A Modica-Mortola approximation for branched transport
Une approximation à la Modica-Mortola pour le transport branché

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Abstract. The M^α energy which is minimized in branched transport problems among singular 1-dimensional rectifiable vector measures with prescribed divergence is approximated by means of a sequence of elliptic energies, defined on more regular vector fields. The procedure recalls that of Modica-Mortola to approximate the perimeter, and the double-well potential is replaced by a concave power.

Résumé. L'énergie M^α qui est minimisée dans les problèmes de transport branché parmi les mesures vectorielles (singulières et supportées sur des ensembles rectifiables de dimension 1) à divergence fixée est approximée par une suite d'énergies elliptiques, définies sur des champs de vecteurs plus réguliers. La procédure rappelle celle de Modica et Mortola pour le périmètre, et le potentiel à double puits est remplacé par une puissance concave.

Version française abrégée

Le nom "trasport branché" s'est récemment affirmé pour appeler tous ces problèmes de transport où le coût pour une masse m qui parcourt une longueur l n'est pas proportionnel à la masse mais sous-additif et, typiquemment, proportionnel à une puissance m^α ($0 < \alpha < 1$). De cette manière le transport conjoint est favori, avec des effets de branchement vers les différentes destinations. Dans le cas des graphes finis ce genre de problèmes remonte aux années '60 (dans un cadre de recherche opérationnelle), alors que ses généralisations au continu viennent de la communauté du transport optimal et sont beaucoup plus récentes.

Un traitement numérique satisfaisant de ces questions est encore loin d'être réalisé et cela pose le problème de l'approximation de ce type d'énergie : cette note présente un résultat dans ce sens, dans le langage de la $\Gamma-$convergence (voir [6]). La formulation continue du problème de transport branché passe par une minimisation sous contraintes de divergence (voir (1.2)) et il est naturel de l'approcher par des problèmes qui portent sur des champs de vecteurs plus réguliers (qui ont une densité différentiable et qui tendent à se concentrer sur un graphe, sans être des mesures de type $\mathcal{H}^1$).

Des énergies avec un terme concave et un terme de Dirichlet (dont l'importance devient de plus en plus faible le long de la convergence) sont proposées en (3.2). L'idée de cette approximation est issue de discussions avec J.-M. Morel pour aboutir à des méthode numériques efficaces ; la démonstration du résultat de convergence contenu dans cette note (disponible en ligne, [11]) sera présentée dans un papier avec E. Oudet [10]. Le même papier illustrera également les

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résultats numériques qui peuvent être obtenus à l'aide de cette approximation, ainsi que les modifications à apporter à l'énoncé et à la preuve pour gérer le cas des exposants $\alpha \leq 1 - \frac{1}{d}$ (ce qui inclut des applications au problème de Steiner, correspondant à $\alpha = 0$). Minimiser ces énérgies permet de passer par des méthodes de gradient (maîtrisées de manière opportune pour éviter dans la mesure du possible les minima locaux), plutôt que par les méthodes combinatoires très compliquées, qui caractérisent les approximations par les réseaux finis.

Au délà des applications numériques, l'intérêt de ce résultat vient aussi de sa comparaison avec la théorie de l'approximation elliptique des énergies singulières. Le cas de référence est l'approximation de la fonctionnelle perimètre par voie d'un potentiel à double et d'un terme de Dirichlet (voir (3.1)), un résultat obtenu par Modica et Mortola, [8], lors des premières années de la théorie de la $\Gamma-$convergence.

Les problèmes variationnelles à considérer pour bien approcher celui du transport branché seraient de la forme (4.1). Pourtant, dans cette note on ne présentera que le résultat de $\Gamma-$convergence des énergies (par rapport à la convergence faible des mesures et de leurs divergences), et ceci en dimension 2. Il reste à montrer que les contraintes de divergence peuvent être incorporées et que les minimiseurs des problèmes approchés satisfont des hypothèses de compacité, ce qui est plus délicat. De toute manière, on suggère une méthode pénalisée qui résout ces deux problèmes, tout en laissant ouvertes les deux questions. Également, le cas des dimensions supérieures reste ouvert.

Dans un vieil article, [4], dédié à des applications différentes, Bouchitté et al. donnent un résultat d'approximation très similaire pour des énergies définies sur les mesures atomiques. On peut dire que dans cette étude les réseaux 1D jouent le même rôle en 2D que les points jouaient en 1D en [4]. De plus, les mêmes auteurs sont en train d'étudier, par slicing, l'extension au transport branché : les techniques sont différentes de celles qui sont utilisés ici et en [10], mais les difficultés rencontrées sont plus ou moins les mêmes.

L'idée de cette note est donc de présenter un type d'approximation, intéressant en soi et très promettant en ce qui concerne les techniques et les applications, avec un résultat précis qui peut déjà s'appliquer dans certains cadres, mais qui demande aussi une étude ultérieure. Pour tous les détails et les nouveaux développements on renvoie à [10, 11].

Les détails de la preuve (voir le Théorème 3.2 et les commentaires juste après) montreront également l'analogie des techniques avec les résultats de [8], analogie qui s'ajoute au fait que, dans les deux cas, on à affaire à des énergies intégrales qui concentrent à la limite sur une structure de dimension plus basse (dans ce cas, sur un graphe rectifiable qui constitue le réseau de transport).

1 Branched transport via divergence constraints

Let $\mathcal{M}(\Omega)$ be the set of finite vector measures on $\Omega \subset \mathbb{R}^d$ with values in $\mathbb{R}^d$ and such that their divergence is a finite scalar measure. On this space we consider the following convergence: we write $u_\varepsilon \rightharpoonup u$ if u_ε and $\nabla \cdot u_\varepsilon$ weakly converge as measures to u and $\nabla \cdot u$, respectively. When a function is considered as an element of this space, or a functional space as a subset of it, we always think at absolutely continuous measures (with respect to the Lebesgue measure on Ω) and the functions represent their densities. When we take $u \in \mathcal{M}(\Omega)$ and we write $u = U(M, \theta, \xi)$ we mean that u is a rectifiable vector measure (i.e. the translation in the language of vector measures of the concept of rectifiable currents) of the form $\theta\xi \cdot \mathcal{H}^1_{\lfloor M}$: the real multiplicity $\theta : M \to \mathbb{R}^+$ multiplied times the orientation $\xi : M \to \mathbb{R}^d$ is its density with respect to the $\mathcal{H}^1-$Hausdorff measure on the $1-$rectifiable set M, and ξ is a measurable vector field of unit vectors belonging to the (approximate) tangent space to M at $\mathcal{H}^1-$almost any

point.

For $0 < \alpha < 1$, we consider the energy

$$M^\alpha(u) = \begin{cases} \int_M \theta^\alpha d\mathcal{H}^1 & \text{if } u = U(M,\theta,\xi), \\ +\infty & \text{otherwise.} \end{cases} \tag{1.1}$$

The problem of branched transport, introduced in a continuous setting by Q. Xia in [12] and then studied by many authors (see for instance [2] for a whole presentation of the theory), amounts to minimizing M^α under a divergence constraint:

$$\min\left\{M^\alpha(u) \ : \ \nabla\cdot u = f := f^+ - f^-\right\}. \tag{1.2}$$

The constraint is intended in weak form and means $\int \nabla\phi\cdot du = \int \phi\, d(f^- - f^+)$ for all $\phi \in C^1(\Omega)$, which actually corresponds to Neumann boundary conditions.

Problem (1.2) stands for the minimization of the energy of a movement bringing the mass from f^+ to f^-, where moving a mass m on a length l costs $m^\alpha l$. It admits a solution with finite energy for any pair of probability measures $(f^+, f^-) \in \mathcal{P}(\overline{\Omega})^2$, provided $\alpha > 1 - \frac{1}{d}$ (see [12]).

2 Variational approximation, preliminaries

The constraint being expressed in differential form it is natural to wonder whether we can approximate this minimization problem by means of more regular ones, where only "regular" measures u, with differentiable densities, are allowed.

In this paper we present a $\Gamma-$convergence result for a sequence of energies M^α_ε approximating M^α. Among possible applications, numerical procedures will be easier to perform on the approximated problems: this aspect, which is only the first interest for the results presented in this paper, will be developed in [10].

In order to precise what we mean by "approximating the energy" and how to use the result, let us briefly sketch the main outlines of $\Gamma-$convergence's theory, as introduced by De Giorgi (see [7] and [6]).

Definition 2.1. *On a metric space X let $F_n : X \to \mathbb{R}\cup\{+\infty\}$ be a sequence of functions. We define F^-, the $\Gamma - \liminf$, and F^+, the $\Gamma - \limsup$ of this sequence, through*

$$F^-(x) := \inf\{\liminf_{n\to\infty} F_n(x_n) \ : \ x_n \to x\},\ F^+(x) := \inf\{\limsup_{n\to\infty} F_n(x_n) \ : \ x_n \to x\}.$$

Should F^- and F^+ coincide, then we say that F_n is $\Gamma-$converging to the common value $F = F^- = F^+$ and we write $F_n \xrightarrow{\Gamma} F$.

The most interesting properties of $\Gamma-$convergence, for our scopes, are the following:

- if there exists a compact set $K \subset X$ such that $\inf_X F_n = \inf_K F_n$ for any n, then F attains its minimum and $\inf F_n \to \min F$;
- if $(x_n)_n$ is a sequence of minimizers for F_n and a subsequence $(x_{n_k})_k$ converges to x, then x minimizes F
- if $F_n \xrightarrow{\Gamma} F$, then $F_n + G \xrightarrow{\Gamma} F + G$ for any continuous function $G : X \to \mathbb{R}\cup\{+\infty\}$.

3 Elliptic approximation

The result we are going to present is somehow inspired or at least recalls the following (see [8] and [5])

Theorem 3.1. *Define the functional F_ε on $L^1(\Omega)$ through*

$$F_\varepsilon(u) = \begin{cases} \frac{1}{\varepsilon}\int W(u(x))dx + \varepsilon \int |\nabla u(x)|^2 dx & \text{if } u \in H^1(\Omega); \\ +\infty & \text{otherwise.} \end{cases} \tag{3.1}$$

Then, if $W(0) = W(1) = 0$ and $W > 0$ on $\mathbb{R} \setminus \{0, 1\}$, we have $F_\varepsilon \xrightarrow{\Gamma} F$, where F is

$$F(u) = \begin{cases} cPer(A) & \text{if } u = 1 \text{ on } A,\ u = 0 \text{ on } A^c, \text{ and } A \text{ is a finite perimeter set;} \\ +\infty & \text{otherwise.} \end{cases}$$

Here lies the other interesting point of this paper, i.e. the comparison with the Modica-Mortola $\Gamma-$convergence result (the proof of our theorem will also recall the proof of Theorem 3.1, see [10] and the considerations after Theorem 3.2).

We will consider functionals of the form

$$M_\varepsilon(u) = \varepsilon^{\gamma_1} \int_\Omega |u(x)|^\beta dx + \varepsilon^{\gamma_2} \int_\Omega |\nabla u(x)|^2 dx, \tag{3.2}$$

defined on $u \in H^1(\Omega; \mathbb{R}^2)$ and set to $+\infty$ outside $H^1 \subset \mathcal{M}(\Omega)$.

As one can see these functionals recall those in theorem 3.1, but the double-well potential is replaced with a concave power. Notice that concave powers, in the minimization, prefer either $u = 0$ or $|u|$ being as large as possible when the average value for u is fixed in a region (which is exactly the meaning of weak convergence). Hence, there is sort of a double well at 0 and ∞.

One difference is that here u is a vector, so that one could also evoke Ginzburg-Landau theory with its approximation (see, for instance, [3]). Yet, due to the divergence constraint, the problem stays essentially scalar, as at the limit there will be locally one direction only which will be relevant (actually, a singular vector measure concentrated on a 1D object must be oriented along a tangent direction, if we want its divergence to be a measure). Moreover, in 2D it is possible to take advantage of the usual decomposition of a vector field into a gradient plus a rotated gradient and translate the functionals M_ε into functionals of the form of those studied by Aviles and Giga (Modica-Mortola results for higher order energies, see [1]).

A simple heuristics allows to determine the exponents β, γ_1 and γ_2. Actually, one can consider a measure $U(S, m, \xi)$, concentrated on a segment S with constant multiplicity m, and approximate it with a measure u_A with smooth density, concentrated on a strip of width A around S. Then it is easy to compute that the the value of the functional $M_\varepsilon(u_A)$ is of the order of

$$M_\varepsilon \approx \varepsilon^{\gamma_1} A^{d-1} \left(\frac{m}{A^{d-1}}\right)^\beta + \varepsilon^{\gamma_2} A^{d-1} \left(\frac{m}{A^d}\right)^2.$$

Minimizing over possible widths A gives the optimal values

$$A \approx \varepsilon^{\frac{\gamma_2-\gamma_1}{2d-\beta(d-1)}} m^{\frac{2-\beta}{2d-\beta(d-1)}}; \quad M_\varepsilon \approx \varepsilon^{\gamma_2-(\gamma_2-\gamma_1)\frac{d+1}{2d-\beta(d-1)}} m^{2-(2-\beta)\frac{d+1}{2d-\beta(d-1)}}.$$

The correct choice for a possible convergence result towards the energy (1.1), which is proportional to m^α, is obtained if one takes

$$\beta = \frac{2-2d+2\alpha d}{3-d+\alpha(d-1)}; \quad \frac{\gamma_1}{\gamma_2} = \frac{(d-1)(\alpha-1)}{3-d+\alpha(d-1)} \quad \text{and, for } d=2, \quad \beta = \frac{4\alpha-2}{\alpha+1}; \quad \frac{\gamma_1}{\gamma_2} = \frac{\alpha-1}{\alpha+1}.$$

Notice that γ_1 and γ_2 may not be both determined since one can always replace ε with a power of ε, thus changing the single exponents but not their ratio. Notice also that the exponent β is positive and less than 1 as soon as $\alpha > 1 - \frac{1}{d}$, which is the usual condition.

Hence, we consider the 2D case and denote by M_ε^α the functionals

$$M_\varepsilon^\alpha(u) = \begin{cases} \varepsilon^{\alpha-1} \int_\Omega |u(x)|^\beta dx + \varepsilon^{\alpha+1} \int_\Omega |\nabla u(x)|^2 dx & \text{if } u \in H^1(\Omega), \\ +\infty & \text{otherwise,} \end{cases}$$

for $\beta = \frac{4\alpha-2}{\alpha+1}$, and we can prove:

Theorem 3.2. *If $d = 2$, the functionals M_ε^α $\Gamma-$converge with respect to the convergence of $\mathcal{M}(\Omega)$ to cM^α as $\varepsilon \to 0$, where c is a constant in $]0, +\infty[$ (the value of c is actually $c = \alpha^{-1} (4c_0\alpha/(1-\alpha))^{1-\alpha}$, being $c_0 = \int_0^1 \sqrt{t^\beta - t} dt$).*

This theorem is the main result we present, and gives a new class of integral energies which concentrate at the limit on lower-dimensional structures.

Just to suggest an idea of the techniques, we stress that, similarly to the proof of Theorem 3.1, the main ingredient of the proof is a Cauchy inequality:

$$\varepsilon^{\alpha-1}|u_\varepsilon|^\beta + \varepsilon^{\alpha+1}|\nabla u_\varepsilon|^2 = \varepsilon^{\alpha-1} f_\varepsilon(u_\varepsilon) + \varepsilon^{\alpha-1} L_\varepsilon |u_\varepsilon| + \varepsilon^{\alpha+1}|\nabla u_\varepsilon|^2 \geq 2\varepsilon^\alpha f_\varepsilon(u_\varepsilon)|\nabla u_\varepsilon| + \varepsilon^{\alpha-1} L_\varepsilon |u_\varepsilon|,$$

where $f_\varepsilon(z) := \sqrt{|z|^\beta - L_\varepsilon |z|}$. The main estimate for the $\Gamma - \liminf$ inequality is obtained through a careful choice of L_ε, by estimating $\int f_\varepsilon(u_\varepsilon)|\nabla u_\varepsilon|$ in terms of the total variation of a suitable function of u_ε (primitive of f_ε). For the $\Gamma - \limsup$, an optimal recovery sequence is obtained by selecting a sequence u_ε which guarantees equality in all the estimates providing the lower bound.

4 Questions and perspectives

The idea of the paper is to replace, in Problem (1.2), the energy M^α with M_ε^α. Yet, since $M_\varepsilon^\alpha(u)$ is finite only if $u \in H^1$ (which implies $\nabla \cdot u \in L^2$), one should in general modify the divergence constraint and solve

$$\min \{ M_\varepsilon^\alpha(u) \ : \ \nabla \cdot u = f_\varepsilon \}, \tag{4.1}$$

being f_ε a suitable approximation of $f = f^+ - f^-$, proving that the minimizers of (4.1) converge to the minimizers of (1.2). Theorem 3.2 proves $\Gamma-$convergence, but we did not address the condition $\nabla \cdot u = f_\varepsilon$, nor we discussed the choice of f_ε. This question is quite delicate, and this is why we propose here a different approach.

Replacing constraints with penalizations is quite natural, especially when numerics is concerned. One could decide to get rid of the (quite severe) condition $\nabla \cdot u = f$ considering

$$\min \quad M_\varepsilon^\alpha(u) + G(\nabla \cdot u),$$

where G is a continuous functional on measures, penalizing the distance to f. This would play the game, thanks to the additive property of $\Gamma-$convergence, but it would converge to $\min \quad M^\alpha(u) + G(\nabla \cdot u)$, which in general is not exactly the same as imposing $\nabla \cdot u = f$. The following interesting consideration works quite well if the divergence is given by the difference of two probability measures, which is typically the case. As explained in [12], the quantity

$d_\alpha(\mu,\nu) := \min\{M^\alpha(u) \ : \ \nabla\cdot u = \mu-\nu\}$ is a distance on $\mathcal{P}(\overline{\Omega})$ that metrizes weak convergence. As a consequence, due to the triangular inequality, solving

$$\min\left\{2d_\alpha(f^+,\mu^+) + \min\{M^\alpha(u) \ : \ \nabla\cdot u = \mu^+-\mu^-\} + 2d_\alpha(f^-,\mu^-), \quad \mu^+,\mu^- \in \mathcal{P}(\overline{\Omega})\right\}$$

amount to choosing $\mu^\pm = f^\pm$ and minimizing $M^\alpha(u)$ under $\nabla\cdot u = f^+ - f^-$.

This means that $2d_\alpha$ is a clever choice as a penalization. To be more precise we can introduce the space $Y(\Omega) := \{(u,\mu,\nu) \in \mathcal{M}(\Omega)\times\mathcal{P}(\overline{\Omega})\times\mathcal{P}(\overline{\Omega}) \ : \ \nabla\cdot u = \mu-\nu\}$, endowed with the obvious topology of componentwise weak convergence. On $Y(\Omega)$, we consider the functionals

$$(u,\mu,\nu) \mapsto M^\alpha_\varepsilon(u) + 2d_\alpha(f^+,\mu) + 2d_\alpha(f^-,\mu). \tag{4.2}$$

It is an easy consequence of our previous considerations and of Theorem 3.2 that the minimization problem for these functionals converges to the minimization of $M^\alpha(u)+2d_\alpha(f^+,\mu)+2d_\alpha(f^-,\mu)$ over $Y(\Omega)$, which is equivalent to (1.2).

Nevertheless, this approximation is not satisfactory yet: we are trying to approximate the minimization of M^α and we propose to use the distance d_α itself!! During the computations one should probably solve a problem of the same kind and no progress would have been done.

One can escape by replacing d_α with other quantities, provided they are larger, they also vanish when the two measures coincide, and they are independent of M^α. This is possible for instance thanks to [9], where the inequality $d_\alpha \le CW_1^{2\alpha-1}$ is proven, W_1 being the usual Wasserstein distance on $\mathcal{P}(\overline{\Omega})$. We do not enter into details here but this produces a consistent approximation for branched transport problems.

It is anyway important to stress that the approach on $\mathcal{M}(\Omega)$ without penalization stays useful for a lot of problems where the divergence is not prescribed but enters the optimization (think at $\min_\mu d_\alpha(\mu,\nu)+F(\mu)$). Yet, we find interesting to ask the following question:

Question 4.1. Given f, is it possible to find a suitable sequence $f_\varepsilon \rightharpoonup f$ so that $M^\alpha_\varepsilon + \chi_{\mathrm{div}=f_\varepsilon} \overset{\Gamma}{\to} M^\alpha + \chi_{\mathrm{div}=f}$ (χ being the indicator function of the constraint)? is it possible to find f_ε explicitly, for instance by convolution?

The second issue we want to address deals with the convergence of the minimizers. Actually, $\Gamma-$convergence is quite useless if we cannot deduce that the minimizers u_ε converge, at least up to subsequences, to a minimizer u. Yet, this requires compactness in $\mathcal{M}(\Omega)$, i.e. we want bounds on the mass of $\nabla\cdot u_\varepsilon$ and of u_ε. The first bound is guaranteed in the space $Y(\Omega)$, but the bound on $|u_\varepsilon|(\Omega)$ has to be proven.

Notice that $M^\alpha_\varepsilon(u_\varepsilon) \le C$ will not be sufficient for such a bound, as one can guess looking at the limit functional. Actually, it seems reasonable that "bounded energy configuration have not necessarily bounded mass, but optimal configuration do have" (in the limit problem, the reason for the bound on the mass is the absence of cycles in the optimizers, and analogous estimates should be true for the approximating problems as well).

Here, the suggestion we provide to overcome this problem is quite naif : just take a sufficiently large number K so that every minimizer u of the limit problem satisfies $|u|(\overline{\Omega}) \le K$ and then restrict the analysis to the compact subset $Y_K(\Omega) := \{(u,\mu,\nu) \in Y(\Omega) \ : \ |u|(\overline{\Omega}) \le K\}$. Anyway, the following question stays open

Question 4.2. Prove a bound on the L^1 norm of the minimizers u_ε. If possible, prove it when u_ε minimizes M^α_ε under a divergence constraint $\nabla\cdot u_\varepsilon = f_\varepsilon$. Is it possible to do it simply by writing and exploiting optimality conditions under the form of elliptic PDEs?

To finish this questioning section, here is the last natural one:

Question 4.3. Prove the same results or investigate what happens in $\mathbb{R}^d$, for $d \ge 3$.

References

[1] P. Aviles and Y. Giga, A mathematical problem related to the physical theory of liquid crystal configurations, *Proc. Centre Math. Anal. ANU, 12,* 1–16, 1987.

[2] M. Bernot, V. Caselles and J.-M. Morel, *Optimal transportation networks, Models and Theory*, Lecture Notes in Mathematics, Springer, Vol. 1955 (2008).

[3] F. Bethuel, H. Brezis and F. Hélein, *Ginzburg-Landau Vortices*, Birkhäuser, 1994.

[4] G.Bouchitté, C. Dubs and P. Seppecher, Transitions de phases avec un potentiel dégénéré à l'infini, application à l'équilibre de petites gouttes, *C.R.Acad. Sci.*, 323, série I, p. 1103-1108, 1996.

[5] A. Braides, *Approximation of free-discontinuity problems*. Lecture Notes in Mathematics, 1694, Springer-Verlag, Berlin, 1998.

[6] G. Dal Maso: *An Introduction to* $\Gamma-$*convergence*. Birkhauser, Basel, 1992.

[7] E. De Giorgi and T. Franzoni, Su un tipo di convergenza variazionale, *Atti Acc. Naz. Lincei Rend.*, 58 (8) (1975), pp. 842-850.

[8] L. Modica and S. Mortola, Un esempio di Γ-convergenza. (Italian) *Boll. Un. Mat. Ital. B* (14), no. 1, 285–299, 1977.

[9] J.-M. Morel and F. Santambrogio, Comparison of distances between measures, *Appl. Math. Lett.*, (20), no. 4, 2007, 427–432.

[10] E. Oudet and F. Santambrogio, A Modica-Mortola approximation for branched transport and applications, in preparation.

[11] F. Santambrogio, manuscript available at `http://cvgmt.sns.it/papers/san09/`

[12] Q. Xia, Optimal Paths related to Transport Problems. *Comm. Cont. Math.* (5), no. 2, 251–279, 2003.