Optimal transportation with traffic congestion and Wardrop equilibria

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Abstract

In the classical Monge-Kantorovich problem, the transportation cost only depends on the amount of mass sent from sources to destinations and not on the paths followed by this mass. Thus, it does not allow for congestion effects. Using the notion of traffic intensity, we propose a variant taking into account congestion. This leads to an optimization problem posed on a set of probability measures on a suitable paths space. We establish existence of minimizers and give a characterization. As an application, we obtain existence and variational characterization of equilibria of Wardrop type in a continuous space setting.

Keywords: optimal transportation, traffic congestion, Wardrop equilibria.

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1 Introduction

Given two mass distributions μ_0 and μ_1 on $\mathbb{R}^d$ with equal total mass, the classical Monge-Kantorovich problem consists in finding transportation plans (i.e. measures on $\mathbb{R}^d \times \mathbb{R}^d$ having μ_0 and μ_1 as marginals) with minimal average transportation cost. This old problem can be traced back to Monge [11]. It has received a lot of attention in the recent years since the pathbreaking paper of Brenier [4] who solved the case of a quadratic transportation cost. We refer to the book of Villani [12], the lecture notes of Ambrosio [1] and the references therein for a recent account of this rich mathematical theory and its numerous applications.

An interesting case is when the transportation cost is given by a conformally flat Riemannian distance:

$$d_g(x,y) := \inf\left\{\int_0^1 g(\sigma(t))|\dot{\sigma}(t)|\ dt\ :\ \sigma(0)=x, \sigma(1)=y\right\}$$

the corresponding Monge-Kantorovich problem then reads as

$$\inf_{\gamma\in\Pi(\mu_0,\mu_1)} \int_{\mathbb{R}^d\times\mathbb{R}^d} d_g(x,y)d\gamma(x,y) \tag{1.1}$$

where $\Pi(\mu_0,\mu_1)$ is the set of transportation plans. However, from a traffic planning point of view, problem (1.1) is not totally realistic. On the one hand, (1.1) is *path-independent*: the total transportation cost only depends on the amount of mass transported from the sources x to the destinations y and not on the paths followed by this mass. Put differently, in (1.1), individual's travelling strategies are irrelevant. On the other hand, (1.1) does not take into account congestion effects i.e. the possibility that the cost $g(x)$ for passing through the point x depends on "how crowded" small neighbourhoods of x are. This idea can be made precise thanks to the notion of *traffic intensity* associated to a probability measure on a suitable set of paths. This notion of *traffic intensity* (see paragraph 2.2) is the path-dependent analogue of the well-known notion of *transport density* in Monge's problem (see Bouchitté, Buttazzo and Seppecher [3], Bouchitté and Buttazzo [2], Caffarelli, Feldman and McCann [6]). With this notion at hand, we propose an optimal transportation problem with congestion. This variant of (1.1) takes the form of a relatively simple convex optimization problem posed on a set of probability measures on a suitable set of paths. We obtain existence of minimizers (Theorem 2.9) and a characterization (Theorem 3.8).

Researchers in the field of applied traffic modelling have long emphasized the role of congestion in networks. In the early 50's, Wardrop (see [13])

considered the situation where a large number of vehicles have to go from one location to another, connected by a finite number of different roads. Each vehicle has to choose one road (or a probability on the set of possible roads) to minimize some transportation cost which depends not only on the road chosen but also on the total flow of vehicles on this road. Wardrop gave a minimal stability requirement for transportation strategies: the cost of every actually used road should be equal and less than that which would be experienced by a single vehicle on any unused road. This natural equilibrium concept has been very popular since its introduction because of applications to networks of course but also due to the development of non-cooperative game theory in the meanwhile. From the best of our knowledge, the study of Wardrop equilibria have mainly been restricted to the case where admissible roads are given by a finite graph. A secondary contribution of the present paper is to introduce an equilibrium concept of Wardrop type in a continuous state setting, to prove the existence of such equilibria and to relate it to the optimal transportation problem with congestion (Theorem 4.2).

2 Optimal transportation with congestion

2.1 Notations

Given a locally compact separable metric space X, we will denote respectively by $\mathcal{M}_+(X)$ and $\mathcal{M}_+^1(X)$ the set of positive and finite Radon measures on X and the set of Radon probability measures on X. If X and Y are locally compact separable metric spaces, $\mu \in \mathcal{M}_+^1(X)$, and $f : X \to Y$ is a Borel map we shall denote by $f\sharp\mu$ the push forward of μ through f i.e. the element of $\mathcal{M}_+^1(Y)$ defined by $f\sharp\mu(B) = \mu(f^{-1}(B))$ for every Borel subset B of Y.

In the sequel, $\mathcal{L}^d$ denotes the d-dimensional Lebesgue measure. If μ and ν are in $\mathcal{M}_+^1(\mathbb{R}^d)$ then $\frac{d\mu}{d\nu}$ denotes the Radon-Nikodym derivative of μ with respect to ν. We shall write $\mu << \nu$ to express that μ is absolutely continuous with respect to ν, in which case, slightly abusing notations, we will identify μ with the Radon-Nikodym derivative $\frac{d\mu}{d\nu}$.

The data of our problem are Ω (its closure $\overline{\Omega}$ modelling the city, say) which is some open bounded convex subset of $\mathbb{R}^2$, two probability measures, μ_0 and μ_1 in $\mathcal{M}_+^1(\overline{\Omega})$, giving respectively the distribution of residents and services in the city $\overline{\Omega}$. The set of transportation plans associated to μ_0 and μ_1 will be denoted $\Pi(\mu_0, \mu_1)$ it consists of the probability measures on $\overline{\Omega} \times \overline{\Omega}$ having μ_0 and μ_1 as marginals:

$$\Pi(\mu_0, \mu_1) := \{\gamma \in \mathcal{M}_+^1(\overline{\Omega} \times \overline{\Omega}) : \pi_0\sharp\gamma = \mu_0,\ \pi_1\sharp\gamma = \mu_1\} \tag{2.1}$$

where $(\pi_0(x,y),\pi_1(x,y)) := (x,y)$, stand for the canonical projections (x and y in $\overline{\Omega}$).

Introducing congestion naturally leads to consider spaces of paths, lengths of such paths and sets of probability measures on sets of paths. From now, on we shall denote:

- $C := W^{1,\infty}([0,1],\overline{\Omega})$, viewed as a subset of $C^0([0,1],\mathbb{R}^2)$,
- $C^{x,y} := \{\sigma \in C :\ \sigma(0) = x,\ \sigma(1) = y\}$ ($x,\ y$ in $\overline{\Omega}$),
- $l(\sigma) := \int_0^1 |\dot{\sigma}(t)|\ dt$, the length of $\sigma \in C$,
- for $\sigma \in C$, $\widetilde{\sigma}$ denotes the arclength reparameterization of σ belonging to C, hence $|\dot{\widetilde{\sigma}}(t)| = l(\sigma) = l(\widetilde{\sigma})$ for a.e. $t \in [0,1]$,
- $\widetilde{C} := \{\sigma \in C\ :\ |\dot{\sigma}| \text{ is constant}\} = \{\widetilde{\sigma},\ \sigma \in C\}$,
- for $Q \in \mathcal{M}_+^1(C)$, we define $\widetilde{Q} \in \mathcal{M}_+^1(\widetilde{C})$ as the push forward of Q through the map $\sigma \mapsto \widetilde{\sigma}$,
- for $\varphi \in C^0(\overline{\Omega},\mathbb{R})$ and $\sigma \in C$, we define

$$L_\varphi(\sigma) := \int_0^1 \varphi(\sigma(t))|\dot{\sigma}(t)|dt = l(\sigma)\int_0^1 \varphi(\widetilde{\sigma}(t))dt,$$

- $e_0(\sigma) := \sigma(0)$, $e_1(\sigma) := \sigma(1)$, for all $\sigma \in C^0([0,1],\mathbb{R}^2)$.

2.2 Traffic congestion modelling

The classical Monge-Kantorovich optimal transportation problem for a given cost function $c \in C^0(\overline{\Omega}\times\overline{\Omega},\mathbb{R})$ is:

$$\inf\left\{\int_{\overline{\Omega}\times\overline{\Omega}} c(x,y)d\gamma(x,y)\ :\ \gamma \in \Pi(\mu_0,\mu_1)\right\}. \tag{2.2}$$

Note that, in the linear problem (2.2), the cost of transporting one unit of mass from x to y, $c(x,y)$, is given and does not depend on the path(s) followed by the mass from x to y. In order to take into account congestion effects, we explicitely introduce probabilities over $C^{x,y}$ as part of the optimization problem. More precisely, the overall transportation cost will depend not only on the transportation plan $\gamma \in \Pi(\mu_0,\mu_1)$ but also on the way travelers commuting from x to y use the different possible paths $\sigma \in C^{x,y}$. In the sequel, the way commuters from x to y are split according to the different

paths will be given by a probability measure $p^{x,y}$ on $C^{x,y}$. Put differently, $p^{x,y}(\Sigma)$ is the proportion of travelers from x to y using a path $\sigma \in \Sigma \subset C^{x,y}$. This naturally leads to the following definition:

Definition 2.1. *A transportation strategy consists of a pair (γ, p) with $\gamma \in \Pi(\mu_0, \mu_1)$ and where $p = (p^{x,y})_{(x,y)\in\overline{\Omega}\times\overline{\Omega}}$ is a Borel family of probability measures on C such that $p^{x,y}(C^{x,y}) = 1$ for γ-a.e. $(x, y) \in \overline{\Omega} \times \overline{\Omega}$.*

There results, from the use of a transportation strategy (γ, p), an overall *traffic intensity* $I_{\gamma,p} \in \mathcal{M}_+(\overline{\Omega})$ defined by

$$\begin{aligned}\int_{\overline{\Omega}} \varphi(x) dI_{\gamma,p}(x) &:= \int_{\overline{\Omega}\times\overline{\Omega}} \left(\int_{C^{x,y}} \left(\int_0^1 \varphi(\sigma(t))|\dot{\sigma}(t)|dt \right) dp^{x,y}(\sigma) \right) d\gamma(x, y) \\ &= \int_{\overline{\Omega}\times\overline{\Omega}} \left(\int_{C^{x,y}} L_\varphi(\sigma) dp^{x,y}(\sigma) \right) d\gamma(x, y), \quad \forall \varphi \in C^0(\overline{\Omega}, \mathbb{R})\end{aligned} \tag{2.3}$$

and an overall probability over paths $Q_{\gamma,p} \in \mathcal{M}_+^1(C)$ given by $Q_{\gamma,p} = p^{x,y} \otimes \gamma$, i.e.:

$$\int_C F(\sigma) dQ_{\gamma,p}(\sigma) = \int_{\overline{\Omega}\times\overline{\Omega}} \left(\int_{C^{x,y}} F(\sigma) dp^{x,y}(\sigma) \right) d\gamma(x, y) \quad \forall F \in C^0(C, \mathbb{R}). \tag{2.4}$$

One could consider the probability $Q_{\gamma,p}$ as if it represented the total number of travelers that use a path $\sigma \in \Sigma$ given the global transportation strategy (γ, p).

Let us remark that if we set $Q := Q_{\gamma,p} \in \mathcal{M}_+^1(C)$ then $I_{\gamma,p}$ only depends on Q, and can be written as $I_{\gamma,p} = i_Q \in \mathcal{M}_+(\overline{\Omega})$ where i_Q is defined for every $Q \in \mathcal{M}_+^1(C)$ by:

$$\int_{\overline{\Omega}} \varphi(x) di_Q(x) = \int_C L_\varphi(\sigma) dQ(\sigma), \quad \forall \varphi \in C^0(\overline{\Omega}, \mathbb{R}). \tag{2.5}$$

Let us also remark that since $L_\varphi(\sigma) = L_\varphi(\widetilde{\sigma})$ one has $i_Q = i_{\widetilde{Q}}$, for all $Q \in \mathcal{M}_+^1(C)$. Finally, let us note that the total mass of i_Q is the average length with respect to Q:

$$i_Q(\overline{\Omega}) = \int_C l(\sigma) dQ(\sigma). \tag{2.6}$$

If the probability Q is concentrated on injective curves, one can also express the measure i_Q through $\mathcal{H}^1-$integrals as explained in the next remark.

Remark 2.2. If a curve σ is injective, then one could also write $l(\sigma) = \mathcal{H}^1(\sigma([0,1]))$ and $L_\varphi(\sigma) = \int_{\sigma([0,1])} \varphi \, d\mathcal{H}^1$. Moreover, if for $\gamma-$a.e. (x,y) the probability $p^{x,y}$ is concentrated on the set of injectives curves from x to y, one could also define the measure $I_{\gamma,p}$ by replacing the integral with respect to $|\dot\sigma(t)|dt$ in (2.3) with an integral in $d\mathcal{H}^1$. Notice moreover that for every Borel subset $A \subset \overline{\Omega}$ one would have:

$$I_{\gamma,p}(A) = \int_{\overline{\Omega}\times\overline{\Omega}} \left(\int_{C^{x,y}} \mathcal{H}^1(A \cap \sigma) dp^{x,y}(\sigma) \right) d\gamma(x,y) = \int_C \mathcal{H}^1(A \cap \sigma) dQ_{\gamma,p}(\sigma).$$

If we imagine that for each $\sigma \in C^{x,y}$, the mass of travelers commuting on σ is uniformly distributed on σ, this means that $I_{\gamma,p}(A)$ represents the cumulative traffic through the region A. The same formula stays true, under no injectivity assumption, if we replace $\mathcal{H}^1(A \cap \sigma)$ with $L_{I_A}(\sigma)$ and in this case the cumulative traffic takes into account the number of times a path σ passes through the points of A.

In the sequel, it will be convenient to formulate our optimization problem in terms of $Q = Q_{\gamma,p}$ rather than in the transportation strategy (γ, p). To that end, we shall use the following:

Lemma 2.3. *Let us define*

$$\mathcal{Q}(\mu_0,\mu_1) := \{Q_{\gamma,p} \ : \ (\gamma,p) \text{ transportation strategy}\}$$

then one has

$$\mathcal{Q}(\mu_0,\mu_1) = \{Q \in \mathcal{M}^1_+(C) \ : \ e_0\sharp Q = \mu_0, \ e_1\sharp Q = \mu_1\}.$$

Proof. If (γ,p) is a transportation strategy then $e_0\sharp Q_{\gamma,p} = \pi_0\sharp\gamma = \mu_0$, and $e_1\sharp Q_{\gamma,p} = \pi_1\sharp\gamma = \mu_1$. Now let $Q \in \mathcal{M}^1_+(C)$ be such that $e_0\sharp Q = \mu_0$, $e_1\sharp Q = \mu_1$. If we define $\gamma := (e_0,e_1)\sharp Q$, we have $\gamma \in \Pi(\mu_0,\mu_1)$. It then follows from the disintegration theorem (see [8]) that there exists $p = (p^{x,y})_{(x,y)\in\overline{\Omega}\times\overline{\Omega}}$ a Borel family of probability measures on C such that $p^{x,y}(C^{x,y}) = 1$ for γ-a.e. $(x,y) \in \overline{\Omega}\times\overline{\Omega}$ and $Q = p^{x,y} \otimes \gamma$. Hence $Q = Q_{\gamma,p}$ for a transportation strategy (γ,p). □

At this point, a natural way to model traffic congestion is, for a given transportation strategy (γ,p), to consider that the transportation cost per unit of mass between x and y is given by

$$c_{\gamma,p}(x,y) = \int_{C^{x,y}} L_{G_{I_{\gamma,p}}}(\sigma) dp^{x,y}(\sigma) \tag{2.7}$$

where $G_{I_{\gamma,p}}$ is a nonnegative function which depends (in a way that will be specified later on) on the traffic intensity $I_{\gamma,p}$. The optimal transportation with traffic congestion then takes the form (to be compared with the usual Monge-Kantorovich problem (2.2)):

$$\inf\left\{\int_{\overline{\Omega}\times\overline{\Omega}} c_{\gamma,p}(x,y)d\gamma(x,y) \ : \ (\gamma,p) \text{ transportation strategy}\right\}. \tag{2.8}$$

Setting $Q = Q_{\gamma,p}$ and using formally (2.5), we see that the total transportation cost in (2.8) can be rewritten as:

$$\int_{\overline{\Omega}\times\overline{\Omega}} c_{\gamma,p}(x,y)d\gamma(x,y) = \int_C L_{G_{i_Q}}(\sigma)dQ(\sigma) = \int_{\overline{\Omega}} G_{i_Q}(x)di_Q(x).$$

Hence using lemma 2.3, we can reformulate (2.8) in terms of Q only:

$$\inf\left\{\int_{\overline{\Omega}} G_{i_Q}(x)di_Q(x) \ : \ Q \in \mathcal{Q}(\mu_0,\mu_1)\right\} \tag{2.9}$$

Note that in the definition (2.7), it is required that $G_{I_{\gamma,p}}$ is continuous (or at least l.s.c.) whereas the form (2.9) allows for more general forms of congestion through $i \mapsto G_i$. From now on, we assume that G has the following local form:

$$G_i(x) = g\left(\frac{di}{d\mathcal{L}^2}(x)\right), \tag{2.10}$$

where $\frac{di}{d\mathcal{L}^2}$ is the Radon-Nikodym derivative of i with respect to the Lebesgue measure and g is a nondecreasing function $\mathbb{R}_+ \to \mathbb{R}_+$ such that the function H defined by $H(z) = zg(z)$ for all $z \in \mathbb{R}_+$ is convex and superlinear (i.e. $\lim_{z\to+\infty} g(z) = +\infty$).

The optimization problem we shall study now then reads as:

$$\inf_{Q\in\mathcal{Q}(\mu_0,\mu_1)} \mathcal{H}(i_Q) \text{ where } \mathcal{H}(i) = \begin{cases} \int_\Omega H(i(x))dx \text{ if } i << \mathcal{L}^2, \\ +\infty \text{ otherwise.} \end{cases} \tag{2.11}$$

In the sequel, we shall say that a transportation strategy (γ, p) is optimal if $Q_{\gamma,p}$ solves (2.11).

Remark 2.4. It will be clear in the sequel that the probability $Q_{\gamma,p}$ associated to an optimal transportation strategy (γ, p) will be concentrated on injective curves, so that the interpretation in terms of $\mathcal{H}^1-$integrals (see Remark 2.2) may apply.

2.3 Existence of minimizers

From now on, we make the following assumptions:

- H is convex and nondecreasing on $\mathbb{R}_+$ with $H(0) = 0$,
- there exists $q > 1$, and positive constants a and b such that $az^q \leq H(z) \leq b(z^q + 1)$ for all $z \in \mathbb{R}_+$,
- H is differentiable on $\mathbb{R}_+$, and there exists a positive constant c such that $0 \leq H'(z) \leq c(z^{q-1} + 1)$, for all $z \in \mathbb{R}_+$,
- the following set

$$\mathcal{Q}^q(\mu_0, \mu_1) := \{Q \in \mathcal{Q}(\mu_0, \mu_1) \ : \ i_Q \in L^q\} \tag{2.12}$$

  is nonempty.

These assumptions enable us to simply rewrite (2.11) as:

$$\inf_{Q \in \mathcal{Q}^q(\mu_0, \mu_1)} \int_\Omega H(i_Q(x))dx. \tag{2.13}$$

Remark 2.5. Let us discuss the assumption that $\mathcal{Q}^q(\mu_0, \mu_1) \neq \emptyset$ which, at first glance, may seem difficult to check. In order to have the existence of a $Q \in \mathcal{Q}(\mu_0, \mu_1)$ such that $i_Q \in L^q$ it is sufficient that μ_0 and μ_1 are in L^q. This result, which is not obvious, follows from the regularity results of De Pascale and Pratelli (see [9] and [10]) who proved that L^q regularity of μ_0 and μ_1 implies that for γ solving the Monge-Kantorovich problem (2.2) with $c(x, y) = |x - y|$ and $p^{x,y} = \delta_{[x,y]}$ (the Dirac mass at the segment $[x, y]$) for every x and y the corresponding traffic density $I_{\gamma,p}$ is L^q. It is not necessary however that μ_0 and μ_1 are absolutely continuous for the assumption to be satisfied: let us consider for instance the case where $\overline{\Omega} = [0, 1]^2$ and μ_0 and μ_1 are respectively the one-dimensional Hausdorff measures of the segments $[(0, 0), (0, 1)]$ and $[(1, 0), (1, 1)]$. If we define $\gamma := (\mathrm{id}, \mathrm{id} + (1, 0))\sharp\mu_0$ and $p^{x,y} = \delta_{[x,y]}$ then a straightforward computation shows that $I_{\gamma,p}$ is uniform on $[0, 1]^2$.

Under the assumptions above, we are going prove that (2.13) admits a solution. The proof of existence involves some preliminary lemmas.

Lemma 2.6. *For any $\varphi \in C^0(\overline{\Omega}, \mathbb{R}_+)$, L_φ is lower semi-continuous on C for the uniform topology, indeed for any $\sigma \in C$, one has:*

$$L_\varphi(\sigma) = \sup \Big\{ \sum_{i=1}^{n} \left(\inf_{[t_i, t_{i+1}]} (\varphi \circ \sigma) \right) |\sigma(t_{i+1}) - \sigma(t_i)| : \\ ([t_i, t_{i+1}])_i \text{ is a subdivision of } [0, 1] \Big\}. \tag{2.14}$$

Proof. For any subdivision $([t_i, t_{i+1}])_{i=1,\dots n}$, we have:

$$\begin{aligned} L_\varphi(\sigma) &= \sum_{i=1}^{n} \int_{t_i}^{t_{i+1}} \varphi(\sigma(t))|\dot\sigma(t)|\ dt \\ &\geq \sum_{i=1}^{n} \inf_{[t_i,t_{i+1}]}(\varphi\circ\sigma) \int_{t_i}^{t_{i+1}} |\dot\sigma(t)|\ dt \\ &\geq \sum_{i=1}^{n} \inf_{[t_i,t_{i+1}]}(\varphi\circ\sigma)|\sigma(t_{i+1})-\sigma(t_i)|. \end{aligned}$$

Taking the supremum over all such divisions, we get:

$$L_\varphi(\sigma) \geq \sup\Big\{ \sum_{i=1}^{n} \inf_{[t_i,t_{i+1}]}(\varphi\circ\sigma)|\sigma(t_{i+1})-\sigma(t_i)| : ([t_i,t_{i+1}])_i \text{ is a subdivision of } [0,1]\Big\}.$$

Let us prove the converse inequality. Let $\varepsilon > 0$, since $\varphi\circ\sigma$ is uniformly continuous, there is a $\delta > 0$ such that:

$$\forall t, t' \in [0,1]^2, \quad (|t-t'| \leq \delta \Rightarrow |\varphi(\sigma(t)) - \varphi(\sigma(t'))| \leq \varepsilon)\,.$$

For any subdivision $([t_i, t_{i+1}])_{i=1,\dots n}$ such that $|t_i - t_{i+1}| \leq \delta$ for all i, we have:

$$\begin{aligned} L_\varphi(\sigma) &\leq \sum_{i=1}^{n} (\inf_{[t_i,t_{i+1}]}(\varphi\circ\sigma) + \varepsilon) \int_{t_i}^{t_{i+1}} |\dot\sigma(t)|\ dt \\ &= \sum_{i=1}^{n} (\inf_{[t_i,t_{i+1}]}(\varphi\circ\sigma) + \varepsilon) \sup\Big\{ \sum_j |\sigma(\tau_j) - \sigma(\tau_{j+1})| : \\ &\qquad\qquad ([\tau_j,\tau_{j+1}])_j \text{ is a subdivision of } [t_i,t_{i+1}]\Big\} \\ &\leq \sup\Big\{ \sum_i \sum_j (\inf_{[\tau_j,\tau_{j+1}]}(\varphi\circ\sigma) + \varepsilon)|\sigma(\tau_j) - \sigma(\tau_{j+1})| : \\ &\qquad\qquad ([\tau_j,\tau_{j+1}])_j \text{ is a subdivision of } [t_i,t_{i+1}]\Big\} \\ &= \sup\Big\{ \sum_{i=1}^{n} (\inf_{t\in[t_i,t_{i+1}]}(\varphi\circ\sigma) + \varepsilon)|\sigma(t_{i+1}) - \sigma(t_i)| : \\ &\qquad\qquad ([t_i,t_{i+1}])_i \text{ is a subdivision of } [0,1]\Big\}. \end{aligned}$$

As this last inequality is true for any $\varepsilon > 0$ we get (2.14). The lower semi-continuity is then obvious since, by (2.14), L_φ is the supremum of family of lower semi-continuous functions on $C^0([0,1], \overline{\Omega})$. □

Lemma 2.7. *Let $(Q_n)_n \in \mathcal{M}_+^1(C^0([0,1],\mathbb{R}^2))^{\mathbb{N}}$ such that $Q_n(C) = 1$ for all n and there exists a constant $M > 0$ such that:*

$$\sup_n \int_C l(\sigma)\ dQ_n(\sigma) \leq M.$$

Then the sequence $(\widetilde{Q}_n)_n$ is tight and admits a subsequence that converges weakly $*$ to a probability Q such that $Q(C) = 1$.

Proof. The tightness of $(\widetilde{Q}_n)_n$ easily follows from the inequality:

$$\begin{aligned} \widetilde{Q}_n\left(\{\sigma \in \widetilde{C} :\ |\dot{\sigma}| > K\}\right) &= Q_n\left(\{\sigma \in C :\ l(\sigma) > K\}\right) \\ &\leq \frac{1}{K}\int_C l(\sigma)\ dQ_n(\sigma). \end{aligned} \tag{2.15}$$

By Prokhorov theorem, we may therefore assume, passing to a subsequence if necessary, that $(\widetilde{Q}_n)_n$ converges weakly $*$ to $Q \in \mathcal{M}_+^1(C^0([0,1],\mathbb{R}^2)$. It remains to show that $Q(C) = 1$. For $K > 0$ let us define $C_K := \{\sigma \in C :\ |\dot{\sigma}| \leq K\}$, then inequality (2.15) and the fact that the measures $\widetilde{Q}_n$ are concentrated on $\widetilde{C}$ yield

$$\sup_n \widetilde{Q}_n(C \backslash C_K) = \sup_n \widetilde{Q}_n(\widetilde{C} \backslash C_K) \leq \frac{M}{K},$$

for every $K > 0$, which implies

$$\begin{aligned} 1 = \limsup_n \widetilde{Q}_n(C) &\leq \limsup_n \widetilde{Q}_n(C_K) + \limsup_n \widetilde{Q}_n(C \setminus C_K) \\ &\leq Q(C_K) + \frac{M}{K}. \end{aligned}$$

Letting K tend to ∞, we then get $Q(C) = \sup_K Q(C_K) = 1$. □

Lemma 2.8. *Let $(Q_n)_n$ be a sequence in $\mathcal{M}_+^1(C)$ that converges weakly $*$ to some $Q \in \mathcal{M}_+^1(C)$. If there exists $i \in \mathcal{M}_+(\overline{\Omega})$ such that i_{Q_n} converges weakly $*$ to i in $\mathcal{M}_+(\overline{\Omega})$ then we have $i_Q \leq i$.*

Proof. Let $\varphi \in C^0(\overline{\Omega}, \mathbb{R}_+)$, we first have:

$$\int_{\overline{\Omega}} \varphi di = \lim_n \int_{\overline{\Omega}} \varphi di_{Q_n} = \lim_n \int_C L_\varphi dQ_n$$

it easily follows from lemma 2.6 that $Q \mapsto \int_C L_\varphi dQ$ is l.s.c. for the weak $*$ topology of $\mathcal{M}_+^1(C)$, we then have:

$$\int_{\overline{\Omega}} \varphi di \geq \int_C L_\varphi dQ = \int_{\overline{\Omega}} \varphi di_Q.$$ □

Now, we are in position to prove:

Theorem 2.9. *The minimization problem (2.13) admits a solution.*

Proof. Our assumptions imply that the value of (2.13) is finite. Let $(Q_n)_n$ be some minimizing sequence of (2.13). From the identity $i_Q = i_{\widetilde{Q}}$, we may assume $Q_n = \widetilde{Q}_n$ for all n. We deduce from our growth condition on H, that $(i_{Q_n})_n$ is bounded in L^q. On the one hand, extracting a subsequence if necessary, we may therefore assume that $(i_{Q_n})_n$ converges weakly in L^q to some i. On the other hand, since i_{Q_n} is bounded in L^q and hence in L^1 we have

$$\sup_n \int_C l(\sigma) dQ_n(\sigma) = \sup_n \int_\Omega i_{Q_n} < +\infty.$$

Moreover $Q_n = \widetilde{Q}_n$ and we deduce from lemma 2.7 that (up to some subsequence) $(Q_n)_n$ weakly $*$ converges to some Q in $\mathcal{M}^1_+(C)$. Since $\mathcal{Q}(\mu_0, \mu_1)$ is obviously weakly $*$ closed, we have $Q \in \mathcal{Q}(\mu_0, \mu_1)$ and lemma 2.8 implies that $i_Q \leq i$ (consequently to this enaquality i_Q is absolutely continuous). From the monotonicity and convexity of H we then have:

$$\int_\Omega H(i_Q(x))dx \leq \int_\Omega H(i(x))dx \leq \liminf_n \int_\Omega H(i_{Q_n}(x))dx$$

which proves that Q solves (2.13). □

Let us remark that if H is furthermore assumed to be strictly convex then if Q_1 and Q_2 solves (2.13) then $i_{Q_1} = i_{Q_2}$ so that the optimal traffic intensity is unique (of course, this does not imply in general that $Q_1 = Q_2$ or that the corresponding optimal transportation strategy is unique).

3 Characterization of the minimizers

In the sequel, we shall denote by q^* the conjugate exponent of q, given by $q^* = q/(q-1)$.

3.1 Optimality conditions

The variational inequalities characterizing solutions of the convex problem (2.13) can be expressed as follows

Proposition 3.1. $\overline{Q} \in \mathcal{Q}^q(\mu_0, \mu_1)$ *solves (2.13) if and only if*

$$\int_\Omega \overline{\xi} i_{\overline{Q}} = \inf \left\{ \int_\Omega \overline{\xi} i_Q \ : \ Q \in \mathcal{Q}^q(\mu_0, \mu_1) \right\} \text{ with } \overline{\xi} := H'(i_{\overline{Q}}) \in L^{q^*}. \quad (3.1)$$

Proof. Assume that $\overline{Q}$ solves (2.13), then for every $Q \in \mathcal{Q}^q(\mu_0, \mu_1)$, one has:

$$0 \leq \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon}[\mathcal{H}(i_{\overline{Q}+\varepsilon(Q-\overline{Q})}) - \mathcal{H}(i_{\overline{Q}})] = \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon}[\mathcal{H}(i_{\overline{Q}} + \varepsilon(i_Q - i_{\overline{Q}})) - \mathcal{H}(i_{\overline{Q}})]$$
$$= \int_\Omega H'(i_{\overline{Q}})(i_Q - i_{\overline{Q}}) = \int_\Omega \overline{\xi}(i_Q - i_{\overline{Q}})$$

which proves (3.1). Conversely, if $\overline{Q} \in \mathcal{Q}^q(\mu_0, \mu_1)$ satisfies (3.1), then by convexity of H, for every every $Q \in \mathcal{Q}^q(\mu_0, \mu_1)$, one has:

$$\mathcal{H}(i_Q) - \mathcal{H}(i_{\overline{Q}}) \geq \int_\Omega \overline{\xi}(i_Q - i_{\overline{Q}}) \geq 0. \qquad \square$$

The next paragraphs will be devoted to investigate the precise meaning of (3.1). Before going further, let us do some formal manipulations to give a formal interpretation of (3.1) in terms of optimal transportation strategy. Let us assume that $\overline{Q}$ solves (2.13) and let us write $\overline{Q} = Q_{\overline{\gamma},\overline{p}}$ for some (optimal) transportation strategy $(\overline{\gamma}, \overline{p})$ and define $\overline{\xi} := H'(i_{\overline{Q}})$, then (3.1) formally can be rewritten as:

$$\begin{aligned}
\int_\Omega \overline{\xi} i_{\overline{Q}} &= \int_C L_{\overline{\xi}}(\sigma) d\overline{Q}(\sigma) \\
&= \int_{\overline{\Omega}\times\overline{\Omega}} \left(\int_{C^{x,y}} L_{\overline{\xi}}(\sigma) d\overline{p}^{x,y}(\sigma) \right) d\overline{\gamma}(x,y) \\
&= \inf_{(\gamma,p) \text{ transp. strategy}} \int_{\overline{\Omega}\times\overline{\Omega}} \left(\int_{C^{x,y}} L_{\overline{\xi}}(\sigma) dp^{x,y}(\sigma) \right) d\gamma(x,y) \\
&= \inf_{\gamma \in \Pi(\mu_0,\mu_1)} \int_{\overline{\Omega}\times\overline{\Omega}} \left(\inf_{p \in \mathcal{M}_+^1(C^{x,y})} \int_{C^{x,y}} L_{\overline{\xi}}(\sigma) dp(\sigma) \right) d\gamma(x,y) \\
&= \inf_{\gamma \in \Pi(\mu_0,\mu_1)} \int_{\overline{\Omega}\times\overline{\Omega}} \left(\inf_{\sigma \in C^{x,y}} L_{\overline{\xi}}(\sigma) \right) d\gamma(x,y)
\end{aligned}$$

defining (again formally) the transportation cost:

$$c_{\overline{\xi}}(x,y) = \inf_{\sigma \in C^{x,y}} L_{\overline{\xi}}(\sigma),$$

we then firstly have:

$$\int_{\overline{\Omega}\times\overline{\Omega}} c_{\overline{\xi}}(x,y) d\overline{\gamma}(x,y) \leq \int_C L_{\overline{\xi}} d\overline{Q} = \inf_{\gamma \in \Pi(\mu_0,\mu_1)} \int_{\overline{\Omega}\times\overline{\Omega}} c_{\overline{\xi}}(x,y) d\gamma(x,y)$$

so that $\overline{\gamma}$ solves the Monge-Kantorovich problem:

$$\inf_{\gamma \in \Pi(\mu_0,\mu_1)} \int_{\overline{\Omega}\times\overline{\Omega}} c_{\overline{\xi}}(x,y) d\gamma(x,y).$$

Secondly:

$$\int_C L_{\overline{\xi}}(\sigma) d\overline{Q}(\sigma) = \int_{\overline{\Omega}\times\overline{\Omega}} c_{\overline{\xi}}(x,y) d\overline{\gamma}(x,y)$$
$$= \int_C c_{\overline{\xi}}(\sigma(0),\sigma(1)) d\overline{Q}(\sigma)$$

and since $L_{\overline{\xi}}(\sigma) \geq c_{\overline{\xi}}(\sigma(0),\sigma(1))$, we get

$$L_{\overline{\xi}}(\sigma) = c_{\overline{\xi}}(\sigma(0),\sigma(1)) \quad \text{for } \overline{Q}\text{-a.e. } \sigma.$$

or, in an equivalent way, for $\overline{\gamma}$-a.e. (x,y) one has:

$$L_{\overline{\xi}}(\sigma) = c_{\xi}(x,y) \quad \text{for } \overline{p}^{x,y}\text{-a.e. } \sigma.$$

Since $\overline{\xi}$ is only L^{q^*}, L_ξ and $c_{\overline{\xi}}$ are not well-defined and the previous arguments are purely formal. In paragraph 3.2, we will extend the definition of c_ξ to the case where ξ is only L^{q^*} under the additional assumption $q < 2$. This will enable us to make the formal argument above rigorous and to characterize optimal transportation strategies in paragraph 3.3.

3.2 The transportation cost $\overline{c}_\xi$ when ξ is L^{q^*}

For a non-negative function $\xi \in C^0(\Omega)$ we define

$$c_\xi(x,y) = \inf\{L_\xi(\sigma) \,:\, \sigma \in C^{x,y}\}.$$

Proposition 3.2. *Let us assume that $q < 2$ and define $\alpha := 1 - 2/q^*$, then there exists a non-negative constant C such that for every $\xi \in C^0(\overline{\Omega}, \mathbb{R}^+)$ and every $(x_1,y_1,x_2,y_2) \in \Omega^4$, one has:*

$$|c_\xi(x_1,y_1) - c_\xi(x_2,y_2)| \leq C\|\xi\|_{L^{q^*}(\Omega)} \left(|x_1-x_2|^\alpha + |y_1-y_2|^\alpha\right). \tag{3.2}$$

Consequently, if $(\xi_n)_n \in C^0(\overline{\Omega}, \mathbb{R}^+)^{\mathbb{N}}$ is bounded in L^{q^*}, then $(c_{\xi_n})_n$ admits a subsequence that converges in $C^0(\overline{\Omega}\times\overline{\Omega}, \mathbb{R}_+)$.

Proof. Let $\xi \in C^0(\overline{\Omega}, \mathbb{R}^+)$ and $x,\ y \in \Omega^2$. For $k > 0$ let $\sigma_k \in C^{x,y}$ be such that

$$\int_0^1 \xi(\sigma_k(t))|\dot{\sigma}_k(t)|\, dt \leq c_\xi(x,y) + \frac{1}{k}.$$

Then for all $\varepsilon > 0$ and z such that $y + \varepsilon z \in \Omega$ and $t_0 \in (0,1)$ we consider the following element of $C^{x,y+\varepsilon z}$:

$$\sigma_{k,t_0}(t) := \begin{cases} \sigma_k\left(\frac{t}{t_0}\right) & \text{if } t \in [0,t_0] \\ y + \left(\frac{t-t_0}{1-t_0}\right)\varepsilon z & \text{if } t \in [t_0,1]. \end{cases}$$

We then have, for all $k > 0$:

$$\begin{aligned} c_\xi(x, y+\varepsilon z) &\leq \int_0^1 \xi(\sigma_{k,t_0}(t))|\dot{\sigma}_{k,t_0}(t)|\ dt \\ &= \int_0^1 \xi(\sigma_k(t))|\dot{\sigma}_k(t)|\ dt + \int_0^1 \xi(y+t\varepsilon z)\varepsilon|z|\ dt \\ &\leq c_\xi(x,y) + \varepsilon|z|\int_0^1 \xi(y+t\varepsilon z)dt + \frac{1}{k}. \end{aligned}$$

Now we let k tend to ∞ and we get

$$\frac{c_\xi(x, y+\varepsilon z) - c_\xi(x,y)}{\varepsilon} \leq |z|\int_0^1 \xi(y+t\varepsilon z)\ dt,$$

and by a similar argument

$$\frac{c_\xi(x, y) - c_\xi(x,y+\varepsilon z)}{\varepsilon} \leq |z|\int_0^1 \xi(y+(1-t)\varepsilon z)\ dt.$$

This implies that $c_\xi(x,.) \in W^{1,\infty}$ and:

$$|\nabla_y c_\xi(x,.)| \leq |\xi(.)|, \quad \text{for all } x. \tag{3.3}$$

By symmetry we also have

$$|\nabla_x c_\xi(.,y)| \leq |\xi(.)|, \quad \text{for all} y. \tag{3.4}$$

Since $q^* > 2$, we deduce from (3.3), (3.4) and Morrey's Theorem (see [5], Chapter IX), that there is a constant C such that:

$$\begin{aligned} |c_\xi(x,y_1) - c_\xi(x,y_2)| &\leq C\|\xi\|_{L^{q^*}}|y_1 - y_2|^\alpha, \text{ for all } x, y_1, y_2 \text{ in } \Omega, \\ |c_\xi(x_1,y) - c_\xi(x_2,y)| &\leq C\|\xi\|_{L^{q^*}}|x_1 - x_2|^\alpha, \text{ for all } x_1, x_2, y \text{ in } \Omega. \end{aligned}$$

This proves (3.2). The second claim in the proposition then follows from (3.2), the identity $c_{\xi_n}(x,x) = 0$ and Ascoli's theorem. □

From now on, we further assume that $q < 2$. For a non-negative function $\xi \in L^{q^*}(\Omega)$ we then define

$$\overline{c}_\xi(x,y) = \sup\left\{c(x,y)\ : c \in \mathcal{A}(\xi)\right\},$$

where

$$\mathcal{A}(\xi) = \left\{\lim_n c_{\xi_n} \text{ in } C^0(\overline{\Omega}\times\overline{\Omega})\ :\ (\xi_n)_n \in C^0(\overline{\Omega}),\ \xi_n \geq 0,\ \xi_n \to \xi \text{ in } L^{q^*}\right\}.$$

Remark 3.3. The definition of $\overline{c}_\xi$ is unchanged if one replaces $\xi_n \to \xi$ in L^{q^*} by $\xi_n \rightharpoonup \xi$ in L^{q^*} in the definition of $\mathcal{A}(\xi)$. Indeed, if we do so, we obviously obtain a function which is larger than $\overline{c}_\xi$. Now, let us assume that $\xi_n \rightharpoonup \xi$ in L^{q^*}, and c_{ξ_n} converges to c in $C^0(\overline{\Omega} \times \overline{\Omega})$, using Mazur's Lemma there exists a sequence η_n which converges strongly to ξ and such that each η_n is in the convex hull of $\{\xi_k,\ k \leq n\}$. It is clear that for fixed x, y, $\xi \to c_\xi(x,y)$ is concave hence $c(x,y) = \lim c_{\xi_n}(x,y) \leq \limsup\ c_{\eta_n}(x,y) \leq \overline{c}_\xi(x,y)$.

When ξ is continuous, one has:

Lemma 3.4. *If ξ is continuous and non-negative, then $\overline{c}_\xi = c_\xi$.*

Proof. The inequality $\overline{c}_\xi \geq c_\xi$ is obvious, as one can always choose the constant sequence $\xi_n = \xi$ in the definition of $\overline{c_\xi}$. Let us show now the opposite inequality. Take $x, y \in \Omega$, $\varepsilon > 0$ and $\sigma \in C^{x,y}$ such that $L_\xi(\sigma) < c_\xi(x,y) + 1/k$. We can choose σ so that it is piecewise linear, by density of this kind of curves and using the continuity of ξ. Let $(S_i)_{i=0,\dots,m-1}$ be the segments which compose σ with $S_i = x_i x_{i+1}$, $x_0 = x$ and $x_m = y$. Let us fix, moreover, a sequence $\xi_n \to \xi$ such that $c_{\xi_n} \to c$. Now, we want to prove $c \leq c_\xi$. Fix a small number $\delta > 0$ and for any $\alpha \in [0, \delta]$ let us define a curve σ^α in the following way: let R be the clockwise 90 degrees rotation in the plane; let $x_i'(\alpha)$ and $x_i''(\alpha)$ be the only points such that $x_i x_i'(\alpha) = \alpha R e_i$ and $x_{i+1} x_i''(\alpha) = \alpha R e_i$, where e_i is the tangent unit vector to σ in the S_i part; define σ^α by linking any point $x_i'(\alpha)$ to $x_i''(\alpha)$ by some segments $S_i'(\alpha)$ and $x_i''(\alpha)$ to $x_{i+1}'(\alpha)$ by some arcs $A_{i+1}(\alpha)$ with center x_{i+1} and radius α. In this way we have $\sigma^\alpha \in C^{x_\alpha, y_\alpha}$, where $x_\alpha = x_0'(\alpha)$ and $y_\alpha = x_m''(\alpha)$. Let $R_i(\delta)$ be the rectangle whose vertices are the points x_i, $x_i'(\delta)$, $x_i''(\delta)$ and x_{i+1} and let $B_i(\delta)$ be the circular sector centered at x_i and whose vertices are $x_{i-1}''(\delta)$ and $x_i'(\delta)$.

If we compute $\int_0^\delta L_{\xi_n}(\sigma_\alpha)\, d\alpha$ it is not difficult to see that we get

$$\int_0^\delta L_{\xi_n}(\sigma_\alpha)\, d\alpha = \sum_{i=0}^{m-1} \left(\int_{R_i(\delta)} \xi_n d\mathcal{L}^2 \right) + \sum_{i=1}^{m-1} \left(\int_{B_i(\delta)} \xi_n d\mathcal{L}^2 \right).$$

Moreover it holds $c_{\xi_n}(x^\alpha, y^\alpha) \leq L_{\xi_n}(\sigma^\alpha)$, hence we get

$$\int_0^\delta c_{\xi_n}(x^\alpha, y^\alpha)\, d\alpha \leq \sum_{i=0}^{m-1} \left(\int_{R_i(\delta)} \xi_n d\mathcal{L}^2 \right) + \sum_{i=1}^{m-1} \left(\int_{B_i(\delta)} \xi_n d\mathcal{L}^2 \right).$$

If we pass to the limit as $n \to \infty$ we get, by using the uniform convergence of c_{ξ_n} to c on the left hand side and the L^{q^*} convergence of ξ_n to ξ on the right hand side,

$$\int_0^\delta c(x^\alpha, y^\alpha)\, d\alpha \leq \sum_{i=0}^{m-1} \left(\int_{R_i(\delta)} \xi d\mathcal{L}^2 \right) + \sum_{i=1}^{m-1} \left(\int_{B_i(\delta)} \xi d\mathcal{L}^2 \right).$$

Then we divide by δ and we pass to the limit as $\delta \to 0$. Using the fact that c is continuous we have

$$\lim_{\delta\to 0} \frac{1}{\delta} \int_0^\delta c(x^\alpha, y^\alpha)\, d\alpha = c(x, y).$$

On the other side, we may notice that the areas of the sectors $B_i(\delta)$ may be estimated by $C\delta^2$ and hence we have, for $\delta \to 0$,

$$\frac{1}{\delta} \sum_{i=1}^{m-1} \left(\int_{B_i(\delta)} \xi d\mathcal{L}^2 \right) \leq mC|\xi|\delta \to 0.$$

On the contrary the integrals over $R_i(\delta)$, when divided by δ, converge on the integrals on the segments S_i, which give exactly the integral over the curve σ, i.e. $L_\xi(\sigma)$. We have consequently

$$\lim_{\delta\to 0} \frac{1}{\delta} \left(\sum_{i=0}^{m-1} \left(\int_{R_i(\delta)} \xi d\mathcal{L}^2 \right) + \sum_{i=1}^{m-1} \left(\int_{B_i(\delta)} \xi d\mathcal{L}^2 \right) \right) = L_\xi(\sigma) < c_\xi(x, y) + \frac{1}{k}.$$

This gives

$$c(x, y) < c_\xi(x, y) + \frac{1}{k}$$

and, k being arbitrary, we also get $c \leq c_\xi$ and the thesis. □

Lemma 3.5. *Let us assume that $q < 2$ and let ξ be non-negative function belonging to L^{q^*}, then there exists a sequence $(\xi_n)_n \in C^0(\Omega)$, $\xi_n \geq 0$, $\xi_n \to \xi$ in L^{q^*}, such that c_{ξ_n} converges uniformly to $\overline{c}_\xi$ on Ω.*

Proof. It is easy to see that for every $(x, y) \in \Omega^2$ there exists a sequence of non-negative continuous functions $(\xi_n)_n$ converging to ξ in $L^{q^*}(\Omega)$ such that c_{ξ_n} converges in C^0 and $\overline{c}_\xi(x, y) = \lim_n c_{\xi_n}(x, y)$. Let I be a finite set, $(x_i, y_i) \in \Omega^2$ for all $i \in I$ and for every i, let (ξ_n^i) be a sequence of non-negative continuous functions converging to ξ in $L^{q^*}(\Omega)$ such that $\overline{c}_\xi(x_i, y_i) = \lim_n c_{\xi_n^i}(x_i, y_i)$. Let us set $\xi_n := \max_{i\in I} \xi_n^i$, we then have $(\xi_n)_n$ converging to ξ in $L^{q^*}(\Omega)$, and

$$\overline{c}_\xi(x_i, y_i) \leq \liminf_n\ c_{\xi_n}(x_i, y_i) \leq \limsup_n\ c_{\xi_n}(x_i, y_i) \leq \overline{c}_\xi(x_i, y_i).$$

We thus have $\overline{c}_\xi(x_i, y_i) = \lim_n c_{\xi_n}(x_i, y_i)$ for every $i \in I$. Now, let $(x_i, y_i)_{i\in\mathbb{N}}$ be a dense sequence of points of Ω^2. From what preceeds, for every n, there exists a continuous non-negative ξ_n such that

$$\|\xi_n - \xi\|_{L^{q^*}} \leq \frac{1}{n},\ |\overline{c}_\xi(x_k, y_k) - c_{\xi_n}(x_k, y_k)| \leq \frac{1}{n},\ \forall k \leq n.$$

By the Hölder estimate of proposition 3.2 and Ascoli's theorem, passing to a subsequence if necessary, we may assume that c_{ξ_n} converges in C^0 to some c. Since obviously $c(x_k, y_k) = \overline{c}_\xi(x_k, y_k)$ for all k , we deduce $c = \overline{c}_\xi$ and the desired result follows. □

The next lemma enables us to extend L_ξ in some sense when $\xi \geq 0$ is only L^{q^*}:

Lemma 3.6. *Let us assume that $q < 2$. Let $Q \in \mathcal{Q}^q(\mu_0, \mu_1)$, ξ be a non-negative element of L^{q^*}, and $(\xi_n)_n$ be a sequence of non-negative continuous functions that converges to ξ in L^{q^*}, then we have the following:*

(i) $(L_{\xi_n})_n$ converges strongly in $L^1(C, Q)$ to some limit which is independent of the approximating sequence $(\xi_n)_n$ and which will again be denoted L_ξ.

(ii) The following equality holds:

$$\int_\Omega \xi(x) i_Q(x)\ dx = \int_C L_\xi(\sigma)\ dQ(\sigma). \tag{3.5}$$

(iii) The following inequality holds for Q-a.e. $\sigma \in C$:

$$L_\xi(\sigma) \geq \overline{c}_\xi(\sigma(0), \sigma(1)). \tag{3.6}$$

Proof. For all n and m in $\mathbb{N}$ we have:

$$\begin{aligned}
\int_C |L_{\xi_n}(\sigma) - L_{\xi_m}(\sigma)|\ dQ(\sigma) &= \int_C \left| \int_0^1 (\xi_n(\sigma(t)) - \xi_m(\sigma(t)))\, |\dot{\sigma}(t)|\ dt \right| dQ(\sigma) \\
&\leq \int_\Omega |\xi_n(x) - \xi_m(x)| i_Q(x)\ dx \\
&\leq \|\xi_n - \xi_m\|_{L^{q^*}} \|i_Q\|_{L^q}.
\end{aligned}$$

This implies that $(L_{\xi_n})_n$ is a Cauchy sequence in $L^1(C, Q)$ and it is obvious, from the previous inequality, that its $L^1(C, Q)$ limit does not depend on the approximating sequence $(\xi_n)_n$.

The proof of (ii) follows from (i):

$$\begin{aligned}
\int_\Omega \xi(x) i_Q(x)\ dx &= \lim_n \int_\Omega \xi_n(x) i_Q(x)\ dx \\
&= \lim_n \int_C L_{\xi_n}(\sigma)\ dQ(\sigma) \\
&= \int_C L_\xi(\sigma)\ dQ(\sigma).
\end{aligned}$$

To prove (iii) we choose an approximating sequence $(\xi_n)_n$ as in lemma 3.5 and pass to the limit in

$$L_{\xi_n}(\sigma) \geq c_{\xi_n}(\sigma(0), \sigma(1)). \qquad \square$$

3.3 Characterization of optimal transport strategies

In this paragraph, our aim is to make the formal arguments of paragraph 3.1 rigorous in order to characterize optimal transport strategies. This can be done under the additional assumption that H is strictly convex. First, we relate the optimality condition (3.1) to the Monge-Kantorovich problem with cost $\overline{c}_{\overline{\xi}}$:

Proposition 3.7. *Let us assume that $q < 2$ and that H is strictly convex. If $\overline{Q}$ solves (2.13) and $\overline{\xi} := H'(i_{\overline{Q}})$ then we have:*

$$\int_\Omega \overline{\xi} i_{\overline{Q}} = \inf_{Q \in \mathcal{Q}^q(\mu_0,\mu_1)} \int_\Omega \overline{\xi} i_Q = \inf_{\gamma \in \Pi(\mu_0,\mu_1)} \int_{\overline{\Omega}\times\overline{\Omega}} \overline{c}_{\overline{\xi}}(x,y) d\gamma(x,y). \tag{3.7}$$

Proof. Let us recall that from proposition 3.1, we have:

$$\int_\Omega \overline{\xi} i_{\overline{Q}} = \inf_{Q \in \mathcal{Q}^q(\mu_0,\mu_1)} \int_\Omega \overline{\xi} i_Q. \tag{3.8}$$

Let ξ be a non-negative element of L^{q^*} and let $Q \in \mathcal{Q}^q(\mu_0, \mu_1)$, using Lemma 3.6 and the definition of $\mathcal{Q}^q(\mu_0, \mu_1)$ yields:

$$\begin{aligned} \int_\Omega \xi i_Q = \int_C L_\xi dQ &\geq \int_C \overline{c}_\xi(\sigma(0)), \sigma(1)) dQ(\sigma) \\ &\geq \inf_{\gamma \in \Pi(\mu_0,\mu_1)} \int_{\overline{\Omega}\times\overline{\Omega}} \overline{c}_\xi(x,y) d\gamma(x,y). \end{aligned}$$

We then have, for all $\xi \in L^{q^*}$, $\xi \geq 0$:

$$\inf_{Q \in \mathcal{Q}^q(\mu_0,\mu_1)} \int_\Omega \xi i_Q \geq \inf_{\gamma \in \Pi(\mu_0,\mu_1)} \int_{\overline{\Omega}\times\overline{\Omega}} \overline{c}_\xi(x,y) d\gamma(x,y). \tag{3.9}$$

By a similar argument and using Lemma 3.4, we also have, for all $\xi \in C^0(\overline{\Omega}, \mathbb{R}_+)$:

$$\inf_{Q \in \mathcal{Q}(\mu_0,\mu_1)} \int_{\overline{\Omega}} \xi di_Q \geq \inf_{\gamma \in \Pi(\mu_0,\mu_1)} \int_{\overline{\Omega}\times\overline{\Omega}} \overline{c}_\xi(x,y) d\gamma(x,y). \tag{3.10}$$

Let $\xi \in C^0(\overline{\Omega}, \mathbb{R}_+)$, and $\varepsilon > 0$, for every x and y in $\overline{\Omega}$, there exists $\sigma_\varepsilon^{x,y} \in C^{x,y}$ such that $(x,y) \mapsto \sigma_\varepsilon^{x,y}$ is measurable (see for instance [7]) and by Lemma 3.4

$$L_\xi(\sigma_\varepsilon^{x,y}) \leq c_\xi(x,y) + \varepsilon = \overline{c}_\xi(x,y) + \varepsilon. \tag{3.11}$$

Let $\gamma \in \Pi(\mu_0, \mu_1)$ and let us define the element of $\mathcal{Q}(\mu_0, \mu_1)$: $Q_\varepsilon := \delta_{\sigma_\varepsilon^{x,y}} \otimes \gamma$, we then have:

$$\int_{\overline{\Omega}} \xi di_{Q_\varepsilon} = \int_{\overline{\Omega}\times\overline{\Omega}} L_\xi(\sigma_\varepsilon^{x,y}) d\gamma(x,y) \leq \int_{\overline{\Omega}\times\overline{\Omega}} \overline{c}_\xi(x,y) d\gamma(x,y) + \varepsilon.$$

Since γ and ε are abitrary, using (3.10) we obtain

$$\inf_{Q\in\mathcal{Q}(\mu_0,\mu_1)} \int_{\overline{\Omega}} \xi di_Q = \inf_{\gamma\in\Pi(\mu_0,\mu_1)} \int_{\overline{\Omega}\times\overline{\Omega}} \overline{c}_\xi(x,y) d\gamma(x,y), \ \forall \xi \in C^0(\overline{\Omega}, \mathbb{R}_+). \tag{3.12}$$

In what follows, for every $\mu \in \mathcal{M}_+(\overline{\Omega})$, we extend μ by 0 outside $\overline{\Omega}$. Let $(\rho_n)_n$ be a standard mollifying sequence. For $n \in \mathbb{N}^*$ let us consider the regularized problem:

$$\inf_{Q\in\mathcal{Q}(\mu_0,\mu_1)} \int_{\mathbb{R}^2} H(\rho_n \star i_Q) \tag{3.13}$$

The existence of a solution $\overline{Q}_n$ of (3.13) can be obtained by similar arguments as in theorem 2.9 (using lemma 2.7 and the fact that the L^1 norm of $\rho_n \star i_Q$ equals the total mass of i_Q). Proceeding as in proposition 3.1 and defining $j_n := \rho_n \star i_{\overline{Q}_n}$, $\xi_n := H'(j_n)$, $\eta_n := \rho_n \star \xi_n$, we have:

$$\int_{\mathbb{R}^2} H'(\rho_n \star i_{\overline{Q}_n})(\rho_n \star i_{\overline{Q}_n}) = \int_{\mathbb{R}^2} \xi_n j_n = \int_{\mathbb{R}^2} \eta_n di_{\overline{Q}_n} = \inf_{Q\in\mathcal{Q}(\mu_0,\mu_1)} \int_{\mathbb{R}^2} \eta_n di_Q \tag{3.14}$$

With (3.12), we then get:

$$\int_{\mathbb{R}^2} \eta_n di_{\overline{Q}_n} = \int_{\mathbb{R}^2} \xi_n j_n = \inf_{\gamma\in\Pi(\mu_0,\mu_1)} \int_{\overline{\Omega}\times\overline{\Omega}} \overline{c}_{\eta_n}(x,y) d\gamma(x,y). \tag{3.15}$$

By convexity of H, we also have:

$$\int_{\mathbb{R}^2} H(j_n) \leq \int_{\mathbb{R}^2} H(\rho_n \star i_{\overline{Q}}) \leq \int_{\mathbb{R}^2} \rho_n \star H(i_{\overline{Q}}) \tag{3.16}$$

which implies that j_n is bounded in L^q. Passing to subsequences, we may therefore assume:

$$j_n \rightharpoonup j \text{ in } L^q, \ \xi_n \rightharpoonup \xi \text{ in } L^{q^*}, \ \eta_n \rightharpoonup \xi \text{ in } L^{q^*}. \tag{3.17}$$

Since the total mass of $i_{\overline{Q}_n}$ is the same of j_n and j_n is bounded in L^q (and hence in L^1), we get a bound on $i_{\overline{Q}_n}(\Omega)$ and, from lemma 2.7, we may also assume:

$$\overline{Q}_n \overset{*}{\rightharpoonup} Q \text{ in } \mathcal{M}_+(C),\ i_{\overline{Q}_n} \overset{*}{\rightharpoonup} i \text{ in } \mathcal{M}_+(\overline{\Omega}). \tag{3.18}$$

It is obvious that $i = j$ and lemma 2.8 implies $j \geq i_Q$. With (3.16) and the monotonicity of H, we then get:

$$\int_\Omega H(i_Q) \leq \int_\Omega H(j) \leq \liminf_n \int_{\mathbb{R}^2} H(j_n) \leq \int_\Omega H(i_{\overline{Q}}). \tag{3.19}$$

With the strict convexity of H and the optimality of $\overline{Q}$, this also yields

$$i_{\overline{Q}} = i_Q = j \in L^q \text{ and } \liminf_n \int_{\mathbb{R}^2} H(j_n) = \int_{\mathbb{R}^2} H(i_{\overline{Q}}). \tag{3.20}$$

Up to some subsequence (as $\overline{\xi} = H'(i_{\overline{Q}}) \in L^{q^*}$), this also implies

$$H(j_n) - H(i_{\overline{Q}}) - \overline{\xi}(j_n - i_{\overline{Q}}) \to 0 \text{ a.e. and in } L^1$$

and using the strict convexity of H, we deduce that j_n converges a.e. to $i_{\overline{Q}}$. This implies that ξ_n converges a.e. to $\overline{\xi} = H'(i_{\overline{Q}})$ and that $\xi = \overline{\xi}$. With Fatou's Lemma and (3.15), we therefore obtain:

$$\begin{aligned}
\int_\Omega \overline{\xi} i_{\overline{Q}} &= \int_{\mathbb{R}^2} H'(i_{\overline{Q}}) i_{\overline{Q}} \\
&\leq \liminf_n \int_{\mathbb{R}^2} \xi_n j_n \\
&= \liminf_n \inf_{\gamma \in \Pi(\mu_0,\mu_1)} \int_{\overline{\Omega}\times\overline{\Omega}} \overline{c}_{\eta_n}(x,y) d\gamma(x,y).
\end{aligned}$$

Using $\eta_n \rightharpoonup \overline{\xi}$ and remark 3.3, from the uniform convergence of c_{η_n} to a cost $c \leq c_{\overline{\xi}}$, we get

$$\int_\Omega \overline{\xi} i_{\overline{Q}} \leq \inf_{\gamma \in \Pi(\mu_0,\mu_1)} \int_{\overline{\Omega}\times\overline{\Omega}} \overline{c}_{\overline{\xi}}(x,y) d\gamma(x,y)$$

together with (3.8) and (3.9), this completes the proof. □

The characterization of optimal transport strategies then reads as:

Theorem 3.8. *Let us assume that $q < 2$ and that H is strictly convex. A transportation strategy $(\overline{\gamma}, \overline{p})$ is optimal if and only if, setting $\overline{Q} := Q_{\overline{\gamma},\overline{p}}$ and $\overline{\xi} := H'(i_{\overline{Q}})$, one has:*

1. $\overline{\gamma}$ solves the Monge-Kantorovich problem:

$$\inf_{\gamma\in\Pi(\mu_0,\mu_1)} \int_{\overline{\Omega}\times\overline{\Omega}} \overline{c}_{\overline{\xi}}(x,y)d\gamma(x,y), \tag{3.21}$$

2. for $\overline{Q}$-a.e. $\sigma \in C$, one has:

$$L_{\overline{\xi}}(\sigma) = \overline{c}_{\overline{\xi}}(\sigma(0),\sigma(1)). \tag{3.22}$$

Proof. Let us assume first that the transportation strategy $(\overline{\gamma},\overline{p})$ is optimal and set $\overline{Q} := Q_{\overline{\gamma},\overline{p}}$ and $\overline{\xi} := H'(i_{\overline{Q}})$. From Proposition 3.7 and Lemma 3.6, we get:

$$\begin{aligned}\int_{\overline{\Omega}\times\overline{\Omega}} \overline{c}_{\overline{\xi}}(x,y)d\overline{\gamma}(x,y) &= \int_C \overline{c}_{\overline{\xi}}(\sigma(0),\sigma(1))d\overline{Q}(\sigma)\\ &\leq \int_C L_{\overline{\xi}}d\overline{Q} = \int_\Omega \overline{\xi}i_{\overline{Q}}\\ &= \inf_{\gamma\in\Pi(\mu_0,\mu_1)} \int_{\overline{\Omega}\times\overline{\Omega}} \overline{c}_{\overline{\xi}}(x,y)d\gamma(x,y)\end{aligned}$$

this proves that $\overline{\gamma}$ solves (3.21) and implies that the inequalities above are equalities. We therefore deduce (3.22) from the inequality $\overline{c}_{\overline{\xi}}(\sigma(0),\sigma(1)) \leq L_{\overline{\xi}}(\sigma)$.

Conversely, assume that the transportation strategy $(\overline{\gamma},\overline{p})$ satisfies the two conditions of the theorem. Condition (3.22) firstly yields:

$$\int_\Omega \overline{\xi}i_{\overline{Q}} = \int_C L_{\overline{\xi}}d\overline{Q} = \int_C \overline{c}_{\overline{\xi}}(\sigma(0),\sigma(1))d\overline{Q}(\sigma) = \int_{\overline{\Omega}\times\overline{\Omega}} \overline{c}_{\overline{\xi}}(x,y)d\overline{\gamma}(x,y)$$

Secondly, if $Q = Q_{\gamma,p} \in \mathcal{Q}^q(\mu_0,\mu_1)$, one has:

$$\int_\Omega \overline{\xi}i_Q = \int_C L_{\overline{\xi}}dQ \geq \int_C \overline{c}_{\overline{\xi}}(\sigma(0),\sigma(1))dQ(\sigma) = \int_{\overline{\Omega}\times\overline{\Omega}} \overline{c}_{\overline{\xi}}(x,y)d\gamma(x,y)$$

and since $\overline{\gamma}$ solves (3.21), we finally have

$$\int_\Omega \overline{\xi}i_{\overline{Q}} \leq \int_\Omega \overline{\xi}i_Q,\ \forall Q \in \mathcal{Q}^q(\mu_0,\mu_1)$$

which, with proposition 3.1, proves that $(\overline{\gamma},\overline{p})$ is optimal.

□

Remark 3.9. Let us remark that $\overline{\xi} = H'(i_{\overline{Q}})$ (with $\overline{Q}$ solving (2.13)) solves the following (dual of (2.13)) problem:

$$\sup_{\xi \in L^{q^*},\, \xi \geq 0} W(\xi) - \int_\Omega H^*(\xi(x))dx \tag{3.23}$$

where H^* is the Fenchel transform of H and:

$$W(\xi) := \inf_{\gamma \in \Pi(\mu_0,\mu_1)} \int_{\overline{\Omega}\times\overline{\Omega}} \overline{c}_\xi(x,y)d\gamma(x,y).$$

Indeed, it follows from proposition 3.7 and $\overline{\xi} = H'(i_{\overline{Q}})$ that

$$W(\overline{\xi}) - \int_\Omega H^*(\overline{\xi}(x))dx = \int_\Omega (\overline{\xi} i_{\overline{Q}} - H^*(\overline{\xi})) = \int_\Omega H(i_{\overline{Q}}).$$

If $\xi \in L^{q^*}$ is non-negative, we deduce from (3.9) and Young's inequality:

$$W(\xi) \leq \int_\Omega \xi i_{\overline{Q}} \leq \int_\Omega H(i_{\overline{Q}}) + \int_\Omega H^*(\xi)$$

which proves that $\overline{\xi}$ solves (3.23).

Let us see, through an easy example, an application of Theorem 3.8.

Exemple 3.10. Suppose that $\overline{\Omega}$ contains the two segments $A = \{0\}\times[0,1]$ and $B = \{1\} \times [0,1]$ and the square $S = [0,1] \times [0,1]$ which is their convex hull. Set $\mu_1 = \mathcal{H}^1 \llcorner A$ and $\mu_2 = \mathcal{H}^1 \llcorner B$ and denote by $T : A \to C$ the map that associates to every point $(0,x) \in A$ the curve $T(x)$ given by $T(x)(t) = (t,x)$, i.e. the horizontal segment from A to B starting from x. Set $Q = T\sharp\mu_1$. It is clear that Q comes from an admissible transportation strategy linking μ_1 to μ_2 and it is not difficult to see that the traffic intensity i_Q has constant density 1 on S and 0 elsewhere. We consider two particular cases only: we claim that Q is optimal if $\Omega =]0,1[\times]0,1[$ while it is not if S is compactly contained in Ω. Indeed, if $\overline{\Omega} = S$, the metric induced by i_Q is the euclidean metric, the paths $T(x)$ are geodesic and the transport plan induced by Q is optimal according to this metric. On the other hand, if Ω is larger than S, then all the segments $T(x)$ that are very close to the upper or lower boundary of S are not geodesic accoding to this metric, because they could be improved by non-straight line paths which arrive up to zone $\Omega \setminus S$ where $i_Q = 0$ and the transportation is cheaper. In the former case, consequently, the sufficient optimality conditions are satisfied, while in the latter the geodesic conditions on the paths (Wardrop condition, see the next section) is not and prevents optimality.

4 Application to equilibria of Wardrop type

In this final section, we relate the results of the previous sections to some concepts of equilibria of Wardrop type. Modelling congestion as in paragraph 2.2 enables us to extend the concept of Wardrop equilibrium to a continuous setting.

Let us consider a congestion function $g : \mathbb{R}_+ \to \mathbb{R}_+$ which is continuous increasing and satisfies $az^{q-1} \leq g \leq b(z^{q-1}+1)$ for all $z \in \mathbb{R}_+$ and some $q \in (1,2)$ and non-negative constants a and b. Then for any transportation strategy (γ, p) such that $I_{\gamma,p}$ (defined by (2.3)) belongs to L^q, the transportation cost function resulting from the strategy (γ, p) is $\overline{c}_\xi$ for $\xi := g \circ I_{\gamma,p} \in L^{q^*}$. Roughly speaking, an equilibrium is then a transportation strategy (γ, p) that satisfies Wardrop stability condition (i.e. $Q_{\gamma,p}$ gives full mass to the set of geodesics for the metric $\xi = g \circ I_{\gamma,p}$) and the additional requirement that γ is an optimal transportation plan between μ_0 and μ_1 for the cost resulting from (γ, p). This leads to the following

Definition 4.1. *A transportation strategy $(\overline{\gamma}, \overline{p})$ is said to be an equilibrium if $I_{\overline{\gamma},\overline{p}} \in L^q$ and, setting $\overline{\xi} := g \circ I_{\overline{\gamma},\overline{p}}$ one has*

1. *$L_{\overline{\xi}}(\sigma) = \overline{c}_{\overline{\xi}}(\sigma(0), \sigma(1))$ for $Q_{\overline{\gamma},\overline{p}}$-a.e. $\sigma \in C$,*
2. *$\overline{\gamma}$ solves the Monge-Kantorovich problem:*

$$\inf_{\gamma \in \Pi(\mu_0,\mu_1)} \int_{\overline{\Omega}\times\overline{\Omega}} \overline{c}_{\overline{\xi}}(x,y) d\gamma(x,y).$$

Only the first condition above is linked to Wardrop's original equilibrium concept. Imagine that some social planner chooses the transportation plan γ, then the second equilibrium condition expresses that γ is optimal for the transportation cost resulting from γ itself and the traveler's individual behavior. Our notion of equilibrium can therefore be viewed as a refinement of the Wardrop equilibrium or its generalization to the case where the transportation plan is not given a priori.

A direct application of theorems 2.9 and 3.8 then gives the existence of equilibria together with a variational characterization:

Theorem 4.2. *Under the assumptions of this paragraph, there exists an equilibrium. Moreover $(\overline{\gamma}, \overline{p})$ is an equilibrium if and only if $\overline{Q} := Q_{\overline{\gamma},\overline{p}}$ solves the minimization problem:*

$$\inf_{Q \in \mathcal{Q}^q(\mu_0,\mu_1)} \int_\Omega H_g(i_Q(x)) dx \quad \text{with } H_g(z) := \int_0^z g(s) ds, \ \forall z \in \mathbb{R}_+. \tag{4.1}$$

Remark 4.3. A slightly different situation, which can be relevant in some applications, occurs when the transportation plan $\overline{\gamma} \in \Pi(\mu_0, \mu_1)$ is fixed and not only the marginals μ_0 and μ_1. In this case, one defines equilibria as the set of $\overline{p}$'s such that $(\overline{\gamma}, \overline{p})$ satisfies the first condition (Wardrop) of definition 4.1. If the set:

$$\mathcal{Q}^q(\overline{\gamma}) := \{Q \in \mathcal{M}^1_+(C) : (e_0, e_1)\sharp Q = \overline{\gamma},\ i_Q \in L^q\}$$

is nonempty, then slightly adapting our arguments, we have existence of equilibria and $\overline{p}$ is an equilibrium if and only if $\overline{Q} := Q_{\overline{\gamma},\overline{p}}$ solves the minimization problem:

$$\inf_{Q \in \mathcal{Q}^q(\overline{\gamma})} \int_\Omega H_g(i_Q(x))dx.$$

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