LOWER SEMICONTINUITY FOR POLYCONVEX INTEGRALS WITHOUT COERCIVITY ASSUMPTIONS

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Abstract. We prove a lower semicontinuity theorem for a polyconvex functional of integral form, related to maps $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ in $W^{1,n}(\Omega;\mathbb{R}^m)$ with $n \geq m \geq 2$, with respect to the weak $W^{1,p}$-convergence for $p > m-1$, without assuming any coercivity condition.

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1. Introduction

In this paper we deal with a classical problem in the Calculus of Variations concerning the lower semicontinuity of integrals of the form

$$\mathcal{I}(u) = \int_\Omega g(x, u(x), Du(x))\, dx\,, \tag{1.1}$$

where $u : \Omega \to \mathbb{R}^m$ is a *vector-valued map* defined in an open set $\Omega \subset \mathbb{R}^n$ and Du is the $m \times n$ *Jacobian matrix* of its partial derivatives.
This problem has a long story, which started from the famous paper of Serrin in 1961 (see [25]), where the first crucial result for the scalar case was obtained. In particular, Serrin pointed out the fundamental role of the convexity with respect to the gradient variable in order to get the lower semicontinuity of the integral $\mathcal{I}(u)$ in (1.1). The convexity assumption plays a relevant role in modeling, too (see, e.g. [8] and [14]).
On the other hand, it is well known that in the vectorial case the convexity is a condition very far from being necessary for the lower semicontinuity and also non realistic in many physical applications (see, e.g. [8]). For instance, in the theory of nonlinear elasticity, the functional $\mathcal{I}$ represents the stored energy of the material, whose position vector is given by the function u; in this case, the convexity of the integrand g with respect to the deformation gradient conflicts with the natural requirement of frame-indifference. For this reason, there is a large literature which investigates on assumptions which can be alternative to convexity and there is no agreement as to which are the most appropriate. In this regard, it was widely investigated by Morrey and Ball (see [24] and [6]) that, when $m > 1$, either the *quasiconvexity* or the *polyconvexity* of g with respect to the gradient variable ξ play a role. We recall, in particular, that the function $g(x, s, \xi)$ is said *polyconvex*

with respect to the gradient variable ξ if it can be represented in the form

$$g(x,s,\xi) = f(x,s,M(\xi)) \,, \tag{1.2}$$

where f is a *convex* function with respect to its last variable and, for every $m \times n$ matrix $\xi \in \mathbb{M}^{m\times n}$, $M(\xi)$ denotes the vector

$$M(\xi) = \left(\mathrm{adj}_1\,\xi, \mathrm{adj}_2\,\xi, \ldots, \mathrm{adj}_i\,\xi, \ldots, \mathrm{adj}_{\min\{m,n\}}\,\xi\right) . \tag{1.3}$$

Here $\mathrm{adj}_1\,\xi = \xi$ while $\mathrm{adj}_i\,\xi$, for $i = 2, \ldots, \min\{m,n\}$, stands as the vector of the determinants of the $i \times i$ minors of the matrix $\xi \in \mathbb{M}^{m\times n}$. Thus $M(\xi)$ is a vector in $\mathbb{R}^\tau$, with

$$\tau = \tau(n,m) = \sum_{i=1}^{\min(m,n)} \binom{n}{i}\binom{m}{i} . \tag{1.4}$$

It is well known that polyconvexity implies quasi-convexity and this last property (at least for smooth integrands) implies the well-known *Legendre-Hadamard* or *ellipticity* condition. Moreover, quasi-convexity (under additional regularity and growth conditions) is a necessary and sufficient condition in order to obtain the $W^{1,p}$-weak lower semicontinuity of the functional $\mathcal{I}$. However, the quasi-convexity property is not so easy to be checked, while polyconvexity is handier and also often satisfied in the applications. Hence, in the present paper, we will consider the case of polyconvex functionals, which is worthwhile to point out, are not necessarily convex, up to the one-dimensional case; i.e. $n = 1$ or $m = 1$.
To this purpose in the following, we will assume that (1.2) holds with $f : \Omega \times \mathbb{R}^m \times \mathbb{R}^\tau \to [0, +\infty)$ a nonnegative function, convex in $\mathbb{R}^\tau$ with respect to the last variable.
The lower semicontinuity for polyconvex integrals has been investigated by several authors in the past years, starting from the results by Morrey and Ball cited above (see, for instance, [9] and the references therein). In the quoted papers the weak topology of $W^{1,p}(\Omega;\mathbb{R}^m)$ is considered, for some $p > \min\{m,n\}$. A more recent counterexample by Maly [21] shows that the semicontinuity in the weak topology of $W^{1,p}(\Omega;\mathbb{R}^m)$ is generically not true if $p < \min\{m,n\} - 1$. Marcellini [22], [23] and Dacorogna-Marcellini [10] proved a lower semicontinuity result when $p > n-1$, $m = n$ and $f = f(M(\xi))$ or $f = f(x, M(\xi))$. The limit case $p = n - 1$ has been studied in the same context by Acerbi-Dal Maso [2], Dal Maso-Sbordone [12], Celada-Dal Maso [7], Fusco-Hutchinson [19] and, more recently, by Focardi-Fusco-Leone-Marcellini-Mascolo-Verde [15]. See also Acerbi-Buttazzo-Fusco [1] for lower semicontinuity with respect to the L^∞-topology.
The situation changes considerably when a dependence on s is also allowed, since the presence of the (x,s)-variables cannot be treated as a simple perturbation. Results in this more general context are due to Gangbo [20], under the structure assumptions that $f(x,s,M(\xi)) = a(x,s)h(x,M(\xi))$, and to Fonseca-Leoni [16], under a coercivity condition.
In a recent paper (see [3]), a lower semicontinuity theorem without coercivity assumptions and under general dependence of the integrand on the (x,s)-variable is proved. However, the lack of coercivity and the general form of the integrand force additional regularity assumptions with respect to x (for instance, a kind of Lipschitz continuity condition).

The present work originates from that paper as a natural improvement of the results contained there. Indeed, we investigate here if the additional regularity assumptions mentioned above can be weakened. We start from the easy remark that, if $f(x,s,M(\xi)) = a(x)h(s,M(\xi))$, the lower semicontinuity of the integral functional can be obtained only by assuming that a is a lower semicontinuous function (see Theorem 3.2). In this direction we prove that this result holds even if f is not splitted, if we assume that the function $x \mapsto f(x,s,M(\xi))$ satisfies a suitable lower semicontinuity condition uniformly with respect to $(s,M(\xi))$ (see condition (2.3) and Theorem 3.3). Notice that this uniform condition has been widely considered in the literature, in different contexts of semicontinuity problems (see, for instance, [4], [5], [11], [16], [17], [18]). We emphasize that, as in [3], in the present paper no assumptions on the strict positivity of the integrand are required; insomuch as the integrand could be zero in a large part of the domain Ω (as we will see in Section 4). The proof of the theorem relies on a suitable revision of an approximation result for convex functions due to Fusco, Gori and Maggi (see [18]), which has been already used in some papers dealing with the scalar case in the context of BV-functions (see also [4] and [5]). Finally, we mention that the last section of this paper is devoted to present some examples of functionals where the integrand is very far from being continuous and it is zero on sets of positive Lebesgue measure. The results proved in this paper assure the lower semicontinuity of these functionals.

2. Preliminaries

Throughout the paper $n, m > 1$ are fixed integers and τ is given in (1.4). For every matrix $\xi \in M\!I^{m\times n}$, the vector $M(\xi)$, defined in (1.3), belongs to $\mathbb{R}^\tau$. Moreover, Ω will be an open subset of $\mathbb{R}^n$. The letter c will denote a strictly positive constant, whose value may change from line to line.
For every open set $A \subseteq \Omega$ and every $u \in W^{1,m}(\Omega;\mathbb{R}^m)$, we set

$$\mathcal{F}_f(u,A) = \int_A f(x,u,M(\nabla u))\,dx\,, \tag{2.1}$$

where $f : \Omega \times \mathbb{R}^m \times \mathbb{R}^\tau \to [0,+\infty)$ is a Borel function, convex in the last variable. When $A = \Omega$ we simply write $\mathcal{F}_f(u)$ instead of $\mathcal{F}_f(u,\Omega)$.

In the following, we collect some preliminary results.

Lemma 2.1. *Let $f, f_j : \Omega \times \mathbb{R}^m \times \mathbb{R}^\tau \to [0,+\infty)$, $j \in \mathbb{N}$, be Borel functions convex in the last variable and such that*

$$f(x,s,\zeta) = \sup_{j\in\mathbb{N}} f_j(x,s,\zeta) \qquad \text{for all } (x,s,\zeta) \in (\Omega \setminus N_0) \times \mathbb{R}^m \times \mathbb{R}^\tau\,,$$

where $N_0 \subset \Omega$ is a Borel set with $|N_0| = 0$. If the functionals $\mathcal{F}_{f_j}(\cdot,\Omega)$ defined by

$$\mathcal{F}_{f_j}(u,\Omega) := \int_\Omega f_j(x,u,M(\nabla u))\,dx$$

are lower semicontinuous along all the sequences $\{u_k\} \in W^{1,m}(\Omega;\mathbb{R}^m)$ *with* $u_k \rightharpoonup u \in W^{1,m}(\Omega;\mathbb{R}^m)$ *weakly in* $W^{1,p}(\Omega;\mathbb{R}^m)$, $p > m-1$, *then* $\mathcal{F}_f(\cdot,\Omega)$ *is lower semicontinuous along the same sequences, too.*

Proof. It is a direct consequence of Lemma 6 proven in [13]. □

Lemma 2.2 (See Lemma 3.2 in [4])**.** *Let* X *be a* σ*-compact metric space and* $\mathcal{G}$ *a family of lower semicontinuous functions* $g : X \to \mathbb{R}$. *Then, there exists a finite or countable subset* $\mathcal{G}' \subset \mathcal{G}$ *such that*

$$\sup_{g\in\mathcal{G}'} g(x) = \sup_{g\in\mathcal{G}} g(x) \qquad \text{for all } x \in X\,.$$

Following the outlines of [18, Lemma 8 (c)], we state an approximation result for functions $f(x,s,\zeta)$, convex with respect to ζ and which satisfy a suitable lower semicontinuity with respect to the spatial variable $x \in \Omega$, by means of splitted regular functions. In that paper, the variable s was treated jointly with x, while in our framework the role of x and s must be different, since we need to separate these two variables.

Theorem 2.3. *Let* $f : \Omega\times\mathbb{R}^m\times\mathbb{R}^\tau \to [0,+\infty)$ *be a function such that* $f(x,s,\cdot)$ *is convex for all* $(x,s) \in (\Omega\setminus N_0)\times\mathbb{R}^m$, $f(x,\cdot,\zeta)$ *is continuous for all* $(x,\zeta) \in (\Omega\setminus N_0)\times\mathbb{R}^\tau$ *and*

$$0 \le f(x,s,\zeta) \le \Lambda(1+|\zeta|) \qquad \text{for all } (x,s,\zeta) \in (\Omega\setminus N_0)\times\mathbb{R}^m\times\mathbb{R}^\tau\,, \tag{2.2}$$

where $N_0 \subset \Omega$ *is a Borel set with* $|N_0| = 0$ *and* Λ *is a positive constant.*
Let us assume that, for all $x_0 \in \Omega\setminus N_0$ *and* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that*

$$f(x_0,s,\zeta) \le f(x,s,\zeta) + \varepsilon(1+f(x,s,\zeta)) \qquad \text{for all } (x,s,\zeta) \in (\Omega\setminus N_0)\times\mathbb{R}^M\times\mathbb{R}^\tau \text{ such that } |x-x_0|<\delta\,. \tag{2.3}$$

Then there exist $\{a_j\} \subset C_0^\infty(\mathbb{R}^n)$ *and* $\{\psi_j\} \subset C^0(\mathbb{R}^m\times\mathbb{R}^\tau)$ *such that, for all* $j \in \mathbb{N}$, $0 \le a_j \le 1$, $\psi_j(s,\cdot)$ *is a convex function satisfying* (2.2), *and*

$$f(x,s,\zeta) = \sup_{j\in\mathbb{N}} [a_j(x)\psi_j(s,\zeta)] \qquad \text{for all } (x,s,\zeta) \in (\Omega\setminus N_0)\times\mathbb{R}^m\times\mathbb{R}^\tau\,.$$

Proof. Fix $\varepsilon > 0$ and $x_0 \in \Omega\setminus N_0$ and choose δ so that (2.3) holds. Let $\alpha_{\varepsilon,x_0} \in C_c^\infty(B_\delta(x_0),[0,1])$ be such that $\alpha_{\varepsilon,x_0}(x_0) = 1$ and $\rho \in C_c^\infty(B_1(0),[0,\infty))$ with $\rho(\zeta) = \rho(-\zeta)$, $\int_{\mathbb{R}^\tau}\rho(\zeta)d\zeta = 1$. We define

$$\begin{aligned} \rho_\varepsilon(\zeta) &:= \varepsilon^{-\tau}\rho(\zeta/\varepsilon),\\ h_{\varepsilon,x_0}(s,\zeta) &:= \frac{1}{1+\varepsilon}\int_{\mathbb{R}^\tau}\rho_\varepsilon(\eta)f(x_0,s,\zeta+\eta)d\eta - \frac{\varepsilon}{1+\varepsilon} - \varepsilon\Lambda,\\ g_{\varepsilon,x_0}(x,s,\zeta) &:= \alpha_{\varepsilon,x_0}(x)[h_{\varepsilon,x_0}(s,\zeta)]^+\,. \end{aligned} \tag{2.4}$$

We have that $g_{\varepsilon,x_0} : \mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^\tau \to [0,+\infty)$ is smooth with respect to x and convex with respect to ζ. Moreover, it is continuous and non-negative on $\Omega\times\mathbb{R}^m\times\mathbb{R}^\tau$. Indeed, the function $s \mapsto f(x_0,s,\zeta+\eta)$ is continuous, for every fixed $x_0 \in \Omega$ and $\zeta,\eta \in \mathbb{R}^\tau$. Moreover, by the convexity of $f(x_0,s,\cdot)$ and (2.2), we obtain the Lipschitz continuity of f in the last variable, uniformly with respect to the other ones. This implies that the function

$f(x_0,\cdot,\cdot)$ is continuous on $\mathbb{R}^m\times\mathbb{R}^\tau$. Hence, as a consequence of (2.2) and Dominated Convergence Theorem, we obtain that also h_{ε,x_0} is continuous on $\mathbb{R}^m\times\mathbb{R}^\tau$. Notice that, by construction, we get easily that

$$f(x,s,\zeta)=\sup_{\varepsilon>0}\sup_{x_0\in\Omega\setminus N_0} g_{\varepsilon,x_0}(x,s,\zeta)\qquad \text{for all } (x,s,\zeta)\in(\Omega\setminus N_0)\times\mathbb{R}^m\times\mathbb{R}^\tau, \tag{2.5}$$

once we prove that $f\geq g_{\varepsilon,x_0}$. In turn, this is a consequence of the fact $f(x,s,\zeta)\geq h_{\varepsilon,x_0}(s,\zeta)$ for $x\in B_\delta(x_0)\setminus N_0$. Indeed, by (2.3) and (2.2), we have

$$\begin{aligned} h_{\varepsilon,x_0}(s,\zeta)&=\frac{1}{1+\varepsilon}\int\limits_{\mathbb{R}^\tau}\rho_\varepsilon(\eta)f(x_0,s,\zeta+\eta)d\eta-\frac{\varepsilon}{1+\varepsilon}-\varepsilon\Lambda\\ &\leq\frac{1}{1+\varepsilon}\int\limits_{\mathbb{R}^\tau}\rho_\varepsilon(\eta)\left\{(1+\varepsilon)f(x,s,\zeta+\eta)+\varepsilon\right\}d\eta-\frac{\varepsilon}{1+\varepsilon}-\varepsilon\Lambda\\ &\leq\int\limits_{\mathbb{R}^\tau}\rho_\varepsilon(\eta)\left\{f(x,s,\zeta)+\Lambda|\eta|\right\}d\eta-\varepsilon\Lambda\leq f(x,s,\zeta). \end{aligned} \tag{2.6}$$

Thus (2.5) is established. The result then follows by observing that, since all functions $g_{\varepsilon,z_0}(x,s,\zeta)$ are continuous, by Lemma 2.2 there exists a sequence $\gamma_j\in(0,\infty)\times(\Omega\setminus N_0)$ such that

$$f(x,s,\zeta)=\sup_{j\in\mathbb{N}} g_{\gamma_j}(x,s,\zeta)\qquad \text{for all } (x,s,\zeta)\in(\Omega\setminus N_0)\times\mathbb{R}^m\times\mathbb{R}^\tau.$$

□

3. The lower semicontinuity results

In this section, we will assume that $\Omega\subseteq\mathbb{R}^n$ is an open set, $m\leq n$ and τ is defined in formula (1.4) of the Introduction.
Our aim is to prove a lower semicontinuity result for the functional $\mathcal{F}_f$ defined in (2.1), under suitable hypotheses on the integrand $f:\Omega\times\mathbb{R}^m\times\mathbb{R}^\tau\to[0,+\infty)$ (convex in the last variable), without requiring any coercivity or structure conditions. Moreover, no continuity assumptions on the dependence of f with respect to x is considered.

We start recalling the following theorem, contained in [3], where the lower semicontinuity inequality along sequences $\{u_k\}\subseteq W^{1,m}(\Omega;\mathbb{R}^m)$ with $u_k\rightharpoonup u\in W^{1,m}(\Omega;\mathbb{R}^m)$ weakly in $W^{1,p}(\Omega;\mathbb{R}^m)$, $p>m-1$, is obtained by assuming the Lipschitz continuity of the function f with respect to x.

Theorem 3.1. *Let $f:\Omega\times\mathbb{R}^m\times\mathbb{R}^\tau\to[0,+\infty)$ is a continuous function such that*

$$\text{for a.e. } x\in\Omega \text{ and for all } s\in\mathbb{R}^m\quad f(x,s,\cdot) \text{ is convex}; \tag{3.1}$$

$$\begin{cases}\forall K\subset\subset\Omega\times\mathbb{R}^m\times\mathbb{R}^\tau\quad\exists c_K>0\ \text{ such that}\\ |f(x,s,\zeta)-f(y,s,\zeta)|\leq c_K|x-y|\quad\forall(x,s,\zeta),(y,s,\zeta)\in K\,;\end{cases} \tag{3.2}$$

$$\forall(x,\zeta)\in\Omega\times\mathbb{R}^\tau\quad D_xf(x,\cdot,\zeta) \text{ is a continuous function}. \tag{3.3}$$

Then for every $u_k, u \in W^{1,m}(\Omega;\mathbb{R}^m)$ with $u_k \rightharpoonup u$ weakly in $W^{1,p}(\Omega;\mathbb{R}^m)$, $p > m-1$, we have

$$\mathcal{F}_f(u) = \int_\Omega f\Big(x, u, M(Du)\Big)\,dx \le \liminf_{k\to+\infty} \int_\Omega f\Big(x, u_k, M(Du_k)\Big)\,dx = \liminf_{k\to+\infty} \mathcal{F}_f(u_k)\,,$$

where, for every $\xi \in \mathbb{M}^{m\times n}$, $M(\xi)$ is defined as in (1.3).

3.1. **The case of separated variables.** We firstly consider the simple model case, where $f(x,s,\zeta) = a(x)h(s,\zeta)$.

Proposition 3.2. *Let $a : \Omega \to [0,+\infty)$ be a locally bounded lower semicontinuous function and $h : \mathbb{R}^m \times \mathbb{R}^\tau \to [0,+\infty)$ be a continuous function, convex with respect to the last variable. Then for every $u_k, u \in W^{1,m}(\Omega;\mathbb{R}^m)$ with $u_k \rightharpoonup u$ weakly in $W^{1,p}(\Omega;\mathbb{R}^m)$, $p > m-1$, we have*

$$\int_\Omega a(x)h\Big(u, M(Du)\Big)\,dx \le \liminf_{k\to+\infty} \int_\Omega a(x)h\Big(u_k, M(Du_k)\Big)\,dx\,.$$

Proof. Since a is a non negative lower semicontinuous function, it is well known that there exists a sequence $\{a_j\}_{i\in\mathbb{N}}$ of Lipschitz continuous functions $a_j : \Omega \to [0,+\infty)$ such that for every $x \in \Omega$

$$a(x) = \sup_{j\in\mathbb{N}} a_j(x).$$

By Theorem 3.1, for every $j \in \mathbb{N}$ and for every $u_k, u \in W^{1,m}(\Omega;\mathbb{R}^m)$ with $u_k \rightharpoonup u$ weakly in $W^{1,p}(\Omega;\mathbb{R}^m)$, we have

$$\int_\Omega a_j(x)h\Big(u, M(Du)\Big)\,dx \le \liminf_{k\to+\infty} \int_\Omega a_j(x)h\Big(u_k, M(Du_k)\Big)\,dx\,.$$

Finally the conclusion follows by Lemma 2.1. □

3.2. **The general case.** Now we are in the position to prove the main result of the paper; i.e., the lower semicontinuity inequality for polyconvex integrals of the form

$$\mathcal{F}_f(u) = \int_\Omega f\big(x, u, M(Du)\big)\,dx$$

along sequences $\{u_k\} \subset W^{1,m}(\Omega;\mathbb{R}^m)$ converging to $u \in W^{1,m}(\Omega;\mathbb{R}^m)$ weakly in $W^{1,p}(\Omega;\mathbb{R}^m)$, for $p > m-1$. This result is established without any hypothesis of coercivity or continuity with respect to x. More precisely, we assume only lower semicontinuity with respect to the spatial variable, with a suitable uniformity condition (see (2.3) above).

Theorem 3.3. *Let $f : \Omega \times \mathbb{R}^m \times \mathbb{R}^\tau \to [0, +\infty)$ be a Borel function such that $f(x, s, \cdot)$ is convex for every $(x, s) \in (\Omega \setminus N_0) \times \mathbb{R}^m$ and $f(x, \cdot, \zeta)$ is continuous for every $(x, \zeta) \in (\Omega \setminus N_0) \times \mathbb{R}^\tau$, where $N_0 \subset \Omega$ is a Borel set with $|N_0| = 0$. Assume that f satisfies (2.2) and (2.3). Then for every $u_k, u \in W^{1,m}(\Omega; \mathbb{R}^m)$ with $u_k \rightharpoonup u$ weakly in $W^{1,p}(\Omega; \mathbb{R}^m)$, $p > m - 1$, we have*

$$\mathcal{F}_f(u) = \int_\Omega f\Big(x, u, M(Du)\Big)\, dx \le \liminf_{k\to+\infty} \int_\Omega f\Big(x, u_k, M(Du_k)\Big)\, dx = \liminf_{k\to+\infty} \mathcal{F}_f(u_k)\,,$$

where, for every $\xi \in \mathbb{M}^{m\times n}$, $M(\xi)$ is defined as in (1.3).

Proof. By Theorem 2.3, we have that

$$f(x, s, \zeta) = \sup_{j\in\mathbb{N}} f_j(x, s, \zeta) \qquad \text{for all } (x, s, \zeta) \in (\Omega \setminus N_0) \times \mathbb{R}^m \times \mathbb{R}^\tau, \tag{3.4}$$

where, for $j \in \mathbb{N}$, $f_j(x, s, \zeta) : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^\tau \to [0, +\infty)$ is defined by $f_j(x, s, \zeta) := a_j(x)\psi_j(s, \zeta)$. Notice that each function f_j satisfies the assumptions of Theorem 3.1. Therefore, the functionals $\mathcal{F}_{f_j}$ are all lower semicontinuous along the sequences $\{u_k\} \subseteq W^{1,m}(\Omega; \mathbb{R}^m)$ such that $u_k \rightharpoonup u \in W^{1,m}(\Omega; \mathbb{R}^m)$ weakly in $W^{1,p}(\Omega; \mathbb{R}^m)$, $p > m - 1$. Therefore the thesis follows by Lemma 2.1. □

Notice that inequality (2.3) is not so easy to be checked; hence, in the following, we find some conditions, easier to be verified, which imply (2.3) and hence the lower semicontinuity of the corresponding functional.

Theorem 3.4. *Let $f : \Omega \times \mathbb{R}^m \times \mathbb{R}^\tau \to [0, +\infty)$ be a locally bounded Borel function such that $f(x, s, \cdot)$ is convex for every $(x, s) \in (\Omega \setminus N_0) \times \mathbb{R}^m$, $f(x, \cdot, \zeta)$ is continuous for every $(x, \zeta) \in (\Omega \setminus N_0) \times \mathbb{R}^\tau$, where $N_0 \subset \Omega$ is a Borel set with $|N_0| = 0$. Assume also that $f(\cdot, \cdot, \zeta)$ is lower semicontinuous on $(\Omega \setminus N_0) \times \mathbb{R}^m$, for every $\zeta \in \mathbb{R}^\tau$, and that the following condition holds:*

$$\begin{cases} f(x, s, \cdot) \ \text{ is positively 1-homogeneous and} \\ f(x, s, \zeta) > 0 \ \text{ for all } (x, s, \zeta) \in (\Omega \setminus N_0) \times \mathbb{R}^m \times (\mathbb{R}^n \setminus \{0\}) \ \text{ with } |N_0| = 0. \end{cases} \tag{3.5}$$

Then for every $u_k, u \in W^{1,m}(\Omega; \mathbb{R}^m)$ with $u_k \rightharpoonup u$ weakly in $W^{1,p}(\Omega; \mathbb{R}^m)$, $p > m - 1$, we have

$$\int_\Omega f\Big(x, u, M(Du)\Big)\, dx \le \liminf_{k\to+\infty} \int_\Omega f\Big(x, u_k, M(Du_k)\Big)\, dx\,. \tag{3.6}$$

Proof. Notice that, since f is locally bounded in $(\Omega \setminus N_0) \times \mathbb{R}^m \times \mathbb{R}^\tau$ and positively 1-homogeneous with respect to ζ, for any open set $\Omega' \subset\subset \Omega$ and every radius $r > 0$, there exists a constant Λ' such that

$$0 \le f(x, s, \zeta) \le \Lambda'|\zeta| \qquad \text{for all } (x, s, \zeta) \in (\Omega' \setminus N_0) \times \overline{B}(0, r) \times \mathbb{R}^\tau\,. \tag{3.7}$$

This estimate, together with the convexity of f with respect to ζ immediately yields that, for all $(x, s, \zeta_1), (x, s, \zeta_2) \in (\Omega' \setminus N_0) \times \overline{B}(0, r) \times \mathbb{R}^\tau$,

$$|f(x, s, \zeta_1) - f(x, s, \zeta_2)| \le \Lambda'|\zeta_1 - \zeta_2|\,. \tag{3.8}$$

Hence, one easily gets that f is lower semicontinuous in $(\Omega' \setminus N_0) \times \overline{B}(0,r) \times \mathbb{R}^\tau$ and $f(x,\cdot,\cdot)$ is continuous in $\overline{B}(0,r) \times \mathbb{R}^\tau$.
We firstly need to show that, given $x_0 \in \Omega' \setminus N_0$, for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$f(x_0, s, \zeta) \leq (1+\varepsilon) f(x, s, \zeta) \tag{3.9}$$

for all $(x, s, \zeta) \in (\Omega' \setminus N_0) \times B(0,r) \times \mathbb{R}^\tau$ such that $|x - x_0| < \delta$. To prove this, we argue by contradiction, assuming that there exist $x_0 \in \Omega' \setminus N_0$ and $\varepsilon_0 > 0$ such that there exist sequences $\{x_k\} \subseteq \Omega' \setminus N_0$, with $|x_k - x_0| < 1/k$ for any $k \in \mathbb{N}$, and $\{(s_k, \zeta_k)\} \subseteq B(0,r) \times \mathbb{R}^\tau$ such that

$$f(x_0, s_k, \zeta_k) > (1+\varepsilon_0) f(x_k, s_k, \zeta_k)\,. \tag{3.10}$$

Clearly, by the positive 1-homogeneity of $f(x,s,\cdot)$, we may assume that $|\zeta_k| = 1$, for every $k \in \mathbb{N}$; hence, up to a subsequence, there exists $\zeta_0 \in \mathbb{S}^{N-1}$ such that $\zeta_k \to \zeta_0$. Moreover, since $\{s_k\} \subseteq B(0,r)$, we may assume that also $s_k \to s_0 \in \overline{B}(0,r)$. Then, passing to the limit when $k \to +\infty$ in (3.10) and using the lower semicontinuity of f and the continuity of $f(x_0,\cdot,\cdot)$, we get that

$$f(x_0, s_0, \zeta_0) = \lim_{k\to+\infty} f(x_0, s_k, \zeta_k) \geq (1+\varepsilon_0) \liminf_{k\to+\infty} f(x_k, s_k, \zeta_k) \geq (1+\varepsilon_0) f(x_0, s_0, \zeta_0)\,.$$

Hence, $f(x_0, s_0, \zeta_0) = 0$, which is a contradiction, since $x_0 \in \Omega' \setminus N_0$; therefore (3.9) is in force.
Fixed $i \in \mathbb{N}$, we consider a Lipschitz continuous function $\phi_i : [0, +\infty) \to \mathbb{R}$ such that

$$\phi_i(t) = \begin{cases} 1 & t \leq i-1, \\ -t+i & i-1 < t \leq i, \\ 0 & t > i. \end{cases}$$

Set $f_i(x,s,\zeta) = \phi_i(|s|) f(x,s,\zeta)$. By condition (3.7) and (3.9), we have that f_i satisfies (2.2) and (2.3) and all the other assumptions of Theorem 3.3, with Ω replaced by Ω'. Hence, we obtain that for every sequence $\{u_k\} \subseteq W^{1,m}(\Omega';\mathbb{R}^m)$ such that $u_k \to u \in W^{1,m}(\Omega';\mathbb{R}^m)$ weakly in $W^{1,p}(\Omega';\mathbb{R}^m)$, $p > m-1$, we have

$$\int_{\Omega'} \phi_i(|u|) f\Big(x, u, M(Du)\Big)\,dx \leq \liminf_{k\to+\infty} \int_{\Omega'} \phi_i(|u_k|) f\Big(x, u_k, M(Du_k)\Big)\,dx\,. \tag{3.11}$$

Therefore

$$\begin{aligned} \liminf_{k\to+\infty} \int_{\Omega'} f\Big(x, u_k, M(Du_k)\Big)\,dx &\geq \liminf_{k\to+\infty} \int_{\Omega'} \phi_i(|u_k|) f\Big(x, u_k, M(Du_k)\Big)\,dx \\ &\geq \int_{\Omega'} \phi_i(|u|) f\Big(x, u, M(Du)\Big)\,dx\,. \end{aligned}$$

By letting $i \to \infty$, since $\phi_i \nearrow 1$, by the Beppo Levi's Lemma we have

$$\liminf_{k\to+\infty} \int_{\Omega'} f\Big(x, u_k, M(Du_k)\Big)\,dx \geq \int_{\Omega'} f\Big(x, u, M(Du)\Big)\,dx\,.$$

This proves that $\mathcal{F}_f(\cdot, \Omega')$ satisfies inequality (3.6). By letting $\Omega' \uparrow \Omega$, the thesis follows. □

4. Examples

In this section we present some examples of functionals where the integrand is very far from being continuous (as in the first example) and it is zero on sets of positive Lebesgue measure (as in the first and the second example). The theorems proved in the previous section assure the lower semicontinuity of these functionals, while the results already available in the literature cannot recover these cases.

Example 4.1. *Let $n > 1$ and $K \subset \mathbb{R}^n$ be a Cantor set with positive measure in $\mathbb{R}^n$; i.e., K is the Cartesian n-product of a standard one-dimensional Cantor set having positive Lebesgue measure. Set*

$$f(x, s, \zeta) = \big(1 - \chi_K(x)\big) h(s, \zeta)\,,$$

where h is a nonnegative function continuous in s and convex in ζ.
Then f satisfies all the assumptions of Proposition 3.2, hence the functional

$$u \mapsto \int_\Omega \big(1 - \chi_K(x)\big) h(s, M(Du))\, dx$$

is lower semicontinuous along sequences $\{u_k\} \subset W^{1,m}(\Omega; \mathbb{R}^m)$ converging to $u \in W^{1,m}(\Omega; \mathbb{R}^m)$ weakly in $W^{1,p}(\Omega; \mathbb{R}^m)$, for $p > m - 1$.

Example 4.2. *Let $\Omega = B(0, 1) \subseteq \mathbb{R}^n$ be the ball centered at the origin, with radius 1. Consider the function $f : \Omega \times \mathbb{R}^m \times \mathbb{R}^\tau \to [0, +\infty)$ defined by*

$$f(x, s, \zeta) = |\sum_{i=1}^{\tau} a_i(x, s)\zeta_i| + |x|\,|\zeta|$$

where, for every $i = 1, \dots, \tau$, a_i are (only) continuous and locally bounded functions on $\Omega \times \mathbb{R}^m$. In particular, we admit also functions a_i such that $a_i(0, s) \equiv 0$, for $i = 1, \dots, \tau$.
Then f satisfies all the assumptions of Theorem 3.4, hence the functional

$$u \mapsto \int_\Omega \big(|\sum_{i=1}^{\tau} a_i(x, u)\, \mathrm{adj}_i(Du)| + |x|\,|M(Du)|\big)\, dx$$

is lower semicontinuous along sequences $\{u_k\} \subset W^{1,m}(\Omega; \mathbb{R}^m)$ converging to $u \in W^{1,m}(\Omega; \mathbb{R}^m)$ weakly in $W^{1,p}(\Omega; \mathbb{R}^m)$, for $p > m - 1$.

Example 4.3. *Let $\Omega = \mathbb{R}^n$ and consider the function $f : \Omega \times \mathbb{R}^m \times \mathbb{R}^\tau \to [0, +\infty)$ defined by*

$$\begin{aligned} f(x,s,\zeta) &= \left(\cos^2|s| + \sin^2|x|\right)\left(\Big[\sum_{i=1}^{\tau} a_i(x)\zeta_i\Big]^+ + |\zeta|\right) \\ &= \left(\cos^2|s| + \sin^2|x|\right)\Big[\sum_{i=1}^{\tau} a_i(x)\zeta_i\Big]^+ + (\cos|s|)^2|\zeta| + (\sin|x|)^2|\zeta| \\ &=: f_1(x,s,\zeta) + f_2(s,\zeta) + f_3(x,\zeta) = f_1(x,s,\zeta) + \widetilde{f}(x,s,\zeta)\,, \end{aligned}$$

where, for every $i = 1, \dots, \tau$, a_i are (only) continuous and locally bounded functions on Ω. Notice that, because of the term $f_3(x,\zeta) = (\sin|x|)^2|\zeta|$, $f(x,s,\zeta) > 0$ for every $(x,s,\zeta) \in (\mathbb{R}^n \setminus N_0) \times \mathbb{R}^m \times (\mathbb{R}^\tau \setminus \{0\})$, where $N_0 = \{x \in \mathbb{R}^n \ : \ |x| = 2k\pi\,,\ k \in \mathbb{N}\}$ has zero Lebesgue measure. Then f satisfies all the assumptions of Theorem 3.4, hence the functional

$$u \mapsto \int_\Omega \left(\cos^2|u| + \sin^2|x|\right)\left(\Big[\sum_{i=1}^{\tau} a_i(x)\,\mathrm{adj}_i(Du)\Big]^+ + |M(Du)|\right) dx$$

is lower semicontinuous along sequences $\{u_k\} \subset W^{1,m}(\Omega;\mathbb{R}^m)$ converging to $u \in W^{1,m}(\Omega;\mathbb{R}^m)$ weakly in $W^{1,p}(\Omega;\mathbb{R}^m)$, for $p > m-1$.
However, even if $f(x,s,\zeta)$ has the special structure given by $f(x,s,\zeta) = a(x,s)h(x,\zeta)$, the results in Gangbo (see [20]*) cannot be applied in order to achieve the l.s.c., since the function $a(x,s)$ does not satisfies the inequality $a(x,s) \geq C > 0$.*
Moreover, if we consider the function

$$\widetilde{f}(x,s,\zeta) = f_2(s,\zeta) + f_3(x,\zeta) = (\cos|s|)^2|\zeta| + (\sin|x|)^2|\zeta|\,,$$

we obtain that f_2 satisfies all the assumptions of Theorem 3.1 and f_3 satisfies all the assumptions of Theorem 3.4, hence the corresponding integral functionals $\mathcal{F}_{f_2}$ and $\mathcal{F}_{f_3}$ are lower semicontinuous. Nevertheless, to our knowledge there is no result which permits to obtain the l.s.c. of the functional associated to the integrand $f_1(x,s,\zeta)$, separately.

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