# Banach-like distances and metric spaces of compact sets.

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**Abstract**

In the first part we study general properties of the metrics obtained by isometrically identifying a generic metric space with a subset of a Banach space; we obtain a rigidity result. We then discuss the Hausdorff distance, proposing some less–known but important results: a closed–form formula for geodesics; generically two compact sets are connected by a continuum of geodesics.

In the second part we present and study a family of distances on the space of compact subsets of $\mathbb{R}^N$ (that we call "shapes"). These distances are "geometric", that is, they are independent of rotation and translation; and the resulting metric spaces enjoy many interesting properties, as, for example, the existence of geodesics. We view our metric space of shapes as a subset of Banach (or Hilbert) spaces: so we can define a "tangent manifold" to shapes, and (in a very weak form) talk of a "Riemannian Geometry" of shapes. Some of the metrics that we propose are topologically equivalent to the Hausdorff distance.

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# 1 Introduction

A wide interest for the study of *Shape Spaces* arose in recent years, in particular inside the Computer Vision community.

There are two different (but interconnected) fields of applications for a good Shape Space in Computer Vision:

**Shape Optimization** where we want to find the shape that best satisfies a design goal; a topic of interest in Engineering at large;

**Shape Analysis** where we study a family of Shapes for purposes of statistics, (automatic) cataloging, probabilistic modeling, among others; possibly also to create an a-priori model for a better Shape Optimization.

To achieve the above, some structure is clearly needed on the Shape Space so that our goals can be studied and the problem can be solved.

## 1.1 Shape spaces

A common way to model shapes is by **representation/embedding**:

- we **represent** the shape $A$ by a function $u_A$
- and then we **embed** this representation in a space $E$ so that we can operate on the shapes $A$ by operating on the representations $u_A$.

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Most often, this representation/embedding scheme does not directly provide a Shape Space satisfying all desired properties. In particular, in many cases it happens that the representation is "redundant", that is, the same shape has many different possible representations. An appropriate **quotient** is then introduced.

There are many examples of Shape Spaces in the literature that are studied by means of the *representation/embedding/quotient* scheme. We review two such examples.

1. The space of embedded curves. When studying embedded curves, usually, for the sake of mathematical analysis, the curves are modeled as smooth immersed parametric curves; a quotient *w.r.t.* the group of possible reparametrizations of the curve $c$ (that coincides with the group of smooth diffeomorphisms $\mathrm{Diff}(S^1)$) is applied afterward to all the mathematical structures that are defined (such as the manifold of curves, the Riemannian metric, the induced distance, etc.). The resulting Riemannian spaces of embedded curves have been studied by Michor–Mumford *et al* [18, 19, 31] and Yezzi–Mennucci [30, 29]; more recently by Mennucci *et al* [17] and Sundaramoorthi *et al* [24, 21, 25, 26, 27, 28, 22, 23].

2. The family $\mathcal{M}$ of all non-empty compact subsets of $\mathbb{R}^N$. This is the shape space that we will study in the main part of this paper.

   A standard representation is obtained by associating a closed subset $A$ to the **distance function**

   $$u_A(x) \stackrel{\text{def}}{=} \inf_{y \in A} |x - y| \tag{1}$$

   or the **signed distance function**

   $$b_A(x) \stackrel{\text{def}}{=} u_A(x) - u_{\mathbb{R}^N \setminus A}(x) \quad . \tag{2}$$

   See Sec. 4 for a list of properties of $u_A$.

   We may then define a *topology of shapes* by deciding that $A_n \to A$ when $u_{A_n} \to u_A$ uniformly on compact sets. This convergence coincides with the Kuratowski topology of closed sets (see Sec. 5.3).

   We may also operate *"linearly"* on shapes by operating on $u_A$ or $b_A$. So we may define *shape averages* and *shape principal component analysis*; see [12] and (4) here.

   When this Shape Space is used for shape analysis, a *registration* of the shapes to a common pose is often performed; or a quotient is enacted, as explained in Sec. 2.1.1.

## 1.2 Goals

To a certain degree, our theory should be independent of rotation and translation; that is, whatever we do with shapes should not depend on "where in the space" we do it.

In the rest of the paper we will denote by $\mathcal{M}$ the family of the nonempty compact sets in $\mathbb{R}^N$ and we will build many examples of metrics $d$ on $\mathcal{M}$. We will always require these metrics to be **Euclidean invariant**: if $R$ is a Euclidean transformation of the space (a rigid transformation), then

$$d(R\Omega_1, R\Omega_2) = d(\Omega_1, \Omega_2) \ . \tag{3}$$

What other properties and operations may be interesting for applications?

### 1.2.1 Means and averages

As mentioned before, a goal of Shape Analysis is to define *shape metrics*, *shape averages*, *shape principal component analysis*, *shape probabilities*...

For example, if we represent shapes $A_j, j = 1 \dots n$ by their *signed distance function* $b_{A_j}$, then we may define *Signed Distance Level Set Averaging*

$$\bar{A} = \Big\{ x \,|\, f(x) \leq 0 \Big\}, \text{ where } f(x) = \frac{1}{N} \sum_{n=1}^{N} b_{A_n}(x) \tag{4}$$

(Note that in general a linear combination of (signed) distance functions will not be a (signed) distance function). A benefit of this definition is that it is easily computable; a defect is that, if the shapes are far away, then $\bar{A}$ will be empty. Another defect is that this definition is quite *ad hoc*: it is not coupled with any other structure that we may wish to add to the Shape Space, such as a metric $d$. We may then look at the problem from a different point of view.

Considering a generic metric space $(M, d)$, define the *Distance Based Averaging* [1] of any given collection $a_1 \dots a_n \in M$, as a minimum point $\bar{a}$ of the sum of its squared distances:

$$\bar{a} = \arg\min_{a} \sum_{j=1}^{n} d(a, a_j)^2 \tag{5}$$

Supposing now that the Shape Space $\mathcal{M}$ is given a metric $d$, we can use the abstract definition above to define *shape averages*; this definition has many advantages.

- It comes from a minimality criterion, so it is *"optimal"* in a certain sense (contrary to the definition in eqn. (4)).
- If the distance is invariant *w.r.t.* a group action, then the *shape average* is invariant as well (see Sec. 2.1.1). For example, in the case of *geometric curves*, where the distance is independent of parameterization, then the *shape average* will be independent of the parameterization of $a_1 \dots a_n$.
- Suppose that $\mathcal{M}$ is a smooth Riemannian manifold and that $d$ is the distance derived from the metric; then, when $a_1 \dots a_n$ are near enough, $\bar{a}$ exists and is unique [11].
- It coincides with the arithmetic mean in Euclidean spaces; more generally, when $\mathcal{M}$ is a smooth submanifold of a Hilbert space and $a_1 \dots a_n$ are near enough, then $\bar{a}$ is an approximation of the arithmetic mean.

### 1.2.2 Averages, midpoints and geodesics

Let $(M, d)$ be a metric space. A *geodesic* is a continuous path connecting $x$ to $y$ that has minimum length in the class of all such paths. The metric space $(M, d)$ is *intrinsic* if the distance $d(x, y)$ between $x, y \in M$ is equal to the infimum of the length of all continuous paths connecting $x$ to $y$. See Sec. 2.1 for details.

**Definition 1.1** (Midpoint)**.** Let $x, y \in M$; a point $z \in M$ such that

$$d(z, y) = d(z, x) = \frac{1}{2} d(x, y)$$

is called a *midpoint*.

It is easily verified that, if $(M, d)$ is *intrinsic* and $x, y$ are connected by a geodesic, a point halfway through the geodesic is a midpoint; see Lemma 2.4.8 in Sec. 2.4.3 in [5]. Vice versa, suppose $(M, d)$ is complete and that for every $x, y \in M$ there exists a midpoint $z$, then $d$ is intrinsic, and every two points in $M$ may be joined by a geodesic; see Thm. 2.4.16 in Sec. 2.4.4 in [5].

Consider now a Shape Space that is a complete finite dimensional Riemannian metric; let $d$ be the distance between shapes; then the average shape $A$ of two shapes $A_1, A_2$, (as defined in eqn. (5) above) is also a *midpoint*.

The above shows that averages, midpoints and geodesics are deeply linked.
For this reason, we will end up studying whether the Shape Space admits geodesics.

### 1.2.3 Motions and tangent spaces

Many operations performed in Shape Optimization may be related to these concepts and operations:

- given the motion of a shape, we would like to define its derivative, that is the *infinitesimal motion*,

[1] Also known as *Karcher mean* due to the seminal work [11], but it is also sometimes attributed to Fréchet, in 1948

- given a vector field of *infinitesimal motions*, we would like to be able to flow shapes according to the field,
- the family of all such *infinitesimal motions* should define a *tangent space* to the Shape Space.

One easy way to define all the above is again by representation/embedding: if we embed the Shape Space in a vector space $E$, then we can define the *infinitesimal motions* as vectors in $E$. At the same time, if $E$ is a Banach space with norm $\|\cdot\|$, we can define a *distance of shapes* simply by $d(A,B)\stackrel{\text{def}}{=}\|u_A - u_B\|$ (so that the embedding $A \mapsto u_A$ is isometric). For this reason, in the first part of the paper we will study general properties of isometric embeddings of metric spaces into Banach spaces.

## 1.3 The proposed framework

In this paper we will study the family $\mathcal{M}$ of all non empty compact subsets of $\mathbb{R}^N$.

Having fixed a decreasing smooth function $\varphi : [0,\infty) \to (0,\infty)$, we will define in eqn. (31) the $L^p$–like distance of compact sets by

$$d_{p,\varphi}(A,B)\stackrel{\text{def}}{=}\|\varphi \circ u_A - \varphi \circ u_B\|_{L^p} \quad .$$

Under appropriate hypotheses on $\varphi$, we will prove that this distance satisfies the requirements listed in the previous sections, and some more, as follows.

- The metric space $(\mathcal{M}, d_{p,\varphi})$ is complete (Prop. 6.12).
- The mapping $A \mapsto \varphi \circ u_A$ associates isometrically $\mathcal{M}$ to a closed subset of $L^p(\mathbb{R}^N)$.
- $d_{p,\varphi}$ is Euclidean invariant.
- $d_{p,\varphi}$ induces a well-defined distance on the Shape Space of compact sets up to Euclidean transformation (Prop. 6.14).
- Compact sets can be connected by minimizing geodesics; the *Geodesic Distance Based Averaging* of shapes exists (Thm. 6.20).
- The metric spaces $(\mathcal{M}, d^g_{p,\varphi})$, $(\mathcal{M}, d_{p,\varphi})$ and $(\mathcal{M}, d_H)$ have the same topology. Here $d^g_{p,\varphi}$ is the distance induced by the length of paths in $(\mathcal{M}, d_{p,\varphi})$ and $d_H$ is the well-known Hausdorff distance of compact sets (Thm. 6.11 and Thm. 6.30).
- Certain motions can be infinitesimally represented by vectors in $L^p(\mathbb{R}^N)$. In particular, to any Lipschitz path $\gamma(t) : \mathbb{R} \to \mathcal{M}$ of compact sets in $(\mathcal{M}, d_{p,\varphi})$ we can associate the path $f : \mathbb{R} \to L^p(\mathbb{R}^N)$ by $f(t,x) = \varphi(u_{\gamma(t)}(x))$; and then we can represent (for almost all $t$) the motion of $\gamma(t)$ by the weak partial derivative $\partial_t f$ (see Sec. 6.4).
- In the case $p = 2, N = 2$, for compact sets with smooth boundary, the metric can be explained as a Riemannian metric of deformations of the boundary (see Sec. 6.6).

The last two properties are what distinguishes this framework from the Hausdorff distance of compact sets.

## 1.4 Plan of the paper

The plan of the paper is as follows.

In Sec. 2 we foremost review the theory of metric spaces, provide definitions of the length of a path, of the metric derivative, of the *induced distance* $d^g$, of geodesics. We define the action of a group on a metric space and the properties of quotient distances. We propose some properties of metric spaces isometrically embedded in Banach spaces.

In Sec. 3 we list definitions and notations.

In Sec. 4 we review properties of the distance function $u_A$, and the *fattening* of sets.

Considering the space $\mathcal{M}$ of nonempty compact subsets of $\mathbb{R}^N$, in Sec. 5 we define on $\mathcal{M}$ the renowned *Hausdorff distance* $d_H$, review some of its properties; we also provide some original results, such as a closed form formula for geodesics and a generic condition for non-uniqueness of geodesics.

In Sec. 6 we discuss the main subject of this paper. We define in eqn. (31) the $L^p$–like distance of compact sets. We prove many properties regarding the metric space $(\mathcal{M}, d_{p,\varphi})$: we prove in Prop. 6.11 that this metric space has the same topology as $(\mathcal{M}, d_H)$, in Prop. 6.12 that it is complete; adding some more hypotheses on $\varphi$, we prove in Thm. 6.20 that any two compact sets may be joined by a geodesic in $(\mathcal{M}, d_{p,\varphi})$. In §6.4 we show a variational description of geodesics and in §6.6 (when $p = 2, N = 2$) we use it to describe $(\mathcal{M}, d_{p,\varphi})$ as a weak kind of Riemannian manifold, that has an explicit description for sets with smooth boundary. In §6.7, assuming that $\varphi$ is convex and $\varphi(|x|) \in W^{1,p}(\mathbb{R}^N)$, we show that the metric space $(\mathcal{M}, d^g_{p,\varphi})$ has the same topology as $(\mathcal{M}, d_{p,\varphi})$ and $(\mathcal{M}, d_H)$; where $d^g_{p,\varphi}$ is the distance induced by the length of paths in $(\mathcal{M}, d_{p,\varphi})$. In §6.8 we present a simple numerical method for computing geodesics and two results.

We conclude in Sec. 7 showing some possible further expansions of the presented framework.

# 2 Metric spaces and embeddings in Banach spaces

## 2.1 Metric spaces

We recall some basilar definitions and results in the abstract theory of metric spaces.

Suppose that $(M, d)$ is a metric space. We will denote with

$$\mathbb{B}(x, \rho) \stackrel{\text{def}}{=} \{x \mid d(x, y) < \rho\}, \tag{6}$$

$$\mathbb{D}(x, \rho) \stackrel{\text{def}}{=} \{x \mid d(x, y) \leq \rho\} \tag{7}$$

the open ball and the closed disc in this metric space; note that in general $\mathbb{D}$ contains the closure of $\mathbb{B}$, but it may be strictly larger.

**Definition 2.1.** *We induce from $d$ the length $\operatorname{Len}^d \gamma$ of a continuous path*

$$\gamma : [\alpha, \beta] \to M$$

*by using the* ***total variation***

$$\operatorname{Len}^d \gamma \stackrel{def}{=} \sup_T \sum_{i=1}^{n} d\big(\gamma(t_{i-1}), \gamma(t_i)\big), \tag{8}$$

*where the supremum is computed over all finite subsets $T = \{t_0, \cdots, t_n\}$ of $[\alpha, \beta]$ and $t_0 \leq \cdots \leq t_n$. When $\operatorname{Len}^d \gamma < \infty$ we will say that $\gamma$ is* ***rectifiable****.*

**Definition 2.2.** *We define the* ***metric derivative*** *[3, 2]*

$$|\dot\gamma|(t) \stackrel{def}{=} \lim_{s \to 0} \frac{d\big(\gamma(t+s), \gamma(t)\big)}{s} \ . \tag{9}$$

(The above notation does not imply that there is an actual object "$\dot\gamma$" and that the metric derivative is the "norm" of this object — the symbol $|\dot\gamma|$ is atomic).

The metric derivative enjoys the following properties.

**Lemma 2.3.**

- *If $\gamma$ is absolutely continuous, then the above limit (9) exists for almost all $t$.*
- *For any absolutely continuous $\gamma$, let*

$$\operatorname{len}^d \gamma \stackrel{def}{=} \int_\alpha^\beta |\dot\gamma|(t)\, \mathrm{d}t \ ; \tag{10}$$

  *then*

$$\operatorname{Len}^d \gamma = \operatorname{len}^d \gamma \ .$$

- *If $\operatorname{Len}^d \gamma < \infty$ then there exists a continuous monotonic $\theta : [0, \operatorname{Len}^d(\gamma)] \to [\alpha, \beta]$ such that for $c = \gamma \circ \theta$ we have $|\dot c| = 1$ for almost all $\theta$. Such a path $c$ is called the* ***reparameterization to arc parameter*** *of $\gamma$.*

See Thm. 1.1.2, Lemma 1.1.4 in [2], and Thm. 4.1.1 in [3].

**Definition 2.4.** *We define the* ***induced distance*** $d^g$ *by*

$$d^g(x,y) \stackrel{def}{=} \inf_{\gamma} \mathrm{Len}^d\, \gamma, \tag{11}$$

*where the infimum is taken in the class of all continuous paths $\gamma$ connecting $x$ to $y$. If the infimum is a minimum, the path providing the minimum is called a* ***geodesic****.*

Note that it may be the case that $d^g(x,y) = \infty$ for some choices of $x, y$. Note also that $d^g \geq d$.
The topology of $(M,d)$ and $(M,d^g)$ may be quite different, as we see in this example.

*Example* 2.5. Consider

$$M = \{x \in \mathbb{R}^2 \mid 0 \leq x_1 \leq 1, x_2 = 0\} \cup \bigcup_{n \geq 1} \{x \in \mathbb{R}^2 \mid 0 \leq x_1 \leq 1, \quad x_2 = x_1/n\}$$

and $d$ the Euclidean distance (see fig. 1). Then $(M,d)$ is compact but $(M,d^g)$ is not.



Figure 1: Example 2.5

When $d = d^g$, we will say that the metric space is **path–metric**, or that $d$ is **intrinsic**.
The following results hold.

**Proposition 2.6.**
- *A path $\gamma : [a,b] \to M$ is continuous and rectifiable in $(M,d)$* iff *it is continuous and rectifiable in $(M,d^g)$.*
- *The length $\mathrm{Len}^{d^g}$ defined by $d^g$ coincides with $\mathrm{Len}^d$ on all such paths.*
- *$d^g = (d^g)^g$, that is, the space $(M,d^g)$ is always intrinsic.*

These results are found in [5] (for the first point, look at Exercises 2.1.4 and 2.1.5 in [5]).

We will use the following propositions.

**Proposition 2.7.** *If for a choice of $\rho > 0$*

$$\mathbb{D}^g(x,\rho) \stackrel{def}{=} \{x \mid d^g(x,y) \leq \rho\} \tag{12}$$

*is compact in the $(M,d)$ topology, then $x$ and any $y \in \mathbb{D}^g(x,\rho)$ may be connected by a geodesic.*

The proof is simply obtained by the direct method in the Calculus of Variations (see Thm. 9.2 in [16]).

**Proposition 2.8.** *Suppose that $a_1 \dots a_n \in M$ are given; a sufficient condition for the existence of the* Geodesic Distance Based Averaging $\bar{a}$ *of $a_1 \dots a_n$*

$$\bar{a} = \mathrm{argmin}_a\, \tau(a) \ , \ \textit{where } \tau(a) \stackrel{def}{=} \sum_{j=1}^{n} d^g(a,a_j)^2 \tag{13}$$

*is that, defining*

$$\rho^* = \min_{i=1,\dots n} \tau(a_i)$$

*we have that $\rho^* < \infty$ and that $\mathbb{D}^g(a_1, 2\sqrt{\rho^* + \varepsilon})$ is compact in the $(M,d)$ topology, for $\varepsilon > 0$ small.*

The proof is in Sec. A.2.

A similar proposition can be stated for $d$.

**Proposition 2.9.** *Suppose that $a_1 \dots a_n \in M$ are given; let*

$$\rho^* = \min_i \sum_{j=1}^{n} d(a_i, a_j)^2 \tag{14}$$

*and $i^*$ the index that achieves the above minimum. Suppose that $\mathbb{D}(a_{i^*}, \sqrt{\rho^* + \varepsilon})$ is compact for $\varepsilon > 0$ small. Then there exists a point $\bar{a}$ that is the* Distance Based Averaging *of $a_1 \dots a_n$, as defined in (5).*

The proof is similar so we omit it.

An intrinsic space such that any disc $\mathbb{D}(x, \rho)$ is compact is called **finitely compact** in [6]. Such a space satisfies the hypotheses of the previous propositions. A classical example is given by finite dimensional complete Riemannian manifolds.

### 2.1.1 Distances, quotients and groups

Let $d_M(x, y)$ be a distance on a space $M$ and $G$ a group acting on $M$. We suppose that $d_M$ is **invariant *w.r.t.*** $G$, i.e.

$$d_M(gx, gy) = d_M(x, y) \quad \forall g \in G \quad .$$

(This generalizes the idea of eqn. (3)).

A distance $d_B$ may be defined on $B = M/G$ by

$$d_B([x], [y]) = \inf_{x \in [x], y \in [y]} d_M(x, y) = \inf_{g,h \in G} d_M(gx, hy)$$

that is the lowest distance between two orbits; we write $d_B(x, y)$ for simplicity. Since $d_M$ is invariant *w.r.t.* the group action, $d_B$ coincides with

$$d_B(x, y) = \inf_{g \in G} d_M(gx, y) \ . \tag{15}$$

It is easy to see that $d_B$ satisfies the triangle inequality; but it may be the case that $d_B(x, y) = 0$ even when $x \neq y$. We state a simple sufficient condition.

**Lemma 2.10.** *If the orbits are compact, then $d_B$ is a distance.*

When studying metrics $d$ on a Shape Space the quotient is particularly useful in at least two cases.

- For the purpose of Shape Analysis, shapes are usually intended "up to rotation, translation and scaling", while in Shape Optimization each shape has a distinctive position and orientation. For this reason, when we wish to distinguish between the two different ideas of "Shape Spaces", we will call a space for Shape Optimization a "*preshape space*".

  When we want to pass from a *preshape space* to a *shape space*, we will apply the quotient above by choosing $G$ to be the Euclidean group of rotations and translation (and sometimes of scaling).

- When the representation is redundant. In the example 1 of embedded curves we proposed in the introduction, we would set $G = \mathrm{Diff}(S^1)$, the family of reparametrizations of the circle.

## 2.2 Embeddings in Banach spaces

In most of what follows, we will be able to identify $M$ (using an isometry $i$) with a subset of a Banach space $E$ with norm $\|\cdot\|$. We remark that in the following an "isometry" is a map $i$ such that $d(x, y) = \|i(x) - i(y)\|$ (and this should not be confused with the concept of "isometrical immersions of Riemannian manifolds").

### 2.2.1 Radon-Nikodym property

The following result from [1] will come handy.

**Theorem 2.11.** *Suppose that $E$ is the dual of a separable Banach space $F$. Let $\gamma : [a,b] \to E$ be a Lipschitz path. By Theorem 8.1 in [1], for almost all $t$ there exists the derivative $\dot\gamma(t)$ that is defined as*

$$\dot\gamma(t) \stackrel{def}{=} w\text{-}\lim_{\tau\to 0} \frac{\gamma(t+\tau)-\gamma(t)}{\tau} \quad , \tag{16}$$

*where the limit is according to the weak-* topology; moreover*

$$\|\dot\gamma(t)\| = \lim_{\tau\to 0} \left\| \frac{\gamma(t+\tau)-\gamma(t)}{\tau} \right\| \quad , \tag{17}$$

*so $\|\dot\gamma(t)\|$ coincides with the* metric derivative (9). [2] *We can then define (following (10)) the length of $\gamma$ using the integral*

$$\operatorname{len}\gamma \stackrel{def}{=} \int_a^b \|\dot\gamma(t)\|\,\mathrm{d}t \quad . \tag{18}$$

*It follows easily (by applying duality* w.r.t. *$F$ in eqn. (16)) that*

$$\gamma(b)-\gamma(a) = \int_a^b \dot\gamma(t)\,\mathrm{d}t \quad , \tag{19}$$

*and, by Lemma 2.3,*

$$\operatorname{Len}^{d_E}\gamma = \operatorname{len}\gamma \quad , \tag{20}$$

*where $\operatorname{Len}^{d_E}\gamma$ is the total variation length (8) of paths in $E$ computed using the usual distance $d_E(x,y) = \|x-y\|$.*

Here is a simple example where the above Theorem does not apply. Consider the map $t \mapsto \mathbf{1}_{[t,t+1]}$ in $L^1(\mathbb{R})$. It is Lipschitz, but its derivative should be $t \mapsto \delta_{t+1} - \delta_t$. [3]

### 2.2.2 The Radon-Nikodym property

It is common to say that $E$ enjoys the *Radon-Nikodym property*, when the limit in (16) exists in the strong sense and for almost all $t$.

We now recall this basilar definition.

**Definition 2.12.** A Banach space $E$ is **uniformly convex** if $\forall\varepsilon>0 \exists\delta>0$,

$$\forall x,y\in E, \|x\|\le 1, \|y\|\le 1, \|x-y\|\ge\varepsilon \Longrightarrow \|(x+y)/2\| < (1-\delta) \ .$$

Examples of uniformly convex Banach spaces include $L^p(\Omega,\mathcal{A},\mu)$ for $p\in(1,\infty)$. Uniformly convex Banach spaces have many interesting properties: for example, they are reflexive (Milman–Pettis Theorem, 3.31 in [4]); moreover, if $x_n \to x$ in weak sense and $\limsup \|x_n\| \le \|x\|$ then $x_n \to x$ in the strong sense (prop. III.32 in [4]).

So we obtain a sufficient condition.

**Corollary 2.13.** *If $E$ is uniformly convex and separable, then it enjoys the* Radon-Nikodym Property *(indeed eqn. (16) and eqn. (17) imply that the limit in (16) is valid also in the strong sense).*

[2] So in this case the metric derivative is the norm of an actual vector.

[3] $\mathbf{1}_A$ is the characteristic, or indicator, function of the set $A$.

### 2.2.3 Embeddings in uniformly convex Banach spaces

If $E$ is uniformly convex then in particular the closed ball $\{x \in E : \|x\| \leq 1\}$ is strictly convex; this has an interesting implication.

**Lemma 2.14.** *Suppose the closed balls in $E$ are strictly convex. Consider $E$ as a metric space, with distance $d_E(x,y) = \|x-y\|$. The segment connecting $x, y \in E$ is the unique (up to reparameterization) geodesic.*

*Proof.* We will prove that, for $x, y$, for any geodesic $\gamma : [0,1] \to M$ connecting $x$ to $y$, if $\gamma$ is reparameterized to arc parameter then $\gamma(1/2) = (x+y)/2$; iterating this reasoning with finer subdivision we obtain that $\gamma(t) = (tx + (1-t)y)$.

With no loss of generality, up to translation and scaling, suppose $y = -x$ and $\|x\| = 1$. The segment $t \mapsto tx$ is a geodesic for $t \in [-1,1]$, by Theorem 2.11 and its length is 2. Suppose now that $\gamma : [-1,1] \to M$ is another geodesic: then the length of $\gamma$ is 2 and, up to reparameterization, $\|\dot\gamma\| = 1$ at almost all points. In particular, setting $z = \gamma(0)$, $\|z-y\| \leq 1$ and $\|x-z\| \leq 1$; but then, by the triangle inequality,

$$\|z-y\| = \|z+x\| = \|x-z\| = 1 \quad .$$

Suppose that $z \neq 0$; then $\|(z+x) - (x-z)\| > 0$; by strict convexity, though, this implies that $\|((z+x)+(x-z))/2\| = \|x\| < 1$ and this is a contradiction. □

**Theorem 2.15.** *Suppose that $(M,d)$ is a complete space and that $i : M \to E$ is an isometrical immersion in a uniformly convex Banach space $E$. If, given $x, y \in M$, $d(x,y) = d^g(x,y)$, then the segment connecting $i(x), i(y)$ is all contained in $i(M)$.*

*In particular, if $(M,d)$ is intrinsic then $i(M)$ is convex and then any two points in $M$ can be joined by a unique geodesic (unique up to reparameterization).*

*Proof.* Note that $i(M)$ is complete and then it is closed in $E$. We will prove that, for any $x, y \in i(M)$, $(x+y)/2 \in i(M)$. We can iterate this idea to further subdivide. Since $i(M)$ is closed then this proves the whole segment connecting $x, y$ is in $i(M)$. By the above lemma the segment is the unique geodesic.

We now fix $x, y \in i(M)$: there must be paths $\gamma_n : [-1,1] \to i(M)$ connecting $x$ to $y$ with length $\mathrm{Len}^d(\gamma_n) < L_n \stackrel{\text{def}}{=} \|x-y\| + 2/n$.

As in the lemma before, we suppose for simplicity that $y = -x$ and $\|x\| = 1$ (so $L_n = 2 + 2/n$); and we reparameterize so that $\|\dot\gamma_n\| le 1 + 1/n$: hence setting $z_n = \gamma_n(0)$

$$\|z_n - y\| = \|z_n + x\| \leq 1 + 1/n \quad , \quad \|x - z_n\| \leq 1 + 1/n \ .$$

and then by the triangle inequality $\|z_n + x\| \to 1$, $\|z_n - x\| \to 1$. Setting

$$w_n = (z_n + x)/\|z_n + x\| \quad , \quad v_n = (x - z_n)/\|z_n - x\|$$

we can prove that $\|(w_n + v_n)/2\| \to 1$ hence by the uniform convexity of $E$ we obtain that $w_n - v_n \to 0$ and $z_n \to 0$. Since $z_n \in i(M)$ and $i(M)$ is closed then $0 \in i(M)$. □

The above is a "rigidity theorem", in that it restricts the class of metric spaces that can be isometrically embedded in a uniformly convex Banach space $E$.

**Corollary 2.16.** *A compact finite dimensional Riemannian manifold $M$ cannot be isometrically embedded*[4] *in a uniformly convex Banach space $E$: indeed in this space $M$ there are two points that can be joined by more than one geodesic.*

When $E$ is not uniformly convex, on the other hand, strange behavior arises.

**Proposition 2.17.** *Let $L^\infty = L^\infty(\Omega, \mathcal{A}, \mu)$ and suppose $\Omega$ is not an atom of $\mu$, that is, suppose the dimension of $L^\infty$ is greater than 1. Given generic $f, g \in L^\infty$, there is an uncountable number of geodesics connecting them.*

[4] In the sense explained at the beginning of section 2.2.

*Proof.* We can assume without loss of generality that $g = 0$ and that $\|f\| = 1$. We abbreviate

$$\{|f| = 1\} = \{x \in \Omega : |f(x)| = 1\}$$

and similarly for similar expressions. Let $A = \{|f| = 1\}$. We will prove that if there is only one geodesic then $|f| = \mathbf{1}_A$. Indeed if $|f| \neq \mathbf{1}_A$ then $\mu\{|f| < 1\} > 0$. Let $0 < t < 1$ be such that $\mu\{|f| < t\} > 0$; obviously $\mu\{|f| \geq t\} > 0$ since $\|f\| = 1$; let $A' = \{|f| \geq t\}$ and $A'' = \{|f| < t\}$. Given any increasing diffeomorphism $b : [0,1] \to [0,1]$ with $b'(s) \leq 1/t$,

$$\gamma(t) \stackrel{\text{def}}{=} tf\mathbf{1}_{A'} + b(t)f\mathbf{1}_{A''}$$

is a geodesic. Indeed its derivative is

$$\gamma'(t) \stackrel{\text{def}}{=} f\mathbf{1}_{A'} + b'(t)f\mathbf{1}_{A''}$$

and $\|\gamma'(t)\| = 1$ by construction.

The family of $f$ s.t. $|f| = \lambda\mathbf{1}_A$ is closed and has empty interior. □

The idea of *isometrical embedding* is quite powerful: indeed any separable metric space may be isometrically embedded in $\ell^\infty$ (that is the dual of the separable space $\ell^1$): so the breadth of application of the Theorem 2.11 is general and is at the basis of many results in [1]. But the embedding in $\ell^\infty$ that is studied in [1] is not suited for our practical applications.

- It would not respect the geometric properties of the space (as we discussed in Sec. 1.2).
- It would be too difficult to find a satisfactory notion of *"shooting of geodesics"* using this embedding: that is, to define a way, given a point $p$ and a direction $v$, to find a (possibly unique) geodesic starting from $p$ and with first derivative $v$ in $p$.

For all above reasons, we will consider *isometrical embeddings* in this paper as well but we will (for the most interesting applications) use an explicitly chosen embedding in uniformly convex Banach spaces.

# 3 Definitions

We introduce some definitions that will be used in the rest of the paper

- We will denote by $e_1, \dots e_n$ the canonical basis of $\mathbb{R}^N$.
- We will write $s^+ = \max\{s, 0\}$, for $s \in \mathbb{R}$.
- We will write $B(x,r)$ or $B_r(x)$ for the **open ball**

  $$B_r(x) = \{y \in \mathbb{R}^N : |x - y| < r\}$$

  of center $x$ and radius $r > 0$ in $\mathbb{R}^N$; we will write $B_r$ for $B_r(0)$. Similarly $D_r(x)$ will be the **disk** (or **closed ball**)

  $$D_r(x) = \{y \in \mathbb{R}^N : |x - y| \leq r\} \tag{21}$$

  of center $x$ and radius $r > 0$ in $\mathbb{R}^N$ and $D_r = D_r(0)$.
- We will say that a family $A_{i \in I}$ of sets in $\mathbb{R}^N$ is **equibounded** if there is a $R > 0$ such that $A_i \subseteq D_R$ for all $i$.
- We denote by $\mathcal{L}^N$ the $N$ dimensional Lebesgue measure, and $\omega_N \stackrel{\text{def}}{=} \mathcal{L}^N(B_1)$; we write $\int_A f(x)\,\mathrm{d}x$ for the Lebesgue integral.

# 4 Distance function and fattening

Let $A \subseteq \mathbb{R}^N$ be a closed set. We here recall some useful properties of the *distance function* $u_A$ that was defined in eqn. (2).

- $u_A$ is the viscosity solution of the *eikonal equation*

$$|\nabla f(x)| - 1 = 0$$

in $\mathbb{R}^N \setminus A$, with boundary condition that $f = 0$ on $A$; the viscosity solution is unique in the class of continuous function $f$ that are bounded from below.

- $u_A$ is Lipschitz of constant 1, hence it is differentiable almost everywhere.
- Suppose that $u_A$ is differentiable at $x$; when $x \notin A$ we have $|\nabla u(x)| = 1$, otherwise $\nabla u(x) = 0$.
- Fix $x \in \mathbb{R}^N, x \notin A$, we will call **projection point** a point $y \in A$ of minimum distance, *i.e.* a point such that $u_A(x) = |x - y|$. There is always at least one projection point.

  The two following facts are equivalent:

  1. $u_A$ is differentiable at $x$;
  2. there is a unique *projection point* $y \in A$;

  and when both hold

$$\nabla u(x) = \frac{x - y}{|x - y|} \quad . \tag{22}$$

- $u_A$ is convex if and only if $A$ is convex.
- If $\lambda > 0$

$$u_{\lambda A}(\lambda z) = \lambda u_A(z) \quad , \tag{23}$$

where $\lambda A \stackrel{\text{def}}{=} \{\lambda z : z \in A\}$ is the rescaled set.

For all of the above see [14] (where the above properties are discussed for general Riemannian manifolds) and references therein.

For $A \subseteq \mathbb{R}^N$ a closed set and $r \geq 0$, we define the **fattened set** to be

$$A + D_r = \{x + y \mid x \in A, |y| \leq r\} = \bigcup_{x \in A} D_r(x) = \{y \mid u_A(y) \leq r\}.$$

The fattened set is closed. The fattening operation is a semi group, in the sense that $A + D_0 = A$ and for $r, s > 0$

$$(A + D_r) + D_s = A + D_{r+s} \quad ;$$

similarly the distance function satisfies

$$u_{A+D_r}(x) = (u_A(x) - r)^+ \quad . \tag{24}$$

We also present this Lemma that will be useful in the following.

**Lemma 4.1.** *Let $r > 0$. Let $F = A + D_r$ and $E = \{x : u_A(x) = r\}$ for convenience.*

- *The boundary $\partial F$ of $F$ is contained in the set $E$.*
- *$E$ is Lebesgue negligible.*

*Proof.*

- If $u_A(z) < r$ then $z$ is in the topological interior of $F$.
- Let $z \in E$ *s.t.* $u_A(z) = r$; let $x \in A$ be a projection point of $z$; then the ball $D_r(x)$ is contained in $F$ and $z$ is in its boundary. Setting $y = (x + z)/2$, the ball $D_{r/2}(y)$ is contained in $F$ and $z$ is in its boundary; but moreover for all points $w \in D_{r/2}(y)$ with $w \neq z$ we have $|w - x| < r$, hence $u_A(w) < r$. We conclude that $D_{r/2}(y)$ intersects $E$ only in $z$. This proves that the Lebesgue density of the set $E$ in the point $z$ cannot be one, so $E$ is negligible.

(The above proves also that $F$ satisfies an *interior sphere condition*). □

# 5 Hausdorff distance

Let again $\mathcal{M}$ be the family of the nonempty compact sets in $\mathbb{R}^N$. A fundamental example of metric on $\mathcal{M}$ is the **Hausdorff distance**

$$d_H(A,B) \stackrel{\text{def}}{=} \inf\{\delta > 0 \mid B \subseteq (A + D_\delta), A \subseteq (B + D_\delta)\} \quad .$$

The Hausdorff distance may be defined in many equivalent ways,

$$d_H(A,B) = \max\{\max_A u_B, \max_B u_A\} \tag{25}$$

$$= \sup_{x \in \mathbb{R}^N} |u_A(x) - u_B(x)| \quad , \tag{26}$$

as shown in §C in Chap. 4 in [20] and §2.2 in Chap. 4 in [7].

This metric enjoys many important properties.

**Theorem 5.1.** *The metric space $(\mathcal{M}, d_H)$ satisfies:*

1. *given $r > 0$, the family of equibounded compact sets*

$$\{A \in \mathcal{M} \mid A \subseteq D_r\}$$

*is compact; in particular, given $B \in \mathcal{M}$, the set*

$$\{A \in \mathcal{M} \mid d_H(A,B) \leq \rho\}$$

*is compact.*

2. *$(\mathcal{M}, d_H)$ is complete.*
3. *$(\mathcal{M}, d_H)$ is intrinsic (that is, $d_H = (d_H)^g$).*
4. *Any two $A, B \in \mathcal{M}$ may be joined by a geodesic $\gamma$.*

The first statement *(1)* is a well-known property of the Hausdorff distance, see e.g. Example 4.13 and Theorem 4.18 in [20]. By exploiting the characterization (26), it also follows from a diagonal/compactness argument and the results presented in Sec. 5.3. The second follows from the first. The third and fourth property in 5.1 derive from the proposition 5.3 below.

We complement the above with this family of nice properties.

**Proposition 5.2.**

1. For any fixed $A \in \mathcal{M}$, the fattening map $\lambda \mapsto A + D_\lambda$ is Lipschitz (of constant one) as a map from $[0, \infty)$ to $(\mathcal{M}, d_H)$.
2. For any fixed $\lambda > 0$, the *"fattened area map"* $L_\lambda : \mathcal{M} \to \mathbb{R}$ defined as $L_\lambda(A) \stackrel{\text{def}}{=} \mathcal{L}^N(A + D_\lambda)$ is continuous on $(\mathcal{M}, d_H)$.
3. The area map $L(A) \stackrel{\text{def}}{=} \mathcal{L}^N(A)$ is upper semi continuous.
4. Let
$$\# : \mathcal{M} \to \mathbb{N} \cup \infty$$
be the number $\#\Omega$ of connected components of a compact set $\Omega$. Then $\#$ is lower semi continuous in the metric space $(\mathcal{M}, d_H)$. [5]

   As a corollary, the family of connected compact sets is a closed family in $(\mathcal{M}, d_H)$.
5. Let $A \in \mathcal{M}$ and $x$ in the topological interior of $A$, and $R > 0$ be *s.t.* $B_R(x) \subseteq A$, then we define the **carving motion** $\gamma : [0, R] \to \mathcal{M}$ as $\gamma(0) = A$ and
$$\gamma(t) = A \setminus B_t(x) \tag{27}$$
for $t \in (0, R]$. Then this motion is an arc parameterized [6] geodesic. See also Example 6.23.

[5] This result does not hold for closed sets.

[6] In the sense that its metric derivative is 1 for all t, see lemma 2.3.

6. Let $\Phi : [-T, T] \times \mathbb{R}^N \to \mathbb{R}^N$ be a locally Lipschitz map. Given $A$ compact, let $A_t = \Phi(t, A)$ be the image: then the path $t \mapsto A_t$ is Lipschitz in $(\mathcal{M}, d_H)$.

Suppose that $G$ is a group of diffeomorphisms of $\mathbb{R}^N$. This group $G$ acts on $\mathcal{M}$; for any $g \in G$, the action of $g$ on $A \in \mathcal{M}$ is $g(A) = \{g(x) : x \in A\}$. The last result shows that any such action is locally Lipschitz.

*Proof.* 1. Follows from (24).

2. If $A_n \to A$ then for fixed $\varepsilon > 0$ there exists $N$ such that for all $n \geq N$,

$$A_n \subseteq A + D_\varepsilon \quad , \quad A \subseteq A_n + D_\varepsilon$$

and then

$$A_n + D_\lambda \subseteq A + D_{\varepsilon+\lambda} \quad , \quad A + D_{\lambda-\varepsilon} \subseteq A_n + D_\lambda \quad .$$

Passing to Lebesgue measures,

$$\mathcal{L}^N(A + D_{\lambda-\varepsilon}) \leq \liminf_n \mathcal{L}^N(A_n + D_\lambda) \leq \limsup_n \mathcal{L}^N(A_n + D_\lambda) \leq \mathcal{L}^N(A + D_{\varepsilon+\lambda}) \quad .$$

We let $\varepsilon \to 0$: the LHS converges to the measure of the set $\{u_A < \lambda\}$ and the RHS converges to the measure of the set $\{u_A \leq \lambda\}$: by Lemma 4.1 they are equal.

3. Since it is the pointwise limit $L_\lambda(A) \downarrow L(A)$ for $\lambda \to 0$.

4. See Thm. 2.3 in Chap. 4 in [7].

5. For $0 \leq s < t \leq R$ we have $\gamma(t) \subseteq \gamma(s)$ and $\gamma(s) \subseteq \gamma(t) + D_{(t-s)}$.

6. Let $R > 0$ *s.t.* $A \subseteq D_R$. Let $L$ be the Lipschitz constant of $\Phi$ on $[-T, T] \times A$, then $A_t \subseteq D_{R+LT}$ for all $t$. Fix $s, t \in [-T, T]$ , given $y \in A_t$ let $x \in A$ *s.t.* $y = \Phi(t, x)$ then consider $z = \Phi(s, x) \in A_s$, by Lipschitzianity $|y - z| \leq L|t - s|$: so $A_t \subseteq A_s + D_{L|t-s|}$.

□

## 5.1 The maximal geodesic

In this section we describe an explicit formula to compute the geodesic connecting two compact sets $A$ to $B$.

**Proposition 5.3** (Maximal geodesic)**.** *Let $A, B \in \mathcal{M}$ be two compact sets, let $\mu = d_H(A, B)$. For all $t \in [0, \mu]$ we define the set*

$$C_t \stackrel{def}{=} \{z : u_A(z) \leq t, u_B(z) \leq (\mu - t)\}$$

*then $t \mapsto C_t$ is an arc parameterized geodesic connecting $A$ to $B$; and in particular its length is $\mu$.*

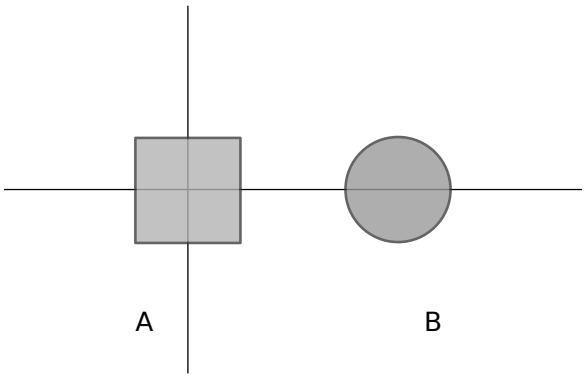
*Moreover $C$ is* maximal *in the sense that, for any arc parameterized geodesic $\gamma : [0, \mu] \to \mathcal{M}$ connecting $A$ to $B$ we have $\gamma(t) \subseteq C_t \quad \forall t \in [0, \mu]$.*

The proof is in Sec. A.3.

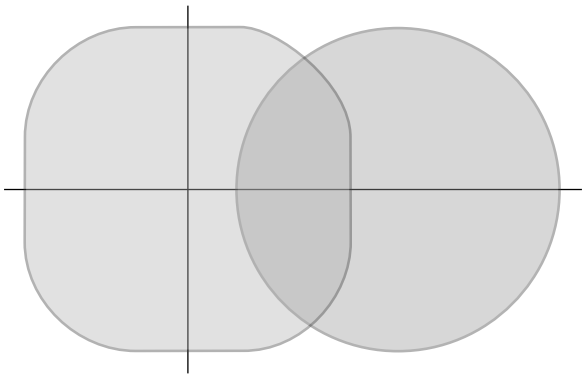
*Remark* 5.4. *The above results still hold for the Hausdorff metric space of compact subsets of a finitely compact intrinsic metric space — in particular they hold for finite dimensional Riemannian manifolds.*

We provide an explicit example.

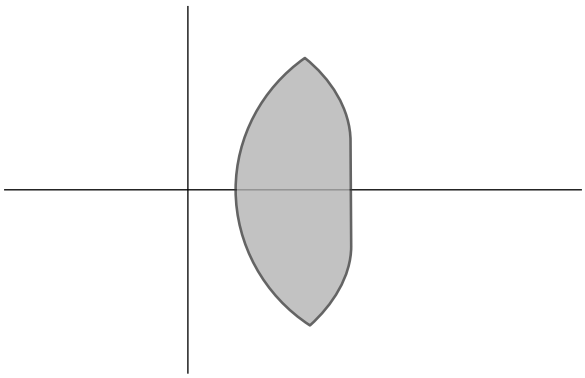
*Example* 5.5. Let $A$ be a square of side 2 centered at the origin, and $B$ a disc of radius 1 centered at $(4, 0)$. (In the figure cartesian axes are drawn for easy comparison with the following steps).




The Hausdorff distance is $\mu = d_H(A, B) = \sqrt{26} - 1$. Let us fix $t = \mu/2$. We now fatten the set $A$ by $t$ and the set $B$ by $\mu - t$ (that in this case is again $t$), and obtain these shapes.



Eventually we intersect the two fattenings to obtain $C_t$.



The above maximal geodesic enjoys some properties.

**Corollary 5.6.**

- *Let $A, B, \tilde{A}, \tilde{B} \in \mathcal{M}$ with $A \subseteq \tilde{A}$ and $B \subseteq \tilde{B}$ and suppose that*

$$\mu = d_H(A, B) = d_H(\tilde{A}, \tilde{B}) \quad .$$

  *Let $C_t, \tilde{C}_t$ be maximal geodesics connecting $A$ to $B$ and respectively $\tilde{A}$ to $\tilde{B}$: then $C_t \subseteq \tilde{C}_t \quad \forall t \in [0, \mu]$.*

- *Let $E$ be the convex hull of $A \cup B$, let $\mu = d_H(A, B)$. For all $t \in [0, \mu]$ we define the set*

$$\tilde{C}_t = C_t \cap E = \{z \in E : u_A(z) \leq t, u_B(z) \leq (\mu - t)\}$$

  *then $t \mapsto \tilde{C}_t$ is another arc parameterized geodesic connecting $A$ to $B$.*

- *If $A, B$ are convex sets, $C_t$ and $\tilde{C}_t$ are convex for all $t \in [0, \mu]$.*

*Proof.* The proof of the first result follows immediately from the definition of $C_t$. For the second we reread the proof in Sec. A.3, noting that, if $z \in E$ then $y \in E$ as well. For the third we remind that the two distance functions are convex. □

The maximal geodesic defined in above Prop. 5.3 may not be suited for applications in Computer Vision. Consider this example.

*Example* 5.7. Let $A \subset \mathbb{R}^2$ be a square of unit side, and $B = A + (4, 0)$ be its translation; then $d_H(A, B) = 4$. The map

$$\gamma : [0, 4] \to \mathcal{M} \quad , \quad \gamma(t) = A + (t, 0)$$

that translates $A$ to $B$ is an arc parameterized geodesic, but is not the maximal geodesic. The maximal geodesic $C_t$ is much larger, the set $C_2$ (at time $t = 2$) is depicted in Figure 2.

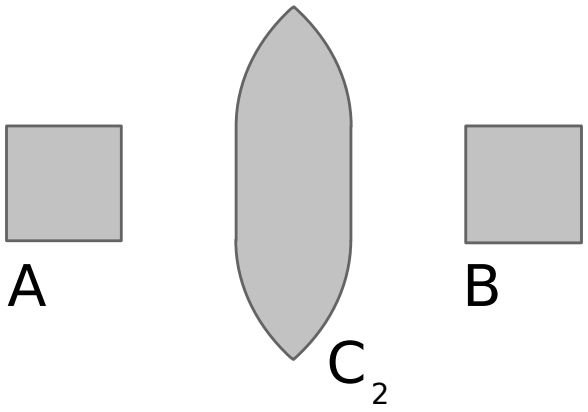


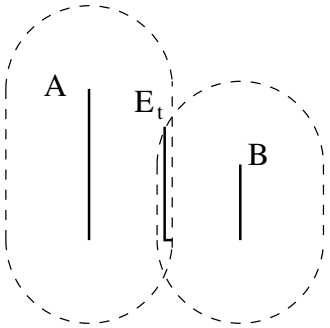

Figure 2: Example 5.7

## 5.2 Multiple geodesics

Unfortunately $(\mathcal{M}, d_H)$ is quite "unsmooth"; we will indeed prove that generically a pair $A, B \in \mathcal{M}$ may be joined by a continuum of geodesics.

*Example* 5.8. *We first provide an example.*

$$A = \{x = 0, 0 \le y \le 2\}$$
$$B = \{x = 2, 0 \le y \le 1\}$$
$$E_t = \left\{x = 1, 0 \le y \le \tfrac{3}{2}\right\} \cup \{y = 0, 1 \le x \le t\}$$
$$\text{with } 1 \le t \le \sqrt{5}/2;$$




*and in the picture we represent (dashed) the fattened sets $A + D_{\sqrt{5}/2}$ and $B + D_{\sqrt{5}/2}$. Note that $d_H(A, B) = \sqrt{5}$ while $d_H(A, E_t) = d_H(B, E_t) = \sqrt{5}/2$. So for all $t \in [1, \sqrt{5}/2]$, $E_t$ is a midpoint of a distinct geodesic between $A$ and $B$.*

The above idea is generalized in the following results.

**Lemma 5.9.** *Let $A, B \in \mathcal{M}$, suppose $A \setminus B$ has non-empty topological interior, let $\theta = \max_A u_B(x) > 0$. For any non-empty open $G \subseteq A \setminus B$ such that $G \cap \{u_B = \theta\} = \emptyset$, setting $E = A \setminus G$ then $d_H(A, B) = d_H(E, B)$.*

The proof is in Sec. A.4. Note that such sets $G$ exist, since the set $\{u_B = \theta\}$ is compact and negligible, due to Lemma 4.1.

**Lemma 5.10.** *Let $A, B \in \mathcal{M}$, with $A \neq B$, let $\mu = d_H(A, B)$. Let $C_t$ the maximal geodesic. Suppose that for a $t \in (0, \mu)$ the set $C_t \setminus (A \cup B)$ has non-empty interior: then there is a continuum of different*[7] *geodesics connecting $A$ to $B$.*

The proof is in Sec. A.5.

We then propose a general Theorem.

**Theorem 5.11.** *Let $A, B \in \mathcal{M}$, with $A \neq B$, let $\mu = d_H(A, B)$; suppose that there is an $x$ in the boundary of $A$* s.t. *$0 < u_B(x) < \mu$: then there is a continuum of different geodesics connecting $A$ to $B$.*

*Proof.* Let $C_t$ be the maximal geodesic connecting $A$ to $B$, as defined in Prop. 5.3. Choose $\varepsilon > 0$ small such that $2\varepsilon < u_B(x) < \mu - 2\varepsilon$; there is a point $y$ near $x$ such that $|x - y| < \varepsilon$ but $y \notin A$: such point satisfies $u_A(y) < \varepsilon$ and $\varepsilon < u_B(y) < \mu - \varepsilon$ hence it is in the topological interior of $C_t$ when $t = \varepsilon$, but is outside of $A$ and $B$. So we can apply the previous Lemmas. □

(Note the similarity of the above arguments to the proof 2.17 – and for a reason!).

**Lemma 5.12.** *Let $(A_n), A \subseteq \mathcal{M}$. Suppose that $A_n \to A$ in the sense of Hausdorff convergence, then for any $x \in \partial A$ there exists a sequence with $x_n \in \partial A_n$ such that $x_n \to x$.*

**Theorem 5.13.** *Generically any pair $A, B \in \mathcal{M}$ is connected by a continuum of different geodesics.*

*Proof.* If a pair $A, B \in \mathcal{M}$ does not satisfy the hypothesis of Thm. 5.11, then

$$\partial A \subseteq \{u_B = \mu\} \cup B \quad , \quad \partial B \subseteq \{u_A = \mu\} \cup A \quad , \tag{28}$$

where $\mu = d_H(A, B)$. Let $\mathcal{U}$ in $\mathcal{M}^2$ be the set of all such pairs; note that any pair $(A, A)$ is in $\mathcal{U}$.

We will prove that $\mathcal{U}$ is closed and has empty interior (*w.r.t.* the Hausdorff convergence).

The fact that $\mathcal{U}$ is closed follows from the previous Lemma and eqn. (26).

Fix $(A, B) \in \mathcal{U}$.

We choose an "exposed boundary point" $x \in A \cup B$; to fix ideas, we let $a = \max\{x_1 : x \in A \cup B\}$ (where $x_1$ is the first component of $x$) and $x$ be a point providing the above maximum; then $A \cup B$ is contained in the half space $H = \{z : z_1 \leq a\}$.

Suppose *wlog* that $x \in A$; let $y = x + \varepsilon e_1$ with $\varepsilon > 0$ small; then add to $A$ the segment $xy$ to create $\tilde{A}$. The pair $(\tilde{A}, B) \notin \mathcal{U}$, since the segment $xy$ is contained in the boundary of $\tilde{A}$, but $xy$ is not contained in $B$ (since by construction $xy$ is outside of $H$) and $u_B$ is not constant on it.

Since this construction holds for any $\varepsilon > 0$, then $\mathcal{U}$ has empty interior. □

To end the section, we review some examples of pairs $A, B$ that are connected by a unique geodesic. Note how the condition (28) is satisfied in these examples.

**Definition 5.14.** Following [8], $F$ is a set of **positive reach** if there exists $r > 0$ such that for any $x$ with $u_F(x) < r$ there exists a unique projection point $y \in F$; this defines a projection map $\pi_F(x) = y$. The *reach* is the largest such $r$. For any $0 < s < r$ the projection $\pi_F$ is Lipschitz on $\{u_F \leq s\}$.

Examples of sets of positive reach are:

- convex sets (in this case $r = \infty$);
- compact sets with non-empty interior and $C^2$-regular boundary;
- compact submanifolds of $\mathbb{R}^N$ that are $C^2$-regular.

*Example* 5.15. Suppose that $F$ is a set of *positive reach* $r$, choose $\mu \in (0, r)$. Suppose that $A$ is a compact subset of $\partial F$, and $B$ is a compact subset of $\{u_F = \mu\}$ *s.t.* $\pi_F(B) = A$. Then $d_H(A, B) = \mu$; there is a unique geodesic $C_t$ connecting $A$ to $B$, and $C_t$ is given by all interpolated points

$$\frac{tx + (\mu - t)\pi_F(x)}{\mu}$$

for all $x \in B$.

[7] That is, not "equal up to reparameterization".

When $A = \partial F$, the above is known as *grassfire evolution*.

The above encompasses many examples.

*Example* 5.16 (Orthogonal translation). Let $A \in \mathcal{M}$, suppose that there is an $N-1$ dimensional affine space $H$ containing $A$, choose a vector $v$ orthogonal to $H$ and define $B = v + A$ ($B$ is the translation of $A$ by the vector $v$); then $d_H(A, B) = |v|$ and the translation $C_t = A + tv/|v|$ is the unique geodesic connecting $A$ to $B$.

*Example* 5.17 (Fattening). Suppose that $A$ is a set of *positive reach* $r$, let $\mu \in (0, r)$ and $B = A + D_\mu$ be a fattening, then $d_H(A, B) = \mu$ and the fattening $C_t = A + D_t$ is the unique geodesic connecting $A$ to $B$.

Note that in general the "fattening" is a geodesic, but it may fail to be unique.

## 5.3 Hausdorff and Kuratowski convergence

We provide some extra definitions.

**Definition 5.18** (Kuratowski convergence)**.** Let $\Omega, \Omega_n$ be nonempty closed sets in $\mathbb{R}^N$. We will say that $\Omega_n \to \Omega$ in the Kuratowski sense if these equivalent facts hold:

- $u_{\Omega_n} \to u_\Omega$ pointwise;
- $u_{\Omega_n} \to u_\Omega$ pointwise on a dense subset of $\mathbb{R}^N$;
- $u_{\Omega_n} \to u_\Omega$ uniformly on compact subsets of $\mathbb{R}^N$.

This definition is not the standard one, but it is equivalent, see §4.B in [20]. The equivalence of the statements in the above definition is due to the fact that distance functions are 1-Lipshitz functions. The above three equivalent facts express a "rigidity" of distance functions, that is again seen in the following.

**Lemma 5.19.** *Let $\Omega_n$ be nonempty closed sets and suppose that $\lim_n u_{\Omega_n}(x)$ exists and is finite, for all $x$ in a dense subset $D$ of $\mathbb{R}^N$; call $f(x) = \lim_n u_{\Omega_n}(x)$. Then there is a nonempty closed set $\Omega$ such that $\Omega_n \to \Omega$ in the Kuratowski sense and $u_\Omega(x) = f(x)$ for all $x \in D$.*

*Proof.* The proof may follow from the theory of Viscosity Solutions: as remarked in Sec. 4, indeed $u_\Omega$ is the unique solution to the *eikonal equation*; moreover viscosity solutions do enjoy the required rigidity property. The proof is anyway easily derived by a direct argument and the Ascoli–Arzelà theorem (similarly to the arguments of Chap. 2 in [7] and of Chap. 4 in [20] [8]). We propose a direct proof in Sec. A.6. □

The Kuratowski convergence and the Hausdorff convergence coincide for equibounded families.

**Lemma 5.20.** *Suppose that $\Omega$ is compact and non-empty and that $\Omega_n$ are non-empty closed sets. These facts are equivalent.*

- *$\Omega_n$ is equibounded and $\Omega_n \to \Omega$ in the Kuratowski sense;*
- *$d_H(\Omega_n, \Omega) \to 0$;*
- *$u_{\Omega_n} \to u_\Omega$ uniformly.*

*Proof.* The equivalence of the first two is proved in Sect. C in Chap. 4 [20]. Eqn. (26) shows that the third condition is equivalent to the second. □

[8] But, see an important correction in Remark 2.7 in [15].

# 6 $L^p$–like metrics of shapes

The definition of the Hausdorff distance by eqn. (26) leads us back to the paradigm of *representation/embedding*; but in this case it is unfortunately not precise, since the Banach metric that we use, namely

$$\|f\| = \|f\|_\infty \stackrel{\text{def}}{=} \sup_x |f(x)| \quad ,$$

is usually associated to the space of continuous bounded functions — whereas the distance function $u_A$ is not bounded! What follows is a simple yet effective workaround.

**Hypotheses 6.1.** *We fix $p \in [1, \infty]$; we fix a function $\varphi : [0, \infty) \to (0, \infty)$ monotonically strictly decreasing and of class $C^1$, such that*

$$\varphi(|x|) \in L^p(\mathbb{R}^N). \tag{29}$$

*When $p = \infty$ we also ask that $\lim_{t\to\infty} \varphi(t) = 0$ as an extra hypothesis.*

In the rest of the paper $\varphi$ will always satisfy the above assumption (and possibly others, that will be specified when needed).

Note that, when $p < \infty$, the above (29) is equivalent to asking that

$$\int_0^\infty t^{N-1} \varphi(t)^p \, \mathrm{d}t < \infty \tag{30}$$

and this implies that $\lim_{t\to\infty} \varphi(t) = 0$.

An example of such a function is $\varphi(t) = \exp(-t)$, or $\varphi = (1 + t)^{-(N+1)/p}$.

We will often write

$$v_A \stackrel{\text{def}}{=} \varphi \circ u_A$$

for simplicity.

**Lemma 6.2.** *Let $\Omega \subseteq \mathbb{R}^N$ be closed and non-empty; suppose $p < \infty$; then the following are equivalent.*

*(a) $v_\Omega \in L^p(\mathbb{R}^N)$.*

*(b) $\Omega$ is bounded (and then $\Omega$ is compact).*

*Proof.* We first prove that $(a) \implies (b)$ by contradiction. Let us assume that $\Omega$ is unbounded. Then there exists a sequence $\{x_k\} \subseteq \Omega$ such that $|x_k| \to \infty$ and $|x_k - x_q| > 2$ for all $k, q \in \mathbb{N}, k \neq q$. The sequence of sets $B_1(x_k)$ is disjoint. It is easy to see that $v_\Omega(x) > \varphi(1)$ for $x \in \bigcup_k B_1(x_k)$ and then $v \notin L^p$.

Then we prove that $(b) \implies (a)$. If $\Omega$ is bounded we can find a disk $D_R$ such that $\Omega \subseteq D_R$. Then easily we have $u_\Omega \geq u_{D_R} \implies v_\Omega \leq v_{D_R}$, but $v_{D_R} \in L^p$ (as is easily proved by $v_{D_R}(x) = \varphi((|x| - R)^+)$ and by (30)) so that $v_\Omega \in L^p$ as well. □

Let again $\mathcal{M}$ be the family of the nonempty compact sets in $\mathbb{R}^N$.

**Definition 6.3.** Given $A, B \in \mathcal{M}$, we define

$$d_{p,\varphi}(A, B) \stackrel{\text{def}}{=} \|\varphi \circ u_A - \varphi \circ u_B\|_{L^p(\mathbb{R}^N)} \quad . \tag{31}$$

By the above lemma, this distance is finite. We will often write $d$ for $d_{p,\varphi}$ in the following, for simplicity. Similarly we will write $d^g$ for the induced distance $d^g_{p,\varphi}$.

The above distance is obtained by the *representation* of a shape $A$ as $v_A$, combined with the *embedding* of $v_A$ in $L^p(\mathbb{R}^N)$. For this reason, we may identify our shape space with

$$\mathcal{N} \stackrel{\text{def}}{=} \{v_\Omega \mid \Omega \in \mathcal{M}\} \tag{32}$$

that is a subset of $L^p$.

*Remark* 6.4. *By the definition of $d$, the map $\Omega \mapsto v_\Omega$ is an isometrical embedding of $\mathcal{M}$ inside $L^p$ and the image is $\mathcal{N}$; $\mathcal{N}$ is a closed subset of $L^p$, by the completeness result 6.12 that we will prove in the following. We will exploit this embedding in the following, in particular in §6.6.*

It is immediate to verify that $d_{p,\varphi}$ satisfies these properties.

- The embedding $A \mapsto v_A$ is injective: if $v_A = v_B$ *a.e.* then $u_A = u_B$ *a.e.* (since $\varphi$ is strictly decreasing, and so it is injective); since distance functions are continuous, this implies that $u_A = u_B$ and then $A = B$. Consequently, for all $A, B \in \mathcal{M}$, $d_{p,\varphi}(A, B) = 0$ iff $A = B$.
- $d_{p,\varphi}$ is Euclidean invariant, as we requested in sec. 1.2.
- When $p < \infty$, for any $A, B$ compact there holds

$$d_{p,\varphi}(A, B) < \sqrt[p]{\|v_A\|_{L^p}^p + \|v_B\|_{L^p}^p}. \tag{33}$$

  Indeed we note that for $a, b > 0$ we have

$$|a - b|^p < \max\{a^p, b^p\} < a^p + b^p \quad ;$$

  so

$$d_{p,\varphi}(A, B)^p = \int_{\mathbb{R}^N} |v_A(x) - v_B(x)|^p \, \mathrm{d}x < \int_{\mathbb{R}^N} v_A(x)^p + v_B(x)^p \, \mathrm{d}x \quad .$$

- When $p = \infty$ instead

$$d_{\infty,\varphi}(A, B) < \varphi(0) \ .$$

- *(Separation at infinity)* Given two bounded sets $A, B$ and $\tau \in \mathbb{R}^N$ we have

$$\lim_{|\tau| \to \infty} d_{p,\varphi}(A, B + \tau) = \sqrt[p]{\|v_A\|_{L^p}^p + \|v_B\|_{L^p}^p} \tag{34}$$

  for $p < \infty$, while

$$\lim_{|\tau| \to \infty} d_{\infty,\varphi}(A, B + \tau) = \varphi(0) \quad . \tag{35}$$

  *Proof.* For the case $p = \infty$ it derives from the hypothesis $\lim_{t \to \infty} \varphi(t) = 0$. When $p < \infty$, it comes from a general result for $L^p$ functions, see Sec. A.1. □

- *(Scaling)* If $p < \infty$ and $\lambda > 0$ is a rescaling of the space, then the rescaled distance may be expressed as

$$d_{p,\varphi}(\lambda A, \lambda B) = \lambda^{N/p} d_{p,\tilde{\varphi}}(A, B) \tag{36}$$

  where $\tilde{\varphi}(r) = \varphi(\lambda r)$; indeed

$$\begin{aligned} d_{p,\varphi}(\lambda A, \lambda B)^p &= \int |v_{\lambda A}(x) - v_{\lambda B}(x)|^p \, \mathrm{d}x && (37)\\ &= \lambda^N \int |v_{\lambda A}(\lambda z) - v_{\lambda B}(\lambda z)|^p \, \mathrm{d}z && (38)\\ &= \lambda^N \int |\varphi(\lambda u_A(z)) - \varphi(\lambda u_B(z))|^p \, \mathrm{d}z && (39)\\ &= \lambda^N d_{p,\tilde{\varphi}}(A, B)^p \end{aligned}$$

  where to go from (37) to (38) we used the change of variable $x = \lambda z$ and the property (23) of the distance function to change (38) to (39).

*Remark* 6.5. *The inequality (33) easily implies that the closed balls of the distance $d$ in general are not compact sets. Indeed it is enough to consider a compact set $\Omega$ and the closed ball*

$$\mathbb{D} = \{A \mid d(A, \Omega) \leq 2r\}$$

*with $r = \|v_\Omega\|_{L^p}$. Then the sequence $\{\Omega + n\tau\}_{n \in \mathbb{N}}$ with $\tau \in \mathbb{R}^N \setminus \{0\}$ is contained in $\mathbb{D}$ and it does not have any convergent subsequence.*

We will nonetheless prove in the following that the metric space $(\mathcal{M}, d)$ is locally compact.

To continue with our study of $d$, we prove this fundamental inequality.

**Lemma 6.6** (Local equiboundedness)**.** *There is a continuous and increasing function $b : \mathbb{R}^+ \to \mathbb{R}^+$ with $b(0) = 0$ and $\lim_{r\to\infty} b(r) = \|\varphi(|x|)\|_{L^p}$ such that, for any $\Omega, \Omega' \in \mathcal{M}$ satisfying*

$$\|v_\Omega - v_{\Omega'}\|_{L^p} < b(r)$$

*we have $\Omega' \subseteq \Omega + D_r$.*

*Proof.* Set $K \stackrel{\text{def}}{=} \Omega + D_r$. By the relation in eqn. (24)

$$v_K(x) = \varphi\big((u_\Omega(x) - r)^+\big).$$

To prove the proposition for $p \in [1, \infty)$, suppose that $x_0 \in \Omega'$, but $x_0 \notin K$; for $y \in B(x_0, r/2)$ recall the simple triangle inequality

$$u_\Omega(y) \geq r - |x_0 - y| \geq |x_0 - y| \geq u_{\Omega'}(y)$$

hence

$$v_\Omega(y) \leq \varphi(r - |x_0 - y|) \leq \varphi(|x_0 - y|) \leq v_{\Omega'}(y)$$

so

$$\begin{aligned} \|v_\Omega - v_{\Omega'}\|_{L^p}^p &\geq \int_{B(x_0,r/2)} |v_{\Omega'} - v_\Omega|^p \,\mathrm{d}x \geq \\ &\geq \int_{B(x_0,r/2)} |\varphi(|x_0 - y|) - \varphi(r - |x_0 - y|)|^p \,\mathrm{d}x = b(r)^p \end{aligned}$$

where

$$b(r)^p \stackrel{\text{def}}{=} \omega_N N \int_0^{r/2} t^{N-1} (\varphi(t) - \varphi(r - t))^p \,\mathrm{d}t$$

and where $\omega_N$ is the N-volume of the ball $B_1$ in $\mathbb{R}^N$. It is easy to prove that $b$ is continuous and increasing (by direct derivation); that $b(0) = 0$ and that $\lim_{r\to\infty} b(r) = \|\varphi(|x|)\|_{L^p}$. With some calculus it is also possible to prove that

$$b(r) \sim |\varphi'(0)| r^{1+N/p} \tag{40}$$

for $r$ small. (This estimate is sharp, see Example 6.23).

The case $p = \infty$ is simpler: in this case we can note that

$$\|v_\Omega - v_{\Omega'}\|_\infty \geq v_{\Omega'}(x_0) - v_\Omega(x_0) \geq \varphi(0) - \varphi(r)$$

and set $b(r) = \varphi(0) - \varphi(r)$. □

**Corollary 6.7.** *For $d(\Omega, \Omega')$ small enough* [9]

$$d_H(\Omega, \Omega') \leq b^{-1}\Big(d(\Omega, \Omega')\Big) \,.$$

*Remark* 6.8. The above does not hold for arbitrarily large distance $d(\Omega, \Omega')$: indeed, let $\Omega = \{0\}$ and $\Omega_n = \{ne_1\}$: then $d(\Omega, \Omega_n) \to 2^{1/p}\|\varphi(|x|)\|_{L^p}$ (as we mentioned in eqn. (34)).

We can also obtain a converse inequality, as follows.

**Lemma 6.9.** *There is a family of continuous functions $f_R : \mathbb{R}^+ \to \mathbb{R}^+$ with $f_R(0) = 0$ such that for any $\Omega, \Omega' \in \mathcal{M}$ with $\Omega, \Omega' \subseteq B_R$*

$$d(\Omega, \Omega') \leq f_R\Big(d_H(\Omega, \Omega')\Big) \,.$$

[9] Precisely, for $d(\Omega, \Omega') < \lim_{r\to\infty} b(r) = \|\varphi(|x|)\|_{L^p}$.

*Proof.* Note that $\varphi$ is uniformly continuous; so when $p = \infty$ we choose $f_1$ to be a (continuous and increasing) modulus of continuity for $\varphi$, and then for any $R > 0$ we set $f_R \equiv f_1$, and use eqn. (26).

We now provide the proof for $p < \infty$. Since $B_R$ contains both $\Omega$ and $\Omega'$, then $u_\Omega, u_{\Omega'} \geq u_{B_R}$ and then

$$v_\Omega(x), v_{\Omega'}(x) \leq \varphi((|x| - R)^+)$$

so for $r \geq R$

$$\int_{\mathbb{R}^N \setminus B_r} |\max\{v_\Omega(x), v_{\Omega'}(x)\}|^p \, \mathrm{d}x \leq a_R(r)$$

where

$$a_R(r) \stackrel{\text{def}}{=} \int_{\mathbb{R}^N \setminus B_r} \varphi(|x| - R)^p \, \mathrm{d}x = \omega_N N \int_r^\infty t^{N-1} \varphi(t - R)^p \, \mathrm{d}t \tag{41}$$

(where $\omega_N$ is the N-volume of the ball $B_1$, and $\omega_N N$ is the $(N-1)$-volume of its boundary) and note that $a_R(r) \to 0$ for $r \to \infty$. At the same time, let $l(r) = \sup_{[0,r]} |\varphi'|$: then

$$\forall x \in B_r, \quad |v_\Omega(x) - v_{\Omega'}(x)| \leq l(r + 2R)|u_\Omega(x) - u_{\Omega'}(x)|$$

so

$$\int_{B_r} |v_\Omega(x) - v_{\Omega'}(x)|^p \, \mathrm{d}x \leq \omega_N r^N l(r + 2R)^p \sup_{x \in B_r} |u_\Omega(x) - u_{\Omega'}(x)|^p \leq$$
$$\leq \omega_N r^N l(r + 2R)^p d_H(\Omega, \Omega')^p$$

Summarizing,

$$d(\Omega, \Omega')^p = \int_{\mathbb{R}^N \setminus B_r} |v_\Omega(x) - v_{\Omega'}(x)|^p \, \mathrm{d}x + \int_{B_r} |v_\Omega(x) - v_{\Omega'}(x)|^p \, \mathrm{d}x \leq$$
$$\leq a_R(r) + \omega_N r^N l(r + R)^p d_H(\Omega, \Omega')^p$$

Let eventually, for $s \geq 0$,

$$g_R(s) = \inf_{r \geq R} \left[ a_R(r) + \omega_N r^N l(r + 2R)^p \; s \right] \tag{42}$$

and note that it is concave and monotonically increasing, and that $\lim_{s \to 0} g_R(s) = 0$; and let $f_R(s) = \sqrt[p]{g_R(s^p)}$. □

*Remark* 6.10. Suppose that $\varphi$ is convex and that $\varphi'(0) = -1$, then $l \equiv 1$ in the above proof. When $s > 0$ the minimum in (42) is obtained by $r = R$ if $s > \varphi(0)^p$, otherwise by

$$r = R + \varphi^{-1}(\sqrt[p]{s}) \quad .$$

A special case is $\varphi(t) = \exp(-t)$, $N = 2$, we obtain $f_R(s) \sim s|R - \log s|^{2/p}$ for $s$ small.

Another interesting case is $\varphi(t) = (1 + t)^{-(N+1)/p}$, in this case, for $R > 1$, $N \geq 2$, we obtain $f_R(s) \sim s^{\frac{1}{N+1}}$ for $s$ small.

Combining the two lemmas 6.9 and 6.7, we obtain this result.

**Theorem 6.11.** *The topology induced by $d_{p,\varphi}$ over the space $\mathcal{M}$ coincides with the topology induced by $d_H$.*

This implies that all topological properties of the Hausdorff distance are valid for the distance $d$ as well.

The two distances though are not equivalent, since

$$\lim_{|\tau| \to \infty} d_H(A, B + \tau) = \infty$$

while equations (34) and (35) show that the limit is finite for $d_{p,\varphi}$. When $p < \infty$ then $d_{p,\varphi}$ and $d_H$ are also not locally equivalent, as seen in the examples 6.23 below.

## 6.1 Completeness

By Thm. 6.11, we know that $(\mathcal{M}, d)$ is locally compact.

We now prove that it is complete.

**Proposition 6.12.** *The space $(\mathcal{M}, d)$ is complete.*

*Proof.* Let $\Omega_n$ be a Cauchy sequence; this means that $\{v_{\Omega_n}\}_n$ is a Cauchy sequence in $L^p$. Since $L^p$ is complete, $v_{\Omega_n} \to g$ in $L^p$.

By lemma 6.6 we know that there exists a compact set $K$ such that the sets $\Omega_n \subseteq K$ for all $n$. In particular, for any $x \in \mathbb{R}^N$, $u_{\Omega_n}(x) \leq \max_{y \in K} |y - x|$ and then

$$v_{\Omega_n}(x) \geq h(x) \stackrel{\text{def}}{=} \min_{y \in K} \varphi(|y - x|) \ ;$$

note that $h(x) > 0$ at all points.

It is well-known (see e.g. thm 4.9 in [4]) that, up to a subsequence that we indicate with $\{v_k\}_k$, there is also convergence $v_k(x) \to g(x)$ for almost all $x$. By the above reasoning, $g(x) \geq h(x) > 0$ in all points of convergence.

Let $u_k(x) \stackrel{\text{def}}{=} \varphi^{-1} v_k(x)$ and $u = \varphi^{-1} g$; then $u_k(x) \to u(x)$ on a dense subset, so by Lemmas 5.19 and 5.20, $u = u_\Omega$ where $\Omega \stackrel{\text{def}}{=} \{u = 0\}$. □

Summarizing, this proposition together with Theorems 5.1 and 6.11 imply that $\mathcal{N}$ is a complete (that is, closed) and locally compact subset of $L^p$. ($\mathcal{N}$ was defined in eqn. (32) as the family of all functions $v_A$ for $A$ compact sets).

*Remark* 6.13. The above implies an interesting property of the subset $\mathcal{N}$ of $L^p$: it admits a small neighbourhood $U$ on $L^p$ such that, for $f \in U$, there is at least a $v \in \mathcal{N}$ providing the minimum of the distance $\inf_{v \in \mathcal{N}} \|f - v\|$. As far as we know, this minimum may fail to be unique.

## 6.2 Shape Analysis

The family of distances is suitable for Shape Analysis.

**Proposition 6.14.** *Let $G = \mathcal{O}(N) \ltimes \mathbb{R}^N$ be the Euclidean group (*i.e. *the group generated by rotations, translations and reflections); as in (15), we can define the quotient metric by*

$$d_q([A], [B]) = \inf_{g \in G} d(gA, B). \tag{43}$$

*Then the above infimum is a minimum; so $d_q([A], [B]) > 0$ when $[A] \neq [B]$.*

*Proof.* Choose a minimizing sequence $\{g_n = (R_n, T_n)\}_{n \in \mathbb{N}}$, that is

$$\inf_{g \in G} d(gA, B) = \lim_{n \to \infty} d(g_n A, B) = \lim_{n \to \infty} d(R_n A + T_n, B).$$

Then $\{T_n\}_{n \in \mathbb{N}}$ must be bounded; we prove this by contradiction. Let us assume that $|T_n| \to \infty$, then by (33) we would have that

$$d(A, B) < \sqrt[p]{\|v_A\|_{L^p}^p + \|v_B\|_{L^p}^p}$$

and by Lemma A.1 that

$$\lim_{n \to \infty} d(R_n A + T_n, B) = \sqrt[p]{\|v_A\|_{L^p}^p + \|v_B\|_{L^p}^p},$$

so $\{g_n\}$ is not a minimizing sequence.

So the translation part of every minimizing sequence of (43) must be bounded. By compactness we have that there exists a limit transformation $g = (R, T) \in K$ and subsequence such that $g_{n_k} \to_k g$; by continuity of $d(fA, B)$ with respect to $f \in G$, we have that $d(gA, B) = d_q([A], [B])$. □

## 6.3 $d^g$ and geodesics

Let $d = d_{p,\varphi}$ in the following.

**Proposition 6.15.** *Given any two $A, B \in \mathcal{M}$ with $A \neq B$ then for all $\lambda \in (0,1)$, $\lambda v_A + (1-\lambda)v_B \notin \mathcal{N}$.*

*Proof.* It is easy to show that $f_\lambda = \lambda v_A + (1-\lambda)v_B$ assumes the value $\varphi(0)$ only on the intersection of the two sets $A \cap B$, for any $\lambda \in (0,1)$. Then $f_\lambda \in \mathcal{N}$ implies that $f_\lambda = v_{A\cap B}$. If $A \cap B = \emptyset$ the proof ends. Suppose wlog that $A \setminus B \neq \emptyset$. Let $x \in A \setminus B$ then $\varphi(0) = v_A(x) > v_B(x) \geq v_{A\cap B}(x)$; so $f_\lambda(x) > v_{A\cap B}(x)$ achieving a contradiction. □

Similarly, the convex combination $\lambda u_A + (1-\lambda)u_B$ of two distance functions $u_A, u_B$ is not a distance function (but for the special cases $\lambda \in \{0,1\}$ or $A = B$).

**Corollary 6.16.** *Suppose that $p \in (1,\infty)$. Given any two $A, B \in \mathcal{M}$ with $A \neq B$ we have that $d(A,B) < d^g(A,B)$.*

The result follows from the previous proposition and Thm. 2.15.

In the above cases $d$ is not intrinsic. So, to prove that the metric $d$ admits geodesics, we have to study $d^g$ as well.

We will need the following extra hypotheses in many results following.

**Hypotheses 6.17.** *Let $\varphi$ be as defined in 6.1. We moreover suppose that there is a constant $T > 0$ such that $\varphi(t)$ is convex for $t \geq T$. When $p < \infty$, we suppose that*

$$\varphi'(|x|) \in L^p(\mathbb{R}^N). \tag{44}$$

The above implies that $\lim_{t\to\infty} \varphi'(t) = 0$. Note also that (44) is equivalent to asking that

$$\int_0^\infty t^{N-1} |\varphi'(t)|^p \, \mathrm{d}t < \infty \quad . \tag{45}$$

**Proposition 6.18.** *If 6.17 holds then the space $(\mathcal{M}, d_{p,\varphi})$ is Lipschitz–arc connected.*

The proof is in Sec. A.7.

When $\mathcal{M}$ is Lipschitz-arcwise connected, the induced metric $d^g = (d_{p,\varphi})^g$ is a finite metric, that is, $d^g(A,B) < \infty$ for all $A, B$ compact.

We can prove an equiboundedness result for $d^g$ (that is stronger than the one in Prop. 6.6).

**Proposition 6.19.** *Suppose that 6.17 holds. Fix a compact nonempty set $\Omega$ and an $r > 0$; then there is a $K$ compact such that for any closed set $\Omega'$ satisfying $d^g(\Omega, \Omega') < r$ we have $\Omega' \subseteq K$.*

*Proof.* Let $b(r)$ be defined by Prop. 6.6. Let $d^g(\Omega, \Omega') < r$ and $\gamma : [0,1] \to \mathcal{N}$ be a Lipschitz path (of constant $L$) connecting $\gamma(0) = \Omega$ to $\gamma(1) = \Omega'$ such that

$$\mathrm{Len}^d \gamma \leq d^g(\Omega, \Omega') + 1 \, .$$

Up to reparameterization, we also assume that $L \leq r + 2$. Let $n$ be large so that $(r+2)/n \leq b(r)$, and let $K = \Omega + D_{rn}$ (note that $n$ only depends on $r$). Let $A_i = \gamma(i/n)$ for $i = 0, \ldots, n$; we know that

$$d(A_i, A_{i+1}) \leq d^g(A_i, A_{i+1}) \leq L/n < (r+2)/n \leq b(r)$$

since $\gamma$ is L-Lipschitz; so we apply recursively the proposition 6.6 on each $A_i$: we obtain that

$$A_{i+1} \subseteq A_i + D_r$$

hence $\Omega' \subseteq \Omega + D_{rn} = K$. □

The above results have many interesting consequences.

**Theorem 6.20.** *Suppose that 6.17 holds. For any $\rho > 0$,*

$$\mathbb{D}^g(A, \rho) \stackrel{def}{=} \{A \mid d^g(A,B) \leq \rho\}$$

*is compact in the $(\mathcal{M}, d)$ topology; so*

- *we obtain by Prop. 2.7 and Prop. 6.18 that geodesics do exist;*
- *and by Prop. 2.8 that the* Geodesic Distance Based Averaging

$$\bar{A} = \operatorname{argmin}_A \sum_{j=1}^{n} d^g(A, A_j)^2 \tag{46}$$

*of any given collection $A_1, \dots A_n$ exists.*

Two examples of geodesics are in Fig. 3 on page 32 and in Fig. 4 on page 33.

## 6.4 Variational description of geodesics

In this section we restrict $p \in (1, \infty)$.

We first state these general results, based on well-known $L^p$ theory.

**Proposition 6.21.** *Suppose that $t \mapsto f(t, \cdot)$ is a Lipschitz path from $t \in [0, 1]$ to $L^p(\mathbb{R}^N)$; then, for almost all $t$, $f$ admits a strong derivative $\frac{df}{dt}$ that is the limit*

$$\frac{df}{dt}(t, \cdot) \stackrel{def}{=} \lim_{\tau \to 0} \frac{f(t + \tau, \cdot) - f(t, \cdot)}{\tau} \tag{47}$$

*in $L^p(\mathbb{R}^N)$. This follows from Corollary 2.13. Moreover*

- *$f$ admits a weak partial derivative $\partial_t f$, and $\partial_t f = \frac{df}{dt}$ for almost all $t$.*
- *If $f$ admits a pointwise partial derivative in $t$ for almost all $t, x$, that we will call $h$, then $\partial_t f = h$.*

The proof is in Sec. A.8.

If $\gamma(t)$ is a Lipschitz path in $(\mathcal{M}, d)$, then it is associated to a function $f(t, x) = \varphi(u_{\gamma(t)}(x))$. If $\gamma$ is Lipschitz then $f(t, \cdot)$ satisfies the hypotheses of the above proposition. The first point means that we can represent the "abstract" derivative $\frac{d\gamma}{dt}$ by means of the weak derivative $\partial_t f(t, \cdot) \in L^p(\mathbb{R}^N)$. The second point is used to compute the derivative in practical cases, such as the following examples.

The above proposition can also be used to provide a variational description of geodesics. Let $\gamma : [0, 1] \to \mathcal{M}$ be a path; by Lemma 2.3 the variational length $\operatorname{Len}^d \gamma$ can be computed using the integral length $\operatorname{len}^d \gamma$ of the metric derivative $|\dot{\gamma}(t)|$; by Theorem 2.11 the metric derivative $|\dot{\gamma}(t)|$ coincides with the norm $\|\dot{v}_{\gamma(t)}\|_{L^p}$ of the derivative $\dot{v}_{\gamma(t)}$ in the Banach space $L^p$; by the above result, $\|\dot{v}_{\gamma(t)}\|_{L^p} = \|\partial_t f(t, x)\|_{L^p}$. Summarizing

$$\operatorname{Len}^d \gamma = \int_0^1 \|\partial_t f(t, \cdot)\|_{L^p} \, \mathrm{d}t \quad . \tag{48}$$

So to find the geodesic between two compact sets $A, B$, we need to minimize the above, with the following constraints

- $f(0, \cdot) = v_A$, $f(1, \cdot) = v_B$
- for any fixed t, $\varphi^{-1} \circ f(t, \cdot)$ is a distance function.

It is possible to prove (using a reparameterization lemma and Hölder inequality) that the geodesic is also the minimum of the *action*

$$J(\gamma) = \int_0^1 \|\partial_t f(t, x)\|_{L^p}^p \, \mathrm{d}t = \int_0^1 \int_{\mathbb{R}^N} |\partial_t f(t, x)|^p \, \mathrm{d}x \, \mathrm{d}t \; . \tag{49}$$

Equivalently, setting $g(t, x) = u_{\gamma(t)}(x)$, to find geodesics we can minimize

$$J(\gamma) = \int_a^b \int_{\mathbb{R}^N} |\varphi'(g) \partial_t g(t, x)|^p \, \mathrm{d}x \mathrm{d}t$$

with the constraint that $g(0, \cdot) = u_A$, $g(1, \cdot) = u_B$, and, for any fixed $t$, $g(t, \cdot)$ is a distance function.

### 6.4.1 Examples

*Example* 6.22. Let $N = 2$, $p = 2$, $\varphi(t)$ smooth, $r \geq 0$. Consider the path $\gamma(t)$ of disks of center $(t, 0) \in \mathbb{R}^2$ and radius $r$ in $\mathbb{R}^2$, for $t \in [0, 1]$. We want to compare the length of this path as computed using the Hausdorff distance and using the distance $d$.

- We have $d_H(\gamma(0), \gamma(1)) = 1$, that is also the length $\mathrm{Len}^{d_H} \gamma$ of the path $\gamma$.
- We use the expression (48). We have

$$v_{\gamma_t}(x_1, x_2) = \begin{cases} \varphi(0) & \text{if } x_2^2 + (x_1 - t)^2 \leq r^2 \ , \\ \varphi\Big(\sqrt{x_2^2 + (x_1 - t)^2} - r\Big) & \text{otherwise} \ . \end{cases}$$

  Upon derivation with respect to t,

$$\partial_t v_{\gamma_t}(x_1, x_2) = \begin{cases} 0 & \text{if } x_2^2 + (x_1 - t)^2 \leq r^2 \ , \\ \frac{t - x_1}{\sqrt{x_2^2 + (x_1 - t)^2}} \varphi'(\sqrt{x_2^2 + (x_1 - t)^2} - r) & \text{otherwise} \end{cases}$$

  so (using polar coordinates around $(t, 0)$)

$$\|\partial_t v_{\gamma_t}\|^2_{L^2(\mathbb{R}^2)} = \int_{\mathbb{R}^2} |\partial_t v_{\gamma_t}(x_1, x_2)|^2 \, \mathrm{d}x = \pi \int_r^\infty \Big(\varphi'(\rho - r)\Big)^2 \rho \, \mathrm{d}\rho = \pi b + ra\pi$$

  where

$$a = \int_0^\infty \big(\varphi'(s)\big)^2 \, \mathrm{d}s \ , \quad b = \int_0^\infty \big(\varphi'(s)\big)^2 s \, \mathrm{d}s \ .$$

  For example, when $\varphi(t) = e^{-t}$ then the length $\mathrm{Len}^d \gamma$ is

$$\frac{\sqrt{\pi(1 + 2r)}}{2} \ .$$

A more general computation of the length of motions of convex bodies will be performed in Sec. 6.6.1.

*Example* 6.23. Let $A$ be compact. We consider again the *carving* motion that we saw in 5.2.(5); to simplify the matter, suppose that the origin is in the topological interior of $A$; let $R > 0$ *s.t.* $B_R \subseteq A$; for $t \in [0, R]$ let $\gamma(t) = A_t = A \setminus B_t$, be the carving of a small ball from $A$.

We suppose $p \in (1, \infty)$, and we also suppose that $\varphi'(0) \neq 0$, for simplicity.

We can explicitly compute (for $r > 0, s > 0$ with $r + s \leq R$)

$$f(r, x) = v_{\gamma(r)}(x) = \begin{cases} \varphi(r - |x|) & \text{if } |x| \leq r \\ v_A(x) & \text{if } |x| > r \end{cases}$$

hence

$$\begin{aligned} \|v_{A_s} - v_{A_{r+s}}\|^p_{L^p} &= \omega_N N \int_s^{r+s} t^{N-1} \Big(\varphi(0) - \varphi(s + r - t)\Big)^p dt + \\ &+ \omega_N N \int_0^s t^{N-1} \Big(\varphi(s - t) - \varphi(s + r - t)\Big)^p dt \leq \\ &\leq \omega_N r^p L^p (r + s)^N \end{aligned}$$

where $L$ is the Lipschitz constant of $\varphi(t)$ for small $t$. We have thus proved that the *carving motion* is Lipschitz for the distance $d_{p,\varphi}$.

Note that $d_H(A, A_t) = t$, but

$$d_{p,\varphi}(A, A_t) \sim t^{1+N/p} \ ;$$

so the two distances are not locally equivalent when $p \in [1, \infty)$; moreover the estimate (40) is sharp.

Suppose now moreover that $p \in (1, \infty)$. Using Prop. 6.21 we can compute the metric derivative of $\gamma$

$$|\dot\gamma|(r) = \|\partial_r f(r, \cdot)\|_{L^p} = \sqrt[p]{\omega_N N \int_0^r s^{N-1} \left|\varphi'(r-s)\right|^p \, \mathrm{d}s}$$

and using Theorem 2.11 we obtain that the length of $\gamma_t$ for $t \in [a, b]$ is

$$\mathrm{Len}^d \gamma|_{[a,b]} = \int_a^b \sqrt[p]{\omega_N N \int_0^r s^{N-1} \Big(\varphi'(r-s)\Big)^p \, \mathrm{d}s} \, \mathrm{d}r \quad ;$$

in particular for $r$ small we obtain

$$\mathrm{Len}^d \gamma|_{[0,r]} \sim r^{1+N/p} \quad . \tag{50}$$

So $d^g(A, A_r) \le O(r^{1+N/p})$, hence the two distances $d^g$ and $d_H$ are not locally equivalent when $p \in (1, \infty)$;

## 6.5 Tangent bundle

Let $p \in (1, \infty)$. We identify $\mathcal{M}$ with $\mathcal{N} \subseteq L^p$, as by remark 6.4.

Given a $v \in \mathcal{N}$, let $T_v\mathcal{N} \subseteq L^p$ be the **contingent cone**

$$\begin{aligned} T_v\mathcal{N} &\stackrel{\text{def}}{=} \left\{ \lim_n t_n(v_n - v) \mid t_n > 0, v_n \in \mathcal{N}, v_n \to v \right\} = \\ &= \left\{ \lambda \lim_n \frac{v_n - v}{\|v_n - v\|_{L^p}} \mid \lambda \ge 0, v_n \to v \right\} \quad , \end{aligned}$$

where we consider all sequences $t_n, v_n$ such that the limit exists; it is intended that the above limits are in the sense of strong convergence in $L^p$.

According to Cor. 2.13 if $\gamma : [a, b] \to \mathcal{N}$ is a Lipschitz path then $\dot\gamma(t)$ exists (in the strong sense) in $L^p(\mathbb{R}^N)$ for almost all $t$; so $\dot\gamma(t) \in T_\gamma\mathcal{N}$ for almost all $t$.

In the following example we write explicitly the element of the contingent cone relative to a particular path.

*Example* 6.24. We fix $\Omega \in \mathcal{M}$. We define the fattening $\Omega_t = \Omega + D_t$ for $t \ge 0$. We are interested in evaluating the derivative $\dot\gamma(t)$. As previously done, we use the relationship (24) namely

$$u_{\Omega_t}(x) = (u_\Omega(x) - t)^+ \tag{51}$$

and note that this map is jointly Lipschitz in $(t, x)$: hence both $u_{\Omega_t}(x)$ and $v_{\Omega_t}(x)$ are almost everywhere differentiable. The pointwise derivative is given by:

$$w = \lim_{\tau \to 0} \frac{1}{\tau} \left[ v_{\Omega_{t+\tau}} - v_{\Omega_t} \right] = \begin{cases} -\varphi'(u_\Omega(x) - t) & \text{for } \ x \notin \Omega_t, \\ 0 & \text{for } \ x \in \mathring{\Omega}_t. \end{cases} \tag{52}$$

where $\mathring{\Omega}_t$ is the topological interior. Note that the derivative may not exist for $x \in \partial\Omega_t$. If $\varphi'(|x|) \in L^p$ then $w \in L^p$, and it can be shown that

$$w = \lim_{\tau \to 0} \frac{1}{\tau} \left[ v_{\Omega_{t+\tau}} - v_{\Omega_t} \right]$$

in the $L^p$ sense; then $w$ is in the contingent cone. In particular, by Rem. 1.1.3 in [2], we obtain that the path $\gamma$ is Lipschitz for $t \in [0, T]$.

Unfortunately the contingent cone is not capable of expressing some shape motions.

*Example* 6.25. We consider again the *carving* motion that we saw in 5.2.(5) and in Example 6.23. We define for convenience the functions

$$w_t \stackrel{\text{def}}{=} \frac{v_{A_t} - v_A}{\|v_{A_t} - v_A\|_{L^p}} \quad .$$

These do not admit a limit in $L^p(\mathbb{R}^N)$ when $t \to 0+$.

Suppose by contradiction that $\lim_{t\to 0+} w_t = w$ in $L^p(\mathbb{R}^N)$. Let us fix $r > t > 0$, then for any $x$ outside of $B_r$ we have that $v_{A_t}(x) = v_A(x)$ and consequently $w_t(x) = 0$, hence $w(x) = 0$ for almost any $x$ outside of $B_r$. By arbitrariness of $r$ this would imply that $w = 0$ for almost every $x$. At the same time since $\|w_t\|_{L^p} = 1$ for all $t > 0$ then $\|w\|_{L^p} = 1$; so $w$ cannot exist.

We conclude that the "velocity" of the carving motion does not admit a representation in $T_v\mathcal{N}$ at the time $t = 0$.

## 6.6 Riemannian metric

Let now $p = 2$. The set $\mathcal{N}$ may fail to be a smooth submanifold of $L^2$; yet we will, as much as possible, pretend that it is, in order to induce a sort of "Riemannian metric" on $\mathcal{N}$ from the standard $L^2$ metric.

We define the "Riemannian metric" on $\mathcal{N}$ simply by

$$\langle h, k\rangle \stackrel{\text{def}}{=} \langle h, k\rangle_{L^2}$$

for $h, k \in T_v\mathcal{N}$ and correspondingly a norm by

$$|h| \stackrel{\text{def}}{=} \sqrt{\langle h, h\rangle} \quad .$$

*Remark* 6.26. We also argue that the distance induced by this "Riemannian metric" coincides with the geodesically induced distance $d^g$. Indeed let $\gamma : [a, b] \to M$ be a Lipschitz path in $\mathcal{N}$; by Cor. 2.13 the derivative $\dot\gamma(t)$ exists in $L^2(\mathbb{R}^N)$ for almost all $t$; so we may define the "Riemannian length" of the path as in eqn. (18), namely

$$\operatorname{len}\gamma \stackrel{\text{def}}{=} \int_a^b \|\dot\gamma(\theta)\|_{L^2}\, \mathrm{d}\theta \quad .$$

Then we define the "Riemannian distance" $d^R(x, y)$ as the infimum of $\operatorname{len}\gamma$ for all $\gamma$ connecting $x$ to $y$. But by Theorem 2.11 $\operatorname{len}\gamma = \operatorname{Len}^d\gamma$ and $d^R = d^g$.

### 6.6.1 Riemannian metric for smooth convex sets

We propose an explicit computation of the Riemannian metric. We fix $p = 2$, $N = 2$. Let $\Omega \subseteq \mathbb{R}^2$ be a convex set with smooth boundary of length $L$. Let $y(\theta) : [0, L] \to \partial\Omega$ be a parameterization of the boundary, $\nu(\theta)$ the unit vector normal to $\partial\Omega$ and pointing external to $\Omega$: then the following *"polar"* change of coordinates holds:

$$\psi : \mathbb{R}^+ \times [0, L] \to \mathbb{R} \setminus \Omega \quad , \quad \psi(\rho, \theta) = y(\theta) + \rho\nu(\theta)$$

We suppose that $y(\theta)$ moves on $\partial\Omega$ in anticlockwise direction; so

$$\nu = J\partial_s y \quad , \quad \partial_{ss} y = -\kappa\nu \quad , \quad \partial_s \nu = \kappa\partial_s y$$

where $J$ is the rotation matrix (of angle $-\pi/2$), $\kappa$ is the curvature, and $\partial_s y$ is the tangent vector (obtained by deriving $y$ with respect to arc parameter).

We can then express a generic integral through this change of coordinates as

$$\int_{\mathbb{R}^2\setminus\Omega} f(x)\ \mathrm{d}x = \int_{\mathbb{R}^+}\int_{\partial\Omega} f(\psi(\rho, s))|1 + \rho\kappa(s)|\ \mathrm{d}\rho\,\mathrm{d}s$$

where $s$ is arc parameter, and $\mathrm{d}s$ is integration in arc parameter.

We want to study a smooth deformation of $\Omega$, that we call $\Omega_t$; then the boundary parameterization $y(\theta, t)$ depends on a time parameter $t$. Suppose also that $\kappa(\theta) > 0$, that is, that the set is strictly convex: then for small smooth deformations, the set $\Omega_t$ will still be strictly convex ("small" is intended in the $C^2$ norm). By deriving

$$\partial_t\partial_s y = \partial_s(\partial_t y) - \partial_s y\langle\partial_s y, \partial_s(\partial_t y)\rangle = \pi_\nu(\partial_s(\partial_t y))$$

where

$$\pi_\nu(w) \stackrel{\text{def}}{=} \nu\langle\nu, w\rangle = w - \partial_s y\langle\partial_s y, w\rangle$$

is the projection of $w$ on the line generated by $\nu$. Supposing now that $\rho = \rho(t)$ as well, we can express the point $\psi(\rho, \theta)$ in a first order approximation *wrt* changes in $t, \theta$ as

$$d\psi = \Big(\partial_t y + \rho'\nu + \rho J\pi_\nu(\partial_s(\partial_t y))\Big)dt + \Big(\partial_\theta y + \rho\partial_\theta\nu\Big)d\theta$$

where moreover

$$\Big(\partial_\theta y + \rho\partial_\theta\nu\Big)d\theta = \Big(\partial_s y + \rho\partial_s\nu\Big)ds = \Big(1+\rho\kappa\Big)\partial_s y ds \ .$$

If $y(\theta,t), \rho(t)$ are expressing a constant point $x = \psi(\rho,\theta)$, then $d\psi = 0$; we apply scalar products *w.r.t.* $\nu$ and $\partial_s y$ to the above relations

$$\langle\nu, (\partial_t y)\rangle + \rho' = 0 \quad , \quad \langle\partial_s y, \partial_t y\rangle dt - \rho\langle\nu, \partial_s(\partial_t y)\rangle dt + (1+\rho\kappa)ds = 0 \ .$$

We assume now that each point of $\partial\Omega_t$ moves orthogonally to it; this means that $\partial_t y \perp \partial_s y$; so we can express the motion using a scalar field $\alpha = \alpha(\theta,t) \in \mathbb{R}$, by setting $\partial_t y = \alpha\nu$. So we simplify the above to obtain the relationships

$$\rho' = -\alpha \quad , \quad \frac{ds}{dt} = \frac{\rho\langle\nu, \partial_s(\alpha\nu)\rangle}{(1+\rho\kappa)} = \frac{\rho\ \partial_s\alpha}{(1+\rho\kappa)} \ .$$

Let now

$$h_\alpha(x) \stackrel{\text{def}}{=} \partial_t v_{\Omega_t}(x) \quad ,$$

so $h_\alpha$ is the vector in $T_v\mathcal{N}$ that is associated to the velocity field $\alpha$ that is moving the border of $\Omega$.

Now, for $x \notin \Omega_t$ we can write

$$x = \psi(\rho,\theta) = y(\theta,t) + \rho(t)\nu(\theta,t)$$

so by following the above relations we know that $u_{\Omega_t}(x) = \rho(t)$ hence

$$h_\alpha(x) = -\varphi'(\rho(t))\alpha(\theta,t)$$

whereas $h_\alpha(x) = 0$ for $x \in \mathring{\Omega}_t$ (the topological interior of $\Omega_t$).

We now wish to use the above computation to pull back the "Riemannian Metric" that we presented in the beginning of Sec. 6.6 to the family of orthogonal deformations of $\partial\Omega$. So let us fix two smooth vector fields $\alpha(s)\nu(s)$ and $\beta(s)\nu(s)$, each orthogonal to $\partial\Omega$; these represent two possible infinitesimal deformations of $\partial\Omega$; those correspond to two vectors $h_\alpha, h_\beta \in T_v\mathcal{N}$. By our initial definition

$$\langle h,k\rangle \stackrel{\text{def}}{=} \langle h,k\rangle_{L^2} = \int_{\mathbb{R}^2} h_\alpha(x)h_\beta(x)\,\mathrm{d}x$$

so by pull back we impose that

$$\langle\alpha,\beta\rangle \stackrel{\text{def}}{=} \int_{\mathbb{R}^2} h_\alpha(x)h_\beta(x)\,\mathrm{d}x \ .$$

Using the previous computation we can then expand and obtain that

$$\begin{aligned}\langle\alpha,\beta\rangle &= \int_{\mathbb{R}^2\setminus\Omega} h_\alpha(x)h_\beta(x)\,\mathrm{d}x = \\ &= \int_{\partial\Omega}\left[\int_{\mathbb{R}^+}(\varphi'(\rho))^2(1+\rho\kappa(s))\ d\rho\right]\alpha(s)\beta(s)\,\mathrm{d}s\end{aligned}$$

that is,

$$\langle\alpha,\beta\rangle = \int_{\partial\Omega}(a+b\kappa(s))\alpha(s)\beta(s)\,\mathrm{d}s \tag{53}$$

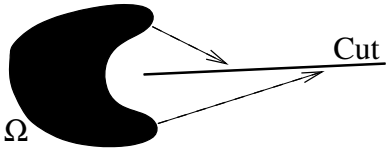
with

$$a = \int_{\mathbb{R}^+}(\varphi'(\rho))^2\ d\rho \quad , \quad b = \int_{\mathbb{R}^+}(\varphi'(\rho))^2\rho\ d\rho \ .$$

### 6.6.2 Riemannian metric for smooth sets

If $\Omega$ is smooth but not convex, then the above formula holds up to the cutlocus. We define a function $R(s) : [0, L] \to \mathbb{R}^+$ that spans the cutlocus, that is,

$$\text{Cut} = \{\psi(R(s), s), s \in [0, L]\} \quad .$$




$\psi$ is a diffeomorphism between the sets

$$\{(\rho, s)\ s \in [0, L], 0 < \rho < R(s)\} \leftrightarrow \mathbb{R}^2 \setminus (\Omega \cup \text{Cut})$$

moreover $R(s)$ is Lipschitz (by results in [10],[13]).

In this case the metric has the form

$$\langle \alpha, \beta \rangle = \int_{\partial\Omega} \left[ \int_0^{R(s)} (\varphi'(\rho))^2 (1 + \rho\kappa(s))\ d\rho \right] \alpha(s)\beta(s)\, \mathrm{d}s \quad . \tag{54}$$

*Remark* 6.27. The above metric (53) is resemblant of the metric presented in [18] for the motion of planar curves, that had though the form

$$\int_{\partial\Omega} (a + b\kappa^2(s))\ k(s) \cdot h(s)\, \mathrm{d}s \quad , \tag{55}$$

where $h(s), k(s)$ are vectors that represent infinitesimal displacements of the curve (not necessarily orthogonal to the curve).

In Sec. 3.6 in [18] it is proved that the completion of the space of smooth curves according to the distance derived from the metric (55) is contained in the space of Lispchitz curves.

Let now $\Xi$ be the family of all connected compact sets in $\mathbb{R}^2$, and $\Xi^\infty$ be the subfamily of all connected compact sets whose boundary is a smooth curve. It is known (see Prop. 5.2) that $\Xi$ is a closed subset of $\mathcal{M}$, according to the Hausdorff distance and it is reasonably easy to show that $\Xi^\infty$ is dense in $\Xi$. Since these are topological results, they hold also for the metrics presented in this paper, by Theorems 6.11 and 6.30.

So there is a fundamental difference between the metric in (53) and (55).

Let us discuss intuitively what happens when a family of sets $(A_n)_n \subset \Xi^\infty$ approximates a generic connected compact set.

- On one hand, when the set $A_n$ is not convex the metric (53) is substituted by the metric (54), where the cutlocus plays an important part in reducing the cost of moving the boundary.
- But, even more importantly, the term $\kappa$ in (54) allows for the formation of kinks in the boundaries of $A_n$ so that $A_n$ can approximate $A$, whereas the term $\kappa^2$ in (55) is stronger and the boundaries are not allowed to form singularities.

## 6.7 Bounds on $d^g$

We can propose a converse of Prop. 6.19 when additional assumptions hold. In this section we will assume that 6.17 holds, and also that $\varphi$ is convex (for simplicity).

**Lemma 6.28.** *We note that*

$$|\nabla v_A(x)| = |\varphi'(u_A(x))| \tag{56}$$

*holds for almost all $x \notin A$; indeed we remarked in Sec. 4 that $u_A$ is Lipschitz (hence almost everywhere differentiable) and $|\nabla u_A(x)| = 1$ almost everywhere.*

**Lemma 6.29.** *Let $p \in [1,\infty)$. Assume that 6.17 holds and that $\varphi$ is convex. Let $r > 0$ and*

$$c^1_{p,\varphi}(r) \stackrel{def}{=} \sup_{A \subset D_r} \|\nabla v_A\|_{L^p} \ . \tag{57}$$

*Then*

$$c^1_{p,\varphi}(r)^p = \omega_N r^N |\varphi'(0)|^p + \omega_N N \int_r^\infty t^{N-1} |\varphi'(t-r)|^p \, \mathrm{d}t \ . \tag{58}$$

*Proof.* By convexity $\varphi'$ is monotonically non decreasing and $\varphi'(0) \leq \varphi'(t) < 0$. Let $A \subset D_r$ compact, then for $x \in D_r$, if $u_A$ is differentiable at $x$, we have

$$|\nabla v_A(x)| = |\varphi'(u_A(x))||\nabla u_A(x)| \leq |\varphi'(0)| \ \ ;$$

whereas for $x \notin D_r$

$$u_A(x) \geq u_{D_r}(x) = |x| - r \Longrightarrow \varphi'(u_A(x)) \geq \varphi'(|x| - r) \Longrightarrow |\varphi'(u_A(x))| \leq |\varphi'(|x| - r)| \ .$$

Equality is obtained by choosing $A = A_\varepsilon$ to be finite collections of points that are $\varepsilon$–nets in $D_r$ (*i.e.* $A_\varepsilon + D_\varepsilon \supseteq D_r$) and letting $\varepsilon \to 0$. □

**Theorem 6.30.** *Let $p \in (1,\infty)$. Assume again that 6.17 holds and that $\varphi$ is convex. Then for any continuous path $\gamma$ and any $r > 0$ such that $\forall t, \gamma(t) \subset D_r$ we have*

$$\mathrm{Len}^d(\gamma) \leq c^1_{p,\varphi}(r) \ \mathrm{Len}^{d_H}(\gamma) \ \ , \tag{59}$$

*and then*

$$\forall A, B \subset D_r, \quad d^g(A,B) \leq c^1_{p,\varphi}(2r) \ d_H(A,B) \ \ .$$

As a corollary, the topology induced by $d^g$ on $\mathcal{M}$ coincides with the topology induced by $d$ and by $d_H$. Note though the intrinsic distances $d_H$ and $d^g$ are not equivalent, see Remark 6.31 below.

*Proof.* Let $\gamma$ be a path as above. Up to reparametrization 2.3 we assume that $\gamma : [0,l] \to \mathcal{M}$ with $l = \mathrm{Len}^{d_H} \gamma$ and that the metric derivative is $|\dot\gamma| \equiv 1$: hence

$$d_H(\gamma(t), \gamma(s)) \leq |t - s|$$

that means that

$$\forall x, |u_{\gamma(t)}(x) - u_{\gamma(s)}(x)| \leq |t - s|$$

so $u_{\gamma(t)}(x)$ is jointly Lipshitz continuous and whenever it is differentiable we have that

$$\left| \frac{\partial}{\partial t} u_{\gamma(t)}(x) \right| \leq 1$$

and then

$$\left| \frac{\partial}{\partial t} v_{\gamma(t)}(x) \right| \leq |\varphi'(u_{\gamma(t)}(x))| \left| \frac{\partial}{\partial t} u_{\gamma(t)}(x) \right| \ \ .$$

By Thm. 2.11 and Prop. 6.21 and eqn. (48)

$$\mathrm{Len}^d \gamma = \int_0^l \| \frac{\partial}{\partial t} v_{\gamma(t)} \|_{L^p} \, \mathrm{d}t$$

and we use the previous Lemma and (56).

For the second inequality, let $A, B \in \mathcal{M}$ and $D_r$ as above, we use Thm. 5.1 to obtain a geodesic for the Hausdorff metric connecting $A$ to $B$; then, by triangle inequality, $\gamma(t) \subset D_{2r}$ for all $t$; and again apply the same reasoning. □

*Remark* 6.31. The example 6.23 shows that there is no constant $c$ such that

$$\mathrm{Len}^d(\gamma) \geq c \ \mathrm{Len}^{d_H}(\gamma)$$

(*i.e.* the reverse of eqn. (59) does not hold). Indeed in that example we noted in eqn. (50) that $\mathrm{Len}^d \gamma|_{[0,r]} \sim r^{1+N/p}$ but $\mathrm{Len}^{d_H} \gamma|_{[0,r]} = r$ for $r$ small.

By the same equation (50) we also obtain that $d^g(A, A_r) \leq c r^{1+N/p}$ for $r > 0$ small and $c > 0$ a constant depending on $\varphi$. Indeed we recall the definition eqn. (11) and remark that the path $\gamma$ is one of the possible paths that connect $A$ to $A_r$, so $d^g(A, A_r) \leq \mathrm{Len}^d \gamma|_{[0,r]}$. At the same time $d^g(A, A_r) = r$. This shows that the intrinsic distances $d_H$ and $d^g$ are not equivalent.

## 6.8 Numerical Approximation

In this section we explain a simple method to numerically approximate the geodesic between two given compact sets $A, B$. We assume that $p \in (1, \infty)$. Two examples of geodesics computed with this method (choosing $p = 2, \varphi(t) = \exp(-t)$) are in Fig. 3 on the following page and in Fig. 4 on page 33.

**Definition 6.32** (Cube)**.** We let $Q = [-1, 1]^N$ be the closed cube of center in the origin and side equal to 2. Let $Q(x, r) \stackrel{\text{def}}{=} x + rQ$ the cube of center $x$ and side $2r > 0$.

**Definition 6.33** (Discretization grids)**.** Let us fix $n_t, n_s$ large and define $\delta_s > 0, \delta_t = 1/n_t$ small (the "thinness" parameters); consider the following equispaced partitions

$$R_{\delta_s, n_s} \stackrel{\text{def}}{=} \{i\delta_s : i = -n_s, \dots, n_s\}^N \subseteq \mathbb{R}^N \quad , \tag{60}$$

$$T_{n_t} \stackrel{\text{def}}{=} \{i\delta_t : i = 0, \dots, n_t\} = \{0, \delta_t, 2\delta_t, \dots 1\} \subseteq [0, 1] \quad ; \tag{61}$$

for simplicity in the following we call $T = T_{n_t}$ the *time grid* and $R = R_{\delta_s, n_s}$ the *space grid*.

Note that $R \subset Q(0, \delta_s n_s)$.

**Definition 6.34** (Pixelization)**.** Given $A$ closed, the *pixelization* of $A$ to $R$ is

$$\Pi_R(A) \stackrel{\text{def}}{=} \{x \in R : A \cap Q(x, \delta_s/2) \neq \emptyset\} \ .$$

Note that $\Pi_R \circ \Pi_R = \Pi_R$. Note also that $\Pi_R$ is not continuous.[10]

**Definition 6.35.** We also define the *discretized (pseudo) distance*

$$d_R(A, B) \stackrel{\text{def}}{=} \left[ \delta_s^N \sum_{x \in R} |v_A(x) - v_B(x)|^p \right]^{1/p} \tag{62}$$

and the *time–discretized length* of a path

$$\text{len}_T^d(C) \stackrel{\text{def}}{=} \sum_{t \in T, t < 1} d\big(C(t), C(t + \delta_t)\big) \quad . \tag{63}$$

Combining the two we obtain a time-and-space–discretized length $\text{len}_T^{d_R}(C)$.

*Remark* 6.36. We called $d_R$ a *pseudo distance* since it is not guaranteed that $d_R(A, B) = 0 \implies A = B$. This can be seen setting $\delta_s < 1/2$, $A = Q(0, 1)$ and $B = A \setminus E$ where $E = B(e_1 \delta_s/4, \delta_s/8)$ is a small open ball contained in $A$ and that does not intersect $R$. ($e_1$ is the first vector of the canonical basis.) At the same time though $d_R$ is symmetric and satifies the triangle equation.

### 6.8.1 Finding numerical geodesics

We fix $A, B \subseteq \mathbb{R}^N$ compact. We assume that $p \in (1, \infty)$. We assume that $n_s$ is large so that

$$A \subseteq Q(0, n_s\delta_s) \quad , \quad B \subseteq Q(0, n_s\delta_s) \quad .$$

We let $T = T_{n_t}$ and $R = R_{\delta_s, n_s}$. We define $\mathcal{C}(T, R)$ the space of all the *discretized paths* $C : T \to \mathcal{P}(R)$ such that

$$C(0) = \Pi_R(A) \quad , \quad C(1) = \Pi_R(B) \quad .$$

To find a numerical approximation to the geodesic connecting $A$ to $B$, we solve the problem

$$d_{T,R}^g(A, B) \stackrel{\text{def}}{=} \min_{C \in \mathcal{C}(T,R)} \text{len}_T^{d_R}(C) \ . \tag{64}$$

[10] As map from $(\mathcal{M}, d_H)$ into itself, where again $\mathcal{M}$ is the family of the nonempty compact sets in $\mathbb{R}^N$ and $d_H$ is the Hausdorff distance.

The complexity of the minimization problem is exponential, [11] thus we reduce it using an iterative method. To this end we define

$$P^+_{C(t)} = \{x \in R : x \notin C(t), \mathcal{B}(x) \cap C(t) \neq \emptyset\} \tag{65}$$

$$P^-_{C(t)} = \{x \in R : x \in C(t), \mathcal{B}(x) \setminus C(t) \neq \emptyset\} \tag{66}$$

where $\mathcal{B}(x)$ are the $(2N)$ points (at most) that are nearest neighbours to $x$ in the grid $R$. We notice that for any $t \in T$ both $P^+_{C(t)}$ and $P^-_{C(t)}$ are discretized version of the boundary of $C(t)$, the first one from the outside and second from the inside.

For any $t \in T, t \neq 0, 1$ and $x \in R$ we define the one–point–variation

$$V_{x,t} : \mathcal{C}(T, R) \to \mathcal{C}(T, R)$$

by

$$(V_{x,t}C)(t') = \begin{cases} \{x\} \Delta C(t') & \text{if } t = t' \\ C(t') & \text{otherwise} \end{cases} \tag{67}$$

where $\Delta$ is the set symmetric difference.

Let $C_0 \in \mathcal{C}(T, R)$ be a starting path; in our experiment we defined it by level set of the linear interpolation of signed distance functions (2), that is

$$C_0(t) = \{x \in R : tb_B(x) + (1-t)b_A(x) \leq 0\} \quad .$$

We then evolve it in such a way that at any step $n \in \mathbb{N}$ we decrease the quantity $\mathrm{len}^{d_R}_T(C_n)$. Let

$$(\hat{x}, \hat{t}) = \mathrm{argmin}_{(x,t) \in P^+_{C_s} \cup P^-_{C_s}} \mathrm{len}^{d_R}_T \left(V_{x,t} C_n\right) \tag{68}$$

then we define $C_{n+1} = V_{\hat{x},\hat{t}} C_n$.

We tested this algorithm on some simple shapes; two examples of geodesics (choosing $p = 2, \varphi(t) = \exp(-t)$) are in Fig. 3 and in Fig. 4 on the next page. To produce the examples below, we iterated the above algorithm until the energy seemed to stabilize.

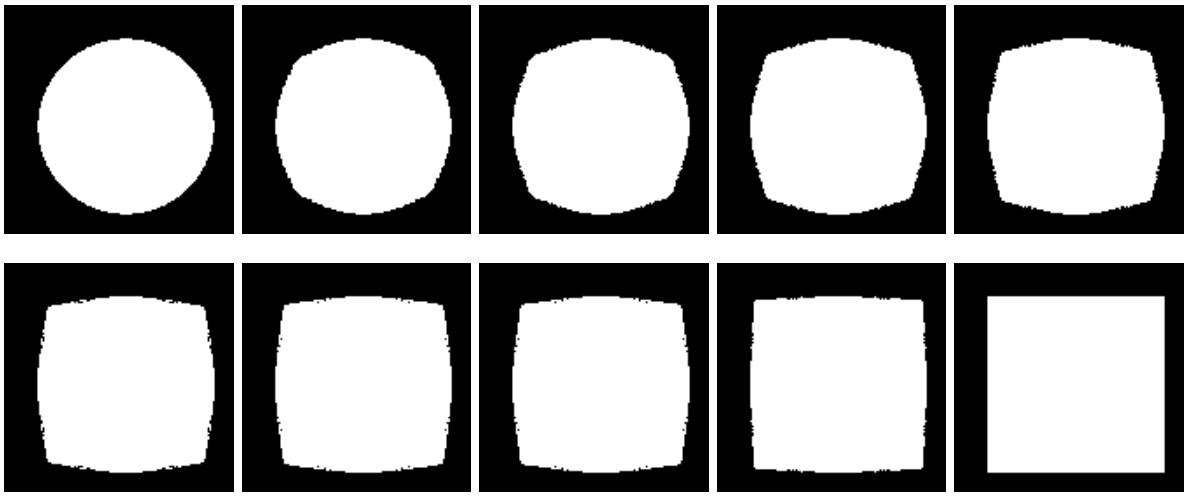


Figure 3: Example of geodesics connecting a disk and a square.

*Remark* 6.37. This numerical method is presented only for the sake of exemplification. No study of the actual convergence of this algorithm has been performed (yet). The approximation step may be ameliorated, in its current form it does not explore carving motions (altough it allows for changes in topology).

[11] There are indeed $2^{(2n_s+1)^N (n_t-1)}$ elements in $\mathcal{C}(T, R)$.

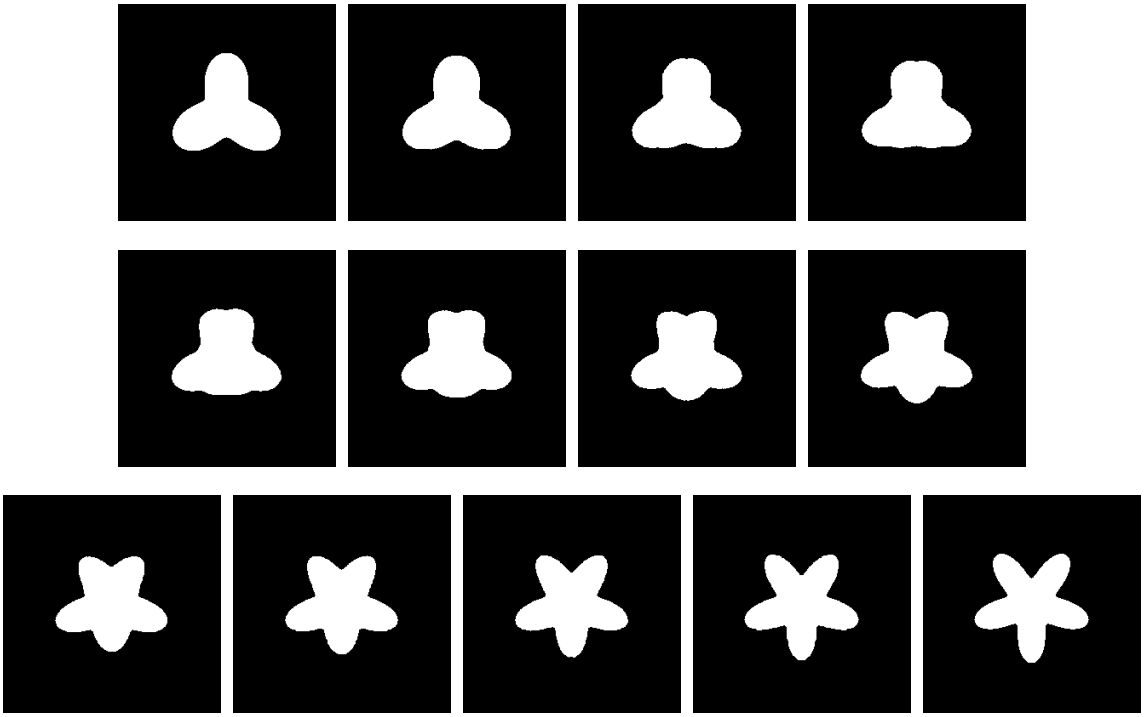


Figure 4: Example of a geodesic connecting non-convex sets

# 7 Other Banach–like metrics of shapes

The paradigm that we presented in the previous section may be exploited in other similar ways; to conclude the paper, we shortly present some different embeddings (leaving to a possible future paper the detailed study of their properties).

## 7.1 Signed distance based representation

We may use the signed distance function $b_A$, that was defined in (2), to define a metric of shapes:

$$d'(A,B) \stackrel{\text{def}}{=} \|\varphi(b_A) - \varphi(b_B)\|_{L^p(\mathbb{R}^N)}$$

in this case, we require that the function $\varphi : \mathbb{R} \to (0,\infty)$ is monotonically decreasing and of class $C^1$ and such that

$$\varphi(|x| - t) \in L^p(\mathbb{R}^N) \quad \forall t. \tag{69}$$

The resulting metric is slightly stronger than the one we studied in the preceding sections; in particular,

*Remark* 7.1. Let $\mathcal{F}$ be the class of all finite subsets of $\mathbb{R}^N$; this class is dense in $\mathcal{M}$ when we use the metric $d_{p,\varphi}$, or the Hausdorff metric; but it is not dense when we use the metric $d'$.

## 7.2 $W^{1,p}$ metrics

Another interesting choice of metric is obtained by embedding the representation in $W^{1,p}$, for $p \in (1,\infty)$

We require that all hypotheses in 6.1 and 6.17 hold. Namely, $\varphi : [0,\infty) \to (0,\infty)$ is Lipschitz, $C^1$ and monotonically decreasing, and $\varphi(|x|) \in W^{1,p}(\mathbb{R}^N)$; for the case $p < \infty$ we are equivalently asking that

$$\int_0^\infty t^{N-1}(\varphi(t)^p + |\varphi'(t)|^p)\, dt < \infty$$

and this implies that $\lim_{t\to\infty} \varphi(t) = 0 = \lim_{t\to\infty} \varphi'(t)$. We also assume that there is a $T > 0$ s.t. $\varphi(t)$ is convex for $t \in [T,\infty)$.

**Proposition 7.2.** *For any $A$ compact we have $v_A \in W^{1,p}(\mathbb{R}^N)$.*

*Proof.* We already know by Lemma 6.2 that $v_A \in L^p(\mathbb{R}^N)$.

By hypotheses above, $v_A$ is Lipschitz; and then, for almost all $x$, $\nabla v_A = \varphi'(u_A)\nabla u_A$; where $|\nabla u_A| = 1$ for almost all $x \notin A$, while $\nabla u_A = 0$ for almost all $x \in A$. We also know that when $t > T$, $\varphi'(t) < 0$, $\varphi'$ is increasing and $\varphi'(t) \uparrow 0$.

Let $R > 0$ be large so that $A \subseteq B_R$, then

$$u_A(x) \geq |x| - R$$

and then when $|x| \geq R + T$ we obtain that

$$\varphi'(u_A(x)) \geq \varphi'(|x| - R)$$

that is

$$\int_{\mathbb{R}^N \setminus B_{R+T}} |\varphi'(u_A(x))|^p \ \mathrm{d}x \leq \int_{\mathbb{R}^N \setminus B_{R+T}} |\varphi'(|x| - R)|^p \ \mathrm{d}x < \infty \quad .$$

At the same time, since $v_A$ is Lipschitz, then $\int_{B_{R+T}} |\nabla v_A| \, \mathrm{d}x$ is finite. □

**Definition 7.3.** Given $A, B \in \mathcal{M}$, we define

$$d_{1,p,\varphi}(A, B) \stackrel{\text{def}}{=} \|\varphi(u_A) - \varphi(u_B)\|_{W^{1,p}(\mathbb{R}^N)}$$

We just state a simple property of this metric.

**Proposition 7.4.** *Let again $\mathcal{F}$ be the class of all finite subsets of $\mathbb{R}^N$: this class is dense in $\mathcal{M}$ if and only if $\varphi'(0) = 0$.*

The proof is in Sec. A.9.

We just conclude with one last remark.

*Remark* 7.5. The embedding of $\varphi \circ u_A$ in $W^{2,p}$ is not feasible: if $A$ is smooth but is not convex, the second derivative of $u_A$ along the cutlocus is expressed by a measure (see 4.13 in [14]) and then $\varphi \circ u_A \notin W^{2,p}$.

# 8 On the choice of hypotheses

Using arguments in Geometric Measure Theory, it is possible to prove this result.

**Proposition 8.1.** *Let $f : [0, \infty) \to [0, \infty)$ be a measurable function such that $\int_0^\infty f(t)^p t^{N-1} \, \mathrm{d}t$. Suppose that $A \subset \mathbb{R}^N$ is compact, let $u_A$ be its distance function. Then $f \circ u_A \in L^p(\mathbb{R}^N)$.*

Using this result, it is possible to prove Prop. 6.18 and Prop. 7.2 without using the *"convexity"* hypothesis listed in 6.17; hence this hypothesis may be dropped in many other results. Similarly it is possible to prove Lemma 6.2 without using the fact that $\varphi$ is "strictly decreasing" (as listed in 6.1). The above Proposition 8.1 though requires a long proof; and relaxing requirements in 6.1 and 6.17 would require longer and more complex proofs in many other propositions. At the same time these generalizations would not improve the usefullness of this theory in applications. So we decided to omit them.

# 9 Conclusions

We have studied a metric space of shapes $(\mathcal{M}, d_{p,\varphi})$; this space has a "weak distance", in that it has many compact sets, and geodesics do exist; but it can be associated in some cases to a smooth Riemannian metric, as we saw in eqn. (53). Moreover, by the properties that we saw in sec. 2.2 (and in particular, by the properties of $L^p$ spaces for $p \in (1, \infty)$ that we proved in Thm. 2.15) we can also hope that geodesics can be studied in the O.D.E. sense (although possibly in a very weak sense).

# 10 Acknowledgments

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# A Proofs

## A.1 Proof of eqn. (34)

**Lemma A.1.** *Let $p \in [1,\infty)$. Suppose that $f, g \in L^p(\mathbb{R}^N)$; let $\tau \in \mathbb{R}^N$ and define the translates $g_\tau(x) = g(x-\tau)$; let moreover $\sigma \in O(N)$ be an orthogonal transformation and $f_\sigma(x) = f(\sigma(x))$ be a rotation of $f$; then*

$$\lim_{|\tau|\to\infty} \|f_\sigma - g_\tau\|_{L^p} = \sqrt[p]{\|f\|_{L^p}^p + \|g\|_{L^p}^p} \tag{70}$$

*where the limit is uniform in $\sigma$.*

*Proof.* The result is obviously true if $f, g \in C_c(\mathbb{R}^N)$. We will prove that the set of $f, g$ such that eqn. (70) holds is closed; since $C_c$ is dense in $L^p$, this will prove QED. Choose sequences $(f_n)_n, (g_n)_n \subset L^p$ such that $f_n \to f, g_n \to g$ in $L^p(\mathbb{R}^N)$; define the translates $g_{n,\tau}(x) = g_n(x-\tau)$, and the rotated versions $\hat{f}_n(x) = f_n(\sigma(x))$ where $\sigma \in O(N)$; suppose moreover that $f_n, g_n$ satisfy eqn. (70) that is

$$\lim_{|\tau|\to\infty} \|\hat{f}_n - g_{n,\tau}\|_{L^p} = \sqrt[p]{\|f_n\|_{L^p}^p + \|g_n\|_{L^p}^p} \quad , \tag{71}$$

where the limit, for each fixed $n$, is uniform *wrt* the choice of $\sigma \in O(N)$. We estimate

$$\begin{aligned} &\Big| \|f_\sigma - g_\tau\|_{L^p} - \|\hat{f}_n - g_{n,\tau}\|_{L^p} \Big| \le \|f_\sigma - g_\tau - \hat{f}_n + g_{n,\tau}\|_{L^p} \le \\ &\le \|f_\sigma - \hat{f}_n\|_{L^p} + \|g_\tau - g_{n,\tau}\|_{L^p} = \|f - f_n\|_{L^p} + \|g - g_n\|_{L^p} \end{aligned}$$

(the last equality derives from Euclidean invariance of the Lebesgue measure). This proves that the term $\|\hat{f}_n - g_{n,\tau}\|_{L^p}$ converges to $\|f_\sigma - g_\tau\|_{L^p}$ as $n \to \infty$ and uniformly *w.r.t.* $\tau$ and $\sigma$. Passing to limits in eqn. (71) on the LHS we can write

$$\lim_{n\to\infty} \lim_{|\tau|\to\infty} \|\hat{f}_n - g_{n,\tau}\|_{L^p} = \lim_{|\tau|\to\infty} \lim_{n\to\infty} \|\hat{f}_n - g_{n,\tau}\|_{L^p} = \lim_{|\tau|\to\infty} \|f_\sigma - g_\tau\|_{L^p} \quad ,$$

whereas clearly the RHS of eqn. (71) converges to the RHS of eqn. (70). □

## A.2 Proof of 2.8

*Proof.* Note first that the infimum of $\tau(a)$ is finite, since it does not exceed $\rho^*$. Recall that

$$d^g(a, a_j) = \inf_{\gamma_j} l_j$$

where $l_j$ is the length of a Lipschitz path $\gamma_j$ connecting $a, a_j$. So we can rewrite the problem (13) as

$$\inf_{\gamma_1 \dots \gamma_n} \theta(\gamma_1 \dots \gamma_n), \quad \text{where } \theta(\gamma_1 \dots \gamma_n) \stackrel{\text{def}}{=} \sum_{j=1}^{n} (l_j)^2 \quad ,$$

where the infimum is computed on all choices of Lipschitz paths $\gamma_1 \dots \gamma_n$ of length $l_1 \dots l_n$ connecting $a_i$ to a common point $x \in M$; for simplicity we represent them as $\gamma_i : [0, l_i] \to M$ parameterized by arc parameter. By the triangle inequality

$$d^g(a_i, \gamma_j(t)) \le d^g(a_i, x) + d^g(x, \gamma_j(t)) \le l_i + l_j \quad .$$

Let then $\gamma_{i,k}$ be a sequence of choices that converges to the infimum:

$$\theta(\gamma_{1,k} \dots \gamma_{n,k}) \to_k \inf_{\gamma_1 \dots \gamma_n} \theta(\gamma_1 \dots \gamma_n)$$

so for large $k$,

$$\theta(\gamma_{1,k} \dots \gamma_{n,k}) \leq \rho^* + \varepsilon \quad ,$$

but then in particular $l_{i,k} \leq \sqrt{\rho^* + \varepsilon}$. Hence

$$d^g(a_1, \gamma_{j,k}(t)) \leq 2\sqrt{\rho^* + \varepsilon}$$

for all $j = 1, \dots n$ and $t \in [0, l_{j,k}]$. So all the paths are contained in a compact set. By Ascoli–Arzelà theorem, we can then extract a uniformly convergent subsequence and use the fact that the length is lower semi continuous. □

## A.3 Proof of 5.3

*Proof.*

- Obviously $C_t$ is compact and $C_0 = A, C_\mu = B$.
- We prove that $C_t$ is not empty. Let $z \in B$; if $u_A(z) = 0$ then $z \in A$ so $z \in C_t$. If $u_A(z) > 0$, let $x \in A$ be a projection point of $z$ so that $u_A(z) = |x - z| = l > 0$, let

  $$y = x + \min\{t, l\} \frac{(z - x)}{|z - x|} \quad .$$

  Obviously $u_A(y) \leq |x - y| \leq t$. If $t \geq l$ then $y = z$ so $u_B(y) = 0$ and $y \in C_t$. If $t < l$ then $|z - y| = l - t \leq \mu - t$ so $u_B(y) \leq \mu - t$ and again $y \in C_t$.
- We prove that for all $0 \leq s < t \leq \mu$

  $$d_H(C_s, C_t) \leq t - s \quad .$$

  The figure 5 may help in reading the following step. [12]

  Let $z \in C_t$, we will prove that $u_{C_s}(z) \leq t - s$. Since $u_A(z) \leq t$, there is a $x \in A$ *s.t.* $|x - z| \leq t$; let $y$ be the interpolated point

  $$y = x + \frac{s\,(z - x)}{|z - x|}$$

  so that $|y - x| = s$ and $|z - y| = |x - z| - s \leq t - s$ and then $u_A(y) \leq s$. Since $z \in C_t$ then $u_B(z) \leq \mu - t$, also $|z - y| \leq t - s$ so by triangle inequality $u_B(y) \leq \mu - s$; we already noted $u_A(y) \leq s$, so we proved that $y \in C_s$; eventually $u_{C_s}(z) \leq |z - y| \leq t - s$.

  Working symmetrically we can prove that for any $z \in C_s$ we have that $u_{C_t}(z) \leq t - s$.
- We prove that for all $0 \leq s < t \leq \mu$

  $$d_H(C_s, C_t) = t - s \quad ;$$

  indeed

  $$\mu = d_H(A, B) \leq d_H(A, C_s) + d_H(C_s, C_t) + d_H(C_t, B) \leq \mu \quad ,$$

  but then the inequalities must be equalities.
- $d_H(A, \gamma(s)) \leq s$ implies that for all $x \in \gamma(s)$, $u_A(x) \leq s$; and similarly for $d_H(\gamma(s), B) \leq \mu - s$; so $x \in C_s$.

□

[12] Note that the drawings of $C_s$ and $C_t$ in the figure are not faithful to the actual $C_s$ and $C_t$ — the maximal geodesic is much larger than that, and corners are rounded out. Exact examples of maximal geodesics are in Sec. 5.1

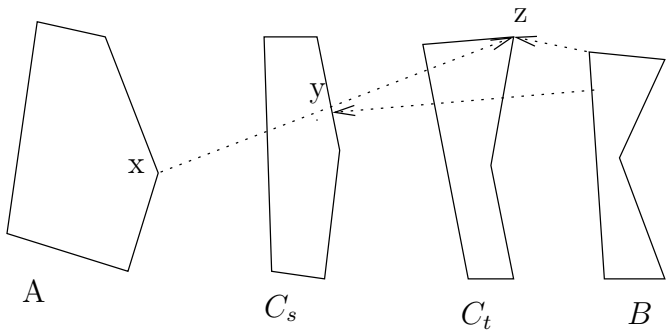



Figure 5: Artistic depiction to guide into the proof of Prop. 5.3.

## A.4 Proof of 5.9

*Proof.* We provide a detailed proof for convenience of the reader.

- We prove that $\max_A u_B(x) = \max_E u_B(x)$. We foremost note that $\max_A u_B(x) \geq \max_E u_B(x)$ since $E \subseteq A$. From $G \cap \{u_B = \theta\} = \emptyset$ we obtain $A \cap \{u_B = \theta\} = E \cap \{u_B = \theta\}$, so we conclude that $\max_A u_B(x) = \max_E u_B(x)$.
- We prove that $\max_B u_A(x) = \max_B u_E(x)$ by proving that $u_A(z) = u_E(z) \forall z \in B$. We foremost note that $u_A \leq u_E$. Let $z \in B$. We have that $z \in A$ *iff* $z \in E$, in that case $u_A(z) = u_E(z) = 0$. Otherwise let $x \in A$ be a projection point; necessarily $x$ is a boundary point of $A$, so $x \in E$, so $u_A(z) \geq u_E(z)$, hence they are equal.

□

## A.5 Proof of 5.10

*Proof.* Let $\theta_B = \max_{C_t} u_B(x)$, $\theta_A = \max_{C_t} u_A(x)$, for any non-empty open $G$ contained in $C_t \setminus (A \cup B)$ and such that $G \cap \{u_B = \theta_B\} = \emptyset$ and $G \cap \{u_A = \theta_A\} = \emptyset$ we set $E = C_t \setminus G$, by Lemma 5.9

$$d_H(A, E) = t \quad , \quad d_H(E, B) = \mu - t \quad ;$$

so we can build a geodesic from $A$ to $B$ that passes through $E$. □

## A.6 Proof of 5.19

*Proof.* We set $u_n \overset{\text{def}}{=} u_{\Omega_n}$; since $u_n$ is 1-Lipschitz, that is

$$\forall x, y \in \mathbb{R}^N \quad |u_n(x) - u_n(y)| \leq |x - y| \quad , \tag{72}$$

passing to the limit in the above (72), we obtain

$$\forall x, y \in D \quad |f(x) - f(y)| \leq |x - y| \quad . \tag{73}$$

So there is a unique extension of $f$ to a positive function $g : \mathbb{R}^N \to \mathbb{R}$ that is again 1-Lipschitz, that is,

$$\forall x, y \in \mathbb{R}^N \quad |g(x) - g(y)| \leq |x - y| \quad . \tag{74}$$

It is easy to prove that $u_n(x) \to g(x)$ for all $x$; then (by imitating the proof of Ascoli–Arzelà theorem) we prove that $u_n \to g$ uniformly on compact sets.

Let $\Omega = \{g = 0\}$; to prove that $\Omega$ is nonempty, let $x_n$ be such that $u_n(x_n) = 0$; let $y \in D$, then $u_n(y)$ is a bounded sequence, hence $x_n$ is bounded, since $|y - x_n| \leq u_n(y)$; so up to a subsequence, $x_n$ converges to a point $x$ such that $g(x) = 0$.

To conclude the proof, we need to prove that $g = u_\Omega$. To this end, we first prove that $g \geq u_\Omega$: indeed, fixing $x$, $u_n(x) = |x - y_n|$ for at least one point $y_n \in \Omega_n$; since $u_n(x) \to g(x)$, then the sequence $\{y_n\}$ is

bounded, so (up to a subsequence $n_k$) it converges to a point $y$; since the family $u_n$ is 1-Lipschitz and $u_n(y_n) = 0$ then $g(y) = 0$, that is $y \in \Omega$: hence

$$g(x) = \lim_k u_{n_k}(x) = \lim_k |y_{n_k} - x| = |y - x| \geq u_\Omega(x) \ .$$

Conversely, let $y \in \Omega$ be such that $u_\Omega(x) = |x - y|$; then by (74) $g(x) \leq g(y) + |x - y| = |x - y| = u_\Omega(x)$. □

## A.7 Proof of 6.18

*Proof.* For $t \in [0, 1]$, let $\gamma(t) = t\Omega$ be the path that connects the singleton $\{0\}$ to $\Omega$ by rescaling; we prove that $\gamma$ is Lipschitz.

Let $R > 0$ be such that $\Omega \subseteq D_R$, where $D_R$ is the disk centered at zero (see eqn. (21)).

Since $\varphi(x)$ is convex for $x$ large and $\lim_{x\to\infty} \varphi'(x) = 0$ then $\varphi$ is Lipschitz. Let $V$ be the Lipschitz constant of $\varphi$.

Let for convenience $f(t, x) = u_{t\Omega}(x)$.

It is not difficult to prove that the map $f(t, x)$ is jointly Lipschitz. Let $F$ be the Lipschitz constant.

This proves the result when $p = \infty$, indeed

$$d_{\infty,\varphi}(s\Omega, t\Omega) \leq FV|t - s| \quad .$$

When $p < \infty$, we proceed as follows.

As a first step we study the pointwise time derivative of $u_{t\Omega}(x)$. By Rademacher's Theorem $u_{t\Omega}(x)$ is differentiable at almost all $t, x$. Fix such a $t, x$; note that

$$u_{t\Omega}(x) = t u_\Omega\left(\frac{x}{t}\right) \tag{75}$$

(as in eqn. (23)); hence, taking derivatives *w.r.t.* $x$ we obtain

$$\nabla u_{t\Omega}(x) = \nabla u_\Omega\left(\frac{x}{t}\right)$$

while taking derivatives *w.r.t.* $t$ we obtain

$$\partial_t u_{t\Omega}(x) = u_\Omega\left(\frac{x}{t}\right) - \frac{1}{t}\langle \nabla u_\Omega\left(\frac{x}{t}\right) \cdot x\rangle = \frac{1}{t}\left(u_{t\Omega}(x) - \langle \nabla u_{t\Omega}(x) \cdot x\rangle\right) \ .$$

Suppose moreover that $x \notin t\Omega$ and let $y \in t\Omega$ be the minimum distance point from $x$: then (as remarked in Sec. 4)

$$u_{t\Omega}(x) = |x - y| \quad , \quad \nabla u_{t\Omega}(x) = \frac{x - y}{|x - y|}$$

so

$$\begin{aligned} \partial_t u_{t\Omega}(x) &= \frac{1}{t}\left(|x - y| - \left\langle \frac{x - y}{|x - y|} \cdot x \right\rangle\right) = \\ &= -\frac{1}{t|x - y|}\langle x - y \cdot y\rangle = -\langle \frac{x - y}{|x - y|} \cdot \frac{y}{t}\rangle \end{aligned} \tag{76}$$

so if $\Omega \subseteq D_R$ we obtain that $|\partial_t u_{t\Omega}(x)| \leq R$.

If instead $x \in t\Omega$ and $u_{t\Omega}(x)$ is differentiable at $(t, x)$ then $\nabla u_{t\Omega}(x) = 0$ and $\partial_t u_{t\Omega}(x) = 0$; indeed $u_{t\Omega}(x) = 0$ and $u \geq 0$ everywhere.

As a second step we remark that the time derivative of $v_{t\Omega}(x)$ exists as a strong limit in $L^p$. For the case $p > 1$ this may follow from Prop. 6.21. Since this would not cover the case $p = 1$, we provide a direct proof, that is based on the above computation, and on Lebesgue dominated convergence theorem. Indeed

$$\left|\frac{v_{t\Omega}(x) - v_{s\Omega}(x)}{t - s}\right| = |\varphi'(\xi)| \left|\frac{f(t, x) - f(s, x)}{t - s}\right| \leq |\varphi'(\xi)|F \quad ,$$

where $\xi = \xi(t,x)$ is a value intermediate between $f(t,x)$ and $f(s,x)$. Clearly $\xi \geq (|x| - R)^+$ hence

$$|\varphi'(\xi)| \leq \left|\varphi'\Big((|x| - R)^+\Big)\right|$$

for $|x|$ large; so by eqn. (45) we obtain that $|\varphi'(\xi)| \in L^p(\mathbb{R}^N)$. (See the similar proof of Prop. 7.2 for more details). To conclude we compute

$$\|\partial_t v_{t\Omega}(x)\|_{L^p}^p = \int_{\mathbb{R}^N} |\varphi'(u_{t\Omega}(x))|^p |\partial_t u_{t\Omega}(x)|^p \ \mathrm{d}x \leq R^p \int_{\mathbb{R}^N} |\varphi'(u_{t\Omega}(x))|^p \, \mathrm{d}x \quad , \tag{77}$$

and we argue as above to state that this quantity is finite, and bounded uniformly in $t$. By Rem. 1.1.3 in [2], we conclude that $\gamma$ is Lipschitz. □

*Remark* A.2. Asking that $\varphi$ satisfy both (29) and (44) is equivalent to asking that $\varphi(|x|) \in W^{1,p}$. By using the equality in (77) and in (76), it is possible to show that, for most compact sets, the rescaling is a Lipschitz path if and only if $\varphi(|x|) \in W^{1,p}$. [13]

## A.8 Proof of 6.21

*Proof.* • We extend $f(t,x) = f(1,x)$ for $t > 1$ and $f(t,x) = f(0,x)$ for $t < 0$. Note that the extended map is still a Lipschitz map $t \mapsto f(t,\cdot)$ with values in $L^p(\mathbb{R}^N)$; let $c$ be its Lipschitz constant. We define

$$g_\tau(t,x) \stackrel{\text{def}}{=} \frac{f(t+\tau,x) - f(t,x)}{\tau}$$

so

$$\|g_\tau(t,x)\|_{L^p(\mathbb{R}^N)} \leq c \quad .$$

Hence

$$\int_0^1 \int_{\mathbb{R}^N} |g_\tau(t,x)|^p \ \mathrm{d}x \, \mathrm{d}t \leq c^p \quad .$$

This means that the family $g_\tau$ is bounded in $L^p([0,1] \times \mathbb{R}^N)$, so we can find a sequence $\tau_n \to 0$ such that $g_{\tau_n} \to w$ weakly, i.e.

$$\lim_n \int_0^1 \int_{\mathbb{R}^N} g_{\tau_n}(t,x)\psi(t,x) \ \mathrm{d}x \, \mathrm{d}t = \int_0^1 \int_{\mathbb{R}^N} w(t,x)\psi(t,x) \ \mathrm{d}x \, \mathrm{d}t \tag{78}$$

for all $\psi \in C_c^\infty([0,1] \times \mathbb{R}^N)$. For all such $\psi$ (extending $\psi(t,x) = 0$ when $t \notin [0,1]$)

$$\int_0^1 \int_{\mathbb{R}^N} g_\tau(t,x)\psi(t,x) \ \mathrm{d}x \, \mathrm{d}t = \int_0^1 \int_{\mathbb{R}^N} f(t,x) \frac{\psi(t-\tau,x) - \psi(t,x)}{\tau} \ \mathrm{d}x \, \mathrm{d}t$$

hence

$$\lim_n \int_0^1 \int_{\mathbb{R}^N} g_{\tau_n}(t,x)\psi(t,x) \ \mathrm{d}x \, \mathrm{d}t = -\int_0^1 \int_{\mathbb{R}^N} f(t,x)\partial_t\psi(t,x) \ \mathrm{d}x \, \mathrm{d}t$$

by dominated convergence, so we conclude that $f$ admits a weak derivative and the derivative is $w$. The relationship (19) in $L^p(\mathbb{R}^N)$, that is

$$f(b,\cdot) - f(a,\cdot) = \int_a^b \frac{df}{\mathrm{d}t} \, \mathrm{d}t$$

implies that

$$\int_a^b \xi \frac{df}{dt} \ \mathrm{d}t = -\int_a^b \frac{d\xi}{dt} f \ \mathrm{d}t$$

for all $\xi \in C_c^\infty([0,1])$; but then setting $\psi(t,x) = \xi(t)$, we obtain that $\frac{df}{dt} = \partial_t f$.

[13] It is moreover plausible that, if (44) does not hold, then in general two compact sets may not be connected by a rectifiable continuous path.

- Suppose that the pointwise limit

$$\lim_{\tau\to 0} g_\tau(t,x) = \lim_{\tau\to 0} \frac{f(t+\tau,x)-f(t,x)}{\tau} \tag{79}$$

exists for almost all $t, x$; we call $h(t,x)$ this limit. We reason as above in eqn. (78), by dominated convergence we obtain that

$$\lim_{\tau\to 0} \int_0^1 \int_{\mathbb{R}^N} g_\tau(t,x)\psi(t,x)\ \mathrm{d}x\,\mathrm{d}t = \int_0^1 \int_{\mathbb{R}^N} h(t,x)\psi(t,x)\ \mathrm{d}x\,\mathrm{d}t \tag{80}$$

so $h$ is a representative of the weak partial derivative.

□

## A.9 Proof of 7.4

First of all, we remark that $\mathcal{F}$ is dense in $\mathcal{M}$ according to the Hausdorff distance. In particular it is easy to build approximating sequences. One method is as follows. Let $A$ compact; let $\{x_n\}_n$ be a dense subset of $A$; let $A_n \stackrel{\text{def}}{=} \{x_k \mid k \le n\}$; then $d_H(A, A_n) \to_n 0$ ($d_H$ being the Hausdorff distance).

*Proof.* When $\varphi'(0) < 0$ it is easy to find examples to show that $\mathcal{F}$ is not dense in $\mathcal{M}$. Let $N = 1$, $A = [0,1]$, suppose by contradiction that there exists $(A_n)_n \subset \mathcal{F}$ such that $A_n \to_n A$ according to $d_{1,p,\varphi}$; then $A_n \to_n A$ according to $d_{p,\varphi}$ and hence $d_H(A, A_n) \to_n 0$, by Thm. 6.11. By direct inspection we observe that $u_{A_k}$ is a piecewise linear function and, for all $x \in (0,1)$ but for finitely many choices, $u_{A_k}$ is differentiable and $|u'_{A_k}(x)| = 1$; we also know that $u_{A_n} \to_n u_A$ uniformly; so we obtain that

$$|v'_{A_k}(x)| = |\varphi'(u_{A_k}(x))\ u'_{A_k}(x)| \to |\varphi'(0)|$$

for almost all $x \in [0,1]$ — whereas $v'_A(x) = 0$ for all $x \in (0,1)$.

Vice versa, suppose now that $\varphi'(0) = 0$. Fix $A$ compact, let $\{x_n\}_n$ be a dense subset of $A$ and $A_n \stackrel{\text{def}}{=} \{x_k \mid k \le n\}$; we will prove that $A_n \to_n A$ according to $d_{1,p,\varphi}$.

We remarked above that $d_H(A, A_n) \to_n 0$, and (by Thm. 6.11) that $v_{A_n} \to_n v_A$ in $L^p$. We need to prove that $\nabla v_{A_n} \to_n \nabla v_A$ in $L^p$.

For any $x \in A$, $\nabla v_A(x)$ exists and is zero (this is obviously true in the topological interior, whereas when $x$ is in the boundary it derives from $\varphi'(0) = 0$). At the same time, for almost every $x \in A$ and for all n we know (see Lemma 6.28) that $\nabla v_{A_n}$ exists and

$$|\nabla v_{A_n}(x)| = |\varphi'(u_{A_n}(x))|$$

so we can write

$$\int_A |\nabla v_{A_n}(x) - \nabla v_A(x)|^p\,\mathrm{d}x = \int_A |\varphi'(u_{A_n}(x))|^p\,\mathrm{d}x \to_n 0$$

exploiting the fact that $u_{A_n} \to_n u_A$ uniformly.

We now consider the complement $A^c$ of $A$. We will argue that $\nabla v_{A_n} \to_n \nabla v_A$ in $L^p(A^c)$.

Let $L$ be the Lipschitz constant of $\varphi$: then $L$ is the Lipschitz constant of any function $v_B$ for $B$ compact. Working as in Prop. 7.2, we select $R > 0$ large so that $A, A_n \subseteq B_R$. For all $n$ and almost every x, when $|x| \le R + T$ we know that $|\nabla v_{A_n}(x)| \le L$; whereas when $|x| \ge R + T$ we know that

$$|\nabla v_{A_n}(x)| = |\varphi'(u_{A_n}(x))| \le |\varphi'(|x| - R)|$$

where the last function is in $L^p$. So, by dominated convergence, it suffice to show that $\nabla v_{A_n} \to_n \nabla v_A$ pointwise. We sketch the argument. Suppose now that $x \notin A$ and that, for all n, $u_{A_n}$ and $u_A$ are differentiable at $x$: we will argue that $\nabla u_{A_n}(x) \to_n \nabla u_A(x)$. By the results presented in Sec. 4 we will prove convergence of the projection points. Let $y_n$ the projection of $x_n$ to $A_n$ and $y$ the projection of $x$ to $A$: then $y_n \to y$. If this is not the case, then there is a subsequence and a $z \neq y$ such that $y_{n_k} \to_k z$: but then $z \in A$ is another projection point of $x$, contradicting the fact that $u_A$ is differentiable at x.

□

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