EXISTENCE RESULTS FOR DOUBLE-PHASE PROBLEMS VIA MORSE THEORY

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Abstract. We obtain nontrivial solutions for a class of double-phase problems using Morse theory. In the absence of a direct sum decomposition, we use a cohomological local splitting to get an estimate of the critical groups at zero.

1. Introduction

The study of energy functionals of the form

$$u \mapsto \int_\Omega \mathcal{H}(x, |\nabla u(x)|)dx, \qquad \mathcal{H}(x,t) = t^p + a(x)t^q, \quad q > p > 1, \;\; a(\cdot) \geq 0, \tag{1.1}$$

where the integrand $\mathcal{H}$ switches between two different elliptic behaviors has been intensively studied since the late eighties. This class of energies was introduced by Zhikov to provide models of *strongly anisotropic* materials, see e.g. [19–21] or the monograph [22]. Also, the integrals of (1.1) settle in the framework of the so-called functionals with non-standard growth conditions, according to a terminology introduced by Marcellini [9,14,15]. In [22], energies of the form (1.1) are used in the context of homogenization and elasticity and $a(\cdot)$ drives the geometry of a composite of two different materials with hardening powers p and q.

Significant progresses were recently achieved by Mingione et al. in the framework of regularity theory for minimizers of (1.1), see e.g. [1–3, 6, 7]. More recently, in [5], a complete study on the existence and properties of a sequence of variational eigenvalues related to $\mathcal{H}$, including a Weyl type estimate for their growth has been performed.

The purpose of this paper is to investigate the existence of solutions to the double phase problem

$$\begin{cases} -\operatorname{div}\left(|\nabla u|^{p-2}\nabla u + a(x)\,|\nabla u|^{q-2}\nabla u\right) = f(x,u) & \text{in } \Omega \\ \qquad\qquad\qquad\qquad\qquad\qquad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with Lipschitz boundary, $N \geq 2$, $1 < p < q < N$,

$$q/p < 1 + 1/N, \quad a : \overline{\Omega} \to [0,\infty) \text{ is Lipschitz continuous}, \tag{1.3}$$

and f is a Carathéodory function on $\Omega \times \mathbb{R}$ satisfying the growth condition

$$|f(x,t)| \leq C\left(|t|^{r-1} + 1\right) \quad \text{for a.a. } x \in \Omega \text{ and all } t \in \mathbb{R}, \tag{1.4}$$

for some $1 < r < p^*$ and $C > 0$, being $p^* = Np/(N-p)$ the critical Sobolev exponent of $W_0^{1,p}(\Omega)$. Assuming that $f(x,0) \equiv 0$, problem (1.2) has the trivial solution $u(x) = 0$ and we study the critical groups of the associated variational functional at 0, obtaining a nontrivial solution using Morse theory. In the absence of a direct sum decomposition, we use a cohomological local splitting to get an estimate of the critical groups.

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Our main result is for the q-superlinear case for the problem

$$(P) \qquad \begin{cases} -\operatorname{div}\left(|\nabla u|^{p-2}\nabla u + a(x)\,|\nabla u|^{q-2}\nabla u\right) = \lambda\,|u|^{p-2}u + |u|^{r-2}u + h(x,u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\lambda \in \mathbb{R}$ is a parameter, $q < r < p^*$ and h is a Carathéodory function on $\Omega \times \mathbb{R}$ satisfying

$$|h(x,t)| \le C\left(|t|^{\rho-1} + |t|^{\sigma-1}\right), \quad \text{for a.a. } x \in \Omega \text{ and all } t \in \mathbb{R}, \tag{1.5}$$

for some $p < \sigma < \rho < r$ and $C > 0$. The notion of weak solution for problem (P) is formulated in a suitable Orlicz Sobolev space $W_0^{1,\mathcal{H}}(\Omega)$ that will be introduced in Section 2.

Let $(\lambda_k) \subset \mathbb{R}^+$ be the sequence of (variational) eigenvalues of the p-Laplacian operator defined via cohomological index, cf. formula (2.13). Let us set

$$G(x,t) := \frac{|t|^r}{r} + H(x,t), \qquad x \in \Omega, \;\; t \in \mathbb{R},$$

being $H(x,t) = \displaystyle\int_0^t h(x,\tau)d\tau$. The following is the main result of the paper.

Theorem 1.1. *Assume that conditions (1.3) and (1.5) hold. Then problem (P) has a nontrivial weak solution $u \in W_0^{1,\mathcal{H}}(\Omega)$ in each of these cases:*

(1) $\lambda \notin \{\lambda_k\}_{k\ge1}$;
(2) *for some* $\delta > 0$, $G(x,t) \le 0$ *for a.a.* $x \in \Omega$ *and* $|t| \le \delta$;
(3) $G(x,t) \ge c\,|t|^s$ *for a.a.* $x \in \Omega$ *and all* $t \in \mathbb{R}$ *for some* $s \in (p,q)$ *and* $c > 0$.

To our knowledge this is the first existence result for double-phase problems (1.2) in the framework of Morse theory and it is obtained by analyzing the critical groups $H^q(\Phi^0 \cap U, \Phi^0 \cap U \setminus \{0\})$ of the associated energy functional Φ at zero, $q \in \mathbb{N}$.

2. Preliminaries and proof

2.1. **Variational setting.** The Musielak-Orlicz space $L^{\mathcal{H}}(\Omega)$ associated with the function

$$\mathcal{H} : \Omega \times [0,\infty) \to [0,\infty), \quad (x,t) \mapsto t^p + a(x)\,t^q$$

consists of all measurable functions $u : \Omega \to \mathbb{R}$ with the $\mathcal{H}$-modular

$$\rho_{\mathcal{H}}(u) := \int_\Omega \mathcal{H}(x,|u|)\,dx < \infty,$$

endowed with the Luxemburg norm

$$\|u\|_{\mathcal{H}} := \inf\left\{\gamma > 0 : \rho_{\mathcal{H}}(u/\gamma) \le 1\right\}.$$

The space $L^{\mathcal{H}}(\Omega)$ is a uniformly convex, and hence reflexive, Banach space. Denoting by $\|\cdot\|_p$ the norm in $L^p(\Omega)$ and by $L^q_a(\Omega)$ the space of all measurable functions $u : \Omega \to \mathbb{R}$ with the seminorm

$$\|u\|_{q,a} := \left(\int_\Omega a(x)\,|u|^q\,dx\right)^{1/q} < \infty,$$

we have the continuous embeddings

$$L^q(\Omega) \hookrightarrow L^{\mathcal{H}}(\Omega) \hookrightarrow L^p(\Omega) \cap L^q_a(\Omega),$$

see [5, Proposition 2.15 (i), (iv), (v)]. Since $\rho_{\mathcal{H}}(u/\|u\|_{\mathcal{H}}) = 1$ whenever $u \ne 0$, we have

$$\min\left\{\|u\|_{\mathcal{H}}^p, \|u\|_{\mathcal{H}}^q\right\} \le \|u\|_p^p + \|u\|_{q,a}^q \le \max\left\{\|u\|_{\mathcal{H}}^p, \|u\|_{\mathcal{H}}^q\right\}, \quad \forall u \in L^{\mathcal{H}}(\Omega). \tag{2.1}$$

The related Sobolev space $W^{1,\mathcal{H}}(\Omega)$ consists of all functions u in $L^{\mathcal{H}}(\Omega)$ with $|\nabla u| \in L^{\mathcal{H}}(\Omega)$, normed by

$$\|u\|_{1,\mathcal{H}} := \|u\|_{\mathcal{H}} + \|\nabla u\|_{\mathcal{H}},$$

where $\|\nabla u\|_{\mathcal{H}} = \||\nabla u|\|_{\mathcal{H}}$. The completion of $C_0^\infty(\Omega)$ in $W^{1,\mathcal{H}}(\Omega)$ is denoted by $W_0^{1,\mathcal{H}}(\Omega)$ and it can be equivalently renormed by

$$\|u\| := \|\nabla u\|_{\mathcal{H}}$$

via a Poincaré-type inequality, cf. [5, Proposition 2.18, (iv)], under assumption (1.3). The spaces $W^{1,\mathcal{H}}(\Omega)$ and $W_0^{1,\mathcal{H}}(\Omega)$ are uniformly convex, and hence reflexive, Banach spaces. The Sobolev embedding $W_0^{1,\mathcal{H}}(\Omega) \hookrightarrow L^r(\Omega)$ is compact since $r < p^*$, cf. [5, Proposition 2.15, (iii)]. We have

$$\min\{\|u\|^p, \|u\|^q\} \le \|\nabla u\|_p^p + \|\nabla u\|_{q,a}^q \le \max\{\|u\|^p, \|u\|^q\}, \quad \forall u \in W_0^{1,\mathcal{H}}(\Omega), \tag{2.2}$$

by virtue of (2.1). A weak solution of problem (1.2) is a function $u \in W_0^{1,\mathcal{H}}(\Omega)$ satisfying

$$\int_\Omega \left(|\nabla u|^{p-2} + a(x)\,|\nabla u|^{q-2}\right) \nabla u \cdot \nabla v \, dx = \int_\Omega f(x,u)\, v \, dx, \quad \forall v \in W_0^{1,\mathcal{H}}(\Omega).$$

Weak solutions coincide with critical points of the functional

$$\Phi(u) = \int_\Omega \left[\frac{1}{p}\,|\nabla u|^p + \frac{a(x)}{q}\,|\nabla u|^q - F(x,u)\right] dx, \quad u \in W_0^{1,\mathcal{H}}(\Omega),$$

where $F(x,t) = \displaystyle\int_0^t f(x,s)\,ds$, by the following proposition.

Proposition 2.1 (C^1 energy)**.** *Assume that* (1.4) *holds. Then* Φ *is of class* C^1 *with*

$$\langle \Phi'(u), v\rangle = \int_\Omega \left[\left(|\nabla u|^{p-2} + a(x)\,|\nabla u|^{q-2}\right) \nabla u \cdot \nabla v - f(x,u)\, v\right] dx, \tag{2.3}$$

for every $u, v \in W_0^{1,\mathcal{H}}(\Omega)$.

Proof. In view of the embeddings mentioned above, (2.3) is clear. To see that Φ' is continuous, suppose that $u_j \to u$ in $W_0^{1,\mathcal{H}}(\Omega)$. For all $v \in W_0^{1,\mathcal{H}}(\Omega)$ with $\|v\| = 1$, by the Hölder inequality,

$$\begin{aligned}|\langle \Phi'(u_j) - \Phi'(u), v\rangle| &\le \left\| |\nabla u_j|^{p-2}\nabla u_j - |\nabla u|^{p-2}\nabla u\right\|_{p'} \|\nabla v\|_p \\ &\quad + \left\| a(x)^{1/q'} \left||\nabla u_j|^{q-2}\nabla u_j - |\nabla u|^{q-2}\nabla u\right|\right\|_{q'} \|\nabla v\|_{q,a} + \|f(x,u_j) - f(x,u)\|_{r'} \|v\|_r,\end{aligned}$$

where $s' = s/(s-1)$ is the Hölder conjugate of s. Since $L^{\mathcal{H}}(\Omega) \hookrightarrow L^p(\Omega) \cap L_a^q(\Omega)$ and $W_0^{1,\mathcal{H}}(\Omega) \hookrightarrow L^r(\Omega)$, $\nabla u_j \to \nabla u$ in $L^p(\Omega) \cap L_a^q(\Omega)$, $u_j \to u$ in $L^r(\Omega)$, and $\|\nabla v\|_p$, $\|\nabla v\|_{q,a}$ and $\|v\|_r$ are uniformly bounded, the assertion follows from the dominated convergence theorem and (1.4). □

2.2. **Palais-Smale condition.** The operator $A : W_0^{1,\mathcal{H}}(\Omega) \to (W_0^{1,\mathcal{H}}(\Omega))'$ defined by

$$\langle A(u), v\rangle := \int_\Omega \left[\left(|\nabla u|^{p-2} + a(x)\,|\nabla u|^{q-2}\right) \nabla u \cdot \nabla v\right] dx, \quad u, v \in W_0^{1,\mathcal{H}}(\Omega),$$

where $(W_0^{1,\mathcal{H}}(\Omega))'$ is the dual space of $W_0^{1,\mathcal{H}}(\Omega)$, has the following important property.

Proposition 2.2. *If* $u_j \rightharpoonup u$ *in* $W_0^{1,\mathcal{H}}(\Omega)$ *and* $A(u_j)\,(u_j - u) \to 0$, *then* $u_j \to u$ *in* $W_0^{1,\mathcal{H}}(\Omega)$.

Proof. Noting that

$$\langle A(u), v\rangle \le \|\nabla u\|_p^{p-1} \|\nabla v\|_p + \|\nabla u\|_{q,a}^{q-1} \|\nabla v\|_{q,a} \quad \forall u, v \in W_0^{1,\mathcal{H}}(\Omega)$$

by the Hölder inequality, and the equality holds when $u = v$, we have

$$0 \le \big(\|\nabla u_j\|_p^{p-1} - \|\nabla u\|_p^{p-1}\big)\big(\|\nabla u_j\|_p - \|\nabla u\|_p\big) + \big(\|\nabla u_j\|_{q,a}^{q-1} - \|\nabla u\|_{q,a}^{q-1}\big)\big(\|\nabla u_j\|_{q,a} - \|\nabla u\|_{q,a}\big) \le \langle A(u_j) - A(u), u_j - u\rangle \to 0,$$

so that $\|\nabla u_j\|_p \to \|\nabla u\|_p$ and $\|\nabla u_j\|_{q,a} \to \|\nabla u\|_{q,a}$. Then $\nabla u_j \to \nabla u$ in $L^p(\Omega) \cap L_a^q(\Omega)$ by uniform convexity, and hence the conclusion follows from from (2.2). □

Recall that the functional Φ satisfies the Palais-Smale compactness condition at the level $c \in \mathbb{R}$, or $(\mathrm{PS})_c$ for short, if every sequence $(u_j) \subset W_0^{1,\mathcal{H}}(\Omega)$ such that $\Phi(u_j) \to c$ and $\Phi'(u_j) \to 0$, called a $(\mathrm{PS})_c$ sequence, has a convergent subsequence. We say that Φ satisfies the (PS) condition if it satisfies the $(\mathrm{PS})_c$ condition for all $c \in \mathbb{R}$. When verifying these conditions, it suffices to show that (u_j) is bounded by the following proposition.

Proposition 2.3 (Bounded Palais-Smale condition)**.** *Every bounded sequence $(u_j) \subset W_0^{1,\mathcal{H}}(\Omega)$ such that $\Phi'(u_j) \to 0$ has a convergent subsequence.*

Proof. Since (u_j) is bounded, a renamed subsequence converges to some u weakly in $W_0^{1,\mathcal{H}}(\Omega)$ and strongly in $L^r(\Omega)$. Then

$$\langle A(u_j), u_j - u\rangle = \langle \Phi'(u_j), u_j - u\rangle + \int_\Omega f(x, u_j)\,(u_j - u)\,dx \to 0$$

since

$$\left|\int_\Omega f(x, u_j)\,(u_j - u)\,dx\right| \le C\left(\|u_j\|_r^{r-1} + 1\right)\|u_j - u\|_r$$

by (1.4) and the Hölder inequality, so the conclusion follows from Proposition 2.2. □

2.3. **Regularity estimates.** For $f \in L^m(\Omega)$ with $m > 1$, solutions of

$$\int_\Omega \left(|\nabla u|^{p-2} + a(x)\,|\nabla u|^{q-2}\right)\nabla u \cdot \nabla v\,dx = \int_\Omega f(x)\,v\,dx \quad \forall v \in W_0^{1,\mathcal{H}}(\Omega) \tag{2.4}$$

enjoy the natural L^m-estimates given in the following proposition.

Proposition 2.4. *Let $f \in L^m(\Omega)$, $1 < m \le \infty$ and let $u \in W_0^{1,\mathcal{H}}(\Omega)$ satisfy (2.4). Then*

$$\|u\|_r \le C\,\|f\|_m^{1/(p-1)}, \tag{2.5}$$

where we have set

$$r = \begin{cases} \dfrac{N\,(p-1)\,m}{N - pm}, & 1 < m < \dfrac{N}{p} \\[2ex] \infty, & m > \dfrac{N}{p} \end{cases}$$

and $C = C(N, \Omega, p, m) > 0$.

Proof. For $k, \alpha > 0$ and $t \in \mathbb{R}$, set $t_k = \max\{-k, \min\{t, k\}\}$ and consider the nondecreasing function $g(t) = t_k^\alpha$ (with the agreement $a^\alpha := |a|^{\alpha-1}a$, for $a \in \mathbb{R}$). Testing equation (2.4) with the $g(u) \in W_0^{1,\mathcal{H}}(\Omega)$ provides the inequality

$$\|\nabla G(u)\|_p^p \le \int_\Omega f(x)\,g(u)\,dx,$$

where

$$G(t) := \int_0^t g'(s)^{1/p}\,ds = \frac{\alpha^{1/p}\,p}{\alpha+p-1}\,t_k^{(\alpha+p-1)/p}, \quad t \in \mathbb{R}.$$

Using the Sobolev inequality on the left and the Hölder inequality on the right now gives

$$\left\|u_k^{(\alpha+p-1)/p}\right\|_{p^*}^p \le C\,\|f\|_m\,\|u_k^\alpha\|_{m'}\,. \tag{2.6}$$

If $1 < m < N/p$, take

$$\alpha = \frac{(p-1)\,p^*}{pm'-p^*} = \frac{N\,(p-1)\,(m-1)}{N-pm} > 0,$$

so that

$$\frac{(\alpha+p-1)\,p^*}{p} = \alpha m' =: r.$$

Then $r = N\,(p-1)\,m/(N-pm)$ and (2.6) gives $\|u_k\|_r^{pr/p^*} \le C\,\|f\|_m\,\|u_k\|_r^{r/m'}$, so

$$\|u_k\|_r \le C\,\|f\|_m^{1/(p-1)}\,.$$

Letting $k \to \infty$ gives (2.5) for this case. If $N/p < m \le \infty$, arguing as in [5, Section 3.2] gives

$$\|u\|_\infty \le C\,\|f\|_m^{1/(p-1)}\,. \tag{2.7}$$

This concludes the proof. □

2.4. **Critical groups at zero.** In this subsection we consider the problem

$$\begin{cases} -\operatorname{div}\left(|\nabla u|^{p-2}\,\nabla u + a(x)\,|\nabla u|^{q-2}\,\nabla u\right) = \lambda\,|u|^{p-2}\,u + g(x,u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.8}$$

where $\lambda \in \mathbb{R}$ is a parameter and g is a Carathéodory function on $\Omega \times \mathbb{R}$ satisfying

$$|g(x,t)| \le C\left(|t|^{r-1} + |t|^{\sigma-1}\right) \quad \text{for a.a. } x \in \Omega \text{ and all } t \in \mathbb{R} \tag{2.9}$$

for some $p < \sigma < r < p^*$ and $C > 0$. Problem (2.8) has the trivial solution $u = 0$, and we study the critical groups at 0 of the associated functional

$$\Phi(u) = \int_\Omega \left[\frac{1}{p}\,|\nabla u|^p + \frac{a(x)}{q}\,|\nabla u|^q - \frac{\lambda}{p}\,|u|^p - G(x,u)\right] dx, \quad u \in W_0^{1,\mathcal{H}}(\Omega),$$

where $G(x,t) = \int_0^t g(x,s)\,ds$. Let us recall that the critical groups of Φ at 0 are given by

$$C^q(\Phi,0) := H^q(\Phi^0 \cap U, \Phi^0 \cap U \setminus \{0\}), \quad q \in \mathbb{N}, \tag{2.10}$$

where $\Phi^0 = \left\{u \in W_0^{1,\mathcal{H}}(\Omega) : \Phi(u) \le 0\right\}$, U is any neighborhood of 0, and H denotes Alexander-Spanier cohomology with $\mathbb{Z}_2$-coefficients. They are independent of U by the excision property of the cohomology groups. They are also invariant under homotopies that preserve the isolatedness of the critical point by the following proposition (see Chang and Ghoussoub [4] or Corvellec and Hantoute [8]).

Proposition 2.5 (Homotopical invariance)**.** *Let Φ_τ, $\tau \in [0,1]$ be a family of C^1-functionals on a Banach space W such that 0 is a critical point of each Φ_τ. If there is a closed neighborhood U of 0 such that*

1. *each Φ_τ satisfies the* (PS) *condition over U,*
2. *U contains no other critical point of any Φ_τ,*
3. *the map $[0,1] \to C^1(U,\mathbb{R})$, $\tau \mapsto \Phi_\tau$ is continuous,*

then $C^q(\Phi_0,0) \approx C^q(\Phi_1,0)$ for all q.

First we show that the critical groups of Φ at 0 depend only on the values of $g(x,t)$ for small $|t|$.

Lemma 2.6. *Let $\delta > 0$ and let $\vartheta : \mathbb{R} \to [-\delta, \delta]$ be a smooth nondecreasing function such that*

$$\vartheta(t) = -\delta \quad \text{for } t \le -\delta, \qquad \vartheta(t) = t \quad \text{for } -\delta/2 \le t \le \delta/2, \qquad \vartheta(t) = \delta \quad \text{for } t \ge \delta.$$

Let us set

$$\Phi_1(u) = \int_\Omega \left[\frac{1}{p} |\nabla u|^p + \frac{a(x)}{q} |\nabla u|^q - \frac{\lambda}{p} |u|^p - G(x, \vartheta(u)) \right] dx, \quad u \in W_0^{1,\mathcal{H}}(\Omega).$$

If 0 is an isolated critical point of Φ, then it is also an isolated critical point of Φ_1 and

$$C^q(\Phi, 0) \approx C^q(\Phi_1, 0), \quad \text{for all } q.$$

Proof. We apply Proposition 2.5 to the family of functionals, for $u \in W_0^{1,\mathcal{H}}(\Omega)$ and $\tau \in [0,1]$,

$$\Phi_\tau(u) := \int_\Omega \left[\frac{1}{p} |\nabla u|^p + \frac{a(x)}{q} |\nabla u|^q - \frac{\lambda}{p} |u|^p - G(x, (1-\tau)\, u + \tau\, \vartheta(u)) \right] dx,$$

in a ball $B_\varepsilon(0) = \{u \in W_0^{1,\mathcal{H}}(\Omega) : \|u\| \le \varepsilon\}$ for $\varepsilon > 0$ small, after noting that $\Phi_0 = \Phi$. Proposition 2.3 implies that each Φ_τ satisfies the Palais-Smale condition over the ball $B_\varepsilon(0)$ and it is readily seen that the map $[0,1] \ni \tau \mapsto \Phi_\tau \in C^1(B_\varepsilon(0), \mathbb{R})$ is continuous, so it only remains to show that for sufficiently small $\varepsilon > 0$, $B_\varepsilon(0)$ contains no critical point of any Φ_τ other than 0. Suppose $u_j \to 0$ in $W_0^{1,\mathcal{H}}(\Omega)$, $\Phi'_{\tau_j}(u_j) = 0$, $\tau_j \in [0,1]$ and $u_j \ne 0$. Then u_j is a weak solution to

$$\begin{cases} -\operatorname{div}\left(|\nabla u_j|^{p-2} \nabla u_j + a(x) |\nabla u_j|^{q-2} \nabla u_j \right) = \lambda |u_j|^{p-2} u_j + g_j(x, u_j) & \text{in } \Omega \\ u_j = 0 & \text{on } \partial\Omega, \end{cases}$$

where we have set

$$g_j(x,t) = (1 - \tau_j + \tau_j\, \vartheta'(t))\, g(x, (1-\tau_j)\, t + \tau_j\, \vartheta(t)).$$

Since $(1-\tau_j)\, t + \tau_j\, \vartheta(t) = t$ for $|t| \le \delta/2$ and $|(1-\tau_j)\, t + \tau_j\, \vartheta(t)| \le |t| + \delta < 3\,|t|$ for $|t| > \delta/2$, the growth estimate (2.9) implies that, for some $C > 0$ independent of j,

$$|g_j(x,t)| \le C \left(|t|^{r-1} + |t|^{\sigma-1} \right) \quad \text{for a.a. } x \in \Omega \text{ and all } t \in \mathbb{R}.$$

Then $u_j \in L^\infty(\Omega)$ (cf. [5, Section 3.2]) with L^∞-bound independent of j. Since $u_j \to 0$ in $W_0^{1,p}(\Omega)$, it follows $\|u_j\|_\ell \to 0$ for any $\ell \ge 1$, as $j \to \infty$. By Proposition 2.4, applied with the choice

$$f_j(x) := \lambda |u_j(x)|^{p-2} u_j(x) + g_j(x, u_j(x)), \quad j \in \mathbb{N},\ x \in \Omega,$$

we get $\|u_j\|_\infty \to 0$ since for a fixed $m_0 > N/p$ we have, for $j \to \infty$,

$$\int_\Omega |f_j|^{m_0} dx \le C \|u_j\|_{m_0(p-1)}^{m_0(p-1)} + C \|u_j\|_{m_0(r-1)}^{m_0(r-1)} + C \|u_j\|_{m_0(\sigma-1)}^{m_0(\sigma-1)} \to 0.$$

For sufficiently large j we thus have $|u_j(x)| \le \delta/2$ for a.e. $x \in \Omega$ and, hence, $\Phi'(u_j) = \Phi'_{\tau_j}(u_j) = 0$, contradicting the assumption that 0 is an *isolated* critical point of Φ. □

In the absence of a direct sum decomposition, the main technical tool to get an estimate of the critical groups is the notion of cohomological local splitting introduced in Perera, Agarwal and O'Regan [18], which is a variant of the homological linking of Perera [17] (see [13]). The following slightly different form of this notion was given in Degiovanni, Lancelotti, and Perera [11].

Definition 2.7. We say that a C^1-functional Φ on a Banach space W has a cohomological local splitting near 0 in dimension $k \geq 1$ if there are symmetric cones $W_\pm \subset W$ with $W_+ \cap W_- = \{0\}$ and $\rho > 0$ such that

$$i(W \setminus W_+) = i(W_- \setminus \{0\}) = k$$

and

$$\Phi(u) \geq \Phi(0) \quad \forall u \in B_\rho \cap W_+, \qquad \Phi(u) \leq \Phi(0) \quad \forall u \in B_\rho \cap W_-, \tag{2.11}$$

where i denotes the $\mathbb{Z}_2$-cohomological index and $B_\rho = \{u \in W : \|u\| \leq \rho\}$.

We recall the definition of the cohomological index (see Fadell and Rabinowitz [12]). For a symmetric subset M of $W \setminus \{0\}$, let $\overline{M} = M/\mathbb{Z}_2$ be the quotient space of M with each u and $-u$ identified, let $f : \overline{M} \to \mathbb{R}\mathrm{P}^\infty$ be the classifying map of $\overline{M}$, and let $f^* : H^*(\mathbb{R}\mathrm{P}^\infty) \to H^*(\overline{M})$ be the induced homomorphism of the Alexander-Spanier cohomology rings. Then the cohomological index of M is defined by

$$i(M) = \begin{cases} \sup\left\{m \geq 1 : f^*(\omega^{m-1}) \neq 0\right\}, & M \neq \emptyset \\ 0, & M = \emptyset, \end{cases}$$

where $\omega \in H^1(\mathbb{R}\mathrm{P}^\infty)$ is the generator of the polynomial ring $H^*(\mathbb{R}\mathrm{P}^\infty) = \mathbb{Z}_2[\omega]$. For example, the classifying map of the unit sphere S^{m-1} in $\mathbb{R}^m$, $m \geq 1$ is the inclusion $\mathbb{R}\mathrm{P}^{m-1} \subset \mathbb{R}\mathrm{P}^\infty$, which induces isomorphisms on H^q for $q \leq m-1$, so $i(S^{m-1}) = m$.

Proposition 2.8 (Degiovanni, Lancelotti, and Perera [11, Proposition 2.1])**.** *Assume that* 0 *is an isolated critical point of* Φ *and that* Φ *has a cohomological local splitting near* 0 *in dimension* k*. Then it holds* $C^k(\Phi, 0) \neq 0$.

In order to give sufficient conditions for Φ to have a cohomological local splitting near 0, and hence a nontrivial critical group by Proposition 2.8, consider the asymptotic eigenvalue problem

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2} u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.12}$$

Let

$$I(u) = \int_\Omega |\nabla u|^p \, dx, \quad J(u) = \int_\Omega |u|^p \, dx, \quad u \in W_0^{1,p}(\Omega),$$

and set

$$\Psi(u) = \frac{1}{J(u)}, \quad u \in \mathcal{M} = \left\{u \in W_0^{1,p}(\Omega) : I(u) = 1\right\}.$$

Then eigenvalues of problem (2.12) coincide with critical values of Ψ. Let $\mathcal{F}$ denote the class of symmetric subsets of $\mathcal{M}$, and set

$$\lambda_k := \inf_{\substack{M \in \mathcal{F} \\ i(M) \geq k}} \sup_{u \in M} \Psi(u), \quad k \geq 1. \tag{2.13}$$

Then $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \nearrow +\infty$ is a sequence of eigenvalues of (2.12) and

$$\lambda_k < \lambda_{k+1} \implies i(\{u \in \mathcal{M} : \Psi(u) \leq \lambda_k\}) = i(\{u \in \mathcal{M} : \Psi(u) < \lambda_{k+1}\}) = k \tag{2.14}$$

(see [18, Propositions 3.52 and 3.53]). The main result of this subsection is the following theorem.

Theorem 2.9 (Critical groups at 0)**.** *Assume that* g *satisfies* (2.9) *and* 0 *is an isolated critical point of* Φ.

(1) $C^0(\Phi, 0) \approx \mathbb{Z}_2$ *and* $C^q(\Phi, 0) = 0$ *for* $q \geq 1$ *in the following cases:*
   (a) $\lambda < \lambda_1$*;*

(b) $\lambda = \lambda_1$ *and, for some* $\delta > 0$, $G(x,t) \le 0$ *for a.a.* $x \in \Omega$ *and* $|t| \le \delta$.

(2) $C^k(\Phi, 0) \ne 0$ *in the following cases:*

(a) $\lambda_k < \lambda < \lambda_{k+1}$;

(b) $\lambda_k < \lambda = \lambda_{k+1}$ *and, for some* $\delta > 0$, $G(x,t) \le 0$ *for a.a.* $x \in \Omega$ *and* $|t| \le \delta$;

(c) $\lambda_k = \lambda < \lambda_{k+1}$ *and* $G(x,t) \ge c\,|t|^s$ *for a.a.* $x \in \Omega$ *and all* $t \in \mathbb{R}$ *for some* $s \in (p,q)$ *and* $c > 0$.

Proof. We have

$$\Phi(u) = \frac{1}{p}\left[I(u) - \lambda\, J(u)\right] + \int_\Omega \left[\frac{a(x)}{q}\,|\nabla u|^q - G(x,u)\right] dx. \tag{2.15}$$

By (2.9) and the Sobolev embedding, we have

$$\int_\Omega G(x,u)\,dx = \mathrm{o}\big(\,\|\nabla u\|_p^p\,\big), \quad \text{as } \|\nabla u\|_p \to 0, \tag{2.16}$$

and in view of Lemma 2.6, without loss of generality, we may assume that the sign conditions on G in (1)(*b*) and (2)(*b*) hold for every $t \in \mathbb{R}$.

(1) We show that 0 is a local minimizer of Φ. Since $\Psi(u) \ge \lambda_1$ for all $u \in \mathcal{M}$, we have

$$I(u) \ge \lambda_1\, J(u), \quad \forall u \in W_0^{1,p}(\Omega). \tag{2.17}$$

• (*a*) By (2.15)–(2.17), we get

$$\Phi(u) \ge \frac{1}{p}\left(1 - \frac{\lambda_+}{\lambda_1} + \mathrm{o}(1)\right)\|\nabla u\|_p^p, \quad \text{as } \|\nabla u\|_p \to 0,$$

where $\lambda_+ = \max\{\lambda, 0\}$. So $\Phi(u) \ge 0$ for all $u \in B_\rho$ for sufficiently small $\rho > 0$ by (2.2).

• (*b*) By (2.15) and (2.17), we get

$$\Phi(u) \ge -\int_\Omega G(x,u)\,dx \ge 0, \quad \forall u \in W_0^{1,\mathcal{H}}(\Omega).$$

(2) We show that Φ has a cohomological local splitting near 0 in dimension k and then apply Proposition 2.8. In light of [10, Theorem 2.3], the set $\{u \in W_0^{1,p}(\Omega) : I(u) \le \lambda_k\, J(u)\}$ contains a *symmetric cone* W_- with $i(W_- \setminus \{0\}) = k$ and $\{u \in W_- : \|u\|_p = 1\}$ is bounded in $C^1(\Omega)$, so that, in particular, we have the inequality

$$\int_\Omega \frac{a(x)}{q}\,|\nabla u|^q\,dx \le C\,\|u\|_p^q, \qquad \forall u \in W_-, \tag{2.18}$$

for some $C > 0$. Since $W_0^{1,\mathcal{H}}(\Omega)$ is embedded in $W_0^{1,p}(\Omega)$ as a *dense linear subspace*, the inclusion

$$\big\{u \in W_0^{1,\mathcal{H}}(\Omega) : I(u) < \lambda_{k+1}\, J(u)\big\} \subset \big\{u \in W_0^{1,p}(\Omega) : I(u) < \lambda_{k+1}\, J(u)\big\}$$

is a *homotopy equivalence* by Palais (cf. [16, Theorem 17]), so

$$i\big(\big\{u \in W_0^{1,\mathcal{H}}(\Omega) : I(u) < \lambda_{k+1}\, J(u)\big\}\big) = i\big(\big\{u \in W_0^{1,p}(\Omega) : I(u) < \lambda_{k+1}\, J(u)\big\}\big) = k$$

by virtue of (2.14). We take now

$$W_+ := \big\{u \in W_0^{1,\mathcal{H}}(\Omega) : I(u) \ge \lambda_{k+1}\, J(u)\big\}.$$

It only remains to show that (2.11) holds for sufficiently small $\rho > 0$.

• (*a*) For $u \in W_+$, we obtain

$$\Phi(u) \ge \frac{1}{p}\left(1 - \frac{\lambda}{\lambda_{k+1}} + \mathrm{o}(1)\right)\|\nabla u\|_p^p, \quad \text{as } \|\nabla u\|_p \to 0$$

by virtue of (2.15) and (2.16). So $\Phi(u) \geq 0$ for all $u \in B_\rho \cap W_+$ for sufficiently small $\rho > 0$ by (2.2). For $u \in W_-$,

$$\Phi(u) \leq -\frac{1}{p}\left(\frac{\lambda}{\lambda_k} - 1 + \mathrm{o}(1)\right) \|\nabla u\|_p^p \quad \text{as } \|\nabla u\|_p \to 0$$

by (2.15), (2.16), and (2.18) since $q > p$. So $\Phi(u) \leq 0$ for all $u \in B_\rho \cap W_-$ for small $\rho > 0$ by (2.2).

• (b) For $u \in W_+$, we have

$$\Phi(u) \geq -\int_\Omega G(x,u)\,dx \geq 0$$

by (2.15), and $\Phi(u) \leq 0$ for all $u \in B_\rho \cap W_-$ for small $\rho > 0$ as in (a).

• (c) We have $\Phi(u) \geq 0$ for all $u \in B_\rho \cap W_+$ for sufficiently small $\rho > 0$ as in (i). For $u \in W_-$,

$$\Phi(u) \leq C\,\|u\|_p^q - \frac{\|u\|_p^s}{C}$$

for some $C > 0$ by (2.15), (2.18), and since $s > p$. Since $s < q$, then $\Phi(u) \leq 0$ for all $u \in B_\rho \cap W_-$ for sufficiently small $\rho > 0$ by (2.2). □

2.5. **Nontrivial solutions.** In this subsection we obtain a nontrivial solution of the problem

$$\begin{cases} -\operatorname{div}\left(|\nabla u|^{p-2}\,\nabla u + a(x)\,|\nabla u|^{q-2}\,\nabla u\right) = \lambda\,|u|^{p-2}\,u + |u|^{r-2}\,u + h(x,u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.19}$$

where $\lambda \in \mathbb{R}$ is a parameter, $r \in (q, p^*)$, and h is a Carathéodory function on $\Omega \times \mathbb{R}$ satisfying

$$|h(x,t)| \leq C\left(|t|^{\rho-1} + |t|^{\sigma-1}\right) \quad \text{for a.a. } x \in \Omega \text{ and all } t \in \mathbb{R} \tag{2.20}$$

for some $p < \sigma < \rho < r$ and $C > 0$. First we verify that the associated functional

$$\Phi(u) = \int_\Omega \left[\frac{1}{p}\,|\nabla u|^p + \frac{a(x)}{q}\,|\nabla u|^q - \frac{\lambda}{p}\,|u|^p - \frac{1}{r}\,|u|^r - H(x,u)\right] dx, \quad u \in W_0^{1,\mathcal{H}}(\Omega),$$

where $H(x,t) = \int_0^t h(x,s)\,ds$, satisfies the (PS) condition. We note that

$$\begin{aligned} q\,\Phi(u) - \langle \Phi'(u), u\rangle = \left(\frac{q}{p} - 1\right)\int_\Omega \left(|\nabla u|^p - \lambda\,|u|^p\right) dx + \left(1 - \frac{q}{r}\right)\int_\Omega |u|^r\,dx \\ + \int_\Omega \left(h(x,u)\,u - q\,H(x,u)\right) dx. \end{aligned} \tag{2.21}$$

Lemma 2.10 (Palais-Smale condition)**.** *Every sequence $(u_j) \subset W_0^{1,\mathcal{H}}(\Omega)$ such that $(\Phi(u_j))$ is bounded and $\Phi'(u_j) \to 0$ has a convergent subsequence.*

Proof. It suffices to show that (u_j) is bounded by Proposition 2.3. Since $p < q < r$ and $\sigma < \rho < r$, it follows from (2.21), (2.20) and the Hölder and Young's inequalities that $\|u_j\|_r^r \leq C + \mathrm{o}(\|u_j\|)$ for some $C > 0$. Then

$$\int_\Omega \left[\frac{1}{p}\,|\nabla u_j|^p + \frac{a(x)}{q}\,|\nabla u_j|^q\right] dx = \Phi(u_j) + \int_\Omega \left[\frac{\lambda}{p}\,|u_j|^p + \frac{1}{r}\,|u_j|^r + H(x,u_j)\right] dx \leq C + \mathrm{o}(\|u_j\|),$$

which together with (2.2) gives the desired conclusion. □

Next we study the structure of the sublevel sets of Φ at infinity.

Lemma 2.11. *There exists $\alpha < 0$ such that the sublevel set*

$$\Phi^\alpha := \left\{u \in W_0^{1,\mathcal{H}}(\Omega) : \Phi(u) \le \alpha\right\}$$

is contractible in itself.

Proof. Since $p < q < r$ and $\sigma < \rho < r$, it follows from (2.21), (2.20), and the Young's inequality that $\langle \Phi'(u), u\rangle - q\Phi(u)$ is bounded from above, so for $\alpha < 0$ with $|\alpha|$ sufficiently large,

$$\langle \Phi'(u), u\rangle < 0, \quad \forall u \in \Phi^\alpha. \tag{2.22}$$

For $u \in W_0^{1,\mathcal{H}}(\Omega) \setminus \{0\}$, taking into account that $\Phi(tu) \to -\infty$ as $t \to +\infty$, set

$$t(u) = \inf\left\{t \ge 1 : \Phi(tu) \le \alpha\right\},$$

and note that the function $u \mapsto t(u)$ is continuous by (2.22) and the implicit function theorem. Then the map $u \mapsto t(u)\, u$ is a *retraction* of $W_0^{1,\mathcal{H}}(\Omega) \setminus \{0\}$ onto Φ^α, and the conclusion follows since the former is contractible in itself. □

2.6. **Proof of Theorem 1.1 completed.** We are now ready to prove the main result. Let (λ_k) be the sequence of eigenvalues of problem (2.12) defined in (2.13). Suppose that 0 is the only critical point of Φ. Taking $U = W_0^{1,\mathcal{H}}(\Omega)$ in (2.10), we have

$$C^q(\Phi, 0) = H^q(\Phi^0, \Phi^0 \setminus \{0\}).$$

Let $\alpha < 0$ be as in Lemma 2.11. Since Φ has no other critical points and satisfies the (PS) condition by Lemma 2.10, Φ^0 is a deformation *retract* of $W_0^{1,\mathcal{H}}(\Omega)$ and Φ^α is a deformation *retract* of $\Phi^0 \setminus \{0\}$ by the second deformation lemma. So

$$C^q(\Phi, 0) \approx H^q(W_0^{1,\mathcal{H}}(\Omega), \Phi^\alpha) = 0 \quad \forall q \in \mathbb{N},$$

since Φ^α is contractible in itself, contradicting Theorem 2.9 in each of the cases (1)–(3). □

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