Sharp isoperimetric inequalities for small volumes in complete noncompact Riemannian manifolds of bounded geometry involving the scalar curvature

Stefano Nardulli, Luis Eduardo Osorio Acevedo*

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ABSTRACT. We provide an isoperimetric comparison theorem for small volumes in an n-dimensional Riemannian manifold (M^n, g) with strong bounded geometry, as in Definition 2.3, involving the scalar curvature function. Namely in strong bounded geometry, if the supremum of scalar curvature function $S_g < n(n-1)k_0$ for some $k_0 \in \mathbb{R}$, then for small volumes the isoperimetric profile of (M^n, g) is less then or equal to the isoperimetric profile of $\mathbb{M}^n_{k_0}$ the complete simply connected space form of constant sectional curvature k_0. This work generalizes Theorem 2 of [Dru02b] in which the same result was proved in the case where (M^n, g) is assumed to be just compact. As a consequence of our result we give an asymptotic expansion in Puiseux's series up to the second nontrivial term of the isoperimetric profile function for small volumes. Finally, as a corollary of our isoperimetric comparison result, it is shown, in the special case of manifolds with strong bounded geometry, and $S_g < n(n-1)k_0$ that for small volumes the Aubin-Cartan-Hadamard's Conjecture in any dimension n is true.

Key Words: Isoperimetric comparison, isoperimetric inequalities, small volumes, bounded geometry, Aubin-Cartan-Hadamard's conjecture, finite perimeter sets, metric geometry, calculus of variations, geometric measure theory, Sobolev's inequalities on manifolds, scalar curvature, partial differential equations on manifolds, smoothing Riemannian manifolds, Ricci flow.

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1 Introduction

For the rigorous technical meaning of the concepts involved in this informal introductory section we refer the reader to Sections 2 and 3. Let us start recalling what is the isoperimetric problem. The isoperimetric problem in a Riemannian manifold (M^n, g) of dimension n, consists in the study of the existence and geometric characterization of isoperimetric regions (see Definition 2.8), i.e., domains which minimizes area under a fixed volume constraint. Together with this problem comes naturally the study of the *isoperimetric profile* function $I_{M,g}$ assigning to every $v \in [0, V_g(M)[$ the infimum of the areas of open bounded sets with smooth boundary and fixed volume v (which is well known to be equal to I_M of Definition 2.8, as could be seen comparing with Theorem 1 of [MFN15]). A natural question arises when dealing with isoperimetric profiles.

Question 1. *It is possible to compare the isoperimetric profile of differents Riemannian manifolds just comparing the way in which they are curved?*

During the lasts decades various attempts were done to answer to this question and several partial results were obtained. We outline here the more relevant (at our knowledge) to the present work. It is well known that the isoperimetric profile $I_{\mathbb{R}^n}(v) = c_n v^{\frac{n-1}{n}}$, for some explicit positive constant c_n depending only on the dimension n of the manifold. We call c_n the *Euclidean isoperimetric constant*. Furthermore it is a classical result that the isoperimetric regions (see Definition 2.8) in Euclidean space are the geodesic balls. so the constant c_n is known explicitly. The Aubin-Cartan-Hadamard's conjecture states that for any Cartan-Hadamard manifold (M^n, g) of dimension n (i.e., a simply connected manifold with sectional curvature less than or equal to 0), every finite perimeter set $\Omega \subseteq M$ satisfies the inequality $\mathcal{P}_g(\Omega) \geq c_n V_g(\Omega)^{\frac{n-1}{n}}$. This conjecture was proved to be true if $n = 2$ in [Wei26], if $n = 3$ first in [Kle92] and later in [Rit05] and [Sch08], if $n = 4$ in [Cro84], finally as a corollary of Theorem 2 of [Dru02b] a weaker form of the conjecture follows for any dimension n assuming M to be compact (of course when M is compact we do not have that the manifold is simply connected and with nonpositive curvature) and Ω having small volume. It is worth noting that in [Cro84] it is shown that $\mathcal{P}_g(\Omega) \geq \tilde{c}_n V_g(\Omega)^{\frac{n-1}{n}}$ is true in any dimension but with a positive constant $\tilde{c}_n$ that is not sharp in general. However the proof of C. Croke gives in the special case $n = 4$

that $\tilde{c}_4 = c_4$. In dimension $n \geq 5$ the Aubin-Cartan-Hadamard conjecture is still open. At our knowledge the only previous partial results in any dimension n with the sharp constant, but restricted to the small volume regime, are Theorem 4.3 of [MJ00] which require additional assumptions on the Gauss-Bonnet-Chern integrand in even dimension, and Theorem 2 of [Dru02b] in case of compact manifolds and Corollary 2 of [MFN15] in case of noncompact manifolds with C^2-locally asymptotically bounded geometry at infinity (compare Definition 2.7) that is the noncompact version of Theorem 4.4 of [MJ00], but this requires a bound on the sectional curvature. Our Corollary 1 extends these partial results in any dimension to domains Ω of small volume inside a Cartan-Hadamard manifold (M, g) having strong bounded geometry. The difference between our result and Theorem 2 of [Dru02b] is that we relax the assumption on the manifold M of being compact and replace it by just requiring that M have C^2-locally asymptotically bounded geometry at infinity (see Definition 2.7) or requiring just strong bounded geometry but loosing a little bit on the strict inequality sign in (9). For a more exhaustive treatment about the state of the art of the Aubin-Cartan-Hadamard's conjecture we suggest the reading of the very good surveys of Olivier Druet [Dru10] available online, Section 3.2 of Manuel Ritoré in [RS10], and the very recent and interesting paper [BK13]. Let us state here Theorem 1 of [Dru02b].

Theorem 1.1 (Theorem 1 of [Dru02b])**.** *Let (M^n, g) be a complete Riemannian manifold, with $n \geq 2$, $x \in M$ such that there exists $k_0 \in \mathbb{R}$ satisfying $Sc_g(x) < n(n-1)k_0$. Then there exists $r_x > 0$ such that for every finite perimeter set Ω contained in the geodesic ball of center x and radius r_x,*

$$\mathcal{P}_g(\Omega) > \mathcal{P}_{g_{k_0}}(B), \tag{1}$$

where B is a ball enclosing a volume v in the model simply connected space form $(\mathbb{M}^n_{k_0}, g_{k_0})$ of constant sectional curvature k_0.

It should be seen from the proof of the preceding result and Theorems 1-5 of this paper that a lower bound of the optimal r_x is continuous with respect to $Sc_g(x)$ and C^0 convergence of metrics, so if M is compact there exists $r := \inf\{r_x : x \in M\} > 0$ such that the conclusion of the Theorem 1.1 holds for any Ω contained in a ball of radius r. Unfortunately the radius r_x could go to zero when x tends to infinity in an arbitrary noncompact complete Riemannan manifold. Hence some extra assumptions on the geometry at infinity of M are needed to allow us to find such a positive uniform lower bound r. Actually using the

last equation at page 2353 of [Dru02b] and reasoning by contradiction it appears evident from the proof that to have C^2-locally asymptotically strong bounded geometry smooth at infinity, gives such a lower bound. A necessary condition to have $r > 0$ is that the volume of balls of a fixed radius for example $r/2$ does not vanish when the centers go to infinity. This is a non collapsing condition that for example follows assuming $Ricci_g \geq (n-1)kg$ for some $k \in \mathbb{R}$ and positivity of the injectivity radius. Thus it seems natural to make these assumptions in our Theorem 1. Actually, in other parts of the proof we will need to strengthen a little more our assumptions on the geometry of M and we are lead to assume that M have strong bounded geometry in the sense of our Definition 2.3. To obtain our main result about small volumes, in first we prove a global isoperimetric comparison for small diameters in Theorem 1 when M has C^2-locally asymptotically bounded geometry at infinity, then we refine it in Theorem 3 requiring only that M have strong bounded geometry, using smoothing of the metric via Ricci flow. Our Theorem 1 replaces the local isoperimetric comparison given by Theorem 1 of [Dru02b] with a global one under the assumptions of C^2-locally asymptotically bounded geometry at infinity (see Definition 2.3). Given granted the proof of Theorem 1 we then prove Theorem 2 by geometric measure theory and Gromov-Hausdorff convergence of manifolds. The proof of Theorem 1 goes along the same lines of Theorem 1 of [Dru02b]. So the main ingredients used in its proof are results about local optimal Sobolev inequalities in $W^{1,p}$ via PDE techniques when $p > 1$ which are easier to obtain than when $p = 1$. After the limit problem when $p \to 1^+$ is studied. These local optimal Sobolev inequalities in $W^{1,p}$ are combined with an asymptotic analysis of solutions of quasi-elliptic equations involving the p-Laplacian when the parameter $p \to 1^+$. The importance of the scalar curvature when studying sharp Sobolev inequalities on Riemannian manifolds was first observed by Olivier Druet in [Dru98], later by Hebey in [Heb02] and appears evident when deducing Theorem 1 [Dru02b] from Proposition 1 of [Dru02b]. The modifications required to achieve our goals are nontrivial, so to make the paper self-contained we wrote the entire proof of Theorem 1 in Section 3. Theorem 2 is a consequence of Theorem 1 using techniques of geometric measure theory, say the theory of sets of finite perimeter, comparison geometry, and Gromov-Hausdorff convergence of manifolds. The proof follows the scheme traced by the proof of Theorem 2 of [Dru02b], however the required changes in the proof are highly nontrivial and original. The two main difficulties that are encountered when one tries to apply the proof of Theorem 2 of [Dru02b] (working only for compact manifolds) to our

more general context consist in the fact that existence of isoperimetric regions for every volume in a noncompact Riemannian manifold is no longer guaranteed and that one needs to prove that isoperimetric regions of small volumes are also of small diameter. For an account of results on the problem of existence of isoperimetric regions (see Definition 2.8) in complete Riemannian manifolds the reader is referred to [Nar14a], [MN16] and the references therein. Our approach to solve this difficulty is to use the theory of generalized existence and generalized compactness developed by the first author in [Nar14a], [MN15], and replace genuine isoperimetric regions in M by generalized isoperimetric regions lying in some pointed limit manifold. This is possible because by a compactness theorem of the theory of convergence of manifolds namely Theorem 76 of [Pet06] the hypotheses of Theorem 1 of [MN15] are automatically fulfilled in the context of C^2-locally asymptotic strong bounded geometry smooth at infinity that we consider here. To finish the proof of Theorem 2 we need to prove that in C^2-locally asymptotically strong bounded geometry (compare Definition 2.3) for an isoperimetric region having small volumes implies small diameter. With this aim in mind we replace the proof of [MJ00] based on Nash's isometric embeddings by another intrinsic one. We carry out this task proving a little more general result in Lemma 4.9, which asserts that if just *Ricci* is bounded below and the injectivity radius is positive, isoperimetric regions of small volumes are of small diameter. In this proof we don't need to use any monotonicity formula; this fact constitutes a novelty with respect to the classical extrinsic proof of [MJ00]. Our proof is completely intrinsic and uses a cut and paste argument inspired by Proposition 2.5 of [Nar14b] (which works only for manifolds of strong bounded geometry) adapted to the case of weak bounded geometry (see Definition 2.1) joint with others non trivial intrinsic arguments aimed to encompass some technical difficulties of geometric measure theory, which arise when passing from the *Euclidean space* $\mathbb{R}^n$ to an arbitrary Riemannian manifold (M^n, g) without using Nash's isometric embedding theorem. The arguments of the proof permit also to give an effective estimate of the constants of Lemma 4.9 as functions of the bounds of the geometry of (M^n, g). The main result of this paper is Theorem 5. To prove it we need to prove beforehand Theorem 2 and then apply Theorem 2 to a suitable smoothing (M, g_t) with initial data (M, g) along the Ricci flow. The main reason is that for every $m \in \mathbb{N} = \{0, 1, 2, ...\}$ the smoothed manifolds (M, g_t) have $C^{m,\alpha}$ bounded geometry in the sense of Definition 2.6 and so at infinity are more regular than the original (M, g). This permits the use of the arguments of the proof of Theorem 1 to (M, g_t) which does not work

if applied directly to (M, g). Hence passing to the limit when $t \to 0^+$, and using the results of [Shi89], [Kap05], we can transport the isoperimetric comparison from (M, g_t) to (M, g) without any further difficulty. As corollaries of our main Theorem 5 we get immediately Corollary 2 that is a special case of the Aubin-Cartan-Hadamard's conjecture and the expansion of the isoperimetric profile in Puiseux's series given by Corollary 3. As a final remark we have that all the constants involved in our statements of Section 2 are effectively computed in terms of the minimal bounds on the geometry that we are assuming.

2 Main results

In the sequel we always assume that all the Riemannian manifolds M^n considered are smooth with smooth Riemannian metric g. We denote by V_g the canonical Riemannian measure induced on M by g, and by A_g the $(n-1)$-Hausdorff measure associated to the canonical Riemannian length space metric d of M, that we also denote by $\mathcal{H}^{n-1}_g$. When it is already clear from the context, explicit mention of the metric g will be suppressed. We will denote by Ric_g the Ricci tensor of (M, g), by Sec_g the sectional curvature of (M, g), Sc_g the scalar curvature function, $S_g := \sup_{x\in M} \{Sc_g(x)\}$ and by $\mathbb{M}^n_k$ the simply connected space form endowed with the standard metric of constant sectional curvature $k \in \mathbb{R}$ that we denote by g_k, by $inj_{(M,g)}$ the injectivity radius of M, for any $D \subseteq M$, $diam_g(D)$ the diameter of D in the metric space (M, g), dv_g the Riemannian measure with respect to the metric g. In what follows we will consider as a key object the set of all finite perimeter sets (see Definition 3.2) of M that we will denote by $\tilde{\tau}$. So a little technical discussion is in order here. By classical results of geometric measure theory (see Proposition 12.19 and Formula (15.3) of [Mag12]) we know that if E is a set of locally finite perimeter in M, then $spt(\nabla\chi_E) = \{x \in M : 0 < V_g(E \cap B(x, r)), \forall r > 0\} \subseteq \partial E$, furthermore there exists an equivalent Borel set F (i.e., $V_g(E\Delta F) = 0$) such that $spt(\nabla\chi_F) = \partial F = \overline{\partial^* F}$, where $\partial^* F$ is the reduced boundary of F. It is not too hard to show that if E has C^1 boundary, then $\partial^* E = \partial E$, where ∂E is the topological boundary of E. De Giorgi's structure theorem (compare Theorem 15.9 of [Mag12]) guarantees that for every set E of locally finite perimeter, $A_g(\partial^* E) = \mathcal{H}^{n-1}_g(\partial^* E) = \mathcal{P}_g(E)$. Hence without loss of generality we will adopt the assumption that all the locally finite perimeter sets considered in this text satisfy $\overline{\partial^* E} = \partial E$. It is worth to mention that the results in the book [Mag12] are stated and proved in

$\mathbb{R}^n$ but they are valid mutatis mutandis also in an arbitrary complete Riemannian manifold, the required details could be easily provided using the work about BV-functions on a Riemannian manifold accomplished in [MPPP07].

Definition 2.1. *A complete Riemannian manifold (M, g), is said to have **weak bounded geometry**, if there exists a constant $k \in \mathbb{R}$, such that $Ric_M \geq k(n-1)$ (i.e., $Ric_M \geq k(n-1)g$ in the sense of quadratic forms) and $V(B_{(M,g)}(p, 1)) \geq v_0 > 0$, for some positive constant v_0, where $B_{(M,g)}(p, r)$ is the geodesic ball of M centered at p and of radius $r > 0$.*

Remark 2.1. *In this paper we differ from the nomenclature used by the first author in his preceding works. What we call here weak bounded geometry is what is called, in all previous articles, just bounded geometry.*

Definition 2.2. *A complete Riemannian manifold (M, g), is said to have **mild bounded geometry**, if there exists a constant $k \in \mathbb{R}$, such that $Ric_M \geq k(n-1)$ (i.e., $Ric_M \geq k(n-1)g$ in the sense of quadratic forms) and $inj_M > 0$, where inj_M is the injectivity radius of M.*

Remark 2.2. *It is known that mild bounded geometry implies weak bounded geometry, but the converse is not true. For more details about this point the reader is referred to Remark 2.5 of [MN16] and to the references therein.*

Definition 2.3. *A complete Riemannian manifold (M, g), is said to have **strong bounded geometry**, if there exists a positive constant $K > 0$, such that $|Sec_M| \leq K$ and $inj_M \geq i_0 > 0$ for some positive constant i_0. Sometimes we will use the condition $\Lambda_1 \leq Sec_M \leq \Lambda_2$, for some given constants $\Lambda_1, \Lambda_2 \in \mathbb{R}$ instead of $|Sec_M| \leq K$ to express that M have a two sided bound on the sectional curvature.*

Remark 2.3. *It turns out that it is easy to check that strong bounded geometry implies mild bounded geometry, with the converse being not true in general.*

Definition 2.4. *For any $m \in \mathbb{N}$, $\alpha \in [0, 1]$, a sequence of pointed smooth complete Riemannian manifolds is said to **converge in the pointed $C^{m,\alpha}$, respectively C^m topology to a smooth manifold** M (denoted by $(M_i, p_i, g_i) \to (M, p, g)$), if for every $R > 0$ we can find a domain Ω_R with $B(p, R) \subseteq \Omega_R \subseteq M$, a natural number $\nu_R \in \mathbb{N}$, and C^{m+1} embeddings $F_{i,R} : \Omega_R \to M_i$, for large $i \geq \nu_R$ such that $B(p_i, R) \subseteq F_{i,R}(\Omega_R)$ and $F_{i,R}^*(g_i) \to g$ on Ω_R in the $C^{m,\alpha}$, respectively C^m topology.*

Definition 2.5. *For any $m \in \mathbb{N}$ and $\alpha \in [0,1]$, we say that a smooth Riemannian manifold (M^n, g) has $C^{m,\alpha}$-**locally asymptotic bounded geometry**, if it is with weak bounded geometry and if for every diverging sequence of points (p_j), there exists a subsequence (p_{j_l}) and a pointed smooth manifold $(M_\infty, g_\infty, p_\infty)$ with g_∞ a smooth Riemannian metric such that the $C^{m,\alpha}$ norm is finite and the sequence of pointed manifolds $(M, p_{j_l}, g) \to (M_\infty, g_\infty, p_\infty)$, in $C^{m,\alpha}$-topology. When $\alpha = 0$ we write C^m instead of $C^{m,0}$.*

Definition 2.6 (Page 308 of [Pet06])**.** *A subset A of a Riemannian n-manifold M has **bounded $C^{m,\alpha}$ norm on the scale of** r, $||A||_{C^{m,\alpha},r} \leq Q$, if every point p of M lies in an open set U with a chart ψ from the Euclidean r-ball into U such that*

(i): For all $p \in A$ there exists U such that $B(p, \frac{1}{10}e^{-Q}r) \subseteq U$.

(ii): $|D\psi| \leq e^Q$ on $B(0,r)$ and $|D\psi^{-1}| \leq e^Q$ on U.

(iii): $r^{|j|+\alpha}||D^j g||_\alpha \leq Q$ for all multi indices j with $0 \leq |j| \leq m$, where g is the matrix of functions of metric coefficients in the ψ coordinates regarded as a matrix on $B(0,r)$.

We write that $(M, g, p) \in \mathcal{M}^{m,\alpha}(n, Q, r)$, if $||M||_{C^{m,\alpha},r} \leq Q$.

Remark 2.4. *The condition of being smooth at infinity is used just in the last equation (164) of the proof of Theorem 2 when we apply Theorem 1 to a possibly limit manifold (M_∞, g_∞) that even in strong bounded geometry is a $C^{3,\beta}$ differentiable manifold but with a metric that is just $C^{1,\beta}$ and no more regular. There are examples of this phenomenon as explained in Example 1.8 of [Pet87]. Actually the limit metric is $W^{2,p}$ for any $p > 1$, as showed in [Nik91]. This last regularity result is not enough strong to allow the use the arguments of the proof of Theorem 1 in (M_∞, g_∞).*

This last remark justifies the following definitions.

Definition 2.7. *We say that a smooth Riemannian manifold (M^n, g) is **smooth at infinity**, if for every diverging sequence of points (p_j), there exists a subsequence (p_{j_l}) and a pointed smooth manifold $(M_\infty, g_\infty, p_\infty)$ with g_∞ of class C^∞. We say that a smooth Riemannian manifold (M^n, g) has **strong bounded geometry smooth at infinity**, if it is of strong bounded geometry and is smooth at infinity. We say that (M^n, g) has $C^{m,\alpha}$-locally asymptotically strong bounded geometry smooth at infinity, if it is of strong bounded geometry, smooth at infinity, and has $C^{m,\alpha}$-**locally asymptotic bounded geometry**.*

Remark 2.5. *Observe that by Theorems* 76 *and* 72 *of [Pet06] or Theorem* 4.4 *of [Pet87] it is easily seen that to have strong bounded geometry smooth at infinity implies to have $C^{1,\beta}$-locally asymptotic bounded geometry, for any β.*

We have now all the definitions needed to state our results.

Theorem 1 (Small diameters in C^2-locally asymptotically strong bounded geometry smooth at infinity)**.** *Let (M^n, g) be a complete Riemannian manifold with $n \geq 2$ and with C^2-locally asymptotically strong bounded geometry smooth at infinity. Let us assume that there exists a real constant $k_0 \in \mathbb{R}$ such that $S_g < n(n-1)k_0$. Then there exists $d = d(n, k, k_0, inj_M, S_g) > 0$, which depends only on n, k, k_0, inj_M, S_g such that for every $\Omega \subseteq M^n$ finite perimeter set with diameter $\operatorname{diam}_g(\Omega) \leq d$ holds*

$$\mathcal{P}_g(\Omega) > \mathcal{P}_{g_{k_0}}(B), \tag{2}$$

where $B \subseteq \mathbb{M}^n_{k_0}$ is a geodesic ball having $V_{g_{k_0}}(B) = V_g(\Omega)$. Moreover we have the following lower bound on the greatest d for which (2) holds, namely $d = d(n, k, k_0, inj_M, S_g)$ could be chosen to be equal to

$$C(n,k)^{-\frac{1}{n}} \left\{ \frac{(n+2)K(n,1)^2}{nC_0(n,k_0)} \left[n(n-1)k_0 - S_g \right] \right\}^{\frac{1}{4}}, \tag{3}$$

see equation (99) for the exact meaning of the constants involved here. Notice that the dependence of d on $inj_M > 0$ appears just to ensure that such a d exists and is positive.

Remark 2.6. *Strict inequality is necessary because as pointed out in [Dru02b] Theorem 1 is false if we have just $Ric_g \leq (n-1)k_0$ and not $S_g < n(n-1)k_0$, as pointed out in [Dru02b]. The comparison result is false also on $S^2 \times S^2$, as noticed in [MJ00], compare again [Dru02b] page* 2352*.*

In the next theorem we refine the results contained in Theorem 1. The price to pay to have this stronger result is that the proof of Theorem 2 is much more involved.

Theorem 2 (Sharp small volumes in C^2-locally asymptotically strong bounded geometry smooth at infinity)**.** *Let (M^n, g) be a complete Riemannian manifold, $n \geq 2$, with C^2-locally asymptotically strong bounded geometry smooth at infinity. Let us assume that there exists a real constant $k_0 \in \mathbb{R}$ such that $S_g < n(n-1)k_0$. Then there exists a positive*

constant $\tilde{v}_0 = \tilde{v}_0(n, k, k_0, inj_{M,g}, S_g) > 0$ *such that for every* $\Omega \subseteq M$ *finite perimeter set with* $V_g(\Omega) \leq \tilde{v}_0$ *it holds*

$$\mathcal{P}_g(\Omega) > \mathcal{P}_{g_{k_0}}(B), \tag{4}$$

where $B \subseteq \mathbb{M}^n_{k_0}$ *is a geodesic ball having* $V_{g_{k_0}}(B) = V_g(\Omega)$. *Moreover* $\tilde{v}_0$ *can be chosen as an arbitrary number*

$$0 < \tilde{v}_0 \leq \left\{ \frac{2(n+2)K(n,1)^2}{nC_0} \left[n(n-1)k_0 - S_g \right] \right\}^{\frac{n}{4}}, \tag{5}$$

where $C_0 = C_0(n, k_0) > 0$.

Theorem 3 (Small diameters in strong bounded geometry)**.** *Let* (M^n, g) *be a complete Riemannian manifold with* $n \geq 2$ *and with strong bounded geometry. Let us assume that there exists a real constant* $k_0 \in \mathbb{R}$ *such that* $S_g < n(n-1)k_0$. *Then there exists* $d = d(n, k, k_0, inj_M, S_g) > 0$ *such that for every* $\Omega \subseteq M^n$ *finite perimeter set with diameter* $\operatorname{diam}_g(\Omega) \leq d$ *holds*

$$\mathcal{P}_g(\Omega) > \mathcal{P}_{g_{k_0}}(B), \tag{6}$$

where $B \subseteq \mathbb{M}^n_{k_0}$ *is a geodesic ball having* $V_{g_{k_0}}(B) = V_g(\Omega)$. *Moreover we have the following lower bound on the greatest* d *for which* (2) *holds, namely* $d = d(n, k, k_0, inj_M, S_g)$ *could be chosen to be equal to*

$$C(n,k)^{-\frac{1}{n}} \left\{ \frac{(n+2)K(n,1)^2}{nC_0(n,k_0)} \left[n(n-1)k_0 - S_g \right] \right\}^{\frac{1}{4}}, \tag{7}$$

see equation (99) *for the exact meaning of the constants involved here. Notice that the dependence of* d *on* $inj_M > 0$ *appears just to ensure that such a* d *exists and is positive.*

A first consequence of Theorem 3 is the following result whose proof is much more simpler than that of our main Theorem 5.

Theorem 4 (Small volumes à la Bérard-Meyer)**.** *Let* (M^n, g) *be a complete Riemannian manifold,* $n \geq 2$, *and with strong bounded geometry. Let us assume that there exists a real constant* $k_0 \in \mathbb{R}$ *such that* $S_{sup} < n(n-1)k_0$. *Then for every* $\varepsilon > 0$ *there exists a positive constant* $\tilde{v}_0 = \tilde{v}_0(M, \varepsilon) > 0$ *such that for every* $\Omega \subseteq M$ *finite perimeter set with* $V_g(\Omega) \leq \tilde{v}_0$ *holds*

$$\mathcal{P}_g(\Omega) > (1-\varepsilon)\mathcal{P}_{g_{k_0}}(B), \tag{8}$$

where $B \subseteq \mathbb{M}^n_{k_0}$ *is a geodesic ball having* $V_{g_{k_0}}(B) = V_g(\Omega)$.

Remark 2.7. *This gives a refinement of the classical result of Bérard-Meyer in [BM82]. Of course Theorem 4 follows immediately from the stronger Theorem 5 below.*

Theorem 5 (Sharp small volumes in strong bounded geometry)**.** *Let (M^n, g) be a complete Riemannian manifold, $n \geq 2$, with strong bounded geometry. Let us assume that there exists a real constant $k_0 \in \mathbb{R}$ such that $S_g < n(n-1)k_0$. Then there exists a positive constant $\tilde{v}_0 = \tilde{v}_0(n, k, k_0, inj_{M,g}, S_g) > 0$ such that for every $\Omega \subseteq M$ finite perimeter set with $V_g(\Omega) \leq \tilde{v}_0$ it holds*

$$\mathcal{P}_g(\Omega) \geq \mathcal{P}_{g_{k_0}}(B), \tag{9}$$

where $B \subseteq \mathbb{M}^n_{k_0}$ is a geodesic ball having $V_{g_{k_0}}(B) = V_g(\Omega)$, and $\tilde{v}_0$ can be chosen as an arbitrary real number satisfying

$$0 < \tilde{v}_0 \leq \left\{ \frac{(n+2)K(n,1)^2}{nC_0} \left[n(n-1)k_0 - S_g \right] \right\}^{\frac{n}{4}}, \tag{10}$$

where $C_0 = C_0(n, k_0) > 0$. Moreover up to take a smaller $\tilde{v}_1 = \tilde{v}_1(n, k, k_0, inj_M, S_g, I_{M,g}) \leq \tilde{v}_0$ we have that for any finite perimeter set Ω with $V_g(\Omega) \leq \tilde{v}_1$ we have

$$\mathcal{P}_g(\Omega) > \mathcal{P}_{g_{k_0}}(B). \tag{11}$$

Remark 2.8. *Observe that the upper bound on $\tilde{v}_0$ of Theorem 5 is strictly less than the corresponding upper bound on $\tilde{v}_0$ of Theorem 2 as it could be noted looking at their respective expressions they differ by the multiplication of a factor $2^{\frac{n}{4}}$.*

A particular case of the more general situation considered in Theorem 2 gives a positive answer to a special case of Aubin-Cartan-Hadamard's conjecture for small volumes as stated in the following corollaries.

Corollary 1 (Aubin's Conjecture in C^2-locally asymptotically strong bounded geometry smooth at infinity for small volumes)**.** *Let (M^n, g) be a Cartan-Hadamard manifold, $n \geq 2$ with C^2-locally asymptotically strong bounded geometry smooth at infinity, and $S_g < 0$. Then there exists a positive constant $\tilde{v}_0 = \tilde{v}_0(n, k, k_0, inj_{M,g}, S_g) > 0$ such that for every $\Omega \subseteq M$ finite perimeter set with $V_g(\Omega) \leq \tilde{v}_0$ it holds*

$$\mathcal{P}_g(\Omega) > \mathcal{P}_{g_{k_0}}(B), \tag{12}$$

where $B \subseteq \mathbb{M}^n_{k_0}$ is a geodesic ball having $V_{g_{k_0}}(B) = V_g(\Omega)$. Moreover $\tilde{v}_0$ can be chosen as an arbitrary number

$$0 < \tilde{v}_0 \leq \left\{ \frac{2(n+2)K(n,1)^2}{nC_0} [-S_g] \right\}^{\frac{n}{4}}, \tag{13}$$

where $C_0 = C_0(n, k_0) > 0$.

As a corollary of Theorem 5 we have the following statement.

Corollary 2 (Aubin's Conjecture in strong bounded geometry for small volumes)**.** *Let (M^n, g) be a complete Riemannian manifold, $n \geq 2$, with strong bounded geometry, and $S_g < 0$. Then there exists a positive constant $\tilde{v}_0 = \tilde{v}_0(n, k, k_0, inj_{M,g}, S_g) > 0$ such that for every $\Omega \subseteq M$ finite perimeter set with $V_g(\Omega) \leq \tilde{v}_0$ it holds*

$$\mathcal{P}_g(\Omega) \geq \mathcal{P}_{g_{k_0}}(B), \tag{14}$$

where $B \subseteq \mathbb{M}^n_{k_0}$ is a geodesic ball having $V_{g_{k_0}}(B) = V_g(\Omega)$. Moreover $\tilde{v}_0$ can be chosen as an arbitrary number

$$0 < \tilde{v}_0 \leq \left\{ \frac{2(n+2)K(n,1)^2}{nC_0} [-S_g] \right\}^{\frac{n}{4}}, \tag{15}$$

where $C_0 = C_0(n, k_0) > 0$. Moreover the all the conclusions of Theorem 5 holds.

As a last consequence of Theorem 5 we get Corollary 3 which gives an asymptotic expansion of the isoperimetric profile in Puiseux's series up to the second non trivial order generalizing previous results of [Nar14b]. Before to state the corollary we recall here the definition of the isoperimetric profile.

Definition 2.8. *Let (M, g) be an arbitrary Riemannian manifold. For every $v \in]0, V(M)[$ we define $I_{M,g}(v) := \inf\{\mathcal{P}_g(\Omega)\}$, where the infimum is taken over the family of finite perimeter subsets $\Omega \subseteq M$ having fixed volume $V(\Omega) = v$ that will be denoted in the sequel $\tilde{\tau}_v$. If there exists a finite perimeter set Ω satisfying $V(\Omega) = v$, $I_M(V(\Omega)) = A(\partial\Omega) = \mathcal{P}(\Omega)$ such an Ω will be called an **isoperimetric region**, and we say that $I_M(v)$ is **achieved**.*

Corollary 3 (Asymptotic expansion of the isoperimetric profile)**.** *If (M, g) have strong bounded geometry, then*

$$I_{M,g}(v) = c_n v^{\frac{(n-1)}{n}} \left(1 - \gamma_n S_g v^{\frac{2}{n}}\right) + O\left(v^{\frac{4}{n}}\right),$$

when $v \to 0^+$, *where* $S_g := \sup_{x\in M}\{Sc_g(x)\}$ *and* $\gamma_n = \frac{1}{2n(n+2)\omega_n^{\frac{2}{n}}}$ *is a positive dimensional constant. Here* ω_n *is the volume of a geodesic ball of radius* 1 *in* $\mathbb{R}^n$.

Remark 2.9. *The preceding corollary roughly speaking means that up to the second nontrivial term the asymptotic expansion of* I_M *coincides with* $I_{\mathbb{M}^n_k,g_k}$, *where* $n(n-1)k = S_g$.

Remark 2.10. *Via the same smoothing results of [Shi89], [Kap05] one can prove the preceding asymptotic expansion just using the theory of pseudo-bubbles developped by the first author in [Nar14b]. However the theory of pseudo bubbles does not give the sharp isoperimetric comparison of Theorem 5.*

2.1 Aknowledgments

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3 Sobolev inequalities and the proof of Theorem 3 in strong bounded geometry

In this section we closely follow the proof of Theorem 1 of [Dru02b]. We just make the needed changes to get the proof of our Theorem 1. First we set some notations and make the definitions that will be required in the sequel. By ξ we denote the standard Euclidean metric of $\mathbb{R}^n$. For every $1 \leq p < n$, $K(n,p) > 0$ is the best constant in the Sobolev inequalities on $(\mathbb{R}^n, \xi)$ defined as

$$K(n,p)^{-p} := \inf_{u\neq 0, u\in C_c(\mathbb{R}^n)} \left\{ \frac{\int_{\mathbb{R}^n} |\nabla u|^p_\xi dv_\xi}{\left(\int_{\mathbb{R}^n} |u|^{p^*} dv_\xi\right)^{\frac{p}{p^*}}} \right\}, \tag{16}$$

where $p^* := \frac{np}{n-p}$ is the critical Sobolev's exponent. The explicit value of $K(n,p)$ is computed in [Aub76], [Tal76] namely

$$\begin{aligned} K(n,1) &:= \frac{1}{n}\left(\frac{n}{\omega_{n-1}}\right)^{1/n} = c_n^{-1}, \\ K(n,p) &:= \frac{1}{n}\left(\frac{n(p-1)}{n-p}\right)^{1-1/p}\left(\frac{\Gamma(n+1)}{\Gamma(n/p)\Gamma(n+1-n/p)\omega_{n-1}}\right)^{1/n}. \end{aligned}$$

However the only property of $K(n,p)$ that we will use is that

$$\lim_{p\to 1^+} K(n,p) = K(n,1) = \frac{1}{n}\left(\frac{n}{\omega_{n-1}}\right)^{1/n}.$$

We will use frequently the L^p and the $W^{1,p}$ norm on M defined by

$$||u||_{p,g} := \left(\int_M |u|^p dv_g\right)^{\frac{1}{p}},$$

$$||u||_{1,p,g} := ||u||_{p,g} + ||\nabla_g u||_{p,g},$$

For any function u belonging respectively to $L^p(M)$ and $W^{1,p}(M)$. When $1 \le p < n$ we will need to work inside $W^{1,p}(\mathbb{R}^n)$ that will denote the standard Sobolev space defined as the completion of $C_c^\infty(\mathbb{R}^n)$ with respect to the norm

$$||u||_{1,p,\xi} := \left(\int_{\mathbb{R}^n} |\nabla_\xi u|_\xi^p dv_\xi\right)^{\frac{1}{p}}. \tag{17}$$

Definition 3.1. *Let (M,g) be a Riemannian manifold of dimension n, $U \subseteq M$ an open subset, $\mathfrak{X}_c(U)$ the set of smooth vector fields with compact support on U. Given a function $u \in L^1(M)$, define the variation of u by*

$$|Du|(M) := sup\left\{\int_M u div_g(X) dV_g : X \in \mathfrak{X}_c(M), ||X||_\infty \le 1\right\}, \tag{18}$$

*where $||X||_\infty := \sup\left\{|X_p|_{g_p} : p \in M\right\}$ and $|X_p|_{g_p}$ is the norm of the vector X_p in the metric g_p on T_pM. We say that a function $u \in L^1(M)$, has **bounded variation**, if $|Du|(M) < \infty$ and we define the set of all functions of bounded variations on M by $BV(M) := \{u \in L^1(M) : |Du|(M) < +\infty\}$. A function $u \in L^1_{loc}(M)$ has **locally bounded variation in** M, if for each open set $U \subset\subset M$,*

$$|Du|(U) := \sup\left\{\int_U u div_g(X) dV_g : X \in \mathfrak{X}_c(U), \|X\|_\infty \le 1\right\} < \infty,$$

and we define the set of all functions of locally bounded variations on M *by* $BV_{loc}(M) := \{u \in L^1_{loc}(M) : |Du|(U) < +\infty, U \subset\subset M\}$.

Definition 3.2. *Let* (M, g) *be a Riemannian manifold of dimension* n, $U \subseteq M$ *be an open subset,* $\mathfrak{X}_c(U)$ *the set of smooth vector fields with compact support in* U. *Given* $E \subset M$ *measurable with respect to the Riemannian measure, the* ***perimeter of*** E ***in*** U, $\mathcal{P}(E, U) \in [0, +\infty]$, *is*

$$\mathcal{P}(E, U) := sup\left\{\int_U \chi_E div_g(X) dV_g : X \in \mathfrak{X}_c(U), ||X||_\infty \leq 1\right\}, \quad (19)$$

where $||X||_\infty := \sup\left\{|X_p|_{g_p} : p \in M\right\}$ *and* $|X_p|_{g_p}$ *is the norm of the vector* X_p *in the metric* g_p *on* T_pM. *If* $\mathcal{P}(E, U) < +\infty$ *for every open set* $U \subset\subset M$, *we call* E *a* ***locally finite perimeter set***. *Let us set* $\mathcal{P}(E) := \mathcal{P}(E, M)$. *Finally, if* $\mathcal{P}(E) < +\infty$ *we say that* E ***is a set of finite perimeter***.

Before to prove Theorem 1 we prove Proposition 1 which is sufficient to prove Theorem 1. We postpone the proof of this last fact to the end of this section. In the proof of Proposition 1 we make frequent use of de Moser's iterative scheme, so we give an ad-hoc version of it in the following lemma which is suitable for our applications.

Lemma 3.1 (Ad-hoc De Giorgi-Nash-Moser)**.** *Let* (M^n, g) *be a complete Riemannian manifold,* $n \geq 2$, $1 < p < n$, *and* $v \in W^{1,p}_g(M)$ *with* $0 \leq v \leq 1$, *satisfying*

$$\Delta_{p,g} v \leq \Lambda v^{p^*-1}, \quad (20)$$

where Λ *only depends on* n. *Then for any* x *in* M, *for any* $\delta > 0$

$$\sup_{y \in M \cap B_g(x, \delta/2)} \{v(y)\} \leq C \left(\int_{M \cap B_g(x,\delta)} v^{p^*} dv_g\right)^{\frac{1}{p^*}}, \quad (21)$$

where $C > 0$ *does not depend on* p.

Remark 3.1. *Substituting the condition* $0 \leq v \leq 1$, *by* $\|v\|_{L^q(B(x,2\delta))} < K$ *for a suitable value of* K, *and* $q > p^*$, *get the same result, the proof is based on the Moser iterative scheme applied to* (20). *See for example Lemmas* 3.1 *and* 3.2 *of* [*AL99*].

Proof. Consider the inequality $\Delta_{p,g} v \leq \Lambda v^{p^*-1}$ in M, and $v \leq 1$, for some positive constant Λ independent of p. Consider a non-negative $\eta \in C^\infty_c(B_g(x, \delta))$ such that for $0 < r < s \leq \delta$ satisfies

i. $0 \leq \eta \leq 1$,

ii. $\eta \equiv 1$, in $B_g(x,r)$,

iii. $\eta \equiv 0$, in $B_g(x,\delta) \setminus B_g(x,s)$,

iv. $|\nabla_g \eta|_g \leq \frac{C_0}{s-r}$, where C_0 depends only on the geometry of (M,g) or on the bounds of the geometry in case M satisfy some condition of bounded geometry, for example instrong bounded geometry C_0 depends on $n, \Lambda_1, \Lambda_2, inj_M$.

Multiplying Equation (20) by $\eta^p v^{k+1}$, for $0 < k \leq p^* - p$, and integrating by parts over $B_g(x,s)$ leads to

$$\int_{B_g(x,s)} |\nabla_g v|_g^{p-2} \langle \nabla_g v, \nabla_g(\eta^p v^{k+1}) \rangle dv_g \leq \int_{B_g(x,s)} \Lambda v^{p^*+k} \eta^p dv_g. \quad (22)$$

Let $w = v^{\frac{k+p}{p}}$, then $|\nabla_g w|_g^p = \left(\frac{k+p}{p}\right)^p v^k |\nabla_g v|_g^p$.

We observe that

$$\begin{aligned}
&\int_{B_g(x,s)} |\nabla_g v|_g^{p-2} \langle \nabla_g v, \nabla_g(\eta^p v^{k+1}) \rangle dv_g \\
&\quad = (k+1) \int_{B_g(x,s)} |\nabla_g v|_g^p v^k \eta^p dv_g + \int_{B_g(x,s)} |\nabla_g v|_g^{p-2} v^{k+1} \langle \nabla_g v, \nabla_g(\eta^p) \rangle dv_g \\
&\quad = (k+1) \left(\frac{k+p}{p}\right)^{-p} \int_{B_g(x,s)} |\eta \nabla_g w|_g^p dv_g + \int_{B_g(x,s)} |\nabla_g v|_g^{p-2} v^{k+1} \langle \nabla_g v, \nabla_g(\eta^p) \rangle dv_g \\
&\quad \geq (k+1) \left(\frac{k+p}{p}\right)^{-p} \int_{B_g(x,s)} |\eta \nabla_g w|_g^p dv_g - \int_{B_g(x,s)} |\nabla_g v|_g^{p-1} v^{k+1} |\nabla_g \eta^p|_g dv_g \\
&\quad = (k+1) \left(\frac{k+p}{p}\right)^{-p} \int_{B_g(x,s)} |\eta \nabla_g w|_g^p dv_g - p \int_{B_g(x,s)} |\nabla_g v|_g^{p-1} v^{k+1} \eta^{p-1} |\nabla_g \eta|_g dv_g \\
&\quad = (k+1) \left(\frac{k+p}{p}\right)^{-p} \int_{B_g(x,s)} |\eta \nabla_g w|_g^p dv_g - p \int_{B_g(x,s)} \left(|\nabla_g \eta|_g v^{\frac{k+p}{p}}\right) \left(|\nabla_g v|_g^{p-1} \eta^{p-1} v^{\frac{k(p-1)}{p}}\right) dv_g \\
&\quad = (k+1) \left(\frac{k+p}{p}\right)^{-p} \int_{B_g(x,s)} |\eta \nabla_g w|_g^p dv_g - p \left(\frac{k+p}{p}\right)^{1-p} \int_{B_g(x,s)} (w |\nabla_g \eta|_g)(\eta |\nabla_g w|_g)^{p-1} dv_g
\end{aligned}$$

where we applied the Cauchy-Schwarz's inequality. Later we have to use in the second integral on the right the Young's inequality in the following form

$$ab \leq \frac{(\theta^{-1} a)^p}{p} + \frac{(p-1)(\theta b)^{\frac{p}{p-1}}}{p},$$

with $\theta \in]0, +\infty[$. Set $a = w|\nabla_g \eta|_g$, $b = (\eta|\nabla_g w|_g)^{p-1}$, and choose $\theta > 0$ such that

$$\left(\frac{k+p}{p}\right)^{1-p}(p-1)\theta^{\frac{p}{p-1}} = \frac{1}{2}(k+1)\left(\frac{k+p}{p}\right)^{-p}, \tag{23}$$

we get

$$\begin{aligned}\int_{B_g(x,s)} |\nabla_g v|^{p-2}\langle \nabla_g v, \nabla_g(\eta^p v^{k+1})\rangle dv_g \quad &\geq \quad \frac{(k+1)}{2}\left(\frac{k+p}{p}\right)^{-p}\int_{B_g(x,s)} |\eta\nabla_g w|^p dv_g \\ &- \quad 2^{p-1}\left(\frac{p-1}{k+1}\right)^{p-1}\int_{B_g(x,s)} |w\nabla_g \eta|^p dv_g.\end{aligned} \tag{24}$$

Combining (22) and (24) leads to

$$\begin{aligned}\int_{B_g(x,s)} |\eta\nabla_g w|_g^p dv_g \leq & C_1\int_{B_g(x,s)} \eta^p v^{k+p^*} dv_g + C_2\int_{B_g(x,s)} |w\nabla_g \eta|_g^p dv_g \\ = & C_1\int_{B_g(x,s)} (\eta w)^p v^{p^*-p} dv_g + C_2\int_{B_g(x,s)} |w\nabla_g \eta|_g^p dv_g,\end{aligned} \tag{25}$$

where $C_1(p) = \frac{2}{k+1}\left(\frac{k+p}{p}\right)^p \Lambda$, and $C_2(p) = 2^p(p-1)^{(p-1)}$.

Independently we have the following computations

$$\int_{B_g(x,s)} |\nabla_g(\eta w)|_g^p dv_g \leq 2^{p-1}\left(\int_{B_g(x,s)} |\eta\nabla_g w|_g^p dv_g + \int_{B_g(x,s)} |w\nabla_g \eta|_g^p dv_g\right), \tag{26}$$

and by the Sobolev embedding we get

$$\begin{aligned}\left(\int_{B_g(x,s)} |\eta w|^{p^*} dv_g\right)^{\frac{p}{p^*}} \quad &\leq \quad C(n,p)\int_{B_g(x,s)} |\nabla_g(\eta w)|_g^p dv_g, \\ &\overset{(26)}{\leq} \quad C_3\int_{B_g(x,s)} |\eta\nabla_g w|_g^p dv_g + C_3\int_{B_g(x,s)} |w\nabla_g \eta|_g^p dv_g \\ &\overset{(25)}{\leq} \quad C_4\int_{B_g(x,s)} (\eta w)^p v^{p^*-p} dv_g + C_5\int_{B_g(x,s)} |w\nabla_g \eta|_g^p dv_g,\end{aligned}$$

where $C_3 = 2^{p-1}C(n,p)$, $C_4 = C_1C_3$, and $C_5 = C_2C_3 + C_3$.

But since $v \leq 1$, $0 \leq \eta \leq 1$ over $B_g(x,s)$, and $|\nabla_g \eta|_g \leq \frac{C_0}{s-r}$ we have that

$$\left(\int_{B_g(x,s)} |\eta w|^{p^*} dv_g\right)^{\frac{p}{p^*}} \leq \left[C_4 + C_5\left(\frac{C_0}{s-r}\right)^p\right] \int_{B_g(x,s)} |w|^p dv_g.$$

On the other hand we have $\left(\frac{k+p}{p}\right)^p \leq (k+1)^p$ for $k > 0$ and $1 < p < n$, then $C_1 \leq 2(k+1)^{n-1}\Lambda$, $C_2 \leq 2^n(n-1)^{n-1}$, $C_3 \leq 2^{n-1}C(n)$, $C_4 \leq 2^n(k+1)^{n-1}\Lambda C(n)$, $C_5 \leq 2^{n-1}(2^n(n-1)^{n-1}(k+1)^{n-1}+1)C(n)$. Then

$$C_4 + C_5\left(\frac{C_0}{s-r}\right)^p \leq 2^{n-1}C(n)\left(2(k+1)^{n-1}\Lambda + (2^n(n-1)^{n-1}+1)\left(\frac{C_0}{s-r}\right)^p\right).$$

Thus setting

$$B_0 = 2^{n-1}C(n)\left(2(k+1)^{n-1}\Lambda + (2^n(n-1)^{n-1}+1)\left(\frac{C_0}{s-r}\right)^p\right),$$

we get

$$\left(\int_{M\cap B_g(x,r)} |v|^{\frac{p^*(k+p)}{p}} dv_g\right)^{\frac{p}{p^*(k+p)}} \leq B_0^{\frac{1}{k+p}} \left(\int_{M\cap B_g(x,s)} |v|^{k+p} dv_g\right)^{\frac{1}{k+p}}. \tag{27}$$

Now we want to use the Moser's iterative scheme. Let us call

$$F(t,\rho) = \left(\int_{M\cap B_g(x,\rho)} v^t dv_g\right)^{1/t},$$

by the inequality (27) we get that

$$F\left((k+p)\frac{p^*}{p}, r\right) \leq B_0^{\frac{1}{k+p}} F((k+p), s). \tag{28}$$

Choose k_0 such that $(k_0+p) = p^*$, $s_0 = \delta$ and define for every $i \geq 1$

$$(k_i+p) = \frac{p^*}{p}(k_{i-1}+p) = \left(\frac{p^*}{p}\right)^i (k_0+p), s_i = \frac{\delta}{2} + \frac{\delta}{2^{i+1}}.$$

Make $k = k_i$, $s = s_i$ and $r = s_{i+1}$. Note that $s_i - s_{i+1} = \frac{\delta}{2^{i+2}}$, furthermore we get $k_i \to +\infty$ when $i \to +\infty$, because

$$k_{i+1} - k_i = (k_0+p)\left(\frac{p^*}{p}\right)^i \left[\frac{p^*}{p} - 1\right] > 0.$$

Now we apply this to (28), and we obtain

$$F\left(\frac{k_i+p}{p}p^*, s_{i+1}\right) = F((k_{i+1}+p), s_{i+1}) \leq B_i^{\frac{1}{k_i+p}} F((k_i+p), s_i).$$

Then making the iteration yields

$$F((k_{i+1}+p), s_{i+1}) \leq \prod_{j=0}^{i} B_j^{\frac{1}{k_j+p}} F((k_0+p), s_0) = \prod_{j=0}^{i} B_j^{\frac{1}{k_j+p}} F(p^*, \delta).$$

Taking $i \to \infty$ the expression above becomes

$$\|v\|_{L_g^\infty(B_g(x,\delta/2))} \leq \prod_{i=0}^{+\infty} B_i^{\frac{1}{k_i+p}} \left(\int_{M\cap B_g(x,\delta)} v^{p^*} dv_g\right)^{\frac{1}{p^*}}. \tag{29}$$

It remains to prove the convergence of $\prod_{i=0}^{+\infty} B_i^{\frac{1}{k_i+p}}$ to a constant independent of p.

Since for p sufficiently close to 1 we can get that $\frac{p^*}{p} = \frac{n}{n-p} \leq 2$, we have

$$(k_i+1) < (k_i+p) = \left(\frac{p^*}{p}\right)^i (k_0+p) \leq 2^i(k_0+n),$$

and making the choice $\tilde{\delta} = \min\{1, \delta\}$, we get

$$\begin{aligned}
B_i &= 2^{n-1}C(n)\left(2(k_i+1)^{n-1}\Lambda + (2^n(n-1)^{n-1}+1)\left(\frac{C_0}{s_{i+1}-s_i}\right)^p\right) \\
&\leq 2^{n-1}C(n)\left(2(k_0+n)^n 2^{in}\Lambda + (2^n(n-1)^{n-1}+1)\left(\frac{2^{(i+2)p}C_0^p}{\delta^p}\right)\right) \\
&\leq 2^{in}2^{n-1}C(n)\left(2(k_0+n)^n\Lambda + (2^n(n-1)^{n-1}+1)2^{2n}C_0^n\tilde{\delta}^n\right) \\
&= 2^{in}\tilde{C}(n).
\end{aligned}$$

As it is easy to see from the definition of k_i we have

$$\frac{1}{2^i(k_0+n)} \leq \frac{1}{k_i+p} \leq \frac{1}{k_i+1}.$$

Let us define $\alpha_i := \frac{1}{2^i(k_0+n)}$, if $B_i < 1$ and $\alpha_i := \frac{1}{k_i+1}$, if $B_i \geq 1$, in any case we have

$$B_i^{\frac{1}{k_i+p}} \leq B_i^{\alpha_i}.$$

Then passing to the infinite products

$$\begin{aligned}\prod_{i=0}^{+\infty} B_i^{\frac{1}{k_i+p}} &\leq \prod_{i=0}^{+\infty} B_i^{\alpha_i} \leq \left(\prod_{i=0}^{+\infty} \tilde{C}^{\alpha_i}\right)\left(\prod_{i=0}^{+\infty} \left(2^{in}\right)^{\alpha_i}\right)\\ &= \left(\tilde{C}^{\sum_{i=0}^{+\infty}\alpha_i}\right)\left(2^{n\sum_{i=0}^{+\infty} i\alpha_i}\right).\end{aligned}$$

Notice that

$$\sum_{i=0}^{+\infty} \alpha_i \quad \text{and} \quad \sum_{i=0}^{\infty} i\alpha_i,$$

are convergent series. Then for values of p close to 1 we have

$$\|v\|_{L_g^\infty(B_g(x,\delta/2))} \leq C\left(\int_{M\cap B_g(x,\delta)} v^{p^*} dv_g\right)^{\frac{1}{p*}}, \tag{30}$$

where C does not depend on p. □

Proposition 3.1 (Proposition [Dru02b] page 2353)**.** *Let (M^n, g) be a complete Riemannian manifold of dimension $n \geq 2$. Let $x_0 \in M$, and define $\alpha_\varepsilon := \frac{n}{n-2} Sc_g(x_0) + \epsilon$. Then for any $\varepsilon > 0$, there exists $r_\varepsilon > 0$ such that for any u in $C_c^\infty(B_g(x_0, r_\epsilon))$ we have that*

$$\|u\|^2_{\frac{n}{n-1},g} \leq K(n,1)^2 \left(\|\nabla u\|^2_{1,g} + \alpha_\varepsilon \|u\|^2_{1,g}\right). \tag{31}$$

The preceding proposition justifies the following definition.

Definition 3.3. *Let (M, g) be a Riemannian manifold. Let us define $r_\varepsilon^*(M, g, x) \in [0, +\infty]$ as the supremum of all $r > 0$ such that (33) is satisfied. Of course $r_\varepsilon^*(M, g, x) = 0$, if there is no such positive r_ε. We call $r_\varepsilon^*(M, g, x)$ the **Druet's radius of** (M, g) **at** x. Let us define $r_\varepsilon^*(M, g) \in [0, +\infty]$ as the infimum of $r_\varepsilon^*(M, g, x)$ taken over all $x \in M$. We call $r_\varepsilon^*(M, g)$ the **Druet's radius of** (M, g).*

By Proposition 3.1, if (M^n, g) is complete then for any $x \in M$ we have $r_\varepsilon^*(M, g, x) > 0$. By Proposition 1 if (M^n, g) has C^2-locally asymptotic strong bounded geometry smooth at infinity, then we have $r_\varepsilon^*(M, g) > 0$. We want to study now a little of stability properties of Druet's radius with respect to the convergence of manifolds.

Lemma 3.2. *Suppose to have a sequence of pointed complete smooth Riemannian manifolds $(M, g_i, p_i) \to (M_\infty, g_\infty, p_\infty)$ in C^0 topology with $(M_\infty, g_\infty, p_\infty)$ smooth and $Sc_{g_i}(p_i) \to Sc_{g_\infty}(p_\infty)$. Then*

$$\varinjlim_{i\to+\infty} r_\varepsilon^*(M, g_i, p_i) \leq r_\varepsilon^*(M_\infty, g_\infty, p_\infty). \tag{32}$$

Proposition 1. *Let (M,g) be a complete Riemannian manifold of dimension $n \geq 2$ with C^2-locally asymptotic strong bounded geometry smooth at infinity. For any $\varepsilon > 0$ there exists $r_\varepsilon = r_\varepsilon(M,g) > 0$ such that for any point $x_0 \in M$, any function $u \in C_c^\infty(B_g(x_0, r_\varepsilon))$, we have*

$$||u||^2_{\frac{n}{n-1},g} \leq K(n,1)^2 \left(||\nabla_g u||^2_{1,g} + \alpha_\varepsilon ||u||^2_{1,g} \right), \tag{33}$$

where $\alpha_\varepsilon = \frac{n}{n+2} S_g(x_0) + \varepsilon$.

Remark 3.2. *We notice that the constant $r_\varepsilon = r_\varepsilon(M) > 0$ is obtained by contradiction and that the proof does not give an explicit effective lower bound on it.*

We state here a Corollary of Lemma 3.2 that could be used to give a slightly different proof of Theorem 3.

Corollary 3.1. *Let (M,g) be a complete Riemannian manifold of dimension $n \geq 2$ with C^2-locally asymptotic strong bounded geometry smooth at infinity, $p_i \to \infty$ and $(M,g,p_i) \to (M_\infty, g_\infty, p_\infty)$. Then for every $\varepsilon > 0$ we have $r^*_\varepsilon(M_\infty, g_\infty) \geq r^*_\varepsilon(M,g) > 0$.*

Proof of Proposition 1. For any $x_0 \in M$, for any $r > 0$, any $p > 1$ and any $\varepsilon > 0$, set

$$\lambda_{p,r,g}(x_0) := \inf_{\substack{u \in C_c^\infty(B_g(x_0,r)) \\ u \not\equiv 0}} \frac{\left(\int_{B_g(x_0,r)} |\nabla_g u|^p dv_g \right)^{2/p} + \alpha_\varepsilon \left(\int_{B_g(x_0,r)} |u|^p dv_g \right)^{2/p}}{\left(\int_{B_g(x_0,r)} |u|^{p^*} dv_g \right)^{2/p^*}},$$

where $B_g(x_0,r) \subseteq M$ is the geodesic ball (M,g) centered at $x_0 \in M$ and of radius $r > 0$. We will argue the theorem by contradiction. With this aim in mind suppose that there exists $\varepsilon_0 > 0$ such that for every $r > 0$ there exists a point $x_{0,r}$ depending on r such that it holds

$$\lambda_{1,r,g}(x_{0,r}) < K(n,1)^{-2}.$$

As it is easy to check from the very definition of $\lambda_{p,r,g}(x_{0,r})$, we have that $\overrightarrow{\lim}_{p \to 1^+} \lambda_{p,r,g}(x_{0,r}) \leq \lambda_{1,r,g}(x_{0,r})$, which implies that for any $r > 0$, there exists $p_r(x_{0,r}) > 1$ such that

$$\lambda_{p_r,r,g}(x_{0,r}) < K(n,1)^{-2} \left(\frac{n - p_r(x_{0,r})}{p_r(x_{0,r})(n-1)} \right), \ \lambda_{p_r(x_{0,r}),r,g} < K(n, p_r(x_{0,r}))^{-2}. \tag{34}$$

We may assume that $r \searrow 0$ and we may choose $p_r(x_{0,r})$ decreasing when r is decreasing. Then inverting this sequence we get a sequence $p > 1$ going to 1^+ a sequence $r_p > 0$ going to 0^+ as p goes to 1^+, and a sequence of points $x_{0,p} := x_{0,r_p} \in M$ which verify (34). Notice here

that in general the sequence of points $x_{0,p}$ could go to infinity when $p \to 1^+$. This is the main difficulty that we encounter in adapting the original proof of Theorem 1 of [Dru02b] in case of noncompact ambient manifolds. Set $\alpha_p := \dfrac{n}{n+2} S_g(x_{0,p}) + \varepsilon_0$. Now up to a subsequence we can assume that

$$\lim_{p \to 1^+} \alpha_p = \frac{n}{n+2} l_1 + \varepsilon_0, \tag{35}$$

for some $l_1 \in [S_{inf,g}, S_g]$, where $S_{inf,g} := \inf\{Sc_g(x) : x \in M\}$ and $S_g := \sup\{Sc_g(x) : x \in M\}$. It is worth to note that $S_{inf,g}$ and S_g are finite real numbers, because S_g is bounded from below by $n(n-1)k$ and from above by $n(n-1)k_0$. The second equation in (34) can be written like $\lambda_{p,r_p}(x_{0,p}) < K(n,p)^{-2}$, and by Theorems 1.1, 1.3, 1.5 of [Dru00], we have the existence of a minimizer u_p which satisfies

$$\begin{aligned} C_p \Delta_{p,g} u_p + \alpha_p \|u_p\|_{p,g}^{2-p} u_p^{p-1} = \lambda_p u_p^{p^*-1}, \quad \text{in } B_g(x_{0,p}, r_p), \\ u_p \in C^{1,\eta}(B_g(x_{0,p}, r_p)), \quad \text{for some } \eta > 0, \\ u_p > 0, \quad \text{in } B_g(x_{0,p}, r_p), \quad u_p = 0, \quad \text{in } \partial B_g(x_{0,p}, r_p), \end{aligned} \tag{36}$$

$$\int_{B_g(x_{0,p}, r_p)} u_p^{p^*} dv_g = 1, \tag{37}$$

$$\lambda_p < K(n,p)^{-2}, \quad \lambda_p < K(n,1)^{-2} \left(\frac{n-p}{p(n-1)} \right)^2, \tag{38}$$

$$C_p := \left(\int_{B_g(x_{0,p}, r_p)} |\nabla_g u_p|_g^p dv_g \right)^{\frac{2-p}{p}}, \tag{39}$$

where $\lambda_p := \lambda_{p,r_p}(x_{0,r_p})$, $\Delta_{p,g}$ is the p-Laplacian with respect to g, defined by $\Delta_{p,g} u := -div_g(|\nabla_g u|^{p-2} \nabla_g u)$, with $\nabla_g u$ being the gradient of u with respect to the metric g. The strategy that will adopt to go head in this proof is concerned with the study of the sequence (u_p) as $p \to 1^+$. With this aim in mind, let x_p be a point in $B_g(x_{0,p}, r_p)$ where u_p achieves its maximum (x_p tends to infinity, iff $x_{0,p}$ tends to infinity) and we define

$$u_p(x_p) = \mu_p^{1-\frac{n}{p}}.$$

Observing that $u_p(x_p)^{p^*} = \mu_p^{-n}$ we get

$$1 = \int_{B_g(x_{0,p}, r_p)} u_p^{p^*} dv_g \leq V_g(B_g(x_{0,p}, r_p)) \mu_p^{-n} \leq C_0(n,k) r_p^n \mu_p^{-n}, \tag{40}$$

where the last inequality is due to Bishop-Gromov. Since r_p goes to 0, μ_p goes to 0 as p goes to 1^+, moreover $\mu_p = O(r_p)$ and the constant $C_0 = C_0(n,k)$ is uniform with respect to p, i.e., is uniform with respect to the location of $x_{0,p}$ inside M. Analogously, applying Hölder's inequalities, with $q = \frac{n}{n-p} > 1$ yields

$$\lim_{p\to 1^+} \int_{B_g(x_{0,p},r_p)} u_p^p dv_g \leq \lim_{p\to 1^+} \left\{ \int_{B_g(x_{0,p},r_p)} u_p^{p^*} dv_g \right\}^{\frac{1}{q}} V_g(B_g(x_{0,p},r_p))^{\frac{1}{q'}} = 0, \tag{41}$$

here q' denotes the conjugate exponent of q, i.e., $\frac{1}{q} + \frac{1}{q'} = 1$.

Step 1 In this first step we want to show the validity of the two following equations

$$\lim_{p\to 1^+} \lambda_p = K(n,1)^{-2}, \tag{42}$$

and

$$\lim_{p\to 1^+} \int_{B_g(x_{0,p},r_p)} |\nabla_g u_p|_g^p dv_g = K(n,1)^{-1}. \tag{43}$$

By Theorem 7.1 of [Heb99], it follows that for all $\varepsilon > 0$ there exists $B_\varepsilon = B_\varepsilon(n,k,inj_{M,g}) > 0$ such that for any $p > 1$,

$$\begin{aligned} \left(\int_{B_g(x_{0,p},r_p)} u_p^{p^*} dv_g \right)^{2\frac{n-1}{n}} &\leq (K(n,1)+\varepsilon)^2 \left(\int_{B_g(x_{0,p},r_p)} \left| \nabla_g \left(u_p^{\frac{p(n-1)}{n-p}} \right) \right|_g dv_g \right)^2 \\ &+ B_\varepsilon \left(\int_{B_g(x_{0,p}0,r_p)} u_p^{\frac{p(n-1)}{n-p}} dv_g \right)^2, \end{aligned}$$

which gives with (36), (37) and Hölder's inequalities

$$\begin{aligned} 1 &\leq (K(n,1)+\varepsilon)^2 \left(\frac{p(n-1)}{n-p} \right)^2 (\lambda_p - \alpha_p \|u_p\|_p^2) + B_\varepsilon \|u_p\|_p^2 \\ &\leq (K(n,1)+\varepsilon)^2 \left(\frac{p(n-1)}{n-p} \right)^2 (\lambda_p - \alpha_0 \|u_p\|_p^2) + B_\varepsilon \|u_p\|_p^2, \end{aligned}$$

where $\alpha_0 := n(n-1)k + \varepsilon_0 \leq \alpha_p$. This combined with (41) give

$$1 \leq \left(1 + \varepsilon K(n,1)^{-1}\right)^2 \varinjlim_{p\to 1^+} \left(\lambda_p K(n,1)^2\right).$$

Since this inequality is valid for every $\varepsilon > 0$, letting $\varepsilon \to 0$ we obtain that $\varinjlim_{p\to 1^+} \lambda_p \geq K(n,1)^{-2}$. Using the fact that $\lambda_p < K(n,p)^{-2}$, and

$\lambda_p < K(n,1)^{-2}\left(\dfrac{n-p}{p(n-1)}\right)^2$, we conclude that

$$K(n,1)^{-2} \leq \varinjlim_{p\to 1^+}\lambda_p \leq \varinjlim_{p\to 1^+} K(n,p)^{-2},$$

and thus (42) is proved.

Remark 3.3. *Until this point we have just used the assumption of mild bounded geometry, i.e., Ricci bounded below and positive injectivity radius.*

Now, it easily seen that (43) is an obvious consequence of (36), (37), (41), and (42).

Remark 3.4. *To prove rigorously* (43) *we have used that the scalar curvature is bounded from both sides that Ricci is bounded below and the injectivity radius is positive.*

To prove the equality (43), we observe that as a consequence of (36) we have

$$\|\nabla_g u_p\|^2_{p,g} + \alpha_p \|u_p\|^2_{p,g} = \lambda_p \|u_p\|^{p^*}_{p^*,g}, \tag{44}$$

and that under the assumptions of Proposition 1, $n(n-1)k \leq Sc_g \leq n(n-1)k_0$. This yields immediately that (α_p) is a bounded sequence, i.e., $\alpha_p = O(1)$. Taking $p \to 1^+$, in (44) and (37) we obtain that

$$\lim_{p\to 1^+} \|\nabla_g u_p\|^2_{p,g} + \alpha_p \|u_p\|^2_{p,g} = \lim_{p\to 1^+} \lambda_p \|u_p\|^{p^*}_{p^*,g} = \lim_{p\to 1^+} \lambda_p,$$

then using the above result, we can conclude that

$$\lim_{p\to 1^+} \|\nabla_g u_p\|_{p,g} = K(n,1)^{-1}.$$

Step 2 Let $\Omega_p := \mu_p^{-1}\exp_{x_p}^{-1}(B_g(x_{0,p}, r_p)) \subset T_{x_p}M \cong \mathbb{R}^n$, the metric $g_p(x) := \exp_{x_p}^* g(\mu_p x)$ for $x \in \Omega_p$, and the function given by $v_p(x) = \mu_p^{\frac{n}{p}-1} u_p(\exp_{x_p}(\mu_p x))$ for $x \in \Omega_p$, $v_p(x) = 0$ in $x \in \mathbb{R}^n \setminus \Omega_p$. It is worth to mention that the definition of v_p for p close to 1 is well posed and does not give any problem, since we suppose that the injectivity radius of M is strictly positive and $r_p \to 0$ as $p \to 1^+$. Then making the substitution in (36) we obtain that v_p satisfies

$$C_p \Delta_{p,g_p} v_p + \alpha_p \mu_p^2 \|v_p\|^{2-p}_{p,g_p} v_p^{p-1} = \lambda_p v_p^{p^*-1}, \text{in } \Omega_p, \tag{45}$$

with $v_p = 0$ in $\partial\Omega_p$, and

$$\int_{\Omega_p} v_p^{p^*} dv_g = 1. \tag{46}$$

Thus v_p satisfy also

$$\|\nabla_{g_p} v_p\|^2_{p,g_p} + \alpha_p \|v_p\|^2_{p,g_p} = \lambda_p \|v_p\|^{p^*}_{p^*,g_p}. \tag{47}$$

Unfortunately the sequence (v_p) is not bounded in $W^{1,1}_{\xi}(\mathbb{R}^n)$ so we need to pass to another auxiliary sequence $(\tilde{v}_p)$ related in some way to the preceding one and that is bounded in $W^{1,1}_{\xi}(\mathbb{R}^n)$. We do this because we are interested in a limit function v_0 that realizes the minimum of the problem at infinity and that we expect to be the characteristic function of a ball. To realize this strategy we look for powers of the function v_p. As we will see later a suitable choice is the following

$$\tilde{v}_p(x) = v_p(x)^{\frac{p(n-1)}{n-p}}. \tag{48}$$

It is useful to recall here that for every $x \in M$ the exponential map exp_x is a bi-Lipschitz map of an open geodesic ball centered at x having radius inj_x over a ball of $\mathbb{R}^n$ having the same radius, with Lipschitz constant L_x that in general depend on x, however by the Rauch's comparison Theorem we know that L_x can be bounded by a constant that depends just on $\Lambda_1, \Lambda_2, inj_M$, which in turn permit to conclude that under our assumption of strong bounded geometry the constants L_x are uniformly bounded with respect to x by a positive constant that depends only on the bounds on the geometry, namely $\Lambda_1, \Lambda_2, inj_M$. Hence using the Cartan's expansion of the metric g_p close to x_p we can show the existence of a positive constant $C = C(\Lambda_1, \Lambda_2, inj_M) > 0$, such that for any $x \in \Omega_p$,

$$(1 - C\mu_p^2|x|^2) dv_{g_p} \leq dv_{\xi} \leq (1 + C\mu_p^2|x|^2) dv_{g_p}. \tag{49}$$

From this we conclude that there exists another constant again denoted by $C = C(\Lambda_1, \Lambda_2, inj_M) > 1$ such that

$$dv_{g_p} \geq \left(1 - \frac{1}{C}\mu_p^2\right) dv_{\xi}, \tag{50}$$

$$|\nabla v_p|^p_{g_p} dv_{g_p} \leq (1 + C\mu_p^2) |\nabla_{\xi} v_p|^p_{\xi} dv_{\xi}, \tag{51}$$

where ξ is the Euclidean metric. Equations (49), (50), (51) with (43), (46) and Hölder's inequalities leads to

$$\lim_{p\to 1^+} \frac{\int_{\mathbb{R}^n} |\nabla_\xi \tilde{v}_p|_\xi dv_\xi}{\left(\int_{\mathbb{R}^n} \tilde{v}_p^{\frac{n}{n-1}} dv_\xi\right)^{\frac{n-1}{n}}} = K(n,1)^{-1}. \tag{52}$$

To show this observe that by (49), (50), (51) $\left(\int_{\mathbb{R}^n} \tilde{v}_p^{\frac{n}{n-1}} dv_\xi\right)^{\frac{n-1}{n}} \sim \int_{\Omega_p} v_p^{p^*} dv_g = 1$, when $p \to 1^+$. To see what happens to the numerator of (52) just look at (53) below

$$\begin{aligned}
\int_{\mathbb{R}^n} |\nabla_\xi \tilde{v}_p|_\xi dv_\xi &= \int_{\mathbb{R}^n} \frac{p(n-1)}{n-p} v_p^{\frac{n(p-1)}{n-p}} |\nabla_\xi v_p|_\xi dv_\xi \\
&\leq \frac{p(n-1)}{n-p} \left\{\int_{\mathbb{R}^n} v_p^{p^*} dv_\xi\right\}^{\frac{p}{p-1}} \left\{\int_{\mathbb{R}^n} |\nabla_\xi v_p|_\xi^p dv_\xi\right\}^{\frac{1}{p}} \\
&\leq \frac{p(n-1)}{n-p}(1+C\mu_p^2)||\nabla_{g_p} v_p||_{p,g_p} \\
&= \frac{p(n-1)}{n-p}(1+C\mu_p^2)||\nabla_g u_p||_{p,g} \to K(n,1)^{-1}. \qquad (53)
\end{aligned}$$

The last equality is a consequence of (43) and the following calculation

$$\begin{aligned}
\int_{\Omega_p} |\nabla_{g_p} v_p(x)|_{g_p}^r dv_{g_p} &= \int_{\Omega_p} |\nabla_{g_p}(\mu_p^{\frac{n-p}{p}} u_p(\exp_{x_p}(\mu_p x)))|_{g_p}^r dv_{g_p}(x) \\
&= \mu_p^{(\frac{n-p}{p})r} \int_{\mu_p^{-1}\exp_{x_p}^{-1}(B_g(x_{0,p},r_p))} |\nabla_{g_p} u_p(\exp_{x_p}(\mu_p x))|_{g_p}^r dv_{g_p}(x) \\
&= \mu_p^{\frac{n-p}{p}r} \mu_p^{-n} \mu_p^r \int_{\exp_{x_p}^{-1}(B_g(x_{0,p},r_p))} |\nabla_{g_p} u_p(\exp_{x_p}(x))|_{g_p}^r dv_{g_p}(\mu_p^{-1}x) \\
&= \mu_p^{\frac{n(r-p)}{p}} \int_{\exp_{x_p}^{-1}(B_g(x_{0,p},r_p))} |\nabla_{g_p} u_p(\exp_{x_p}(x))|_{g_p}^r dv_{g_p}(\mu_p^{-1}x) \\
&= \mu_p^{\frac{n(r-p)}{p}} \int_{B_g(x_{0,p},r_p)} |\nabla_g u_p(x)|_g^r dv_g,
\end{aligned}$$

from which follows

$$\|\nabla_{g_p} v_p\|_{r,g_p}^r = \mu_p^{\frac{n(r-p)}{p}} \|\nabla_g u_p\|_{r,g}^r.$$

Remember here that $r_p \to 0$ as $p \to 1^+$. Notice that by (17), $(\tilde{v}_p)$ is bounded in $W_\xi^{1,1}(\mathbb{R}^n)$. Thus there exists $v_0 \in BV_{loc}(\mathbb{R}^n)$ such that

$$\lim_{p\to 1^+} \tilde{v}_p = v_0, \quad \text{strictly in } BV_{loc}(\mathbb{R}^n),$$

this means that $\tilde{v}_p \to v_0$ in $L^1_{loc}(\mathbb{R}^n)$ e $||\nabla \tilde{v}_p||_{1,\xi}(K) \to |Dv_0|(K)$, $\forall K \subset\subset \mathbb{R}^n$. For a proof of this fact see Thm. 3.23 of [AFP00]. If we apply the concentration-compactness principle of P.L. Lions ([Lio84], [Lio85], see also [Str08] for an exposition in book form) to $|v_p|^{p^*} dv_\xi$, four situations may occur: compactness, concentration, dichotomy or vanishing. Dichotomy is classically forbidden by (52). To be convinced of this fact the reader could mimic the proof of Theorem 4.9 of [Str08]. Concentration without compactness cannot happen since $\sup_{\Omega_p} v_p = v_p(0) = 1$. As for vanishing, since v_p is bounded in L^∞, by applying Moser's iterative scheme (see for instance Theorem 1 [Ser64]) to (45), one gets the existence of some $C = C(n, \alpha_p C_p^{-1} \mu_p^2 ||v_p||_p^{2-p}, C_p^{-1}\lambda_p, ||g||_{0,r}) > 0$ such that for any $p > 1$,

$$1 = \sup_{\Omega_p \cap B_{g_p}(0,1/2)} v_p \leq C \left(\int_{\Omega_p \cap B_{g_p}(0,1)} v_p^{p^*} dv_{g_p} \right)^{\frac{1}{p^*}}, \tag{54}$$

where $||g||_{0,r}$ is the norm defined at page 308 of [Pet06] (see Definition 2.6). Since a careful analysis of the proof of Theorem 1 of [Ser64] combined with (39), (41), (42), (43), (which imply, by a change of variables in the integrals, that $\alpha_p \mu_p^2 ||v_p||_p^{2-p} = \alpha_p \mu_p^p ||u_p||^{2-p} \to 0$, thanks to the fact that $\alpha_p \to l_1 + \varepsilon_0 \in \mathbb{R}$, hence α_p is uniformly bounded), and the C^0 convergence of the metric tensor due to Theorems 72 and 76 of [Pet06], when $p \to 1^+$, shows that C is uniformly bounded with respect to p. Thus vanishing cannot happen. Another way to see that our problem have no vanishing is to apply directly Lemma 3.1 with $g = g_p$ and $v = v_p$, this is justified because in equation (45) we know that $\alpha_p \mu_p^2 \|v_p\|_{p,g_p}^{2-p} \to 0$, and $C_p^{-1}\lambda_p \to K(n,1)^{-1}$. Then for p close to 1, we can consider that v satisfies the following inequality

$$\Delta_{p,g} v_p \leq \Lambda v_p^{p^*-1},$$

where Λ depends only on n. Compactness implies that $|v_p|^{p^*} dv_\xi \to |v_0|^{\frac{n}{n-1}} dv_\xi$ that is $\|\tilde{v}_p\|_{\frac{n}{n-1}} \to \|v_0\|_{\frac{n}{n-1}}$. To see this we observe that by the compactness case of the concentration-compactness principle we have that for all $\varepsilon > 0$ there exist $R_\varepsilon > 0$ and $p_\varepsilon > 0$ such that

$$1 - \varepsilon \leq \int_{B_\xi(0,R_\varepsilon)} v_p^{p^*} dv_\xi \leq 1 + \varepsilon, \quad p \leq p_\varepsilon,$$

passing to the limit when $p \to 1^+$ yields $\int_{\mathbb{R}^n} v_0^{\frac{n}{n-1}} = 1$, since

$$1 - \varepsilon \leq \int_{B_\xi(0,R_\varepsilon)} v_0^{\frac{n}{n-1}} dv_\xi = \lim_{p \to 1^+} \int_{B_\xi(0,R_\varepsilon)} v_p^{p^*} dv_\xi \leq 1 + \varepsilon,$$

and

$$\int_{\mathbb{R}^n} v_0^{\frac{n}{n-1}} dv_\xi = \lim_{\varepsilon \to 0^+} \int_{B_\xi(0,R_\varepsilon)} v_0^{\frac{n}{n-1}} dv_\xi. \tag{55}$$

It is clear that $\|\tilde{v}_p\|_{\frac{n}{n-1}}$ is bounded by all $p > 1$, on the other hand, as is well known $L^{\frac{n}{n-1}}(\mathbb{R}^n)$ is a reflexive Banach space thus $\tilde{v}_p \rightharpoonup v_0$ weakly in $L^{\frac{n}{n-1}}(\mathbb{R}^n)$. A classical result ensures that weak convergence and convergence of norms as in (55) gives $\tilde{v}_p \to v_0$ strongly in $L^{\frac{n}{n-1}}(\mathbb{R}^n)$.

Since we have that $\int_{\mathbb{R}^n} |\nabla \tilde{v}_p| dv_\xi \to \int_{\mathbb{R}^n} |\nabla v_0| dv_\xi = K(n,1)^{-1}$. Then v_0 is a minimizer for the $W^{1,1}$ Euclidean Sobolev inequality which verifies

$$\int_{\mathbb{R}^n} v_0^{\frac{n}{n-1}} dv = 1.$$

Thus there exists $y_0 \in \mathbb{R}^n$, $\lambda_0 > 0$ and $R_0 > 0$ such that

$$v_0 = \lambda_0 \mathbf{1}_{B_\xi(y_0,R_0)}, \tag{56}$$

where $\mathbf{1}_{B_\xi(y_0,R_0)}$ denotes the characteristic function of the Euclidean ball $B_\xi(y_0,R_0)$, and moreover, since $v_p \leq 1$ in Ω_p we obtain by pointwise convergence a.e. dv_ξ that $0 \leq \lambda_0 \leq 1$. On the other hand $v_p \leq 1$ and the strong convergence in $L^{\frac{n}{n-1}}(\mathbb{R}^n)$ give that for all $q \geq \frac{n}{n-1}$, $\tilde{v}_p \to v_0$ strongly in $L^q(\mathbb{R}^n)$. Therefore

$$\lambda_0^q V_\xi(B_\xi(y_0,R_0)) = \lim_{p \to 1^+} \int_{\mathbb{R}^n} \tilde{v}_p^{\frac{n}{n-1}} dv_\xi = 1, \ \forall q \geq \frac{n}{n-1}.$$

Taking the limit when $q \to +\infty$ we deduce that λ_0 cannot be strictly less than 1, thus we get $\lambda_0 = 1$. So we have

$$V_\xi(B_\xi(y_0,R_0)) = \frac{\omega_{n-1}}{n} R_0^n = 1. \tag{57}$$

Up to changing x_p into $\exp_{x_p}(\mu_p y_0)$ in the definition of v_p, Ω_p and g_p, we may assume that $y_0 = 0$. In particular we have

$$\lim_{p \to 1} \tilde{v}_p = \mathbf{1}_{B_\xi(0,R_0)}, \quad \text{strongly in } L^{\frac{n}{n-1}}(\mathbb{R}^n), \tag{58}$$

and that for any $\varphi \in C_c^\infty(\mathbb{R}^n)$,

$$\lim_{p \to 1} \int_{\mathbb{R}^n} |\nabla \tilde{v}_p|_\xi^p \varphi dv_\xi = \int_{\partial B_\xi(0,R_0)} \varphi d\sigma_\xi, \tag{59}$$

where $d\sigma_\xi$ is the $(n-1)$-dimensional Riemannian measure of $\partial B_\xi(0,R_0)$ induced by the metric ξ of $\mathbb{R}^n$. Consider the extremal functions $V_p \in W^{1,p}(\mathbb{R}^n)$ for $K(n,p)^{-p}$ defined below

$$V_p(x) = \left(1 + \left(\frac{|x|}{R_0}\right)^{\frac{p}{p-1}}\right)^{1-\frac{n}{p}}, \quad x \in \mathbb{R}^n, \tag{60}$$

a simple application of the concentration-compactness principle, using (59), gives

$$\lim_{p\to 1^+} \int_{\mathbb{R}^n} |\nabla_\xi(\tilde{v}_p - V_p)|_\xi dv_\xi = 0. \tag{61}$$

Applying again the Moser's iterative scheme Lemma 3.1 to (45) with the help of (58), we also get that for any $R > R_0$,

$$\lim_{p\to 1^+} \sup_{\Omega_p \setminus B_{g_p}(0,R)} v_p = 0. \tag{62}$$

The application of Moser's iterative scheme is possible in strong bounded geometry because of the same arguments leading to (54).

Step 3 In this step we want to obtain from the $L^{\frac{n}{n-1}}$-estimate (58) the pointwise estimates (74), (75) which gives estimates on the decay rate to zero of $v_p(z)$ when $|z| \to +\infty$. For more details one can see for instance [Dru02a] and [Dru99]. With this aim in mind let us define

$$w_p(x) = |x|^{\frac{n}{p}-1} v_p(x), \tag{63}$$

and let $z_p \in \Omega_p$ be a point where w_p attains its maximum, i.e.,

$$w_p(z_p) = \|w_p\|_\infty. \tag{64}$$

Suppose by contradiction that

$$\lim_{p\to 1} \|w_p\|_\infty = \lim_{p\to 1} w_p(z_p) = +\infty. \tag{65}$$

Now we set

$$\nu_p^{1-\frac{n}{p}} = v_p(z_p),$$

this implies by (65) that

$$\lim_{p\to 1} |z_p| v_p(z_p)^{\frac{p}{n-p}} = \lim_{p\to 1} \frac{|z_p|}{\nu_p} = +\infty. \tag{66}$$

Using the fact that $v_p \leq 1$ in Ω_p and (63), we conclude that

$$\lim_{p\to 1} |z_p| = +\infty. \tag{67}$$

Consider $\exp_{g_p,z_p}$ the exponential map associated to g_p at z_p, let $\tilde{\Omega}_p = \nu_p^{-1} \exp^{-1}_{g_p,z_p}(\Omega_p)$, the metric $\tilde{g}_p(x) = \exp^*_{g_p,z_p} g_p(\nu_p x)$ for $x \in \tilde{\Omega}_p$, and the function given by

$$\phi_p(x) = \nu_p^{\frac{n}{p}-1} v_p(\exp_{z_p}(\nu_p x)) \text{ for } x \in \tilde{\Omega}_p,\ \phi_p(x) = 0 \text{ in } x \in \tilde{\Omega}_p^c.$$

Then for $x \in B(0,1)$, by (66), and (67), we can prove that ϕ_p is uniformly bounded in $B(0,1)$, and verifies (70).

In fact, for $x \in B(0,1)$, and by the definition of the $\exp_{g_p,z_p}$ map we have

$$\nu_p \quad \geq \quad d_{g_p}(z_p, \exp_{z_p}(\nu_p x)),$$

using the triangular inequality get

$$\begin{aligned} |\exp_{z_p}(\nu_p x)| &\geq |z_p| - d_{g_p}(z_p, \exp_{z_p}(\nu_p x)), \\ &\geq |z_p| - \nu_p \\ &= |z_p| - \left(w_p(z_p)|z_p|^{1-\frac{n}{p}}\right)^{\frac{p}{p-n}} \\ &= \left(1 - w_p(z_p)^{\frac{p}{p-n}}\right)|z_p|. \end{aligned} \tag{68}$$

Since $w_p(z_p) \to +\infty$ when $p \to 1$ and $\frac{p}{p-n} < 0$ for values of p very close to 1, and since (64) and (65), are valid for $x \in B(0,1)$ we obtain

$$|\exp_{z_p}(\nu_p x)| \geq \frac{1}{2}|z_p|. \tag{69}$$

Rewriting in terms of w_p we get

$$\begin{aligned} \phi_p(x) &= \nu_p^{\frac{n}{p}-1} v_p(\exp_{z_p}(\nu_p x)) \\ &= \nu_p^{\frac{n}{p}-1} w_p(\exp_{z_p}(\nu_p x))|\exp_{z_p}(\nu_p x)|^{1-\frac{n}{p}}. \end{aligned}$$

Since $1 - \frac{n}{p} < 0$ for values of p close to 1, we obtain

$$\phi_p(x) \quad \leq \quad \nu_p^{\frac{n}{p}-1} w_p(\exp_{z_p}(\nu_p x)) \left(\frac{1}{2}|z_p|\right)^{1-\frac{n}{p}},$$

and since z_p is the maximum of w_p, we have $w_p(\exp_{z_p}(\nu_p x)) \leq w_p(z_p)$, thus

$$\phi_p(x) \leq 2^{\frac{n}{p}-1}|z_p|^{1-\frac{n}{p}}\nu_p^{\frac{n}{p}-1}w_p(z_p),$$

and we know by definition that $v_p(z_p)^{-1}w_p(z_p) = |z_p|^{\frac{n}{p}-1}$, we are lead to

$$\phi_p(x) \leq 2^{\frac{n}{p}-1},$$

that is, $\|\phi_p\|_{L^\infty(B_\xi(0,1))} \leq 2^{\frac{n}{p}-1}$. Making the needed substitution in (45) a straightforward computation gives that ϕ_p satisfies

$$C_p\Delta_{p,\tilde{g}_p}\phi_p + \alpha_p\mu_p^2\nu_p^2\|\phi_p\|_p^{2-p}\phi_p^{p-1} = \lambda_p\phi_p^{p^*-1}, \text{ in } \tilde{\Omega}_p, \tag{70}$$

and $\phi_p = 0$ in $\partial\tilde{\Omega}_p$. Since ϕ_p is uniformly bounded we can apply Moser's iterative scheme Lemma 3.1 to the equation (70) and to get the existence of some $C > 0$ independent of p such that

$$1 = \phi_p(0) \leq \sup_{\tilde{\Omega}_p\cap B(0,1/2)} \phi_p \leq C\left(\int_{\tilde{\Omega}_p\cap B(0,1)} \phi_p^{p^*}\, dv_{\tilde{g}_p}\right)^{\frac{1}{p^*}}. \tag{71}$$

For a subsequent use remember that

$$\int_{\tilde{\Omega}_p\cap B_\xi(0,1)} \phi_p^{p^*}\, dv_{\tilde{g}_p} = \int_{\Omega_p\cap B_{g_p}(z_p,\nu_p)} v_p^{p^*}\, dv_{g_p}. \tag{72}$$

Again an application of the Moser's iterative scheme Lemma 3.1 is legitimate by the same arguments leading to (54). Therefore by (58) we get immediately that

$$\varinjlim_{p\to 1} \int_{B_{g_p}(z_p,\nu_p)\cap\Omega_p} v_p^{p^*}\, dv_{g_p} > 0. \tag{73}$$

By (71) and (72) given $R > 0$, we get $B_{g_p}(0,R)\cap B_{g_p}(z_p,\nu_p) = \emptyset$ because $|z_p| \to \infty$ when $p \to 1$. Furthermore

$$1 = \int_{\Omega_p} v^{p^*}\, dv_{g_p} = \int_{\Omega_p\cap B_{g_p}(0,R)} v^{p^*}\, dv_{g_p} + \int_{\Omega_p\setminus B_{g_p}(0,R)} v^{p^*}\, dv_{g_p},$$

on the other hand by (49), (50) we have

$$\begin{aligned}\left(1 - C\mu_p^2\right)\int_{B_\xi\left(0,\frac{R}{\mu_p}\right)} v_p^{p^*}\, dv_\xi &\leq \int_{\Omega_p\cap B_{g_p}(0,R)} v^{p^*}\, dv_{g_p} \\ &\leq \left(1 + C\mu_p^2\right)\int_{B_\xi\left(0,\frac{R}{\mu_p}\right)} v^{p^*}\, dv_\xi,\end{aligned}$$

taking the limit when $p \to 1$ in the last two equations, using (55), (58), (73) and remembering that $\mu_p \to 0$ when $p \to 1^+$ we get easily

$$0 < \varinjlim_{p\to 1^+} \int_{B_{g_p}(z_p,\nu_p)\cap\Omega_p} v_p^{p^*} dv_{g_p} \leq \varinjlim_{p\to 1^+} \int_{\Omega_p \setminus B_{g_p}(0,R)} v^{p^*} dv_{g_p} = 0,$$

which is the desired contradiction. Since this contradition comes from taking for granted (65), we are lead to negate (65) and to have the existence of some $C > 0$ such that for any $p > 1$, and for all $x \in \Omega_p$

$$w_p(x) = |x|^{\frac{n}{p}-1} v_p(x) \leq C. \tag{74}$$

In the same way, using (74), one proves thanks to (62) that for any $R > R_0$,

$$\lim_{p\to 1} \sup_{\Omega_p \setminus B_{g_p}(0,R)} |x|^{\frac{n}{p}-1} v_p(x) = 0. \tag{75}$$

To prove (75) we argue by contradiction so we suppose that there exist $y_p \in \Omega_p$ and $\delta > 0$ such that

$$\lim_{p\to 1} |y_p| = +\infty, \qquad \text{and} \quad w_p(y_p) \geq \delta.$$

Define $v_p(y_p) = \nu_p^{1-\frac{n}{p}}$, and $\tilde{\Omega}_p = \nu_p^{-1} \exp_{y_p}^{-1}(\Omega_p)$. Observe that $w_p(y_p) = |y_p|^{\frac{n}{p}-1} \nu_p^{1-\frac{n}{p}} \geq \delta$. For $x \in \tilde{\Omega}_p$, let $\phi_p(x) = \nu_p^{\frac{n}{p}-1} v_p(\exp_{y_p}(\nu_p x))$ and $\phi_p(x) = 0$ in $x \in \tilde{\Omega}_p^c$, and $\tilde{g}_p(x) = \exp_{y_p}^* g_p(\nu_p x)$.

Now for any $x \in B_\xi\left(0, \frac{1}{2}\delta^{\frac{p}{n-p}}\right)$, by the same arguments that above, we get that $|\exp_{y_p}(\nu_p x)| \geq \frac{1}{2}|y_p|$. Then using (74), we get that

$$\begin{aligned} \phi_p(x) &= \nu_p^{\frac{n}{p}-1} v_p(\exp_{y_p}(\nu_p x)) = \nu_p^{\frac{n}{p}-1} w_p(\exp_{y_p}(\nu_p x)) |\exp_{y_p}(\nu_p x)|^{1-\frac{n}{p}} \\ &\leq C 2^{\frac{n}{p}-1} |y_p|^{1-\frac{n}{p}} \nu_p^{\frac{n}{p}-1} \leq C 2^{\frac{n}{p}-1} \delta^{-1}. \end{aligned}$$

That is $\|\phi_p\|_{L^\infty\left(B_\xi\left(0,\frac{1}{2}\delta^{\frac{p}{n-p}}\right)\right)} \leq C 2^{\frac{n}{p}-1}\delta^{-1}$, and by Moser's iterative scheme Lemma 3.1 we get that

$$1 = \phi_p(0) \leq \sup_{\tilde{\Omega}_p \cap B_\xi\left(0,\frac{1}{4}\delta^{\frac{p}{n-p}}\right)} \phi_p \leq C \left(\int_{\tilde{\Omega}_p \cap B_\xi\left(0,\frac{1}{2}\delta^{\frac{p}{n-p}}\right)} \phi_p^{p^*} dv_{\tilde{g}_p} \right)^{\frac{1}{p^*}}.$$

On the other hand, since

$$\int_{\tilde{\Omega}_p\cap B_\xi\left(0,\frac{1}{2}\delta^{\frac{p}{n-p}}\right)}\phi_p^{p^*}dv_{\tilde{g}_p}=\int_{\Omega_p\cap B_{g_p}\left(y_p,\frac{1}{2}\delta^{\frac{p}{n-p}}\nu_p\right)}v_p^{p^*}dv_{g_p},$$

using the same arguments as above we get that for $R>0$ for p close to 1

$$B_{g_p}\left(y_p,\frac{1}{4}\delta^{\frac{p}{n-p}}\nu_p\right)\cap B_{g_p}(0,R)=\emptyset.$$

But for $R>R_0$, by (62) we have $\lim_{p\to 1}\sup_{\Omega_p\setminus B_{g_p}(0,R)}v_p=0$, and

$$\Omega_p\cap B_{g_p}\left(y_p,\frac{1}{2}\delta^{\frac{p}{n-p}}\nu_p\right)\subset\Omega_p\setminus B_{g_p}(0,R),$$

thus

$$1\leq\lim_{p\to 1}\sup_{\tilde{\Omega}_p\cap B_\xi\left(0,\frac{1}{2}\delta^{\frac{p}{n-p}}\right)}\phi_p=0,$$

which is a contradiction.

Step 4 Unfortunately the pointwise estimates that we obtained in (74) is not enough to prove our crucial (93). For this reasons we need to improve it. This is the goal to achieve in this step 4, which culminate in the proof of (76) below. Consider the following operator

$$L_pu:=C_p\Delta_{p,g_p}u+\alpha_p\mu_p^2\|v_p\|_{p,g_p}^{2-p}u^{p-1}-\lambda_pv_p^{p^*-p}u^{p-1}.$$

Choose $0<\nu<n-1$ and put

$$G_p(x)=\theta_p|x|^{-\frac{n-p-\nu}{p-1}},$$

where θ_p is some positive constant to be fixed later.

We will use the following relation for the p-Laplacian for radial functions that could be found in Lemma 1.2 of [Bie03] for an arbitrary Riemannian metric h

$$-\Delta_{p,h}u=-\Delta_{p,\xi}u+O(r)|\partial_ru|^{p-2}\partial_ru.$$

and we obtain

$$|x|^p\frac{L_pG_p(x)}{G_p(x)^{p-1}}\geq C_p\nu\left(\frac{n-p-\nu}{p-1}\right)^{p-1}-C\mu_p^2|x|^2+\alpha_p\mu_p^2\|v\|_p^{2-p}|x|^p-\lambda_p|x|^pv_p^{p^*-p},$$

in $\Omega_p \setminus \{0\}$. Here C denotes some constant independent of p. Thanks to (41), (42), (43), (75) and the fact that $r_p \to 0$ as $p \to 1$, one gets that for any $R > R_0$, $L_p G_p(x) \geq 0$ in $\Omega_p \setminus B_{g_p}(0,R)$ for p close enough to 1. On the other hand,

$$L_p v_p = 0 \quad \text{in } \Omega_p.$$

At last, it is not too hard to check with (62) that

$$v_p \leq G_p \quad \text{on } \partial B_{g_p}(0,R),$$

if we take $\theta_p = R^{\frac{n-p-\nu}{p-1}}$. Now we may apply the maximum principle as stated for instance in Lemma 3.4 of [AL99] to get,

$$v_p(y) \leq \left(\frac{R}{|y|}\right)^{\frac{n-p-\nu}{p-1}} \quad \text{in } \Omega_p \setminus B_{g_p}(0,R),$$

for p close enough to 1. This inequality obviously holds on $B_{g_p}(0,R)$ and so we have finally obtained that for any $n-1 > \nu > 0$ and any $R > R_0$, there exists $C(R,\nu) > 0$ such that for any $p > 1$ and any $y \in \Omega_p$,

$$\left(\frac{|y|}{R}\right)^{\frac{n-p-\nu}{p-1}} v_p(y) \leq C(R,\nu). \tag{76}$$

Step 5 We give in this Step the final arguments to conclude the proof of our Proposition 1. We apply the $W^{1,1}_{\xi}(\mathbb{R}^n)$ Euclidean Sobolev inequality to $\tilde{v}_p$:

$$\left(\int_{\Omega_p} \tilde{v}_p^{\frac{n}{n-1}} dv_\xi\right)^{\frac{n-1}{n}} \leq K(n,1) \int_{\Omega_p} |\nabla_\xi \tilde{v}_p|_\xi dv_\xi. \tag{77}$$

Recalling the Cartan expansion of g_p around 0, we have

$$dv_\xi = \left(1 + \frac{1}{6}\mu_p^2 Ric_g(x_p)_{ij} x^i x^j + o(\mu_p^2 |x|^2)\right) dv_{g_p}, \tag{78}$$

where Ric_g denotes the Ricci curvature of g in the $\exp_{x_p}$ -map. This last formula is true because $Ric_{g_p}(0) = \mu_p^2 Ric_g(\exp_{x_p}(0)) = \mu_p^2 Ric_g(x_p)$. Thus, by (46) we obtain

$$\int_{\Omega_p} \tilde{v}_p^{\frac{n}{n-1}} dv_\xi = 1 + \frac{1}{6}\mu_p^2 Ric_g(x_p)_{ij} \int_{\Omega_p} x^i x^j v_p^{p^*} dv_{g_p} + o\left(\mu_p^2 \int_{\Omega_p} |x|^2 v_p^{p^*} dv_{g_p}\right).$$

To estimate the last term on the right hand side of the preceding equality we need to prove (79) and (80) below

$$\begin{aligned}\int_{B_\xi(0,R_0)} x^i x^j dv_\xi &= \frac{\delta^{ij}}{n}\int_{B_\xi(0,R_0)} |x|^2 dv_\xi = \frac{\delta^{ij}}{n}\int_0^{R_0}\int_{\partial B_\xi(0,r)} r^2 d\sigma_\xi dr \\ &= \frac{\delta^{ij}}{n}\int_0^{R_0} r^{n+1}dr \int_{\partial B_\xi(0,1)} d\sigma_\xi \\ &= \frac{\delta^{ij}}{n(n+2)}\omega_{n-1}R_0^{n+2}. \end{aligned} \tag{79}$$

Let $\beta_p = \frac{n-p-\nu}{p-1}$ and $R > \max\{1, R_0\}$, by (76) we obtain that

$$\begin{aligned}\int_{\Omega_p\setminus B_\xi(0,R)} |x|^2 v_p^{p^*} dv_{g_p} &\leq CR^{p^*\beta_p}\int_{\Omega_p\setminus B_\xi(0,R)} |x|^{2-p^*\beta_p} dv_\xi \\ &\leq C(1+C\mu_p^2)R^{p^*\beta_p}\int_{\mathbb{R}^n} |x|^{2-p^*\beta_p} dv_\xi \\ &\leq C(1+C\mu_p^2)\omega_{n-1}R^{p^*\beta_p}\int_R^\infty \rho^{2-p^*\beta_p}\rho^{n-1}d\rho \\ &\leq C(1+C\mu_p^2)\omega_{n-1}R^{p^*\beta_p}\left(\frac{\rho^{n+2-p^*\beta_p}}{n+2-p^*\beta_p}\right)\Big|_R^\infty \\ &= C(1+C\mu_p^2)\omega_{n-1}R^{n+2}\tilde{\gamma}_{p,n} \to 0, \end{aligned} \tag{80}$$

where $\tilde{\gamma}_{p,n} := \dfrac{1}{p^*\beta_p - n - 2}$. Using (58), (79), and (80) we conclude that

$$\int_{\Omega_p} \tilde{v}_p^{\frac{n}{n-1}} dv_\xi = 1 + \frac{Sc_g(x_p)}{6n(n+2)}\omega_{n-1}R_0^{n+2}\mu_p^2 + o(\mu_p^2),$$

and the expression on the right hand side of (77) becomes

$$\left(\int_{\Omega_p} \tilde{v}_p^{\frac{n}{n-1}} dv_\xi\right)^{\frac{n-1}{n}} = 1 + \frac{(n-1)Sc_g(x_p)}{6n^2(n+2)}\omega_{n-1}R_0^{n+2}\mu_p^2 + o(\mu_p^2). \tag{81}$$

Denote by l_2 the limit of the scalar curvature function at x_p, i.e.,

$$l_2 := \lim_{p\to 1^+} Sc_g(x_p) \in \mathbb{R}, \tag{82}$$

which exists and is finite because in strong bounded geometry $|Sc_g(x)|$ is uniformly bounded with respect to $x \in M$. A fact that will be used

often in the sequel is that thanks to the hypothesis of C^2 convergence of the metric to the metric at infinity we have

$$\lim_{p\to 1^+} Sc_g(x_p) = \lim_{p\to 1^+} Sc_g(x_{0,p}) = l_1, \tag{83}$$

since $d_g(x_p, x_{0,p}) \leq r_p \to 0$, when $p \to 1^+$. But observe that in the proof of Proposition 2 this no longer true. By the Cartan expansion of g_p at 0, since $r_p \to 0$ as $p \to 1$, we also have

$$|\nabla_\xi \tilde{v}_p|^p_\xi = |\nabla_{g_p}\tilde{v}|^p_{g_p}\left(1 + \frac{\mu_p^2}{6}|\nabla_{g_p}\tilde{v}_p|^{-2}_{g_p} Rm_g(x_p)(\nabla_{g_p}\tilde{v}_p, x, x, \nabla_{g_p}\tilde{v}_p) + o(\mu_p^2|x|^2)\right),$$

where Rm_g denotes the Riemann curvature of g in the $\exp_{x_p}$-map. Then, using (78), we get

$$\begin{aligned}\int_{\Omega_p} |\nabla_\xi \tilde{v}_p|_\xi dv_\xi = &\int_{\Omega_p} |\nabla_{g_p}\tilde{v}_p|_{g_p} dv_{g_p} + \frac{\mu_p^2}{6} Ric_g(x_p)_{ij}\int_{\Omega_p} x^i x^j |\nabla_\xi \tilde{v}_p|_\xi dv_\xi \\ &+ \frac{\mu_p^2}{6}\int_{\Omega_p} |\nabla_{g_p}\tilde{v}_p|^{-1}_{g_p} Rm_g(x_p)(\nabla_{g_p}\tilde{v}_p, x, x, \nabla_{g_p}\tilde{v}_p) dv_{g_p} \\ &+ o\left(\mu_p^2\int_{\Omega_p}|x|^2|\nabla_{g_p}\tilde{v}_p|_{g_p} dv_{g_p}\right).\end{aligned} \tag{84}$$

Let us now estimate the different terms of (84). First, by equation (45) and relation (38), we have

$$\begin{aligned}\int_{\Omega_p} |\nabla_{g_p}\tilde{v}_p|_{g_p} dv_{g_p} &= \tilde{\gamma}^*_{p,n}\int_{\Omega_p} v_p^{\frac{n(p-1)}{n-p}} |\nabla_{g_p} v_p|_{g_p} dv_{g_p} \\ &\leq \tilde{\gamma}^*_{p,n}\left(\int_{\Omega_p} v_p^{p^*} dv_{g_p}\right)^{\frac{p-1}{p}}\left(\int_{\Omega_p}|\nabla_{g_p}v_p|^p_{g_p} dv_{g_p}\right)^{\frac{1}{p}} \\ &\leq \tilde{\gamma}^*_{p,n}\left(\lambda_p - \alpha_p\mu_p\|v\|_p^2\right)^{\frac{1}{2}} \\ &= \tilde{\gamma}^*_{p,n}\lambda_p^{\frac{1}{2}}\left(1 - \alpha_p\mu_p\lambda_p^{-1}\|v\|_p^2\right)^{\frac{1}{2}} \\ &\leq K(n,1)^{-1}\left(1 - \alpha_p\mu_p^2\lambda_p^{-1}\|v_p\|_p^2\right)^{\frac{1}{2}},\end{aligned}$$

where $\tilde{\gamma}^*_{p,n} := \frac{p(n-1)}{n-p}$. Since, by (58) and (76), $\|v_p\|_{g_p,p} = 1 + o(1)$, we get

$$\int_{\Omega_p} |\nabla_{g_p}\tilde{v}_p|_{g_p} dv_{g_p} \leq K(n,1)^{-1} - \frac{\alpha_p}{2}K(n,1)\mu_p^2 + o(\mu_p^2). \tag{85}$$

By Holder's inequalities, we have

$$\begin{aligned}\int_{\Omega_p}|x|^2|\nabla_{g_p}\tilde{v}_p|_{g_p}dv_{g_p} &= \tilde{\gamma}^*_{p,n}\int_{\Omega_p}v_p^{\frac{n(p-1)}{n-p}}|x|^2|\nabla_{g_p}v_p|_{g_p}dv_{g_p}\\ &\leq \tilde{\gamma}^*_{p,n}\left(\int_{\Omega_p}v_p^{p^*}dv_{g_p}\right)^{\frac{p-1}{p}}\left(\int_{\Omega_p}|x|^{2p}|\nabla_{g_p}v_p|^p_{g_p}dv_{g_p}\right)^{\frac{1}{p}}\\ &= \tilde{\gamma}^*_{p,n}\left(\int_{\Omega_p}|x|^{2p}|\nabla_{g_p}v_p|^p_{g_p}dv_{g_p}\right)^{\frac{1}{p}}.\end{aligned}$$

Multiplying the equation (45) by $|x|^{2p}v_p$ and integrating by parts, one gets

$$\begin{aligned}\int_{\Omega_p}|\nabla_{g_p}v_p|^{p-2}\langle\nabla_{g_p}(|x|^{2p}v_p),\nabla_{g_p}v_p\rangle dv_{g_p} &= C_p^{-1}\lambda_p\int_{\Omega_p}|x|^{2p}v_p^{p^*}dv_{g_p}\\ &- C_p^{-1}\alpha_p\mu_p^2\|v_p\|_p^{2-p}\int_{\Omega_p}|x|^{2p}v_p^p dv_{g_p}.\end{aligned}$$

By (76), every term on the right hand side of the preceding inequality is uniformly bounded with respect to p, then we conclude that

$$\int_{\Omega_p}|\nabla_{g_p}v_p|^{p-2}\langle\nabla_{g_p}(|x|^{2p}v_p),\nabla_{g_p}v_p\rangle dv_{g_p}\leq C, \tag{86}$$

for some $C>0$ that does not depend on p. Furthermore by Cauchy-Schwarz's inequality we get that

$$\begin{aligned}\int_{\Omega_p}|\nabla_{g_p}v_p|^{p-2}\langle\nabla_{g_p}(|x|^{2p}v_p),\nabla_{g_p}v_p\rangle dv_{g_p} &= \int_{\Omega_p}|x|^{2p}|\nabla_{g_p}v_p|^p\\ &+ \int_{\Omega_p}|\nabla_{g_p}v_p|^{p-2}v_p\langle\nabla_{g_p}(|x|^{2p}),\nabla_{g_p}v_p\rangle dv_{g_p}\\ &\geq \int_{\Omega_p}|x|^{2p}|\nabla_{g_p}v_p|^p dv_{g_p}\\ &- 2p\int_{\Omega_p}|x|^{2p-1}\nabla_{g_p}(|x|)|\nabla_{g_p}v_p|^{p-1}v_p dv_{g_p}\\ &= \int_{\Omega_p}|x|^{2p}|\nabla_{g_p}v_p|^p dv_{g_p}\\ &- 2p\int_{\Omega_p}|x|^{2p-1}|\nabla_{g_p}v_p|^{p-1}v_p dv_{g_p}.\end{aligned}$$

Therefore we are lead to

$$\begin{aligned}\int_{\Omega_p}|x|^{2p}|\nabla_{g_p}v_p|^p_{g_p}dv_{g_p} &\leq \int_{\Omega_p}|\nabla_{g_p}v_p|^{p-2}_{g_p}\langle\nabla_{g_p}(|x|^{2p}v_p),\nabla_{g_p}v_p\rangle_{g_p}dv_{g_p}\\ &+ C\int_{\Omega_p}|x|^{2p-1}|\nabla_{g_p}v_p|^{p-1}_{g_p}v_p dv_{g_p}\\ &\leq C\\ &+ C\left(\int_{\Omega_p}|x|^{2p}|\nabla_{g_p}v_p|^p_{g_p}dv_{g_p}\right)^{\frac{p-1}{p}}\left(\int_{\Omega_p}|x|^p v_p^p dv_{g_p}\right)^{\frac{1}{p}},\end{aligned}$$

where C denotes some constants independent of p. By (76) we see easily that $\int_{\Omega_p}|x|^p v_p^p dv_{g_p}$, is uniformly bounded with respect to p. Then by Young's inequalities, one deduces that

$$\begin{aligned}\int_{\Omega_p}|x|^{2p}|\nabla_{g_p}v_p|^p_{g_p}dv_{g_p} \leq& C + C\left(\int_{\Omega_p}|x|^{2p}|\nabla_{g_p}v_p|^p_{g_p}dv_{g_p}\right)^{\frac{p-1}{p}}\\ \leq& C+\frac{C^p}{p}+\frac{p-1}{p}\left(\int_{\Omega_p}|x|^{2p}|\nabla_{g_p}v_p|^p_{g_p}dv_{g_p}\right)^{\frac{p-1}{p}\frac{p}{p-1}},\end{aligned}$$

and so

$$\int_{\Omega_p}|x|^{2p}|\nabla_{g_p}v_p|^p_{g_p}dv_{g_p}\leq\left(1-\frac{p-1}{p}\right)^{-1}\left(C+\frac{C^p}{p}\right)\leq\tilde{C},$$

with $\tilde{C}>0$ independent of p. That is

$$\int_{\Omega_p}|x|^{2p}|\nabla_{g_p}v_p|^p_{g_p}dv_{g_p}=O(1). \tag{87}$$

Now for some $R>R_0$, we get readily by (59) that

$$\begin{aligned}\int_{\Omega_p}|\nabla_\xi\tilde{v}_p|_\xi x^i x^j dv_\xi &= O\left(\int_{\Omega_p\setminus B_\xi(0,R)}|x|^2|\nabla_\xi\tilde{v}_p|_\xi dv_\xi\right)\\ &+ \int_{\partial B_\xi(0,R_0)}x^i x^j d\sigma_\xi+o(1).\end{aligned}$$

By Hölder inequality we obtain that

$$\int_{\Omega_p\setminus B_\xi(0,R)}|x|^2|\nabla_{g_p}\tilde{v}_p|_{g_p}dv_{g_p}\leq\frac{p(n-1)}{n-p}\left(\int_{\Omega_p\setminus B_\xi(0,R)}|x|^{2p}|\nabla_{g_p}v_p|^p_g dv_{g_p}\right)^{\frac{1}{p}}.$$

Multiplying the equation (45) by $|x|^{2p}v_p$, integrating over $\Omega_p\backslash B_\xi(0,R) := \Omega_p^*$, and using Cauchy-Schwarz, Hölder inequality and later by Young inequality we obtain

$$\begin{aligned}
\int_{\Omega_p^*}|x|^{2p}|\nabla_{g_p}v_p|_{g_p}^p dv_{g_p} &\leq \int_{\Omega_p^*}|\nabla_{g_p}v_p|_{g_p}^{p-2}\langle\nabla_{g_p}(|x|^{2p}v_p),\nabla_{g_p}v_p\rangle_{g_p}dv_{g_p}\\
&+ 2p\int_{\Omega_p^*}|x|^{2p-1}|\nabla_{g_p}v_p|_{g_p}^{p-1}v_p dv_{g_p}\\
&\leq \int_{\Omega_p^*}|\nabla_{g_p}v_p|_{g_p}^{p-2}\langle\nabla_{g_p}(|x|^{2p}v_p),\nabla_{g_p}v_p\rangle_{g_p}dv_{g_p}\\
&+ 2p\left(\int_{\Omega_p^*}|x|^{2p}|\nabla_{g_p}v_p|_{g_p}^p dv_{g_p}\right)^{\frac{p-1}{p}}\left(\int_{\Omega_p^*}|x|^p v_p^p dv_{g_p}\right)^{\frac{1}{p}}\\
&\leq \int_{\Omega_p^*}|\nabla_{g_p}v_p|_{g_p}^{p-2}\langle\nabla_{g_p}(|x|^{2p}v_p),\nabla_{g_p}v_p\rangle_{g_p}dv_{g_p}\\
&+ 2(p-1)\int_{\Omega_p^*}|x|^{2p}|\nabla_{g_p}v_p|_{g_p}^p dv_{g_p} + 2\int_{\Omega_p^*}|x|^p v_p^p dv_{g_p}.
\end{aligned}$$

At last we obtain that

$$\begin{aligned}
0\leq(3-2p)\int_{\Omega_p^*}|x|^{2p}|\nabla_{g_p}v_p|_{g_p}^p dv_{g_p} &\leq \int_{\Omega_p^*}|\nabla_{g_p}v_p|_{g_p}^{p-2}\langle\nabla_{g_p}(|x|^{2p}v_p),\nabla_{g_p}v_p\rangle_{g_p}dv_{g_p}\\
&+ 2\int_{\Omega_p^*}|x|^p v_p^p dv_{g_p}.
\end{aligned}$$

But when $p\to 1$, by (76), the terms on the right hand side go to 0, then we can conclude that

$$\int_{\Omega_p\backslash B_\xi(0,R)}|x|^2|\nabla_\xi v_p|_\xi dv_\xi \to 0. \tag{88}$$

Thus for the second term on the right hand side of (84) we see that

$$\lim_{p\to 1}Ric_g(x_p)_{ij}\int_{\Omega_p}|\nabla_\xi\tilde{v}_p|_\xi x^i x^j dv_\xi = \frac{\omega_{n-1}}{n}R_0^{n+1}l_1. \tag{89}$$

Now, we look at the third term on the right hand side of (84). Since ∇V_p, V_p as in (60), and x are pointwise colinear vector fields, we have

$$Rm_g(x_p)(\nabla_{g_p}\tilde{v}_p,x,x,\nabla_{g_p}\tilde{v}_p)\leq C|x|^2|\nabla_\xi\tilde{v}_p|_\xi|\nabla_\xi(\tilde{v}_p-V_p)|_\xi. \tag{90}$$

Now by (90), integrating over $\Omega_p \cap B_\xi(0,R) := \hat{\Omega}_p$ we have

$$\begin{aligned}\int_{\hat{\Omega}_p} |\nabla_{g_p}\tilde{v}_p|_{g_p}^{-1} Rm_g(x_p)(\nabla_{g_p}\tilde{v}_p, x, x, \nabla_{g_p}\tilde{v}_p)dv_{g_p} &\leq \int_{\hat{\Omega}_p} C|\nabla_{g_p}\tilde{v}_p|_{g_p}^{-1}|x|^2|\nabla_\xi\tilde{v}_p|_\xi|\nabla_{g_p}(\tilde{v}_p - V_p)|_\xi dv_{g_p} \\ &\leq C_R\left(1 + C\mu_p^2\right)\int_{\hat{\Omega}_p}|\nabla_\xi(\tilde{v}_p - V_p)|_\xi dv_\xi.\end{aligned}$$

This last inequality combined with (61) yields to

$$\lim_{p\to 1}\int_{\Omega_p\cap B_\xi(0,R)} |\nabla_{g_p}\tilde{v}_p|_{g_p}^{-1} Rm_g(x_p)(\nabla_{g_p}\tilde{v}_p, x, x, \nabla_{g_p}\tilde{v}_p)dv_{g_p} = 0. \quad (91)$$

We want to estimate the integral of the same integrand function of (91) but outside $B_\xi(0,R)$, for this we have

$$\begin{aligned}\int_{\Omega_p\setminus B_\xi(0,R)} |\nabla_{g_p}\tilde{v}_p|_{g_p}^{-1} Rm_g(x_p)(\nabla_{g_p}\tilde{v}_p, x, x, \nabla_{g_p}\tilde{v}_p)dv_{g_p} &\leq \int_{\Omega_p\setminus B_\xi(0,R)} \Lambda_2|x|^2|\nabla_{g_p}\tilde{v}_p|_{g_p}^{-1}|\nabla_{g_p}\tilde{v}_p|_{g_p}^2 dv_{g_p} \\ &\leq \Lambda_2(1+C\mu_p^2)\int_{\Omega_p\setminus B_\xi(0,R)}|x|^2|\nabla_{g_p}\tilde{v}_p|_\xi dv_\xi \\ &\overset{(88)}{\to} 0. \quad (92)\end{aligned}$$

Combining (91) and (92) we conclude that

$$\lim_{p\to 1}\int_{\Omega_p} |\nabla_{g_p}\tilde{v}_p|_{g_p}^{-1} Rm_g(x_p)(\nabla_{g_p}\tilde{v}_p, x, x, \nabla_{g_p}\tilde{v}_p)dv_{g_p} = 0. \quad (93)$$

Finally, substituting in (77), using (83), and (81)-(87), we obtain,

$$\begin{aligned}1 + \frac{(n-1)l_2}{6n^2(n+2)}\omega_{n-1}R_0^{n+2}\mu_p^2 &+ o(\mu_p^2) \\ &\leq K(n,1)\left[K(n,1)^{-1} - \frac{\alpha_p}{2}K(n,1)\mu_p^2\right] \\ &+ K(n,1)\left[\frac{\omega_{n-1}}{6n}R_0^{n+1}l_2\mu_p^2\right] + o(\mu_p^2) \\ &= 1 - \frac{\alpha_p}{2}K(n,1)^2\mu_p^2 \\ &+ \frac{K(n,1)\omega_{n-1}}{6n}R_0^{n+1}l_2\mu_p^2 + o(\mu_p^2).\end{aligned}$$

Since $\frac{\omega_{n-1}}{n} = \frac{1}{R_0^n}$, and $K(n,1) = \frac{1}{n}\left(\frac{n}{\omega_{n-1}}\right)^{\frac{1}{n}} = \frac{R_0}{n}$, a straightforward computation leads to

$$\frac{K(n,1)^2}{2}\left(\alpha_p - \frac{n}{n+2}l_2\right)\mu_p^2 + o(\mu_p^2) \leq 0.$$

This gives the desired contradiction by letting p go to 0, recalling here that $l_1 = l_2$ by (82) it holds

$$\frac{n}{n+2}l_1 - \varepsilon_0 + \frac{n}{n+2}l_2 = \lim_{p\to 1^+}\alpha_p - \frac{n}{n+2}l_2 = \varepsilon_0 > 0. \tag{94}$$

This ends the proof of Proposition 1. □

We are now ready to accomplish the proof of our global comparison theorem for small diameters in C^2-locally asymptotically strong bounded geometry. We use the same argument used in [Dru02b], for completeness's sake we write the details here as pointed out to us by Olivier Druet in a private communication.

Proof of Theorem 1.1. The Proposition at page 2353 of [Dru02b] rewritten in this text as Proposition 3.1 says that for any $\varepsilon > 0$, there exists $r_\varepsilon = r_\varepsilon(x_0, M, g) > 0$ such that if $\Omega \subset B_g(x_0, r_\varepsilon)$, then

$$V_g(\Omega)^{2\frac{n-1}{n}} \leq K(n,1)^2 A_g(\partial\Omega)^2 + K(n,1)^2\left(\frac{n}{n+2}S_g(x_0) + \varepsilon\right)V_g(\Omega)^2.$$

By assumption we know that $S_g(x_0) < n(n-1)k_0$, so that applying the preceding inequality with

$$\varepsilon = \frac{n}{2(n+2)}\left[n(n-1)k_0 - S_g(x_0)\right],$$

we get that there exists $r > 0$, $r_\varepsilon(x_0, M, g) \geq r > 0$ such that if $\Omega \subset B_g(x_0, r)$, then

$$\begin{aligned} V_g(\Omega)^{2\frac{n-1}{n}} &\leq K(n,1)^2 A_g(\partial\Omega)^2 \\ &+ K(n,1)^2\left(\frac{n}{n+2}n(n-1)k_0 - \frac{n}{2(n+2)}\varepsilon_0\right)V_g(\Omega)^2. \end{aligned} \tag{95}$$

where $\varepsilon_0 = n(n-1)k_0 - S_g(x_0) > 0$ fixed. Now let B_v be a small ball in the model space $(\mathbb{M}_{k_0}, g_{k_0})$ of constant sectional curvature k_0 and of volume v, for any $V_0 > 0$ (small enough in the case of the sphere, i.e., $k_0 > 0$) there exists $C_0 = C_0(n, k, V_0) > 0$ such that for balls of volume $0 \leq v \leq V_0$ it holds

$$V_{g_{k_0}}(B_v)^{2\frac{n-1}{n}} \geq K(n,1)^2 A_{g_{k_0}}(\partial B_v)^2 \tag{96}$$

$$+ K(n,1)^2\frac{n}{n+2}n(n-1)k_0 V_{g_{k_0}}(B_v)^2 \tag{97}$$

$$- C_0 v^{2\frac{n}{n+2}}. \tag{98}$$

If we assume that $V_g(\Omega) = V_{g_0}(B_v) = v \leq V_0$, we get that

$$\begin{aligned} K(n,1)^2 A_{g_0}(\partial B_v)^2 &+ K(n,1)^2 \frac{n}{n+2} n(n-1)k_0 v^2 - C_0 v^{2\frac{n+2}{n}} \\ &\leq v^{2\frac{n-2}{n}} \\ &\leq K(n,1)^2 A_g(\partial\Omega)^2 \\ &+ K(n,1)^2 \left(\frac{n}{n+2} n(n-1)k_0 - \frac{n}{2(n+2)} \varepsilon_0 \right) v^2 \end{aligned}$$

that is,

$$A_{g_0}(\partial B_v)^2 \leq A_g(\partial\Omega)^2 + C_0 K(n,1)^{-2} v^{2\frac{n+2}{n}} - \frac{n}{2(n+2)} \varepsilon_0 v^2.$$

If we choose $v < V_1 < V_0 < \min\{1, V_{g_{k_0}}\left(\mathbb{M}^n_{k_0}\right)\}$ [1], with the property that

$$C_0(n,k_0) K(n,1)^{-2} V_1^{2\frac{n+2}{n}} - \frac{n}{2(n+2)} \varepsilon_0 V_1^2 < 0,$$

which is always possible to find, then we get

$$A_{g_{k_0}}(\partial B_v) < A_g(\partial\Omega).$$

Thus there exists $V_1 = V_1(n, k_0, V_0, Sc_g(x_0)) > 0$ such that if $\Omega \subset B_g(x_0, r)$ with volume $V_{g_0}(B_v) = v < V_1$, then the comparison inequality (1) of the theorem holds. Now, up to lower a little bit r (depending on curvatures of M), we are sure that any domain $\Omega \subset B_g(x_0, r)$ has volume less than that of the ball $B_g(x_0, r)$ which can be chosen to be less than V_1 and the theorem is proved. At the end of this proof we understand the subtle fact that a lower bound on r_x of Theorem 1 of [Dru02b] does not depend explicitly on the r_ε of the Proposition at page 2353 but we need to prove the existence of r_ε to prove the existence of r_x. □

We are thus led to the following version of Theorem 1 of [Dru02b], namely our Theorem 1, in which an uniform estimate of a lower bound on r_x is obtained provided M is of C^2-locally asymptotically strong bounded geometry at infinity.

Proof of Theorem 1. We proceed as in the proof of the preceding theorem using our Proposition 1 instead of the Proposition at page 2353

[1] $\min\{1, +\infty\}$ is assumed to be equal at 1.

of [Dru02b]. This gives the existence of an uniform $r_\varepsilon(M,g) > 0$ independent of x_0. With the help of Bishop-Gromov we have for every $0 \leq r \leq 1$.

$$V_g(B_g(x_0,r)) \leq V_{g_k}(B_{g_k}(x_0,r)) \leq C(n,k)r^n,$$

where k is a lower bound on the Ricci curvature of (M,g). So we can take a radius $r = r(n,k,k_0,V_0,S_g) > 0$ such that

$$r < \sqrt[n]{\frac{V_1(n,k_0,V_0,S_g)}{C(n,k)}}, \tag{99}$$

where V_1 is as in the proof of Theorem 1.1 with ε_0 replaced by $\tilde{\varepsilon}_0 = n(n-1)k_0 - S_g > 0$. Observing that we can take for example $V_0 < \min\{1, V_{g_{k_0}}\left(\mathbb{M}^n_{k_0}\right)\}$ fixed we obtain $r = r(n,k,k_0,S_g) > 0$. □

The following proposition have a weaker conclusion with respect to the previous one but that holds for manifolds with weaker assumptions, namely for manifolds with just strong bounded geometry.

Proposition 2. *Let (M,g) be a complete Riemannian manifold of dimension $n \geq 2$ with strong bounded geometry. For any $\varepsilon > 0$ there exists $r_\varepsilon = r_\varepsilon(M,g) > 0$ such that for any point $x_0 \in M$, any function $u \in C^\infty_c(B_g(x_0,r_\varepsilon))$, we have*

$$||u||^2_{\frac{n}{n-1},g} \leq K(n,1)^2 \left(||\nabla_g u||^2_{1,g} + \alpha_\varepsilon ||u||^2_{1,g}\right), \tag{100}$$

where $\alpha_\varepsilon = \frac{n}{n+2}S_g + \varepsilon$ with $S_g := \sup_{x\in M}\{Sc_g(x)\} \in \mathbb{R}$.

Remark 3.5. *We notice that the constant $r_\varepsilon = r_\varepsilon(M) > 0$ is obtained by contradiction and that the proof does not give an explicit effective lower bound on it.*

First Proof of Proposition 2. This proof is obtained from the proof of Proposition 1 replacing in that proof α_p by the constant independent of p defined by $\alpha_{\varepsilon_0} := \dfrac{n}{n+2}S_g + \varepsilon_0$. Then we find a sequence $(\tilde{u}_p)$ solving the same partial differential equations that the analogous u_p solve in the proof of Proposition 1 but replacing α_p with α_{ε_0}, after we take as x_p a point of maximum of $\tilde{u}_p$ and observe that up to a subsequence we can assume

$$\lim_{p\to 1^+} Sc_g(x_p) = \frac{n}{n+2}l_2 \leq S_g, \tag{101}$$

for some $l_2 \in [S_{inf}, S_g]$, where $S_{inf,g} := \inf\{S_g(x) : x \in M\}$. It is worth to note that $S_{inf,g}$ and S_g are finite real numbers, because S_g is bounded from below by $n(n-1)k$ and from above by $n(n-1)k_0$. Then exactly in the same way we are lead to the analog of (94), that is

$$\alpha_{\varepsilon_0} - \frac{n}{n+2} l_2 = \frac{n}{n+2} S_g + \varepsilon_0 - \frac{n}{n+2} l_2 \geq \varepsilon_0 > 0. \tag{102}$$

This last inequality giving the desired contradiction proves our Proposition 2. □

We can now proceed to the proof of Theorem 3.

Proof of Theorem 3. At this point it is easy to remark that Theorem 3 follows from Proposition 2 exactly in the same way as Theorem 1 follows from Proposition 1. We leave the details of the proof to the reader. □

In the remaining part of this section we give an alternative proof of Proposition 2, using smoothing via the Ricci flow and results of [Shi89] and [Kap05]. Let us denote $(M, \tilde{g})$ a complete Riemannian manifold with $|Sec_{\tilde{g}}| \leq \Lambda$. Consider the Ricci flow of $\tilde{g}_t$ with initial data $\tilde{g} = \tilde{g}_0$

$$\frac{\partial \tilde{g}_t}{\partial t} = -2Ric(\tilde{g}_t). \tag{103}$$

Theorem 3.1 (see Theorem 1.1 of [Shi89], compare also [BMOR84] and [Ham82])**.** *Under the above assumptions on $(M, \tilde{g})$ there exists a constant $T = T(n, \Lambda) > 0$ such that (103) with initial condition $\tilde{g}_0 = \tilde{g}$ has a smooth solution $\tilde{g}_t$ for a short time $0 \leq t \leq T$. Moreover $\tilde{g}_t$ satisfies the following estimates*

$$e^{-c(n,\Lambda)t}\tilde{g} \leq \tilde{g}_t \leq e^{c(n,\Lambda)t}\tilde{g}, \tag{104}$$

$$|\nabla - \nabla_t| \leq c(n, \Lambda)t, \tag{105}$$

and for any integer $m \geq 0$, there exists constants $c_m = c(n, m, \Lambda) > 0$ such that

$$\sup_{x \in M} |\nabla^m R_{ijkl}(x, t)| \leq c_m t^{-\frac{m}{2}}, \quad 0 < t \leq T(n, \Lambda). \tag{106}$$

Moreover the sectional curvature of $\tilde{g}(t)$ satisfies

$$|Sec_{\tilde{g}_t}| \leq C(n, T), \quad 0 \leq t \leq T(n, \Lambda). \tag{107}$$

for some constant $C(n, T) > 0$.

Then we can assume true the hypotheses of the existence of the constant $C(n,\Lambda,T)$ in the theorem below.

Theorem 3.2 (Proposition at page 260 of [Kap05])**.** *Let $(M^n,\tilde{g})$ be a complete Riemannian manifold and suppose that $|Sec_{\tilde{g}}|\leq\Lambda$, for some positive constant Λ. Consider the Ricci flow of $\tilde{g}$ given by*

$$\frac{\partial\tilde{g}_t}{\partial t}=-2Ric(\tilde{g}_t). \tag{108}$$

Then sectional curvatures satisfy the following relation

$$\inf_{x\in M}\{Sec_{\tilde{g}_t}(x)\}-C(n,T)t\leq Sec_{\tilde{g}_t}(x)\leq\sup_{x\in M}\{Sec_{\tilde{g}_t}(x)\}+C(n,T)t, \tag{109}$$

where the scalar curvature satisfies $|Sec_{\tilde{g}_t}|\leq C(n,t)$.

Remark 3.6. *If (M,g) have C^2-locally asymptotically strong bounded geometry, then using Proposition 1 it is trivial to check that (M,g) satisfies (100).*

Second Proof of Proposition 2. For an arbitrary Riemannian metric g, over M we define

$$\lambda_{1,r,x_0,g}:=\inf_{\substack{u\in C_c^\infty(B_g(x_0,r))\\ u\not\equiv 0}}\left\{\frac{\|\nabla_g u\|_{1,g}^2+\left(\frac{n}{n+2}S_g+\varepsilon_0\right)\|u\|_{1,g}^2}{\|u\|^2_{\frac{n}{n-1},g}}\right\}. \tag{110}$$

By reduction to the absurd, we suppose that (100) above is false. Then there exists $\varepsilon_0>0$ such that for all $r>0$ there exist $x_{0,r}\in M$, $u_{x_{0,r}}\in C_c^\infty(B_g(x_{0,r},r))$ such that

$$\|u\|^2_{\frac{n}{n-1},g}>K(n,1)^2\left[\|\nabla_g u\|_{1,g}^2+\left(\frac{n}{n+2}S_g+\varepsilon_0\right)\|u\|_{1,g}^2\right].$$

The last inequality is equivalent to say that

$$\lambda_{1,r,x_{0,r},g}<K(n,1)^{-2}. \tag{111}$$

Take the Ricci flow $(M,\tilde{g}_t)$ with the initial data $\tilde{g}_0:=g$. Then by Theorem 3.1, i.e., Theorems 1.1 and 1.2 of [Shi89] we get that $(M,\tilde{g}_t)$ is a smooth manifold with smooth metric $\tilde{g}_t$ that by (106) satisfies $|\nabla^m_{\tilde{g}_t}R_{ijkl,\tilde{g}_t}|_{\tilde{g}_t},|\nabla^m_{g_t}R_{ijkl,\tilde{g}_t}|_{g_t}\leq c_m t^{-\frac{m}{2}}$, for any integer $m\geq 0$. Moreover, by (104) we have $\tilde{g}_t\to\tilde{g}_0=g$ in C^0 topology. This combined with Klingenberg's Lemma (compare for instance Theorem III.2.4 of [Cha06])

guarantees that for any $t \in [0,T]$ holds $inj_{M,\tilde{g}_t} \geq i_0 = i_0(n,\Lambda_2,g) > 0$ where Λ_2 is an upper bound on the sectional curvature, and so the manifold $(M,\tilde{g}_t)$ have strong bounded geometry. Furthermore for any integer $m \geq 0$ we have that $(M,\tilde{g}_t)$ belongs to $\mathcal{M}(n,m,\Lambda,r = r(n,\Lambda_2,i_0))$ (see Definition 2.6 or page 308 of [Pet06]). Hence $(M,\tilde{g}_t)$ satisfy the hypothesis of Theorems 76, 72 of [Pet06], Theorems 1.2, 1.3 of [Heb99], and thus $(M,\tilde{g}_t)$ is smooth at infinity and have $C^{m,\alpha}$-locally asymptotically bounded geometry smooth at infinity, for any $m \in \mathbb{N}$. This implies that $(M,\tilde{g}_t)$ satisfies the assumptions of Proposition 1 and so trivially also the conclusion (100) of Proposition 2. Independently we observe that by Theorem 3.2, we have that there exists a sufficiently small $t_r \in]0,T]$ such that $\lambda_{1,r,x_{0,r},\tilde{g}_{t_r}} < K(n,1)^{-2}$, but this in turns means that $(M,\tilde{g}_{t_r})$ does not satisfies the conclusion (100) of Proposition 2. In this way we get a contradiction, which indeed completes the proof of the proposition. □

4 In mild bounded geometry isoperimetric regions of small volume are of small diameter

In this section we work with just a fixed Riemannian metric g defined on M.

Lemma 4.1 (Lemma 3.2 of [Heb99])**.** *Let (M,g) be a smooth, complete Riemannian n- dimensional manifold with weak bounded geometry. There exist two positive constants $C_{Heb} = C_{Heb}(n,k,v_0) > 0$ and $\bar{v} := \bar{v}(n,k,v_0) > 0$, depending only on n,k, and v_0, such that for any open subset Ω of M with smooth boundary and compact closure, if $V_g(\Omega) \leq \bar{v}$, then $C_{Heb} V_g(\Omega)^{\frac{n-1}{n}} < A_g(\partial\Omega)$.*

Remark 4.1. *After Theorem 1 of [MFN15] we know that we can extend the preceding lemma to an arbitrary finite perimeter set simply by approximating with open bounded with smooth boundary subsets having the same volume.*

Let us introduce a crucial notion for the remaining part of this section.

Definition 4.1. *We say that a sequence (D_j) of finite perimeter sets, $D_j \subseteq M$, with finite volume $V_g(D_j) \to 0$, is called an **approximate isoperimetric sequence**, if*

$$\lim_{j\to\infty} \frac{\mathcal{P}_g(D_j)}{V_g(D_j)^{\frac{n-1}{n}}} = \lambda,$$

where $\lambda := \varinjlim_{v\to 0^+} \frac{I_{M,g}(v)}{v^{(n-1)/n}}$.

Remark 4.2. *Comparing with geodesic balls we have clearly that* $\lambda \leq c_n$*, where* c_n *is the Euclidean isoperimetric constant defined by* $I_{\mathbb{R}^n}(v) = c_n v^{\frac{n-1}{n}}$, $\forall v \in]0, V(M)[$.

Remark 4.3. *When* (M, g) *have weak bounded geometry then* $\lambda \geq C_{Heb}(n, k, v_0) > 0$*, because of Lemma* 3.2 *of [Heb99] reported here in Lemma 4.1 and the related Remark 4.1.*

Remark 4.4. *When* (M, g) *have strong bounded geometry then* $\lambda = c_n$*, this is an easy consequence of the Théoréme of Appendice C at page 531 of [BM82]. We wrote an alternative proof of this last fact, based on Theorems 3 and 4, in our Theorem 4.1 below.*

We recall here three well known lemmas (see for instance Corollary 2.1 of [Nar14a]) that we will use often in the sequel.

Lemma 4.2. *Let* M^n *be a complete Riemannian manifold with weak bounded geometry. Then for each* $r > 0$ *there exists* $c_1 = c_1(n, k, r) > 0$ *such that* $V_g(B_M(p, r)) > c_1(n, k, r)v_0$*, where* $c_1(n, k, r) = \min\left\{\frac{r^n}{e^{\sqrt{(n-1)|k|}}}, 1\right\}$.

Lemma 4.3. *Let* M *with weak bounded geometry. Then there exist two positive constants* $C_1 = C_1(n, k) > 0$, $C_2 = C_2(n, k) > 0$ *such that for every* $0 < r < \bar{r} = \bar{r}(n, k) := \min\left\{1, e^{\frac{\sqrt{(n-1)|k|}}{n}}\right\}$ *we have*

$$v_0 C_1 r^n \overset{doubling+noncollapsing}{\leq} V_g(B_M(x, r)) \overset{Bishop-Gromov}{\leq} C_2 r^n, \quad (112)$$

where $C_1 = C_1(n, k) = \frac{1}{e^{\sqrt{(n-1)|k|}}}$.

Lemma 4.4. *Let* M *with weak bounded geometry. Then there exist two positive constants* $\bar{v}_1 = \bar{v}_1(n, k, v_0) > 0$ *and* $C_3 = C_3(n, k) > 0$*, such that for every* $0 < v < \bar{v}_1$ *we have*

$$\lambda \leq \frac{I_M(v)}{v^{\frac{n-1}{n}}} \leq C_3(n, k). \quad (113)$$

Here $\bar{v}_1 := \min\{1, \bar{v}\}$.

Lemma 4.5. *Let* M^n *be a complete Riemannian manifold weak with bounded geometry. There exists a positive constant* $N = N(n, k, v_0) > 0$ *such that, whenever* D *is a finite perimeter set with finite volume and*

$0 < R < \bar{R} = \bar{R}(n,k) := \min\left\{1, 2e^{\frac{\sqrt{(n-1)|k|}}{n}}, \frac{2}{7}\bar{r}\right\}$ *there exists a partition* $(D_l)_l$ *of* D*, i.e,* $D = \mathring{\cup}_l D_l$*, where every* D_l *is a set of finite perimeter contained in a ball of radius* R *and such that*

$$\left(\sum_l \mathcal{P}(D_l)\right) - \mathcal{P}(D) \leq N(n,k,v_0)\frac{V(D)}{R}. \tag{114}$$

Proof. Let $(p_l)_{l\in\mathbb{N}}$ be a sequence of points of M such that $\left\{B_M(p_l, \frac{R}{4})\right\}$ is a maximal set of disjoint balls. It is straightforward to show that

$$M = \bigcup_l B_M\left(p_l, \frac{R}{2}\right).$$

Set $\mathcal{A} := \{p_l\}_{l\in\mathbb{N}}$. By coarea formula we can cut D with a ball of radius r_l centered at p_l, such that $\frac{R}{2} < r_l < R$ and

$$A(D \cap \partial B_M(p_l, r_l)) \leq \frac{2V(D)}{R}. \tag{115}$$

Consider $D \setminus (\bigcup_l \partial B_M(p_l, r_l)) = \mathring{\bigcup}_l D_l$. Then there exists a constant $\tilde{N} = \tilde{N}(n,k,v_0) > 0$ such that

$$\left(\sum_l \mathcal{P}(D_l)\right) - \mathcal{P}(D) \leq 4\tilde{N}\frac{V(D)}{R}.$$

Note that by a simple combinatorial argument, $\tilde{N}$ could be taken as an upper bound of the greatest number of disjoint balls of radius $\frac{R}{4}$ contained in a ball of radius $\frac{7}{4}R$. This upper bound depends only on n, k, v_0 since for every $x \in M$ by our assumption $R < \bar{R}$ it holds

$$\begin{aligned}
\tilde{N}C_1(n,k)\left(\frac{R}{4}\right)^n v_0 &\leq \sum_{p_i \in B_M(x, \frac{7}{4}R)} V_g(B_M(p_i, \frac{R}{4})) \\
&\leq V_g\left(B_M\left(x, \frac{7}{4}R\right)\right) \\
&\overset{Bishop-Gromov}{\leq} V_{g_k}\left(B_{\mathbb{M}^n_k}\left(\frac{7}{4}R\right)\right) \leq C_2(n,k)\left(\frac{7}{4}R\right)^n,
\end{aligned}$$

where $C_1(n,k) = \frac{1}{e^{\sqrt{(n-1)|k|}}}$. Setting $N = 4\tilde{N}$ we finish the proof of the lemma. □

Lemma 4.6. *Let M^n be a complete Riemannian manifold with weak bounded geometry, and $D_j \subset M$ be a sequence of finite perimeter sets with finite volume. Then there exist a partition of D_j by finite perimeter sets of $D_j = \mathring{\bigcup}_l D_{j,l}$ and a sequence of radii R_j, such that* $\operatorname{diam}(D_{j,l}) \leq 2R_j$, *with* $\lim_{j\to\infty} \dfrac{V(D_j)^{1/n}}{R_j} = 0$, $R_j \to 0$, *and*

$$\overrightarrow{\lim}_{j\to\infty} \left[\left(\sum_l \mathcal{P}(D_{j,l}) \right) - P(D_j) \right] \frac{1}{V(D_j)^{\frac{n-1}{n}}} = 0. \tag{116}$$

Proof. It is enough to apply (114) with $D = D_j$ and $R = R_j := V(D_j)^\alpha$, with $0 < \alpha < \frac{1}{n}$. □

We are now ready to prove Theorem 4.

Proof of Theorem 4. Consider an arbitrary sequence of finite perimeter sets Ω_j such that $v_j := V_g(\Omega_j) \to 0$. By Lemma 4.6 we can find a partition of Ω_j satisfying (116). For sufficiently large j we have $R_j \leq r$ where $r := \frac{d}{2}$ and d is given by Theorem 3. Set $\eta_j := N(n,k,v_0)\dfrac{v_j}{R_j}$, with $R_j >> v_j^{\frac{1}{n}}$, i.e., $\frac{v_j}{R_j} \to 0$, when $j \to +\infty$, we obtain

$$\begin{aligned}
\frac{I_{\mathbb{M}^n_{k_0}}(v_j)}{v_j^{\frac{n-1}{n}}} - \frac{\eta_j}{v_j^{\frac{n-1}{n}}} &\leq \frac{\sum_l I_{\mathbb{M}^n_{k_0}}(v_{j,l})}{v_j^{\frac{n-1}{n}}} - \frac{\eta_j}{v_j^{\frac{n-1}{n}}} && (117)\\
&\leq \frac{\sum_l \mathcal{P}_g(\Omega_{j,l})}{v_j^{\frac{n-1}{n}}} - \frac{\eta_j}{v_j^{\frac{n-1}{n}}} && (118)\\
&\leq \frac{\mathcal{P}_g(\Omega_j)}{v_j^{\frac{n-1}{n}}}, && (119)
\end{aligned}$$

where the first inequality is due to the strict subadditivity of $I_{\mathbb{M}^n_{k_0}}$, the second is due to Theorem 1 (because $\operatorname{diam}(\Omega_{j,l}) < d$ for j large enough), and the last inequality is due to Lemma 4.6. For all j large enough we have that

$$(1-\varepsilon)\frac{I_{\mathbb{M}^n_{k_0}}(v_j)}{v_j^{\frac{n-1}{n}}} \leq \frac{I_{\mathbb{M}^n_{k_0}}(v_j)}{v_j^{\frac{n-1}{n}}} - \frac{\eta_j}{v_j^{\frac{n-1}{n}}}, \tag{120}$$

thus

$$(1-\varepsilon)\frac{I_{\mathbb{M}^n_{k_0}}(v_j)}{v_j^{\frac{n-1}{n}}} \leq \frac{\mathcal{P}_g(\Omega_j)}{v_j^{\frac{n-1}{n}}},$$

The last inequality combined with (117)-(119) easily establish the validity of (8) and complete the proof of the theorem. □

Corollary 4.1. *Let M be a complete Riemannian manifold with strong bounded geometry. Then*

$$\varinjlim_{v\to 0^+} \frac{I_M(v)}{v^{\frac{(n-1)}{n}}} = c_n,$$

where c_n is the Euclidean isoperimetric constant defined by $I_{\mathbb{R}^n}(v) = c_n v^{\frac{n-1}{n}}$.

Proof. Take an arbitrary sequence $v_j \to 0$ and a sequence of positive real numbers $\varepsilon_j \to 0$, by the definition of $I_{M,g}$ we know that we can take a sequence of finite perimeter sets Ω_j, such that $V_g(\Omega_j) = v_j$ and $I_{M,g}(v_j) \leq \mathcal{P}_g(\Omega_j) \leq I_{M,g}(v_j) + \varepsilon_j$. Passing to the limit in (8) or using (120) combined with the asymptotic expansion of the perimeter of geodesic balls in the model simply connected space forms in function of the volume enclosed it follows that

$$c_n \leq \varinjlim_{j\to+\infty} \frac{\mathcal{P}_g(\Omega_j)}{v_j^{\frac{n-1}{n}}}. \tag{121}$$

With this last inequality and Inequality (121) in mind the theorem follows without any further complications. □

Before to continue, let us state some results of independent interest that will be crucial in the proof of Theorem 2.

Theorem 4.1 (Selecting a large subdomain non effective)**.** *Let (M^n, g) be a complete Riemannian manifold with weak bounded geometry, and $(D_j)_j$ is an approximate isoperimetric sequence. Then there exists another approximate isoperimetric sequence (D'_j) such that $\lim_{j\to\infty} V_g(D_j \triangle D'_j) = 0$, $\lim_{j\to\infty} \frac{V_g(D'_j)}{V_g(D_j)} = 1$, $\lim_{j\to\infty} \frac{\mathcal{P}_g(D'_j)}{\mathcal{P}_g(D_j)} = 1$ and $diam(D'_j) \to 0$, when $j \to \infty$.*

Proof. We perform the same construction of a partition as in the proof of Lemma 4.5 applied to any D_j with a suitable radius R_j that we will choose later, and obtain a suitable partition $D_{j,l}$ of D_j a maximal family of points $\mathcal{A}_j$ such that

$$\left(\sum_l \mathcal{P}_g(D_{j,l})\right) - \mathcal{P}_g(D_j) \leq N(n,k,v_0)\frac{V_g(D_j)}{R_j}. \tag{122}$$

Set $v_j := V_g(D_j)$, by the definition of λ and of I_M, we get that for large j it holds

$$\mathcal{P}(D_{j,l}) \geq I_{M,g}(V(D_{j,l})) \geq \lambda v_{j,l}^{(n-1)/n}, \tag{123}$$

where $v_{j,l} := V_g(D_{j,l})$. Trivially for large j we have $V_g(D_{j,l}) \leq v_j \leq \bar{v}$ and the Euclidean type isoperimetric inequality for small volumes holds. This implies by Lemma 4.5 that

$$\frac{\sum_l \lambda V_g(D_{j,l})^{(n-1)/n}}{v_j^{(n-1)/n}} \leq \frac{\sum_l \mathcal{P}_g(D_{j,l})}{v_j^{(n-1)/n}} \leq \frac{\mathcal{P}_g(D_j)}{v_j^{\frac{n-1}{n}}} + N\frac{v_j^{\frac{1}{n}}}{R_j}. \tag{124}$$

Using the arguments of the combinatorial Lemma 2.3 of [Nar14b] applied to $f_{j,l} := \dfrac{V_g(D_{j,l})}{v_j}$, we get that $f_j^* := \max\{f_{j,1}, ..., f_{j,l_j}\}$ satisfies

$$\sum_l f_{j,l} f_j^{*-\frac{1}{n}} \leq \sum_l f_{j,l} f_{j,l}^{-\frac{1}{n}} = \sum_l f_{j,l}^{\frac{n-1}{n}} \leq \frac{1}{\lambda}\left[\frac{\mathcal{P}_g(D_j)}{v_j^{\frac{n-1}{n}}} + N\frac{v_j^{\frac{1}{n}}}{R_j}\right],$$

hence

$$f_j^* \geq \left[\frac{\lambda}{\frac{\mathcal{P}(D_j)}{v_j^{\frac{n-1}{n}}} + N\frac{v_j^{\frac{1}{n}}}{R_j}}\right]^n, \tag{125}$$

which implies

$$V_g(D'_{j,1}) \geq v_j \left[\frac{\lambda}{\frac{\mathcal{P}(D_j)}{v_j^{\frac{n-1}{n}}} + N\frac{v_j^{\frac{1}{n}}}{R_j}}\right]^n. \tag{126}$$

On the other hand we recall that by construction there exists a point $p'_j \in \mathcal{A}_j$ depending on D_j such that $D'_{j,1} \subseteq B_M(p_{D_j}, R_j)$. Fix an arbitrary sequence $\mu_j \to +\infty$ and set $R_j := \mu_j v_j^{\frac{1}{n}}$, $D'_j := B_M(p_{D_j}, R_j) \cap D_j$, $v'_j := V(D'_j)$, $l_{1,j} := \mathcal{P}_g(D_j, B_M(p_{D_j}, R_j))$, $l_{2,j} := \mathcal{P}(D_j) - l_{1,j}$, $A_j := \mathcal{P}_g(D'_j) = l_{1,j} + \frac{\Delta v_j}{R_j}$, and $\Delta v_j := v_j - v'_j$ we have $D_{1,j} \subseteq D'_j \subseteq D_j$ thus

$$\lim_{j\to+\infty} \frac{v'_j}{v_j} = 1, \tag{127}$$

$$\frac{\Delta v_j}{v_j} \leq 1 - \left[\frac{\lambda}{\frac{\mathcal{P}(D_j)}{v_j^{\frac{n-1}{n}}} + N\frac{v_j^{\frac{1}{n}}}{R_j}}\right]^n = 1 - \left[\frac{\lambda}{\frac{\mathcal{P}(D_j)}{v_j^{\frac{n-1}{n}}} + \frac{N}{\mu_j}}\right]^n, \tag{128}$$

$$\frac{\Delta v_j}{v_j} \to 0, \tag{129}$$

$$\frac{A_j}{\mathcal{P}_g(D_j)} \to 1, \tag{130}$$

$$\frac{l_{1,j}}{\mathcal{P}_g(D_j)} \to 1, \tag{131}$$

$$\frac{l_{2,j}}{v_j^{\frac{n-1}{n}}} \to 0. \tag{132}$$

□

Essentially Theorem 4.1 says that for small volumes approximate isoperimetric sequences have all the mass and perimeter that stay inside a ball of small radius. What will be proved further in Lemma 4.9 is that in fact the part outside this latter ball in fact does not give any contribution and actually for small volumes is empty.

Definition 4.2. *Let (M^n, g) be a Riemannian manifold. We say that (M, g)* ***satisfy*** *(H), if there exists*

$$\lim_{v\to 0^+} \frac{I_M(v)}{v^{\frac{n-1}{n}}} = \varliminf_{v\to 0^+} \frac{I_M(v)}{v^{\frac{n-1}{n}}} = \varlimsup_{v\to 0^+} \frac{I_M(v)}{v^{\frac{n-1}{n}}} = \lambda. \tag{133}$$

In the next theorem we will give a little more refined proof of Theorem 4.1 having the advantage of being effective.

Theorem 4.2 (Selecting a large subdomain effective)**.** *Let M^n be a complete Riemannian manifold with weak bounded geometry and $\mu > 0$. Then there exists $\bar{v}_2 = \bar{v}_2(n, k, v_0, \mu) > 0$ such that for any finite perimeter set D with volume $v \leq \bar{v}_2$ there exists $p_D \in M$ another finite perimeter set $D' := B_M(p_D, \mu v^{\frac{1}{n}}) \cap D \subseteq D$ such that*

$$V_g(D \triangle D') \leq 1 - \left[\frac{\lambda}{\frac{\mathcal{P}_g(D)}{v^{\frac{n-1}{n}}} + \frac{N}{\mu}}\right]^n, \tag{134}$$

$$\frac{V_g(D')}{V_g(D)} \geq \left[\frac{\lambda}{\frac{\mathcal{P}_g(D)}{v^{\frac{n-1}{n}}} + \frac{N}{\mu}} \right]^n. \tag{135}$$

In particular $diam_g(D') \leq 2\mu v^{\frac{1}{n}}$.

Proof. First of all choose $\bar{v}_2 \leq \min\{\frac{\bar{R}}{\mu}, \bar{v}_1\}$. Then perform the same construction of a partition as in the proof of Lemma 4.5 applied to D with radius $R := \mu v^{\frac{1}{n}}$, and obtain a suitable partition $\{D_l\}_l$ containing a finite number l_D of of components $D_j = \mathring{\cup} D_l$ joint with a maximal family of points $\mathcal{A}$ such that

$$\left(\sum_{l=1}^{l_D} \mathcal{P}_g(D_l) \right) - \mathcal{P}_g(D) \leq N(n, k, v_0) \frac{V_g(D)}{R}. \tag{136}$$

Set $v_l := V_g(D_l)$, by the definition of λ and of I_M, it holds

$$\mathcal{P}_g(D_l) \geq I_{M,g}(V_g(D_l)) \geq \lambda v_l^{\frac{n-1}{n}}. \tag{137}$$

Since $v \leq \bar{v}_2$ we have $V_g(D_l) \leq v \leq \bar{v}_2$ and the Euclidean type isoperimetric inequality for small volumes holds. This implies by Lemma 4.5 that

$$\frac{\sum_l \lambda V_g(D_l)^{(n-1)/n}}{v^{(n-1)/n}} \leq \frac{\sum_l \mathcal{P}_g(D_l)}{v^{(n-1)/n}} \leq \frac{\mathcal{P}_g(D)}{v^{\frac{n-1}{n}}} + N \frac{v^{\frac{1}{n}}}{R}. \tag{138}$$

Using the arguments of the combinatorial Lemma 2.3 of [Nar14b] applied to $\gamma_l := \dfrac{V_g(D_l)}{v}$, we get that $\gamma^* := \max\{\gamma_1, ..., \gamma_{l_D}\}$ satisfies

$$\sum_l \gamma_l \gamma^{*-\frac{1}{n}} \leq \sum_l \gamma_l \gamma_l^{-\frac{1}{n}} = \sum_l \gamma_l^{\frac{n-1}{n}} \leq \frac{1}{\lambda} \left[\frac{\mathcal{P}_g(D)}{v^{\frac{n-1}{n}}} + N \frac{v^{\frac{1}{n}}}{R} \right],$$

hence

$$\gamma^* \geq \left[\frac{\lambda}{\frac{\mathcal{P}_g(D)}{v^{\frac{n-1}{n}}} + N \frac{v^{\frac{1}{n}}}{R}} \right]^n, \tag{139}$$

which implies

$$V(\tilde{D}_1') \geq v \left[\frac{\lambda}{\frac{\mathcal{P}_g(D)}{v^{\frac{n-1}{n}}} + N \frac{v^{\frac{1}{n}}}{R}} \right]^n, \tag{140}$$

where $\tilde{D}'_1$ is one of the connected components of the partition $\{D_l\}_l$ of D that satisfy $\frac{V_g(\tilde{D}'_1)}{v} = \gamma^*$. On the other hand we recall that by construction there exists a point $p_D \in \mathcal{A}$ depending on D such that $\tilde{D}'_1 \subseteq B_M(p_D, R)$. Set $D' := B_M(p_D, R) \cap D$, $v' := V_g(D')$, and $\Delta v := v - v'$ we have $\tilde{D}'_1 \subseteq D' \subseteq D$ thus by (140)

$$\frac{v'(D)}{v} \geq \left[\frac{\lambda}{\frac{\mathcal{P}_g(D)}{v^{\frac{n-1}{n}}} + \frac{N}{\mu}} \right]^n, \tag{141}$$

uniformly with respect to $D \in \tilde{\tau}_v$ ($\tilde{\tau}$ is defined in Definition). Furthermore, it is also easily seen by (140) that

$$\frac{\Delta v}{v} \leq 1 - \left[\frac{\lambda}{\frac{\mathcal{P}_g(D)}{v^{\frac{n-1}{n}}} + \frac{N}{\mu}} \right]^n. \tag{142}$$

Remark 4.5. *At this stage we made the choice of not controlling the perimeter added cutting with a ball of radius R by a coarea formula argument. We recall that this is always possible (by coarea formula) up to take a slightly larger radius $R + \eta_D R$ for a suitable $0 < \eta_D < 1$.*

□

The following lemma have its own interest. Its proof is based on the adaptation of the arguments of the Deformation Lemma 4.5 of [GR13] and of formula (1.10) of [GMT83] also named Almgren's Lemma in some literature see for instance the book [Mag12].

Definition 4.3. *For every $f \in BV(M)$ every $D \subset M$ with Lipschitz continuous boundary ∂D we define the **trace of f on the boundary of** D as the function $f_{|\partial D} : \Sigma \subseteq \partial D \to \mathbb{R}$, where $\mathcal{H}^{n-1}(\partial D \setminus \Sigma) = 0$ and defined by*

$$f_{|\partial B} : x \mapsto \lim_{r \to 0^+} \frac{1}{B_M(x,r)} \int_{B_M(x,r)} f(y) dV_g(y).$$

Lemma 4.7 (Theorem 2.10 of [Giu84])**.** *Let Ω be a bounded open set in M^n with Lipschitz continuous boundary $\partial\Omega$ and let $f \in BV(\Omega)$. Then there exists a function $\phi \in L^1(\partial\Omega)$ such that for $\mathcal{H}^{n-1}$-almost all $x \in \partial\Omega$*

$$\lim_{\rho \to 0^+} \rho^{-n} \int_{B_M(x,\rho) \cap \Omega} |f(z) - \phi(x)| dz = 0. \tag{143}$$

In particular such a $\phi = f_{|\partial D}$ *belongs to* $L^1(\partial\Omega)$. *Moreover, for every* $X \in \mathfrak{X}_0^1(M)$,

$$\int_\Omega f div X dx = -\int_\Omega \langle X, Df\rangle + \int_{\partial\Omega} \phi\langle X, \nu\rangle, \tag{144}$$

where ν *is the unit outer normal to* $\partial\Omega$.

Lemma 4.8 (Deformation Lemma)**.** *Let* (M^n, g) *be a Riemannian manifold,* $p \in M$, $B := B_M(p, r)$ *a geodesic ball with* $0 < r < inj_M$, $k \in \mathbb{R}$, $(n-1)k$ *a lower bound on the Ricci curvature tensor inside* B, $u : B \to [0, +\infty[$, $u : x \mapsto d_M(p, x)$, *is the distance function to the point* p, $E \subseteq B$ *a set of locally finite perimeter in* B. *Then it holds*

$$\mathcal{P}_g(\partial B, (M \setminus E)^{(1)}) \leq \mathcal{P}_g(E, B) + \frac{c}{r} V_g(B \setminus E), \tag{145}$$

where $c = c(n, k, inj_M) := 1 + (n-1)c_k(inj_M) > 0$ *is a positive constant, and* $(M \setminus E)^{(1)}$ *are the points of density* 1 *of* $(M \setminus E)$.

Proof. Applying Lemma 4.7 with $f = \chi_{E^c}$ and $X := \varphi \frac{u}{r} \nabla u$, where $E^c := B \setminus E$ and $\varphi \in C_0^1(B_M(p, inj_M))$ with the property that $\{x \in M : d_M(x, B) \leq \varepsilon\} \subseteq \varphi^{-1}(1) \subseteq B_M(p, inj_M)$ for some small $\varepsilon > 0$ (observe that this choice of φ yields $X \in \mathfrak{X}_0^1(M)$) leads to

$$\begin{aligned} \int_B \chi_{E^c} div_g \left(\frac{u}{r}\nabla u\right) dV_g &= \int_{\partial B} \chi_{E^c}|_{\partial B} \frac{u}{r} \langle \nabla u, \nu_{ext}\rangle d\mathcal{H}^{n-1} \\ &- \int_B \left\langle \nabla \chi_{E^c}, \frac{u\nabla u}{r} \right\rangle dV_g, \end{aligned}$$

then

$$\begin{aligned} \int_B \chi_{E^c} div_g \left(\frac{u}{r}\nabla u\right) dV_g &= \int_B \chi_{E^c} \left(\frac{\|\nabla u\|^2}{r} + \frac{u}{r} div_g(\nabla_g u)\right) dV_g \\ &\leq \int_B \chi_{E^c} \left(\frac{1}{r} + \frac{u(x)(n-1)}{r} \cot_k(u(x))\right) dV_g(x) \\ &\leq \int_B \chi_{E^c} \left(\frac{1}{r} + \frac{n-1}{r} c_k(r)\right) dV_g(x) \\ &\leq \frac{1}{r}\left(1 + (n-1)c_k(r)\right) V_g(E^c). \end{aligned}$$

On the other hand

$$-\int_B \left\langle \nabla \chi_{E^c}, \frac{u}{r}\nabla u \right\rangle = -\mathcal{P}(E, B).$$

Hence

$$\mathcal{P}(B \setminus E, (M \setminus E)^{(1)}) - \mathcal{P}(E, B) \leq \frac{1}{r}\left(1 + (n-1)c_k(r)\right) V_g(E^c).$$

From the last inequality it is easy to deduce (145), after the simple observation that $c_k(r) := r\cot_k(r)$ is a strictly increasing function in particular is bounded in $[0, inj_M]$. □

Remark 4.6. *It is worth to recall here that by Theorem* 1 *of [GMT83] (which immediately could be adapted to the Riemannian manifold because is a local theorem) an isoperimetric region have always nonempty interior as well as its complement but a lot of proofs of regularity do not give a satisfying and uniform estimates of the radius of the balls contained inside.*

The following lemma have its own interest. Its proof is based on the adaptation of the arguments of the Deformation Lemma 4.5 of [GR13] which in this context are given by our Lemma 4.8 combined with the arguments of Section 2 of [Nar14b] that are adapted here in Theorem 3.2 with the use of the Heintze-Karcher comparison Theorem of [HK78] combined with the proof of Lemma 3.8 of [Nar15] and Theorem 3 of [Nar14a].

Lemma 4.9. *Let (M^n, g) be a complete Riemannian manifold with mild bounded geometry satisfying (H). Then there exist two positive constants $\mu^* = \mu^*(n, k, v_0, \lambda) > 0$ and $v^* = v^*(n, k, v_0, \lambda) > 0$ such that whenever $\Omega \subseteq M$ is an isoperimetric region of volume $0 \leq v \leq v^*$ it holds that*

$$\operatorname{diam}_g(\Omega) \leq \mu^* v^{\frac{1}{n}}.$$

Remark 4.7. *In mild bounded geometry v_0 depends on k and inj_M so in the preceding lemma we have that $\mu^* = \mu^*(n, k, inj_M, \lambda)$ and $v^* = v^*(n, k, inj_M, \lambda)$. In strong bounded geometry condition (H) is always fulfilled, moreover it is known that $\lambda = c_n$, hence in the preceding lemma when specialised to the case of strong bounded geometry we have actually $\mu^* = \mu^*(n, k, inj_M)$ and $v^* = v^*(n, k, inj_M)$. The construction made to prove the preceding lemma it is possible only because we assume positive injectivity radius. So the injectivity radius is hidden inside μ^* and v^* although it is tempting to prove Lemma 4.9 just assuming M with weak bounded geometry instead of mild bounded geometry.*

Remark 4.8. *As already observed, if M have strong bounded geometry, then always exists $\lim_{v\to 0^+} \frac{I_M(v)}{v^{\frac{n-1}{n}}} = \lambda$ and so in particular Lemma 4.9*

applies to manifold with strong bounded geometry. Unfortunately, we still do not know wether the existence of $\lim_{v\to 0^+} \frac{I_M(v)}{v^{\frac{n-1}{n}}} = \lambda$ *could be dropped or not in the statement of the preceding lemma. Obviously in weak bounded geometry or in mild bounded geometry, if one proves that* $\lambda = c_n$*, then automatically condition* (H) *is fulfilled.*

In fact the following questions are still open at the present stage of our knowledge.

Question 2. *If* (M^n, g) *is with weak bounded geometry, then* M *satisfy* (H)*?*

Question 3. *If* (M^n, g) *is with mild bounded geometry, then* M *satisfy* (H)*?*

Question 4. *If* (M^n, g) *is with weak bounded geometry or with mild bounded geometry, what is the sharp value of* λ*?*

Remark 4.9. *The main reason to assume positive injectivity radius in the preceding lemma is that we make a crucial use of Lemma 4.8 which in turn uses radial deformations which are well defined only locally at a point* $x \in M$ *inside a ball of radius less than* inj_x*. We will see later in the proof of Lemma 4.9 that we want to apply radial deformations with center at the point* p^*_Ω *defined further, but if* $inj_M = 0$ *we have no control about the size of* $inj_{p^*_\Omega}$ *(remember that* p^*_Ω *could go to infinity) and the volume that we can put inside* $B_M(p^*_\Omega, inj_{p^*_\Omega})$*. To avoid this problem of course it is enough to assume positive injectivity radius, but we still do not know whether this assumption could be dropped and replaced just by the noncollapsing of the volume of balls of radius* 1*.*

Remark 4.10. *It is well known by that in mild bounded geometry* $v_0 = v_0(n, k, inj_M) > 0$*. Thus in the statement of Lemma 4.9 we can suppress the dependence of* v^* *on* v_0*.*

The geometric idea of the proof is not too complicated but unfortunately the writing turns out to be technical, because of the effective calculations of the constants. In first by an application of Theorem 4.1 we find a point $p_\Omega \in M$ and a controlled radius $\mu v^{\frac{1}{n}}$, such that almost all the volume of Ω is recovered inside the ball $B_M(p_\Omega, \mu v^{\frac{1}{n}})$. In second, we take a ball inside Ω of controlled volume and radius and show that these two balls cannot be disjointed when the volume tends to 0, so we take a bigger but still controlled radius. Then we proceed by contradiction and suppose that there are points of Ω very far from p_Ω. Under this

assumption we take a possibly bigger controlled radius such that we can pick a fraction of the volume of Ω that is far from p_Ω and compensate close to p_Ω to readjust volume. In this way a competitor F of the same volume than Ω is constructed provided the volume of Ω is taken small enough. Finally it is shown that this last competitor F have perimeter strictly less than the perimeter of Ω for volumes possibly smaller. This gives a contradiction and so the impossibility to find points of Ω far from p_Ω.

Proof of Lemma 4.9. For simplicity of notations we consider just the case $k \leq 0$. When $k > 0$ the theorem of Bonnet-Myers ensures that M is compact and so the lemma is already proved in Theorem 2.2 of [MJ00]. Our proof works also without the restriction $k \leq 0$. For now on in this proof we assume that $k \leq 0$. The hypothesis (H) permit to us to define the quantities

$$\tilde{f}^*(n,k,v_0,\lambda,\mu) := \lim_{v\to 0^+} \tilde{f}(v,n,k,v_0,\lambda,\mu) = \left[\frac{\lambda}{\lambda + \frac{N}{\mu}}\right]^n, \tag{146}$$

$$f^*(n,k,v_0,\lambda,\mu) := \lim_{v\to 0^+} f(v,n,k,v_0,\lambda,\mu) = 1 - \left[\frac{\lambda}{\lambda + \frac{N}{\mu}}\right]^n, \tag{147}$$

where

$$\tilde{f}(v,n,k,v_0,\lambda,\mu,M) := \left[\frac{\lambda}{\frac{I_M(v)}{v^{\frac{n-1}{n}}} + \frac{N}{\mu}}\right]^n,$$

$$f(v,n,k,v_0,\lambda,\mu,M) := 1 - \left[\frac{\lambda}{\frac{I_M(v)}{v^{\frac{n-1}{n}}} + \frac{N}{\mu}}\right]^n.$$

In the remaining part of this proof we will use frequently the two following crucial properties

$$\lim_{\mu\to+\infty} \tilde{f}^*(n,k,v_0,\lambda,\mu) = 1, \tag{148}$$

$$\lim_{\mu\to+\infty} \tilde{f}^*(n,k,v_0,\lambda,\mu) = 0. \tag{149}$$

Suppose until the end of the proof that Ω is an isoperimetric region of volume v. By Heintze-Karcher's theorem we have that in weak bounded geometry

$$c_7(n,k)v^{\frac{1}{n}} \geq inrad(\Omega) \geq \frac{v}{I_M(v)} \geq \frac{v}{I_{\mathbb{M}^n_k}(v)} \geq \frac{v}{c_2 v^{(n-1)/n}} = \tilde{c}_2 v^{\frac{1}{n}},$$

for some positives constants $c_7 = c_7(n,k) > 0$ and $\tilde{c}_2 = \tilde{c}_2(n,k) > 0$. To see this in details the reader could consult Lemma 3.1 of [Nar15]. In first we observe that the hypothesis that M satisfy (H) permits to have (146)-(149) which in turn allow us to choose $\mu = \mu(n,k,v_0,\lambda) > 0$ large enough to satisfy simultaneously

$$\mu > c_1(n,k)^{1/n}, \tag{150}$$

$$\mu > c_7(n,k), \tag{151}$$

$$\frac{\tilde{c}_2}{2n}\left(2f^*(n,k,v_0,\lambda,\mu)\right)^{1/n} < \frac{C_{Heb}}{4n}, \tag{152}$$

where $C_{Heb} > 0$ is the constant appearing in Lemma 3.2 of [Heb99] .As it is easy to see, using (146) and (147) we can prove the existence of $v^* = v^*(n,k,v_0,inj_M,\lambda) > 0$ such that for every $v \leq v^*$ we have that the following conditions are satisfied

$$v^{\frac{n-1}{n}} f(v,n,k,v_0,\lambda,\mu) < c_1(n,k)v_0, \tag{153}$$

$$f(v,n,k,v_0,\lambda,\mu) \leq 2f^*(n,k,v_0,\lambda,\mu), \tag{154}$$

$$r_v = 4\mu v^{\frac{1}{n}} \leq \frac{1}{4} inj_M, \tag{155}$$

$$c_1(n,k,\mu v^{\frac{1}{n}}) = \tilde{C}_1(n,k)\mu^n v, \tag{156}$$

$$v \leq \min\{1, \overline{v}, \bar{v}_1, \bar{v}_2\}, \tag{157}$$

where $\overline{v}$ is obtained in Lemma 3.2 of [Heb99], i.e., such that for volumes smaller than $\overline{v}$ it holds $I_M(v) \geq \lambda v^{\frac{n-1}{n}}$. In the remaining part of this proof we always assume that $v \leq v^*$. Consider an isoperimetric region Ω of $V(\Omega) = v$, the same construction of Theorem 4.1 applied to Ω gives the existence of $p_\Omega \in M$, (notice that the point p_Ω could be chosen satisfying the condition $p_\Omega \in \mathring{\Omega}$, but this is not relevant for the rest of our discussion) such that

$$\begin{aligned}
\frac{V(B_M(p_\Omega, \mu v^{1/n}) \cap \Omega)}{v} &= \frac{v_1(\Omega)}{v} \\
&\geq \left[\frac{\lambda}{\frac{\mathcal{P}_g(\Omega)}{v^{\frac{n-1}{n}}} + \frac{N}{\mu}}\right]^n \\
&\geq \tilde{f}(v,n,k,v_0,\lambda,\mu,M) = \left[\frac{\lambda}{\frac{I_M(v)}{v^{\frac{n-1}{n}}} + \frac{N}{\mu}}\right]^n.
\end{aligned}$$

Consider $\Delta v = \Delta v(\Omega) := v - v_1(\Omega)$, observe that

$$\frac{\Delta v}{v} = \frac{v - v_1(\Omega)}{v} \leq 1 - \left[\frac{\lambda}{\frac{\mathcal{P}_g(\Omega)}{v^{\frac{n-1}{n}}} + \frac{N}{\mu}}\right]^n \leq 1 - \left[\frac{\lambda}{\frac{I_M(v)}{v^{\frac{n-1}{n}}} + \frac{N}{\mu}}\right]^n. \quad (158)$$

Observe that we can put inside Ω a geodesic ball

$$B_1(\Omega) := B_M(p^*_\Omega, inrad(\Omega)) \subset \Omega.$$

We now show that $B_0(\Omega) := B_M(p_\Omega, \mu v^{1/n})$, cannot be disjoint from $B_1(\Omega)$. We prove this last assertion by contradiction. Indeed if it was the case we would have $B_1(\Omega) \subset \Omega \setminus B_0(\Omega)$, this would implies that $V(B_1(\Omega)) \leq V(\Omega \setminus B_0(\Omega)) = \Delta v$, and in turn by estimative (158)

$$c_1 v_0 v^{\frac{1}{n}} \leq v(B_1(\Omega)) \leq v f_M(v, \mu) \leq v f(v, n, k, v_0, \lambda, \mu),$$

which manifestly contradicts (153). Hence we necessarily have $B_1(\Omega) \cap B_0(\Omega) \neq \emptyset$. Thanks to our choice (151) the ball $B_1(\Omega) \subseteq B_M(p^*_\Omega, \mu v^{\frac{1}{n}})$, since $\mu v^{\frac{1}{n}} > c_7(n,k) v^{\frac{1}{n}} \geq inrad(\Omega)$. Moreover by our choice (150) the $V(B_M(p^*_\Omega, \mu v^{\frac{1}{n}})) > v$, hence there exists a radius $r^*_v < \mu v^{\frac{1}{n}}$ such that the ball $B := B_M(p^*_\Omega, r^*_v) \subset B_2(\Omega) := B_M(p_\Omega, 3\mu v^{\frac{1}{n}})$ have $V(B) = v$. Notice that B is just contained in $B_2(\Omega)$ and cannot be chosen as a proper subset of Ω. This guarantees that $V(B \setminus \Omega) > 0$ and furthermore that

$$V(B \setminus \Omega) = V(B) - V(B \cap \Omega) \geq v - V(B_2 \cap \Omega) \geq \Delta v, \quad (159)$$

because $V(B \setminus \Omega) = v - V(B_M(p^*_\Omega, r^*_v) \cap \Omega)$ but $V(B_M(p^*_\Omega, r^*) \cap \Omega) \leq v_1(\Omega)$ and (159) follows readily. Observe that by our choice (155) we have $B_M(p_\Omega, r_v) \subset B_M(p_\Omega, \frac{1}{4} inj_M)$. The following picture illustrates well our construction

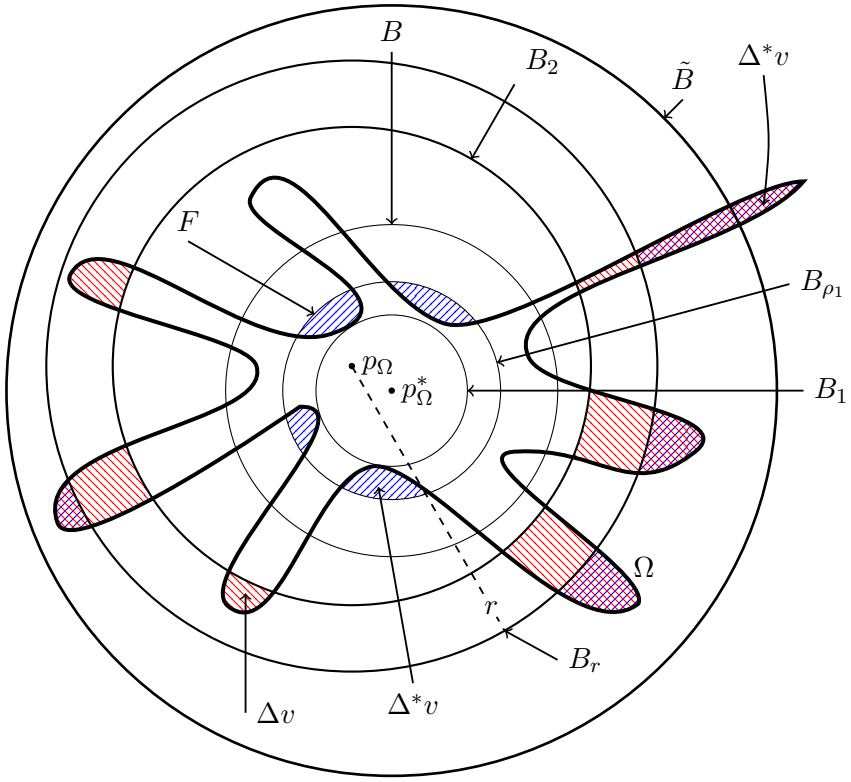


Figure 1: Construction of the competitor $F := (B_3 \cup \Omega) \setminus B_r$ used in the proof of Lemma 4.9. Here $\tilde{B} := B_M(p^*_\Omega, inj_M)$, $B_2 := B_M(p_\Omega, 3\mu v^{\frac{1}{n}})$, $B_r := B_M(p_\Omega, r)$.

Assume the following notations

$$\tilde{d}_\Omega := \sup_{x\in\Omega}\{d(x, p_\Omega)\}, \quad d_\Omega = \tilde{d}_\Omega - r_v, \quad d_v := \sup_{\Omega\in\tilde{\tau}, V(\Omega)=v}\{d_\Omega\}.$$

For any $r > 0$ let us define $V_\Omega(r) := V(\Omega \cap (M \setminus \bar{B}_r)) = V(U_r)$ where $B_r := \{y \in M : d_M(p_\Omega, y) < r\}$, p_Ω is given by Theorem 4.1, and $U_r = \Omega \cap (M \setminus \bar{B}_r)$. The function $V_\Omega(r)$ is monotone decreasing and $V_\Omega(r) \searrow 0$ as $r \to \infty$. Denote by $A_\Omega(r) := A(\partial\Omega \cap (M \setminus \bar{B}_r))$. Coarea formula gives immediately

$$V(\Omega \cap (M \setminus \bar{B}_r)) = \int_r^\infty A(\Omega \cap \partial B_r)dr,$$

then

$$V'_\Omega(r) = -A(\Omega \cap \partial B_r) = -A(\Omega \cap \partial(M \setminus B_r)).$$

Consider any $r \geq 3\mu v^{\frac{1}{n}}$ and put all the volume $\Delta^* v := V_\Omega(r)$ inside B, by choosing a concentric ball B_3 with $B_1 \subset B_3 \subset B \subset B_2$ of radius

$$Cv^{\frac{1}{n}} \leq \rho_1 = \rho_1(v, r) \leq \mu v^{\frac{1}{n}},$$

such that $V(B_3 \setminus \Omega) = \Delta^* v$, then

$$F := (B_3 \cup \Omega) \setminus B_M(p_\Omega, r) = (B_M(p^*_\Omega, \rho_1(v, r)) \cup \Omega) \setminus B_M(p_\Omega, r),$$

satisfy $V(F) = V(\Omega)$. Now, we make an application of Lemma 4.8 with $E = \Omega$ inside the ball B. From this and the fact that Ω is an isoperimetric region follows that $\mathcal{P}(\Omega) \leq \mathcal{P}(F)$ and thus for almost all $r \geq 3\mu v^{\frac{1}{n}}$ it holds

$$l_1(\Omega)(r) + A_\Omega(r) \leq l_1(\Omega)(r) - V'_\Omega(r) + (1 + (n-1)c_k(\rho_1))\frac{1}{\rho_1} V_\Omega(r), \quad (160)$$

where $l_1(\Omega)(r) := \mathcal{P}(\Omega, B_M(p_\Omega, r))$. This easily leads to

$$A_\Omega(r) \leq -V'_\Omega(r) + KV_\Omega(r), \quad (161)$$

where $K = K(n, k, v_0, v, \mu) = \tilde{c}_2(n, k)\frac{1}{v^{\frac{1}{n}}}$. Independently, by the Euclidean type isoperimetric inequality for small volumes of Lemma 3.2 of [Heb99] we have that for small volumes there exists a positive constant $\bar{v}_1 = \bar{v}_1(n, k, v_0) > 0$, such that if $v \leq \bar{v}_1$, then for every $r > 0$ it holds

$$C_{Heb} V_\Omega(r)^{(n-1)/n} \leq A(\partial U_r),$$

where $C_{Heb} = C_{Heb}(n, k, v_0) > 0$ is given by Lemma 3.2 of [Heb99] too. Thus for almost every $r > 0$ we have the following

$$\begin{aligned} -V'_\Omega(r) + A_\Omega(r) =& A(\Omega \cap \partial(M \setminus B_r)) + A(\partial\Omega \cap (M \setminus \bar{B}_r)) \\ \geq& A(\partial(\Omega \cap (M \setminus B_r))) \\ =& A(\partial U) \geq C_{Heb} V_\Omega(r)^{(n-1)/n}. \end{aligned} \quad (162)$$

Adding the two inequalities (161) and (162) we get that

$$2V'_\Omega(r) \leq KV_\Omega(r) - C_{Heb} V_\Omega(r)^{(n-1)/n}.$$

Using the fact that $n(V_\Omega^{1/n})' = V_\Omega^{\frac{1}{n}-1} V'_\Omega$ we can write the preceding inequality as

$$(V_\Omega^{1/n})'(r) \leq \frac{\tilde{c}_2}{2n} \left(\frac{V_\Omega(r)}{v}\right)^{1/n} - \frac{C_{Heb}}{2n},$$

for every $v \leq v^*$ and Ω such that $V(\Omega) = v$, where $C_{Heb} = C_{Heb}(n, k, v_0) > 0$ is the constant appearing in the isoperimetric inequality for small volumes of Lemma 3.2 of [Heb99] reported here as Lemma 4.1. Since $r \geq 3\mu v^{\frac{1}{n}}$, one have

$$(V_\Omega^{1/n})' \leq \frac{\tilde{c}_2}{2n} \left(2f^*(n, k, v_0, \lambda, \mu)\right)^{1/n} - \frac{C_{Heb}}{2n}.$$

By Theorem 4.1 and (152) we argue that

$$(V_\Omega^{1/n}(r))' \leq -C' = -\frac{C_{Heb}}{4n}. \tag{163}$$

It is worth to recall here that by Theorem 3 of [Nar14a], in weak bounded geometry $diam(\Omega) < +\infty$, because Ω is an isoperimetric region, and hence $d_\Omega := essup_{x\in\Omega} d_M(p_\Omega, x) = ||d_M(p_\Omega, \cdot)||_{L^\infty(\Omega)} < +\infty$. Furthermore we have the elementary relation $diam(\Omega) \leq 2d_\Omega$. Now, if we assume $r_v := 3\mu v^{\frac{1}{n}} < d_\Omega$, we can integrate (163) over the interval $[r_v, d_\Omega]$, and noting that $V_\Omega(r_v) \leq V(\Omega) = v$, $V_\Omega(d_\Omega) = 0$, we get

$$d_\Omega \leq \frac{1}{C'} V_\Omega(r_v)^{1/n} + r_v \leq \frac{v^{\frac{1}{n}}}{C'} + r_v = \left(\frac{1}{C'} + 3\mu\right) v^{\frac{1}{n}}.$$

From this last equation we easily a constant μ^* such that for every $v \leq v^*$ results

$$diam_g(\Omega) \leq \mu^*(n, k, v_0, \lambda) v^{\frac{1}{n}},$$

which clearly proves the lemma. □

5 Isoperimetric comparison in strong bounded geometry

Now we are in position to prove Theorem 2.

Proof of Theorem 2. If $|Sec_M| \leq K$ and $inj_M > 0$ then the assumptions of Theorem 76 of [Pet06] holds which implies that also the assumptions of Theorem 72 of [Pet06] are satisfied with $m = 1$, see also Theorem 4.4 of [Pet87]. The problem here is that the limit metric space have an atlas of harmonic coordinates of class $C^{3,\alpha}$ with just a $C^{1,\alpha}$ limit metric. Unfortunately, to apply Theorem 1 to a limit manifold we need to have in the limit a smooth Riemannian manifold with a smooth Riemannian metric, for this reason we make a stronger assumption on (M^n, g) requiring that M have strong bounded geometry smooth at infinity. A fortiori

M have also C^0-locally bounded geometry. This means by Theorem 1 of [Nar14a] or Theorem 1 of [MN15] that for every $v \in]0, V(M)[$ there exists a generalized isoperimetric region $\tilde{\Omega}_v$ contained in some smooth limit manifold (M_∞, g_∞) (that could even coincide with (M, g)). Now if we look at the limit manifold (M, g_∞), by (iv) in Theorem 4.4 of [Pet87] we learn that $inj_{(M_\infty, g_\infty)} \geq inj_{(M,g)} > 0$. Moreover, Theorem 10.7.1 of [BBI01] permits to conclude that (M_∞, g_∞) have sectional curvature bounded below by Λ_1. On the other hand, the property of being a *metric space of curvature* $\leq K$ (see Definition 1.2 at page 159 of [BH99], i.e., being locally a $Cat(K)$ space) pass to the limit in the pointed Gromov-Hausdorff convergence because distances pass to the Gromov-Hausdorff limit. This fact combined with Theorem $1A.6$ at page 173 of [BH99] implies that for a smooth Riemannian manifold to have sectional curvature bounded above by K is equivalent to satisfy the condition of having curvature bounded above in the sense of Alexandrov that is in the sense of Definition 1.2 of page 159 of [BH99]. From this easily follows that the sectional curvature of (M_∞, g_∞) (that exists because g_∞ is assumed at least C^2 or more regular by the assumption of strong bounded geometry smooth at infinity) is bounded from above by the same constant than the sectional curvature of M. Hence (M_∞, g_∞) have strong bounded geometry, in particular have also mild bounded geometry which gives the validity of Theorem 1 and Lemma 4.9 in (M_∞, g_∞), with constants $d(M_\infty, g_\infty)$ and $v^*(M_\infty, g_\infty)$ that in principle depend on (M_∞, g_∞). On the other hand by (3) and the explicit estimates that Lemma 4.9 on v^* we argue that $d(M_\infty, g_\infty) \geq d(M, g) > 0$ and $v^*(M_\infty, g_\infty) \geq v^*(M, g) > 0$. At this stage an application of Lemma 4.9 gives that for $v \leq v^*$ any generalized isoperimetric region $\tilde{\Omega}_v \subseteq M_\infty$ with $V_{g_\infty}(\tilde{\Omega}_v) = v$ have $diam_{g_\infty}(\tilde{\Omega}_v) \leq \mu^* v^{\frac{1}{n}}$. Thus for small values of $v \leq \tilde{v}_0 := \min\{v^*, \left(\frac{d}{\mu^*}\right)^n\}$ we have $\mathrm{diam}_{g_\infty}(\tilde{\Omega}_v) \leq d$, where d is given by Theorem 1. Finally for every finite perimeter set $\Omega \subset M$ such that $V_g(\Omega) = v$, we conclude that

$$\mathcal{P}_g(\Omega) \geq I_{M,g}(v) = I_{M_\infty, g_\infty}(v) = \mathcal{P}_{g_\infty}(\tilde{\Omega}_v) > \mathcal{P}_{g_0}(B) = I_{\mathbb{M}^n_{k_0}}(v), \quad (164)$$

where the first inequality comes from the definition of I_M, the first equality comes from Theorem 1 of [Nar14a] where Ω_v is a generalized isoperimetric region of $V_{g_\infty}(\Omega_v) = v$, the second equality from the definition of Ω_v as a generalized isoperimetric region of volume v, the second inequality is an application of Theorem 1 to (M_∞, g_∞), and the last is simply the fact that the isoperimetric regions in space forms are the geodesic balls. This finish the proof of Theorem 2. □

We have now all the tools needed to prove Theorem 5.

Proof of Theorem 5. Analogously to the second proof of Proposition 2 we deform smoothly the metric g along the Ricci flow $\tilde{g}_t$ with initial data $\tilde{g}_0 = g$. In first we suppose that k_0 is also a strict upper bound for the sectional curvature of (M, g), i.e., $Sec_g < k_0$. Then for any k_0 such that $n(n-1)k_0 > S_g$ we apply Theorem 2 to some manifold $(M, \tilde{g}_t)$ since these manifolds satisfy the required hypothesis of regularity and bounded geometry at infinity. Moreover, thanks to the estimates on the supremum of the sectional curvatures of $(M, \tilde{g}_t)$ given by the Proposition at page 260 of [Kap05] reported here in Theorem 3.2, there exists a sufficiently small t such that $S_{\tilde{g}_t} < n(n-1)k_0$. Hence applying Theorem 2 to $(M, \tilde{g}_t)$ we have that there exists a $v^* = v^*(n, k, k_0, S_g) > 0$ such that $\tilde{v}_0(M, \tilde{g}_t) \geq v^*$ for every $t \in [0, T]$ such that for every Ω with $V_{\tilde{g}_t}(\Omega) \leq v^*(n, k, v_0, \tilde{g}_t)$ we have

$$\mathcal{P}_{\tilde{g}_t}(\Omega) > \mathcal{P}_{g_{k_0}}(B_{k_0}). \tag{165}$$

Consider now a finite perimeter set Ω such that $V_g(\Omega) < v^*$. By C^0 convergence of $\tilde{g}_t$ to $\tilde{g}_0 = g$ as stated in (104), we can find t small enough such that $V_{\tilde{g}_t}(\Omega) < v^*$ and thus (165) hold for all sufficiently small t. Again by C^0 convergence of $\tilde{g}_t$ to $\tilde{g}_0 = g$, taking the limit when $t \to 0^+$ in (165) we obtain

$$\mathcal{P}_g(\Omega) = \mathcal{P}_{\tilde{g}_0}(\Omega) = \lim_{t \to 0^+} \mathcal{P}_{\tilde{g}_t}(\Omega) \geq \mathcal{P}_{g_{k_0}}(B_{k_0}). \tag{166}$$

To settle the general case we need just to observe that it is not too hard to obtain (from the proof of Proposition at page 260 of [Kap05], i.e., Theorem 3.2) the following estimates on the bounds of the scalar curvature is obtained, i.e.,

$$\inf_{x \in M} \{Sc_{\tilde{g}_t}(x)\} - C(n, T)t \leq Sc_{\tilde{g}_t}(x) \leq \sup_{x \in M} \{Sc_{\tilde{g}_t}(x)\} + C(n, T)t. \tag{167}$$

With (167) in mind the same argument used in the special case is valid again in the general case. So (166) is proved in the general case. To finish the proof of the theorem we remark that using just (166) the asymptotic expansion of the isoperimetric profile given in Corollary 3 holds and then reasoning by reduction to the absurd we conclude that there exists $\tilde{v}_1 = \tilde{v}_1(n, k, k_0, inj_M, S_g, I_M) \leq \tilde{v}_0$ such that for every volume $v \leq \tilde{v}_1$ the equality case in (166) cannot happen. Unfortunately we do not still have an effective estimates for $\tilde{v}_1$. From this the theorem follows. □

6 Asymptotic expansion of the isoperimetric profile in strong bounded geometry

We prove in this last section the asymptotic expansion in Puiseux's series up to the second nontrivial term stated in Corollary 3.

Proof of Corollary 3. We use Theorem 2 to prove the first of the following inequalities then we compare with the area of a geodesic ball centered at x_0 and of enclosed volume v, proving the second inequality of

$$\begin{aligned}
P_{g\,\frac{S_g}{n(n-1)}}\left(B_{g\,\frac{S_g}{n(n-1)},v}\right) &= c_n v^{\frac{(n-1)}{n}}\left(1-\gamma_n S_g v^{\frac{2}{n}}\right)+O_{\frac{S_g}{n(n-1)}}\left(v^{\frac{4}{n}}\right)\\
&< P_{g_{k_0}}\left(B_{g_{k_0},v}\right)\\
&= c_n v^{\frac{(n-1)}{n}}\left(1-\gamma_n n(n-1)k_0 v^{\frac{2}{n}}\right)+O_{k_0}\left(v^{\frac{4}{n}}\right)\\
&\leq I_M(v) \qquad (168)\\
&\leq c_n v^{\frac{(n-1)}{n}}\left(1-\gamma_n S c_g(x_0) v^{\frac{2}{n}}\right)+O_{x_0}\left(v^{\frac{4}{n}}\right) \qquad (169)\\
&= P_g(B_g(x_0,v)),
\end{aligned}$$

for every $x_0 \in M$, $k_0 > S_g$, $v \leq \tilde{v}_0$, with $B_{g_{k_0},v}$ a ball in $\mathbb{M}^n_{k_0}$ such that $V_{g_{k_0}}(B_{k_0,v}) = v$ and $B_g(x_0,v)$ a ball of (M^n,g) having $V_g\left(B_g(x_0,v)\right) = v$. Taking the infimum with respect to x_0 and using (168) and (169) we get for a fixed $\tilde{x}_0 \in M$, that

$$O_{\tilde{x}_0}\left(v^{\frac{4}{n}}\right) \geq \eta_1(v) := \inf_{x_0\in M} O_{x_0}\left(v^{\frac{4}{n}}\right) \geq O_{\frac{S_g}{n(n-1)}}\left(v^{\frac{4}{n}}\right).$$

Hence

$$\begin{aligned}
c_n v^{\frac{(n-1)}{n}}\left(1-\gamma_n S_g v^{\frac{2}{n}}\right)+\eta_2(v) &\leq I_M(v) \qquad (170)\\
&\leq c_n v^{\frac{(n-1)}{n}}\left(1-\gamma_n S_g v^{\frac{2}{n}}\right)+\eta_1(v), \qquad (171)
\end{aligned}$$

with $\eta_1(v) = O\left(v^{\frac{4}{n}}\right)$, $\eta_2(v) = O\left(v^{\frac{4}{n}}\right)$. We need that the metric g on the manifold M is at least C^3 because of the Taylor expansion of the metric in normal coordinates. This ensures that $O_{\tilde{x}_0}\left(v^{\frac{4}{n}}\right)$ does not blows up. From this the corollary, indeed follows promptly. □

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Stefano Nardulli
Departamento de Matemática
Instituto de Matemática
UFRJ-Universidade Federal do Rio de Janeiro, Brazil
email: nardulli@im.ufrj.br

Luis Eduardo Osorio Acevedo
Ph.D. student
Instituto de Matemática
UFRJ-Universidade Federal do Rio de Janeiro, Brazil
email: mat.eduardo@gmail.com