction $r_x \geqslant r$ for all $x \in \mathcal{D}$, and since the balls $\{B_{r_x/10}(x)\}_{x \in \mathcal{D}}$ are pairwise disjoint, a standard covering argument shows that for all $x \in B_1(0)$,

$$\mu_r(B_r(x)) \leqslant C_0(m) r^k \,. \tag{10.109}$$

Induction step

The induction step is divided into two parts: first we are going to prove a weak packing bound for balls of radius $2^{j+1} r$. With this estimate, we will be able to apply the discrete Reifenberg theorem, which gives us a uniform scale invariant upper bound for the measure that lets us complete the induction.

COARSE BOUNDS Assuming that the induction step j is proved, we can easily obtain a rough bound for $j+1$. Indeed, let $x \in B_1(0)$ be arbitrary, and consider the ball $B_{2^{j+1}r}(x)$. By covering this ball with $c(m)$ balls of half the radius, and using the induction hypothesis, we can estimate

$$\mu_{2^j r}\left(B_{2^{j+1}r}(x)\right) \leqslant c(m) C_1(m) (2^{j+1} r)^k . \tag{10.110}$$

With a similar covering argument, we can estimate the "new contributions" in $\mu_{2^{j+1}r}$. To be precise, since $\left\{B_{r_x/10}(x)\right\}_{x\in\mathcal{D}}$ are all pairwise disjoint, we have

$$\bar{\mathcal{D}} = \left\{x \in \mathcal{D} \cap B_{2^{j+1}r}(x) \quad \text{with } r_x \in (2^j r, 2^{j+1} r]\right\} \qquad \sum_{x\in\bar{\mathcal{D}}} r_x^k \leqslant C_0(m) (2^{j+1} r)^k . \tag{10.111}$$

Thus, choosing $C_1(m) \geqslant C_0(m)$, we have

$$\mu_{2^{j+1}r}\left(B_{2^{j+1}r}(x)\right) \leqslant c(m) C_1(m) (2^{j+1} r)^k . \tag{10.112}$$

REFINED ESTIMATE In order to refine this last estimate, we need to apply the discrete Reifenberg 10.4.3. An essential tool is given by the estimates in corollary 10.5.3. Fix any $B_{2^{j+1}r}(x)$ for $x \in \mathcal{D}$. For convenience, hereafter we will denote

$$\mu_{2^{j+1}r} \llcorner B_{2^{j+1}r}(x) \equiv \mu . \tag{10.113}$$

Set also for $y \in \mathcal{D}$:

$$\mathcal{W}_{\mathcal{D}}(y,s) = \begin{cases} \theta(y,4s) - \theta(y,2s) & \text{for } s \geqslant r_y/10 , \\ 0 & \text{for } s < r_y/10 . \end{cases} \tag{10.114}$$

By construction, and in particular by the estimate in (10.104) and (10.96), for η sufficiently small we can apply Corollary 10.5.3 to μ and any ball $B_s(x)$ with $x \in \mathcal{D}$ and $s \in [r_x, 1]$, and obtain

$$D_\mu^k(x,s) \leqslant C_1 s^{-k} \int_{B_s(x)} \mathcal{W}_{\mathcal{D}}(y,s)\, d\mu(y) . \tag{10.115}$$

As a corollary of this and (10.39), we can extend this relation for all $s \in [r_x/10, 1]$, up to enlarging C_1 by $c(m)$:

$$D_\mu^k(x,s) \leqslant c(m) C_1 s^{-k} \int_{B_{10s}(x)} \mathcal{W}_{\mathcal{D}}(y,10s)\, d\mu(y) . \tag{10.116}$$

Note that this relation is trivially true also for $s \leqslant r_x/10$, because in this case the support of the measure μ inside the ball $B_{r_x/10}(x)$ is an isolated point.

We can use this estimate to prove (10.40) for the measure μ. Indeed, fix any $y \in B_{2^{j+2}r}(x)$, $t \in (0, 2^{j+1} r]$, and in turn choose any $s \in [0,t]$. For these parameters, we can bound:

$$\int_{B_t(y)} D_\mu^k(z,s)\, d\mu(z) \overset{(10.116)}{\leqslant} C_1 s^{-k} \int_{B_t(y)} \left[\int_{B_{10s}(z)} \mathcal{W}_{\mathcal{D}}(p,10s)\, d\mu(p)\right] d\mu(z) . \tag{10.117}$$

Considering that

$$\{(p,z) \text{ s.t. } |z-y| \leqslant t \text{ and } |p-z| \leqslant 10s\} \subset \{(p,z) \text{ s.t. } |p-y| \leqslant t+10s \text{ and } |p-z| \leqslant 10s\}\,, \tag{10.118}$$

we can exchange the variables of integration and estimate

$$\begin{aligned}\int_{B_t(y)} D_\mu^k(z,s)\,d\mu(z) &\leqslant C_1 \int_{B_{11t}(y)} \frac{\mu(B_{10s}(p))}{s^k} \mathcal{W}_{\mathcal{D}}(p,10s)\,d\mu(p) \\ &\leqslant c(m)C_1C_{II} \int_{B_{11t}(y)} \mathcal{W}_{\mathcal{D}}(p,10s)\,d\mu(p)\,.\end{aligned} \tag{10.119}$$

Recall that by (10.113), $\mu(A) = \mu(A \cap B_{2r^{j+1}r}(x)$. Note that the induction hypothesis and the coarse estimates have been used to obtain the last inequality.

By integrating this inequality on $\int_0^t \frac{ds}{s}$, we get

$$\int_{B_t(y)} \left(\int_0^t D_\mu^k(z,s)\,\frac{ds}{s} \right) d\mu(z) \leqslant c(m)C_1C_{II} \int_{B_{11t}(y)} \left[\int_0^t \mathcal{W}_{\mathcal{D}}(z,10s)\frac{ds}{s} \right] d\mu(z)\,. \tag{10.120}$$

Note that for all $x \in \mathcal{D}$, $\theta(0,1) - \theta(0,r_x) \leqslant \eta$. Thus for $t \leqslant 1/70$ we have

$$\int_0^t \mathcal{W}_{\mathcal{D}}(x,10s)\frac{ds}{s} = \int_{r_x}^t [\theta(x,40s) - \theta(x,20s)]\frac{ds}{s} \tag{10.121}$$

$$= \int_{t/2}^t \theta(x,40s)\frac{ds}{s} + \underbrace{\int_{r_x}^{t/2} \theta(x,40s)\frac{ds}{s} - \int_{2r_x}^{t/2} \theta(x,20s)\frac{ds}{s}}_{=0} - \int_{r_x}^{2r_x} \theta(x,20s)\frac{ds}{s} \tag{10.122}$$

$$= \int_{t/2}^t \left[\theta(x,40s) - \theta\left(x, 40\frac{r_x}{t}s\right) \right] \frac{ds}{s} \leqslant c\eta\,. \tag{10.123}$$

This in turn implies

$$\int_{B_t(y)} \left(\int_0^t D_\mu^k(z,s)\,\frac{ds}{s} \right) d\mu(z) \leqslant c(m)C_1(m,\varepsilon)C_{II}\eta t^k\,. \tag{10.124}$$

By picking η sufficiently small (in turn: by picking $\delta(m,\mathcal{N},Q,\Lambda,\varepsilon,\eta)$ sufficiently small), we can apply the discrete Reifenberg theorem to μ and prove that

$$\mu_{2^{j+1}r}(B_{2^{j+1}r}(x)) \leqslant C_R(m)(2^{j+1}r)^k\,. \tag{10.125}$$

By picking $C_{II}(m) = \max\{C_0(m), C_R(m)\}$, we complete the induction step, and in turn the proof of this proposition.

□

11 CONTINUITY IN NON-POSITIVELY CURVED SPACES

This chapter is devoted to the proof of the following result.

Theorem 11.0.1. *Let $\mathcal{N} \hookrightarrow \mathbb{R}^d$ be a complete, simply connected manifold all of whose sectional curvatures are non-positive. Then, every minimizing harmonic map $u \in W^{1,2}(\Omega, \mathcal{A}_Q(\mathcal{N}))$ satisfies*

$$\operatorname{sing}_H(u) = \emptyset.$$

The proof will be split into two parts. In the first part of the argument we will show a general lemma, Lemma 11.1.1. Then, in Section 11.2 we will show how the lemma implies the theorem.

Observe that in the single-valued case $Q = 1$ the hypothesis that $\pi_1(\mathcal{N}) = \{0\}$ is not necessary: indeed, in Section 11.3 we will show how the same result holds when $Q = 1$ under the weaker assumption that $\mathcal{N}$ is connected. The proof will follow from the simply connected situation by means of lifting of Lipschitz continuous functions into covering spaces. The hypothesis that $\mathcal{N}$ is simply connected, instead, is indispensable when $Q > 1$: in Section 11.4 we will provide an example of a *singular* Q-valued minimizing harmonic map in a flat target manifold $\mathcal{N}$ (cf. Proposition 11.4.1).

11.1 A TECHNICAL LEMMA

Lemma 11.1.1. *Let $f : \mathcal{N} \to \mathbb{R}$ be a C^2-regular function such that $\nabla^2 f \geqslant 0$ on $T_p\mathcal{N}$ for all $p \in N$. Then*

$$f \circ u = \sum_{\ell=1}^{Q} f(u_\ell) = \text{const.}$$

for any 0-homogeneous Dirichlet minimizer $u : \mathbb{R}^m \to \mathcal{A}_Q(\mathcal{N})$.

Proof. We will split the proof of the lemma into two steps:

claim 1: for any Dirichlet minimizer $u : \Omega \to \mathcal{A}_Q(\mathcal{N})$, $\Omega \subset \mathbb{R}^m$ open, we have that $f \circ u \colon \Omega \to \mathbb{R}$ is subharmonic in the sense of distributions i.e.

$$\Delta(f \circ u) \geqslant 0\,; \tag{11.1}$$

claim 2: any 0-homogeneous subharmonic function is constant.

The lemma is an immediate consequence of claim 1 and claim 2.

Proof of claim 1: Let $\hat{f}$ be any extension of f to $\mathbb{R}^d$ such that $\hat{f}$ is C^2 (for instance, take $\hat{f}(p) := \phi(p) f(\Pi(p))$, where $\Pi(p) : \mathbf{U}_\delta(\mathcal{N}) \to \mathcal{N}$ is the nearest point projection from a δ-tubular neighborhood $\mathbf{U}_\delta(\mathcal{N})$ and ϕ is a non-negative smooth bump function supported in

$\mathbf{U}_\delta(\mathcal{N})$ and constantly equal to 1 in a small neighborhood of $\mathcal{N}$). Observe that for every $p \in \mathcal{N}$ we have $\nabla^2 f = (D(D\hat{f})^{T_p})^{T_p}$, where v^{T_p} denotes the orthogonal projection of v onto $T_p\mathcal{N}$. In order to deduce the claim, let $\varphi = \varphi(x) \in C^1_c(\Omega)$ non-negative be given and define the vector field

$$Y(x,p) := \varphi(x)\nabla\hat{f}(p) = \varphi(x)(D\hat{f}(p))^{T_{\pi(p)}}.$$

The outer variation formula (2.39) provides now

$$\begin{aligned} 0 &= \int_\Omega \sum_{i=1}^m \sum_{\ell=1}^Q \left(\langle D_i u_\ell, \nabla\hat{f}(u_\ell)\rangle D_i\varphi + \langle D_i u_\ell, D\nabla\hat{f}\cdot D_i u_\ell\rangle \varphi \right) \\ &= \int_\Omega \sum_{i=1}^m \left(D_i(f\circ u) D_i\varphi + \sum_{\ell=1}^Q \nabla^2 f(u_\ell)(D_i u_\ell, D_i u_\ell)\varphi \right). \end{aligned}$$

In the last line we have used that $D_i u_\ell \in T_{u_\ell}\mathcal{N}$ and so $\langle D_i u_\ell, D\nabla\hat{f}\cdot D_i u_\ell\rangle = \nabla^2 f(u_\ell)(D_i u_\ell, D_i u_\ell)$. Since the last term is non-negative we deduce the claim:

$$\int_\Omega \langle D(f\circ u), D\varphi\rangle \leqslant 0 \quad \text{for all } \varphi \in C^1_c(\Omega), \varphi \geqslant 0.$$

Proof of claim 2: Let $h \in W^{1,2}(\mathbb{R}^m)$ be 0-homogeneous and subharmonic in the sense of distributions i.e.

$$\int \langle Dh, D\varphi\rangle \leqslant 0 \quad \text{for all } \varphi \in C^1_c(\mathbb{R}^m), \varphi \geqslant 0. \tag{11.2}$$

Suppose h is not constant. Then there exists $a > 0$ such that h is not constant on the super-level set $\{x\colon h(x) \geqslant -a\}$, which in turn implies $(h+a)^+$ is not constant. Take any non-negative $\eta(t), \eta(t) = 0$ for $t > R$ (possibly a smooth approximation of $(R-t)^+$), and consider the test function $\varphi(x) = \eta(|x|^2)(h+a)^+$ in (11.2). Observe that $D_i\varphi = \eta(|x|^2)D_i(h+a)^+ + \eta'(|x|^2)2x^i(h+a)^+$. But $\sum_i D_i h(x)x^i = 0$ for a.e. x in $\mathbb{R}^m$ because h is homogeneous. Hence we deduce

$$0 \geqslant \int |D(h+a)^+|^2\eta(|x|^2).$$

But this contradicts the assumption that $(h+a)^+$ is not constant. □

11.2 PROOF OF THEOREM 11.0.1

In this section we conclude the proof of Theorem 11.0.1. Recall that the hypotheses on $\mathcal{N}$ imply by the Cartan-Hadamard Theorem that $\exp_p : T_p\mathcal{N} \to \mathcal{N}$ is a covering map for every $p \in \mathcal{N}$. Furthermore, since $\mathcal{N}$ is assumed to be smooth we have $\mathrm{dist}_{\mathcal{N}}(q,p) = |\exp_p^{-1}(q)|$. As a further consequence we deduce that for each p the map $q \mapsto d_p^2(q) := \mathrm{dist}_{\mathcal{N}}(q,p)^2$ is smooth. By the second variation formula for length we deduce that $\nabla^2 d_p^2 \geqslant 0$.

Proof of theorem 11.0.1. Again we split the proof in two parts:

claim 1: every 0-homogeneous and locally minimizing $u : \mathbb{R}^m \to \mathcal{A}_Q(\mathcal{N})$ is constant;

claim 2: claim 1 implies that every locally minimizing map $u \in W^{1,2}(\Omega, \mathcal{A}_Q(\mathcal{N}))$ is continuous.

Obviously claim 2 is equivalent to the theorem. Let us first show how claim 2 follows from claim 1:
Proof of claim 2: Let $u \in W^{1,2}(\Omega, \mathcal{A}_Q(\mathcal{N}))$ be locally energy minimizing, and suppose by contradiction that $\mathrm{sing}_H(u) \neq \emptyset$. Due to the characterization of the Hölder regular set by means of the tangent maps [Hir16b, Lemma 6.1], there is $y \in \mathrm{sing}_H(u)$ with a non-constant tangent map T_y^u at y. But every tangent map is 0-homogeneous and locally minimizing, and thus constant by claim 1. This is the required contradiction.
Proof of claim 1: Let $u \in W^{1,2}(\mathbb{R}^m, \mathcal{A}_Q(\mathcal{N}))$ be any 0-homogeneous locally minimizing map. As a consequence of the previous discussion, for every $k > 1$ and $p \in \mathcal{N}$ the function $q \in \mathcal{N} \mapsto f(q) := (d_p(q)^2)^k$ is C^2 regular and satisfying $\nabla^2 f \geqslant 0$ on $T_q\mathcal{N}$ since $t \mapsto t^k$ is convex. Hence, we can apply lemma 11.1.1 and deduce that for all $p \in \mathcal{N}$, $k > 1$

$$d_p^{2k} \circ u = \sum_{\ell=1}^{Q} d_p^{2k}(u_\ell) \tag{11.3}$$

is constant. To conclude we need the following small algebraic fact, whose proof we postpone and first show the end of the argument.

Lemma 11.2.1. *Let $\{a_\ell\}_{\ell=1}^Q, \{b_\ell\}_{\ell=1}^Q$ be two families of non-negative real numbers. Suppose that for some sequence $k_i \to \infty$ we have*

$$\sum_{\ell=1}^{Q} a_\ell^{k_i} = \sum_{\ell=1}^{Q} b_\ell^{k_i}.$$

Then, $\{a_\ell\}_{\ell=1}^Q = \{b_\ell\}_{\ell=1}^Q$.

In order to conclude the proof, fix any $x, y \in \mathbb{R}^m$ and let $u(x) = \sum_{\ell=1}^Q [\![p_\ell]\!], u(y) = \sum_{\ell=1}^Q [\![q_\ell]\!]$. For a fixed p_j we have by (11.3) that for all $k > 1$

$$\sum_{\ell=1}^{Q} \mathrm{dist}_{\mathcal{N}}(p_\ell, p_j)^{2k} = \sum_{\ell=1}^{Q} \mathrm{dist}_{\mathcal{N}}(q_\ell, p_j)^{2k}.$$

But so the lemma 11.2.1 implies that the number of zeros of the left- and right-hand side are the same. So we conclude that $\#\{\ell : p_\ell = p_j\} = \#\{\ell : q_\ell = p_j\}$. Since p_j was arbitrary we have $u(x) = u(y)$, that is u is constant.
It remains to give the proof of the lemma.

Proof of lemma 11.2.1. This lemma follows by induction on Q. For $Q = 1$ the claim is obvious.
Suppose the claim is proven for $Q' < Q$. We may assume that the families are ordered, i.e. $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_Q$ and $b_1 \geqslant b_2 \geqslant \cdots b_Q$. If $a_1 = 0$ the claim follows. Hence we may assume $a_1 > 0$. The hypothesis implies that for all k_i

$$\sum_{\ell=1}^{Q} \left(\frac{a_\ell}{a_1}\right)^{k_i} = \sum_{\ell=1}^{Q} \left(\frac{b_\ell}{a_1}\right)^{k_i}.$$

If we consider the limits for $k_i \to \infty$ we deduce that the LHS converges to $\#\{\ell\colon a_\ell = a_1\}$. If $b_1 > a_1$, the RHS converges to $+\infty$. If $b_1 < a_1$, on the other hand, the RHS converges to 0. Hence, $b_1 = a_1$. Furthermore the RHS converges therefore to $\#\{\ell\colon b_\ell = b_1 = a_1\}$ which must be the same number as for the family $\{a_\ell\}_{\ell=1}^{Q}$. Hence we conclude that the assumption can now be written as

$$\sum_{\ell\colon a_\ell = a_1} a_1^{k_i} + \sum_{\ell\colon a_\ell \neq a_1} a_\ell^{k_i} = \sum_{\ell\colon b_\ell = a_1} b_1^{k_i} + \sum_{\ell\colon b_\ell \neq a_1} b_\ell^{k_i}.$$

As we have just shown the first sum on the left agrees with the first sum on the right, hence we deduce equality for the second sums for all k_i. The lemma follows now by induction hypothesis. □

□

11.3 THE IMPROVED RESULT WHEN $Q = 1$

Although it is a known result we want to give a short proof of how the previous implies the following theorem. The important fact to remark is that for the single-valued case the topology of the target does not play a role.

Theorem 11.3.1. *Let $\mathcal{N}$ be a complete, connected manifold all of whose sectional curvatures are non-positive. Then, every locally energy minimizing map $u \in W^{1,2}(\Omega, \mathcal{N})$ is smooth.*

Proof. It is classical that every continuous harmonic map is smooth (cf. § 2.3.1), hence it is sufficient to prove the continuity of the harmonic map. We will show it by induction on the dimension m of the base space $\Omega \subset \mathbb{R}^m$. In fact, we will proceed similarly to the simply connected situation:

claim 1: every 0-homogeneous locally energy minimizer $u : \mathbb{R}^m \to \mathcal{N}$ is constant;

claim 2: every locally energy minimizing map $u \in W^{1,2}(\Omega, \mathcal{N})$ is continuous.

Proof of claim 1: Assume claim 1 is proven for $m' < m$. In a first step we want to show that the map $u|_{S^{m-1}}$ is continuous. For $m \leqslant 3$ this holds true since $\mathcal{H}^{m-2}(\mathrm{sing}(u)) = 0$. Now let $u : \mathbb{R}^m \to \mathcal{N}$ be 0-homogeneous and energy minimizing, but suppose by contradiction that when restricted to the sphere S^{m-1} u is not continuous, i.e. $\mathrm{sing}(u) \cap S^{m-1} \neq \emptyset$. Hence we can find $y \in \mathrm{sing}(u) \cap S^{m-1}$. Since u is singular at both 0 and y, there is a tangent map T to u at y with at least one line of symmetry, i.e. there is $z \in \mathbb{R}^m$ such that $T(x + \lambda z) = T(x)$ for all $\lambda \in \mathbb{R}$, for all x. But this implies that T is a locally energy minimizing 0-homogeneous map from $\mathbb{R}^{m-1}$ to $\mathcal{N}$. By induction hypothesis T must be constant. Hence $\mathrm{sing}(u) \cap S^{m-1} = \emptyset$.
We have thus concluded that $v := u|_{S^{m-1}} : S^{m-1} \to \mathcal{N}$ is continuous and so smooth. Let $P : \tilde{\mathcal{N}} \to \mathcal{N}$ be an isometric covering map e.g. we can take $P = \exp_p : T_p\mathcal{N} \to \mathcal{N}$ by the Cartan-Hadamard Theorem. Since S^{m-1} is simply connected we have that $u_*(\pi_1(S^{m-1})) \subset P_*(\pi_1(\mathbb{R}^n))$ and hence there exists a lift $\tilde{v} : S^{m-1} \to \tilde{\mathcal{N}}$ of v, that is with $P \circ \tilde{v} = v$, compare [Hato2, Proposition 1.33]. The 0-homogeneous extension $\tilde{u}(x) := \tilde{v}(\frac{x}{|x|})$ must be locally

energy minimizing since P is isometric (indeed, if $\tilde{w}$ is a local competitor for $\tilde{u}$ then $w := P \circ \tilde{w}$ is a local competitor for u, and $\int_\Omega |Dw|^2 = \int_\Omega |D\tilde{w}|^2$; hence, $\tilde{u}$ must be locally minimizing if u is). But as proven in the simply connected situation every 0-homogeneous locally energy minimizing map $\tilde{u} : \mathbb{R}^m \to \tilde{\mathcal{N}}$ is constant, compare claim 1 in Section 11.2 with $Q = 1$. This shows the claim.
Proof of claim 2: Assume $\operatorname{sing}(u) \neq \emptyset$. Hence we can find $y \in \operatorname{sing}(u)$ at which there is a non-trivial tangent map T. But the existence of T is ruled out by claim 1. □

11.4 Q-VALUED COUNTEREXAMPLE

In this section we want to present an example that the continuity fails for Q-valued functions if the target is not simply connected. Due to the results in Section 11.2 we already know that the reason must be of topological nature.

Proposition 11.4.1. *There is a 2-valued Dirichlet minimizing map u from $B^3 \subset \mathbb{R}^3$ into the flat torus $\mathbb{T}^2 = \mathbb{C}/\mathbb{Z}^2$ with the property that $u|_{S^2}$ is Lipschitz continuous,* $\operatorname{sing}_H(u) \Subset B^3$ *and* $\operatorname{sing}_H(u) \neq \emptyset$.

Proof. The construction of the example proceeds as follows:

1. we present an explicit example of a branched covering $\pi : \mathcal{V} \to S^2$, where $\mathcal{V}$ is a torus. $\mathcal{V}$ is constructed as a complex variety in $\hat{\mathbb{C}} \times \hat{\mathbb{C}}$;
2. using π we construct a 2-valued, Lipschitz continuous map ν from S^2 into the flat torus $\mathbb{T}^2 = \mathbb{C}/\mathbb{Z}^2$ with finite energy;
3. let u be a minimizer of the Dirichlet energy with respect to $g(x) := \nu(\frac{x}{|x|})$. We will show that u cannot be continuous.

Let us now present the details to the outlined steps:

step 1: Let $\hat{\mathbb{C}}$ be the Riemann sphere. We fix two non zero, unequal complex numbers a, b and define the meromorphic function $m(z) := z\frac{z-a}{z-b}$. Consider the complex variety

$$\mathcal{V} := \left\{ (w, z) \in \hat{\mathbb{C}} \times \hat{\mathbb{C}} \colon w^2 = z\frac{z-a}{z-b} \right\}.$$

Consider the projection $\pi : \hat{\mathbb{C}} \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ onto the second component. Restricted to $\mathcal{V}$, we obtain a ramified covering map

$$\pi : \mathcal{V} \to \hat{\mathbb{C}}.$$

The map π by definition is a two valued covering with ramification points in $P_1 = (0, 0)$, $P_2 = (0, a)$, $P_3 = (\infty, b)$ and $P_4 = (\infty, \infty)$. We claim that π takes the form $\pi(\zeta) = \zeta^2$ at each of the ramifications points P_i. Furthermore this implies that $\mathcal{V}$ is smoothly embedded, i.e. does not have any singular points.
Set $p_1 = 0 = p_4'$, $p_2 = a$, $p_3' = \frac{1}{b}$ ($p_3 = b = \frac{1}{p_3'}$, $p_4 = +\infty = \frac{1}{p_4'}$).
At P_1, P_2 we have $m(z) = (z - p_i)h_i(z - p_i)$ with h_i holomorphic in a neighborhood U_i

of 0 and $h_i(0) \neq 0$. We deduce that $\varphi_i(z) := (z-p_i)h_i(z-p_i) = m(z)$ is locally a holomorphic diffeomorphism between $p_i + U_i$ and a neighborhood $V_i \subset \mathbb{C}$ of 0. Now it is straightforward to check that

$$\Phi_i : \zeta \in V_i \mapsto (\zeta, \varphi_i^{-1}(\zeta^2)),$$

is a local parametrization of $\mathcal{V}$ around P_i, i.e. $\varphi_i \circ \pi \circ \Phi_i(\zeta) = \zeta^2$. Changing U_i we may assume that $V_i = \mathbb{D}_{r_i}$ for each $i = 1, 2$, where $\mathbb{D}_r$ is the disc centered at $0 \in \mathbb{C}$ with radius r. Furthermore since Φ_i is a smooth regular map P_i is not a singular point of $\mathcal{V}$.

To analyze the ramification points P_3, P_4 we use the inversion $I : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ with $I(z) = \frac{1}{z}$. Observe that $(w, z) \in \mathcal{V}$ if and only if $(w' = I(w), z' = I(z))$ is a solution of $(w')^2 = m'(z')$ with $m'(z) = I \circ m \circ I = \frac{b}{a}\, z' \frac{z' - \frac{1}{b}}{z' - \frac{1}{a}}$ or

$$I(\mathcal{V}) = \left\{ (w', z') \in \hat{\mathbb{C}} \times \hat{\mathbb{C}} \colon w'^2 = \frac{b}{a}\, z' \frac{z' - \frac{1}{b}}{z' - \frac{1}{a}} \right\}.$$

Now we can argue for P_3, P_4 as for P_1, P_2 interchanging p_1, p_2 with p_4' and p_3' (and denote with U_i', $i = 3, 4$ the related neighborhoods of 0). As a conclusion we can apply the Riemann-Hurwitz formula, and obtain

$$\chi(\mathcal{V}) = -4 + \sum_{i=1}^{4} (2 - 1) = 0.$$

Hence $\mathcal{V}$ is a torus.

step 2: In the following we equip $\mathcal{V}$ with the pullback metric $\mathbf{g} := \iota^* \delta$ of its immersion $\iota : \mathcal{V} \hookrightarrow \hat{\mathbb{C}} \times \hat{\mathbb{C}}$. Observe that the metric $\mathbf{g}$ is compatible with the conformal structure considered in step 1.

The construction of v will be done in two steps. First, since $\pi : \mathcal{V} \to \hat{\mathbb{C}}$ is a branched conformal covering of degree two there is a natural way to define 2-valued maps with finite energy. These maps are not Lipschitz continuous, in fact only $C^{0,\frac{1}{2}}$, but we are able to find a Lipschitz continuous map with similar properties nearby.

Let $f : \mathcal{V} \to \mathcal{N}$ be any smooth function from the Riemann surface $\mathcal{V}$ into a manifold $\mathcal{N}$. We define a two valued map $u = u_f \colon \hat{\mathbb{C}} \to \mathcal{A}_2(\mathcal{N})$ using the branched covering map $\pi : \mathcal{V} \to \hat{\mathbb{C}}$ as follows

$$u(z) := \sum_{P \in \pi^{-1}(z)} [\![f(P)]\!],$$

counting multiplicities i.e. $u(p_i) = 2\,[\![f(P_i)]\!]$ for $i = 1, \cdots, 4$.

We claim that $u \in W^{1,2}(\mathbb{S}^2, \mathcal{A}_2(\mathcal{N}))$ with

$$\int_{\mathbb{S}^2} |\nabla u|^2 = \int_{\mathcal{V}} |\nabla f|^2. \tag{11.4}$$

Let γ be a smooth path connecting p_1, p_2, p_3, p_4. We obtain a simply connected domain $\Omega \subset \mathbb{C}$ setting

$$\Omega := \hat{\mathbb{C}} \setminus \left(\bigcup_{i=1,2} (p_i + U_i) \cup \bigcup_{i=3,4} I(p_i' + U_i') \cup \gamma \right).$$

Hence there exist two holomorphic maps $\psi_i : \Omega \to \pi^{-1}(\Omega)$ with $\psi_1(\Omega) \cup \psi_2(\Omega) = \pi^{-1}(\Omega)$ such that

$$u(z) = [\![f \circ \psi_1]\!] + [\![f \circ \psi_2]\!] \quad \text{for every } z \in \Omega\,.$$

Since the Dirichlet energy is conformally invariant (cf. [DLS11, Lemma 3.12]), we have

$$\int_\Omega |\nabla u|^2 = \int_{\pi^{-1}(\Omega)} |\nabla f|^2.$$

Now we consider a ramification point, for instance P_1 and the related neighborhood $p_1 + U_1$. Using the previously introduced parameterization Φ_1 we have

$$u \circ \varphi_1^{-1}(\zeta) = \left[\!\!\left[f \circ \Phi_1(\zeta^{\frac{1}{2}}) \right]\!\!\right] + \left[\!\!\left[f \circ \Phi_1(-\zeta^{\frac{1}{2}}) \right]\!\!\right].$$

The maps $\zeta \in \mathbb{D}_{r_1^2} \mapsto \pm\zeta^{\frac{1}{2}}$ both together parametrize $\mathbb{D}_{r_1}$. Hence, as before, due to the conformal invariance of Dirichlet energy we obtain

$$\int_{\varphi_1^{-1}(\mathbb{D}_{r_1^2})} |\nabla u|^2 = \int_{\Phi_1(\mathbb{D}_{r_1})} |\nabla f|^2.$$

Summing up all the pieces and using that $\mathcal{H}^2(\gamma) = 0$ we obtain (11.4).
By step 1 $\mathcal{V}$ is a smoothly embedded torus in $\hat{\mathbb{C}} \times \hat{\mathbb{C}}$; hence, there exists a smooth diffeomorphism $\Phi : \mathcal{V} \to \mathbb{T}^2$. Apply the above construction with the specific choice $f = \Phi$ to obtain

$$\tilde{v}(z) := \sum_{P \in \pi^{-1}(z)} [\![\Phi(P)]\!] \in W^{1,2}(\hat{\mathbb{C}}, \mathcal{A}_2(\mathbb{T}^2)).$$

It remains to show that there is $v \in \mathrm{Lip}(\hat{\mathbb{C}}, \mathcal{A}_2(\mathbb{T}^2))$ nearby. This will be a consequence of the following approximation lemma:

Lemma 11.4.2. *Given $w \in W^{1,2}(\Omega, \mathcal{A}_Q(\mathcal{N})) \cap C^0(\Omega, \mathcal{A}_Q(\mathcal{N}))$, for every $\Omega' \Subset \Omega$ there exists $w_j \in W^{1,2}(\Omega, \mathcal{A}_Q(\mathcal{N})) \cap C^0(\Omega, \mathcal{A}_Q(\mathcal{N}))$ with*

$$w_j \in \mathrm{Lip}(\Omega', \mathcal{A}_Q(\mathcal{N})); \quad w_j = w \text{ in a neighborhood of } \partial\Omega$$
$$\left\|\mathcal{G}(w_j, w)\right\|_{L^\infty(\Omega')} \to 0; \quad \int_{\Omega'} |Dw_j|^2 \to \int_{\Omega'} |Dw|^2 \text{ as } j \to \infty.$$

Before coming to the proof of this lemma let us present how to conclude. Apply the lemma to the 0-homogeneous extension of $\tilde{v}$ in $\Omega := B_2^3(0) \setminus B_{\frac{1}{4}}^3(0)$ to obtain an approximating sequence $v_j \in W^{1,2}(B_2^3(0) \setminus B_{\frac{1}{4}}^3(0), \mathcal{A}_2(\mathbb{T}^2)) \cap \mathrm{Lip}(B_{\frac{3}{2}}^3(0) \setminus B_{\frac{1}{2}}^3(0), \mathcal{A}_2(\mathbb{T}^2))$. Choosing j sufficiently large we can guarantee that for every $p \in \mathbb{T}^2 \setminus \bigcup_{i=1}^4 B_{2^{-2017}}(\Phi(P_i))$ there is precisely one $z \in \hat{\mathbb{C}} \simeq \partial B_1^3(0)$ with $p \in \mathrm{spt}(v_j(z))$. Now fix such j sufficiently large and set $v := v_j\big|_{\hat{\mathbb{C}}}$. The 0-homogeneous extension of v i.e. $g(x) := v(\frac{x}{|x|})$ for $x \in B_1 \subset \mathbb{R}^3$ is an element of $W^{1,2}(B_1, \mathcal{A}_2(\mathbb{T}^2))$ and Lipschitz continuous outside of 0. Now we may apply the direct method to obtain a Dirichlet minimizing map $u : B_1 \to \mathcal{A}_2(\mathbb{T}^2)$ with $u|_{S^2} = g|_{S^2}$, compare [DLS11, Theorem 0.8].

Proof of Lemma 11.4.2. Since $\mathcal{N} \hookrightarrow \mathbb{R}^d$ smooth isometrically there exists a smooth nearest point projection $\Pi : \mathbf{U}_\delta(\mathcal{N}) \to \mathcal{N}$ for some $\delta > 0$. Let $\boldsymbol{\xi}_{BW} : \mathcal{A}_Q(\mathbb{R}^d) \to \mathbb{R}^M$ be the locally isometric "improved" Almgren/B. White embedding of $\mathcal{A}_Q(\mathbb{R}^d)$, cf. [DLS11, Section 2]. We will denote with $\boldsymbol{\rho}_{BW} : \mathbb{R}^M \to \mathcal{A}_Q(\mathbb{R}^d)$ the related Lipschitz retraction, satisfying $\boldsymbol{\rho}_{BW} \circ \boldsymbol{\xi}_{BW} = \mathrm{id}$ on $\mathcal{A}_Q(\mathbb{R}^d)$, [DLS11, Corollary 2.2].
Since w is assumed to be continuous, there exists $\tilde{w}_j$ with $\tilde{w}_j \to \boldsymbol{\xi}_{BW} \circ w$ in $L^\infty(\Omega, \mathbb{R}^M) \cap W^{1,2}(\Omega, \mathbb{R}^M)$, $\tilde{w}_j \in \mathrm{Lip}(\Omega', \mathbb{R}^M)$ for every $\Omega' \Subset \Omega$, and $\tilde{w}_j = \boldsymbol{\xi}_{BW} \circ w$ in a neighborhood of $\partial\Omega$. For instance, one may take $\tilde{w}_j = (1-\theta)\, \boldsymbol{\xi}_{BW} \circ w + \theta\, \eta_{\varepsilon_j} \star (\boldsymbol{\xi}_{BW} \circ w)$, for an appropriate cut-of-function θ and a sequence of mollifiers η_{ε_j}.
Since $\boldsymbol{\rho}_{BW}$ is a Lipschitz-retraction and $\boldsymbol{\xi}_{BW}$ is a local isometry we conclude that the sequence

$$\hat{w}_j := \boldsymbol{\rho}_{BW} \circ \tilde{w}_j : \Omega \to \mathcal{A}_Q(\mathbb{R}^d)$$

has the claimed properties up to the fact that $\hat{w}_j$ does not necessarily take values in $\mathcal{N}$. But for sufficient large j we have $\mathcal{G}(\hat{w}_j(x), w(x)) < \frac{1}{2}\delta$ for all $x \in \Omega$ hence

$$w_j(x) := \Pi \circ \hat{w}_j(x) = \sum_{\ell=1}^{Q} [\![\Pi((\hat{w}_j(x))_\ell)]\!]$$

is well-defined and has all the claimed properties. It is clearly Lipschitz continuous on Ω' since Π is smooth and Lipschitz. The sequence w_j converges uniformly to w since Π is the identity on $\mathcal{N}$ and finally

$$\int_\Omega |\nabla w|^2 \leqslant \liminf_{j\to\infty} \int_\Omega |\nabla w_j|^2 \leqslant \liminf_{j\to\infty} \int_\Omega \frac{|\nabla \hat{w}_j|^2}{(1 - \mathrm{dist}(\hat{w}_j(x), \mathcal{N})C)^2} = \int_\Omega |\nabla w|^2.$$

In the first inequality we used the lower-semicontinuity of the Dirichlet energy, in the second an estimate on the derivative of the nearest point projection Π, compare [Hir16b, Remark 2.1 (iv)]. □

step 3: That $\mathrm{sing}_H(u) \Subset B^3$ follows from the fact that $u|_{S^2}$ is Lipschitz continuous and a boundary regularity result for Q-valued locally energy minimizing maps, which can be obtained from the analogous result of [Hir16a] for "classical" $\mathbb{R}^d$-valued Dir-minimizers modulo slight modifications of the arguments: precisely, this is how to proceed in order to obtain the boundary regularity result [Hir16a, Theorem 0.1] in the manifold valued setting for $s = 1$. Only in the proof of Proposition 3.3, one replaces the application of Lemma B.2. to obtain the interpolation $\varphi(k')$ by the application of the Q-valued Luckhaus lemma, [Hir16b, Lemma 3.1] to obtain $\varphi(k')$. Due to the L^∞-bound in the Luckhaus lemma one can apply the nearest point projection $\Pi : \mathbf{U}_\delta(\mathcal{N}) \to \mathcal{N}$ and obtain an interpolation function $\Pi \circ \varphi(k')$ that satisfies the same bounds.

To show that $\mathrm{sing}_H(u) \neq \emptyset$ the idea is to use the "degree" of $u|_{S^2}$ to show that u cannot be continuous. We will use the notion of "degree" suggested by the theory of Cartesian currents. We will need the tools developed in Chapter 3 about push-forwards of integral currents by Q-valued proper Lipschitz continuous functions. In particular, let $\Omega \subset \mathbb{R}^m$ be open (non necessarily connected) with smooth boundary $\partial\Omega$, $\Sigma \subset \Omega$ any smooth k-dimensional surface, and $f\colon \Omega \to \mathcal{A}_Q(\mathcal{N})$ Lipschitz and proper. Then, the following holds:

- $T := f_\sharp \llbracket \Omega \rrbracket$ is an m-dimensional integer rectifiable current in $\mathcal{N}$, $S := f_\sharp \llbracket \Sigma \rrbracket$ is a k-dimensional integer rectifiable current in $\mathcal{N}$;
- it holds $\partial T = f_\sharp \llbracket \partial \Omega \rrbracket$.

In case Ω is 3-dimensional, Σ and $\mathcal{N}$ are 2-dimensional without boundary, the constancy theorem for integral currents, Theorem 2.1.6, implies that

(i) $T = f_\sharp \llbracket \Omega \rrbracket = 0$ since T is a 3-dimensional current supported in a 2-dimensional manifold;

(i) $S = f_\sharp \llbracket \Sigma \rrbracket = \theta_\Sigma \llbracket \mathcal{N} \rrbracket$ for some $\theta_\Sigma \in \mathbb{Z}$ since S is a 2-dimensional integer rectifiable current without boundary supported in a 2-dimensional manifold;

(iii) the following identity holds true

$$0 = \partial T = f_\sharp \llbracket \partial \Omega \rrbracket = \sum_{j=1}^{J} \theta_{\Sigma_j} \llbracket \mathcal{N} \rrbracket \tag{11.5}$$

where Σ_j are the different components of $\partial\Omega$ i.e. $\partial\Omega = \bigcup_{j=1}^{J} \Sigma_j$.

Now we can conclude *step 3*. Assume by contradiction that u is continuous. First extend u to B_2 setting $u(x) = u(\frac{x}{|x|})$ for $|x| > 1$. Apply the approximation lemma 11.4.2 to u with $\Omega = B_{\frac{3}{2}}$ and $\Omega' = B_1$ to obtain a sequence $u_j \in W^{1,2}(B_{\frac{3}{2}}, \mathcal{A}_2(\mathbb{T}^2))$ with $u_j|_{\partial B_{\frac{3}{2}}} = u|_{\partial B_{\frac{3}{2}}}$ for all j. Since u is Lipschitz continuous on $B_2 \setminus \overline{B_1}$ we have that $u_j \in \mathrm{Lip}(B_{\frac{3}{2}}, \mathcal{A}_2(\mathbb{T}^2))$. Modifying u_j slightly we can assume that u_j is constant in a small ball $B_r(0)$. This can be achieved for instance by composing u_j with a Lipschitz function of the form

$$\psi(x) := \begin{cases} x & \text{for } |x| \geqslant 2r \\ \frac{|x|-r}{r} x & \text{for } r \leqslant |x| < 2r \\ 0 & \text{for } |x| < r. \end{cases}$$

Now consider the set $\Omega = B_{\frac{3}{2}} \setminus B_{\frac{r}{2}}$ with smooth boundary components Σ_1, Σ_2 given by $\llbracket \Sigma_1 \rrbracket = \left\llbracket \partial B_{\frac{3}{2}} \right\rrbracket$ and $\llbracket \Sigma_2 \rrbracket = -\left\llbracket \partial B_{\frac{r}{2}} \right\rrbracket$ in the sense of currents. Since u_j is constant on B_r we have $(u_j)_\sharp \llbracket \Sigma_2 \rrbracket = 0$ by the very definition of push-forward. The identity (11.5) implies that

$$0 = (u_j)_\sharp \llbracket \Sigma_1 \rrbracket = u_\sharp \left\llbracket \partial B_{\frac{3}{2}} \right\rrbracket = u_\sharp \llbracket \partial B_1 \rrbracket .$$

We used that $u_j = u$ on $\partial B_{\frac{3}{2}}$ for all j and u is 0-homogeneous on $B_2 \setminus B_1$. But this is a contradiction since $u_\sharp \llbracket \partial B_1 \rrbracket \neq 0$ by the way u was constructed, compare the choice of the boundary datum in the approximation above. □

11.5 CONCLUDING REMARKS: AN EXAMPLE OF A "NON-CLASSICAL" TANGENT MAP

We want to conclude the chapter observing that tangent maps of Q-valued locally Dirichlet minimizing maps may have different structures than "classical" one-valued tangent

maps.
Following the classical scheme we make the following definition:

Definition 11.5.1. Let $u \in W^{1,2}(\Omega, \mathcal{A}_Q(\mathcal{N}))$ be energy minimizing. A point $x \in \operatorname{sing}_H(u)$ is called a regular-singular point if for every tangent map T at x there are classical one-valued tangent maps $T_\ell : \mathbb{R}^m \to \mathcal{N}$, i.e. 0-homogeneous locally energy minimizing maps, such that

$$T = \sum_{\ell=1}^{Q} [\![T_\ell]\!] .$$

It is worth noting that every continuity point of a locally energy minimizing map has the property above, by the identification of regular points by the existence of a constant tangent map, [Hir16b, Lemma 6.1 (iii)].

We will show the following

Proposition 11.5.2. *Let $u : B_1(0) \subset \mathbb{R}^3 \to \mathcal{A}_2(\mathbb{T}^2)$ be the Dirichlet minimizing map constructed in the previous section. Then, $\operatorname{sing}_H(u)$ does not contain any regular-singular point.*

Proof. It was shown in *step 3* of the previous section that $\operatorname{sing}_H(u) \neq \emptyset$ and $\operatorname{sing}_H(u) \Subset B^3$, hence at every point $x \in \operatorname{sing}_H(u)$ a tangent map exists. Let $T : \mathbb{R}^3 \to \mathcal{A}_2(\mathbb{T}^2)$ be an arbitrary tangent map at some some $y \in \operatorname{sing}_H(u)$. Assume by contradiction that there are "classical" tangent maps $T_1, T_2 : \mathbb{R}^3 \to \mathbb{T}^2$ such that

$$T = [\![T_1]\!] + [\![T_2]\!] .$$

Each T_i is 0-homogeneous and locally energy minimizing. Since $\mathbb{T}^2$ is flat each T_i satisfies the assumptions of claim 1 in the proof of Theorem 11.3.1, hence T_i must be constant. But this contradicts that T is a non-constant tangent map and concludes the proof of the proposition. □

Part IV

Results on real currents and currents with coefficients in groups

12 RELAXATION OF FUNCTIONALS ON REAL POLYHEDRAL CHAINS

In this chapter we present the proof of the representation formula for the lower semi-continuous envelope of a general class of functionals defined on real polyhedral chains obtained in [CDMS17]. After a brief introduction to real currents in Section 12.1, we present the main results of the chapter. The main theorem is Theorem 12.2.4, which is a simple corollary of Proposition 12.2.6 and Proposition 12.2.7. The former is proved in Section 12.3, the latter in Section 12.4. Finally, in the last section we present a simple necessary and sufficient condition on the function H so that the lower semi-continuous envelope F_H cannot be finite on a real flat chain T with finite mass if T is not rectifiable.

12.1 REAL CURRENTS

Let $0 \leqslant m \leqslant d$ be integers, and assume that $E \Subset \mathbb{R}^d$ is a (countably) m-rectifiable set oriented by $\vec{\tau}$ and carrying a multiplicity $\theta \in L^1(\mathcal{H}^m \llcorner E)$. We know from § 2.1.1 that there exists a current $R = [\![E, \vec{\tau}, \theta]\!] \in \mathcal{D}_m(\mathbb{R}^d)$ which is naturally associated to the triple $(E, \vec{\tau}, \theta)$, whose action on forms is given by

$$R(\omega) := \int_E \langle \omega(x), \vec{\tau}(x) \rangle \, \theta(x) \, d\mathcal{H}^m(x) \quad \forall \, \omega \in \mathcal{D}^m(\mathbb{R}^d). \tag{12.1}$$

We have called *integer rectifiable* all currents R which admit a representation formula as above under the additional hypothesis that the multiplicity function θ is integer-valued. We will simply call *rectifiable* any current $R = [\![E, \vec{\tau}, \theta]\!]$ as above, even when the last assumption on the target of the multiplicity function θ fails to hold. Hence, rectifiable currents are simply currents of finite mass which can be represented via integration over a rectifiable set with orientation $\vec{\tau}$ and (possibly) real-valued multiplicity θ according to formula (12.1). Observe that the set of rectifiable m-currents in $\mathbb{R}^d$, denoted $\mathbf{R}_m(\mathbb{R}^d)$, can be naturally endowed with the structure of real vector space, and that $\mathscr{R}_m(\mathbb{R}^d)$ is a $\mathbb{Z}$-submodule of $\mathbf{R}_m(\mathbb{R}^d)$. Also note that if $R = [\![E, \vec{\tau}, \theta]\!]$ is rectifiable then we can assume without loss of generality, modulo changing the orientation $\vec{\tau}$, that $\theta > 0$ $\mathcal{H}^m$-a.e. on E. We will apply this convention in this chapter.

An important subspace of $\mathbf{R}_m(\mathbb{R}^d)$ consists of the *real polyhedral m-chains* $\mathbf{P}_m(\mathbb{R}^d)$. Analogously to the integral case, but obviously allowing real multiplicities, we say that $P \in \mathbf{P}_m(\mathbb{R}^d)$ if P can be written as a linear combination

$$P = \sum_{i=1}^{N} \theta_i [\![\sigma_i]\!], \tag{12.2}$$

where $\theta_i \in (0,\infty)$, the σ_i's are non-overlapping, oriented, m-dimensional, convex polytopes (finite unions of m-simplexes) in $\mathbb{R}^d$ and $[\![\sigma_i]\!] = [\![\sigma_i, \vec{\tau}_i, 1]\!]$, $\vec{\tau}_i$ being a constant m-vector orienting σ_i. If $P \in \mathbf{P}_m(\mathbb{R}^d)$, then its *flat norm* is defined by

$$\mathbb{F}(P) := \inf\{\mathbb{M}(S) + \mathbb{M}(P - \partial S) : S \in \mathbf{P}_{m+1}(\mathbb{R}^d)\}.$$

The $\mathbb{F}$-completion of $\mathbf{P}_m(\mathbb{R}^d)$ in $\mathscr{E}_m(\mathbb{R}^d) := \left\{T \in \mathcal{D}_m(\mathbb{R}^d) : \operatorname{spt}(T) \text{ is compact } \right\}$ is the space of *real flat* m*-chains* in $\mathbb{R}^d$, denoted $\mathbf{F}_m(\mathbb{R}^d)$.

We remark that for the spaces of currents considered above the following chain of inclusions holds:

$$\mathbf{P}_m(\mathbb{R}^d) \subset \mathbf{R}_m(\mathbb{R}^d) \subset \mathbf{F}_m(\mathbb{R}^d) \cap \{T \in \mathscr{E}_m(\mathbb{R}^d) : \mathbb{M}(T) < \infty\}. \tag{12.3}$$

The flat norm $\mathbb{F}$ extends to a functional (still denoted $\mathbb{F}$) on $\mathscr{E}_m(\mathbb{R}^d)$, which coincides on $\mathbf{F}_m(\mathbb{R}^d)$ with the completion of the flat norm on $\mathbf{P}_m(\mathbb{R}^d)$, by setting:

$$\mathbb{F}(T) := \inf\{\mathbb{M}(S) + \mathbb{M}(T - \partial S) : S \in \mathscr{E}_{m+1}(\mathbb{R}^d)\}. \tag{12.4}$$

In the sequel, we will also use the following equivalent characterization of the flat norm of a flat chain (cf. [Fed69, 4.1.12] and [Moro9, 4.5]). If $T \in \mathbf{F}_m(\mathbb{R}^d)$ and $K \subset \mathbb{R}^d$ is a ball such that $\operatorname{spt}(T) \subset K$, then

$$\mathbb{F}(T) = \sup\{\langle T, \omega\rangle : \omega \in \mathcal{D}^m(\mathbb{R}^d) \text{ with } \|\omega\|_{C^0(K;\Lambda^m(\mathbb{R}^d))} \leqslant 1,\ \|d\omega\|_{C^0(K;\Lambda^{m+1}(\mathbb{R}^d))} \leqslant 1\}. \tag{12.5}$$

Many of the results that we have presented in Section 2.1 in the context of currents with integer multiplicities are still valid in the real multiplicities case, and in fact some of them have way simpler proofs. For instance, the Compactness Theorem 2.1.3 for real currents with finite mass and finite mass of the boundary (the so called *normal currents*) is evidently a simple exercise in Functional Analysis. The slicing theory can be extended to normal currents and real flat chains (see [Fed69, Sections 4.2.1 and 4.3]), and a deformation result analogous to Theorem 2.1.10 is available in this context (see [Fed69, Theorem 4.2.9]). There are as well some striking differences. For instance, we remark that a "real coefficients" version of Theorem 2.1.8 does *not* hold: hence, real flat chains with finite mass need not be rectifiable, and hence the last inclusion in (12.3) is strict. In the more general framework of currents with coefficients in a normed abelian group G the remarkable work [Whi99b] by White establishes a simple necessary and sufficient condition on the group G (which, by the discussion above, is not satisfied when $G = \mathbb{R}$) in order for every flat chain with finite mass to be rectifiable: the condition being that G does not contain any non-constant continuous path of finite length.

12.2 SETTING AND MAIN RESULTS

Assumption 12.2.1. We will consider a Borel function $H\colon \mathbb{R} \to [0,\infty)$ satisfying the following hypotheses:

(H1) $H(0) = 0$ and H is even, namely $H(-\theta) = H(\theta)$ for every $\theta \in \mathbb{R}$;

(H2) H is subadditive, namely $H(\theta_1+\theta_2)\leqslant H(\theta_1)+H(\theta_2)$ for every $\theta_1,\theta_2\in\mathbb{R}$;

(H3) H is lower semi-continuous, namely $H(\theta)\leqslant \liminf_{j\to\infty} H(\theta_j)$ whenever θ_j is a sequence of real numbers such that $|\theta-\theta_j|\searrow 0$ when $j\uparrow\infty$.

Remark 12.2.2. Observe that the hypotheses (H2) and (H3) imply that H is in fact countably subadditive, namely

$$H\left(\sum_{j=1}^{\infty}\theta_j\right)\leqslant\sum_{j=1}^{\infty}H(\theta_j),$$

for any sequence $\{\theta_j\}_{j=1}^{\infty}\subset\mathbb{R}$ such that $\sum_{j=1}^{\infty}\theta_j$ converges.

Remark 12.2.3. Let $\tilde{H}\colon[0,\infty)\to[0,\infty)$ be any Borel function satisfying:

($\tilde{H}$1) $\tilde{H}(0)=0$;

($\tilde{H}$2) $\tilde{H}$ is subadditive and monotone non-decreasing, i.e. $\tilde{H}(\theta_1)\leqslant\tilde{H}(\theta_2)$ for any $0\leqslant\theta_1\leqslant\theta_2$;

($\tilde{H}$3) $\tilde{H}$ is lower semi-continuous,

and let $H\colon\mathbb{R}\to[0,\infty)$ be the even extension of $\tilde{H}$, that is set $H(\theta):=\tilde{H}(|\theta|)$ for every $\theta\in\mathbb{R}$. Then, the function H satisfies Assumption 12.2.1.

Let H be as in Assumptions 12.2.1. We define a functional $\Phi_H\colon\mathbf{P}_m(\mathbb{R}^d)\to[0,\infty)$ as follows. Assume $P\in\mathbf{P}_m(\mathbb{R}^d)$ is as in (12.2). Then, we set

$$\Phi_H(P):=\sum_{i=1}^{N}H(\theta_i)\mathcal{H}^m(\sigma_i). \tag{12.6}$$

The functional Φ_H naturally extends to a functional $\mathbb{M}_H$, called the H-mass, defined on $\mathbf{R}_m(\mathbb{R}^d)$ by

$$\mathbb{M}_H(R):=\int_E H(\theta(x))\,d\mathcal{H}^m(x),\qquad\text{for every } R=[\![E,\vec{\tau},\theta]\!]\in\mathbf{R}_m(\mathbb{R}^d). \tag{12.7}$$

We also define the functional $F_H\colon\mathbf{F}_m(\mathbb{R}^d)\to[0,\infty]$ to be the lower semi-continuous envelope of Φ_H. More precisely, for every $T\in\mathbf{F}_m(\mathbb{R}^d)$ we set

$$F_H(T):=\inf\left\{\liminf_{j\to\infty}\Phi_H(P_j):\ P_j\in\mathbf{P}_m(\mathbb{R}^d)\text{ with }\mathbb{F}(T-P_j)\searrow 0\right\}. \tag{12.8}$$

The main result of this chapter is the following theorem.

Theorem 12.2.4. *Let H satisfy Assumption 12.2.1. Then, $F_H\equiv\mathbb{M}_H$ on $\mathbf{R}_m(\mathbb{R}^d)$.*

Remark 12.2.5. The hypotheses in Assumption 12.2.1 are also necessary. Indeed, without (H1) the functional Φ_H would not be well-defined; moreover, without (H2) and (H3) Theorem 12.2.4 would fail, in view of the following counterexamples.

- Assume that (H2) is not satisfied, and let θ_1, θ_2 be such that $H(\theta_1 + \theta_2) > H(\theta_1) + H(\theta_2)$. Let σ be a line segment of unit length in $\mathbb{R}^d$ and

$$P_\infty := (\theta_1 + \theta_2)[\![\sigma]\!]; \quad P_j := \theta_1[\![\sigma]\!] + \theta_2[\![\sigma + v_j]\!],$$

  where $0 \neq v_j \in \mathbb{R}^d$, v_j is not parallel to σ and $|v_j| \to 0$. Then, since $\mathbb{F}(P_j - P_\infty) \to 0$, we have

$$\mathbb{M}_H(P_\infty) = H(\theta_1 + \theta_2) > H(\theta_1) + H(\theta_2) = \Phi_H(P_j) \geqslant F_H(P_\infty).$$

- Assume that (H3) is not satisfied, and let θ, θ_j be such that $\theta_j \to \theta$ and $H(\theta) > \liminf_j H(\theta_j)$. Let σ be as above and

$$P_\infty := \theta[\![\sigma]\!]; \quad P_j := \theta_j[\![\sigma]\!].$$

  Then, since $\mathbb{F}(P_j - P_\infty) \to 0$, we have

$$\mathbb{M}_H(P_\infty) = H(\theta) > \liminf_j H(\theta_j) = \liminf_j \Phi_H(P_j) \geqslant F_H(P_\infty).$$

In order to prove Theorem 12.2.4, we adopt the following strategy. First, we show that the functional $\mathbb{M}_H$ is lower semi-continuous on rectifiable currents with respect to the flat convergence, as in the following proposition with $A = \mathbb{R}^d$.

Proposition 12.2.6. *Let H satisfy Assumption 12.2.1, and let $A \subset \mathbb{R}^d$ be open. Let $T_j, T \in \mathbf{R}_m(\mathbb{R}^d)$ be rectifiable m-currents such that $\mathbb{F}(T - T_j) \searrow 0$ as $j \to \infty$. Then*

$$\mathbb{M}_H(T \llcorner A) \leqslant \liminf_{j\to\infty} \mathbb{M}_H(T_j \llcorner A). \tag{12.9}$$

Next, we observe that, as an immediate consequence of Proposition 12.2.6 and of the properties of the lower semi-continuous envelope, it holds

$$\mathbb{M}_H(R) \leqslant F_H(R) \quad \text{for every } R \in \mathbf{R}_m(\mathbb{R}^d). \tag{12.10}$$

The opposite inequality, which completes the proof of Theorem 12.2.4, is obtained as a consequence of the following proposition, which provides the anticipated polyhedral approximation in flat norm of any rectifiable m-current R with a real polyhedral chain having H-mass and mass close to those of the given R.

Proposition 12.2.7. *Let H be any Borel function satisfying (H1) in Assumption 12.2.1, and let $R \in \mathbf{R}_m(\mathbb{R}^d)$ be rectifiable. For every $\varepsilon > 0$ there exists a polyhedral m-chain $P \in \mathbf{P}_m(\mathbb{R}^d)$ such that*

$$\mathbb{F}(R - P) \leqslant \varepsilon, \qquad \Phi_H(P) \leqslant \mathbb{M}_H(R) + \varepsilon \qquad \text{and} \qquad \mathbb{M}(P) \leqslant \mathbb{M}(R) + \varepsilon. \tag{12.11}$$

Theorem 12.2.4 characterizes the lower semi-continuous envelope F_H on rectifiable currents to be the (possibly infinite) H-mass $\mathbb{M}_H$. Without further assumptions on H, the lower semi-continuous envelope F_H can have finite values on flat chains which are non-rectifiable (for instance, the choice $H(\theta) := |\theta|$ induces the mass functional $F_H = \mathbb{M}$). If instead we add the natural hypothesis that H is monotone non-decreasing on $[0, \infty)$, then there is a simple necessary and sufficient condition which prevents this to happen in the case of flat chains with finite mass, thus allowing us to obtain an explicit representation for F_H on all flat chains with finite mass.

Proposition 12.2.8. *Let* H *be as in Assumption 12.2.1 and monotone non-decreasing on* $[0,\infty)$. *The condition*

$$\lim_{\theta \searrow 0^+} \frac{H(\theta)}{\theta} = +\infty. \tag{12.12}$$

holds if and only if

$$F_H(T) = \begin{cases} \mathbb{M}_H(T) & \text{for } T \in \mathbf{R}_m(\mathbb{R}^d), \\ +\infty & \text{for } T \in (\mathbf{F}_m(\mathbb{R}^d) \cap \{T \in \mathscr{E}_m(\mathbb{R}^d) : \mathbb{M}(T) < \infty\}) \setminus \mathbf{R}_m(\mathbb{R}^d). \end{cases} \tag{12.13}$$

12.3 PROOF OF THE LOWER SEMI-CONTINUITY

This section is devoted to the proof of Proposition 12.2.6. It is carried out by slicing the rectifiable currents T_j and T and reducing the proposition to the lower semi-continuity of 0-dimensional currents. Some of the techniques here adopted are borrowed from [DPHo3, Lemma 3.2.14].

Given $1 \leqslant m \leqslant d$, let $I(d,m)$ be the set of m-tuples $(i_1,\dots,i_m)$ with

$$1 \leqslant i_1 < \dots < i_m \leqslant d.$$

Let $\{e_1,\dots,e_d\}$ be an orthonormal basis of $\mathbb{R}^d$. For any $I = (i_1,\dots,i_m) \in I(d,m)$, let V_I be the m-plane spanned by $\{e_{i_1},\dots,e_{i_m}\}$. Given an m-plane V, we will denote $\mathbf{p}_V$ the orthogonal projection onto V. If $V = V_I$ for some I, we write $\mathbf{p}_I$ in place of $\mathbf{p}_{V_I}$. Given a current $T \in \mathbf{F}_m(\mathbb{R}^d)$, a Lipschitz function $f : \mathbb{R}^d \to \mathbb{R}^k$ for some $k \leqslant m$ and $y \in \mathbb{R}^k$, recall the notation $\langle T, f, y\rangle$ for the $(m-k)$-dimensional slice of T in $f^{-1}(y)$ (cf. § 2.1.2).

Let us denote by $Gr(d,m)$ the Grassmannian of m-dimensional planes in $\mathbb{R}^d$, and by $\gamma_{d,m}$ the Haar measure on $Gr(d,m)$ (see [KPo8, Section 2.1.4]).

In the following lemma, we prove a version of the integral-geometric equality for the H-mass, which is a consequence of [Fed69, 3.2.26; 2.10.15] (see also [DPHo3, (21)]). We observe that the hypotheses (H2) and (H3) on the function H are not needed here, and indeed Lemma 12.3.1 below is valid for any Borel function H for which the H-mass $\mathbb{M}_H$ is well defined.

Lemma 12.3.1. *Let* $E \subseteq \mathbb{R}^d$ *be* m*-rectifiable. Then there exists* $c = c(d,m)$ *such that the following integral-geometric equality holds:*

$$\mathcal{H}^m(E) = c \int_{Gr(d,m)} \int_{\mathbb{R}^m} \mathcal{H}^0(\mathbf{p}_V^{-1}(\{y\}) \cap E)\, d\mathcal{H}^m(y)\, d\gamma_{d,m}(V). \tag{12.14}$$

In particular, if $R \in \mathbf{R}_m(\mathbb{R}^d)$,

$$\mathbb{M}_H(R) = c \int_{Gr(d,m)\times\mathbb{R}^m} \mathbb{M}_H\big(\langle R, \mathbf{p}_V, y\rangle\big)\, d(\gamma_{d,m} \otimes \mathcal{H}^m)(V,y). \tag{12.15}$$

Proof. The equality (12.14) is proved in [Fed69, 3.2.26; 2.10.15]. For any Borel set $A \subset \mathbb{R}^d$, denoting $f = \mathbf{1}_A$, (12.14) implies that

$$\int_E f(x)\, d\mathcal{H}^m(x) = c \int_{Gr(d,m)} \int_{\mathbb{R}^m} \int_E f(x)\, \mathbf{1}_{\mathbf{p}_V^{-1}(\{y\})}(x)\, d\mathcal{H}^0(x)\, d\mathcal{H}^m(y)\, d\gamma_{d,m}(V). \tag{12.16}$$

Since the previous equality is linear in f, it holds also when f is piecewise constant. Since the measure $\mathcal{H}^m \llcorner E$ is σ-finite, the equality can be extended to any measurable function $f \in L^1(\mathcal{H}^m \llcorner E)$. The case $f \notin L^1(\mathcal{H}^m \llcorner E)$ follows from the Monotone Convergence Theorem via a simple truncation argument.

Taking $R = [\![E, \vec{\tau}, \theta]\!]$, and applying (12.16) with $f(x) = H(\theta(x))$, we deduce that

$$\mathbb{M}_H(R) = c \int_{Gr(d,m)} \int_{\mathbb{R}^m} \int_{E \cap p_V^{-1}(\{y\})} H(\theta(x))\, d\mathcal{H}^0(x)\, d\mathcal{H}^m(y)\, d\gamma_{d,m}(V).$$

We observe that the right-hand side coincides with the right-hand side in (12.15) since for $\mathcal{H}^m$-a.e. $y \in \mathbb{R}^m$ the 0-dimensional current $\langle R, \mathbf{p}_V, y\rangle$ is concentrated on the set $E \cap \mathbf{p}_V^{-1}(y)$ and its density at any $x \in E \cap \mathbf{p}_V^{-1}(y)$ is $\theta(x)$. □

We prove the lower semi-continuity in (12.9) by an explicit computation in the case $m = 0$. Then, by slicing, we get the proof for $m > 0$, too.

Proof of Proposition 12.2.6. Step 1: the case $m = 0$. Let $T_j := [\![E_j, \tau_j, \theta_j]\!], T := [\![E, \tau, \theta]\!] \in \mathbf{R}_0(\mathbb{R}^d)$ be such that $\mathbb{F}(T - T_j) \searrow 0$ as $j \to \infty$. Since $T \llcorner A$ is a signed, atomic measure, we write

$$T \llcorner A = \sum_{i \in \mathbb{N}} \tau(x_i)\theta(x_i)\delta_{x_i}$$

for distinct points $\{x_i\}_{i\in\mathbb{N}} \subseteq E \cap A$, orientations $\tau(x_i) \in \{-1, 1\}$, and for $\theta(x_i) > 0$. Fix $\varepsilon > 0$ and let $N = N(\varepsilon) \in \mathbb{N}$ be such that

$$\mathbb{M}_H(T \llcorner A) - \sum_{i=1}^{N} H(\theta(x_i)) \leqslant \varepsilon \qquad \text{if } \mathbb{M}_H(T \llcorner A) < \infty \tag{12.17}$$

and

$$\sum_{i=1}^{N} H(\theta(x_i)) \geqslant \frac{1}{\varepsilon} \qquad \text{otherwise.} \tag{12.18}$$

Since H is positive, even, and lower semi-continuous, for every $i \in \{1, \dots, N\}$ it is possible to determine $\eta_i = \eta_i(\varepsilon, \theta(x_i)) > 0$ such that

$$H(\theta) \geqslant (1 - \varepsilon)H(\theta(x_i)) \quad \text{ for every } |\theta - \tau(x_i)\theta(x_i)| < \eta_i. \tag{12.19}$$

Moreover, for every $i \in \{1, \dots, N\}$ there exists $0 < r_i < \min\{\operatorname{dist}(x_i, \partial A), 1\}$ such that the balls $B(x_i, r_i)$ are pairwise disjoint, and moreover such that for every $\rho \leqslant r_i$ it holds

$$\left| \tau(x_i)\theta(x_i) - \sum_{x \in E \cap B(x_i,\rho)} \tau(x)\theta(x) \right| \leqslant \frac{\eta_i}{2}. \tag{12.20}$$

Our next aim is to prove that in sufficiently small balls and for j large enough, the sum of the multiplicities of T_j (with sign) is close to the sum of the multiplicities of T. In order to do this, we would like to test the current $T - T_j$ with the indicator function of each ball. Since this test is not admissible, we have to consider a smooth and compactly supported extension of it outside the ball, provided we can prove that the flat convergence of T_j to T

localizes to the ball. From this, our claimed convergence of the signed multiplicities follows by the characterization of the flat norm in (12.5).

To make this formal, we define $\eta_0 := \min_{1\leqslant i\leqslant N} \eta_i$ and $r_0 := \min_{1\leqslant i\leqslant N} r_i$. Let j_0 be such that

$$\mathbb{F}(T - T_j) \leqslant \frac{\eta_0 r_0}{16} \quad \text{for every } j \geqslant j_0.$$

By the definition (12.4) of flat norm, there exist $R_j \in \mathscr{E}_0(\mathbb{R}^d)$, $S_j \in \mathscr{E}_1(\mathbb{R}^d)$ such that $T - T_j = R_j + \partial S_j$ with $\mathbb{M}(R_j) + \mathbb{M}(S_j) \leqslant \frac{\eta_0 r_0}{8}$ for every $j \geqslant j_0$. Observe that the mass and the mass of the boundary of both R_j and S_j are finite, and thus by [Fed69, 4.1.12] it holds $R_j \in \mathbf{F}_0(\mathbb{R}^d)$ and $S_j \in \mathbf{F}_1(\mathbb{R}^d)$. We want to deduce that for every $i \in \{1,\dots,N\}$ there exists $\rho_i \in \left(\frac{r_0}{2}, r_0\right)$ such that

$$\mathbb{F}((T - T_j) \llcorner B(x_i, \rho_i)) \leqslant \frac{\eta_0}{2}.$$

Indeed, for any fixed $i \in \{1,\dots,N\}$ one has that for a.e. $\rho \in \left(\frac{r_0}{2}, r_0\right)$

$$\begin{aligned}(T - T_j) \llcorner B(x_i, \rho) &= R_j \llcorner B(x_i, \rho) + (\partial S_j) \llcorner B(x_i, \rho) \\ &= R_j \llcorner B(x_i, \rho) - \langle S_j, d(x_i, \cdot), \rho\rangle + \partial \left(S_j \llcorner B(x_i, \rho)\right),\end{aligned} \tag{12.21}$$

where $d(x_i, z) := |x_i - z|$ and where the last identity holds by the definition of slicing for normal currents (cf. [Fed69, 4.2.1] and observe that it is analogous to the identity in Proposition 2.1.7 (ii)). On the other hand, by [Fed69, 4.2.1] we have

$$\int_{\frac{r_0}{2}}^{r_0} \mathbb{M}(\langle S_j, d(x_i, \cdot), \rho\rangle)\, d\rho \leqslant \mathbb{M}(S_j \llcorner (B(x_i, r_0) \setminus B(x_i, \tfrac{r_0}{2}))) \leqslant \frac{\eta_0 r_0}{8}.$$

Hence, there exists $\rho_i \in \left(\frac{r_0}{2}, r_0\right)$ such that

$$\mathbb{M}(\langle S_j, d(x_i, \cdot), \rho_i\rangle) \leqslant \frac{\eta_0}{4}. \tag{12.22}$$

We conclude from (12.21) that

$$\begin{aligned}\mathbb{F}((T - T_j) \llcorner B(x_i, \rho_i)) &\leqslant \mathbb{M}(R_j \llcorner B(x_i, \rho_i)) + \mathbb{M}(\langle S_j, d(x_i, \cdot), \rho_i\rangle) + \mathbb{M}(S_j \llcorner B(x_i, \rho_i)) \\ &\overset{(12.22)}{\leqslant} \frac{\eta_0 r_0}{4} + \frac{\eta_0}{4} \leqslant \frac{\eta_0}{2}.\end{aligned} \tag{12.23}$$

Using the characterization of the flat norm in (12.5), and testing the currents $(T - T_j) \llcorner B(x_i, \rho_i)$ with any smooth and compactly supported function $\phi_i \colon \mathbb{R}^d \to \mathbb{R}$ which is identically 1 on $B(x_i, \rho_i)$, we obtain

$$\left| \sum_{x \in E \cap B(x_i, \rho_i)} \tau(x)\theta(x) - \sum_{y \in E_j \cap B(x_i, \rho_i)} \tau_j(y)\theta_j(y) \right| \leqslant \frac{\eta_0}{2}. \tag{12.24}$$

Combining (12.24) with (12.20), we deduce by triangle inequality that

$$\left| \tau(x_i)\theta(x_i) - \sum_{y \in E_j \cap B(x_i, \rho_i)} \tau_j(y)\theta_j(y) \right| \leqslant \eta_i. \tag{12.25}$$

Finally, using (12.19) and the fact that H is countably subadditive (cf. Remark 12.2.2), we conclude that for every $j \geqslant j_0$

$$\begin{aligned} H(\theta(x_i)) &\leqslant \frac{1}{1-\varepsilon} H\left(\sum_{y \in E_j \cap B(x_i,\rho_i)} \tau_j(y)\theta_j(y) \right) \\ &\leqslant \frac{1}{1-\varepsilon} \sum_{y \in E_j \cap B(x_i,\rho_i)} H(\theta_j(y)) \\ &= \frac{1}{1-\varepsilon} \mathbb{M}_H(T_j \llcorner B(x_i,\rho_i)). \end{aligned}$$

Summing over i, since the balls $B(x_i,\rho_i)$ are pairwise disjoint, we get that

$$\sum_{i=1}^{N} H(\theta(x_i)) \leqslant \frac{1}{1-\varepsilon} \liminf_{j\to\infty} \sum_{i=1}^{N} \mathbb{M}_H(T_j \llcorner B(x_i,\rho_i)) \leqslant \frac{1}{1-\varepsilon} \liminf_{j\to\infty} \mathbb{M}_H(T_j \llcorner A).$$

By (12.17) (or (12.18) in the case that $\mathbb{M}_H(T \llcorner A) = \infty$) and since ε is arbitrary, we find (12.9).

Step 2 (Reduction to $m = 0$ through integral-geometric equality). We prove now Proposition 12.2.6 for $m > 0$. Up to subsequences, we can assume

$$\lim_{j\to\infty} \mathbb{M}_H(T_j \llcorner A) = \liminf_{j\to\infty} \mathbb{M}_H(T_j \llcorner A).$$

By [Fed69, 4.3.1], for every $V \in \mathrm{Gr}(d,m)$ it holds

$$\int_{\mathbb{R}^m} \mathbb{F}(\langle T_j - T, \mathbf{p}_V, y\rangle)\, dy \leqslant \mathbb{F}(T_j - T), \tag{12.26}$$

Integrating the inequality (12.26) in $V \in \mathrm{Gr}(d,m)$ and using that $\gamma_{d,m}$ is a probability measure on $\mathrm{Gr}(d,m)$ we get

$$\lim_{j\to\infty} \int_{\mathrm{Gr}(d,m)\times\mathbb{R}^m} \mathbb{F}(\langle T_j - T, \mathbf{p}_V, y\rangle) d(\gamma_{d,m} \otimes \mathcal{H}^m)(V,y) \leqslant \lim_{j\to\infty} \mathbb{F}(T_j - T) = 0.$$

Since the integrand $\mathbb{F}(\langle T_j - T, \mathbf{p}_V, y\rangle)$ is converging to 0 in L^1, up to subsequences, we get

$$\lim_{j\to\infty} \mathbb{F}(\langle T_j - T, \mathbf{p}_V, y\rangle) = 0 \quad \text{for } \gamma_{d,m} \otimes \mathcal{H}^m\text{-a.e. } (V,y) \in \mathrm{Gr}(d,m) \times \mathbb{R}^m.$$

We conclude from Step 1 that

$$\mathbb{M}_H(\langle T, \mathbf{p}_V, y\rangle \llcorner A) \leqslant \liminf_{j\to\infty} \mathbb{M}_H(\langle T_j, \mathbf{p}_V, y\rangle \llcorner A) \quad \text{for } \gamma_{d,m} \otimes \mathcal{H}^m\text{-a.e. } (V,y) \in \mathrm{Gr}(d,m) \times \mathbb{R}^m. \tag{12.27}$$

By [AKoo, (5.15)], for every $V \in \mathrm{Gr}(d,m)$ one has $\langle T, \mathbf{p}_V, y\rangle \llcorner A = \langle T \llcorner A, \mathbf{p}_V, y\rangle$ for $\mathcal{H}^m$-a.e. $y \in \mathbb{R}^m$.

In order to conclude, we apply twice the integral-geometric equality (12.15). Indeed, using (12.27) and Fatou's lemma, we get

$$\begin{aligned}\mathbb{M}_H(T \llcorner A) &= c \int_{Gr(d,m)\times\mathbb{R}^m} \mathbb{M}_H\big(\langle T \llcorner A, \mathbf{p}_V, y\rangle\big)\, d(\gamma_{d,m}\otimes\mathcal{H}^m)(V,y)\\ &\leqslant c \int_{Gr(d,m)\times\mathbb{R}^m} \liminf_{j\to\infty} \mathbb{M}_H\big(\langle T_j \llcorner A, \mathbf{p}_V, y\rangle\big)\, d(\gamma_{d,m}\otimes\mathcal{H}^m)(V,y)\\ &\leqslant c \liminf_{d\to\infty} \int_{Gr(d,m)\times\mathbb{R}^m} \mathbb{M}_H\big(\langle T_j \llcorner A, \mathbf{p}_V, y\rangle\big)\, d(\gamma_{d,m}\otimes\mathcal{H}^m)(V,y)\\ &= \liminf_{j\to\infty} \mathbb{M}_H(T_j \llcorner A).\end{aligned} \tag{12.28}$$

This concludes the proof of Step 2, so the proof of Proposition 12.2.6 is complete. □

12.4 PROOF OF THE POLYHEDRAL APPROXIMATION

In order to prove Proposition 12.2.7, we will consider a family of pairwise disjoint balls containing the entire mass of the current R, up to a small error. Then, we replace in every ball the current R with an m-dimensional disc with constant multiplicity. Afterwards, we further approximate each disc with polyhedral chains.

We begin with the following lemma, where we prove that, at many points x in the m-rectifiable set supporting the current R and at sufficiently small scales (depending on the point), R is close in the flat norm to the tangent m-plane at x weighted with the multiplicity of R at x.

In this section, given the m-current $R = [\![E, \vec{\tau}, \theta]\!]$, for a.e. $x \in E$ we denote with π_x the affine m-plane through x spanned by the (simple) m-vector $\vec{\tau}(x)$ and with $S_{x,\rho}$ the m-current

$$S_{x,\rho} := [\![B(x,\rho) \cap \pi_x, \vec{\tau}(x), \theta(x)]\!].$$

Lemma 12.4.1. *Let $\varepsilon > 0$, and let $R = [\![E, \vec{\tau}, \theta]\!]$ be a rectifiable m-current in $\mathbb{R}^d$. There exists a subset $E' \subset E$ such that the following holds:*

(i) $\mathbb{M}(R \llcorner (E \setminus E')) \leqslant \varepsilon$;

(ii) *for every $x \in E'$ there exists $r = r(x) > 0$ such that for any $0 < \rho \leqslant r$*

$$\mathbb{F}(R \llcorner (E' \cap B(x,\rho)) - S_{x,\rho}) \leqslant \varepsilon \mathbb{M}(R \llcorner B(x,\rho)). \tag{12.29}$$

Proof. Since E is countably m-rectifiable, there exist countably many linear m-dimensional planes Π_i and C^1 and globally Lipschitz maps $f_i \colon \Pi_i \to \Pi_i^\perp$ such that

$$E \subset E_0 \cup \bigcup_{i=1}^{\infty} \mathrm{Graph}(f_i),$$

with $\mathcal{H}^m(E_0) = 0$. We will denote $\Sigma_i := \mathrm{Graph}(f_i) \subset \mathbb{R}^d$. For every $x \in \bigcup_{i=1}^\infty \Sigma_i$, we let $i(x)$ be the first index such that $x \in \Sigma_i$. Then, for every $i \geqslant 1$, we define $R_i := [\![E \cap \Sigma_i, \vec{\tau}, \theta_i]\!]$, where

$$\theta_i(x) := \begin{cases} \theta(x) & \text{if } i = i(x)\\ 0 & \text{otherwise.}\end{cases} \tag{12.30}$$

Clearly, $R = \sum_{i=1}^{\infty} R_i$ and $\mathbb{M}(R) = \sum_{i=1}^{\infty} \mathbb{M}(R_i)$. Hence, there exists $N = N(\varepsilon)$ such that

$$\sum_{i \geqslant N+1} \mathbb{M}(R_i) \leqslant \varepsilon. \tag{12.31}$$

Now, recall that x is a Lebesgue point of the function θ_i with respect to the Radon measure $\mathcal{H}^m \llcorner \Sigma_i$ if

$$\lim_{r \to 0} \frac{1}{\mathcal{H}^m(\Sigma_i \cap B(x,r))} \int_{\Sigma_i \cap B(x,r)} |\theta_i(y) - \theta_i(x)| \, d\mathcal{H}^m(y) = 0.$$

We define the set $E' \subset E$ by

$$E' := \Big\{ x \in E \cap \bigcup_{i=1}^{N} \Sigma_i \quad \text{such that } x \text{ is a Lebesgue point of } \theta_i \text{ with respect to } \mathcal{H}^m \llcorner \Sigma_i \text{ for every } i \in \{1, \dots, N\} \Big\}, \tag{12.32}$$

and we observe that (i) follows from (12.31) and [AFPoo, Corollary 2.23].

Let us set

$$L := \max\{\mathrm{Lip}(f_i) : \ i = 1, \dots, N\}. \tag{12.33}$$

Fix $i \in \{1, \dots, N\}$. For every $x \in \Sigma_i$ there exists $r > 0$ such that whenever $j \in \{1, \dots, N\}$ is such that $\Sigma_j \cap B(x, \sqrt{d}r) \neq \emptyset$, then $x \in \Sigma_j$.

Now, fix any point $x \in E'$, and fix an index $j \in \{1, \dots, N\}$ such that $x \in \Sigma_j$. If $j = i(x)$, then $\theta_j(x) = \theta(x) > 0$. Since by the definition of E'

$$\lim_{r \to 0} \frac{\mathbb{M}(R_j \llcorner (\Sigma_j \cap B(x,r)))}{\mathcal{H}^m(\Sigma_j \cap B(x,r))} = \theta_j(x), \tag{12.34}$$

then there exists $r > 0$ such that for any $0 < \rho \leqslant \sqrt{d}r$

$$\frac{\mathbb{M}(R_j \llcorner (\Sigma_j \cap B(x,\rho)))}{\mathcal{H}^m(\Sigma_j \cap B(x,\rho))} \geqslant \frac{\theta_j(x)}{2}. \tag{12.35}$$

Again by [AFPoo, Corollary 2.23] applied with $\mu = \mathcal{H}^m \llcorner \Sigma_j$ and $f = \theta_j$, there exists a radius $r > 0$ (depending on x) such that

$$\begin{aligned} \int_{\Sigma_j \cap B(x,\rho)} |\theta_j(y) - \theta_j(x)| \, d\mathcal{H}^m(y) &\leqslant \varepsilon \frac{\theta_j(x)}{2} \mathcal{H}^m(\Sigma_j \cap B(x,\rho)) \\ &\leqslant \varepsilon \frac{\mathbb{M}(R_j \llcorner (\Sigma_j \cap B(x,\rho)))}{\mathcal{H}^m(\Sigma_j \cap B(x,\rho))} \mathcal{H}^m(\Sigma_j \cap B(x,\rho)) \\ &\leqslant \varepsilon \mathbb{M}(R_j \llcorner B(x,\rho)), \end{aligned} \tag{12.36}$$

for every $0 < \rho \leqslant \sqrt{d}r$.

If, instead, $j \neq i(x)$, then $\theta_j(x) = 0$ and therefore there exists a radius $r > 0$ (depending on x) such that for every $0 < \rho \leqslant \sqrt{d}r$

$$\begin{aligned}\int_{\Sigma_j \cap B(x,\rho)} \theta_j(y)\, d\mathcal{H}^m(y) &\leqslant \frac{\varepsilon \theta_{i(x)}(x)}{N(1+L)^m} \mathcal{H}^m(\Sigma_j \cap B(x,\rho)) \\ &\leqslant \frac{\varepsilon}{N} \theta_{i(x)}(x) \omega_m \rho^m \\ &\overset{(12.35)}{\leqslant} 2\frac{\varepsilon}{N} \mathbb{M}(R_{i(x)} \llcorner B(x,\rho)),\end{aligned} \tag{12.37}$$

where ω_m denotes the volume of the unit ball in $\mathbb{R}^m$.

Fix any point $x \in E'$ and let $i = i(x)$. By possibly reparametrizing $f_i|_{\Pi_i \cap B(x,r)}$ from the m-plane tangent to Σ_i at x, translating and tilting such a plane, we can assume that $x = 0$, $\Pi_i = \{x_{m+1} = \cdots = x_d = 0\}$ and $\nabla f_i(x) = 0$. By possibly choosing a smaller radius $r = r(x) > 0$, we may also assume that

$$|\nabla f_i| \leqslant \varepsilon \quad \text{in } \Pi_i \cap B(x,r). \tag{12.38}$$

With these conventions, the current $S_{x,\rho}$ in the statement reads $S_{x,\rho} = [\![B(0,\rho) \cap \Pi_i, \vec{\tau}(0), \theta_i(0)]\!]$. We let $F_i \colon \Pi_i \times \Pi_i^\perp \to \mathbb{R}^d$ be given by $F_i(z,w) := (z, f_i(z))$, and we set $\tilde{R}_i := (F_i)_\sharp S_{x,\rho} \in \mathbf{R}_m(\mathbb{R}^d)$.

By (12.38) and the homotopy formula (cf. formulae (2.5) and (2.6)) applied with $g = F_i$ and $f(z,w) := (z,0)$, we have, denoting $C(x,\rho) := (B(x,\rho) \cap \Pi_i) \times \Pi_i^\perp$,

$$\begin{aligned}\mathbb{F}(\tilde{R}_i - S_{x,\rho}) &\leqslant C\|g - f\|_{L^\infty(C(x,\rho))} (\mathbb{M}(S_{x,\rho}) + \mathbb{M}(\partial S_{x,\rho})) \\ &\leqslant C\varepsilon\rho\, (\mathbb{M}(S_{x,\rho}) + \mathbb{M}(\partial S_{x,\rho})) \\ &\leqslant C\varepsilon\theta(x)\omega_m\rho^m \\ &\leqslant C\varepsilon\theta(x)\mathcal{H}^m(\Sigma_j \cap B(x,\rho)) \\ &\overset{(12.35)}{\leqslant} C\varepsilon \mathbb{M}(R_i \llcorner B(x,\rho)).\end{aligned} \tag{12.39}$$

Now, observe that, if we denote by ξ_i the orientation of Σ_i induced by the orientation of $\Pi_i \times \Pi_i^\perp$ via F_i, the rectifiable current $\tilde{R}_i$ reads $\tilde{R}_i = [\![\Sigma_i \cap C(x,\rho), \xi_i, \theta_i(x)]\!]$ (cf. [Sim83b, 27.2] or Section 3.1). Therefore, we can compute

$$\begin{aligned}\mathbb{M}(R_i \llcorner B(x,\rho) - \tilde{R}_i) &\leqslant \mathbb{M}(R_i \llcorner B(x,\rho) - \tilde{R}_i \llcorner B(x,\rho)) + \mathbb{M}(\tilde{R}_i \llcorner (C(x,\rho) \setminus B(x,\rho))) \\ &\overset{(12.36)}{\leqslant} \varepsilon \mathbb{M}(R_i \llcorner B(x,\rho)) + \mathbb{M}(\tilde{R}_i \llcorner (C(x,\rho) \setminus B(x,\rho))) \\ &\overset{(12.38)}{\leqslant} \varepsilon \mathbb{M}(R_i \llcorner B(x,\rho)) + C\varepsilon\theta_i(x)\mathcal{H}^m(\Sigma_i \cap B(x,\rho)) \\ &\overset{(12.35)}{\leqslant} C\varepsilon \mathbb{M}(R_i \llcorner B(x,\rho)).\end{aligned} \tag{12.40}$$

Hence, we conclude:

$$\begin{aligned}\mathbb{F}(R \llcorner E' \cap B(x,\rho) - S_{x,\rho}) &\leqslant \mathbb{F}(R_{i(x)} \llcorner B(x,\rho) - S_{x,\rho}) + \sum_{\substack{j=1 \\ j \neq i(x)}}^N \mathbb{M}(R_j \llcorner B(x,\rho)) \\ &\overset{(12.37)}{\leqslant} \mathbb{F}(R_{i(x)} \llcorner B(x,\rho) - \tilde{R}_i) + \mathbb{F}(\tilde{R}_i - S_{x,\rho}) + 2\varepsilon \mathbb{M}(R_{i(x)} \llcorner B(x,\rho)) \\ &\overset{(12.39),(12.40)}{\leqslant} C\varepsilon \mathbb{M}(R \llcorner B(x,\rho)).\end{aligned} \tag{12.41}$$

This proves (12.29). □

A straightforward iteration argument yields the following corollary.

Corollary 12.4.2. *Let* $R = [\![E, \vec{\tau}, \theta]\!]$ *be a rectifiable* m*-current in* $\mathbb{R}^d$*. Then, for* $\mathcal{H}^m$*-a.e.* $x \in E$

$$\lim_{r \to 0} \frac{\mathbb{F}(R \llcorner B(x,r) - S_{x,\rho})}{\mathbb{M}(R \llcorner B(x,r))} = 0. \tag{12.42}$$

Proof. For every $i \in \mathbb{N}$ define the set E_i to be the set E' given by Lemma 12.4.1 applied to R with $\varepsilon = 2^{-i-1}$, and let $F_i \subset E_i$ be the set of Lebesgue points of $\mathbf{1}_{E_i}$ (inside E_i) with respect to $\theta \mathcal{H}^m \llcorner E$. By [AFPoo, Corollary 2.23], the set F_i equals the set E_i up to a set of $\mathcal{H}^m$-measure 0 and for every $x \in F_i$ and for ρ sufficiently small (possibly depending on x) it holds

$$\begin{aligned}\mathbb{M}(R \llcorner B(x,\rho) - R \llcorner (E_i \cap B(x,\rho))) &= \int_{(E \setminus E_i) \cap B(x,\rho)} \theta \, d\mathcal{H}^m \\ &\leqslant 2^{-i-1} \int_{E \cap B(x,\rho)} \theta \, d\mathcal{H}^m = 2^{-i-1} \mathbb{M}(R \llcorner B(x,\rho)).\end{aligned}$$

Hence by Lemma 12.4.1 for every $x \in F_i$ there exists $r_i(x) > 0$ such that for every $0 < \rho < r_i(x)$

$$\begin{aligned}\mathbb{F}(R \llcorner B(x,\rho) - S_{x,\rho}) &\leqslant \mathbb{M}(R \llcorner B(x,\rho) - R \llcorner (E_i \cap B(x,\rho))) + \mathbb{F}(R \llcorner (E_i \cap B(x,\rho)) - S_{x,\rho}) \\ &\leqslant 2^{-i} \mathbb{M}(R \llcorner B(x,\rho))\end{aligned}$$

and

$$\mathbb{M}(R \llcorner (E \setminus F_i)) \leqslant 2^{-i-1}.$$

Denoting $F := \bigcup_{i \in \mathbb{N}} \bigcap_{j \geqslant i} F_j$, and noticing that $E \setminus F = E \cap F^c = E \cap \bigcap_{i \in \mathbb{N}} \bigcup_{j \geqslant i} F_j^c$ is contained in $\bigcup_{j \geqslant i} F_j^c$ for every $i \in \mathbb{N}$, we have

$$\mathbb{M}(R \llcorner (E \setminus F)) \leqslant \lim_{i \to \infty} \sum_{j=i}^{\infty} \mathbb{M}(R \llcorner (E \setminus F_j)) \leqslant \lim_{i \to \infty} \sum_{j=i}^{\infty} \frac{1}{2^j} = 0$$

and this implies that $\mathcal{H}^m(E \setminus F) = 0$. Since every $x \in F$ belongs definitively to every F_j (namely, for every $x \in F$ there exists $i_0(x) \in \mathbb{N}$ such that $x \in F_i$ for every $i \geqslant i_0(x)$), we obtain (12.42). □

Proof of Proposition 12.2.7. Let R be represented by $R = [\![E, \vec{\tau}, \theta]\!]$ with $\theta \in L^1(\mathcal{H}^m \llcorner E; (0, \infty))$. We denote

$$\mu := \theta \mathcal{H}^m \llcorner E.$$

Moreover, if $\mathbb{M}_H(R) < +\infty$, we define the positive finite measure

$$\nu := H(\theta) \mathcal{H}^m \llcorner E.$$

Fix $\varepsilon > 0$. We make the following

Claim: There exists a finite family of mutually disjoint balls $\{B_i\}_{i=1}^N$ with $B_i := B(x_i, r_i)$, such that the following properties are satisfied:

(i)
$$r_i \leqslant \varepsilon \qquad \forall\, i = 1, \dots, N \qquad \text{and} \qquad \mu(\mathbb{R}^d \setminus (\cup_{i=1}^N B_i)) \leqslant \varepsilon;$$

(ii) if we denote $R_i := R \llcorner B_i$ and $S_i := S_{x_i, r_i}$, then
$$\mathbb{F}(R_i - S_i) \leqslant \varepsilon \mu(B_i);$$

(iii)
$$|\mu(B_i) - \theta(x_i)\omega_m r_i^m| \leqslant \varepsilon \mu(B_i), \qquad \forall\, i = 1, \dots, N;$$

(iv) if $\mathbb{M}_H(R) < +\infty$, then
$$H(\theta(x_i))\omega_m r_i^m \leqslant (1+\varepsilon)\nu(B_i), \qquad \forall\, i = 1, \dots, N.$$

Let us for the moment assume the validity of the claim and see how to conclude the proof of the proposition.

By point (iii) in the claim we deduce
$$\mathbb{M}(S_i) \leqslant (1+\varepsilon)\mathbb{M}(R_i). \tag{12.43}$$
and by point (iv) we get
$$\mathbb{M}_H(S_i) \leqslant (1+\varepsilon)\mathbb{M}_H(R_i). \tag{12.44}$$
On the other hand, we can find a polyhedral chain $P_i \in \mathbf{P}_m(\mathbb{R}^d)$ (supported on $\pi_i \cap B_i$, $\pi_i := \pi_{x_i}$), such that
$$\mathbb{F}(P_i - S_i) \leqslant \varepsilon \mu(B_i), \qquad \mathbb{M}_H(P_i) \leqslant \mathbb{M}_H(S_i) \qquad \text{and} \qquad \mathbb{M}(P_i) \leqslant \mathbb{M}(S_i). \tag{12.45}$$
Indeed, it is enough to approximate the m-dimensional current S_i with simplexes with constant multiplicity and supported in $B_i \cap \pi_i$.

To conclude, we denote $P := \sum_{i=1}^N P_i$ and we estimate
$$\begin{aligned}\mathbb{F}(R-P) &\leqslant \sum_{i=1}^N \mathbb{F}(R_i - P_i) + \mathbb{M}(R \llcorner (\mathbb{R}^d \setminus (\cup_{i=1}^N B_i))) \\ &\overset{(i)}{\leqslant} \varepsilon + \sum_{i=1}^N \mathbb{F}(R_i - S_i) + \sum_{i=1}^N \mathbb{F}(S_i - P_i) \overset{(ii),(12.45)}{\leqslant} \varepsilon + 2\sum_{i=1}^N \varepsilon \mu(B_i) \leqslant \varepsilon + 2\varepsilon \mathbb{M}(R).\end{aligned} \tag{12.46}$$

Moreover
$$\mathbb{M}_H(P) = \sum_{i=1}^N \mathbb{M}_H(P_i) \overset{(12.45)}{\leqslant} \sum_{i=1}^N \mathbb{M}_H(S_i) \overset{(12.44)}{\leqslant} (1+\varepsilon) \sum_{i=1}^N \mathbb{M}_H(R_i) \leqslant (1+\varepsilon)\mathbb{M}_H(R) \tag{12.47}$$
and
$$\mathbb{M}(P) = \sum_{i=1}^N \mathbb{M}(P_i) \overset{(12.45)}{\leqslant} \sum_{i=1}^N \mathbb{M}(S_i) \overset{(12.43)}{\leqslant} (1+\varepsilon) \sum_{i=1}^N \mathbb{M}(R_i) \leqslant (1+\varepsilon)\mathbb{M}(R). \tag{12.48}$$

Proof of the Claim: Consider the set F of points $x \in E$ such that the following properties hold:

1. x satisfies
$$\lim_{r\to 0} \frac{\mathbb{F}(R \llcorner B(x,r) - S_{x,r})}{\mathbb{M}(R \llcorner B(x,r))} = 0;$$

2. denoting $\eta_{x,r} : \mathbb{R}^d \to \mathbb{R}^d$ the map $y \mapsto \frac{y-x}{r}$, we have the following convergences of measures for $r \to 0$:
$$\mu_{x,r} := r^{-m}(\eta_{x,r})_\#(\mu \llcorner B(x,r)) \rightharpoonup \theta(x)\mathcal{H}^m \llcorner ((x + \mathrm{span}(\vec{\tau}(x))) \cap B(0,1)), \tag{12.49}$$
and
$$\nu_{x,r} := r^{-m}(\eta_{x,r})_\#(\nu \llcorner B(x,r)) \rightharpoonup H(\theta(x))\mathcal{H}^m \llcorner ((x + \mathrm{span}(\vec{\tau}(x))) \cap B(0,1)). \tag{12.50}$$

We observe that properties (1) and (2) hold for μ-a.e. point. Indeed the fact that (1) holds for μ-a.e. x follows from Corollary 12.4.2, while the fact that (2) holds for μ-a.e. x is a consequence of [DL08, Theorem 4.8]. Moreover, by (12.49) and by (12.50), for every $x \in F$ there exists a radius $r(x) < \varepsilon$ such that
$$|\mu_{x,r}(B(0,1)) - \theta(x)\omega_m| \leqslant \frac{\varepsilon}{2}\theta(x)\omega_m, \qquad \text{for a.e. } r < r(x).$$
This inequality implies that
$$|\mu(B(x,r)) - \theta(x)\omega_m r^m| \leqslant \frac{\varepsilon}{2}\theta(x)\omega_m r^m, \qquad \text{for a.e. } r < r(x), \tag{12.51}$$
so that in particular
$$\theta(x)\left(1 - \frac{\varepsilon}{2}\right)\omega_m r^m \leqslant \mu(B(x,r)), \qquad \text{for a.e. } r < r(x).$$
Plugging the last inequality in the right-hand side of (12.51), we get
$$|\mu(B(x,r)) - \theta(x)\omega_m r^m| \leqslant \frac{\varepsilon}{2-\varepsilon}\mu(B(x,r)) \leqslant \varepsilon\mu(B(x,r)), \qquad \text{for a.e. } r < r(x).$$
which gives condition (iii) of the Claim.

Analogously, we get that
$$|\nu(B(x,r)) - H(\theta(x))\omega_m r^m| \leqslant \varepsilon\nu(B(x,r)), \qquad \text{for a.e. } r < r(x).$$

The validity of the claim is then obtained via the Vitali-Besicovitch covering theorem ([AFP00, Theorem 2.19]).

□

12.5 PROOF OF PROPOSITION 12.2.8

We first observe that the condition (12.12) is necessary for the validity of (12.13). Indeed, consider a map H as in Assumption 12.2.1 for which (12.12) does not hold. It means that there exists a constant $C > 0$ and a sequence $\{\theta_i\}_{i\in\mathbb{N}}$ converging to 0 such that $H(\theta_i) \leqslant C\theta_i$

for every $i \in \mathbb{N}$. We consider now the sequence of polyhedral m-chains $\{P_i\}_{i\in\mathbb{N}}$ supported in the unit cube $[0,1]^d$ and defined as

$$P_i := \sum_{j=1}^{N_i} [\![\pi_i^j \cap [0,1]^d, \vec{\tau}, \theta_i]\!],$$

where for i fixed, π_i^j are m-planes parallel to $\{x_{m+1} = \ldots = x_d = 0\}$ whose last $(d-m)$ coordinates are "uniformly distributed" in $[0,1]^{d-m}$, $\vec{\tau}$ is a fixed orientation for all the m-planes π_i^j not depending on i or j and $N_i := \min\{N \in \mathbb{N} : N\theta_i \geqslant 1\}$. Since $\theta_i \to 0$, then $N_i \to \infty$. For i large enough, so that $\theta_i N_i \leqslant 2$, we can compute

$$\Phi_H(P_i) = \sum_{j=1}^{N_i} \Phi_H([\![\pi_i^j \cap [0,1]^d, \vec{\tau}, \theta_i]\!]) = N_i H(\theta_i) \leqslant C N_i \theta_i \leqslant 2C.$$

Nevertheless, since $\theta_i N_i \to 1$, then the sequence $\{P_i\}_{i\in\mathbb{N}}$ converges in flat norm to the m-current T, acting on m-forms as

$$\langle T, \omega\rangle = \int_{[0,1]^d} \langle \omega(x), \vec{\tau}\rangle \, d\mathcal{L}^d(x),$$

which belongs to $(\mathbf{F}_m(\mathbb{R}^d) \cap \{T \in \mathscr{E}_m(\mathbb{R}^d) : \mathbb{M}(T) < \infty\}) \setminus \mathbf{R}_m(\mathbb{R}^d)$. Clearly, $F_H(T) \leqslant 2C$.

We show now that, if H is also monotone non-decreasing on $[0,\infty)$, then condition (12.12) is also sufficient to the validity of (12.13). The proof is a consequence of the definition of F_H in (12.8) and the following Lemma (see also [CDM17, Lemma 4.5]):

Lemma 12.5.1. *Assume H is as in Assumption 12.2.1, is monotone non-decreasing on $[0,\infty)$, and satisfies (12.12). Let $\{R_j\}_{j\in\mathbb{N}} \subset \mathbf{R}_m(\mathbb{R}^d)$ and let us assume that*

$$\sup_{j\in\mathbb{N}} \mathbb{M}_H(R_j) \leqslant C < +\infty.$$

If $\lim_{j\to\infty} \mathbb{F}(R_j - T) = 0$ for some $T \in \mathbf{F}_m(\mathbb{R}^d)$ with finite mass, then T is in fact rectifiable.

Proof. Step 1. We prove the lemma for $m = 0$, recalling that a 0-dimensional rectifiable current $R = [\![E, \tau, \theta]\!]$, with $\tau(x) = \pm 1$ and $\theta > 0$, is an atomic signed measure (i.e. a measure supported on a countable set).

We observe that (12.12) implies that there exists $\delta_0 > 0$ such that $H(\theta) > 0$ for every $\theta \in (0, \delta_0)$. We define the monotone non-decreasing function $f : [0, \delta_0) \to [0, +\infty)$ given by

$$f(\theta) := \begin{cases} \sup_{t\in(0,\theta]} \dfrac{t}{H(t)} & \text{if } 0 < \theta < \delta_0, \\ 0 & \text{if } \theta = 0. \end{cases}$$

By assumption (12.12), f is continuous in 0 and $H(\theta)f(\theta) \geqslant \theta$. Fix any $\delta \in (0, \delta_0)$. For any $j \in \mathbb{N}$

$$\begin{aligned} \mathbb{M}(R_j \llcorner \{x : \theta_j(x) < \delta\}) &= \int_{E_j \cap \{\theta_j < \delta\}} \theta_j(x) \, d\mathcal{H}^0(x) \leqslant \int_{E_j\cap\{\theta_j<\delta\}} f(\theta_j(x)) H(\theta_j(x)) \, d\mathcal{H}^0(x) \\ &\leqslant f(\delta) \int_{E_j\cap\{\theta_j<\delta\}} H(\theta_j(x)) \, d\mathcal{H}^0(x) \leqslant f(\delta)\mathbb{M}_H(R_j) \leqslant C f(\delta). \end{aligned}$$

Therefore, up to subsequences the sequence $\{R_j \llcorner \{x : \theta_j(x) < \delta\}\}_{j\in\mathbb{N}}$ converges to a signed measure R_2 of mass less than or equal to $Cf(\delta)$. On the other hand, using the upper bound on $\mathbb{M}_H(R_j)$ and the monotonicity of H, we deduce that the measures $R_j \llcorner \{x : \theta_j(x) \geqslant \delta\}$ are supported on a uniformly (with respect to j) bounded number of points, and converge to a discrete measure R_1. Hence, for any $\varepsilon > 0$, the limit T can be written as the sum of a discrete measure R_1 and of an error R_2 with mass less than or equal to ε. Since ε is arbitrary, the statement follows.

Step 2. We prove the claim for $m > 0$.

We apply [Fed69, 4.3.1] to the sequence $\{R_j\}_{j\in\mathbb{N}}$ to deduce that for any $I \in I(d, m)$

$$\lim_{j\to\infty} \int_{\mathbb{R}^m} \mathbb{F}(\langle R_j - T, \mathbf{p}_I, y\rangle)\, dy \leqslant \lim_{j\to\infty} \mathbb{F}(R_j - T) = 0.$$

Since the sequence of non-negative functions $\{\mathbb{F}(\langle R_j - T, \mathbf{p}_I, \cdot\rangle)\}_{j\in\mathbb{N}}$ converges in $L^1(\mathbb{R}^m)$ to 0, up to a (not relabeled) subsequence, we get the pointwise convergence

$$\lim_{j\to\infty} \mathbb{F}(\langle R_j - T, \mathbf{p}_I, y\rangle) = 0 \qquad \text{for } \mathcal{H}^m\text{-a.e. } y \in \mathbb{R}^m, \text{ for every } I \in I(d, m).$$

We apply the Fatou lemma and [DPH03, Corollary 3.2.5(5)] to the sequence $\{R_j\}_{j\in\mathbb{N}}$ to deduce

$$\int_{\mathbb{R}^m} \liminf_{j\to\infty} \mathbb{M}_H(\langle R_j, \mathbf{p}_I, y\rangle)\, dy \leqslant \liminf_{j\to\infty} \int_{\mathbb{R}^m} \mathbb{M}_H(\langle R_j, \mathbf{p}_I, y\rangle)\, dy \leqslant \liminf_{j\to\infty} \mathbb{M}_H(R_j) \leqslant C. \tag{12.52}$$

Hence the integrand in the left-hand side is finite a.e., namely $\liminf_{j\to\infty} \mathbb{M}_H(\langle R_j, \mathbf{p}_I, y\rangle) < \infty$ for $\mathcal{H}^m$-a.e. $y \in \mathbb{R}^m$, for every $I \in I(d, m)$. Hence we are can apply Step 1 to a.e. slice $\langle R_j, \mathbf{p}_I, y\rangle$ to a y-dependent subsequence and deduce that

$$\langle T, \mathbf{p}_I, y\rangle \text{ is } 0\text{-rectifiable for } \mathcal{H}^m\text{-a.e. } y \in \mathbb{R}^m, \text{ for every } I \in I(d, m). \tag{12.53}$$

To conclude the proof we employ Theorem [Whi99b, Rectifiable slices theorem, pp. 166-167], which ensures that a finite mass flat chain T is rectifiable if and only if property (12.53) holds. □

13 THE STRUCTURE OF FLAT CHAINS MODULO p

In this chapter, we turn our attention to the structure of flat chains and integral currents modulo p. In Section 13.1 we provide an introduction to currents modulo p, and, most importantly, we identify two open questions related to their structure. After collecting the known partial answers from the literature, we present in Section 13.2 our contribution to the solution of the aforementioned questions.

13.1 FLAT CHAINS MODULO p

In this section, we recall the definitions of the sets of currents with coefficients in $\mathbb{Z}_p$, and collect some of the most relevant open questions regarding their structure.

13.1.1 Definitions and basic properties

Let $p \geqslant 2$ be a positive integer. Assume $0 \leqslant m \leqslant d$, and let $K \subset \mathbb{R}^d$ be a compact set. For any $T \in \mathscr{F}_{m,K}(\mathbb{R}^d)$, we define

$$\mathbb{F}^p_K(T) := \inf\{\mathbb{M}(R) + \mathbb{M}(S) : R \in \mathscr{R}_{m,K}(\mathbb{R}^d), S \in \mathscr{R}_{m+1,K}(\mathbb{R}^d) \text{ s.t. } T = R + \partial S + pQ \text{ for some } Q \in \mathscr{F}_{m,K}(\mathbb{R}^d)\}. \tag{13.1}$$

Observe that, since $\mathscr{I}_{m,K}(\mathbb{R}^d)$ is flat-dense in $\mathscr{F}_{m,K}(\mathbb{R}^d)$, the infimum is unchanged if we let Q run in $\mathscr{I}_{m,K}(\mathbb{R}^d)$. Also notice that the inequality $\mathbb{F}^p_K(T) \leqslant \mathbb{F}_K(T)$ holds for any $T \in \mathscr{F}_{m,K}(\mathbb{R}^d)$.

Now, we introduce the equivalence relation $\mathrm{mod}(p)$ in $\mathscr{F}_{m,K}(\mathbb{R}^d)$: given $T, \tilde{T} \in \mathscr{F}_{m,K}(\mathbb{R}^d)$, we say that $T = \tilde{T} \,\mathrm{mod}(p)$ in $\mathscr{F}_{m,K}(\mathbb{R}^d)$ if and only if $\mathbb{F}^p_K(T - \tilde{T}) = 0$. The corresponding quotient group will be denoted $\mathscr{F}^p_{m,K}(\mathbb{R}^d)$. As in the classical case, $\mathbb{F}^p_K$ induces a distance $d_{\mathbb{F}^p_K}$ which makes $\mathscr{F}^p_{m,K}(\mathbb{R}^d)$ a complete metric space.

It is evident that if $T - \tilde{T} = pQ$ for some $Q \in \mathscr{F}_{m,K}(\mathbb{R}^d)$, then $T = \tilde{T} \,\mathrm{mod}(p)$ in $\mathscr{F}_{m,K}(\mathbb{R}^d)$, but the converse implication is not known (see Question 13.1.5 below).

We say that two flat m-chains $T, \tilde{T} \in \mathscr{F}_m(\mathbb{R}^d)$ are equivalent $\mathrm{mod}(p)$ in $\mathscr{F}_m(\mathbb{R}^d)$, and we write $T = \tilde{T} \,\mathrm{mod}(p)$ in $\mathscr{F}_m(\mathbb{R}^d)$ if there exists a compact $K \subset \mathbb{R}^d$ such that $\mathbb{F}^p_K(T - \tilde{T}) = 0$. The elements of the corresponding quotient group $\mathscr{F}^p_m(\mathbb{R}^d)$ are called *flat m-chains modulo p* and they will be denoted by $[T]$.

Remark 13.1.1. (i) Note that if $T \in \mathscr{F}_m(\mathbb{R}^d)$ and $\mathrm{spt}(T) \subset K$, then it is false in general that $T \in \mathscr{F}_{m,K}(\mathbb{R}^d)$. The simplest counterexample being the 0-dimensional current obtained as the boundary of (the rectifiable 1-current associated to) a countable union of disjoint intervals S_i contained in $[0,1]$ and clustering only at the origin, when

$K = \{0\} \cup \bigcup_i \partial S_i$. Nevertheless, it is a consequence of the polyhedral approximation theorem 2.1.12 that if $\mathrm{spt}(T) \subset \mathrm{int} K$, then indeed $T \in \mathscr{F}_{m,K}(\mathbb{R}^d)$ (see also [Fed69, Theorem 4.2.22]).

(ii) One would expect that the following property holds. If $T = \tilde{T} \bmod(p)$ in $\mathscr{F}_m(\mathbb{R}^d)$, then $T = \tilde{T} \bmod(p)$ in $\mathscr{F}_{m,K}(\mathbb{R}^d)$, whenever K is a compact set which contains $\mathrm{spt}(T)$ and $\mathrm{spt}(\tilde{T})$, and $T, \tilde{T} \in \mathscr{F}_{m,K}(\mathbb{R}^d)$. Nevertheless, the validity of this property does not appear to be obvious for a general compact set K. On the other hand, if K is also convex, the validity of the property is immediate. Indeed, let K' be a compact set such that $T - \tilde{T} = R_j + \partial S_j + pQ_j$ with $R_j \in \mathscr{R}_{m,K'}(\mathbb{R}^d)$, $S_j \in \mathscr{R}_{m+1,K'}(\mathbb{R}^d)$, $Q_j \in \mathscr{F}_{m,K'}(\mathbb{R}^d)$ and $\mathbb{M}(R_j) + \mathbb{M}(S_j) \leqslant \frac{1}{j}$. Then, denoting by π the (1-Lipschitz) closest-point projection on K, and by $\pi_\sharp$ the push-forward operator through π, we have that

$$T - \tilde{T} = \pi_\sharp T - \pi_\sharp \tilde{T} = \pi_\sharp R_j + \partial \pi_\sharp S_j + p \pi_\sharp Q_j,$$

where $\pi_\sharp R_j \in \mathscr{R}_{m,K}(\mathbb{R}^d)$, $\pi_\sharp S_j \in \mathscr{R}_{m+1,K}(\mathbb{R}^d)$ and $\pi_\sharp Q_j \in \mathscr{F}_{m,K}(\mathbb{R}^d)$. Moreover $\mathbb{M}(\pi_\sharp R_j) + \mathbb{M}(\pi_\sharp S_j) \leqslant \mathcal{M}(R_j) + \mathbb{M}(S_j)$, hence $T = \tilde{T} \bmod(p)$ in $\mathscr{F}_{m,K}(\mathbb{R}^d)$.

(iii) Observe that, using the same argument as in (ii), we are able to conclude that if $T \in \mathscr{F}_m(\mathbb{R}^d)$ and $\mathrm{spt}(T) \subset K$ then $T \in \mathscr{F}_{m,K}(\mathbb{R}^d)$ when K is convex (or, more in general, whenever there exists a Lipschitz projection onto K).

13.1.2 Boundary, mass and support modulo p

It is immediate to see that if $T = \tilde{T} \bmod(p)$ in $\mathscr{F}_m(\mathbb{R}^d)$ (resp. in $\mathscr{F}_{m,K}(\mathbb{R}^d)$), then also $\partial T = \partial \tilde{T} \bmod(p)$ in $\mathscr{F}_{m-1}(\mathbb{R}^d)$ (resp. in $\mathscr{F}_{m-1,K}(\mathbb{R}^d)$), and therefore a boundary operator ∂ can be defined also in the quotient groups $\mathscr{F}^p_m(\mathbb{R}^d)$ (resp. in $\mathscr{F}^p_{m,K}(\mathbb{R}^d)$) in such a way that

$$\partial\,[T] = [\partial T] \qquad \text{for every } T \in \mathscr{F}_m(\mathbb{R}^d). \tag{13.2}$$

For $T \in \mathscr{F}_m(\mathbb{R}^d)$, we also define its *mass modulo* p, or simply p-*mass* $\mathbb{M}^p(T)$, as the least $t \in \mathbb{R} \cup \{+\infty\}$ such that for every $\varepsilon > 0$ there exists a compact $K \subset \mathbb{R}^d$ and a rectifiable current $R \in \mathscr{R}_{m,K}(\mathbb{R}^d)$ satisfying

$$\mathbb{F}^p_K(T - R) < \varepsilon \quad \text{and} \quad \mathbb{M}(R) \leqslant t + \varepsilon. \tag{13.3}$$

One has that $\mathbb{M}^p(T_1 + T_2) \leqslant \mathbb{M}^p(T_1) + \mathbb{M}^p(T_2)$ and $\mathbb{M}^p(T) = \mathbb{M}^p(\tilde{T})$ if $T = \tilde{T} \bmod(p)$ in $\mathscr{F}_m(\mathbb{R}^d)$. This allows to regard $\mathbb{M}^p$ as a functional on the quotient group $\mathscr{F}^p_m(\mathbb{R}^d)$. Such functional is lower semi-continuous with respect to the $\mathbb{F}^p_K$-convergence for every K.

Finally, we denote by $\mathrm{spt}^p([T])$ the *support modulo* p of $[T] \in \mathscr{F}^p_m(\mathbb{R}^d)$, given by

$$\mathrm{spt}^p([T]) := \bigcap \{\mathrm{spt}(\tilde{T}) : \tilde{T} \in \mathscr{F}_m(\mathbb{R}^d),\ \tilde{T} = T \bmod(p) \text{ in } \mathscr{F}_m(\mathbb{R}^d)\}. \tag{13.4}$$

13.1.3 Rectifiable and integral currents modulo p

We define now the group $\mathscr{R}^p_m(\mathbb{R}^d)$ of the *integer rectifiable currents modulo* p by setting

$$\mathscr{R}^p_{m,K}(\mathbb{R}^d) := \{[T] \in \mathscr{F}^p_{m,K}(\mathbb{R}^d) : T \in \mathscr{R}_{m,K}(\mathbb{R}^d)\}. \tag{13.5}$$

As usual, $\mathscr{R}^p_m(\mathbb{R}^d)$ is the union over K compact of $\mathscr{R}^p_{m,K}(\mathbb{R}^d)$. Clearly, not all the elements in a class $[T] \in \mathscr{R}^p_{m,K}(\mathbb{R}^d)$ are classical rectifiable currents, but whenever we write $[T] \in \mathscr{R}^p_{m,K}(\mathbb{R}^d)$ we will always implicitly intend that T is a rectifiable representative of its class.

A current $R = [\![E, \vec{\tau}, \theta]\!] \in \mathscr{R}_{m,K}(\mathbb{R}^d)$ is called *representative modulo* p if and only if

$$\|R\|(A) \leqslant \frac{p}{2}\mathcal{H}^m(E \cap A) \quad \text{for every Borel set } A \subset \mathbb{R}^d.$$

Evidently, this condition is equivalent to ask that

$$|\theta(x)| \leqslant \frac{p}{2} \quad \text{for } \|R\|\text{-a.e. } x.$$

Since obviously for any integer z there exists a (unique) integer $-\frac{p}{2} < \tilde{z} \leqslant \frac{p}{2}$ with $z \equiv \tilde{z} \,(\mathrm{mod}\, p)$, then for any $T \in \mathscr{R}_{m,K}(\mathbb{R}^d)$ there exists an integer rectifiable current $R \in \mathscr{R}_{m,K}(\mathbb{R}^d)$ such that $R = T \,\mathrm{mod}(p)$ in $\mathscr{F}_{m,K}(\mathbb{R}^d)$ and R is representative modulo p. We immediately conclude that any $T \in \mathscr{R}_{m,K}(\mathbb{R}^d)$ can be written as

$$T = R + pQ, \tag{13.6}$$

where $R, Q \in \mathscr{R}_{m,K}(\mathbb{R}^d)$ and R is representative modulo p. It is proved in [Fed69, 4.2.26, p. 430] that

$$\mathbb{M}^p(T) = \mathbb{M}(R), \qquad \mathrm{spt}^p([T]) = \mathrm{spt}(R), \tag{13.7}$$

if R is representative modulo p of the current T.

A modulo p version of Theorem 2.1.8 is contained in [Fed69, (4.2.16)$^\nu$, p. 431]:

Theorem 13.1.2 (Rectifiability of flat chains modulo p)**.**

$$\mathscr{R}^p_{m,K}(\mathbb{R}^d) = \{[T] \in \mathscr{F}^p_{m,K}(\mathbb{R}^d) : \mathbb{M}^p([T]) < \infty\}. \tag{13.8}$$

Hence, if $[T] \in \mathscr{F}^p_{m,K}(\mathbb{R}^d)$ has finite $\mathbb{M}^p$ mass, then there exists $R \in \mathscr{R}_{m,K}(\mathbb{R}^d)$ such that $R = T \,\mathrm{mod}(p)$ in $\mathscr{F}_{m,K}(\mathbb{R}^d)$, $\mathbb{M}(R) = \mathbb{M}^p([T])$, and $\mathrm{spt}(R) = \mathrm{spt}^p([T])$.

Next, we define the group $\mathscr{I}^p_m(\mathbb{R}^d)$ of the *integral currents modulo* p as the union of the groups

$$\mathscr{I}^p_{m,K}(\mathbb{R}^d) := \{[T] \in \mathscr{R}^p_{m,K}(\mathbb{R}^d) : \partial\,[T] \in \mathscr{R}^p_{m-1,K}(\mathbb{R}^d)\}.$$

The conclusions about integer rectifiable currents modulo p deriving from the above discussion allow us to say that if $[T] \in \mathscr{I}^p_{m,K}(\mathbb{R}^d)$ then $\mathbb{M}^p([T]) < \infty$, $\mathbb{M}^p(\partial\,[T]) < \infty$ and that there are currents $R \in \mathscr{R}_{m,K}(\mathbb{R}^d)$ and $S \in \mathscr{R}_{m-1,K}(\mathbb{R}^d)$ such that $R = T \,\mathrm{mod}(p)$ in $\mathscr{F}_{m,K}(\mathbb{R}^d)$ and $S = \partial T \,\mathrm{mod}(p)$ in $\mathscr{F}_{m-1,K}(\mathbb{R}^d)$. In particular, R and S may be chosen to be representative modulo p, so that $\mathbb{M}(R) = \mathbb{M}^p(T)$ and $\mathbb{M}(S) = \mathbb{M}^p(\partial T)$. It is not known whether it is possible to choose $I \in \mathscr{I}_{m,K}(\mathbb{R}^d)$ such that $T = I \,\mathrm{mod}(p)$ in $\mathscr{F}_{m,K}(\mathbb{R}^d)$ (see Question 13.1.7 below).

A modulo p version of the Boundary Rectifiability Theorem can be straightforwardly deduced from Theorem 13.1.2, as we have:

Theorem 13.1.3 (Boundary Rectifiability modulo p, cf. [Fed69, (4.2.16)$^{\nu}$]).

$$\mathscr{I}^{p}_{m,K}(\mathbb{R}^d) = \{[T] \in \mathscr{R}^{p}_{m,K}(\mathbb{R}^d) : \mathbb{M}^p(\partial\,[T]) < \infty\}. \tag{13.9}$$

We conclude with the following modulo p version of the Polyhedral approximation Theorem 2.1.11, which can be deduced from [Fed69, (4.2.20)$^{\nu}$]. Since the statement does not appear in [Fed69], for the reader's convenience we include here the proof.

Theorem 13.1.4 (Polyhedral approximation modulo p). *If* $[T] \in \mathscr{I}^{p}_{m}(\mathbb{R}^d)$, $\varepsilon > 0$, $K \subset \mathbb{R}^d$ *is a compact set such that* $\mathrm{spt}^p([T]) \subset \mathrm{int}K$, *then there exists* $P \in \mathscr{P}_m(\mathbb{R}^d)$, *with* $\mathrm{spt}(P) \subset K$, *such that*

$$\mathbb{F}^{p}_{K}(T - P) < \varepsilon, \quad \mathbb{M}^p(P) \leqslant \mathbb{M}^p(T) + \varepsilon, \quad \mathbb{M}^p(\partial P) \leqslant \mathbb{M}^p(\partial T) + \varepsilon. \tag{13.10}$$

Proof. Let $T \in \mathscr{R}_m(\mathbb{R}^d)$ be a (rectifiable) representative modulo p of $[T]$. In particular, by formula (13.7), T satisfies $\mathrm{spt}(T) = \mathrm{spt}^p([T]) \subset \mathrm{int}K$ and $\mathbb{M}(T) = \mathbb{M}^p(T)$. Fix $\varepsilon > 0$, and let $0 < \delta \leqslant \varepsilon$ be such that $\{x \in \mathbb{R}^d : \mathrm{dist}(x, \mathrm{spt}(T)) < \delta\} \subset K$. By [Fed69, Theorem (4.2.20)$^{\nu}$], there exist $P \in \mathscr{P}_m(\mathbb{R}^d)$ with $\mathrm{spt}(P) \subset K$ and a diffeomorphism $f \in C^1(\mathbb{R}^d, \mathbb{R}^d)$ such that:

(i) $\mathbb{M}^p(P - f_\sharp T) + \mathbb{M}^p(\partial P - f_\sharp \partial T) \leqslant \delta$;

(ii) $\mathrm{Lip}(f) \leqslant 1 + \delta$, and $\mathrm{Lip}(f^{-1}) \leqslant 1 + \delta$;

(iii) $|f(x) - x| \leqslant \delta$ for $x \in \mathbb{R}^d$, and $f(x) = x$ if $\mathrm{dist}(x, \mathrm{spt}(T)) \geqslant \delta$.

From (i) it readily follows that

$$\mathbb{M}^p(P) \leqslant \delta + \mathbb{M}^p(f_\sharp T) \leqslant \delta + (1+\delta)^m \mathbb{M}^p(T), \tag{13.11}$$

and analogously

$$\mathbb{M}^p(\partial P) \leqslant \delta + \mathbb{M}^p(f_\sharp \partial T) \leqslant \delta + (1+\delta)^{m-1} \mathbb{M}^p(\partial T). \tag{13.12}$$

In order to prove the estimate on the $\mathbb{F}^p_K$ distance, let h be the affine homotopy from the identity map to f, i.e. $h(t,x) := (1-t)x + tf(x)$, and observe that the homotopy formula (2.5) yields

$$P - T = P - f_\sharp T + \partial\left(h_\sharp(\llbracket(0,1)\rrbracket \times T)\right) + h_\sharp(\llbracket(0,1)\rrbracket \times \partial T). \tag{13.13}$$

Now, since $\mathbb{M}^p(\partial T) < \infty$, there exists a rectifiable current $Z \in \mathscr{R}_{m-1,K}(\mathbb{R}^d)$ such that

$$\mathbb{F}^p_K(\partial T - Z) \leqslant \delta \quad \text{and} \quad \mathbb{M}(Z) \leqslant \mathbb{M}^p(\partial T) + \delta. \tag{13.14}$$

In particular, this implies the existence of $R \in \mathscr{R}_{m-1,K}(\mathbb{R}^d)$, $S \in \mathscr{R}_{m,K}(\mathbb{R}^d)$ and $Q \in \mathscr{I}_{m-1,K}(\mathbb{R}^d)$ with $\mathbb{M}(R) + \mathbb{M}(S) \leqslant 2\delta$ such that $\partial T - Z = R + \partial S + pQ$. If combined with (13.13), this gives

$$P - T = P - f_\sharp T + \partial h_\sharp(\llbracket(0,1)\rrbracket \times T) + h_\sharp(\llbracket(0,1)\rrbracket \times Z) + h_\sharp(\llbracket(0,1)\rrbracket \times (R + \partial S + pQ)). \tag{13.15}$$

Since, again by the homotopy formula,

$$h_\sharp(\llbracket(0,1)\rrbracket \times \partial S) = f_\sharp S - S - \partial h_\sharp(\llbracket(0,1)\rrbracket \times S),$$

we can finally re-write equation (13.15) as follows:

$$\begin{aligned} P - T = P - f_\sharp T \\ + h_\sharp(\llbracket (0,1) \rrbracket \times (Z + R)) + f_\sharp S - S \\ + \partial h_\sharp(\llbracket (0,1) \rrbracket \times (T - S)) \\ + p h_\sharp(\llbracket (0,1) \rrbracket \times Q). \end{aligned} \tag{13.16}$$

Therefore, we can finally estimate

$$\begin{aligned} \mathbb{F}^p_K(P - T) &\leqslant \mathbb{F}^p_K(P - f_\sharp T) \\ &\quad + \mathbb{M}(h_\sharp(\llbracket (0,1) \rrbracket \times (Z + R))) + \mathbb{M}(f_\sharp S) + \mathbb{M}(S) \\ &\quad + \mathbb{M}(h_\sharp(\llbracket (0,1) \rrbracket \times (T - S))) \\ &\leqslant 3\delta + 2\delta(1+\delta)^m + \delta(2+\delta)(\mathbb{M}^p(T) + \mathbb{M}^p(\partial T) + 3\delta), \end{aligned} \tag{13.17}$$

where we have used (2.6) to estimate the first and last addenda in the second line.

The conclusion, formula (13.10), clearly follows from (13.11), (13.12) and (13.17) for a suitable choice of $\delta = \delta(\varepsilon, m, \mathbb{M}^p(T), \mathbb{M}^p(\partial T))$. □

13.1.4 Questions on the structure of flat chains and integral currents modulo p

As already anticipated, two very natural questions arise about the structure of flat chains and integral currents modulo p (see [Fed69, 4.2.26]).

We fix a compact subset $K \subset \mathbb{R}^d$.

Question 13.1.5. Given $T, \tilde{T} \in \mathscr{F}_{m,K}(\mathbb{R}^d)$, is it true that $T = \tilde{T} \bmod(p)$ in $\mathscr{F}_{m,K}(\mathbb{R}^d)$ if and only if $T - \tilde{T} = pQ$ for some $Q \in \mathscr{F}_{m,K}(\mathbb{R}^d)$? In other words, using the density of $\mathscr{I}_{m,K}(\mathbb{R}^d)$ in $\mathscr{F}_{m,K}(\mathbb{R}^d)$, the problem is to prove or disprove the following statement. Given three sequences $\{R_j\} \subset \mathscr{R}_{m,K}(\mathbb{R}^d)$, $\{S_j\} \subset \mathscr{R}_{m+1,K}(\mathbb{R}^d)$, $\{Q_j\} \subset \mathscr{I}_{m,K}(\mathbb{R}^d)$ such that

$$T - \tilde{T} = R_j + \partial S_j + p Q_j \quad \forall j, \tag{13.18}$$

and

$$\lim_{j\to\infty} \big(\mathbb{M}(R_j) + \mathbb{M}(S_j)\big) = 0, \tag{13.19}$$

then $T - \tilde{T} = pQ$ for some $Q \in \mathscr{F}_{m,K}(\mathbb{R}^d)$.

Remark 13.1.6. As we shall soon see, the answer to the above question is affirmative if the class $\mathscr{F}_{m,K}(\mathbb{R}^d)$ is replaced by the class $\mathscr{R}_{m,K}(\mathbb{R}^d)$: in other words, given integer rectifiable currents $T, \tilde{T}$ one has that $T = \tilde{T} \bmod(p)$ in $\mathscr{F}_{m,K}(\mathbb{R}^d)$ if and only if $T - \tilde{T} = pQ$ for some $Q \in \mathscr{R}_{m,K}(\mathbb{R}^d)$. As a corollary, Question 13.1.5 admits affirmative answer for $m = d$, since $\mathscr{F}_{d,K}(\mathbb{R}^d) = \mathscr{R}_{d,K}(\mathbb{R}^d)$. For $0 \leqslant m \leqslant d-1$, the question is widely open.

Question 13.1.7. Given $[T] \in \mathscr{I}^p_{m,K}(\mathbb{R}^d)$, does there exist an *integral* current $I \in \mathscr{I}_{m,K}(\mathbb{R}^d)$ such that $I = T \bmod(p)$ in $\mathscr{F}_{m,K}(\mathbb{R}^d)$? In other words: is it true that

$$\mathscr{I}^p_{m,K}(\mathbb{R}^d) = \{[T] \,:\, T \in \mathscr{I}_{m,K}(\mathbb{R}^d)\}?$$

Remark 13.1.8. The answer is trivial for $m = 0$, since integral and integer rectifiable 0-dimensional currents are the same class. In [Fed69, 4.2.26, p. 426], Federer does not really present this issue as a "question", but he rather claims that the answer is negative, in general dimension and codimension. Nevertheless, the counterexample he suggests (an infinite sum of disjoint $\mathbf{RP}^2$ in $\mathbb{R}^6$ with the property that the sum of the areas is finite but the sum of the lengths of the bounding projective lines is infinite) is not fully satisfactory (cf. [ope86, Problem 3.3]). Indeed, it allows one to negatively answer the question only for very special choices of the set K (in particular, the question remains open when K is a convex set).

13.1.5 Some partial answers from the literature

An immediate consequence of (13.7) is the following: if $T, \tilde{T} \in \mathscr{R}_{m,K}(\mathbb{R}^d)$ are such that $T = \tilde{T} \bmod(p)$ in $\mathscr{F}_{m,K}(\mathbb{R}^d)$, then evidently $\mathbb{M}^p(T - \tilde{T}) = \mathbb{M}^p(0) = 0$, and hence the representative modulo p of $T - \tilde{T}$ is $R = 0$ because of (13.7). Therefore, equation (13.6) yields $T - \tilde{T} = pQ$ for some integer rectifiable current $Q \in \mathscr{R}_{m,K}(\mathbb{R}^d)$. In conclusion, we have the following

Proposition 13.1.9. *The answer to Question 13.1.5 is affirmative in the class of integer rectifiable currents. Therefore:*

$$\mathscr{R}_{m,K}(\mathbb{R}^d) \cap \{T \in \mathscr{F}_{m,K}(\mathbb{R}^d) : T = 0 \bmod(p) \text{ in } \mathscr{F}_{m,K}(\mathbb{R}^d)\} = \{pR : R \in \mathscr{R}_{m,K}(\mathbb{R}^d)\}. \tag{13.20}$$

In particular, the following corollary holds true:

Corollary 13.1.10. *Let $T, \tilde{T} \in \mathscr{R}_m(\mathbb{R}^d)$. Then, $T = \tilde{T} \bmod(p)$ in $\mathscr{F}_m(\mathbb{R}^d)$ if and only if $T = \tilde{T} \bmod(p)$ in $\mathscr{F}_{m,K}(\mathbb{R}^d)$ for every K compact with $\operatorname{spt}(T) \cup \operatorname{spt}(\tilde{T}) \subset K$.*

Proof. The "if" implication is trivial. For the converse, assume $T = \tilde{T} \bmod(p)$ in $\mathscr{F}_m(\mathbb{R}^d)$ and fix any compact set K such that $\operatorname{spt}(T) \cup \operatorname{spt}(\tilde{T}) \subset K$. By definition, there exists a compact set K' such that $\mathbb{F}^p_{K'}(T - \tilde{T}) = 0$, which, by the above proposition, implies

$$T - \tilde{T} = pQ$$

for some $Q \in \mathscr{R}_{m,K'}(\mathbb{R}^d)$. Note that since $T - \tilde{T}$ is supported in K, so is Q, and thus $\mathbb{F}_K(T - \tilde{T}) = 0$, i.e. $T = \tilde{T} \bmod(p)$ in $\mathscr{F}_{m,K}(\mathbb{R}^d)$. □

From now on, by virtue of the previous corollary, for rectifiable currents T and $\tilde{T}$ in $\mathscr{F}_m(\mathbb{R}^d)$ which are equivalent modulo p we will just write $T = \tilde{T} \bmod(p)$ without specifying in which class the equivalence relation is meant.

In codimension 0, B. White [Whi79] gave an affirmative answer to Question 13.1.7.

Theorem 13.1.11 (cf. [Whi79, Proposition 2.3])**.** *Let $T \in \mathscr{R}_{d,K}(\mathbb{R}^d)$. Then, $[T] \in \mathscr{I}^p_{d,K}(\mathbb{R}^d)$ if and only if the select representative modulo p of T is an integral current.*

The *select* representative modulo p of a rectifiable current $T = [\![E, \vec{\tau}, \theta]\!]$ is the unique $T' = [\![E, \vec{\tau}, \theta']\!]$ representative modulo p of T with multiplicity $\theta' \in \left(-\frac{p}{2}, \frac{p}{2}\right]$.

White's proof relies on the following:

Proposition 13.1.12. *If $T \in \mathscr{R}_{d,K}(\mathbb{R}^d)$ is a select representative modulo p, then*

$$\mathbb{M}(\partial T) \leqslant (p-1)\mathbb{M}^p(\partial T). \tag{13.21}$$

We sketch the proof of Theorem 13.1.11, having shown Proposition 13.1.12. Take $[T] \in \mathscr{I}^p_{d,K}(\mathbb{R}^d)$, and let T' be the unique select representative modulo p of T. A priori, T' is just an integer rectifiable current. On the other hand, since $[T]$ is integral, $\mathbb{M}^p(\partial T)$ is finite by (13.8). Then Proposition 13.1.12 implies that $\mathbb{M}(\partial T')$ is finite. Hence, T' is integral because of (2.1).

Unfortunately, in order to carry on the argument that White uses to prove Proposition 13.1.12, the codimension 0 assumption is indispensable. The idea is the following. Firstly, Theorem 13.1.4 allows one to reduce the problem to the case of polyhedral chains. Now, for any given polyhedral chain T which is a select representative modulo p one writes $T = [\![\mathbb{R}^d, \vec{\mathbf{e}}_d, \theta]\!]$, where $\vec{\mathbf{e}}_d$ is the constant standard orientation of $\mathbb{R}^d$ and θ is a summable, piecewise constant, integer-valued function with values in $\left(-\frac{p}{2}, \frac{p}{2}\right]$. Then, White makes the following key observation: since the codimension is 0, if Z is a polyhedron in ∂T then for $\mathcal{H}^{d-1}$-a.e. $x \in Z$ the multiplicity at x is the difference of the values that the function θ takes on the two sides of Z (with the correct sign), whose absolute value is in fact bounded by $p-1$ (because T is a select representative modulo p).

In the next section, we will show that the validity of a statement like the one in Proposition 13.1.12 is in fact the key not only for giving an affirmative answer to Question 13.1.7, but also for positively answering Question 13.1.5. Furthermore, we will answer Question 13.1.7 in dimension $m = 1$.

13.2 MAIN RESULTS

In this section, we will further analyze Questions 13.1.5 and 13.1.7. First, we point out that the two questions are, in fact, connected.

13.2.1 Connection between Questions 13.1.5 and 13.1.7

For every $K \subset \mathbb{R}^d$ compact, consider the following family of statements $\mathcal{S}_m$, for $m = 1, \dots, d$.

Statement $\mathcal{S}_m$. *There exists a constant $C = C(m, d, p, K)$ with the following property. For any $[S] \in \mathscr{R}^p_{m,K}(\mathbb{R}^d)$ there exists a current $\tilde{S} \in \mathscr{R}_{m,K}(\mathbb{R}^d)$ with $\tilde{S} = S \bmod(p)$ and such that*

$$\mathbb{M}(\partial \tilde{S}) \leqslant C\mathbb{M}^p(\partial S).$$

Using Theorem 13.1.4, it is easy to see that the validity of Statement $\mathcal{S}_m$ follows from the validity of a slightly stronger property for polyhedral chains, which, on the other hand, might be easier to check.

Statement $\mathcal{P}_m$. *There exists a constant $C = C(m, d, p)$ independent of K with the following property. For any $P \in \mathscr{P}_m(\mathbb{R}^d)$ with $\operatorname{spt}(P) \subset K$, there exists a current $\tilde{P} \in \mathscr{P}_m(\mathbb{R}^d)$, with $\tilde{P} = P \bmod(p)$ and $\operatorname{spt}(\tilde{P}) \subset K$ such that*

$$\mathbb{M}(\partial \tilde{P}) \leqslant C\mathbb{M}^p(\partial P); \quad \mathbb{M}(\tilde{P}) \leqslant C\mathbb{M}^p(P).$$

Proposition 13.2.1. *The validity of Statement $\mathcal{P}_m$ implies that of Statement $\mathcal{S}_m$.*

Proof. Let $[S] \in \mathscr{R}^p_{m,K}(\mathbb{R}^d)$. We can assume that $\mathbb{M}^p([\partial S])$ is finite, otherwise the conclusion of Statement $\mathcal{S}_m$ is trivial. By Theorem 13.1.4, for every $j = 1, 2, \dots$ there exists $P_j \in \mathscr{P}_m(\mathbb{R}^d)$ such that, denoting

$$K_j := \left\{ x \in \mathbb{R}^d : \operatorname{dist}(x, K) \leqslant \frac{1}{j} \right\},$$

one has $\operatorname{spt}(P_j) \subset K_j$ and

$$\mathbb{F}^p_{K_j}(S - P_j) < \frac{1}{j}, \quad \mathbb{M}^p(P_j) \leqslant \mathbb{M}^p(S) + \frac{1}{j}, \quad \mathbb{M}^p(\partial P_j) \leqslant \mathbb{M}^p(\partial S) + \frac{1}{j}. \tag{13.22}$$

Now, by Statement $\mathcal{P}_m$ there exist a constant C (which does not depend on j) and a sequence $\{\tilde{P}_j\}$ of polyhedral chains with $\tilde{P}_j = P_j \bmod(p)$ and $\operatorname{spt}(\tilde{P}_j) \subset K_j$ such that

$$\mathbb{M}(\partial \tilde{P}_j) \leqslant C\mathbb{M}^p(\partial P_j); \quad \mathbb{M}(\tilde{P}_j) \leqslant C\mathbb{M}^p(P_j).$$

Combining this with (13.22), we get

$$\sup_{j \geqslant 1} \{\mathbb{M}(\tilde{P}_j) + \mathbb{M}(\partial \tilde{P}_j)\} \leqslant C(\mathbb{M}^p(S) + \mathbb{M}^p(\partial S) + 2) < \infty.$$

Then, by the Compactness Theorem 2.1.3 there exist $\tilde{S} \in \mathscr{I}_{m,K_1}(\mathbb{R}^d)$ and a subsequence $\{\tilde{P}_{j_h}\}$ such that

$$\lim_{h \to \infty} \mathbb{F}_{K_1}(\tilde{S} - \tilde{P}_{j_h}) = 0. \tag{13.23}$$

Moreover by the lower semi-continuity of the mass, it holds

$$\mathbb{M}(\partial \tilde{S}) \leqslant C\mathbb{M}^p(\partial S); \quad \mathbb{M}(\tilde{S}) \leqslant C\mathbb{M}^p(S)$$

and we claim that $\operatorname{spt}(\tilde{S}) \subset K$. Indeed, take $x \in \mathbb{R}^d \setminus K$. We will prove that there exists a closed set C such that $x \notin C$ and $\langle \tilde{S}, \omega \rangle = 0$ whenever $\omega \equiv 0$ on C, which implies that $x \notin \operatorname{spt}(\tilde{S})$. Fix ℓ such that $x \notin K_{j_\ell}$ and let $C := K_{j_\ell}$. Let ω be an m-form with $\omega \equiv 0$ on C. Since for every $h \geqslant \ell$ it holds $\operatorname{spt}(\tilde{P}_{j_h}) \subset C$, we have

$$\langle \tilde{P}_{j_h}, \omega \rangle = 0, \quad \text{for every } h \geqslant \ell. \tag{13.24}$$

On the other hand, by (13.23), for every $\varepsilon > 0$ there exists $h \geqslant \ell$ such that we can write $\tilde{S} - \tilde{P}_{j_h} = R + \partial Q$ for some $R \in \mathscr{R}_{m,K_1}(\mathbb{R}^d)$ and $Q \in \mathscr{R}_{m+1,K_1}(\mathbb{R}^d)$ with $\mathbb{M}(R) + \mathbb{M}(Q) \leqslant \varepsilon$. Hence it holds

$$\langle \tilde{S} - \tilde{P}_{j_h}, \omega \rangle = \langle R, \omega \rangle + \langle \partial Q, \omega \rangle \leqslant \mathbb{M}(R)\|\omega\|_\infty + \mathbb{M}(Q)\|d\omega\|_\infty \leqslant \varepsilon(\|\omega\|_\infty + \|d\omega\|_\infty).$$

Hence by (13.24) $\langle \tilde{S}, \omega \rangle = 0$, which completes the proof of the claim.

Finally, we show that $\tilde{S} = S \bmod(p)$. To this aim, for every $h = 1, 2, \ldots$, we compute

$$\mathbb{F}^p_{K_1}(\tilde{S} - S) \leqslant \mathbb{F}^p_{K_1}(\tilde{S} - \tilde{P}_{j_h}) + \mathbb{F}^p_{K_1}(\tilde{P}_{j_h} - S) \leqslant \mathbb{F}_{K_1}(\tilde{S} - \tilde{P}_{j_h}) + \mathbb{F}^p_{K_{j_h}}(\tilde{P}_{j_h} - S),$$

which by (13.22) and (13.23) tends to 0 when h tends to ∞. □

Remark 13.2.2. It follows from the above proof that if the Statement $\mathcal{P}_m$ holds true then the Statement $\mathcal{S}_m$ holds true with the same constant C. In particular, the constant would not depend on the compact set K.

Clearly, if the Statement $\mathcal{S}_m$ is true then every m-dimensional integral current modulo p in K has an integral representative in K, and thus the answer to Question 13.1.7 is affirmative in dimension m. The next theorem shows that, in fact, the validity of $\mathcal{S}_m$ has important consequences on Question 13.1.5 as well.

Theorem 13.2.3. *If $\mathcal{S}_m$ holds true, then Question 13.1.5 has affirmative answer in $\mathscr{F}_{m-1,K}(\mathbb{R}^d)$.*

Proof. It is sufficient to prove that if $T \in \mathscr{F}_{m-1,K}(\mathbb{R}^d)$ is a flat $(m-1)$-chain such that $T = 0 \bmod(p)$ in $\mathscr{F}_{m-1,K}(\mathbb{R}^d)$, then $T = pQ$ for some $Q \in \mathscr{F}_{m-1,K}(\mathbb{R}^d)$. Let $\{R_j\} \subset \mathscr{R}_{m-1,K}(\mathbb{R}^d)$, $\{S_j\} \subset \mathscr{R}_{m,K}(\mathbb{R}^d)$ and $\{Q_j\} \subset \mathscr{I}_{m-1,K}(\mathbb{R}^d)$ be such that

$$T = R_j + \partial S_j + pQ_j \quad \forall j \tag{13.25}$$

and

$$\lim_{j\to\infty} \big(\mathbb{M}(R_j) + \mathbb{M}(S_j)\big) = 0. \tag{13.26}$$

Conditions (13.25) and (13.26) are equivalent to say that the currents pQ_j converge to T in flat norm $\mathbb{F}_K$. We want to conclude from this that $T = pQ$ for some $Q \in \mathscr{F}_{m-1,K}(\mathbb{R}^d)$. In other words, we are looking for a result of closedness of the currents of the form pQ with respect to flat convergence. Now, observe the following. For every j, the current R_j is rectifiable. Therefore, we can write

$$R_j = \tilde{R}_j + pV_j, \tag{13.27}$$

with $V_j \in \mathscr{R}_{m-1,K}(\mathbb{R}^d)$ and $\tilde{R}_j$ representative modulo p. In particular, this implies that

$$\mathbb{M}(\tilde{R}_j) = \mathbb{M}^p(R_j) \leqslant \mathbb{M}(R_j) \to 0. \tag{13.28}$$

Also the currents S_j are rectifiable, and of dimension m. Since $\mathcal{S}_m$ holds true, for every j we can let $\tilde{S}_j$ be the representative of $[S_j]$ given in there, so that

$$S_j = \tilde{S}_j + pZ_j \tag{13.29}$$

with $\tilde{S}_j, Z_j \in \mathscr{R}_{m,K}(\mathbb{R}^d)$, and

$$\mathbb{M}(\partial \tilde{S}_j) \leqslant C(m, d, p, K)\mathbb{M}^p(\partial S_j). \tag{13.30}$$

Now, since $\mathbb{M}^p(T - pQ_j) = \mathbb{M}^p(T) = 0$ for every j and $\mathbb{M}^p(R_j) \to 0$, we deduce from (13.25) that also $\mathbb{M}^p(\partial S_j) \to 0$, and therefore also $\mathbb{M}(\partial \tilde{S}_j) \to 0$.

Thus, the above argument produces the following: modulo replacing $Q_j \in \mathscr{I}_{m-1,K}(\mathbb{R}^d)$ with $\tilde{Q}_j := Q_j + V_j + \partial Z_j \in \mathscr{F}_{m-1,K}(\mathbb{R}^d)$, we can replace (13.25) with

$$T = \tilde{R}_j + \partial \tilde{S}_j + p\tilde{Q}_j \quad \forall j, \tag{13.31}$$

and (13.26) with the stronger

$$\lim_{j\to\infty} \left(\mathbb{M}(\tilde{R}_j) + \mathbb{M}(\partial \tilde{S}_j)\right) = 0, \tag{13.32}$$

that is the currents $p\tilde{Q}_j$ are now approximating T *in mass*.

The problem, now, reduces to proving that the subset of flat chains in $\mathscr{F}_{m-1,K}(\mathbb{R}^d)$ of the form pQ is closed with respect to convergence in mass: this question, though, is evidently much easier than the previous one, and it turns out to always have affirmative answer. Indeed, let $\{Q_j\}_{j=1}^{\infty} \subset \mathscr{F}_{m-1,K}(\mathbb{R}^d)$ be a sequence of flat chains such that $\mathbb{M}(T - pQ_j) \to 0$. In particular, this would imply that the sequence $\{pQ_j\}$ is a Cauchy sequence in mass. Therefore, the sequence $\{Q_j\}$ is also a Cauchy sequence in mass, and in fact also in the flat norm $\mathbb{F}_K$, since $\mathbb{F}_K(T) \leqslant \mathbb{M}(T)$ for any $T \in \mathscr{F}_{m-1,K}(\mathbb{R}^d)$.[1] So, by completeness there is $Q \in \mathscr{F}_{m-1,K}(\mathbb{R}^d)$ such that $\mathbb{F}_K(Q - Q_j) \to 0$. This also implies $\mathbb{F}_K(pQ - pQ_j) \to 0$, since $\mathbb{F}_K(nT) \leqslant n\mathbb{F}_K(T)$ in general. So, pQ is a flat limit of the sequence pQ_j. By uniqueness of the limit, one therefore has to conclude $T = pQ$. □

Corollary 13.2.4. *The answer to Question 13.1.5 is positive for* $m = d - 1$.

Proof. It immediately follows from Theorem 13.2.3, since $\mathcal{S}_d$ is Proposition 13.1.12. □

13.2.2 Answer to Question 13.1.7 in dimension $m = 1$

Theorem 13.2.5. *The answer to Question 13.1.7 is positive for* $m = 1$.

In the proof, we will use the following elementary fact.

Lemma 13.2.6. *Let* $P \in \mathscr{P}_1(\mathbb{R}^d)$ *have positive multiplicities. Let* z *be a point in* $\operatorname{spt}(\partial P)$. *Then one can select a finite sequence of oriented segments* $S_1, \dots, S_N$ *supported in the support of* P *such that:*

1. *the orientation of each segment* S_i *coincides with the orientation of* P *on* S_i*;*
2. *the second extreme of* S_i *coincides with the first extreme of* S_{i+1}*, for* $i = 1, \dots, N - 1$*;*

1 If $\mathbb{M}(T) = \infty$ there is of course nothing to prove. On the other hand, if $\mathbb{M}(T) < \infty$ then T is integer rectifiable, and hence it is a competitor for the decomposition in the definition of the flat norm.

3. *If the multiplicity of* ∂P *at* z *is negative, then the first extreme of* S_1 *is* z *and the second extreme of* S_N *is a point* x *of the support of* ∂P *with positive multiplicity. Vice versa, if the multiplicity of* ∂P *at* z *is positive, then the first extreme of* S_1 *is a point* x *of the support of* ∂P *with negative multiplicity and the second extreme of* S_N *is* z*;*

4. $S_i \neq S_j$ *for* $i \neq j$.

Proof. Assume without loss of generality that the multiplicity of ∂P at z is negative. Since the multiplicities on P are all positive, then among the (finitely many) segments defining the support of P there is at least a segment S_1 whose first extreme is z such that

$$\mathbb{M}(P) = \mathbb{M}(P - [\![S_1]\!]) + \mathbb{M}([\![S_1]\!]), \tag{13.33}$$

If the second extreme y of S_1 is not a point with positive multiplicity of ∂P, it is a point of negative multiplicity of $\partial(P - [\![S_1]\!])$, hence the procedure can be repeated with $P - [\![S_1]\!]$ in place of P and y in place of z. The procedure has to terminate in a finite number of steps, because of (13.33) and the fact that the mass of each $[\![S_i]\!]$ is bounded from below. When the procedure ends, one can easily see that the ordered sequence of segments collected satisfies properties $(1)-(3)$. Property (4) is not necessarily satisfied. If a certain segment S' is repeated in the procedure, it is sufficient to eliminate from the sequence one copy of S' and all the segments appearing between two repetitions of S'. After this elimination, the sequence satisfies also property (4). □

Proof of Theorem 13.2.5. By Proposition 13.2.1 it is sufficient to prove Statement $\mathcal{P}_1$. Consider $P \in \mathscr{P}_1(\mathbb{R}^d)$. Firstly we choose a representative $Q \in \mathscr{P}_1(\mathbb{R}^d)$ modulo p of P with multiplicities in $\{1, \dots, p-1\}$. Clearly we have $\mathbb{M}(Q) \leqslant (p-1)\mathbb{M}^p(P)$, but at the moment we have no control on $\mathbb{M}(\partial Q)$. Hence, we want to replace Q with another representative $\tilde{P} \in \mathscr{P}_1(\mathbb{R}^d)$ of P, for which we can control both the mass and the mass of the boundary. More precisely, we want to find a representative $\tilde{P}$ with multiplicities in $\{1, \dots, p-1\}$ and with the multiplicities of $\partial\tilde{P}$ in $\{-(p-1), \dots, p-1\}$.

Consider a point $z \in \operatorname{spt}(\partial Q)$ with multiplicity θ_z such that $|\theta_z| \geqslant p$. Without loss of generality, we can assume $\theta_z < 0$. Given that the multiplicities on Q are all positive, we can use Lemma 13.2.6 to select a finite sequence of oriented segments $S_1, \dots, S_N$ supported in the support of Q, satisfying properties $(1)-(4)$ (with Q in place of P).

Once we have found such a sequence of segments, denote by Q^1 the polyhedral current obtained from Q by changing on every segment S_i both the orientation and the multiplicity from θ_i to $\theta_i^1 := (p - \theta_i)$. Clearly Q^1 has still multiplicities in $\{1, \dots, p-1\}$. Moreover, if θ_z^1 denotes the multiplicity of ∂Q^1 at z then one has $|\theta_z^1| = |\theta_z| - p$. On the other hand, if x denotes the other endpoint of the chain of segments as in (3) of Lemma 13.2.6 and θ_x, θ_x^1 are the multiplicities of ∂Q and ∂Q^1 at x respectively, then it holds $\theta_x^1 = \theta_x - p$. Now, since by Lemma 13.2.6(3) it holds $\theta_x \geqslant 1$, it follows that $\theta_x^1 = (\theta_x - p) \in [1-p, \theta_x]$. Hence, $|\theta_x^1| \leqslant |\theta_x| + p - 2$.

Therefore, one has

$$\mathbb{M}(\partial Q^1) \leqslant \mathbb{M}(\partial Q) - 2. \tag{13.34}$$

If possible, we repeat the procedure above with Q^1 in place of Q, producing a new polyhedral current Q^2. By formula (13.34), the procedure can be iterated only a finite number

M of times. The corresponding $\tilde{P} := Q^M$ has the required property, because any point $z \in \mathrm{spt}(\partial Q^M)$ has multiplicity $|\theta_z| \leqslant p-1$. Obviously we have

$$\mathbb{M}(\tilde{P}) \leqslant (p-1)\mathbb{M}^p(P) \quad \text{and} \quad \mathbb{M}(\partial\tilde{P}) \leqslant (p-1)\mathbb{M}^p(\partial P),$$

and the proof is complete. □

Since we have actually proved the Statement $\mathcal{P}_1$, it follows from Proposition 13.2.1 that the Statement $\mathcal{S}_1$ holds true. By virtue of Theorem 13.2.3, we can therefore deduce the following

Corollary 13.2.7. *The answer to Question 13.1.5 is positive for* $m = 0$.

13.2.3 Negative answer to Question 13.1.7 in general dimension

It is evident that the choice of the compact set K could be crucial for establishing an answer to Question 13.1.7. In the spirit of the counterexample suggested by Federer in [Fed69, 4.2.26, p. 426] (see Remark 13.1.6 above), we provide a negative answer to the question, proving the existence of a compact subset $K \subset \mathbb{R}^5$ and a current $[T] \in \mathscr{I}^2_{2,K}(\mathbb{R}^5)$ with $\partial T = 0 \bmod(2)$ such that there exists no $I \in \mathscr{I}_{2,K}(\mathbb{R}^5)$ with $I = T \bmod(2)$. Nevertheless, for a different choice of a compact $K' \supset K$ we can exhibit an integral current $I' \in \mathscr{I}_{2,K'}(\mathbb{R}^5)$ with $\partial I' = 0$ and $I' = T \bmod(2)$.

In what follows, we will let $\mathcal{K}$ be the embedded Klein bottle in $\mathbb{R}^4$ (in particular, $\mathcal{K}$ is a non-orientable compact two dimensional surface without boundary in $\mathbb{R}^4$). There exist a closed curve γ and an integral current $S := [\![\mathcal{K}, \vec{\tau}, 1]\!] \in \mathscr{I}_{2,\mathcal{K}}(\mathbb{R}^4)$ such that the set of discontinuity points of $\vec{\tau}$ coincides with γ. In particular, ∂S is the integral current $[\![\gamma, \vec{\tau}_\gamma, 2]\!]$, $\vec{\tau}_\gamma$ being the orientation of γ naturally induced by $\vec{\tau}$. We let $[S] \in \mathscr{I}^2_{2,\mathcal{K}}(\mathbb{R}^4)$ be the associated current $\bmod(2)$. In particular, $\partial\,[S] = 0$ and $\mathbb{M}^2([S]) = \mathcal{H}^2(\mathcal{K})$. We have the following, elementary

Lemma 13.2.8. *There exists a constant* $c = c(\mathcal{K})$ *with the following property. If* $R \in \mathscr{I}_{2,\mathcal{K}}(\mathbb{R}^4)$ *is such that* $R \in [S]$, *one has*

$$\mathbb{M}(\partial R) \geqslant c.$$

Proof. By contradiction, let $\{\alpha_j\}_{j=1}^\infty$ be a sequence of positive numbers with $\alpha_j \searrow 0$, and $\{R_j\}_{j=1}^\infty \subset \mathscr{I}_{2,\mathcal{K}}(\mathbb{R}^4)$ be such that

$$R_j \in [S] \qquad \forall j,$$

and

$$\mathbb{M}(\partial R_j) \leqslant \alpha_j.$$

We write $R_j = [\![\mathcal{K}, \vec{\tau}, \theta_j]\!]$, and we observe that, since $R_j = S \bmod(2)$, from (13.20) and from the definition of S it follows that

$$\theta_j(x) \equiv 1 \,(\mathrm{mod}\, 2) \quad \text{for } \mathcal{H}^2\text{-a.e. } x \in \mathcal{K}. \tag{13.35}$$

We replace every R_j with the integral current $\tilde{R}_j = [\![\mathcal{K}, \vec{\tau}, \tilde{\theta}_j]\!]$, where $\tilde{\theta}_j := \mathrm{sign}(\theta_j)$. Clearly, by (13.35) and the definition of $\tilde{\theta}_j$, $\tilde{R}_j = R_j \bmod(2)$, and thus $\tilde{R}_j \in [S]$ for every j. Notice, furthermore, that $\mathbb{M}(\tilde{R}_j) = \mathcal{H}^2(\mathcal{K})$ for every j, and that

$$\mathbb{M}(\partial\tilde{R}_j) \leqslant \mathbb{M}(\partial R_j) \leqslant \alpha_j. \tag{13.36}$$

In order to show (13.36), let U be any open set in $\mathcal{K}$ homeomorphic to a two-dimensional disc. Let also $\vec{\sigma}$ be a fixed continuous orientation on U. We have that $R_j \llcorner U = [\![U, \vec{\sigma}, \Theta_j]\!]$, where Θ_j is the function defined by

$$\Theta_j(x) := \begin{cases} \theta_j(x) & \text{if } \vec{\tau}(x) = \vec{\sigma}(x) \\ -\theta_j(x) & \text{if } \vec{\tau}(x) = -\vec{\sigma}(x). \end{cases}$$

As a consequence of [Sim83b, Remark 27.7], it holds

$$\mathbb{M}((\partial R_j) \llcorner U) = |D\Theta_j|(U),$$

where $|D\Theta_j|$ is the variation of the BV function Θ_j. Analogously, $\tilde{R}_j \llcorner U = [\![U, \vec{\sigma}, \tilde{\Theta}_j]\!]$, where $\tilde{\Theta}_j$ is the function defined by

$$\tilde{\Theta}_j(x) := \begin{cases} \tilde{\theta}_j(x) & \text{if } \vec{\tau}(x) = \vec{\sigma}(x) \\ -\tilde{\theta}_j(x) & \text{if } \vec{\tau}(x) = -\vec{\sigma}(x). \end{cases}$$

Observe that $\tilde{\Theta}_j \equiv \operatorname{sign}(\Theta_j)$, and hence

$$\mathbb{M}((\partial \tilde{R}_j) \llcorner U) = |D\tilde{\Theta}_j|(U) \leqslant |D\Theta_j|(U) = \mathbb{M}((\partial R_j) \llcorner U),$$

which completes the proof of (13.36).

Now, by the Compactness Theorem 2.1.3 there exists a current $\tilde{R} \in \mathscr{I}_{2,\mathcal{K}}(\mathbb{R}^4)$ and a subsequence (not relabeled) such that

$$\lim_{j\to\infty} \mathbb{F}_{\mathcal{K}}(\tilde{R} - \tilde{R}_j) = 0.$$

Moreover, by the lower semi-continuity of the mass one has $\partial \tilde{R} = 0$. Since the equivalence classes $\operatorname{mod}(2)$ are closed with respect to the flat convergence, $\tilde{R} \in [S]$, which contradicts the fact that $\mathcal{K}$ is not orientable. □

Remark 13.2.9. Observe that if $\mathcal{K}_\lambda$ is a homothetic copy of $\mathcal{K}$ with homothety ratio λ, then $c(\mathcal{K}_\lambda) = \lambda c(\mathcal{K})$.

We finally define the compact set $K \subset \mathbb{R}^5$ and the current $[T] \in \mathscr{I}^2_{2,K}(\mathbb{R}^5)$ as follows. For every $i = 1, 2, \ldots$, we let Λ_i be the homothety on $\mathbb{R}^4$ defined by $\Lambda_i(x) := \frac{x}{i}$, and $\pi_i \colon \mathbb{R}^4 \to \mathbb{R}^5$ be the isometry $\pi_i(x) := \left(\frac{1}{i}, x\right)$. We set

$$K := \{0\} \cup \bigcup_{i=1}^{\infty} \pi_i \circ \Lambda_i(\mathcal{K}),$$

which is evidently compact, and

$$T := \sum_{i=1}^{\infty} (\pi_i \circ \Lambda_i)_\sharp S.$$

We let $[T]$ denote the equivalence class of T modulo 2. Since $\mathbb{M}^2((\pi_i \circ \Lambda_i)_\sharp S) = \frac{1}{i^2}\mathcal{H}^2(\mathcal{K})$, then $[T]$ is well defined, and in particular $\partial\,[T] = 0$. In the following proposition, we show that the choice of K and $[T]$ provides a negative answer to Question 13.1.7.

Proposition 13.2.10. *In general, the answer to Question 13.1.7 is negative.*

Proof. Let K and $[T]$ be as above, and assume by contradiction that there exists $I \in \mathscr{I}_{2,K}(\mathbb{R}^5)$ with $I \in [T]$. Then, the restriction of I to each plane $x_1 = \frac{1}{i}$ belongs to the class $\left[(\pi_i \circ \Lambda_i)_\sharp S\right]$, and thus by Lemma 13.2.8 and Remark 13.2.9 one has

$$\mathbb{M}(\partial I) = c(\mathcal{K}) \sum_{i=1}^{\infty} \frac{1}{i} = \infty,$$

which gives the desired contradiction. □

Remark 13.2.11. Observe that if we replace $\mathcal{K}$ with $\mathcal{K}' := \mathcal{K} \cup D$, where D is a suitable two-dimensional disc, then Lemma 13.2.8 fails, as there exists $R \in \mathscr{I}_{2,\mathcal{K}'}(\mathbb{R}^4)$ such that $R \in [S]$ and $\partial R = 0$. Hence, it is possible to construct an integral representative of $[T]$ with support in

$$K' := \{0\} \cup \bigcup_{i=1}^{\infty} \pi_i \circ \Lambda_i(\mathcal{K}').$$

13.2.4 Concluding remarks

Ambrosio and Wenger proved in [AW11, Theorem 4.1] a statement similar to our Theorem 13.2.5, under the hypothesis that $\partial [T] = 0$. They were motivated by the will to prove the analogue of Theorem 13.1.2 above when the ambient space is a compact convex subset of a Banach space with mild additional assumptions. Even though our theorem covers also the case with boundary, our proof is considerably simpler than theirs, essentially because we can rely on the polyhedral approximation theorem, which is not available in their context. Actually, our result would follow directly from theirs if one could independently guarantee the validity of the following proposition. However, we were not able to devise a proof independent of Theorem 13.2.5.

Proposition 13.2.12. *Let* $[T] \in \mathscr{I}^p_{1,K}(\mathbb{R}^d)$. *Then, for any* $R = \sum_{i=1}^q \theta_i \delta_{x_i} \in \mathscr{R}_{0,K}(\mathbb{R}^d)$ *such that* $R = \partial T \bmod(p)$ *one has:*

$$\sum_{i=1}^{q} \theta_i \equiv 0 \,(\mathrm{mod}\, p). \tag{13.37}$$

Assume the validity of the Proposition. An alternative proof of our Theorem 13.2.5 can be obtained as follows. Let $[T] \in \mathscr{I}^p_{1,K}(\mathbb{R}^m)$ and let $R = \sum_{i=1}^q \theta_i \delta_{x_i} \in \mathscr{R}_{0,K}(\mathbb{R}^d)$ be such that $R = \partial T \bmod(p)$. Fix $x_0 \notin \{x_1, \ldots, x_q\}$ and consider the cone C with vertex x_0 over R, i.e. the integral 1-current

$$C := \sum_{i=1}^{q} [\![S_i, \vec{\tau}_i, \theta_i]\!], \tag{13.38}$$

where S_i is the segment joining x_i to x_0 and $\vec{\tau}_i := \dfrac{x_0 - x_i}{|x_0 - x_i|}$. By (13.37), the multiplicity of ∂C at x_0 is an integer multiple of p, and thus via a simple computation $\partial(T + C) = 0 \bmod(p)$. Applying the result of Ambrosio and Wenger, we finally obtain that there exists an integral

current $J \in \mathscr{I}_{1,K}(\mathbb{R}^d)$ such that $J = T + C \bmod(p)$. Hence, $I := J - C$ is an integral current with $I = T \bmod(p)$.

Although the analogue of Proposition 13.2.12 for classical currents is a well known fact (i.e. the sum of the multiplicities in the boundary of an integral 1-current is zero), the validity of Proposition 13.2.12 does not follow trivially. Nevertheless, it can in fact be deduced as a consequence of our Corollary 13.2.7.

Proof of Proposition 13.2.12. Let T and R be as in the statement. Then, since $\mathbb{F}^p_K(\partial T - R) = 0$, Corollary 13.2.7 implies the existence of currents $Q \in \mathscr{R}_{0,K}(\mathbb{R}^d)$ and $S \in \mathscr{R}_{1,K}(\mathbb{R}^d)$ such that

$$\partial T - R = p(Q + \partial S),$$

that is

$$\partial(T - pS) = R + pQ.$$

In particular, $T - pS$ is a classical integral current, and thus the sum of the multiplicities in R must equal that of $-pQ$, which concludes the proof. □

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