Global existence for a singular phase field system related to a sliding mode control problem*

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Abstract

In the present contribution we consider a singular phase field system located in a smooth bounded domain of $\mathbb{R}^3$. The entropy balance equation is perturbed by a logarithmic nonlinearity and by the presence of an additional term involving a possibly nonlocal maximal monotone operator and arising from a class of sliding mode control problems. The second equation of the system accounts for the phase dynamics and is deduced from a balance law for the microscopic forces that are responsible for the phase transition process. The resulting system is highly nonlinear and the main difficulties lie in the contemporary presence of two nonlinearities, one of which under time derivative, in the entropy balance equation. Consequently, we are able to prove only the existence of solutions. To this aim, we will introduce a backward finite differences scheme and argue on this by proving uniform estimates and passing to the limit on the time step.

Key words: Phase field system; maximal monotone nonlinearities; nonlocal terms; initial and boundary value problem; existence of solutions.

AMS (MOS) subject classification: 35K61, 35K20, 35D30, 80A22.

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1 Introduction

This paper is devoted to the mathematical analysis of a system of partial differential equations (PDE) arising from a thermodynamic model describing phase transitions. The system is written in terms of a rescaled balance of energy and of a balance law for the microforces that govern the phase transition. Moreover, the first equation of the system is perturbed by the presence of an additional maximal monotone nonlinearity. This paper will focus only on analytical aspects and, in particular, will investigate the existence of solutions. In order to make the presentation clear from the beginning, let us briefly introduce the main ingredients of the PDE system and give some comments on the physical meaning.

We consider a two-phase system located in a smooth bounded domain $\Omega \subseteq \mathbb{R}^3$ and let $T > 0$ denote a final time. The unknowns of the problem are the *absolute* temperature ϑ and a phase parameter χ which may represent the local proportion of one of the two phases. To ensure thermomechanical consistency, suitable physical constraints on χ are introduced: if it is assumed, e.g., that the two phases may coexist at each point with different proportions, it turns out to be reasonable to require that χ lies between 0 and 1, with $1 - \chi$ representing the proportion of the second phase. In particular, the values $\chi = 0$ and $\chi = 1$ may correspond to the pure phases, while χ is between 0 and 1 in the regions when both phases are present. Clearly, the model should provide an evolution for χ that complies with the previous physical constraint.

Now, let us state precisely the equations as well as the initial and boundary conditions. The two equations governing the evolution of ϑ and χ are recovered as balance laws. The first equation is obtained as a reduction of the energy balance equation divided by the absolute temperature ϑ (see [5, formulas (2.33)–(2.35)]). Hence, the so-called entropy balance can be written in $\Omega \times (0, T)$ as follows:

$$\partial_t(\ln \vartheta + \ell\chi) - k_0\Delta\vartheta = F, \tag{1.1}$$ II144

where $k_0 > 0$ is a thermal coefficient for the entropy flux, ℓ is a positive parameter and F stands for an external entropy source. We point out that in the previous equation one finds the entropy flux $\mathbf{Q}$, related to the heat flux vector $\mathbf{q}$ by $\mathbf{Q} = \mathbf{q}/\vartheta$, and specified by $\mathbf{Q}(t) = -k_0\nabla\vartheta(t)$, $t \in (0, T)$.

In the present contribution, we assume that the entropy balance equation (1.1) is perturbed by the presence of an additional maximal monotone nonlinearity, i.e.,

$$\partial_t(\ln \vartheta + \ell\chi) - k_0\Delta\vartheta + \zeta = F, \tag{1.2}$$ II1

where

$$\zeta(t) \in A(\vartheta(t) - \vartheta^*) \qquad \text{for a.e. } t \in (0, T). \tag{1.3}$$ II3

Here, ϑ^* is a positive and smooth function ($\vartheta^* \in H^2(\Omega)$ with null outward normal derivative on the boundary) and $A : L^2(\Omega) \to L^2(\Omega)$ is a maximal monotone operator satisfying some conditions, namely: A is the subdifferential of a proper, convex and lower semicontinuous (l.s.c.) function $\Phi : L^2(\Omega) \to \mathbb{R}$ which takes its minimum in 0, and A is linearly bounded in $L^2(\Omega)$. In order to explain the role of this further nonlinearity, we refer to [2], where a class of sliding mode control problems is considered: a state-feedback control

$(\vartheta, \chi) \mapsto u(\vartheta, \chi)$ is added in the balance equations with the purpose of forcing the trajectories of the system to reach the sliding surface (i.e., a manifold of lower dimension where the control goal is fulfilled and such that the original system restricted to this manifold has a desired behavior) in finite time and maintains them on it. As widely described in [2], this study is physically meaningful in the framework of phase transition processes.

Let us mention the contributions [15,16], where standard phase field systems of Caginalp type, perturbed by the presence of nonlinearities similar to (1.3), are considered. In [15,16] the existence of strong solutions, the global well-posedness of the system and the sliding mode property can be proved; unfortunately, here the problem we consider is rather more delicate due to the doubly nonlinear character of equation (1.2) and it turns out that we cannot perform a so complete analysis. On the other hand, we observe that, due to the presence of the logarithm of the temperature in the entropy equation (1.2), in the system we investigate here the positivity of the variable representing the absolute temperature follows directly from solving the problem, i.e., from finding a solution component ϑ to which the logarithm applies. This is important, since we can avoid the use of other methods, or the setting of special assumptions, in order to guarantee the positivity of ϑ in the space-time domain.

The second equation of the system under study accounts for the phase dynamics and is deduced from a balance law for the microscopic forces that are responsible for the phase transition process. According to [18,19], this balance reads

$$\partial_t \chi - \Delta \chi + \beta(\chi) + \pi(\chi) \ni \ell\vartheta, \tag{1.4}$$ II2

where $\beta + \pi$ represents the derivative, or the subdifferential, of a double-well potential $\mathcal{W}$ defined as

$$\mathcal{W} = \tilde{\beta} + \tilde{\pi},$$

where

$$\tilde{\beta} : \mathbb{R} \longrightarrow [0, +\infty] \text{ is proper, l.s.c. and convex with } \tilde{\beta}(0) = 0, \tag{1.5}$$ Intro-beta

$$\tilde{\pi} \in C^1(\mathbb{R}) \text{ and } \pi = \tilde{\pi}' \text{ is Lipschitz continuous in } \mathbb{R}. \tag{1.6}$$ Intro-lambda

Due to (1.5), the subdifferential $\beta := \partial\tilde{\beta}$ is well defined and turns out to be a maximal monotone graph. Moreover, as $\tilde{\beta}$ takes on its minimum in 0, we have that $0 \in \beta(0)$. Note that in (1.4) the inclusion is used in place of the equality in order to allow for the presence of a multivalued β.

We recall that many different choices of $\tilde{\beta}$ and $\tilde{\pi}$ have been introduced in the literature (see, e.g., [3,6,17,21]). In the case of a solid-liquid phase transition, $\mathcal{W}$ may be chosen in such a way that the full potential (cf. (1.4))

$$\chi \mapsto \tilde{\beta}(\chi) + \tilde{\pi}(\chi) - \ell\vartheta\chi$$

exhibits one of the two minima $\chi = 0$ and $\chi = 1$ as global minimum for equilibrium, depending on whether ϑ is below or above a critical value ϑ_c, which may represent a phase change temperature. A sample case is given by $\tilde{\pi}(\chi) = \ell\vartheta_c\chi$ and by the $\tilde{\beta}$ that coincides with the indicator function $I_{[0,1]}$ of the interval $[0, 1]$, that is,

$$\tilde{\beta}(\rho) = I_{[0,1]}(\rho) = \begin{cases} 0 & \text{if } 0 \le \rho \le 1 \\ +\infty & \text{elsewhere} \end{cases}$$

so that $\beta = \partial I_{[0,1]}$ is specified by

$$r \in \beta(\rho) \quad \text{if and only if} \quad r \begin{cases} \leq 0 & \text{if } \rho = 0 \\ = 0 & \text{if } 0 < \rho < 1 \\ \geq 0 & \text{if } \rho = 1 \end{cases} .$$

Of course, this yields a singular case for the potential $\mathcal{W}$, in which $\tilde{\beta}$ is not differentiable, and it is known in the literature as the double obstacle case (cf. [3,6,18])

In the last decades phase field models have attracted a number of mathematicians and applied scientists to describe many different physical phenomena. Let us just recall some results in the literature that are related to our system. Some key references are the papers [4–6]. Besides, we quote [8], where a first simplified version of the entropy system is considered, and [7,9] for related analyses and results. About special choices of the heat flux and phase field models ensuring positivity of the absolute temperature, we aim to quote the papers [12–14], where some Penrose-Fife models have been addressed.

The full problem investigated in this paper consists of equations (1.2)–(1.4) coupled with suitable boundary and initial conditions. In particular, we prescribe a no-flux condition on the boundary for both variables:

$$\partial_\nu \vartheta = 0, \quad \partial_\nu \chi = 0 \quad \text{on } \Gamma \times (0,T), \tag{1.7}$$ II5

where ∂_ν denotes the outward normal derivative on the boundary Γ of Ω. Besides, in the light of (1.3), initial conditions are stated for $\ln \vartheta$ and χ:

$$\ln \vartheta(0) = \ln \vartheta_0, \quad \chi(0) = \chi_0 \quad \text{in } \Omega. \tag{1.8}$$ II6

The resulting system is highly nonlinear and the main difficulties lie in the treatment of the doubly nonlinear equation (1.2). The expert reader can realise that it is not trivial to recover some coerciveness and regularity for ϑ from (1.2), (1.3) and (1.7); morever, the presence of both $\ln \vartheta$ under time derivative and the selection ζ from $A(\vartheta - \vartheta^*)$ complicates possible uniqueness arguments. For the moment, we are just able to prove the existence of solutions for the described problem. To this aim, we introduce a backward finite differences scheme and first examine the solvability of it, for which we have to introduce another approximating problem based on the use of Yosida regularizations for the maximal monotone operators.

As far as the outline of the paper is concerned, we state precisely assumptions and main results in Section 2, then introduce the time-discrete problem (P_τ) in Section 3 and completely prove existence and uniqueness of the solution. Section 4 is devoted to the proof of several uniform estimates, independent of τ, involving the solution of (P_τ). Finally, in Section 5 we pass to the limit as as $\tau \searrow 0$ by means of compactness and monotonicity arguments in order to find a solution to the problem (1.2)–(1.4), (1.7)–(1.8).

2 Main results

2.1 Preliminary assumptions

We assume $\Omega \subseteq \mathbb{R}^3$ to be open, bounded, connected, of class C^1 and we write $|\Omega|$ for its Lebesgue measure. Moreover, Γ and ∂_ν stand for the boundary of Ω and the outward normal derivative, respectively. Given a finite final time $T > 0$, for every $t \in (0, T]$ we set

$$Q_t = \Omega \times (0,t), \quad Q = Q_T, \quad \Sigma_t = \Gamma \times (0,t), \quad \Sigma = \Sigma_T.$$

We also introduce the spaces

$$H = L^2(\Omega), \quad V = H^1(\Omega), \quad W = \{u \in H^2(\Omega) : \ \partial_\nu u = 0 \text{ on } \Gamma\}, \tag{2.1}$$ W

with usual norms $\|\cdot\|_H$, $\|\cdot\|_V$, $\|\cdot\|_W$ and related inner products $(\cdot,\cdot)_H$, $(\cdot,\cdot)_V$, $(\cdot,\cdot)_W$, respectively. We identify H with its dual space H', so that $W \subset V \subset H \subset V' \subset W'$ with dense and compact embeddings. Let $\langle\cdot,\cdot\rangle$ denote the duality pairing between V' and V. The notation $\|\cdot\|_p$ $(1 \le p \le \infty)$ stands for the standard norm in $L^p(\Omega)$. For short, in the notation of norms we do not distinguish between a space and a power thereof.

From now on, we interpret the operator $-\Delta$ as the Laplacian operator from the space W to H, then including the Neumann homogeneous boundary condition. Moreover, we extend $-\Delta$ to an operator from V to V' by setting

$$\langle -\Delta u, v\rangle := \int_\Omega \nabla u \cdot \nabla v, \quad u, v \in V. \tag{2.2}$$ pier4

Throughout the paper, we account for the well-known continuous embeddings $V \subset L^q(\Omega)$, with $1 \le q \le 6$, $W \subset C^0(\overline{\Omega})$ and for the related Sobolev inequalities:

$$\|v\|_q \le C_s \|v\|_V \qquad \text{and} \qquad \|v\|_\infty \le C_s \|v\|_W$$

for $v \in V$ and $v \in W$, respectively, where C_s depends on Ω only, since sharpness is not needed. We will also use a variant of the Poincaré inequality, i.e., there exists a positive constant C_p such that

$$\|v\|_V \le C_p\Big(\|v\|_{L^1(\Omega)} + \|\nabla v\|_H\Big), \qquad v \in V. \tag{2.3}$$ Poincare

Furthermore, we make repeated use of the Hölder inequality, and of Young's inequalities, i.e., for every $a, b > 0$, $\alpha \in (0,1)$ and $\delta > 0$ we have that

$$ab \le \alpha a^{\frac{1}{\alpha}} + (1-\alpha) b^{\frac{1}{1-\alpha}}, \tag{2.4}$$ dis1

$$ab \le \delta a^2 + \frac{1}{4\delta} b^2. \tag{2.5}$$ dis2

Besides, for every a, $b \in \mathbb{R}$ we have that

$$(a-b)a = \frac{1}{2}a^2 - \frac{1}{2}b^2 + \frac{1}{2}(a-b)^2. \tag{2.6}$$ dis777

We also recall the discrete version of the Gronwall lemma (see, e.g., [20, Prop. 2.2.1]).

grondiscr **Lemma 2.1.** *If* $(a_0, \dots, a_N) \in [0, +\infty)^{N+1}$ *and* $(b_1, \dots, b_N) \in [0, +\infty)^N$ *satisfy*

$$a_m \le a_0 + \sum_{n=1}^{m-1} a_n b_n \qquad \text{for} \quad m = 1, \dots, N,$$

then

$$a_m \le a_0 \ exp\left(\sum_{n=1}^{m-1} b_n \right) \qquad \text{for } m = 1, \dots, N. \tag{2.7}$$ pier1

Finally, we state another useful result for the sequel.

superlemma1 **Lemma 2.2.** *Assume that* a, $b \in \mathbb{R}$ *are strictly positive. Then*

$$(a - b) \le \big(\ln a^2 - \ln b^2 \big)(a + b). \tag{2.8}$$ pier2

Proof. We consider $a > b$ (if $b > a$ the technique of the proof is analogous) and obtain

$$(a - b) \le (\ln a^2 - \ln b^2)(a + b) = 2(\ln a - \ln b)(a + b) = 2 \ln \left(\frac{a}{b} \right)(a + b).$$

Then, dividing by b, we have that

$$\left(\frac{a}{b} - 1 \right) \le 2 \ln \left(\frac{a}{b} \right) \left(\frac{a}{b} + 1 \right). \tag{2.9}$$ lemmm12

Letting $x = a/b$, we can rewrite (2.9) as

$$(x - 1) \le 2(x + 1) \ln x \quad \text{ for } x \ge 1.$$

Now, we observe that (2.8) is verified if and only if the function

$$f(x) := 2(x + 1) \ln x - x + 1 \quad \text{is nonnegative for every } x \ge 1. \tag{2.10}$$ lemmm14

Since $f(1) = 0$ and $f'(x) > 0$ for every $x \ge 1$, we conclude that (2.10) holds. Then, the proof of the lemma is complete. □

In the following, the small-case symbol c stands for different constants which depend only on Ω, on the final time T, on the shape of the nonlinearities and on the constants and the norms of the functions involved in the assumptions of our statements. On the contrary, we use different symbols to denote precise constants to which we could refer. The reader should keep in mind that the meaning of c might change from line to line and even in the same chain of inequalities.

2.2 Statement of the problem and results

As far as the data of our problem are concerned, let ℓ and $k_0 > 0$ be two real constants. We also consider the data F, ϑ^*, ϑ_0 and χ_0 such that

$$F \in H^1(0, T; H) \cap L^1(0, T; L^\infty(\Omega)), \tag{2.11}$$ R

$$\vartheta^* \in W, \quad \vartheta^* > 0 \ \text{ in } \Omega, \tag{2.12}$$ kappa

$$\vartheta_0 \in V, \quad \vartheta_0 > 0 \ \text{ a.e. in } \Omega, \quad \ln \vartheta_0 \in H, \tag{2.13}$$ teta0

$$\chi_0 \in W. \tag{2.14}$$ chi0

Moreover, we introduce the functions $\tilde{\beta}$ and $\tilde{\pi}$, satisfying the conditions listed below:

$$\tilde{\beta} : \mathbb{R} \longrightarrow [0, +\infty] \text{ is lower semicontinuous and convex with } \tilde{\beta}(0) = 0, \tag{2.15}$$ beta

$$\tilde{\pi} \in C^1(\mathbb{R}) \text{ and } \pi \text{ is Lipschitz continuous.} \tag{2.16}$$ lambda

Since $\tilde{\beta}$ is proper, l.s.c. and convex, the subdifferential $\beta := \partial\tilde{\beta}$ is well defined. We denote by $D(\beta)$ and $D(\tilde{\beta})$ the effective domains of β and $\tilde{\beta}$, respectively. Thanks to these assumptions, β is a maximal monotone graph. Moreover, as $\tilde{\beta}$ takes on its minimum in 0, we have that $0 \in \beta(0)$. We also assume that

$$\begin{gathered}\chi_0 \in D(\beta) \text{ a.e. in } \Omega, \text{ and there exists } \xi_0 \in H \\ \text{such that } \quad \xi_0 \in \beta(\chi_0) \text{ a.e. in } \Omega,\end{gathered} \tag{2.17}$$ chi0b

whence

$$\tilde{\beta}(\chi_0) \in L^1(\Omega). \tag{2.18}$$ chi0c

Indeed, thanks to the definition of the subdifferential and to (2.15), we have that

$$0 \le \int_\Omega \tilde{\beta}(\chi_0) \le (\xi_0, \chi_0) \le \|\xi_0\|_H \|\chi_0\|_H.$$

In the following, the same symbol β will be used for the maximal monotone operators induced by β on $H \equiv L^2(\Omega)$ and $L^2(0,T;H) \equiv L^2(Q)$.

In our problem, the maximal monotone operator

$$A : H \longrightarrow H$$

also appears. We assume that

$$\begin{gathered}A \text{ is the subdifferential of a convex and l.s.c. function } \Phi : H \longrightarrow \mathbb{R} \\ \text{which takes its minimum in 0 and has at most a quadratic growth.}\end{gathered} \tag{2.19}$$ A1-bis

These properties are related to our assumptions on $A = \partial\Phi$, which read

$$0 \in A(0), \quad \exists\, C_A > 0 \text{ such that } \|y\|_H \le C_A(1 + \|x\|_H) \quad \forall\, x \in H, \ \forall\, y \in Ax. \tag{2.20}$$ stimaA

In the following, the same symbol A will be used for the maximal monotone operator induced on $L^2(0,T;H)$.

Examples of operators A. Let us consider the operator

$$\text{sign} \ : \mathbb{R} \longrightarrow 2^{\mathbb{R}}, \quad \text{sign}(r) = \begin{cases} \frac{r}{|r|} & \text{if } r \neq 0, \\ [-1,1] & \text{if } r = 0, \end{cases}$$

and its nonlocal counterpart in H, that is,

$$\text{Sign} : H \longrightarrow 2^H, \quad \text{Sign}(v) = \begin{cases} \frac{v}{\|v\|_H} & \text{if } v \neq 0, \\ B_1(0) & \text{if } v = 0, \end{cases}$$

where $B_1(0)$ denotes the closed unit ball of H. It is straightforward to check that Sign satisfies (2.19)–(2.20) and turns out to be the subdifferential of the norm function $v \mapsto \|v\|_H$. Concerning the graph sign, it is well known that it induces a maximal monotone operator in H which is the the subdifferential of the convex function $v \mapsto \int_\Omega |v|$.

Main result. Our aim is to find a quadruplet $(\vartheta, \chi, \zeta, \xi)$ satisfying the regularity conditions

$$\vartheta \in L^2(0,T;V), \tag{2.21}$$ reg1

$$\vartheta > 0 \ \text{ a.e. in } Q \quad \text{and} \quad \ln \vartheta \in H^1(0,T;V') \cap L^\infty(0,T;H), \tag{2.22}$$ reg2

$$\chi \in H^1(0,T;H) \cap C^0([0,T];V) \cap L^2(0,T;W), \tag{2.23}$$ reg3

$$\zeta \in L^2(0,T;H), \qquad \xi \in L^2(0,T;H), \tag{2.24}$$ reg4

and solving the Problem (P) defined by

$$\partial_t(\ln \vartheta(t) + \ell\chi(t)) - k_0\Delta\vartheta(t) + \zeta(t) = F(t) \quad \text{in } V', \text{ for a.e. } t \in (0,T), \tag{2.25}$$ iniziale1

$$\partial_t\chi - \Delta\chi + \xi + \pi(\chi) = \ell\vartheta \quad \text{a.e. in } Q, \tag{2.26}$$ iniziale2

$$\zeta(t) \in A(\vartheta(t) - \vartheta^*) \quad \text{for a.e. } t \in (0,T), \tag{2.27}$$ iniziale3

$$\xi \in \beta(\chi) \quad \text{a.e. in } Q, \tag{2.28}$$ iniziale4

$$\partial_\nu\vartheta = 0, \quad \partial_\nu\chi = 0 \quad \text{in the sense of traces on } \Sigma, \tag{2.29}$$ iniziale5

$$\ln \vartheta(0) = \ln \vartheta_0, \qquad \chi(0) = \chi_0 \quad \text{a.e. in } \Omega. \tag{2.30}$$ iniziale6

Here, we pointed out the boundary conditions (2.29) although they are already contained in the specified meaning of $-\Delta$ (cf. (2.2)). By the way, a variational formulation of (2.25) reads

$$\langle \partial_t(\ln \vartheta(t) + \ell\chi(t)) + \zeta(t), v\rangle + k_0 \int_\Omega \nabla\vartheta(t) \cdot \nabla v = \int_\Omega F(t)v \quad \text{for all } v \in V, \text{ for a.e. } t \in (0,T). \tag{2.31}$$ rifff

About the initial conditions in (2.30), note that from (2.22) it follows that $\ln \vartheta$ is at least weakly continuous from $[0,T]$ to H.

The following result is concerned with the existence of solutions to Problem (P).

eo-esistenza **Theorem 2.1.** *Assume* (2.11)–(2.20). *Then the Problem* (P) *stated by* (2.25)–(2.30) *has at least a solution* $(\vartheta, \chi, \zeta, \xi)$ *satisfying* (2.21)–(2.24) *and the regularity properties*

$$\vartheta \in L^\infty(0,T;V), \qquad \zeta \in L^\infty(0,T;H), \tag{2.32}$$ reg1th

$$\chi \in W^{1,\infty}(0,T;H) \cap H^1(0,T;V) \cap L^\infty(0,T;W), \qquad \xi \in L^\infty(0,T;H). \tag{2.33}$$ reg2th

The proof of Theorem 2.1 will be given in the subsequent three sections.

3 The approximating problem (P_τ)

In order to prove the existence theorem, first we introduce a backward finite differences scheme. Assume that N is a positive integer and let Z be any normed space. By fixing the time step

$$\tau = T/N, \quad N \in \mathbb{N},$$

we introduce the interpolation maps from Z^{N+1} into either $L^\infty(0,T;Z)$ or $W^{1,\infty}(0,T;Z)$. For $(z^0, z^1, \dots, z^N) \in Z^{N+1}$, we define the piecewise constant functions $\overline{z}_\tau$ and the piecewise linear functions $\widehat{z}_\tau$, respectively:

$$\begin{aligned}
&\overline{z}_\tau \in L^\infty(0,T;Z), \quad \overline{z}((i+s)\tau) = z^{i+1},\\
&\widehat{z}_\tau \in W^{1,\infty}(0,T;Z), \quad \widehat{z}((i+s)\tau) = z^i + s(z^{i+1} - z^i),\\
&\qquad\qquad \text{if } 0 < s < 1 \text{ and } i = 0, \dots, N-1. \qquad (3.1)
\end{aligned}$$ regtratti1

By a direct computation, it is straightforward to prove that

$$\|\overline{z}_\tau - \widehat{z}_\tau\|_{L^\infty(0,T;Z)} = \max_{i=0,\dots,N-1} \|z_{i+1} - z_i\|_Z = \tau \|\partial_t \widehat{z}_\tau\|_{L^\infty(0,T;Z)}, \qquad (3.2)$$ regtratti5

$$\|\overline{z}_\tau - \widehat{z}_\tau\|^2_{L^2(0,T;Z)} = \frac{\tau}{3} \sum_{i=0}^{N-1} \|z_{i+1} - z_i\|^2_Z = \frac{\tau^2}{3} \|\partial_t \widehat{z}_\tau\|^2_{L^2(0,T;Z)}, \qquad (3.3)$$ regtratti6

$$\begin{aligned}
\|\overline{z}_\tau - \widehat{z}_\tau\|^2_{L^\infty(0,T;Z)} &= \max_{i=0,\dots,N-1} \|z_{i+1} - z_i\|^2_Z\\
&\le \sum_{i=0}^{N-1} \tau^2 \left\| \frac{z_{i+1} - z_i}{\tau} \right\|^2_Z \le \tau \|\partial_t \widehat{z}_\tau\|^2_{L^2(0,T;Z)}. \qquad (3.4)
\end{aligned}$$ regtratti7

Then, we consider the approximating problem (P_τ). We set

$$F^i := \frac{1}{\tau} \int_{(i-1)\tau}^{i\tau} F(s)\, ds, \qquad \text{for } i = 1, \dots, N, \qquad (3.5)$$ Fapp-i

and we look for two vectors $(\vartheta^0, \vartheta^1, \dots, \vartheta^N) \in V^{N+1}$, $(\chi^0, \chi^1, \dots, \chi^N) \in W^{N+1}$ satisfying, for $i = 1, \dots, N$, the system

$$\vartheta^i > 0 \quad \text{a.e. in } \Omega, \quad \ln \vartheta^i \in H, \quad \exists\, \zeta^i,\, \xi^i \in H \quad \text{such that} \qquad (3.6)$$ D-ini-0

$$\tau^{1/2}\vartheta^i + \ln \vartheta^i + \ell\chi^i + \tau\zeta^i - \tau k_0 \Delta\vartheta^i = \tau F^i + \tau^{1/2}\vartheta^{i-1} + \ln \vartheta^{i-1} + \ell\chi^{i-1} \quad \text{a.e. in } \Omega, \qquad (3.7)$$ D-ini-1

$$\chi^i - \tau\Delta\chi^i + \tau\xi^i + \tau\pi(\chi^i) = \chi^{i-1} + \tau\ell\vartheta^i \quad \text{a.e. in } \Omega, \qquad (3.8)$$ D-ini-2

$$\zeta^i \in A(\vartheta^i - \vartheta^*), \qquad (3.9)$$ D-ini-3

$$\xi^i \in \beta(\chi^i) \quad \text{a.e. in } \Omega, \qquad (3.10)$$ D-ini-4

$$\partial_\nu \vartheta^i = \partial_\nu \chi^i = 0 \quad \text{a.e. on } \Gamma, \qquad (3.11)$$ D-ini-5

$$\vartheta^0 = \vartheta_0, \quad \chi^0 = \chi_0 \quad \text{a.e. in } \Omega. \qquad (3.12)$$ D-ini-6

In view of (2.11)–(2.14), we infer that for $i = 1$ the right-hand side of (3.7) is an element of H, and for any given χ^1 (present in the left-hand side) we have to find the corresponding ϑ^1, along with ξ^1, fulfilling (3.6)–(3.7) and (3.9); in case we succeed, from a comparison in (3.7) it will turn out that $\vartheta^1 \in W$. Then, we insert ϑ^1, depending on χ^1, in the right-hand side of (3.8) and we seek somehow a fixed point χ^1 , together with $\xi^1 \in H$, satisfying (3.8) and (3.10). Once we recover χ^1 and the related ϑ^1, we can start again our procedure, and so on. Then, it is important to show that, for a fixed i and known data F^i, ϑ^{i-1}, $\ln \vartheta^{i-1}$, χ^{i-1} we are able to find a pair (ϑ^i, χ^i) solving (3.6)–(3.11).

teotau **Theorem 3.1.** *There exists some fixed value $\tau_1 \le \min\{1, T\}$, depending only on the data, such that for any time step $0 < \tau < \tau_1$ the approximating problem (P_τ) stated by (3.6)–(3.12) has a unique solution*

$$(\vartheta^0, \vartheta^1, \ldots, \vartheta^N) \in V \times W^N, \quad (\chi^0, \chi^1, \ldots, \chi^N) \in W^{N+1}.$$

Let us now rewrite the discrete equation (3.7)–(3.12) by using the piecewise constant and piecewise linear functions defined in (3.1), with obvious notation, and obtain that

$$\tau^{1/2}\partial_t\widehat{\vartheta}_\tau + \partial_t\widehat{\ln\vartheta}_\tau + \ell\partial_t\widehat{\chi}_\tau + \overline{\zeta}_\tau - k_0\Delta\overline{\vartheta}_\tau = \overline{F}_\tau \quad \text{a.e. in } Q, \tag{3.13}$$ appr1

$$\partial_t\widehat{\chi}_\tau - \Delta\overline{\chi}_\tau + \overline{\xi}_\tau + \pi(\overline{\chi}_\tau) = \ell\overline{\vartheta}_\tau \quad \text{a.e. in } Q, \tag{3.14}$$ appr2

$$\overline{\zeta}_\tau(t) \in A(\overline{\vartheta}_\tau(t) - \vartheta^*) \quad \text{for a.e. } t \in (0, T), \tag{3.15}$$ appr3

$$\overline{\xi}_\tau \in \beta(\overline{\chi}_\tau) \quad \text{a.e. in } Q, \tag{3.16}$$ appr4

$$\partial_\nu\overline{\vartheta}_\tau = \partial_\nu\overline{\chi}_\tau = 0 \quad \text{a.e. on } \Sigma, \tag{3.17}$$ appr5

$$\widehat{\vartheta}_\tau(0) = \vartheta_0, \quad \widehat{\chi}_\tau(0) = \chi_0 \quad \text{a.e. in } \Omega. \tag{3.18}$$ appr6

3.1 The auxiliary approximating problem (AP_ε)

In this subsection we introduce the auxiliary approximating problem (AP_ε) obtained by considering the approximating problem (P_τ) at each step $i = 1, \ldots, N$ and replacing the monotone operators appearing in (3.6)–(3.12) with their Yosida regularizations. About general properties of maximal monotone operators and subdifferentials of convex functions, we refer the reader to [1, 10].

Yosida regularization of ln**.** We introduce the Yosida regularization of ln. For $\varepsilon > 0$ we set

$$\ln_\varepsilon : \mathbb{R} \longrightarrow \mathbb{R}, \quad \ln_\varepsilon := \frac{I - (I + \varepsilon\ln)^{-1}}{\varepsilon}. \tag{3.19}$$ richiamolog1

where I denotes the identity. We remark that $\ln_\varepsilon$ is monotone, Lipschitz continuous (with Lipschitz constant $1/\varepsilon$) and satisfies the following properties: denoting by $L_\varepsilon = (I + \varepsilon\ln)^{-1}$ the resolvent operator, we have that

$$\ln_\varepsilon(x) \in \ln(L_\varepsilon x) \quad \text{for all } x \in \mathbb{R},$$
$$|\ln_\varepsilon(x)| \le |\ln(x)|, \quad \lim_{\varepsilon \searrow 0} \ln_\varepsilon(x) = \ln(x) \quad \text{for all } x > 0.$$

We also introduce the nonnegative and convex functions

$$\Lambda(x) = \int_1^x \ln r \, dr, \quad \Lambda_\varepsilon(y) = \int_1^y \ln_\varepsilon r \, dr \quad \text{for all } x > 0 \text{ and } y \in \mathbb{R}. \tag{3.20}$$ defLam

Note that the graph $x \mapsto \ln x$ is nothing but the subdifferential of the convex function Λ extended by lower semicontinuity in 0 and with value $+\infty$ for $x < 0$. On the other hand, Λ_ε coincides with the Moreau–Yosida regularization of Λ and, in particular, we have that

$$0 \le \Lambda_\varepsilon(x) \le \Lambda(x) \quad \text{for every } x > 0. \tag{3.21}$$ propLam

Yosida regularization of A. We introduce the Yosida regularization of A. For $\varepsilon > 0$ we define

$$A_\varepsilon : H \longrightarrow H, \qquad A_\varepsilon = \frac{I - (I + \varepsilon A)^{-1}}{\varepsilon}. \tag{3.22}$$

richiamoA1

Note that A_ε is Lipschitz-continuous (with Lipschitz constant $1/\varepsilon$) and maximal monotone in H. Moreover, A satisfies the following properties: denoting by $J_\varepsilon = (I + \varepsilon A)^{-1}$ the resolvent operator, for all $\delta > 0$ and for all $x \in H$, we have that

$$A_\varepsilon x \in A(J_\varepsilon x), \tag{3.23}$$

AJe

$$\|A_\varepsilon x\|_H \le \|A^\circ x\|_H, \quad \lim_{\varepsilon \searrow 0} \|A_\varepsilon x - A^\circ x\|_H = 0, \tag{3.24}$$

pier5

where $A^\circ x$ is the element of the range of A having minimal norm. Let us point out a key property of A_ε, which is a consequence of (2.20): indeed, there holds

$$\|A_\varepsilon x\|_H \le C_A(1 + \|x\|_H) \quad \text{for all } x \in H. \tag{3.25}$$

Ae

Notice that $0 \in A(0)$ and $0 \in I(0)$: consequently, for every $\varepsilon > 0$ we infer that $J_\varepsilon(0) = 0$. Moreover, since A is maximal monotone, J_ε is a contraction. Then, from (2.20) and (3.23) it follows that

$$\|A_\varepsilon x\|_H \le C_A(\|J_\varepsilon x\|_H + 1) \le C_A(\|J_\varepsilon x - J_\varepsilon 0\|_H + 1) \le C_A(\|x\|_H + 1) \quad \text{for every } x \in H.$$

Yosida regularization of β. We introduce the Yosida regularization of β. For $\varepsilon > 0$ we define

$$\beta_\varepsilon : \mathbb{R} \longrightarrow \mathbb{R}, \qquad \beta_\varepsilon = \frac{I - (I + \varepsilon\beta)^{-1}}{\varepsilon}. \tag{3.26}$$

richiamobeta

We remark that β_ε is Lipschitz continuous (with Lipschitz constant $1/\varepsilon$) and satisfies the following properties: denoting by $R_\varepsilon = (I + \varepsilon\beta)^{-1}$ the resolvent operator, we have that

$$\beta_\varepsilon(x) \in \beta(R_\varepsilon x) \quad \text{for all } x \in \mathbb{R},$$
$$|\beta_\varepsilon(x)| \le |\beta^\circ(x)|, \quad \lim_{\varepsilon \searrow 0} \beta_\varepsilon(x) = \beta^\circ(x) \quad \text{for all } x \in D(\beta),$$

where $\beta^\circ(x)$ is the element of the range of $\beta(x)$ having minimal modulus. We also introduce the Moreau–Yosida regularization of $\tilde\beta$. For $\varepsilon > 0$ and $x \in \mathbb{R}$ we define

$$\tilde\beta_\varepsilon : \mathbb{R} \longrightarrow [0, +\infty], \qquad \tilde\beta_\varepsilon(x) := \min_{y \in \mathbb{R}} \left\{ \tilde\beta(y) + \frac{1}{2\varepsilon}|x - y|^2 \right\}$$

and recall that

$$\tilde\beta_\varepsilon(x) \le \tilde\beta(x) \quad \text{for every } x \in \mathbb{R}.$$

We also observe that β_ε is the derivative of $\tilde\beta_\varepsilon$. Then, for every $x_1, x_2 \in \mathbb{R}$ we have that

$$\tilde\beta_\varepsilon(x_2) = \tilde\beta_\varepsilon(x_1) + \int_{x_1}^{x_2} \beta_\varepsilon(s)\, ds.$$

Definition of the auxiliary approximating problem (AP_ε). We fix τ and specify an auxiliary approximating problem (AP_ε), which is obtained by considering (3.6)–(3.11) for a fixed i and introducing the regularized operators defined above. We set

$$g := \tau F^i + \tau^{1/2}\vartheta^{i-1} + \ln\vartheta^{i-1} + \ell\chi^{i-1}, \qquad h := \chi^{i-1}, \tag{3.27}$$ gh

and note that both g and h are prescribed elements of H (cf. (3.5), (2.11), (2.13), (2.14) and (3.6)). We look for a pair $(\Theta_\varepsilon, X_\varepsilon)$ such that

$$\tau^{1/2}\Theta_\varepsilon + \ln_\varepsilon\Theta_\varepsilon + \tau A_\varepsilon(\Theta_\varepsilon - \vartheta^*) - \tau k_0\Delta\Theta_\varepsilon = -\ell X_\varepsilon + g \quad \text{a.e. in } \Omega, \tag{3.28}$$ E-D-ini-1

$$X_\varepsilon - \tau\Delta X_\varepsilon + \tau\beta_\varepsilon(X_\varepsilon) + \tau\pi(X_\varepsilon) = h + \tau\ell\Theta_\varepsilon \quad \text{a.e. in } \Omega, \tag{3.29}$$ E-D-ini-2

where $\ln_\varepsilon$, A_ε and β_ε are the Yosida regularization of ln, A and β defined by (3.19), (3.22) and (3.26), respectively. Here, according to the extended meaning of $-\Delta$ (see (2.2)), we omit the specification of the boundary conditions as with (3.11).

teotau2 **Theorem 3.2.** *Let g, $h \in H$. Then there exists some fixed value $\tau_2 \le \min\{1, T\}$, depending only on the data, such that for every time step $\tau \in (0, \tau_2)$ and for all $\varepsilon \in (0, 1]$ the auxiliary approximating problem (AP_ε) stated by (3.28)–(3.29) has a unique solution $(\Theta_\varepsilon, X_\varepsilon)$.*

3.2 Existence of a solution for (AP_ε)

In order to prove the existence of the solution for the auxiliary approximating problem (AP_ε) we intend to apply [1, Corollary 1.3, p. 48]. To this aim, we point out that, for τ small enough, the two operators

$$[\tau^{1/2}I + \ln_\varepsilon + \tau A_\varepsilon(\cdot - \vartheta^*) - \tau k_0\Delta] \quad \text{appearing in (3.28),} \quad \boxed{\texttt{oper1}} \tag{3.30}$$ oper1

$$[I + \tau\beta_\varepsilon + \tau\pi - \tau\Delta] \quad \text{appearing in (3.29),} \quad \boxed{\texttt{oper2}} \tag{3.31}$$ oper2

both with domain W and range H, are maximal monotone and coercive. Indeed, they are the sum of a monotone, Lipschitz continuous and coercive operator:

$$\tau^{1/2}I + \ln_\varepsilon + \tau A_\varepsilon(\cdot - \vartheta^*) \quad \text{in (3.30), and} \quad I + \tau\beta_\varepsilon + \tau\pi \quad \text{in (3.31),}$$

and of a maximal monotone operator that is $-\Delta$ with a positive coefficient in front. We now check our first claim. Letting v_1, $v_2 \in H$, we have that

$$\begin{aligned}
&\big((\tau^{1/2}I + \ln_\varepsilon + \tau A_\varepsilon(\cdot - \vartheta^*))(v_1) - (\tau^{1/2}I + \ln_\varepsilon + \tau A_\varepsilon(\cdot - \vartheta^*))(v_2), v_1 - v_2\big) \\
&\ge \tau^{1/2}\|v_1 - v_2\|_H^2 + \big(\ln_\varepsilon(v_1) - \ln_\varepsilon(v_2), v_1 - v_2\big) \\
&\quad + \tau\big(A_\varepsilon(v_1 - \vartheta^*) - A_\varepsilon(v_2 - \vartheta^*), (v_1 - \vartheta^*) - (v_2 - \vartheta^*)\big).
\end{aligned}$$

Due to the monotonicity of $\ln_\varepsilon$ and A_ε, we have that the last two terms on the right-hand side are nonnegative, so that

$$\begin{aligned}
&\big((\tau^{1/2}I + \ln_\varepsilon + \tau A_\varepsilon(\cdot - \vartheta^*))(v_1) - (\tau^{1/2}I + \ln_\varepsilon + \tau A_\varepsilon(\cdot - \vartheta^*))(v_2), v_1 - v_2\big) \\
&\ge \tau^{1/2}\|v_1 - v_2\|_H^2,
\end{aligned} \tag{3.32}$$ monop21

i.e., the operator $\tau^{1/2}I + \ln_\varepsilon + \tau A_\varepsilon(\cdot - \vartheta^*)$ is strongly monotone, hence coercive, in H. Next, for all v_1, $v_2 \in H$ we have that

$$\begin{aligned}&((I + \tau\beta_\varepsilon + \tau\pi)(v_1) - (I + \tau\beta_\varepsilon + \tau\pi)(v_2), v_1 - v_2)\\ &\geq \|v_1 - v_2\|_H^2 + \tau(\beta_\varepsilon(v_1) - \beta_\varepsilon(v_2), v_1 - v_2) - C_\pi\tau\|v_1 - v_2\|_H^2. \end{aligned} \tag{3.33}$$ mon1

where C_π denotes a Lipschitz constant for π. Since β_ε is monotone, it turns out that

$$\tau(\beta_\varepsilon(v_1) - \beta_\varepsilon(v_2), v_1 - v_2) \geq 0$$

and, choosing $\tau_2 \leq 1/2C_\pi$, from (3.33) we infer that

$$((I + \tau\beta_\varepsilon + \tau\pi)(v_1) - (I + \tau\beta_\varepsilon + \tau\pi)(v_1), v_1 - v_2) \geq \frac{1}{2}\|v_1 - v_2\|_H^2, \tag{3.34}$$ maxmonop2

whence the operator $I+\tau\beta_\varepsilon+\tau\pi$ is strongly monotone and coercive in H, for every $\tau \leq \tau_2$.

Now, in order to prove Theorem 3.2, we divide the proof into two steps. In the first step, we fix $\overline{\Theta}_\varepsilon \in H$ in place of Θ_ε on the right-hand side of (3.29) and find a solution X_ε for (3.29). In the second step, we insert on the right-hand side of (3.28) the element X_ε obtained in the first step and find a solution Θ_ε to (3.28). Now, let $\overline{\Theta}_{1,\varepsilon}$ and $\overline{\Theta}_{2,\varepsilon}$ be two different input data. We denote by $X_{1,\varepsilon}$, $X_{2,\varepsilon}$ the corresponding solutions for (3.29) obtained in the first step and by $\Theta_{1,\varepsilon}$, $\Theta_{2,\varepsilon}$ the related solution of (3.28) found in the second step.

Hence, taking the difference between the two equations (3.29) written for $\overline{\Theta}_{1,\varepsilon}$ and $\overline{\Theta}_{2,\varepsilon}$ and testing the result by $(X_{1,\varepsilon} - X_{2,\varepsilon})$, we have that

$$\begin{aligned}&\big((I + \tau\beta_\varepsilon + \tau\pi)(X_{1,\varepsilon}) - (I + \tau\beta_\varepsilon + \tau\pi)(X_{2,\varepsilon}), X_{1,\varepsilon} - X_{2,\varepsilon}\big)\\ &+ \tau\int_\Omega |\nabla(X_{1,\varepsilon} - X_{2,\varepsilon})|^2 \leq \tau\ell\big(\overline{\Theta}_{1,\varepsilon} - \overline{\Theta}_{2,\varepsilon}, X_{1,\varepsilon} - X_{2,\varepsilon}\big).\end{aligned} \tag{3.35}$$ intermezz1

Then, applying (3.34) and (2.5) to the first term on the left-hand side of (3.35) and to the right-hand side of (3.35), respectively, we infer that

$$\frac{1}{2}\|X_{1,\varepsilon} - X_{2,\varepsilon}\|_H^2 + \tau\int_\Omega |\nabla(X_{1,\varepsilon} - X_{2,\varepsilon})|^2 \leq \frac{1}{4}\|X_{1,\varepsilon} - X_{2,\varepsilon}\|_H^2 + \tau^2\ell^2\|\overline{\Theta}_{1,\varepsilon} - \overline{\Theta}_{2,\varepsilon}\|_H^2,$$

whence

$$\|X_{1,\varepsilon} - X_{2,\varepsilon}\|_H^2 \leq 4\tau^2\ell^2\|\overline{\Theta}_{1,\varepsilon} - \overline{\Theta}_{2,\varepsilon}\|_H^2. \tag{3.36}$$ ultimastep1

Now, we take the difference between the corresponding equations (3.28) written for the solutions $X_{1,\varepsilon}$, $X_{2,\varepsilon}$ obtained in the first step and test by $(\Theta_{1,\varepsilon} - \Theta_{2,\varepsilon})$. We obtain that

$$\begin{aligned}&\big((\tau^{1/2}I + \ln_\varepsilon + \tau A_\varepsilon(\cdot - \vartheta^*))(\Theta_{1,\varepsilon}) - (\tau^{1/2}I + \ln_\varepsilon + \tau A_\varepsilon(\cdot - \vartheta^*))(\Theta_{2,\varepsilon}), \Theta_{1,\varepsilon} - \Theta_{2,\varepsilon}\big)\\ &+ \tau k_0\int_\Omega |\nabla(\Theta_{1,\varepsilon} - \Theta_{2,\varepsilon})|^2 \leq \frac{\ell^2}{2\tau^{1/2}}\|X_{1,\varepsilon} - X_{2,\varepsilon}\|_H^2 + \frac{\tau^{1/2}}{2}\|\Theta_{1,\varepsilon} - \Theta_{2,\varepsilon}\|_H^2.\end{aligned} \tag{3.37}$$ ultimastep2

By recalling (3.32) and using it in the left-hand side of (3.37) we infer that

$$\tau^{1/2}\|\Theta_{1,\varepsilon} - \Theta_{2,\varepsilon}\|_H^2 \leq \frac{\ell^2}{\tau^{1/2}}\|X_{1,\varepsilon} - X_{2,\varepsilon}\|_H^2$$

Then, by combining this inequality with (3.36), we deduce that

$$\|\Theta_{1,\varepsilon} - \Theta_{2,\varepsilon}\|_H^2 \le 4\tau\ell^4 \|\overline{\Theta}_{1,\varepsilon} - \overline{\Theta}_{2,\varepsilon}\|_H^2, \tag{3.38}$$ contrazione1

whence we obtain a contraction mapping for every $\tau \le \tau_2$, provided that $\tau_2 \le 1/(8\ell^4)$. Finally, by applying the Banach fixed point theorem, we conclude that there exists a unique solution $(\Theta_\varepsilon, X_\varepsilon)$ to the auxiliary problem (AP_ε).

3.3 A priori estimates on AP_ε

In this subsection we derive a series of a priori estimates, independent of ε, inferred from the equations (3.28)–(3.29) of the auxiliary approximating problem (AP_ε).

First a priori estimate. We test (3.28) by $\tau(\Theta_\varepsilon - \vartheta^*)$ and (3.29) by X_ε, then we sum up. By exploiting the cancellation of the suitable corresponding terms and recalling the definition (3.20) of Λ_ε, we obtain that

$$\begin{aligned}
&\tau^{3/2}\|\Theta_\varepsilon - \vartheta^*\|_H^2 + \tau\Lambda_\varepsilon(\Theta_\varepsilon) + \tau^2(A_\varepsilon(\Theta_\varepsilon - \vartheta^*), \Theta_\varepsilon - \vartheta^*) + \tau^2 k_0 \int_\Omega |\nabla(\Theta_\varepsilon - \vartheta^*)|^2 \\
&+ \big((I + \tau\beta_\varepsilon + \tau\pi)(X_\varepsilon) - (I + \tau\beta_\varepsilon + \tau\pi)(0), X_\varepsilon\big) + \tau \int_\Omega |\nabla X_\varepsilon|^2 \\
&\le -\tau^{3/2}(\vartheta^*, \Theta_\varepsilon - \vartheta^*) + \tau\Lambda_\varepsilon(\vartheta^*) - \tau^2 k_0 \int_\Omega \nabla\vartheta^* \cdot \nabla(\Theta_\varepsilon - \vartheta^*) \\
&\quad + \ell\tau(X_\varepsilon, \vartheta^*) + \tau(g, \Theta_\varepsilon - \vartheta^*) - \tau(\pi(0), X_\varepsilon) + (h, X_\varepsilon).
\end{aligned} \tag{3.39}$$ stma-1

Let us note that all terms on the left-hand side are nonnegative; in particular, recalling (3.34), we have that

$$\big((I + \tau\beta_\varepsilon + \tau\pi)(X_\varepsilon) - (I + \tau\beta_\varepsilon + \tau\pi)(0), X_\varepsilon\big) \ge \frac{1}{2}\|X_\varepsilon\|_H^2, \tag{3.40}$$ stma-1au1

Due to (2.12) and the continuity of the positive function ϑ^*, (3.21) helps us in estimating the second term on the right-hand side of (3.39):

$$\tau\Lambda_\varepsilon(\vartheta^*) \le \tau\Lambda(\vartheta^*) \le c\,\tau. \tag{3.41}$$

Since g, $h \in H$ and (2.12) holds, by applying the Young inequality (2.5) to the other terms on the right-hand side of (3.39), we find that

$$-\tau^{3/2}(\vartheta^*, \Theta_\varepsilon - \vartheta^*) \le \frac{\tau^{3/2}}{4}\|\Theta_\varepsilon - \vartheta^*\|_H^2 + c\,\tau^{3/2}, \tag{3.42}$$

$$-\tau^2 k_0 \int_\Omega \nabla\vartheta^* \cdot \nabla(\Theta_\varepsilon - \vartheta^*) \le \frac{\tau^2 k_0}{2} \int_\Omega |\nabla(\Theta_\varepsilon - \vartheta^*)|^2 + c\,\tau^2, \tag{3.43}$$

$$\ell\tau(X_\varepsilon, \vartheta^*) \le \frac{1}{8}\|X_\varepsilon\|_H^2 + c\,\tau^2, \tag{3.44}$$

$$\tau(g, \Theta_\varepsilon - \vartheta^*) \le \frac{\tau^{3/2}}{4}\|\Theta_\varepsilon - \vartheta^*\|_H^2 + c\,\tau^{1/2}, \tag{3.45}$$

$$-\tau(\pi(0), X_\varepsilon) \le \frac{1}{8}\|X_\varepsilon\|_H^2 + c\,\tau^2, \qquad (h, X_\varepsilon) \le \frac{1}{8}\|X_\varepsilon\|_H^2 + c. \tag{3.46}$$ stma-1au2

Then, in view of (3.40)–(3.46), from (3.39) and (2.12) it is not difficult to infer that

$$\tau^{3/4}\|\Theta_\varepsilon\|_H + \tau\|\nabla\Theta_\varepsilon\|_H + \|X_\varepsilon\|_H + \tau^{1/2}\|\nabla X_\varepsilon\|_H \le c \tag{3.47}$$ stima-1

taking into account that $\tau \le \tau_2$.

Second a priori estimate. We test (3.29) by $\beta_\varepsilon(X_\varepsilon)$ and obtain that

$$\begin{aligned}&\big(X_\varepsilon, \beta_\varepsilon(X_\varepsilon)\big) + \tau\int_\Omega \beta_\varepsilon'(X_\varepsilon)|\nabla X_\varepsilon|^2 + \tau\int_\Omega |\beta_\varepsilon(X_\varepsilon)|^2 \\ &\le -\tau\int_\Omega \pi(X_\varepsilon)\beta_\varepsilon(X_\varepsilon) + \tau\ell\int_\Omega \Theta_\varepsilon\beta_\varepsilon(X_\varepsilon) + \int_\Omega h\beta_\varepsilon(X_\varepsilon).\end{aligned} \tag{3.48}$$ stma-2

Thanks to the monotonicity of β_ε and to the condition $\beta_\varepsilon(0) = 0$, the terms on the left-hand side are nonnegative. As π is Lipschitz continuous, by applying the Young inequality (2.5) to every term on the right-hand side of (3.48) and using (3.47), for $0 < \tau \le 1$ we obtain that

$$-\tau\int_\Omega \pi(X_\varepsilon)\beta_\varepsilon(X_\varepsilon) \le \frac{\tau}{4}\int_\Omega |\beta_\varepsilon(X_\varepsilon)|^2 + c, \tag{3.49}$$ grill1

$$\tau\ell\int_\Omega \Theta_\varepsilon\beta_\varepsilon(X_\varepsilon) \le \frac{\tau}{4}\int_\Omega |\beta_\varepsilon(X_\varepsilon)|^2 + \frac{c}{\tau^{1/2}}, \tag{3.50}$$ grill2

$$\int_\Omega h\beta_\varepsilon(X_\varepsilon) \le \frac{\tau}{4}\int_\Omega |\beta_\varepsilon(X_\varepsilon)|^2 + \frac{c}{\tau}. \tag{3.51}$$ grill3

Then, owing to (3.49)–(3.51), from (3.48) it follows that

$$\tau\|\beta_\varepsilon(X_\varepsilon)\|_H^2 \le c\big(1+\tau^{-1}\big), \quad \text{so that} \quad \tau\|\beta_\varepsilon(X_\varepsilon)\|_H \le c. \tag{3.52}$$ stima-12

Hence, by comparison in (3.29), we conclude that $\tau\|\Delta X_\varepsilon\|_H \le c$ and, from (3.47) and standard elliptic regularity results,

$$\tau\|X_\varepsilon\|_W \le c. \tag{3.53}$$ stima-13

Third a priori estimate. Recalling (3.25), (2.12) and (3.47), we immediately deduce that

$$\tau\|A_\varepsilon(\Theta_\varepsilon - \vartheta^*)\|_H \le \tau\, C_A(1 + \|\Theta_\varepsilon\|_H + \|\vartheta^*\|_H) \le c. \tag{3.54}$$ pier6

Next, we test (3.28) by $\ln_\varepsilon\Theta_\varepsilon$ and obtain that

$$\begin{aligned}&\|\ln_\varepsilon\Theta_\varepsilon\|_H^2 + \tau k_0\int_\Omega \ln_\varepsilon'(\Theta_\varepsilon)|\nabla\Theta_\varepsilon|^2 \le -\tau^{1/2}(\Theta_\varepsilon, \ln_\varepsilon\Theta_\varepsilon) \\ &-\tau\big(A_\varepsilon(\Theta_\varepsilon - \vartheta^*), \ln_\varepsilon\Theta_\varepsilon\big) - \ell(X_\varepsilon, \ln_\varepsilon\Theta_\varepsilon) + (g, \ln_\varepsilon\Theta_\varepsilon).\end{aligned}$$

Then, by applying the Cauchy–Schwarz inequality to every term on the right-hand side and using (3.47) and (3.54), we infer that

$$\|\ln_\varepsilon\Theta_\varepsilon\|_H \le \tau^{1/2}\|\Theta_\varepsilon\|_H + c \le c\big(\tau^{-1/4} + 1\big),$$

whence

$$\tau^{1/4}\|\ln_\varepsilon\Theta_\varepsilon\|_H \le c. \tag{3.55}$$ stima-14

Moreover, due to (3.55) and (3.47), by comparison in (3.28) it is straightforward to see that $\tau^{5/4}\|\Delta\Theta_\varepsilon\|_H \le c$ and consequently

$$\tau^{5/4}\|\Theta_\varepsilon\|_W \le c. \tag{3.56}$$ stima-15

3.4 Passage to the limit as $\varepsilon \searrow 0$

In this subsection we pass to the limit as $\varepsilon \searrow 0$ and prove that the limit of subsequences of solutions $(\Theta_\varepsilon, X_\varepsilon)$ for (AP_ε) (see (3.28)–(3.29)) yields a solution (ϑ^i, χ^i) to (3.6)–(3.10); then, we can conclude that the problem (P_τ) has a solution.

Since the constants appearing in (3.47) and (3.52)–(3.56) do not depend on ε, we infer that, at least for a subsequence, there exist some limit functions $(\vartheta^i, \chi^i, L^i, Z^i, B^i)$ such that

$$\Theta_\varepsilon \rightharpoonup \vartheta^i \quad \text{and} \quad X_\varepsilon \rightharpoonup \chi^i \quad \text{in } W, \tag{3.57}$$ ld1

$$\ln_\varepsilon(\Theta_\varepsilon) \rightharpoonup L^i, \quad A_\varepsilon(\Theta_\varepsilon - \vartheta^*) \rightharpoonup Z^i \quad \text{and} \quad \beta_\varepsilon(X_\varepsilon) \rightharpoonup B^i \quad \text{in } H, \tag{3.58}$$ ld4

as $\varepsilon \searrow 0$. Thanks to the well-known compact embedding $W \subset V$, from (3.57) we infer that

$$\Theta_\varepsilon \to \vartheta^i \quad \text{and} \quad X_\varepsilon \to \chi^i \quad \text{in } V. \tag{3.59}$$ ldf1

Besides, as π is Lipschitz continuous, we have that $|\pi(X_\varepsilon) - \pi(\chi^i)| \le C_\pi |X_\varepsilon - \chi^i|$, whence, thanks to (3.59), we obtain that

$$\pi(X_\varepsilon) \to \pi(\chi^i) \qquad \text{in } H, \tag{3.60}$$ cvgpigr

as $\varepsilon \searrow 0$. Now, we pass to the limit on $\ln_\varepsilon(\Theta_\varepsilon)$, $A_\varepsilon(\Theta_\varepsilon - \vartheta^*)$ and $\beta_\varepsilon(X_\varepsilon)$. In view of a general convergence result involving maximal monotone operators (see, e.g., [1, Proposition 1.1, p. 42]), thanks to the strong convergences in H ensured by (3.59) and to the weak convergences in (3.58), we conclude that

$$L^i \in \ln(\chi^i), \quad Z^i \in A(\vartheta^i - \vartheta^*), \quad B^i \in \beta(\chi^i). \tag{3.61}$$ convoperator

In conclusion, using (3.57)–(3.61) and recalling (3.27), we can pass to the limit as $\varepsilon \searrow 0$ in (3.28)–(3.29) so to obtain (3.6)–(3.10) for the limiting functions ϑ^i and χ^i.

3.5 Uniqueness of the solution of (P_τ)

In this section we prove that the approximating problem (P_τ) stated by (3.6)–(3.12) has a unique solution. Then, the proof of Theorem 3.1 will be complete.

We write problem (P_τ) for two solutions $(\vartheta_1^i, \chi_1^i)$, $(\vartheta_2^i, \chi_2^i)$ and set $\vartheta^i := \vartheta_1^i - \vartheta_2^i$ and $\chi^i := \chi_1^i - \chi_2^i$, $i = 1, \dots, N$. Then, we multiply by $\tau\vartheta^i$ the difference between the corresponding equations (3.7) and by χ^i the difference between the corresponding equations (3.8). Adding the resultant equations, we obtain that

$$\begin{aligned} &\tau^{3/2} \|\vartheta^i\|_H^2 + \tau \big(\ln \vartheta_1^i - \ln \vartheta_2^i, \vartheta_1^i - \vartheta_2^i \big) + \tau^2 \big(\zeta_1^i - \zeta_2^i, \vartheta_1^i - \vartheta^* - (\vartheta_2^i - \vartheta^*) \big) + \tau^2 \int_\Omega |\nabla \vartheta^i|^2 \\ &+ \|\chi^i\|_H^2 + \tau \int_\Omega |\nabla \chi^i|^2 + \tau (\xi_1^i - \xi_2^i, \chi_1^i - \chi_2^i) = - \tau \big(\pi(\chi_1^i) - \pi(\chi_2^i), \chi_1^i - \chi_2^i \big). \end{aligned} \tag{3.62}$$ nuoobvauni1

Since ln, A and β are monotone, in view of (3.9) and (3.10) the second, the third and the seventh term on the left-hand side of (3.62) are nonnegative. Besides, if $\tau \le 1/(2C_\pi)$, thanks to the Lipschitz continuity of π, the right-hand side of (3.62) can be estimated as

$$- \tau \big(\pi(\chi_1^i) - \pi(\chi_2^i), \chi_1^i - \chi_2^i \big) \le \frac{1}{2} \|\chi^i\|_H^2. \tag{3.63}$$ nuoobvauni2

Then, due to (3.63), from (3.62) we infer that

$$\tau^{3/2}\|\vartheta^i\|_H^2 + \tau^2 \int_\Omega |\nabla\vartheta^i|^2 + \frac{1}{2}\|\chi^i\|_H^2 + \tau \int_\Omega |\nabla\chi^i|^2 \le 0, \tag{3.64}$$

whence we easily conclude that $\vartheta^i = \chi^i = 0$, i.e., $\vartheta_1^i = \vartheta_2^i$ and $\chi_1^i = \chi_2^i$ for $i = 1, \dots, N$.

4 A priori estimates on (AP_τ)

In this section we deduce some uniform estimates, independent of τ and inferred from the equations (3.6)–(3.12) of the approximating problem (P_τ).

First uniform estimate. We test (3.7) by ϑ^i and (3.8) by $(\chi^i - \chi^{i-1})/\tau$, then we sum up. Adding $(\chi^i, \chi^i - \chi^{i-1})$ to both sides of the resulting equality and exploiting the cancellation of the suitable corresponding terms, we obtain that

$$\begin{aligned}
&\tau^{1/2}(\vartheta^i - \vartheta^{i-1}, \vartheta^i) + \big(\ln\vartheta^i - \ln\vartheta^{i-1}, \vartheta^i\big) + \tau(\zeta^i, \vartheta^i - \vartheta^*) + \tau k_0 \int_\Omega |\nabla\vartheta^i|^2 \\
&+ \tau \left\| \frac{\chi^i - \chi^{i-1}}{\tau} \right\|_H^2 + (\chi^i, \chi^i - \chi^{i-1}) + (\nabla\chi^i, \nabla\chi^i - \nabla\chi^{i-1}) + (\xi^i, \chi^i - \chi^{i-1}) \\
&= -\tau(\zeta^i, \vartheta^*) + \tau(F^i, \vartheta^i) - \big(\pi(\chi^i) - \chi^i, \chi^i - \chi^{i-1}\big).
\end{aligned} \tag{4.1}$$

stimacomple1

Due to (2.6), we can rewrite the first, the fifth and the sixth term on the left-hand side of (4.1) as

$$\tau^{1/2}(\vartheta^i - \vartheta^{i-1}, \vartheta^i) = \frac{\tau^{1/2}}{2}\|\vartheta^i\|_H^2 - \frac{\tau^{1/2}}{2}\|\vartheta^{i-1}\|_H^2 + \frac{\tau^{1/2}}{2}\|\vartheta^i - \vartheta^{i-1}\|_H^2, \tag{4.2}$$

help1

$$(\chi^i, \chi^i - \chi^{i-1}) + (\nabla\chi^i, \nabla\chi^i - \nabla\chi^{i-1}) = \frac{1}{2}\|\chi^i\|_V^2 - \frac{1}{2}\|\chi^{i-1}\|_V^2 + \frac{1}{2}\|\chi^i - \chi^{i-1}\|_V^2. \tag{4.3}$$

Moreover, since the function $u \longmapsto e^u$ is convex and e^u turns out to be its subdifferential, by setting $u^i = \ln\vartheta^i$ we obtain that

$$\big(\ln\vartheta^i - \ln\vartheta^{i-1}, \vartheta^i\big) = \big(u^i - u^{i-1}, e^{u^i}\big) \ge \int_\Omega e^{u^i} - \int_\Omega e^{u^{i-1}} = \|\vartheta^i\|_{L^1(\Omega)} - \|\vartheta^{i-1}\|_{L^1(\Omega)}. \tag{4.4}$$

Recalling that A is a maximal monotone operator and $0 \in A(0)$, by (3.9) the third term on the left-hand side of (4.1) is nonnegative. We also notice that, since β is the subdifferential of $\tilde\beta$, from (3.10) it follows that

$$(\xi^i, \chi^i - \chi^{i-1}) \ge \int_\Omega \tilde\beta(\chi^i) - \int_\Omega \tilde\beta(\chi^{i-1}), \tag{4.5}$$

while, due to (2.3), (2.5) and the sub-linear growth of A stated by (2.20), we deduce that

$$\begin{aligned}
-\tau(\zeta^i, \vartheta^*) &\le C_A\tau(1 + \|\vartheta^i - \vartheta^*\|_H)\|\vartheta^*\|_H \le c\tau(1 + \|\vartheta^i\|_H) \le c\tau(1 + \|\vartheta^i\|_V) \\
&\le c\tau(1 + \|\vartheta^i\|_{L^1(\Omega)} + \|\nabla\vartheta^i\|_H) \le c\tau + \tau C_1\|\vartheta^i\|_{L^1(\Omega)} + \tau\frac{k_0}{2}\|\nabla\vartheta^i\|_H^2,
\end{aligned} \tag{4.6}$$

where we have applied the Young inequality in the last term and where the constant C_1 depends on C_A, $\|\vartheta^*\|_H$ and C_p. Due to the the boundedness of F^i in $L^\infty(\Omega)$ and the Lipschitz continuity of π, we also infer that

$$\tau(F^i, \vartheta^i) \le \tau \|F^i\|_{L^\infty(\Omega)} \|\vartheta^i\|_{L^1(\Omega)}, \tag{4.7}$$

$$\begin{aligned} -\big(\pi(\chi^i) - \chi^i, \chi^i - \chi^{i-1}\big) &\le c\tau(1 + \|\chi^i\|_H) \left\| \frac{\chi^i - \chi^{i-1}}{\tau} \right\|_H \\ &\le \frac{\tau}{2} \left\| \frac{\chi^i - \chi^{i-1}}{\tau} \right\|_H^2 + \tau C_2 (1 + \|\chi^i\|_H^2), \end{aligned} \tag{4.8}$$

help2

where C_2 depends on C_π, $|\pi(0)|$ and $|\Omega|$. Now, we apply the estimates (4.2)–(4.8) to the corresponding terms of (4.1) and sum up for $i = 1, \dots, n$, letting $n \le N$. We obtain that

$$\begin{aligned} &\frac{\tau^{1/2}}{2} \|\vartheta^n\|_H^2 + \sum_{i=1}^n \frac{\tau^{1/2}}{2} \|\vartheta^i - \vartheta^{i-1}\|_H^2 + \|\vartheta^n\|_{L^1(\Omega)} + \frac{k_0}{2} \sum_{i=1}^n \tau \|\nabla \vartheta^i\|_H^2 \\ &+ \frac{1}{2} \sum_{i=1}^n \tau \left\| \frac{\chi^i - \chi^{i-1}}{\tau} \right\|_H^2 + \frac{1}{2} \|\chi^n\|_V^2 + \frac{1}{2} \sum_{i=1}^n \|\chi^i - \chi^{i-1}\|_V^2 + \int_\Omega \tilde\beta(\chi^n) \\ &\le \frac{\tau^{1/2}}{2} \|\vartheta_0\|_H^2 + \|\vartheta_0\|_{L^1(\Omega)} + \frac{1}{2} \|\chi_0\|_V^2 + \int_\Omega \tilde\beta(\chi_0) + \tau \sum_{i=1}^n \|F^i\|_{L^\infty(\Omega)} \|\vartheta^i\|_{L^1(\Omega)} \\ &\quad + C_1 \sum_{i=1}^n \tau \|\vartheta^i\|_{L^1(\Omega)} + C_2 \sum_{i=1}^n \tau \|\chi^i\|_H^2 + c. \end{aligned} \tag{4.9}$$

stimacomple2

On account of (2.13)–(2.14) and (2.18), the first four terms on the right-hand side of (4.9) are bounded. Now, recalling the definition (3.5) of F^i, we have that

$$\tau \sum_{i=1}^n \|F^i\|_{L^\infty(\Omega)} \|\vartheta^i\|_{L^1(\Omega)} = \|\vartheta^n\|_{L^1(\Omega)} \int_{(n-1)\tau}^{n\tau} \|F(s)\|_{L^\infty(\Omega)} \, ds + \sum_{i=1}^{n-1} \|F^i\|_{L^\infty(\Omega)} \|\vartheta^i\|_{L^1(\Omega)}.$$

Thanks to the absolute continuity of the integral, if τ is small enough (independently of n) we have that

$$\int_{(n-1)\tau}^{n\tau} \|F(s)\|_{L^\infty(\Omega)} \, ds \le \frac{1}{4}, \quad C_1 \tau \le \frac{1}{4}, \quad C_2 \tau \le \frac{1}{4}. \tag{4.10}$$

pier7

Then, on the basis of (4.10), from (4.9) we infer that

$$\begin{aligned} &\frac{\tau^{1/2}}{2} \|\vartheta^n\|_H^2 + \sum_{i=1}^n \frac{\tau^{1/2}}{2} \|\vartheta^i - \vartheta^{i-1}\|_H^2 + \frac{1}{2} \|\vartheta^n\|_{L^1(\Omega)} + \frac{k_0}{2} \sum_{i=1}^n \tau \|\nabla \vartheta^i\|_H^2 \\ &+ \frac{1}{2} \sum_{i=1}^n \tau \left\| \frac{\chi^i - \chi^{i-1}}{\tau} \right\|_H^2 + \frac{1}{4} \|\chi^n\|_V^2 + \frac{1}{2} \sum_{i=1}^n \|\chi^i - \chi^{i-1}\|_V^2 + \int_\Omega \tilde\beta(\chi^n) \\ &\le c + \sum_{i=1}^{n-1} \tau \Big(\|F^i\|_{L^\infty(\Omega)} \|\vartheta^i\|_{L^1(\Omega)} + C_1 \|\vartheta^i\|_{L^1(\Omega)} + C_2 \|\chi^i\|_H^2 \Big). \end{aligned} \tag{4.11}$$

Now, we observe that

$$\sum_{i=1}^{n-1}\tau C_1 \le \sum_{i=1}^{N}\tau C_1 = C_1 T, \qquad \sum_{i=1}^{n-1}\tau C_2 \le \sum_{i=1}^{N}\tau C_2 = C_2 T.$$

and, according to (2.11),

$$\sum_{i=1}^{n-1}\tau\|F^i\|_{L^\infty(\Omega)} \le \sum_{i=1}^{N}\int_{(i-1)\tau}^{i\tau}\|F(s)\|_{L^\infty(\Omega)}\,ds = \int_0^T \|F(s)\|_{L^\infty(\Omega)}\,ds \le c.$$

Then, we can apply Lemma 2.1 and, recalling the notations (3.1), we conclude that

$$\begin{aligned}&\tau^{1/2}\|\overline{\vartheta}_\tau\|^2_{L^\infty(0,T;H)} + \tau^{3/2}\|\partial_t\widehat{\vartheta}_\tau\|^2_{L^2(0,T;H)} + \|\overline{\vartheta}_\tau\|_{L^\infty(0,T;L^1(\Omega))} + \|\nabla\overline{\vartheta}_\tau\|^2_{L^2(0,T;H)}\\ &+ \|\partial_t\widehat{\chi}_\tau\|^2_{L^2(0,T;H)} + \|\overline{\chi}_\tau\|^2_{L^\infty(0,T;V)} + \tau\|\partial_t\widehat{\chi}_\tau\|^2_{L^2(0,T;V)} + \|\tilde\beta(\overline{\chi}_\tau)\|_{L^\infty(0,T;L^1(\Omega))} \le c. \quad (4.12)\end{aligned}$$ gaf1

Since the third and the fourth term of the left-hand side of (4.12) are bounded, owing to (2.3) we also infer that

$$\|\overline{\vartheta}_\tau\|_{L^2(0,T;V)} \le c. \quad (4.13)$$ gaf2

Besides, in view of (3.9) and due to the sub-linear growth of A stated by (2.20) and to (2.12), we deduce that

$$\|\overline{\zeta}_\tau\|_{L^2(0,T;H)} \le c. \quad (4.14)$$ gaf3

Second uniform estimate. We formally test (3.8) by ξ^i and obtain

$$(\chi^i - \chi^{i-1}, \xi^i) + \tau\|\xi^i\|^2_H \le \tau\big(\pi(\chi^i) + \ell\vartheta^i, \xi^i\big). \quad (4.15)$$ formale1

We point out that the previous estimate (4.15) can be rigorously derived by testing (3.29) by $\beta_\varepsilon(X_\varepsilon)$ and then passing to the limit as $\varepsilon \searrow 0$. Since β is the subdifferential of $\tilde\beta$, we have that

$$(\chi^i - \chi^{i-1}, \xi^i) \ge \int_\Omega \tilde\beta(\chi^i) - \int_\Omega \tilde\beta(\chi^{i-1}). \quad (4.16)$$ formale2

Moreover, due to the Lipschitz continuity of π, applying the Young inequality (2.5) to the right-hand side of (4.15), we deduce that

$$\tau\big(\pi(\chi^i) + \ell\vartheta^i, \xi^i\big) \le \frac{1}{2}\tau\|\xi^i\|^2_H + c\,\tau\big(1 + \|\chi^i\|^2_H + \|\vartheta^i\|^2_H\big). \quad (4.17)$$ formale3

Now, combining (4.15)–(4.17) and summing up for $i = 1, \dots, n$, with $n \le N$, we infer that

$$\int_\Omega \tilde\beta(\chi^n) + \frac{1}{2}\sum_{i=1}^{n}\tau\|\xi^i\|^2_H \le \int_\Omega \tilde\beta(\chi_0) + \sum_{i=1}^{n}\tau\big(1 + \|\chi^i\|^2_H + \|\vartheta^i\|^2_H\big), \quad (4.18)$$ formale4

whence, due to (4.12)–(4.13), we obtain that

$$\|\overline{\xi}_\tau\|_{L^2(0,T;H)} \le c. \quad (4.19)$$ formale5

Finally, by comparison in (3.14), we conclude that $\|\Delta\overline{\chi}_\tau\|_{L^2(0,T;H)} \le c$. Then, thanks to (4.12) and elliptic regularity, we find that

$$\|\overline{\chi}_\tau\|_{L^2(0,T;W)} \le c. \quad (4.20)$$ formale7

Third uniform estimate. We introduce the function $\psi_n : \mathbb{R} \longmapsto \mathbb{R}$ obtained by truncating the logarithmic function in the following way:

$$\psi_n(u) = \begin{cases} \ln(u) & \text{if } u \geq 1/n, \\ -\ln(n) & \text{if } u < 1/n. \end{cases}$$

It is easy to see that ψ_n is an increasing and Lipschitz continuous function. Then, defining

$$j_n(u) = \int_1^u \psi_n(s)\, ds, \quad u \in \mathbb{R}, \quad \text{and} \quad j(u) = \int_1^u \ln s\, ds, \quad u > 0, \tag{4.21}$$ j

and testing (3.7) by $\psi_n(\vartheta^i)$, we obtain that

$$\begin{aligned} &\tau^{1/2}\big(\vartheta^i - \vartheta^{i-1}, \psi_n(\vartheta^i)\big) + \big(\ln \vartheta^i - \ln \vartheta^{i-1}, \psi_n(\vartheta^i)\big) + \tau k_0 \int_{\Omega \cap \{\vartheta^i \geq 1/n\}} \frac{|\nabla \vartheta^i|^2}{\vartheta^i} \\ &= -\ell\big(\chi^i - \chi^{i-1}, \psi_n(\vartheta^i)\big) - \tau\big(\zeta^i, \psi_n(\vartheta^i)\big) + \tau\big(F^i, \psi_n(\vartheta^i)\big). \end{aligned} \tag{4.22}$$ sb10

Recalling that j_n is a convex function with derivative ψ_n, we have that

$$\tau^{1/2}\big(\vartheta^i - \vartheta^{i-1}, \psi_n(\vartheta^i)\big) \geq \tau^{1/2} \int_\Omega j_n(\vartheta^i) - \tau^{1/2} \int_\Omega j_n(\vartheta^{i-1}),$$

and consequently from (4.22) we infer that

$$\begin{aligned} &\tau k_0 \int_{\Omega \cap \{\vartheta^i \geq 1/n\}} \frac{|\nabla \vartheta^i|^2}{\vartheta^i} \leq \tau^{1/2} \int_\Omega j_n(\vartheta^{i-1}) - \tau^{1/2} \int_\Omega j_n(\vartheta^i) \\ &- \int_\Omega (\ln \vartheta^i - \ln \vartheta^{i-1}) \psi_n(\vartheta^i) - \int_\Omega \big(\ell(\chi^i - \chi^{i-1}) + \tau\zeta^i - \tau F^i\big)\psi_n(\vartheta^i). \end{aligned} \tag{4.23}$$ sb12

Due to the properties of the subdifferential, we have that

$$0 \leq j(\vartheta^k) \leq j(1) + (\ln \vartheta^k, \vartheta^k - 1) \quad \text{for } k = 0, 1, \ldots, N. \tag{4.24}$$ pier8

Since $\ln \vartheta^k \in H$, $\vartheta^k > 0$ a.e. in Ω and $\vartheta^k \in H$, from (4.24) we infer that $j(\vartheta^k) \in L^1(\Omega)$; consequently, passing to the limit as $n \to +\infty$, we obtain that

$$\begin{aligned} \psi_n(\vartheta^k) &\to \ln \vartheta^k \quad \text{in } H \text{ and a.e. in } \Omega, \\ j_n(\vartheta^k) &\to j(\vartheta^k) \quad \text{in } L^1(\Omega) \text{ and a.e. in } \Omega, \end{aligned}$$

for $k = 0, 1, \ldots, N$. Then, taking the lim inf in (4.23) as $n \to +\infty$ and applying the Fatou Lemma and (2.6), we have that

$$\begin{aligned} &\tau k_0 \int_\Omega \frac{|\nabla \vartheta^i|^2}{\vartheta^i} \leq \tau^{1/2} \int_\Omega j(\vartheta^{i-1}) - \tau^{1/2} \int_\Omega j(\vartheta^i) + \frac{1}{2} \int_\Omega |\ln \vartheta^{i-1}|^2 - \frac{1}{2} \int_\Omega |\ln \vartheta^i|^2 \\ &- \frac{1}{2} \int_\Omega |\ln \vartheta^i - \ln \vartheta^{i-1}|^2 - \int_\Omega \big(\ell(\chi^i - \chi^{i-1}) + \tau\zeta^i - \tau F^i\big) \ln \vartheta^i. \end{aligned} \tag{4.25}$$ sb13

Now, sum up (4.25) for $i = 1, \dots, k$, with $k \le N$, and obtain that

$$\begin{aligned}
&\tau^{1/2} \int_\Omega j(\vartheta^k) + \frac{1}{2} \| \ln \vartheta^k \|_H^2 + \frac{1}{2} \sum_{i=1}^k \tau^2 \left\| \frac{\ln \vartheta^i - \ln \vartheta^{i-1}}{\tau} \right\|_H^2 + k_0 \sum_{i=1}^k \tau \int_\Omega \frac{|\nabla \vartheta^i|^2}{\vartheta^i} \\
&\le \tau^{1/2} \int_\Omega j(\vartheta_0) + \frac{1}{2} \| \ln \vartheta_0 \|_H^2 + \frac{1}{4} \sum_{i=1}^k \tau \| \ln \vartheta^i \|_H^2 + c \sum_{i=1}^k \tau \left\| \frac{\chi^i - \chi^{i-1}}{\tau} \right\|_H^2 \\
&\quad + c \sum_{i=1}^k \tau \| \zeta^i \|_H^2 + c \sum_{i=1}^k \tau \| F^i \|_H^2 . \qquad (4.26)
\end{aligned}$$ sb14

We observe that if $\tau \le 1$ then

$$\frac{1}{4} \sum_{i=1}^k \tau \| \ln \vartheta^i \|_H^2 \le \frac{1}{4} \sum_{i=1}^{k-1} \tau \| \ln \vartheta^i \|_H^2 + \frac{1}{4} \| \ln \vartheta^k \|_H^2 . \qquad (4.27)$$ sb15

We also notice that the fourth and the fifth term on the right-hand side of (4.26) are bounded by a positive constant c, due to (4.12) and (4.14), respectively. Moreover, thanks to (2.11) and to the definition (3.5) of F^i, by using the Hölder inequality the last term on the right-hand side of (4.26) can be estimated as follows:

$$\begin{aligned}
c \sum_{i=1}^k \tau \| F^i \|_H^2 &\le c \sum_{i=1}^k \tau \left\| \frac{1}{\tau} \int_{(i-1)\tau}^{i\tau} F(s) \, ds \right\|_H^2 \\
&\le c \sum_{i=1}^k \int_{(i-1)\tau}^{i\tau} \| F(s) \|_H^2 \, ds \le c \| F \|_{L^2(0,T;H)}^2 . \qquad (4.28)
\end{aligned}$$ sb16

Then, combining (4.26) with (4.27)–(4.28) (see also (2.13) and (4.24)), we infer that

$$\begin{aligned}
&\tau^{1/2} \int_\Omega j(\vartheta^k) + \frac{1}{4} \| \ln \vartheta^k \|_H^2 + \frac{1}{2} \sum_{i=1}^k \tau^2 \left\| \frac{\ln \vartheta^i - \ln \vartheta^{i-1}}{\tau} \right\|_H^2 \\
&+ 4 k_0 \sum_{i=1}^k \tau \int_\Omega \left| \nabla (\vartheta^i)^{1/2} \right|^2 \le c + \frac{1}{4} \sum_{i=1}^{k-1} \tau \| \ln \vartheta^i \|_H^2 ,
\end{aligned}$$

whence, by applying Lemma 2.1, we conclude that

$$\tau^{1/2} \| j(\overline{\vartheta}_\tau) \|_{L^\infty(0,T;L^1(\Omega))} + \| \overline{\ln \vartheta}_\tau \|_{L^\infty(0,T;H)} + \left\| \nabla \overline{\vartheta^{1/2}}_\tau \right\|_{L^2(0,T;H)} \le c . \qquad (4.29)$$ sb17

Moreover, due to (4.12) as well, we also infer that

$$\| \overline{\vartheta^{1/2}}_\tau \|_{L^\infty(0,T;H) \cap L^2(0,T;V)} \le c . \qquad (4.30)$$ sb18

Fourth uniform estimate. We test (3.7) by $(\vartheta^i - \vartheta^{i-1})$. Then, we take the difference between (3.8) written for i and for $i-1$, and test by $(\chi^i - \chi^{i-1})/\tau$. Using (2.19) and

adding, it is note difficult to obtain that

$$\begin{aligned}
&\tau^{1/2}\|\vartheta^i-\vartheta^{i-1}\|_H^2+(\ln\vartheta^i-\ln\vartheta^{i-1},\vartheta^i-\vartheta^{i-1})+\ell(\chi^i-\chi^{i-1},\vartheta^i-\vartheta^{i-1})\\
&+\tau\Phi(\vartheta^i-\vartheta^*)-\tau\Phi(\vartheta^{i-1}-\vartheta^*)+\tau\frac{k_0}{2}\left(\|\nabla\vartheta^i\|_H^2+\|\nabla(\vartheta^i-\vartheta^{i-1})\|_H^2-\|\nabla\vartheta^{i-1}\|_H^2\right)\\
&+\frac{\tau}{2}\left\|\frac{\chi^i-\chi^{i-1}}{\tau}\right\|_H^2+\frac{\tau}{2}\left\|\frac{\chi^i-\chi^{i-1}}{\tau}-\frac{\chi^{i-1}-\chi^{i-2}}{\tau}\right\|_H^2-\frac{\tau}{2}\left\|\frac{\chi^{i-1}-\chi^{i-2}}{\tau}\right\|_H^2\\
&+\tau^2\left\|\nabla\frac{\chi^i-\chi^{i-1}}{\tau}\right\|_H^2+(\xi^i-\xi^{i-1},\chi^i-\chi^{i-1})-\tau\ell\left(\vartheta^i-\vartheta^{i-1},\frac{\chi^i-\chi^{i-1}}{\tau}\right)\\
&\le\tau(F^i,\vartheta^i-\vartheta^{i-1})-\tau\left(\pi(\chi^i)-\pi(\chi^{i-1}),\frac{\chi^i-\chi^{i-1}}{\tau}\right),
\end{aligned}\tag{4.31}$$

sommona1

for $i=2,\dots,N$. Now, we write (3.7) and (3.8) for $i=1$ and test the corresponding equations by $(\vartheta^1-\vartheta^0)$ and $(\chi^1-\chi^0)/\tau$, respectively. Since $\vartheta^0=\vartheta_0$ and $\chi^0=\chi_0$, we have that

$$\begin{aligned}
&\tau^{1/2}\|\vartheta^1-\vartheta^0\|_H^2+(\ln\vartheta^1-\ln\vartheta^0,\vartheta^1-\vartheta^0)+\ell(\chi^1-\chi^0,\vartheta^1-\vartheta^0)+\tau\Phi(\vartheta^1-\vartheta^*)\\
&-\tau\Phi(\vartheta_0-\vartheta^*)+\tau\frac{k_0}{2}\left(\|\nabla\vartheta^1\|_H^2+\|\nabla(\vartheta^1-\vartheta^0)\|_H^2-\|\nabla\vartheta_0\|_H^2\right)+\tau\left\|\frac{\chi^1-\chi^0}{\tau}\right\|_H^2\\
&+\|\nabla(\chi^1-\chi^0)\|_H^2+(\xi^1-\xi_0,\chi^1-\chi_0)\le-\tau\left(\pi(\chi^1)-\pi(\chi^0),\frac{\chi^1-\chi^0}{\tau}\right)\\
&+\tau\ell\left(\vartheta^1-\vartheta^0,\frac{\chi^1-\chi^0}{\tau}\right)+\tau(F^1,\vartheta^1-\vartheta^0)+(\ell\vartheta_0+\Delta\chi_0-\xi_0-\pi(\chi_0),\chi^1-\chi^0).
\end{aligned}\tag{4.32}$$

sommona2

Then, we divide (4.31) and (4.32) by τ and sum up the corresponding equations for $i=1,\dots,n$, with $n\le N$. Since β is maximal monotone and (3.10) and (2.17) hold, then the eleventh term on the left-hand side of (4.31) and the ninth term on the left-hand side of (4.32) are nonnegative. Assuming $\chi^{-1}=\chi_0$, we infer that

$$\begin{aligned}
&\tau^{1/2}\sum_{i=1}^n\tau\left\|\frac{\vartheta^i-\vartheta^{i-1}}{\tau}\right\|_H^2+\sum_{i=1}^n\frac{1}{\tau}(\ln\vartheta^i-\ln\vartheta^{i-1},\vartheta^i-\vartheta^{i-1})+\Phi(\vartheta^n-\vartheta^*)\\
&+\frac{k_0}{2}\|\nabla\vartheta^n\|_H^2+\frac{k_0}{2}\tau\sum_{i=1}^n\tau\left\|\nabla\frac{\vartheta^i-\vartheta^{i-1}}{\tau}\right\|_H^2+\frac{1}{2}\left\|\frac{\chi^n-\chi^{n-1}}{\tau}\right\|_H^2\\
&+\frac{1}{2}\sum_{i=1}^n\left\|\frac{\chi^i-\chi^{i-1}}{\tau}-\frac{\chi^{i-1}-\chi^{i-2}}{\tau}\right\|_H^2+\frac{1}{2}\left\|\frac{\chi^1-\chi^0}{\tau}\right\|_H^2+\sum_{i=1}^n\tau\left\|\nabla\frac{\chi^i-\chi^{i-1}}{\tau}\right\|_H^2\\
&\le\Phi(\vartheta_0-\vartheta^*)+\frac{k_0}{2}\|\nabla\vartheta_0\|_H^2+\|\ell\vartheta_0+\Delta\chi_0-\xi_0-\pi(\chi_0)\|_H^2+\frac{1}{4}\left\|\frac{\chi^1-\chi^0}{\tau}\right\|_H^2\\
&\quad+(F^n,\vartheta^n)-(F^1,\vartheta_0)-\sum_{i=1}^{n-1}(F^{i+1}-F^i,\vartheta^i)+\sum_{i=1}^nC_\pi\tau\left\|\frac{\chi^i-\chi^{i-1}}{\tau}\right\|_H^2.
\end{aligned}\tag{4.33}$$

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In view of (2.16)–(2.17) and noting that $\vartheta_0\in V$, $\chi_0\in W$ and Φ has at most a quadratic growth (see (2.13)–(2.14) and (2.19)), the first three terms on the right-hand side of (4.33) are bounded by a positive constant. Besides, using (2.5), (2.3) and the Hölder inequality

and recalling (2.11) and (4.12), the fifth and the sixth term on the right-hand side of (4.33) can be estimated as follows:

$$|(F^n,\vartheta^n)| \le \|F^n\|_H\|\vartheta^n\|_H \le C_p\|F^n\|_H\Big(\|\vartheta^n\|_{L^1(\Omega)} + \|\nabla\vartheta^n\|_H\Big)$$
$$\le C_p\|F\|_{C^0([0,T];H)}\Big(\|\overline{\vartheta}_\tau\|_{L^\infty(0,T;L^1(\Omega))} + \|\nabla\vartheta^n\|_H\Big) \le \frac{k_0}{4}\|\nabla\vartheta^n\|_H^2 + c, \tag{4.34}$$ xxz1

$$|(F^1,\vartheta_0)| \le \|F^1\|_H\|\vartheta_0\|_H \le \|F\|_{C^0([0,T];H)}\|\vartheta_0\|_H \le c. \tag{4.35}$$ xxz2

With the help of (2.5), Hölder's inequality and (4.13) we also infer that

$$\left|\sum_{i=1}^{n-1}(F^{i+1}-F^i,\vartheta^i)\right| \le \sum_{i=2}^{n}\tau\left\|\frac{F^i-F^{i-1}}{\tau}\right\|_H\|\vartheta^{i-1}\|_H$$
$$\le \frac12\sum_{i=2}^{n}\tau\left\|\frac{F^i-F^{i-1}}{\tau}\right\|_H^2 + \frac12\sum_{i=1}^{n-1}\tau\|\vartheta^i\|_H^2 \le \frac12\sum_{i=2}^{n}\tau\left\|\frac{F^i-F^{i-1}}{\tau}\right\|_H^2 + c. \tag{4.36}$$ xxz3

Recalling (2.11) and the definition of F^i (see (3.5)), we have that

$$\left\|\frac{F^i-F^{i-1}}{\tau}\right\|_H^2 = \left\|\frac{1}{\tau^2}\int_{(i-1)\tau}^{i\tau}F(s)\,ds - \frac{1}{\tau^2}\int_{(i-2)\tau}^{(i-1)\tau}F(s)\,ds\right\|_H^2$$
$$= \left\|\frac{1}{\tau^2}\int_{(i-1)\tau}^{i\tau}\Big(F(s)-F(s-\tau)\Big)ds\right\|_H^2$$
$$\le \frac{1}{\tau^4}\left|\int_{(i-1)\tau}^{i\tau}\|F(s)-F(s-\tau)\|_H\,ds\right|^2 \le \frac{1}{\tau^3}\int_{(i-1)\tau}^{i\tau}\left\|\int_{s-\tau}^{s}\partial_tF(t)\,dt\right\|_H^2 ds$$
$$\le \frac{1}{\tau^2}\int_{(i-1)\tau}^{i\tau}\left(\int_{s-\tau}^{s}\|\partial_tF(t)\|_H^2\,dt\right)ds \le \frac{1}{\tau}\|\partial_tF\|^2_{L^2((i-2)\tau,i\tau;H)},$$

so that

$$\frac12\sum_{i=2}^{n}\tau\left\|\frac{F^i-F^{i-1}}{\tau}\right\|_H^2 \le \|\partial_tF\|^2_{L^2(0,T;H)}. \tag{4.37}$$ xxz4

Next, we take advantage of Lemma 2.2 in order to deal with the second term on the left-hand side of (4.33). Indeed (cf. (2.8)), we realize that

$$\big|(\vartheta^i)^{1/2}-(\vartheta^{i-1})^{1/2}\big|^2 \le (\ln\vartheta^i - \ln\vartheta^{i-1},\vartheta^i-\vartheta^{i-1}),$$

whence

$$\sum_{i=1}^{n}\frac{1}{\tau}(\ln\vartheta^i-\ln\vartheta^{i-1},\vartheta^i-\vartheta^{i-1}) \ge \sum_{i=1}^{n}\tau\left\|\frac{(\vartheta^i)^{1/2}-(\vartheta^{i-1})^{1/2}}{\tau}\right\|_H^2. \tag{4.38}$$ xxz5

Collecting now (4.34)–(4.38), from (4.33) and (4.12) we infer that

$$\tau^{1/4}\|\partial_t\widehat{\vartheta}_\tau\|_{L^2(0,T;H)} + \|\partial_t\widehat{\vartheta^{1/2}}_\tau\|_{L^2(0,T;H)} + \|\Phi(\overline{\vartheta}_\tau-\vartheta^*)\|_{L^\infty(0,T)} + \|\overline{\vartheta}_\tau\|_{L^\infty(0,T;V)}$$
$$+\tau^{1/2}\|\partial_t\widehat{\vartheta}_\tau\|_{L^2(0,T;V)} + \|\partial_t\widehat{\chi}_\tau\|_{L^\infty(0,T;H)} + \|\partial_t\widehat{\chi}_\tau\|_{L^2(0,T;V)} \le c. \tag{4.39}$$ aaa1

Therefore, thanks to (3.15) and using (2.20) and (2.12), we have that

$$\|\overline{\zeta}_\tau\|_{L^\infty(0,T;H)} \le c. \tag{4.40}$$ pier9

Moreover, by comparison in (3.13) and in view of (4.12)–(4.14), (4.19)–(4.20), (4.29)–(4.30) and (4.39), we obtain that

$$\begin{aligned}\|\partial_t\widehat{\ln\vartheta}_\tau\|_{L^2(0,T;V')} &\le c\tau^{1/2}\|\partial_t\widehat{\vartheta}_\tau\|_{L^2(0,T;H)} + c\|\partial_t\widehat{\chi}_\tau\|_{L^2(0,T;H)} \\ &+ c\|\overline{\zeta}_\tau\|_{L^2(0,T;H)} + k_0\|\Delta\overline{\vartheta}_\tau\|_{L^2(0,T;V')} + c\|\overline{F}_\tau\|_{L^2(0,T;H)} \le c. \end{aligned} \tag{4.41}$$ aaa2

Furthermore, recalling (3.14), a comparison of the terms yields the bound

$$\|\Delta\overline{\chi}_\tau + \overline{\xi}_\tau\|_{L^\infty(0,T;H)} \le c.$$

Hence, by arguing as in the Second uniform estimate, we can improve (4.19) and (4.20) to find out that

$$\|\overline{\xi}_\tau\|_{L^\infty(0,T;H)} + \|\overline{\chi}_\tau\|_{L^\infty(0,T;W)} \le c. \tag{4.42}$$ pier10

Summary of the uniform estimates. Let us collect the previous estimates. From (4.12)–(4.14), (4.19)–(4.20), (4.29)–(4.30) and (4.39)–(4.42) we conclude that there exists a constant $c > 0$, independent of τ, such that

$$\begin{aligned}&\|\overline{\vartheta}_\tau\|_{L^\infty(0,T;V)} + \|\widehat{\vartheta}_\tau\|_{L^\infty(0,T;V)} + \tau^{1/4}\|\partial_t\widehat{\vartheta}_\tau\|_{L^2(0,T;H)} \\ &+ \|\overline{\ln\vartheta}_\tau\|_{L^\infty(0,T;H)} + \|\widehat{\ln\vartheta}_\tau\|_{H^1(0,T;V')\cap L^\infty(0,T;H)} \\ &+ \|\overline{\vartheta^{1/2}}_\tau\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} + \|\widehat{\vartheta^{1/2}}_\tau\|_{H^1(0,T;H)\cap L^2(0,T;V)} + \|\overline{\zeta}_\tau\|_{L^\infty(0,T;H)} \\ &+ \|\overline{\chi}_\tau\|_{L^\infty(0,T;W)} + \|\widehat{\chi}_\tau\|_{W^{1,\infty}(0,T;H)\cap H^1(0,T;V)\cap L^\infty(0,T;W)} + \|\overline{\xi}_\tau\|_{L^\infty(0,T;H)} \le c. \end{aligned} \tag{4.43}$$ gransommafin

5 Passage to the limit as $\tau \searrow 0$

Thanks to (4.43) and to the well-known weak or weak* compactness results, we deduce that, at least for a subsequence of $\tau \searrow 0$, there exist ten limit functions ϑ, $\widehat{\vartheta}$, λ, $\widehat{\lambda}$, w, $\widehat{w}$, ζ, χ, $\widehat{\chi}$, and ξ such that

$$\overline{\vartheta}_\tau \rightharpoonup^* \vartheta \quad \text{in} \quad L^\infty(0,T;V), \tag{5.1}$$
$$\widehat{\vartheta}_\tau \rightharpoonup^* \widehat{\vartheta} \quad \text{in} \quad L^\infty(0,T;V), \tag{5.2}$$
$$\tau^{1/4}\widehat{\vartheta}_\tau \rightharpoonup^* 0 \quad \text{in} \quad H^1(0,T;H) \cap L^\infty(0,T;V), \tag{5.3}$$
$$\overline{\ln\vartheta}_\tau \rightharpoonup^* \lambda \quad \text{in} \quad L^\infty(0,T;H), \tag{5.4}$$
$$\widehat{\ln\vartheta}_\tau \rightharpoonup^* \widehat{\lambda} \quad \text{in} \quad H^1(0,T;V') \cap L^\infty(0,T;H), \tag{5.5}$$
$$\overline{\vartheta^{1/2}}_\tau \rightharpoonup^* w \quad \text{in} \quad L^\infty(0,T;H) \cap L^2(0,T;V), \tag{5.6}$$
$$\widehat{\vartheta^{1/2}}_\tau \rightharpoonup \widehat{w} \quad \text{in} \quad H^1(0,T;H) \cap L^2(0,T;V), \tag{5.7}$$
$$\overline{\zeta}_\tau \rightharpoonup^* \zeta \quad \text{in} \quad L^\infty(0,T;H), \tag{5.8}$$
$$\overline{\chi}_\tau \rightharpoonup^* \chi \quad \text{in} \quad L^\infty(0,T;W), \tag{5.9}$$
$$\widehat{\chi}_\tau \rightharpoonup^* \widehat{\chi} \quad \text{in} \quad W^{1,\infty}(0,T;H) \cap H^1(0,T;V) \cap L^\infty(0,T;W), \tag{5.10}$$
$$\overline{\xi}_\tau \rightharpoonup^* \xi \quad \text{in} \quad L^\infty(0,T;H). \tag{5.11}$$

First, we observe that $\vartheta = \widehat{\vartheta}$: indeed, thanks to (3.3) and (5.3), we have that

$$\|\overline{\vartheta}_\tau - \widehat{\vartheta}_\tau\|_{L^2(0,T;H)} \le \frac{\tau}{\sqrt{3}}\|\partial_t\widehat{\vartheta}_\tau\|_{L^2(0,T;H)} \le c\tau^{3/4}$$

and consequently $\overline{\vartheta}_\tau - \widehat{\vartheta}_\tau \to 0$ strongly in $L^2(0,T;H)$. Moreover, it turns out that $\lambda = \widehat{\lambda}$: in fact, on account of (3.4) and (5.5) we have that

$$\|\overline{\ln \vartheta}_\tau - \widehat{\ln \vartheta}_\tau\|_{L^\infty(0,T;V')} \le \tau^{1/2} \|\partial_t \widehat{\ln \vartheta}_\tau\|_{L^2(0,T;V')} \le c\tau^{1/2},$$

whence

$$\lim_{\tau \searrow 0} \|\overline{\ln \vartheta}_\tau - \widehat{\ln \vartheta}_\tau\|_{L^\infty(0,T;V')} = 0. \tag{5.12}$$ pier16

Similarly, thanks to (3.3) and (5.10), we see that

$$\|\overline{\vartheta^{1/2}}_\tau - \widehat{\vartheta^{1/2}}_\tau\|_{L^2(0,T;H)} \le \frac{\tau}{\sqrt{3}} \|\partial_t \widehat{\vartheta^{1/2}}_\tau\|_{L^2(0,T;H)} \le c\tau,$$

which entails

$$\lim_{\tau \searrow 0} \|\overline{\vartheta^{1/2}}_\tau - \widehat{\vartheta^{1/2}}_\tau\|_{L^2(0,T;H)} = 0 \tag{5.13}$$ pier17

and $w = \widehat{w}$. Finally, we check that $\chi = \widehat{\chi}$. In the light of (3.4), we have that

$$\|\overline{\chi}_\tau - \widehat{\chi}_\tau\|_{L^\infty(0,T;V)} \le \tau \|\partial_t \widehat{\chi}_\tau\|_{L^2(0,T;V)} \le c\tau$$

and consequently

$$\lim_{\tau \searrow 0} \|\overline{\chi}_\tau - \widehat{\chi}_\tau\|_{L^\infty(0,T;V)} = 0. \tag{5.14}$$ pier18

Next, in view of the convergences in (5.5), (5.7), (5.10) and owing to the strong compactness lemma stated in [22, Lemma 8, p. 84], we have that

$$\widehat{\ln \vartheta}_\tau \to \lambda \quad \text{in} \quad C^0([0,T];V'), \tag{5.15}$$
$$\widehat{\vartheta^{1/2}}_\tau \to w \quad \text{in} \quad L^2(0,T;H), \tag{5.16}$$
$$\widehat{\chi}_\tau \to \chi \quad \text{in} \quad C^0([0,T];V). \tag{5.17}$$

Then, by (5.12)–(5.14) we can also conclude that

$$\overline{\ln \vartheta}_\tau \to \lambda \quad \text{in} \quad L^\infty(0,T;V'), \tag{5.18}$$
$$\overline{\vartheta^{1/2}}_\tau \to w \quad \text{in} \quad L^2(0,T;H), \tag{5.19}$$
$$\overline{\chi}_\tau \to \chi \quad \text{in} \quad L^\infty(0,T;V). \tag{5.20}$$

Thanks to (5.20) and to the Lipschitz continuity of π, we have that

$$\pi(\overline{\chi}_\tau) \to \pi(\chi) \quad \text{in } L^\infty(0,T;H).$$

Now, we check that $\lambda = \ln \theta$: in fact, due to the weak convergence of $\overline{\vartheta}_\tau$ ensured by (5.1) and to the strong convergence of $\ln(\overline{\vartheta}_\tau)$ in (5.18) (see (5.4) as well), we have that

$$\limsup_{\tau \searrow 0} \int_0^T \int_\Omega \left(\ln \overline{\vartheta}_\tau\right) \overline{\vartheta}_\tau = \lim_{\tau \searrow 0} \int_0^T \langle \ln \overline{\vartheta}_\tau, \overline{\vartheta}_\tau \rangle = \int_0^T \langle \lambda, \vartheta \rangle = \int_0^T \int_\Omega \lambda \vartheta,$$

so that a standard tool for maximal monotone operators (cf., e.g., [1, Lemma 1.3, p. 42]) ensure that $\lambda = \ln \vartheta$. In the light of (3.16) and of the convergences (5.11) and (5.20), it is even simpler to check that ξ and χ satisfy (2.28).

At this point, recalling also (5.4), (5.5), (5.10) and passing to the limit in (3.13) and (3.14), we arrive at (2.25) and (2.26). In addition, note that (3.12) implies that $\widehat{\ln \vartheta}_\tau(0) = \ln \vartheta_0$ and $\widehat{\chi}_\tau(0) = \chi_0$; thus, thanks to (5.18) and (5.20), passing to the limit as $\tau \searrow 0$ leads to the initial conditions (2.30).

It remains to show (2.27). To this aim, we point out that (5.19) implies that, possibly taking another subsequence, $\overline{\vartheta^{1/2}}_\tau \to w$ almost everywhere in Q. Then, using (5.1) and the Egorov theorem, it is not difficult to verify that

$$\overline{\vartheta}_\tau = \left(\overline{\vartheta^{1/2}}_\tau\right)^2 \to w^2 \quad \text{a.e. in } Q \text{ and in } L^2(0,T;H),$$

as well as $\vartheta = w^2$. Details of this argument can be found, for instance, in [11, Exercise 4.16, part 3, p. 123]. Then, as A induces a natural maximal monotone operator on $L^2(0,T;H)$, recalling (3.15) and observing that (cf. (5.8))

$$\limsup_{\tau \searrow 0} \int_0^T \left(\overline{\zeta}_\tau, \overline{\vartheta}_\tau - \vartheta^*\right)_H = \lim_{\tau \searrow 0} \int_0^T \left(\overline{\zeta}_\tau, \overline{\vartheta}_\tau - \vartheta^*\right)_H = \int_0^T \left(\zeta, \vartheta - \vartheta^*\right)_H ,$$

we easily recover (2.27). Therefore, Theorem 2.1 is completely proved.

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