STABLE NORMS AND HODGE FORMS

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Abstract. For closed connected oriented smooth n-manifolds, $n \geq 3$, the volume infimum under finite positive comass bounds on real classes of degrees at least two equals the gauge of a convex body of Euclidean wedge moments, evaluated on normalized top cup products. Comparison of closed representatives controls comass, volume, and quadratic integrals. Parametric energy comparison, duality, and finite-dimensional correction prescribe all compatible intermediate Hodge inner products and positive volume forms in smooth families over compact manifolds. In dimension four, we determine these infima for degree-two classes at prescribed periods and the closure of stable norms on second homology at fixed period and volume. Admissible balanced stable systolic exponents form the convex hull of degree profiles of nonzero top cup products. For fixed $p \geq 2$ and $m \geq 3$, the supremum of stable p-systolic ratios over mp-manifolds with pth Betti number b and a nonzero real m-fold degree-p cup product has order b^{m-1} for all sufficiently large integers b. For eight-manifolds whose integral middle intersection matrix is the identity I_b, $b \geq 1$, the optimal stable four-systolic ratio is 14; a metric attains it precisely when $b = 1$ and a positive parallel Cayley form induces the metric and orientation.

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Introduction

Let M be a closed connected oriented smooth manifold of dimension n, with oriented fundamental class $[M]$. For a smooth Riemannian metric g, write $*_g$ for the Hodge star and $N_g^{(r)}$ for the stable norm on $H_r(M;\mathbb{R})$, $1 \le r \le n$. Comass is the maximum absolute value on orthonormal r-frames. The dual cohomological comass norm of $\alpha \in H^r(M;\mathbb{R})$ is [9, §§1, 3–4]:

$$\|\alpha\|_g^* = \inf_{d\omega=0,\,[\omega]=\alpha} \sup_{x\in M} \operatorname{comass}_g(\omega_x).$$

Here the infimum is over smooth representatives. We study which bounds on these norms can be realized by a metric, and the volume they require. We also prescribe the quadratic forms defined by harmonic representatives. These questions concern the same closed forms, but place different conditions on their pointwise norms and integrals.

There are two immediate restrictions. Exterior products of bounded closed forms constrain the top cup products of their classes and therefore the volume. The Hodge inner products satisfy duality relations determined by the cup pairings in complementary degrees. The problem is to determine when these restrictions suffice. We first express the finite comass problem in terms of Euclidean exterior products; the comparison giving its upper bound also controls quadratic integrals.

Comass bounds and Hodge forms. Assume $n \ge 3$. Choose an ordered list of degrees $2 \le r_i \le n$, $1 \le i \le s$, and put

$$\mathcal{I} = \left\{ a \in \mathbb{Z}^s_{\ge 0} \,:\, \sum_i r_i a_i = n, \quad a_i \le 1 \text{ if } r_i \text{ is odd} \right\}.$$

Let dv_0 be oriented Euclidean volume on $\mathbb{R}^n$. For a tuple $\omega = (\omega_1, \dots, \omega_s)$ of constant forms $\omega_i \in \Lambda^{r_i}(\mathbb{R}^n)^*$, define $t_a(\omega)$ by $\omega_1^{a_1} \wedge \cdots \wedge \omega_s^{a_s} = t_a(\omega)\, dv_0$, with the factors in the indicated order. Define the compact convex moment body $K \subset \mathbb{R}^{\mathcal{I}}$ and its gauge by

$$K = \operatorname{conv}\{(t_a(\omega))_{a\in\mathcal{I}} \,:\, \operatorname{comass}(\omega_i) \le 1\}, \qquad \gamma_K(v) = \inf\{t \ge 0 \,:\, v \in tK\} \quad (v \in \mathbb{R}^{\mathcal{I}}).$$

For real classes $\alpha_i \in H^{r_i}(M;\mathbb{R})$ and positive numbers β_i, set

$$T_{\alpha/\beta} = \left(\frac{\langle \alpha_1^{a_1} \cdots \alpha_s^{a_s}, [M] \rangle}{\prod_i \beta_i^{a_i}} \right)_{a\in\mathcal{I}}.$$

If $\mathcal{I}$ is empty, the moment space is zero and $\gamma_K(0) = 0$.

Theorem A. *For every finite list of real classes and positive bounds as above,*

$$\inf_{\|\alpha_i\|_g^* \le \beta_i \ (1\le i\le s)} \operatorname{Vol}(M,g) = \gamma_K(T_{\alpha/\beta}).$$

The infimum is over smooth Riemannian metrics. Zero and dependent classes are permitted.

Theorem 1.10 proves the formula. The Euclidean forms may have arbitrary real coefficients. For fixed dimension and degrees, K is independent of M; the manifold enters through the top cup evaluations. Integration gives the lower bound. For the upper bound, rational constant forms on flat tori realize finite convex combinations of moments. The following comparison transfers their bounds to M. Let N be another closed connected oriented smooth n-manifold, and let rational classes α_i' on N have the same degrees and top moments as rational classes α_i on M. Given a metric h on N and closed representatives η_i of α_i' with $\sup_N \operatorname{comass}_h(\eta_i) \le \beta_i$, consider the following degreewise Gram matrices. For a metric g on M and closed forms θ_i of degrees r_i, put $G_g^{(r)}(\theta) = (\int_M \theta_i \wedge *_g\, \theta_j)_{r_i=r_j=r}$, with the analogous definition on N. Write $\preceq$ for positive-semidefinite order and I for the identity matrix in each degree. Proposition 1.9 gives, for every $\varepsilon > 0$, a metric g and closed representatives θ_i of α_i such that

$$\sup_M \operatorname{comass}_g(\theta_i) \le \beta_i, \qquad \operatorname{Vol}(M,g) \le \operatorname{Vol}(N,h) + \varepsilon, \qquad G_g^{(r)}(\theta) \preceq G_h^{(r)}(\eta) + \varepsilon I.$$

The required monotone comparison is Brunnbauer's [17, Lemma 3.2]. Equal top evaluations identify the images of the fundamental classes in a product of spheres and James spaces. Closed forms extend across the added complex of dimension at most $n-1$; its contribution to volume tends to zero in the pullback metric. His single-degree invariance theorem determines the stable systolic constant from the integral top cup form for a proper degree dividing n [17, Theorem 1.4 and Corollary 1.5]. Auxiliary rational basis classes give the passage to real markings in Theorem A.

For $b_r(M) = \dim H_r(M;\mathbb{R}) > 0$, put $L_r = H_r(M;\mathbb{Z})/\text{torsion}$, regarded as a lattice in real homology, and

$$\operatorname{stsys}_r(M,g) = \min_{0\neq z\in L_r} N_g^{(r)}(z).$$

For simultaneous thresholds in degrees at least two, Theorem 2.5 gives a finite semialgebraic formula for the volume infimum. Integral cup data and exactly specified positive algebraic thresholds give a computable real algebraic value. Grunewald–Segal's orbit algorithm decides equivalence of top cup arrays under integral changes of basis [33, Algorithm A2, p. 915].

For $v \in V_r := H^r(M;\mathbb{R})$, let ω_v^g be its harmonic representative. For $v, w \in V_r$, define the Hodge inner product by

$$P_{r,g}(v,w) = \int_M \omega_v^g \wedge *_g\, \omega_w^g.$$

Among the Gram matrices of closed representatives of a fixed cohomology basis, the harmonic Gram matrix is minimal in positive-semidefinite order. Applied to a rational basis in degree $r \geq 2$, Proposition 1.9 therefore bounds $P_{r,g}$. In bases (e_i) of V_r and (f_j) of V_{n-r}, put $Q_r = (\langle e_i f_j, [M]\rangle)_{ij}$, with the same basis in middle degree. We denote bilinear forms and their matrices in the chosen bases by the same symbols. Duality gives

$$P_{n-r,g} = Q_r^t P_{r,g}^{-1} Q_r.$$

Write $\operatorname{Sym}^+(V)$ for the positive definite symmetric bilinear forms on V. If $n = 2m$, then $Q_m^t = (-1)^m Q_m$, and set

$$\mathcal{D}(Q_m) = \{P \in \operatorname{Sym}^+(V_m) \,:\, PQ_m^{-1}P = (-1)^m Q_m\}.$$

The space of independent Hodge forms is

$$\mathscr{H}(M) = \begin{cases} \displaystyle\prod_{1\leq r<n/2} \operatorname{Sym}^+(V_r), & n \text{ odd}, \\ \left(\displaystyle\prod_{1\leq r<m} \operatorname{Sym}^+(V_r)\right) \times \mathcal{D}(Q_m), & n = 2m. \end{cases}$$

Factors on zero vector spaces are omitted. Let $\Pi(g)$ denote the corresponding tuple. The compatibility identities are independent of the bases. This tuple and the total volume determine all the Hodge forms. In the bases given by 1 and the integral orientation class, the degree-zero and degree-n Hodge forms are $\operatorname{Vol}(M,g)$ and $\operatorname{Vol}(M,g)^{-1}$.

Theorem B. *Let M be closed, connected, oriented, and smooth, of dimension $n \geq 3$. Let B be a compact smooth manifold, $\mathsf{P} : B \to \mathscr{H}(M)$ a smooth map, and μ_b a smooth family of positive volume forms on M. There is a smooth family of metrics g_b such that*

$$\Pi(g_b) = \mathsf{P}_b, \qquad dv_{g_b} = \mu_b.$$

The differentials of Π at g_b, restricted to the slices of metrics with volume form μ_b, admit smoothly varying linear right inverses.

Theorem 4.2 gives uniform bounds for these right inverses in each fixed spatial C^k norm. It also treats compact parameter manifolds with boundary or corners, when the data extend smoothly to a neighborhood.

Proposition 5.1 gives the parametric energy comparison, with an integral basis in degree one on M and integral corresponding classes on the comparison manifold. In middle degree, integral transvections and isotropic reduction vary the compatible form through positive scaling parameters. In complementary degrees, minimization over auxiliary representatives gives a quadratic form and its inverse. Duality turns the resulting upper estimates into two-sided estimates. Products of at least three classes are realized on additional tori with quadratic energies tending to zero. Connected sums assemble these forms while preserving their cup moments. Total volume is adjusted on a region where the representatives of degrees $1, \dots, n-1$ vanish; the normalized orientation form has squared L^2 norm equal to the reciprocal volume. Uniform derivative bounds give C^1 approximation in the control parameters, and a finite-dimensional inverse argument makes the Hodge data and total volume exact. Moser's isotopy then prescribes the volume form without changing the cohomological data [58, Theorem 2, pp. 292–293].

Let b_+ and b_- be the positive and negative indices of a symmetric middle cup form. On four-manifolds, Gay–Kirby's construction [29, Theorem 1, parts (1) and (8)] yields a self-dual harmonic representative of every integral class of positive square, as formulated by Hamilton. Hamilton deduced density of the period map when $b_+ = 1$ [36, Proposition 1.1 and Theorem 3.1]. Scaduto proved density in every signature and surjectivity when $b_+ = 1$ or $b_- = 1$ [64, Theorems 1.1–1.2]; his Remarks 2.6 and 3.5 give the corresponding $4k$-dimensional results. Theorem B prescribes the full independent Hodge tuple, including every compatible middle form in every even dimension at least four, on compact smooth parameter families.

On four-manifolds with $b_+ \le 3$, Shevchishin–Smirnov prescribed simultaneous anti-self-duality of the duals of disjoint embedded (-1)-spheres [69, Theorem 1], and then of disjoint embedded (-2)-spheres [70, Theorem 1]. These are particular closed conditions in the period domain, whose realization does not follow from density. For a four-manifold with $b_2(M) > 0$, let $H^2(M;\mathbb{Z})_{\mathbb{R}}$ be the image of integral cohomology in real cohomology. Its conformal two-systole is

$$\operatorname{confsys}_2(M,g) = \min_{0 \neq \alpha \in H^2(M;\mathbb{Z})_{\mathbb{R}}} P_{2,g}(\alpha,\alpha)^{1/2}.$$

Density already determines its supremum from the integral intersection form [64, Corollary 1.3]. Bavard's maximum theorem [12, Proposition 4.6 and §4.5], together with exact period prescription, gives a smooth maximizing metric with any positive volume form (Theorem 11.2). In index one, this attainment already follows from the cited results.

Norms at a prescribed period. Let X be a closed connected oriented smooth four-manifold. Put $V = H^2(X;\mathbb{R})$, $Q(v,w) = \langle vw, [X]\rangle$, and $P_g = P_{2,g}$. A metric is regular if the differential of $g \mapsto P_g$ is onto $T_{P_g}\mathcal{D}(Q)$. Fix $P \in \mathcal{D}(Q)$, real classes $\alpha_1, \dots, \alpha_s \in V$, and positive bounds $\beta = (\beta_1, \dots, \beta_s)$. Write $Q_\alpha = (Q(\alpha_i,\alpha_j))$ and $P_\alpha = (P(\alpha_i,\alpha_j))$. Let (e_i) be the standard basis of $\mathbb{R}^s$ and $|\cdot|$ the Euclidean norm on $\mathbb{R}^3$. Identify the spaces of Euclidean self-dual and anti-self-dual two-forms on $\mathbb{R}^4$ isometrically with $\mathbb{R}^3$, and put

$$\mathcal{K}_\beta = \{(A,B) \in \operatorname{Hom}(\mathbb{R}^s,\mathbb{R}^3)^2 \,:\, |Ae_i| + |Be_i| \le \sqrt{2}\,\beta_i \quad (1 \le i \le s)\}.$$

Let $\Gamma(Q_\alpha, P_\alpha; \beta)$ be the infimum of the masses of nonnegative finite Borel measures ν on $\mathcal{K}_\beta$ satisfying

$$\int (A^tA - B^tB)\, d\nu = Q_\alpha, \qquad \int (A^tA + B^tB)\, d\nu \succeq P_\alpha.$$

The second condition is a positive-semidefinite inequality because the energy matrix of closed representatives majorizes their harmonic energy matrix.

Theorem C. *For every such finite list, including empty, zero, or dependent lists,*

$$\inf_{P_g = P,\ \|\alpha_i\|_g^* \le \beta_i} \operatorname{Vol}(X,g) = \Gamma(Q_\alpha, P_\alpha; \beta).$$

The infimum is unchanged if the metrics are required to be regular.

Theorem 9.1 also gives a minimizing measure with at most $1 + s(s+1)$ atoms and a computable algebraic value for exactly specified algebraic data. To prove the upper bound, the constrained forms on the tori are augmented by auxiliary forms corresponding to a cohomology basis of X. These auxiliary forms have energy matrix P and no prescribed comass bounds (Lemma 9.3). After rational approximation and correction of the cup moments, transfer and harmonic minimization (Lemma 9.6), followed by compatibility, give convergence of the entire period to P. For real marked classes, the auxiliary energy bound controls the resulting errors in L^2; a conformal change restores the comass bounds while preserving the period. Theorem 8.1 then makes the period exact while preserving the comass bounds, with added volume bounded linearly by the period error in fixed local coordinates, up to an arbitrarily small error. Its constant is independent of the initial metric.

For a norm N on $H := H_2(X;\mathbb{R})$, write $N_g = N_g^{(2)}$ and put

$$G_N(P) = \inf\{\mathrm{Vol}(X,g) \,:\, P_g = P,\ N_g \geq N\}.$$

If $b_2(X) > 0$ and $v > 0$, Theorem 10.1 gives

$$v \geq G_N(P) \quad \Longleftrightarrow \quad \begin{array}{c} \text{there are smooth regular metrics } g_j \text{ with} \\ P_{g_j} = P, \qquad \mathrm{Vol}(X,g_j) = v, \qquad N_{g_j} \longrightarrow N. \end{array}$$

Convergence is uniform on bounded subsets of H; period and volume are exact for every j. The function $G_N(P)$ is positive, finite, and jointly continuous in N and P. If the dual unit ball

$$C_N = \{\alpha \in V \,:\, |\alpha(h)| \leq N(h) \text{ for all } h \in H\} = \mathrm{conv}\{\pm\alpha_1, \dots, \pm\alpha_s\}$$

is polyhedral, put $\mathbf{1} = (1,\dots,1)$. Then $G_N(P) = \Gamma(Q_\alpha, P_\alpha; \mathbf{1})$; the generators necessarily span V. Comass bounds give lower bounds for the stable norm. Areas of embedded surfaces give upper bounds, obtained by conformal changes that preserve the period. Polyhedral approximation and continuity give the conclusion for arbitrary norms. Above the infimum, volume is increased in a ball disjoint from these surfaces. At the infimum, constant rescalings tending to one give the exact volume and preserve norm convergence. When $b_2(X) = 0$, the norm and period are unique, $G = 0$, and every positive volume is realized.

In degree one and dimension at least three, Babenko–Balacheff realized every centrally symmetric convex polytope with nonempty interior and rational vertex directions as a stable unit ball in any prescribed conformal class [6, Theorem B]. For every norm N on $H_1(M;\mathbb{R})$ and every $\varepsilon > 0$, there is also a metric g in that conformal class with $N \leq N_g^{(1)} \leq (1+\varepsilon)N$ [6, Corollary B]. Jotz gave smooth Hedlund constructions [45, Proposition 3.2 and Theorem 3.5]. Theorem 10.1 concerns the degree-two norm together with its exact period and volume on a four-manifold. Marques–Neves–Sun proved local rigidity in the smooth topology near a flat metric on $T^n = \mathbb{R}^n/\mathbb{Z}^n$ when its degree-d stable norm is preserved, for $1 \leq d \leq n-2$; in codimension one, equal volume is also required [55, Theorems 1.1–1.2]. Their Theorem 1.4 constructs a smooth family on T^3, starting at a flat metric, with fixed degree-two stable norm and volume tending to zero.

Systolic inequalities and equality. The comparison methods also determine which products of stable systoles admit a volume bound, now for $n \geq 1$. Put $D(M) = \{r \,:\, 1 \leq r \leq n,\ b_r(M) > 0\}$. Let $\mathcal{A}_M \subset \mathbb{Z}_{\geq 0}^{D(M)}$ consist of the degree profiles of nonzero rational top cup products: a profile a has a_r factors of degree r and $\sum_r r a_r = n$. Factors of the same degree need not coincide. For $\rho \in \mathbb{R}_{\geq 0}^{D(M)}$ with $\sum_r r\rho_r = n$, Theorem 3.1 gives

$$\left(\exists C > 0\ \forall g, \quad \prod_{r \in D(M)} \mathrm{stsys}_r(M,g)^{\rho_r} \leq C\,\mathrm{Vol}(M,g) \right) \quad \Longleftrightarrow \quad \rho \in \mathrm{conv}\,\mathcal{A}_M.$$

For $\ell \in (0,\infty)^{D(M)}$, define

$$V_M(\ell) = \inf\{\mathrm{Vol}(M,g) \,:\, \mathrm{stsys}_r(M,g) \geq \ell_r\ (r \in D(M))\}, \qquad \ell^a = \prod_r \ell_r^{a_r}.$$

By Corollary 3.3, there are positive constants c_M, C_M such that, for every ℓ,

$$c_M \max_{a\in\mathcal{A}_M} \ell^a \le V_M(\ell) \le C_M \max_{a\in\mathcal{A}_M} \ell^a.$$

Degree one and the top degree are included. Gromov's cup-product inequalities and Newton-polyhedron viewpoint are the antecedents [32, §§7.4.A–C]. Independently scaled factors in a fixed finite complex built from circles, spheres, and James spaces give necessity. The stable systolic category is the largest number of factors in such an inequality with integral exponents. It equals rational cup-length, the largest length of a nonzero product of positive-degree rational classes (Corollary 3.4). Dranishnikov–Rudyak proved this equality for simply connected manifolds of dimension less than eight [27, Theorem 4.6], and Ryu for products of real homology spheres [63, Theorem 3.8]. An integral exponent vector $\rho \in \operatorname{conv}\mathcal{A}_M$ need not belong to $\mathcal{A}_M$, even when $\sum_r \rho_r$ equals the rational cup-length of M. Example 3.5 gives a negative answer to [27, Remark 4.10].

For a closed connected nonorientable smooth n-manifold M, $n \ge 3$, let N_r be arbitrary norms on $H_r(M;\mathbb{R})$ in the nonzero degrees $2 \le r < n$. Suppose first that $b_1(M) > 0$, and set $\sigma_1^{\mathrm{st}}(M) = \inf_g \operatorname{Vol}(M,g)/\operatorname{stsys}_1(M,g)^n$. For every $a > 0$, Theorem A.1 gives

$$\inf_{\substack{\operatorname{stsys}_1(M,g)\ge a \\ N_g^{(r)}\ge N_r\ (2\le r<n,\ b_r(M)>0)}} \operatorname{Vol}(M,g) = \sigma_1^{\mathrm{st}}(M)a^n.$$

Thus the higher norm bounds do not change this infimum. Without the degree-one constraint the infimum is zero, including when $b_1(M) = 0$. Pure higher-degree freedom is already due to Brunnbauer [17, Theorem 1.3]; Appendix A gives the relative comparison needed here.

For a closed connected oriented smooth d-manifold X and $1 \le p \le d$ with $b_p(X) > 0$, write

$$\mathrm{SR}_p(X) = \sup_g \frac{\operatorname{stsys}_p(X,g)^{d/p}}{\operatorname{Vol}(X,g)}.$$

For four-manifolds with odd intersection form of positive index one, Hamilton proved an unconditional lower bound of order $\sqrt{b_2}$ for $\sup_g \operatorname{confsys}_2(X,g)^2$ [36, Theorem 1.1]; Katz's earlier blowup estimate was conditional [47, Theorem 1.2]. For every four-manifold X with indefinite intersection form, Theorem 13.9 gives, with a universal $c > 0$,

$$c\, b_2(X) \le \mathrm{SR}_2(X) \le 2b_2(X).$$

The supremum of $\operatorname{confsys}_2(X,g)^2$ over g also has linear order in $b_2(X)$. More generally, Corollary 13.10 gives $c\, b_{2m}(X) \le \mathrm{SR}_{2m}(X) \le C_m b_{2m}(X)$ for indefinite middle forms in dimension $4m$, with the same lower coefficient. Here C_m depends only on m, through the pointwise mass–comass estimate in that dimension. Large minima of compatible lattice majorants have classical theta-average antecedents [71, Chapter IV, Theorem 12]; the split-even estimate is explicit in [1, §4.5, equation (4.34)]. Finite comass realization gives the metric lower bound. In dimension four the stable norm is controlled at each prescribed period; linear growth holds at a suitable choice of period. For $X_r = \mathbb{CP}^2\#r\overline{\mathbb{CP}^2}$, both optima have order r as $r \to \infty$, answering [47, Questions 1.4–1.5]. In particular, $\mathrm{SR}_2(X_r) \to \infty$, contrary to the uniform bound proposed in [22, Conjecture 5.1].

Fix integers $p \ge 2$ and $m \ge 3$. Let $A_{p,m}(b)$ be the supremum of $\mathrm{SR}_p(M)$ over closed connected oriented smooth mp-manifolds with $b_p(M) = b$ and a nonzero product of m real degree-p classes. Theorem 15.1 gives positive constants $c_{p,m}, C_{p,m}$ such that, for every sufficiently large integer b,

$$c_{p,m} b^{m-1} \le A_{p,m}(b) \le C_{p,m} b^{m-1}.$$

Gromov's critical quotient norms [32, §7.4.A] are the infima of the L^m-comass over closed representatives. Their K-convexity constants are bounded for fixed p, m. Cup products and Banaszczyk's transference theorem [7, Theorem 1] then give the upper bound. Bangert–Katz already proved the power b^2 for $m = 3$; their general estimate has an additional logarithmic factor [9, Theorems 8.1 and 2.7]. For degree two, Goodwillie–Hebda–Katz give explicit bounds, including the coefficient

$m!(3/2)^m$ when $b_2 = 2$ [31, Theorem 3.3 and Corollary 3.4]. The lower examples come from trace tensors over an unramified totally real tower with bounded root discriminant [35, Theorems 2.2, 2.4–2.5] and ideal-lattice minima [60, full version, Lemma 6.2]. Integral realization and framed surgery [57, Lemmas 2, 2′ and Theorem 2] produce the manifolds; the finite comass formula gives the metrics. Connected sums with $S^p \times S^{mp-p}$ give examples for every sufficiently large b. They are $(p-1)$-connected and stably parallelizable, with free integral homology in degree p. Thus the exponent is optimal even under these topological restrictions.

For quaternionic and complex projective space, finite comass realization gives

$$\mathrm{SR}_4(\mathbb{H}P^2) = \mathrm{SR}_4(\mathbb{C}P^4) = 14.$$

Bangert–Katz–Shnider–Weinberger proved that these suprema agree and lie in $[6, 14]$, and proved equality to 14 assuming the existence of a compact Joyce manifold with holonomy Spin(7) and $b_4 = 1$ [13, Theorem 1.2, Proposition 1.4 and Theorem 1.5]. Theorem 12.4 gives metrics on both projective spaces whose ratios tend to 14.

For X^4 with $b = b_2(X) > 0$, put $L = H_2(X;\mathbb{Z})/\text{torsion}$ and $B_g = \{z : N_g^{(2)}(z) \le 1\}$. Let μ be a Lebesgue measure on $H_2(X;\mathbb{R})$, $\mathrm{covol}_\mu(L)$ the measure of a fundamental parallelepiped, and ω_b the Euclidean unit-ball volume. The full-rank Hausdorff–Busemann stable two-systole is [41, §2 and Definition 4.1]

$$\mathrm{stsys}_{2,b}(X,g) = \mathrm{covol}_{\mathrm{HB}}(L, N_g^{(2)})^{1/b} = \left(\frac{\omega_b \,\mathrm{covol}_\mu(L)}{\mu(B_g)}\right)^{1/b}.$$

For $b_\pm \ge 2$, Theorem 16.1 gives an explicit upper bound, depending only on (b_+, b_-), for its square divided by volume, with a strict gap in each conformal class. The bound is sharp and unattained when $b_\pm \in \{2, 3\}$. On T^4 the supremum is $(15\pi/8)^{1/3}$, exceeding Hebda's conformally flat value $(3\pi/4)^{1/3}$ [41, Theorem 6.7 and p. 123].

For a closed connected oriented smooth eight-manifold M whose integral middle intersection matrix is the identity I_b, $b \ge 1$, one has $\mathrm{SR}_4(M) = 14$. A Cayley form is positive if it belongs pointwise to the orbit of the standard Cayley form under orientation-preserving linear isomorphisms. A smooth metric attains the value 14 exactly when $b = 1$ and a positive parallel Cayley form induces the metric and given orientation (Theorems 17.1 and 17.6). The pointwise Cayley geometry is classical [38, Chapter IV, §1.C] [23, Theorem 3.7] [49, Theorem 1.1 and §§3–5]. A closed positive Cayley form is parallel for its induced metric [46, §2, Definition 2.2 and the following paragraph]. The necessity of this equality condition and the rank restriction require the harmonic and spinor arguments in Section 17. For $b \ge 2$, every smooth conformal class has a strict gap; no maximizing sequence has a subsequence converging in C^0 to a smooth positive definite metric after arbitrary diffeomorphisms and positive smooth conformal rescalings (Theorem 17.7 and Corollary 17.9).

Sections 1–2 establish finite comparison, moment formulas, and arithmetic reduction; Section 3 classifies exponents by independent scaling. Sections 4–7 develop parametric energy comparison and exact Hodge prescription. Period correction, finite comass constraints, and surface-area prescription occupy Sections 8–10. Sections 11–15 derive conformal maxima, sharp constants, and Betti-number bounds. Sections 16–17 supply the full-rank and Cayley equality arguments. Appendix A treats nonorientable manifolds by relative comparison.

1. Finite cohomological constraints

Throughout this section, M is a closed connected oriented smooth n-manifold, where $n \ge 3$, and g denotes a smooth Riemannian metric on M. For $1 \le r \le n$, the comass of an alternating r-form on an inner product space is the supremum of its absolute value on orthonormal r-frames. For a smooth r-form ω, put

$$\|\omega\|_{\infty,g} = \sup_{x\in M} \mathrm{comass}_g(\omega_x), \qquad \|\alpha\|_g^* = \inf_{d\omega=0,\,[\omega]=\alpha} \|\omega\|_{\infty,g}, \quad \alpha \in H^r(M;\mathbb{R}),$$

where the infimum is over smooth representatives. Denote by $N_g^{(r)}$ the stable norm on $H_r(M;\mathbb{R})$ and by $\langle \alpha, z\rangle$ the evaluation pairing. Stable–comass duality [9, §4] gives

$$N_g^{(r)}(z) = \max_{\|\alpha\|_g^* \le 1} |\langle \alpha, z\rangle|, \qquad z \in H_r(M;\mathbb{R}). \tag{1.1}$$

The maximum exists because the cohomological unit ball is compact. We identify $H^r(M;\mathbb{Q})$ with its image in $H^r(M;\mathbb{R})$, and integral homology and cohomology modulo torsion with their images over $\mathbb{R}$. The integral lattices are dual under the evaluation pairing [9, Lemma 3.2]. The class $[M]$ is the fundamental class determined by the orientation.

1.1. **The universal moment body.** Let $s \ge 0$ be an integer. Fix an ordered list of degrees $2 \le r_i \le n$, $1 \le i \le s$, and set

$$\mathcal{I} = \left\{ a \in \mathbb{Z}_{\ge 0}^s \,:\, \sum_{i=1}^s r_i a_i = n, \quad a_i \le 1 \text{ if } r_i \text{ is odd} \right\}.$$

All exterior and cup products are taken in the order of the indices; zeroth powers and empty products equal 1. Give $\mathbb{R}^n$ its standard Euclidean metric and orientation, and put $dv_0 = dx_1 \wedge \cdots \wedge dx_n$. For constant forms $\omega_i \in \Lambda^{r_i}(\mathbb{R}^n)^*$, write $\omega = (\omega_1, \dots, \omega_s)$ and define $t(\omega) = (t_a(\omega))_{a\in\mathcal{I}}$ by

$$\omega_1^{a_1} \wedge \cdots \wedge \omega_s^{a_s} = t_a(\omega)\, dv_0.$$

A tuple is rational if all coefficients of its forms in the standard coordinate exterior bases are rational. The moment body and its gauge are

$$\begin{aligned} K = K_{n;r_1,\dots,r_s} &= \operatorname{conv}\{t(\omega) \,:\, \operatorname{comass}(\omega_i) \le 1 \text{ for every } i\}, \\ \gamma_K(v) &= \inf\{u \ge 0 \,:\, v \in uK\}, \qquad v \in \mathbb{R}^{\mathcal{I}}. \end{aligned} \tag{1.2}$$

If $\mathcal{I}$ is empty, the ambient space is $\{0\}$, $K = \{0\}$, and $\gamma_K(0) = 0$. For $\alpha = (\alpha_1, \dots, \alpha_s)$ with $\alpha_i \in H^{r_i}(M;\mathbb{R})$, let

$$T_\alpha = \left(\left\langle \alpha_1^{a_1} \cdots \alpha_s^{a_s}, [M] \right\rangle \right)_{a\in\mathcal{I}}, \qquad \mathcal{V}_M(\alpha) = \inf_{\|\alpha_i\|_g^* \le 1 \ \forall i} \operatorname{Vol}(M, g). \tag{1.3}$$

The infimum is over smooth Riemannian metrics. Its admissible set is nonempty, since $\|\alpha_i\|_{\rho^2 g}^* = \rho^{-r_i}\|\alpha_i\|_g^*$ for $\rho > 0$.

Theorem 1.1. *If $\alpha_i \in H^{r_i}(M;\mathbb{Q})$ for every i, then*

$$\mathcal{V}_M(\alpha) = \gamma_K(T_\alpha). \tag{1.4}$$

The classes may vanish or be linearly dependent in each degree. The Euclidean forms in (1.2) *may have arbitrary real coefficients.*

The proof is given in Subsection 1.4. The infimum need not be attained. Theorem 1.10 gives the corresponding equality for real classes.

Lemma 1.2. *The body K is compact and centrally symmetric. If $\mathcal{I}$ is nonempty, 0 is an interior point; hence γ_K is a continuous norm. The moment vectors of rational tuples with all comasses strictly less than one are dense in the moment image defining K. For each $a \in \mathcal{I}$, this rational image contains $c_a e_a$ and $-c_a e_a$ for some $c_a \in \mathbb{Q}_{>0}$, where e_a is the corresponding coordinate vector.*

Proof. The product of the comass unit balls is compact, and t is polynomial. Its image has compact convex hull [61, Theorem 17.2]. The coordinate reflection $R(x_1, x_2, \dots, x_n) = (-x_1, x_2, \dots, x_n)$ preserves comass and rational coefficients, and $t(R^*\omega) = -t(\omega)$. Thus $K = -K$. Contraction by a factor less than one, followed by rational approximation in the open comass balls, shows that the rational moment vectors are dense.

Assume $\mathcal{I} \neq \emptyset$ and fix $a \in \mathcal{I}$. Partition the ordered coordinate coframe into consecutive blocks, ordered first by i and then by j, with a_i blocks of size r_i for each i. Let η_{ij} be the coordinate volume form of the jth block assigned to i. Choose $c \in \mathbb{Q}_{>0}$ with $c \max_i a_i < 1$, and put

$$\omega_i = c \sum_{j=1}^{a_i} \eta_{ij}.$$

Then $\operatorname{comass}(\omega_i) \leq c a_i < 1$. Since $a_i \leq 1$ whenever r_i is odd,

$$t_a(\omega) = c^{\sum_i a_i} \prod_i a_i! > 0.$$

If $b \in \mathcal{I}$ and $t_b(\omega) \neq 0$, no coordinate block can occur twice, so $b_i \leq a_i$ for every i. The equalities $\sum_i r_i b_i = \sum_i r_i a_i = n$ then give $b = a$. Thus $t(\omega) = c_a e_a$, with $c_a = c^{\sum_i a_i} \prod_i a_i! \in \mathbb{Q}_{>0}$, and reflection gives $-c_a e_a$. The convex hull of these signed coordinate vectors contains a neighborhood of zero. Hence γ_K is a continuous norm by [61, Theorem 15.2]. □

Lemma 1.3. *Let $z \in \mathbb{Q}^{\mathcal{I}}$ and let $v > \gamma_K(z)$ be a positive rational number. There exist an integer $q \geq 1$, rational tuples $\omega^{(1)}, \dots, \omega^{(q)}$ with every comass strictly less than one, and $\lambda_1, \dots, \lambda_q \in \mathbb{Q}_{>0}$ such that*

$$z = \sum_{j=1}^{q} \lambda_j t(\omega^{(j)}), \qquad \sum_{j=1}^{q} \lambda_j = v.$$

Proof. If $\mathcal{I} = \emptyset$, take the zero tuple with weight v. Otherwise, let $S_{\mathbb{Q}}$ be the set of moment vectors of rational tuples with all comasses strictly less than one. Lemma 1.2 gives $\overline{\operatorname{conv} S_{\mathbb{Q}}} = K$ and shows that $\operatorname{conv} S_{\mathbb{Q}}$ has nonempty interior. Hence

$$\operatorname{int}(\operatorname{conv} S_{\mathbb{Q}}) = \operatorname{int} K$$

by [61, Theorem 6.3]. Since $\gamma_K(z/v) < 1$, the point z/v belongs to $\operatorname{conv} S_{\mathbb{Q}}$. Carathéodory's theorem [61, Theorem 17.1] expresses it as a convex combination of affinely independent vectors $t(\omega^{(1)}), \dots, t(\omega^{(q)}) \in S_{\mathbb{Q}}$. After omitting zero coefficients, the barycentric coordinates $\theta_1, \dots, \theta_q$ are positive. They are rational, since they are the unique solution of the rational linear system

$$\sum_{j=1}^{q} \theta_j t(\omega^{(j)}) = z/v, \qquad \sum_{j=1}^{q} \theta_j = 1.$$

Set $\lambda_j = v\theta_j$. □

Lemma 1.4. *For real classes α_i and any metric satisfying $\|\alpha_i\|_g^* \leq 1$, one has $T_\alpha \in \operatorname{Vol}(M, g)K$.*

Proof. For $\delta > 0$, choose smooth closed representatives ω_i of α_i with $\|\omega_i\|_{\infty,g} \leq 1 + \delta$, and put $\eta_i = \omega_i/(1+\delta)$. Define $f_\delta = (f_{\delta,a})_{a \in \mathcal{I}} : M \to \mathbb{R}^{\mathcal{I}}$ by

$$\eta_1^{a_1} \wedge \cdots \wedge \eta_s^{a_s} = f_{\delta,a}\, dv_g,$$

where dv_g is the volume form of g and the given orientation. At each point, an oriented orthonormal frame identifies this coefficient vector with a moment vector of forms of comass at most one. Thus $f_\delta(x) \in K$ for every $x \in M$. By convexity,

$$\frac{1}{\operatorname{Vol}(M, g)} \int_M f_\delta\, dv_g \in K.$$

The de Rham cup product formula gives

$$\int_M f_\delta\, dv_g = \left((1+\delta)^{-\sum_i a_i} (T_\alpha)_a\right)_{a \in \mathcal{I}}.$$

Letting $\delta \downarrow 0$ and using the closedness of K gives $T_\alpha \in \operatorname{Vol}(M, g)K$. □

1.2. **Closed forms on finite complexes.** Let Y be a finite polyhedron with a triangulation. Finite subdivisions are allowed; forms and metrics are compared on a common subdivision. A piecewise smooth form on Y is given on each simplex by a smooth form on a neighborhood in its affine span. The tangential restrictions agree on common faces. Piecewise smooth metrics satisfy the same compatibility condition and are positive definite on each simplex. For a form ω and a metric h, $\|\omega\|_{\infty,h}$ is the maximum of the simplexwise comass norms. If $\dim Y = n$, then $\mathrm{Vol}_n(Y,h)$ is the sum of the volumes of the n-simplices. Path length is denoted by L_h, and d_h is the infimum of lengths of simplexwise piecewise smooth paths between the given endpoints. Triangulations of smooth manifolds are assumed smooth. Integration identifies the cohomology of the piecewise smooth de Rham complex with real cohomology [72, Theorem 7.1 and §7, examples (ii)–(iii)].

Lemma 1.5. *Let A be a subcomplex of a finite polyhedron Y, with $\dim A = n$ and $\dim(Y \setminus A) \le n-1$, and let h be a positive piecewise smooth metric on A. For $1 \le i \le s$, let $r_i \ge 1$ be an integer, let $u_i \in H^{r_i}(Y;\mathbb{R})$, and let η_i be a closed piecewise smooth r_i-form on A representing $u_i|_A$. Suppose that $\beta_i > 0$ and $\|\eta_i\|_{\infty,h} \le \beta_i$. For every $\delta > 0$, there are a positive piecewise smooth metric h_δ on Y and closed piecewise smooth representatives $\widetilde{\eta}_i$ of u_i such that*

$$\|\widetilde{\eta}_i\|_{\infty,h_\delta} \le \beta_i, \qquad \mathrm{Vol}_n(Y,h_\delta) \le \mathrm{Vol}_n(A,h) + \delta.$$

Moreover, $h_\delta|_A \to h$ uniformly as $\delta \downarrow 0$.

Proof. After barycentric subdivision, A is a full subcomplex. Let λ_v denote the barycentric coordinate of the vertex v, and put

$$b = \sum_{v \in A} \lambda_v, \qquad U = \{b > 1/2\}, \qquad r(x) = \frac{\sum_{v\in A} \lambda_v(x) v}{b(x)}.$$

On each simplex meeting U, the map r takes values in its face contained in A. It is a compatible piecewise smooth retraction $U \to A$. The homotopy $H : [0,1] \times U \to U$ given by $H(t,x) = (1-t)r(x) + tx$ remains in the same simplex, and $b(H(t,x)) = 1 - t + tb(x) > 1/2$.

Choose closed representatives θ_i of u_i on Y. The de Rham isomorphism on A gives piecewise smooth forms ζ_i with $\theta_i|_A - \eta_i = d\zeta_i$. For a form α on U, set

$$K\alpha = \int_0^1 \iota_{\partial_t} H^* \alpha \, dt,$$

where ι_{∂_t} denotes contraction with ∂_t. Cartan's formula [77, 2.25(d)] gives $dK\alpha + Kd\alpha = \alpha - r^*(\alpha|_A)$. Hence

$$\xi_i = K\theta_i + r^*\zeta_i, \qquad d\xi_i = \theta_i - r^*\eta_i.$$

Choose piecewise smooth functions $\chi, \psi : Y \to [0,1]$, equal to one near A, with

$$\operatorname{supp}\chi \subset U, \qquad \operatorname{supp}\psi \subset \operatorname{int}\{\chi = 1\}.$$

They may be chosen as smooth functions of b. Extend $\chi\xi_i$ by zero outside U and put

$$\widetilde{\eta}_i = \theta_i - d(\chi\xi_i).$$

These forms are closed, represent u_i, and equal $r^*\eta_i$ near $\operatorname{supp}\psi$.

Fix a positive piecewise smooth metric h_0 on Y. Choose $C > 0$ so that

$$Ch_0 \ge r^*h \quad \text{on } \operatorname{supp}\psi, \qquad C^{-r_i/2}\|\widetilde{\eta}_i\|_{\infty,h_0} \le \beta_i \quad (1 \le i \le s).$$

Such a C exists by compactness and finiteness of the list. For $\tau > 0$, define

$$k_\tau = \psi r^*h + \big(\psi\tau^2 + (1-\psi)C\big)h_0,$$

where ψr^*h is extended by zero outside U. This metric is positive. On $\{\psi > 0\}$,

$$k_\tau - r^*h = (1-\psi)(Ch_0 - r^*h) + \psi\tau^2 h_0 \ge 0.$$

Hence the comass of $\widetilde{\eta}_i = r^*\eta_i$ is at most β_i. On $\{\psi = 0\}$ the same bound follows from $k_\tau = Ch_0$. On A, one has $k_\tau = h + \tau^2 h_0|_A$. Every n-simplex of Y belongs to A, whence

$$\mathrm{Vol}_n(Y, k_\tau) = \mathrm{Vol}_n(A, h + \tau^2 h_0|_A) \longrightarrow \mathrm{Vol}_n(A, h).$$

Choose $0 < \tau_\delta \le \min\{1, \delta\}$ with volume excess at most δ, and set $h_\delta = k_{\tau_\delta}$. □

A zero comass bound forces $\widetilde{\eta}_i = 0$, hence $u_i = 0$. Thus $\beta_i > 0$ is necessary when $u_i \neq 0$, even if $u_i|_A = 0$.

Lemma 1.6. *Let (M, g) be a closed Riemannian manifold with a finite smooth triangulation, and let ω be a closed piecewise smooth r-form. There are linear regularization operators R_τ and constants $C, \tau_0 > 0$, independent of ω and its degree, such that $R_\tau\omega$ is smooth, closed, and cohomologous to ω, with*

$$R_\tau\omega \longrightarrow \omega \quad \text{in } L^2(g) \quad (\tau \downarrow 0), \qquad \|R_\tau\omega\|_{\infty,g} \le (1 + C\tau)^r \|\omega\|_{\infty,g} \quad (0 < \tau < \tau_0).$$

In particular, $\|[\omega]\|_g^* \le \|\omega\|_{\infty,g}$.

Proof. The L^2 de Rham complex consists of L^2 forms whose weak exterior derivative belongs to L^2. Stokes' theorem and the equality of tangential traces on common faces make the inclusion of piecewise smooth forms into the L^2 de Rham complex a morphism of complexes. The inclusions of smooth forms into these two complexes induce isomorphisms by the simplicial de Rham theorem above and [30, Theorem 12.10]. Thus the class of ω is unchanged by the inclusion. De Rham regularization [30, Theorem 12.5], with all Lebesgue exponents equal to 2, gives smooth closed representatives $R_\tau\omega$ converging to ω in $L^2(g)$.

On the compact manifold M, the construction in [30, §§12.1–12.3] gives an integer $m \ge 1$, a smooth nonnegative compactly supported function $\rho : \mathbb{R}^{\dim M} \to \mathbb{R}$ with $\int \rho = 1$, and smooth families of diffeomorphisms $\Phi_{j,v} : M \to M$, $1 \le j \le m$, $v \in \mathbb{R}^{\dim M}$, with $\Phi_{j,0} = \mathrm{id}$, such that

$$S_{j,\tau}\alpha = \int_{\mathbb{R}^{\dim M}} \Phi_{j,\tau y}^*\alpha \, \rho(y)\, dy, \qquad R_\tau = S_{1,\tau} \circ \cdots \circ S_{m,\tau}.$$

Compactness gives constants $c_j \ge 0$ such that

$$\sup_{x \in M,\, y \in \mathrm{supp}\,\rho} \|D\Phi_{j,\tau y}(x)\|_g \le 1 + c_j\tau \quad (0 < \tau < \tau_0).$$

The pullback inequality for comass and the triangle inequality yield

$$\|R_\tau\omega\|_{\infty,g} \le \prod_{j=1}^{m} (1 + c_j\tau)^r \|\omega\|_{\infty,g} \le (1 + C\tau)^r \|\omega\|_{\infty,g}$$

after increasing C. The operators do not depend on ω or its degree. Letting $\tau \downarrow 0$ in the definition of cohomological comass proves the last inequality. □

Lemma 1.7. *Let P be a closed smooth n-manifold, let Y be a finite polyhedron with a positive piecewise smooth metric h, and let $F : P \to Y$ be continuous and smooth up to the faces of each simplex in a finite smooth triangulation of P, with each source simplex mapped into one target simplex on which h is smooth. For every $\eta > 0$, there is a smooth positive metric g on P such that F is nonexpanding and*

$$\mathrm{Vol}(P, g) < \int_P dv_{F^*h} + \eta.$$

The integral is that of the volume density of the semidefinite tensor F^*h; this density is zero where F^*h has rank less than n.

Proof. Fix a smooth positive metric g_0 and put $q_\varepsilon = F^*h + \varepsilon^2 g_0$. This is a positive piecewise smooth metric. Continuity of the determinant and dominated convergence on the finitely many simplices give

$$\mathrm{Vol}(P, q_\varepsilon) \longrightarrow \int_P dv_{F^*h}.$$

Let S be the $(n-1)$-skeleton of the source triangulation, and set $U_a = \{x \in P : d_{g_0}(x,S) < a\}$. For fixed $\varepsilon > 0$, Babenko's metric smoothing [5, proof of Lemma 2.3, pp. 13–14] gives smooth metrics $g_{\varepsilon,a}$ and a constant C_ε, independent of a, such that

$$q_\varepsilon \le g_{\varepsilon,a} \le C_\varepsilon g_0, \qquad g_{\varepsilon,a} = q_\varepsilon \quad \text{on } P \setminus U_a,$$
$$\mathrm{Vol}(P, g_{\varepsilon,a}) \longrightarrow \mathrm{Vol}(P, q_\varepsilon) \quad (a \downarrow 0).$$

Choose ε and then a so that each volume excess is less than $\eta/2$, and put $g = g_{\varepsilon,a}$. Since $F^*h \le g$,

$$L_h(F \circ \gamma) \le L_g(\gamma)$$

for every finite concatenation γ of smooth curves contained in source simplices. These curves compute d_g; taking infima gives $d_h(Fx, Fy) \le d_g(x,y)$. □

1.3. **Marked transfer.** For each integer $r \ge 2$, put $E_r = S^r$ if r is odd and $E_r = J(S^r)$ if r is even, where $J(S^r)$ is the James reduced product. One has $J(S^r) \simeq \Omega S^{r+1}$, where Ω denotes the based loop space. Choose a generator $u_r \in H^r(E_r;\mathbb{Z}) \simeq \mathbb{Z}$, and denote its image in rational cohomology by the same symbol. The standard CW structure has one cell in each of dimensions 0 and r for odd r, and one cell in each dimension kr, $k \ge 0$, for even r. The integral homology is free. The rational cohomology algebra is exterior on u_r for odd r and polynomial on u_r for even r. For every positive integer a, there is a self-map $\mu_a : E_r \to E_r$ with $\mu_a^* u_r = a u_r$; see [17, §4].

Lemma 1.8. *Let X be a finite CW complex, let $r \ge 2$, and let $\alpha \in H^r(X;\mathbb{Q})$. There are a positive integer q and a map $f : X \to E_r$ such that $f^* u_r = q\alpha$ in rational cohomology. Every positive integer multiple of q has the same property.*

Proof. Suppose first that X is connected and $\alpha \ne 0$. If $r = \dim X$, replace X by $X \vee S^{r+1}$ and extend α by zero; restriction to X will give the required map. Thus $r < \dim X$. Write $\alpha = (a/d)e$, where a, d are positive integers and e is primitive in the image of $H^r(X;\mathbb{Z})$ in $H^r(X;\mathbb{Q})$. Complete e to an integral basis of this lattice, of rank b, and let $p_1 : E_r^b \to E_r$ be projection to the first factor. Lemma 5.1 of [17] and its proof, with this prescribed basis, give a map $F : X \to E_r^b$ and a positive integer m such that $F^* p_1^* u_r = me$. For $f = \mu_a \circ p_1 \circ F$, one has

$$f^* u_r = ame = dm\alpha.$$

Postcomposition with μ_ℓ replaces dm by ℓdm for every positive integer ℓ. A zero class is represented by a constant map. For disconnected X, apply the construction to each component and replace the finitely many multipliers by a common positive multiple. □

Proposition 1.9. *Let M, N be closed connected oriented smooth n-manifolds, $n \ge 3$. Suppose that $\alpha_i \in H^{r_i}(M;\mathbb{Q})$ and $\alpha_i' \in H^{r_i}(N;\mathbb{Q})$, $1 \le i \le s$, with $2 \le r_i \le n$, have equal top moments. Let h be a smooth metric on N, and let smooth closed forms η_i represent α_i', with $\|\eta_i\|_{\infty,h} \le \beta_i$ for positive numbers β_i. For every $\delta > 0$, there are a smooth metric g on M and smooth closed forms θ_i representing α_i such that*

$$\|\theta_i\|_{\infty,g} \le \beta_i, \qquad \mathrm{Vol}(M,g) \le \mathrm{Vol}(N,h) + \delta.$$

For any specified sublist of a common degree, let I denote the identity matrix of the corresponding size. These choices can also make the $L^2(g)$ Gram matrix of the θ_i at most the $L^2(h)$ Gram matrix of the corresponding η_i plus δI in positive-semidefinite order.

Proof. Apply Lemma 1.8 on both manifolds and choose common positive multipliers q_i. Put $E = \prod_i E_{r_i}$, and let u_i be the pullback of u_{r_i} from the ith factor. There are maps

$$f : M \longrightarrow E, \qquad j : N \longrightarrow E, \qquad f^* u_i = q_i \alpha_i, \quad j^* u_i = q_i \alpha_i'.$$

For $a = (a_1, \dots, a_s) \in \mathcal{I}$, put $u^a = u_1^{a_1} \cdots u_s^{a_s}$ and $\alpha^a = \alpha_1^{a_1} \cdots \alpha_s^{a_s}$, with the analogous notation $(\alpha')^a$. Equality of the top moments gives

$$\langle u^a, f_*[M] - j_*[N] \rangle = \Big(\prod_i q_i^{a_i} \Big) \big(\langle \alpha^a, [M] \rangle - \langle (\alpha')^a, [N] \rangle \big) = 0.$$

These monomials span $H^n(E;\mathbb{Q})$; hence $f_*[M] = j_*[N]$ in $H_n(E;\mathbb{Q})$. The integral Künneth theorem [39, Theorem 3B.6] makes $H_n(E;\mathbb{Z})$ free abelian, so the equality holds over $\mathbb{Z}$.

For a CW complex Z and an integer $k \geq 0$, write $Z^{(k)}$ for its k-skeleton. By cellular approximation [39, Theorem 4.8], both maps may be taken to have values in $K = E^{(n+1)}$. This is a finite simply connected CW complex, and $H_n(K;\mathbb{Z}) \to H_n(E;\mathbb{Z})$ is an isomorphism. Choose a finite simplicial complex P and homotopy inverse maps $\varphi : K \to P$ and $\psi : P \to K$ [39, Theorem 2C.5]. With a smooth triangulation of N, replace $\varphi \circ j$ by a simplicial map j_P. Let X be its simplicial mapping cylinder [39, §2.C, p. 183], and let $\rho : X \to P$ be its retraction. Thus $N \subset X$ is a subcomplex with a smooth triangulation and $\rho|_N = j_P$.

Let $\iota : N \hookrightarrow X$ be the inclusion, and let $f_X : M \to X$ be the composite of $\varphi \circ f$ with $P \subset X$. Since ρ_* is an isomorphism, $(f_X)_*[M] = \iota_*[N]$ in $H_n(X;\mathbb{Z})$. Moreover, $\pi_1(X) = 0$, so f_X is surjective on fundamental groups. Brunnbauer's lemma [17, Lemma 3.2] gives a simplicial map

$$F : M \longrightarrow Y = N \cup X^{(n-1)}$$

whose composite with $Y \hookrightarrow X$ is homotopic to f_X. The smooth triangulations of M and N may be chosen as in [18, §§2.3–2.5]. The inverse image of each open n-simplex of N is empty or one open n-simplex of M, on which F is a simplex isomorphism. A simplicial map with this property is called $(n,1)$-monotone. Every other n-simplex of M maps to $X^{(n-1)}$. Subdivide Y so that N is full, and triangulate the inverse-image polytopal subdivision of M compatibly, without adding vertices [62, Proposition 2.9 and Lemmas 2.16, 3.3(a)]. On every nondegenerate n-simplex this is the inverse-image triangulation; hence F remains simplicial and $(n,1)$-monotone. For $c : X \xrightarrow{\rho} P \xrightarrow{\psi} K \hookrightarrow E$, put $v_i = q_i^{-1} c^* u_i$. Then

$$v_i|_N = \alpha_i', \qquad F^*(v_i|_Y) = \alpha_i.$$

For each degree r occurring in the list, put $I_r = \{i : r_i = r\}$, and let Id_r be the identity matrix on $\mathbb{R}^{I_r}$. For a family of r-forms ζ_i, $i \in I_r$, on a compact Riemannian n-manifold (P', ℓ), write

$$G_{\ell,r}(\zeta) = \left(\int_{P'} \langle \zeta_i, \zeta_j \rangle_\ell \, dv_\ell \right)_{i,j \in I_r}.$$

The same notation applies to piecewise smooth forms and metrics. All matrix limits below are in the Euclidean operator norm.

The construction in Lemma 1.5, with $A = N$, gives closed piecewise smooth representatives $\widetilde{\eta}_i$ of $v_i|_Y$ and positive piecewise smooth metrics k_ν, $\nu > 0$, such that

$$\widetilde{\eta}_i|_N = \eta_i, \qquad \|\widetilde{\eta}_i\|_{\infty,k_\nu} \leq \beta_i, \qquad k_\nu|_N \longrightarrow h \quad \text{uniformly as } \nu \downarrow 0.$$

Every n-simplex of Y belongs to N. Consequently

$$\mathrm{Vol}_n(Y, k_\nu) \longrightarrow \mathrm{Vol}(N, h), \qquad G_{k_\nu|_N,r}(\eta) \longrightarrow G_{h,r}(\eta),$$

the second limit following from continuity of the inverse metric and volume density.

Fix ν, put $k = k_\nu$ and $\omega_i = F^*\widetilde{\eta}_i$, and choose a smooth positive metric g_0 on M. For $\varepsilon > 0$, set

$$q_\varepsilon = F^*k + \varepsilon^2 g_0.$$

The forms ω_i are closed, represent α_i, and satisfy $\|\omega_i\|_{\infty,q_\varepsilon} \leq \beta_i$. The pullback comparison [17, proof of Lemma 2.2] gives

$$\lim_{\varepsilon \downarrow 0} \mathrm{Vol}(M, q_\varepsilon) \leq \mathrm{Vol}_n(Y, k).$$

For each r, put

$$B_r = \max_{\sigma \subset Y} \sup_{y \in \sigma} \sum_{i \in I_r} |\widetilde{\eta}_i(y)|_k^2,$$

where σ ranges over the simplices of Y and $|\cdot|_k$ is the norm induced by k on exterior forms. This number is finite. The differential of F is a contraction from q_ε to k, as are its exterior powers. Hence, for every $z = (z_i)_{i\in I_r} \in \mathbb{R}^{I_r}$,

$$\left|\sum_{i\in I_r} z_i\omega_i\right|^2_{q_\varepsilon} \le B_r \sum_{i\in I_r} z_i^2.$$

On an n-simplex Δ on which F has rank less than n, $\mathrm{Vol}(\Delta, q_\varepsilon) \to 0$, and therefore

$$\int_\Delta \left|\sum_{i\in I_r} z_i\omega_i\right|^2_{q_\varepsilon} dv_{q_\varepsilon} \le B_r \left(\sum_{i\in I_r} z_i^2\right) \mathrm{Vol}(\Delta, q_\varepsilon) \longrightarrow 0.$$

On every other n-simplex, q_ε converges uniformly to the positive metric F^*k. Change of variables identifies the limiting integral with the integral of $|\sum_{i\in I_r} z_i\eta_i|^2_{k|_N}$ over the image simplex in N. Both limits are uniform for $\sum_i z_i^2 \le 1$. Since each target n-simplex occurs at most once, there is a matrix $G_{0,r}$ with

$$G_{q_\varepsilon,r}(\omega) \longrightarrow G_{0,r} \le G_{k|_N,r}(\eta).$$

For fixed ε and $a > 0$, let U_a be the a-neighborhood, for g_0, of the $(n-1)$-skeleton of M. The construction in Lemma 1.7 gives smooth metrics ℓ_a and a smooth metric H_ε, independent of a, such that

$$q_\varepsilon \le \ell_a \le H_\varepsilon, \qquad \ell_a = q_\varepsilon \text{ on } M \setminus U_a, \qquad \mathrm{Vol}(U_a, H_\varepsilon) \longrightarrow 0 \quad (a \downarrow 0).$$

The comass bounds persist, and $\mathrm{Vol}(M, \ell_a) \to \mathrm{Vol}(M, q_\varepsilon)$. Domination of q_ε by ℓ_a also gives

$$\|G_{\ell_a,r}(\omega) - G_{q_\varepsilon,r}(\omega)\|_{\mathrm{op}} \le 2B_r \,\mathrm{Vol}(U_a, H_\varepsilon) \longrightarrow 0.$$

For fixed a, apply Lemma 1.6 with metric ℓ_a. Let C be its constant, and put

$$\theta_{i,\tau} = R_\tau\omega_i, \qquad t_\tau = 1 + C\tau, \qquad g_\tau = t_\tau^2\ell_a.$$

These forms are smooth, closed, and represent α_i, with

$$\|\theta_{i,\tau}\|_{\infty,g_\tau} = t_\tau^{-r_i}\|R_\tau\omega_i\|_{\infty,\ell_a} \le \beta_i.$$

The $L^2(\ell_a)$ convergence in the lemma and the scaling laws give

$$\begin{aligned}\mathrm{Vol}(M, g_\tau) &= t_\tau^n \,\mathrm{Vol}(M, \ell_a) \longrightarrow \mathrm{Vol}(M, \ell_a),\\ G_{g_\tau,r}(\theta_\tau) &= t_\tau^{n-2r} G_{\ell_a,r}(R_\tau\omega) \longrightarrow G_{\ell_a,r}(\omega).\end{aligned}$$

The preceding limits permit successive choices of $\nu, \varepsilon, a, \tau$ such that

$$\begin{aligned}\mathrm{Vol}(M, g_\tau) &\le \mathrm{Vol}(M, \ell_a) + \delta/4\\ &\le \mathrm{Vol}(M, q_\varepsilon) + \delta/2\\ &\le \mathrm{Vol}_n(Y, k_\nu) + 3\delta/4 \le \mathrm{Vol}(N, h) + \delta\end{aligned}$$

and, simultaneously for every degree r occurring in the list,

$$\begin{aligned}G_{g_\tau,r}(\theta_\tau) &\le G_{\ell_a,r}(\omega) + (\delta/4)\mathrm{Id}_r\\ &\le G_{q_\varepsilon,r}(\omega) + (\delta/2)\mathrm{Id}_r\\ &\le G_{k_\nu|_N,r}(\eta) + (3\delta/4)\mathrm{Id}_r\\ &\le G_{h,r}(\eta) + \delta\mathrm{Id}_r.\end{aligned}$$

Set $g = g_\tau$ and $\theta_i = \theta_{i,\tau}$. Passage to a principal submatrix gives the assertion for every specified sublist. □

1.4. **Rational forms on flat tori.**

Proof of Theorem 1.1. Lemma 1.4 gives $\mathcal{V}_M(\alpha) \geq \gamma_K(T_\alpha)$. Fix a rational $v > \gamma_K(T_\alpha)$ and a number $\delta > 0$. Since $T_\alpha \in \mathbb{Q}^{\mathcal{I}}$, Lemma 1.3 gives rational tuples $\omega^{(1)}, \dots, \omega^{(q)}$, $q \geq 1$, with all comasses less than one, and positive rational numbers $\lambda_1, \dots, \lambda_q$ such that

$$T_\alpha = \sum_{j=1}^{q} \lambda_j t(\omega^{(j)}), \qquad \sum_{j=1}^{q} \lambda_j = v.$$

For each j, let

$$T_j = \mathbb{R}^n/(\lambda_j \mathbb{Z} \oplus \mathbb{Z}^{n-1})$$

have its Euclidean quotient metric g_j and standard orientation. The constant forms $\omega_i^{(j)}$ descend to T_j with the same comasses. In period-one coordinates $y_1 = x_1/\lambda_j$, $y_k = x_k$ for $k \geq 2$, their coefficients and their periods on the coordinate subtori are rational. These subtori give an integral homology basis by the Künneth theorem; the universal coefficient theorem therefore gives rational cohomology classes [39, Theorems 3B.6 and 3.2]. Moreover, for $a \in \mathcal{I}$,

$$\mathrm{Vol}(T_j, g_j) = \lambda_j, \qquad \int_{T_j} (\omega_1^{(j)})^{a_1} \wedge \cdots \wedge (\omega_s^{(j)})^{a_s} = \lambda_j t_a(\omega^{(j)}).$$

If $q = 1$, set

$$N = T_1, \qquad h_\rho = g_1 \quad (\rho > 0), \qquad \widetilde{\omega}_i^{(1)} = \eta_i = \omega_i^{(1)}.$$

For $q \geq 2$, form the oriented connected sum $N = T_1 \# \cdots \# T_q$ along a chain. Choose the $2(q-1)$ attachment balls disjoint and Euclidean, of radius $\rho > 0$. The tori are fixed; hence these balls exist for every sufficiently small ρ. In centered Euclidean coordinates, put

$$B_t = \{x \in \mathbb{R}^n : |x| < t\}, \qquad t > 0.$$

Fix a smooth function $\chi : \mathbb{R}^n \to [0,1]$, equal to one on $B_{1/2}$ and supported in $B_{3/4}$, and put $c = \sup |d\chi|$ and $\chi_\rho(x) = \chi(x/\rho)$. For a constant r-form ω, $2 \leq r \leq n$, of comass at most one, let $R = \sum_{k=1}^n x_k \partial_{x_k}$ and $\xi = r^{-1} \iota_R \omega$. Cartan's formula [77, 2.25(d)] gives $d\xi = \omega$. Let $\|\cdot\|_{\infty, B_\rho}$ denote the supremum of the Euclidean comass on B_ρ. Then

$$\widetilde{\omega} = \omega - d(\chi_\rho \xi) = (1 - \chi_\rho)\omega - d\chi_\rho \wedge \xi, \qquad \|\xi\|_{\infty, B_\rho} \leq \rho/r, \qquad \|\widetilde{\omega}\|_{\infty, B_\rho} \leq 1 + c.$$

Extend $\chi_\rho \xi$ by zero outside B_ρ. The form $\widetilde{\omega}$ is closed, cohomologous to ω, zero on $B_{\rho/2}$, and equal to ω outside $B_{3\rho/4}$. Apply this replacement to every marked form on every attachment ball, and denote the resulting forms on T_j by $\widetilde{\omega}_i^{(j)}$.

Put $L = \sqrt{1+c}$. Multiply metric lengths by L on each $B_{3\rho/4}$, and interpolate smoothly to the original metric outside B_ρ, with conformal length factor between 1 and L. On $B_{3\rho/4}$ the comasses are at most $L^{-r_i}(1+c) \leq 1$, since $r_i \geq 2$; in the interpolation annulus the forms are unchanged and the metric dominates g_j.

Let g_S be the unit round metric on S^{n-1}. Fix a positive smooth function φ on $[1/4, 1/2]$, with $\varphi = 1/4$ near $1/4$ and $\varphi(t) = t$ near $1/2$. In polar coordinates, put $u = |x|$. Remove each $B_{\rho/4}$ and, on the annulus $\rho/4 \leq u \leq \rho/2$, replace the metric by

$$L^2\big(du^2 + \rho^2 \varphi(u/\rho)^2 g_S\big).$$

Join each paired boundary by $[0, \rho] \times S^{n-1}$ with metric

$$L^2\big(dt^2 + (\rho/4)^2 g_S\big),$$

using boundary isometries that reverse the induced boundary orientations. The metrics have identical product germs at the joins and agree with $L^2 g_j$ near $u = \rho/2$. They therefore give a smooth positive metric h_ρ on N. All the modified forms vanish on these collars and extend by zero over the cylinders

to smooth closed forms η_i on N, with $\|\eta_i\|_{\infty,h_\rho} \le 1$. The conformal factors are bounded by L, and the collar and cylinder metrics are ρ^2 times fixed metrics after dilation. Consequently, in both cases, there is a constant $C > 0$, independent of ρ, such that

$$\mathrm{Vol}(N, h_\rho) \le v + C\rho^n.$$

The removed balls and cylinders contribute zero to every top product. Since $[\widetilde{\omega}_i^{(j)}] = [\omega_i^{(j)}]$ on each T_j, for every $a \in \mathcal{I}$ one has

$$\begin{aligned}\int_N \eta_1^{a_1} \wedge \cdots \wedge \eta_s^{a_s} &= \sum_{j=1}^{q} \int_{T_j} (\widetilde{\omega}_1^{(j)})^{a_1} \wedge \cdots \wedge (\widetilde{\omega}_s^{(j)})^{a_s} \\ &= \sum_{j=1}^{q} \lambda_j t_a(\omega^{(j)}) = (T_\alpha)_a.\end{aligned}$$

For $r_i < n$, the connected-sum cohomology decomposition, together with the universal coefficient theorem [39, Theorem 3.2 and §3.3, Exercise 6(a)], identifies $[\eta_i]$ with the tuple of rational classes $([\omega_i^{(j)}])_{j=1}^q$; this decomposition is compatible with the inclusion $\mathbb{Q} \subset \mathbb{R}$. For $r_i = n$, the displayed identity gives $\int_N \eta_i \in \mathbb{Q}$, which is equivalent to rationality of its class. Thus $\alpha_i' = [\eta_i] \in H^{r_i}(N; \mathbb{Q})$ for every i, and $T_{\alpha'} = T_\alpha$.

Choose ρ so that $C\rho^n < \delta$. Proposition 1.9, with $\beta_i = 1$, gives a smooth metric g on M and smooth closed representatives of the α_i with comass at most one and

$$\mathrm{Vol}(M, g) \le \mathrm{Vol}(N, h_\rho) + \delta < v + 2\delta.$$

Hence $\mathcal{V}_M(\alpha) \le v + 2\delta$. Let $\delta \downarrow 0$, then let rational v decrease to $\gamma_K(T_\alpha)$. □

1.5. **Real classes.**

Theorem 1.10 (Finite comass bounds). *For any real classes $\alpha_i \in H^{r_i}(M; \mathbb{R})$, $2 \le r_i \le n$, and any positive bounds β_i,*

$$\inf_{\|\alpha_i\|_g^* \le \beta_i \ \forall i} \mathrm{Vol}(M, g) = \gamma_K\big(T_{(\alpha_i/\beta_i)_i}\big). \tag{1.5}$$

The classes may vanish or be linearly dependent in each degree.

Proof. Replacing α_i by α_i/β_i reduces the proof to $\beta_i = 1$ for every i. The lower bound is Lemma 1.4. For each degree r occurring in the list, choose a basis $e_{r,1}, \dots, e_{r,b_r}$ of $H^r(M; \mathbb{Q})$, where $b_r = \dim H^r(M; \mathbb{Q})$, and write $\alpha_i = \sum_{j=1}^{b_{r_i}} c_{ij} e_{r_i,j}$. For integers $\nu \ge 2$, put $R_\nu = \nu$ and $\varepsilon_\nu = \nu^{-1}$. Choose rational numbers $c_{ij}^{(\nu)}$ such that

$$\sum_{j=1}^{b_{r_i}} |c_{ij}^{(\nu)} - c_{ij}| \le \frac{\varepsilon_\nu}{2R_\nu}, \qquad \alpha_i^{(\nu)} = \sum_{j=1}^{b_{r_i}} c_{ij}^{(\nu)} e_{r_i,j}.$$

Apply Theorem 1.1 to the rational list

$$\frac{\alpha_i^{(\nu)}}{1 - \varepsilon_\nu} \quad (1 \le i \le s), \qquad \frac{e_{r,j}}{R_\nu} \quad (r \in \{r_1, \dots, r_s\},\ 1 \le j \le b_r), \tag{1.6}$$

with the auxiliary entries ordered first by r, then by j. Let $\mathcal{I}'$ and K' be the moment index set and body of this enlarged degree list, and let T_ν' be the moment vector of (1.6). The degree list, $\mathcal{I}'$, and K' are independent of ν. There is a smooth metric g_ν such that

$$\|\alpha_i^{(\nu)}\|_{g_\nu}^* \le 1 - \varepsilon_\nu, \quad \|e_{r,j}\|_{g_\nu}^* \le R_\nu, \quad \mathrm{Vol}(M, g_\nu) \le \gamma_{K'}(T_\nu') + \nu^{-1}.$$

Hence

$$\|\alpha_i\|_{g_\nu}^* \le 1 - \varepsilon_\nu + R_\nu \sum_{j=1}^{b_{r_i}} |c_{ij}^{(\nu)} - c_{ij}| \le 1 - \varepsilon_\nu/2.$$

Let $\pi : \mathbb{R}^{\mathcal{I}'} \to \mathbb{R}^{\mathcal{I}}$ be the coordinate projection deleting the moments that involve auxiliary entries, and let $\iota : \mathbb{R}^{\mathcal{I}} \to \mathbb{R}^{\mathcal{I}'}$ insert zeros in the deleted coordinates. Projection of the Euclidean tuples gives $\pi(K') \subset K$, and adjoining zero forms gives $\iota(K) \subset K'$. Since $\pi\iota$ is the identity, for $v \in \mathbb{R}^{\mathcal{I}}$ and $u \geq 0$ one has $\iota(v) \in uK'$ if and only if $v \in uK$. Thus

$$\gamma_{K'}(\iota(v)) = \gamma_K(v), \qquad v \in \mathbb{R}^{\mathcal{I}}. \tag{1.7}$$

Each moment is polynomial in the coordinates of the classes. Since $\alpha_i^{(v)}/(1-\varepsilon_v) \to \alpha_i$ and $e_{r,j}/R_v \to 0$, one has $T'_v \to \iota(T_\alpha)$. Lemma 1.2 gives continuity of $\gamma_{K'}$, and therefore

$$\mathcal{V}_M(\alpha) \leq \limsup_{v\to\infty} \operatorname{Vol}(M, g_v) \leq \gamma_{K'}(\iota(T_\alpha)) = \gamma_K(T_\alpha).$$

These formulas remain valid for empty bases and empty moment index sets. □

1.6. **Finite algebraic formulas and stable directions.** Real algebraic numbers are specified by nonzero polynomials in $\mathbb{Z}[X]$ and rational isolating intervals.

Lemma 1.11. *For a fixed dimension and degree list, K and the graph of γ_K are semialgebraic over $\mathbb{Q}$. If $v \in \mathbb{R}^{\mathcal{I}}$ has real algebraic coordinates, then $\gamma_K(v)$ is a real algebraic number computable from these coordinates.*

Proof. Let $\delta_{k\ell}$ be the Kronecker delta. In standard exterior coordinates, the comass unit ball in $\Lambda^r(\mathbb{R}^n)^*$ is defined by

$$\omega(v_1, \dots, v_r)^2 \leq 1 \quad \text{for every } (v_1, \dots, v_r) \in (\mathbb{R}^n)^r \text{ with } \langle v_k, v_\ell \rangle = \delta_{k\ell} \text{ for all } k, \ell.$$

This is a first-order formula with rational coefficients. The moment polynomials t_a also have rational coefficients. Put $d = |\mathcal{I}|$. For $x \in \mathbb{R}^{\mathcal{I}}$ and $u \in \mathbb{R}$, let $\Phi(x, u)$ mean that $u \geq 0$ and that there exist $\lambda_0, \dots, \lambda_d \geq 0$ and tuples $\omega^{(0)}, \dots, \omega^{(d)}$ of the prescribed degrees such that

$$\sum_{k=0}^{d} \lambda_k = 1, \qquad \operatorname{comass}(\omega_i^{(k)}) \leq 1 \quad \text{for all } i, k, \qquad x = u \sum_{k=0}^{d} \lambda_k t(\omega^{(k)}).$$

Carathéodory's theorem [61, Theorem 17.1] gives $\Phi(x, u)$ if and only if $u \geq 0$ and $x \in uK$. In particular, $x \in K$ if and only if $\Phi(x, 1)$. By Lemma 1.2, K is compact and absorbing, so $\gamma_K(x)$ is the least $u \geq 0$ with $\Phi(x, u)$. Its graph is therefore defined by

$$\Phi(x, u) \ \wedge \ \neg\exists w \left(0 \leq w < u \ \wedge \ \Phi(x, w)\right).$$

Quantifier elimination [14, Theorem 2.77] shows that K and the graph of γ_K are semialgebraic over $\mathbb{Q}$. For algebraic v, substitution of $x = v$ gives a formula defining the singleton $\{\gamma_K(v)\}$. Effective quantifier elimination and algebraic sampling determine this point [14, Theorem 14.16 and Algorithm 13.2]. □

Theorem 1.12. *Let $0 \neq z_i \in H_{r_i}(M; \mathbb{R})$, $2 \leq r_i \leq n$, and let $a_i > 0$. Then*

$$\inf_{N_g^{(r_i)}(z_i) \geq a_i \ \forall i} \operatorname{Vol}(M, g) = \inf_{\substack{\eta_i \in H^{r_i}(M;\mathbb{R}) \\ \langle \eta_i, z_i \rangle \geq a_i \ \forall i}} \gamma_K(T_\eta). \tag{1.8}$$

Suppose the evaluations on $[M]$ of all top products of rational cohomology basis classes in the occurring degrees are given. If the coordinates of the z_i in the dual homology bases and the a_i are real algebraic numbers, this common value is a finite nonnegative real algebraic number computable from these data.

Proof. For every admissible metric g, (1.1) gives classes η_i, after changing their signs if necessary, such that

$$\|\eta_i\|_g^* \leq 1, \qquad \langle \eta_i, z_i \rangle = N_g^{(r_i)}(z_i) \geq a_i.$$

Lemma 1.4 gives $\gamma_K(T_\eta) \leq \operatorname{Vol}(M, g)$, hence one inequality in (1.8). Conversely, fix a tuple η satisfying the pairing inequalities. For every $\varepsilon > 0$, Theorem 1.10 gives a smooth metric g with

$$\|\eta_i\|_g^* \leq 1 \quad \text{for every } i, \qquad \operatorname{Vol}(M, g) < \gamma_K(T_\eta) + \varepsilon.$$

Equation (1.1) then gives $N_g^{(r_i)}(z_i) \geq \langle \eta_i, z_i \rangle \geq a_i$. Taking the infimum over η and letting $\varepsilon \downarrow 0$ proves equality.

In rational cohomology bases and their dual homology bases, the pairing inequalities are linear in the coordinates of η, and T_η is polynomial with rational coefficients. Under the algebraicity hypotheses, Lemma 1.11 gives a first-order description, with real algebraic coefficients, of

$$E = \{u \geq 0 \,:\, \exists \eta \text{ with } \langle \eta_i, z_i \rangle \geq a_i \text{ for every } i,\ \gamma_K(T_\eta) \leq u\}.$$

The pairing is nondegenerate and each $z_i \neq 0$, so the tuple set is nonempty. Since the gauge is finite, E is nonempty and has a finite infimum $c \geq 0$. This number is characterized by

$$c \geq 0, \qquad (\forall u \in E)\ c \leq u, \qquad (\forall \varepsilon > 0)(\exists u \in E)\ u < c + \varepsilon.$$

Effective quantifier elimination and algebraic sampling compute this singleton [14, Theorem 14.16 and Algorithm 13.2]. Neither boundedness of the tuple set nor attainment of the infimum is required. □

Corollary 1.13. *In addition to the data of Theorem 1.12, fix classes $\theta_j \in H^{q_j}(M;\mathbb{R})$, $2 \leq q_j \leq n$, and numbers $\beta_j > 0$, $1 \leq j \leq t$. Let K^+ be the moment body for the ordered degree list $(r_1, \dots, r_s, q_1, \dots, q_t)$. Then*

$$\inf_{\substack{N_g^{(r_i)}(z_i) \geq a_i\ \forall i \\ \|\theta_j\|_g^* \leq \beta_j\ \forall j}} \mathrm{Vol}(M,g) = \inf_{\substack{\eta_i \in H^{r_i}(M;\mathbb{R}) \\ \langle \eta_i, z_i \rangle \geq a_i\ \forall i}} \gamma_{K^+}\left(T_{(\eta_1,\dots,\eta_s,\theta_1/\beta_1,\dots,\theta_t/\beta_t)}\right).$$

If the top cup-product coefficients for the combined degrees, the coordinates of the z_i, and the a_i are given as in Theorem 1.12, and the coordinates of every θ_j/β_j in the chosen rational cohomology bases are real algebraic, this common value is a finite nonnegative real algebraic number computable from these data.

Proof. Append the fixed classes θ_j/β_j to the tuples η in the proof of Theorem 1.12. Since $\|\theta_j/\beta_j\|_g^* = \beta_j^{-1}\|\theta_j\|_g^*$, their unit comass bounds hold for each admissible metric and are supplied by Theorem 1.10 in the converse direction. The two inequalities in that proof give the displayed equality. Under the stated algebraicity hypotheses, the last paragraph of that proof shows that the common value is computable and real algebraic, since the resulting moment polynomials have real algebraic coefficients. □

2. Integral stable systoles

Let M be as in Section 1, and fix a nonempty set $D \subset \{2, \dots, n\}$. For $r \in D$, write $b_r = \dim H^r(M;\mathbb{R})$ and assume $b_r > 0$. Put

$$L_r = H_r(M;\mathbb{Z})/\operatorname{tors}, \qquad L_r^\vee = H^r(M;\mathbb{Z})/\operatorname{tors},$$

where tors denotes the torsion subgroup. Regard these groups as lattices in real homology and cohomology. The integral universal coefficient theorem identifies $L_r^\vee$ with $\mathrm{Hom}(L_r, \mathbb{Z})$ by evaluation [39, Theorem 3.2]. For a smooth Riemannian metric g on M and $\ell = (\ell_r)_{r \in D} \in (0,\infty)^D$, put

$$\mathrm{stsys}_r(M,g) = \min_{0 \neq z \in L_r} N_g^{(r)}(z), \qquad V_M(\ell) = \inf_{\mathrm{stsys}_r(M,g) \geq \ell_r\ (r \in D)} \mathrm{Vol}(M,g).$$

The minimum exists and is positive, since L_r is a nonzero lattice and $N_g^{(r)}$ is a norm. For $t > 0$, $\mathrm{stsys}_r(M, t^2 g) = t^r\, \mathrm{stsys}_r(M,g)$; hence $V_M(\ell) < \infty$.

For each $r \in D$, choose a $\mathbb{Z}$-basis $e_{r,1}, \dots, e_{r,b_r}$ of $L_r^\vee$. Let T be the array of evaluations on $[M]$ of all products of these classes whose degrees sum to n, with factors ordered first by degree and then by basis index. Changing an integral lift alters a top product by a torsion class in $H^n(M;\mathbb{Z}) \simeq \mathbb{Z}$, so T is independent of the lifts. By graded commutativity [39, Theorem 3.11], a top product with a repeated odd-degree factor is 2-torsion and therefore zero.

2.1. **An integral basis inside a dual norm ball.** Fix an integer $b \geq 1$. On $\mathbb{R}^b$, write $x \cdot z$ for the Euclidean inner product and $\|\cdot\|_p$ for the usual ℓ^p norm, $p \in \{1, 2, \infty\}$. For a matrix A, write A^t for its transpose, $A^{-t} = (A^{-1})^t$ when A is invertible, and $\|A\|_{2\to 2}$ for its Euclidean operator norm. The group $\mathrm{GL}_b(\mathbb{Z})$ consists of the integer $b \times b$ matrices with determinant ± 1. Let $e_1, \dots, e_b$ be the standard basis of $\mathbb{R}^b$, and put

$$B_1^b = \{x \in \mathbb{R}^b \,:\, \|x\|_1 \leq 1\}, \qquad B_\infty^b = [-1, 1]^b.$$

For a nonempty compact convex set $C \subset \mathbb{R}^b$, let

$$h_C(z) = \max_{x \in C} x \cdot z \qquad (z \in \mathbb{R}^b).$$

Lemma 2.1. *Let $C \subset \mathbb{R}^b$ be compact, convex, symmetric about zero, and have nonempty interior. Suppose $h_C(z) \geq s > 0$ for every nonzero $z \in \mathbb{Z}^b$. Set*

$$c_b = (b^2 4^{b-1})^{-1}. \tag{2.1}$$

There is $U \in \mathrm{GL}_b(\mathbb{Z})$ such that

$$s c_b B_\infty^b \subset U^t C. \tag{2.2}$$

Proof. Replacing C by $s^{-1}C$ reduces the proof to $s = 1$. By Auerbach's lemma [65, Lemma 4.2], applied to the norm with unit ball C, there is an invertible real matrix A such that

$$AB_1^b \subset C \subset AB_\infty^b. \tag{2.3}$$

For the Euclidean lattice $\Lambda = A^t\mathbb{Z}^b$, every nonzero vector $A^t z$ has length at least $1/b$, since

$$1 \leq h_C(z) \leq \|A^t z\|_1 \leq b\|A^t z\|_2.$$

Choose a reduced basis $v_1, \dots, v_b$ of Λ in the sense of [53, §1, (1.4)–(1.5)]. Such a basis exists for every real lattice [53, §1, (1.23)]. Let v_i^* be its Gram–Schmidt vectors and put $d_i = \|v_i^*\|_2$. Then

$$v_i = v_i^* + \sum_{j<i} \mu_{ij} v_j^*, \qquad |\mu_{ij}| \leq \tfrac{1}{2}.$$

By [53, Proposition 1.6],

$$d_i \geq 2^{-(i-1)/2} d_1 \geq (b 2^{b-1})^{-1}.$$

Let $W = A^t U$ be the matrix of this basis, where $U \in \mathrm{GL}_b(\mathbb{Z})$. Write $W = O\,\mathrm{diag}(d_i)R$, where O is orthogonal and R is upper triangular with $R_{ii} = 1$ and $|R_{ij}| \leq \frac{1}{2}$ for $i < j$. By [43, Theorem 8.14], $\|R^{-1}\|_{2\to 2} \leq 2^{b-1}$. Hence

$$\|W^{-1}\|_{2\to 2} \leq \|R^{-1}\|_{2\to 2} \max_i d_i^{-1} \leq b 4^{b-1}.$$

If $\|y\|_\infty \leq c_b$, then

$$\|W^{-t} y\|_1 \leq \sqrt{b}\,\|W^{-t}\|_{2\to 2}\|y\|_2 \leq b^2 4^{b-1} c_b = 1.$$

Hence $c_b B_\infty^b \subset W^t B_1^b = U^t A B_1^b \subset U^t C$. Rescaling gives (2.2). □

Lemma 2.2. *Let $s, q > 0$, and let $R \geq 1$ be an integer with $qR \geq s$. Set $Z_R = \{z \in \mathbb{Z}^b \,:\, 0 < \|z\|_\infty \leq R\}$. Suppose a finite set $F \subset \mathbb{R}^b$ contains $qe_1, \dots, qe_b$ and, for every $z \in Z_R$, a vector α_z with $\alpha_z \cdot z \geq s$. Then*

$$\max_{\alpha \in F} |\alpha \cdot z| \geq s \qquad (0 \neq z \in \mathbb{Z}^b).$$

If a compact convex body C contains qB_∞^b and $h_C(z) \geq s$ on the nonzero lattice, all these vectors may be chosen in C.

Proof. Let $0 \neq z \in \mathbb{Z}^b$. If $z \in Z_R$, then $|\alpha_z \cdot z| \geq \alpha_z \cdot z \geq s$. Otherwise, choose j with $|z_j| = \|z\|_\infty \geq R + 1$. Then

$$|(qe_j) \cdot z| = q\|z\|_\infty \geq q(R+1) > s.$$

If $qB_\infty^b \subset C$, then $qe_j \in C$ for every j. By compactness, for each $z \in Z_R$ there is $\alpha_z \in C$ with $\alpha_z \cdot z = h_C(z) \geq s$. □

Lemma 2.3. *Let $\varnothing \neq C \subset [-B,B]^b$ be compact and convex, with $C = -C$ and $B > 0$. The set C may have empty interior. Put*

$$\rho = \min\{\tfrac{1}{2}, (bB)^{-1}\}, \qquad R_0 = b\rho^{1-b} + 1, \qquad R = \lceil R_0 \rceil.$$

Then $h_C(z) \geq 1$ on every nonzero integer vector if and only if it holds whenever $0 < \|z\|_\infty \leq R$.

Proof. Assume $h_C(z) \geq 1$ for $0 < \|z\|_\infty \leq R$, $z \in \mathbb{Z}^b$. Put $P = \{x \in \mathbb{R}^b : h_C(x) \leq 1\}$. Since

$$h_C(x) \leq B\|x\|_1 \leq bB\|x\|_\infty,$$

one has $\rho B_\infty^b \subset P$. The support function is continuous and positively homogeneous, so $\operatorname{int} P = \{x : h_C(x) < 1\}$.

Suppose $x \in P$ and $\|x\|_\infty \geq R_0$. Set $y = R_0 x/\|x\|_\infty$. Then $y \in P$ and $\|y\|_\infty = R_0$. The symmetric convex body

$$K = \operatorname{conv}(\rho B_\infty^b \cup \{y, -y\})$$

lies in $P \cap [-R_0, R_0]^b$. Choose j with $|y_j| = R_0$. The double pyramid with vertices $y, -y$ and base $\{x : x_j = 0,\ |x_i| \leq \rho\ (i \neq j)\}$ lies in K and has volume

$$\frac{2R_0(2\rho)^{b-1}}{b} = 2^b\left(1 + \frac{\rho^{b-1}}{b}\right) > 2^b.$$

For $b = 1$, this is the length of an interval. Minkowski's convex-body theorem [19, Chapter III, §2, Theorem II], applied to $\operatorname{int} K$ and $\mathbb{Z}^b$, gives a nonzero integer point z with

$$z \in \operatorname{int} K \subset \operatorname{int} P, \qquad h_C(z) < 1, \qquad \|z\|_\infty \leq R_0 \leq R,$$

a contradiction. Therefore $P \subset (-R_0, R_0)^b$. Every integer vector outside the prescribed cube is outside P, so its support value exceeds one. The converse implication is immediate. □

2.2. **The finite arithmetic formula.** Write $E_r = (e_{r,1}, \dots, e_{r,b_r})$ as a row, and set

$$\mathcal{J} = \left\{ a = (a_{r,j}) \in \mathbb{Z}_{\geq 0}^{\sum_{r\in D} b_r} : \sum_{r\in D}\sum_{j=1}^{b_r} r a_{r,j} = n, \quad a_{r,j} \leq 1 \text{ if } r \text{ is odd} \right\}.$$

Omitting the zero coordinates with repeated odd-degree factors, write $T = (T_a)_{a\in\mathcal{J}}$, where $T_a = \langle \prod_{r,j} e_{r,j}^{a_{r,j}}, [M] \rangle$, with the order fixed above. Put $k_r(a) = \sum_j a_{r,j}$, $W = \mathbb{Q}^{\mathcal{J}}$, and $W_\mathbb{Z} = \mathbb{Z}^{\mathcal{J}}$. An array $S \in W$ determines, by multilinearity and graded commutativity, the evaluations of all top products of vectors in the spaces $\mathbb{Q}^{b_r}$. These evaluations extend to real vectors.

Let

$$G = \prod_{r\in D} \mathrm{GL}_{b_r}, \qquad \Gamma = \prod_{r\in D} \mathrm{GL}_{b_r}(\mathbb{Z}),$$

where G is an algebraic group over $\mathbb{Q}$. For $g = (g_r)_{r\in D} \in G$, define the left representation $\rho : G \to \mathrm{GL}(W)$ by

$$(\rho(g)S)(v_1, \dots, v_k) = S(g_{r_1}^{-1}v_1, \dots, g_{r_k}^{-1}v_k), \quad v_i \in \mathbb{Q}^{b_{r_i}}, \quad r_i \in D, \quad \sum_i r_i = n.$$

In particular, $\rho(g)T$ is the cup array in the bases $E_r g_r^{-1}$. The matrix entries of $\rho(g)$ are polynomials over $\mathbb{Q}$ in the entries of the g_r^{-1}, and $\rho(\Gamma)W_\mathbb{Z} = W_\mathbb{Z}$. In its block diagonal embedding in $\mathrm{GL}_{\sum_r b_r}$, the group G is defined by the vanishing of the off-block entries. Moreover, $\Gamma = G(\mathbb{Z})$ and $[G(\mathbb{Z}) : \Gamma] = 1$; membership is decided by integral entries and determinant ± 1 in each block.

These equations and rational functions satisfy the hypotheses of the orbit algorithm of Grunewald–Segal [33, Algorithm A2, p. 915], [34]; see also [25, §2, item 2]. For $S, T \in W$, it decides whether $S = \rho(\gamma)T$ for some $\gamma \in \Gamma$ and, if so, produces such a γ. If $\mathcal{J} = \varnothing$, the only array is zero and the identity matrix suffices.

Lemma 2.4. *Let $\beta_i \in H^{p_i}(M;\mathbb{R})$, $1 \le i \le k$, where $p_i \ge 1$ and $\sum_i p_i = n$. Then*

$$|\langle \beta_1 \cdots \beta_k, [M]\rangle| \le n!\,\mathrm{Vol}(M,g) \prod_{i=1}^{k} \|\beta_i\|_g^*. \tag{2.4}$$

Proof. For $\delta > 0$, choose smooth closed representatives ω_i of β_i with $\|\omega_i\|_{\infty,g} \le \|\beta_i\|_g^* + \delta$. The exterior-product estimate underlying [9, §4, (4.4)], applied successively, gives

$$\left| \int_M \omega_1 \wedge \cdots \wedge \omega_k \right| \le \frac{n!}{p_1! \cdots p_k!} \mathrm{Vol}(M,g) \prod_i (\|\beta_i\|_g^* + \delta).$$

Let $\delta \downarrow 0$ and use $p_1! \cdots p_k! \ge 1$. □

Theorem 2.5 (Fixed positive lattice thresholds). *For each $\ell \in (0,\infty)^D$, the value $V_M(\ell)$ is the minimum of finitely many infima of semialgebraic functions on finite-dimensional semialgebraic sets, determined by n, D, $(b_r)_{r\in D}$, T and ℓ. If the thresholds are exactly specified real algebraic numbers, then $V_M(\ell)$ is a computable nonnegative real algebraic number.*

Under this algebraicity hypothesis, for every positive algebraic ε the algorithm also produces a finite list of cohomology classes with real algebraic coordinates in the chosen integral bases, whose comass bounds by one imply all the lattice thresholds and whose constrained volume infimum is less than $V_M(\ell) + \varepsilon$.

Proof. Consider the classes $\ell_r e_{r,j}$, $r \in D$, $1 \le j \le b_r$. If they all have comass at most one, duality (1.1) gives, in the dual integral homology bases,

$$N_g^{(r)}(z) \ge \ell_r \max_j |z_j| \ge \ell_r \qquad (0 \ne z \in \mathbb{Z}^{b_r}).$$

Choose a rational $U \ge 1$ strictly larger than the constrained volume infimum for this list. Then

$$V_M(\ell) < U. \tag{2.5}$$

For algebraic thresholds, Theorem 1.10 and Lemma 1.11 make this choice effective. Put

$$q_r = c_{b_r}\ell_r, \qquad R_r = c_{b_r}^{-1} = b_r^2 4^{b_r-1}. \tag{2.6}$$

Let g satisfy all the lattice thresholds and $\mathrm{Vol}(M,g) \le U$. In the bases E_r, its cohomological comass unit balls C_r are compact convex sets with interior, symmetric about zero, and $h_{C_r}(z) = N_g^{(r)}(z) \ge \ell_r$ for $0 \ne z \in \mathbb{Z}^{b_r}$. Lemma 2.1 gives $U_r \in \mathrm{GL}_{b_r}(\mathbb{Z})$ such that

$$q_r B_\infty^{b_r} \subset C_r' = U_r^t C_r.$$

The bases $E_r' = E_r U_r^{-t}$ are dual to the new integral homology bases, and C_r' are their comass coordinate balls. Their cup array is $T' = \rho((U_r^t)_{r\in D})T$. The classes with new coordinates $q_r e_j$ have comass at most one. Applying Lemma 2.4 to each top product gives

$$|T_a'| \le n!U \prod_{r\in D} q_r^{-k_r(a)} \qquad (a \in \mathcal{J}). \tag{2.7}$$

Define the finite set

$$\mathcal{T} = \left\{ S \in W_{\mathbb{Z}} \cap \rho(\Gamma)T \,:\, |S_a| \le n!U \prod_{r\in D} q_r^{-k_r(a)} \text{ for every } a \in \mathcal{J} \right\}.$$

It is nonempty by (2.5) and the preceding construction. For each $S \in \mathcal{T}$, choose $\gamma_S = (\gamma_{S,r})_{r\in D} \in \Gamma$ with $S = \rho(\gamma_S)T$. For algebraic thresholds, the coordinate bounds are algebraic, so the orbit algorithm computes $\mathcal{T}$ and these matrices. If $\mathcal{J} = \emptyset$, then $\mathcal{T} = \{0\}$.

Set

$$Z_r = \{z \in \mathbb{Z}^{b_r} \,:\, 0 < \|z\|_\infty \le R_r\}, \qquad \alpha = (\alpha_{r,z})_{r\in D,\, z\in Z_r}, \quad \alpha_{r,z} \in \mathbb{R}^{b_r}.$$

Let $\mathcal{P}(\ell)$ be the polyhedron defined by

$$\alpha_{r,z} \cdot z \geq \ell_r \qquad (r \in D,\ z \in Z_r). \tag{2.8}$$

It is nonempty: take $\alpha_{r,z} = \ell_r z/(z \cdot z)$. Let $F(\alpha)$ be the ordered list consisting, in each degree r, of the coefficient vectors $q_r e_j$, $1 \leq j \leq b_r$, followed by the vectors $\alpha_{r,z}$, $z \in Z_r$. The degrees are in increasing order, the coordinate vectors are in their index order, and Z_r has the lexicographic order. The number of entries in degree r is

$$b_r + (2R_r + 1)^{b_r} - 1 = b_r + (2b_r^2 4^{b_r - 1} + 1)^{b_r} - 1.$$

Let K_F be the moment body for this degree list. For $S \in \mathcal{T}$, write $T_{F(\alpha),S}$ for the top product evaluations of the vectors in $F(\alpha)$, calculated by multilinearity from S. Its coordinates are polynomials in α with coefficients determined by S and the q_r. Put

$$f_S(\alpha) = \gamma_{K_F}(T_{F(\alpha),S}), \qquad \Phi_S(\ell) = \inf_{\alpha \in \mathcal{P}(\ell)} f_S(\alpha).$$

By Lemma 1.11, f_S is semialgebraic. The gauge is finite and nonnegative, so $0 \leq \Phi_S(\ell) < \infty$. We have

$$V_M(\ell) = \min_{S \in \mathcal{T}} \Phi_S(\ell). \tag{2.9}$$

Fix $S \in \mathcal{T}$ and $\alpha \in \mathcal{P}(\ell)$. Replace each degree-$r$ coefficient vector v in $F(\alpha)$ by the cohomology class $E_r \gamma_{S,r}^{-1} v$. This tuple has top moment vector $T_{F(\alpha),S}$. If its classes have comass at most one, then Lemma 2.2 and (1.1) give $N_g^{(r)}(z) \geq \ell_r$ for every $0 \neq z \in L_r$, since $q_r R_r = \ell_r$. Theorem 1.10 identifies the constrained volume infimum of this tuple with $f_S(\alpha)$. Thus $V_M(\ell) \leq f_S(\alpha)$, and taking infima proves $V_M(\ell) \leq \min_{S \in \mathcal{T}} \Phi_S(\ell)$.

Conversely, let g satisfy the lattice thresholds and $\mathrm{Vol}(M, g) \leq U$, and choose the bases E_r' and balls C_r' above. For every $z \in Z_r$, compactness gives $\alpha_{r,z} \in C_r'$ with $\alpha_{r,z} \cdot z = h_{C_r'}(z) \geq \ell_r$. Hence $\alpha \in \mathcal{P}(\ell)$, and every class $E_r' v$ with v in the degree-r part of $F(\alpha)$ has comass at most one. Their top moment vector is $T_{F(\alpha),T'}$, where $T' \in \mathcal{T}$. Lemma 1.4 yields

$$\min_{S \in \mathcal{T}} \Phi_S(\ell) \leq \Phi_{T'}(\ell) \leq f_{T'}(\alpha) \leq \mathrm{Vol}(M, g).$$

By (2.5), restricting the defining infimum for $V_M(\ell)$ to metrics of volume at most U leaves it unchanged. This proves (2.9).

Assume now that the thresholds are algebraic. By Lemma 1.11, f_S and $\mathcal{P}(\ell)$ are semialgebraic with real algebraic coefficients. The set

$$E_S = \{u \geq 0 \,:\, \exists \alpha \in \mathcal{P}(\ell),\ f_S(\alpha) \leq u\}$$

is nonempty and semialgebraic with real algebraic coefficients, and its finite infimum is $\Phi_S(\ell)$. The last paragraph of the proof of Theorem 1.12 computes this infimum as a real algebraic number. Taking the minimum over $\mathcal{T}$ computes $V_M(\ell)$.

Choose $S \in \mathcal{T}$ with $\Phi_S(\ell) = V_M(\ell)$. For a positive algebraic ε, the set

$$\{\alpha \in \mathcal{P}(\ell) \,:\, f_S(\alpha) < V_M(\ell) + \varepsilon\}$$

is nonempty and semialgebraic over the real algebraic numbers. Effective quantifier elimination and algebraic sampling produce a point in this set [14, Theorem 14.16 and Algorithm 13.2]. The corresponding classes $E_r \gamma_{S,r}^{-1} v$ have algebraic coordinates in the original integral bases. Their unit comass bounds imply the lattice thresholds, and their constrained volume infimum is $f_S(\alpha) < V_M(\ell) + \varepsilon$. □

Corollary 2.6. *On every compact box in $(0, \infty)^D$ with positive algebraic endpoints, the graph of V_M is semialgebraic, and a quantifier-free formula for it is computable from the cup data and the endpoints. The function V_M is continuous on the positive orthant.*

Proof. Write the box as $\prod_{r\in D}[a_r, b_r']$, with $0 < a_r \le b_r'$. Choose the rational bound U at the upper thresholds (b_r'), as in the preceding proof. Then $V_M(\ell) \le V_M((b_r')) < U$ throughout the box. Choose fixed rational numbers q_r with $0 < q_r \le c_{b_r} a_r$, and set $R_r = \lceil b_r'/q_r \rceil$. For every ℓ in the box,

$$q_r \le c_{b_r} a_r \le c_{b_r} \ell_r, \qquad q_r R_r \ge b_r' \ge \ell_r.$$

Define $\mathcal{T}$, Z_r, $F(\alpha)$ and K_F by the preceding construction with these fixed U, q_r and R_r. The first inequality gives $q_r B_\infty^{b_r} \subset C_r'$ after the integral basis change; the second gives the implication from the finite comass bounds to all the lattice thresholds. Thus both inequalities in (2.9) hold for every ℓ in the box.

Only the right sides of (2.8) depend on ℓ. The infimum characterization in the proof of Theorem 1.12, with ℓ as free variables, therefore defines the graph of each Φ_S by a first-order formula with real algebraic coefficients. Quantifier elimination and the finite minimum in (2.9) give the stated formula for the graph of V_M [14, Corollary 2.78 and Theorem 14.16].

For continuity, let $\ell, \ell' \in (0,\infty)^D$ and put

$$\lambda_- = \min_{r\in D} (\ell_r'/\ell_r)^{1/r}, \qquad \lambda_+ = \max_{r\in D} (\ell_r'/\ell_r)^{1/r}.$$

If g satisfies the thresholds ℓ, then $\lambda_+^2 g$ satisfies the thresholds ℓ'. Applying this also with ℓ and ℓ' interchanged gives

$$\lambda_-^n V_M(\ell) \le V_M(\ell') \le \lambda_+^n V_M(\ell).$$

Both factors tend to one as $\ell' \to \ell$. Since V_M is finite, these inequalities prove continuity, including at points where its value is zero. □

Corollary 2.7. *For $2 \le r < n$ and $b_r(M) > 0$, the stable systolic constant*

$$\sigma_r^{\mathrm{st}}(M) = \inf_g \frac{\mathrm{Vol}(M,g)}{\mathrm{stsys}_r(M,g)^{n/r}}$$

is a computable real algebraic number determined by the integral degree-r top cup tensor and n.

Proof. Multiplying lengths by $\mathrm{stsys}_r(M,g)^{-1/r}$ identifies this infimum with $V_M(1)$ for $D = \{r\}$. Apply Theorem 2.5. □

When $r \mid n$, dependence on the integral degree-r top cup form was proved by Brunnbauer [17, Corollary 1.5]. If $r \nmid n$, then $\sigma_r^{\mathrm{st}}(M) = 0$ [17, Theorem 1.2].

3. Cup products and stable systolic exponents

Let M be a closed connected oriented smooth manifold of dimension $n \ge 1$, with integral fundamental class $[M]$. For $1 \le r \le n$, put

$$V_r = H_r(M;\mathbb{R}), \qquad L_r = H_r(M;\mathbb{Z})/\text{ torsion},$$
$$b_r(M) = \dim V_r, \qquad D(M) = \{r \,:\, 1 \le r \le n,\ b_r(M) > 0\}.$$

Identify L_r with its image in V_r under the coefficient map. For a smooth Riemannian metric g, let $N_g^{(r)}$ denote the stable norm on V_r, and put

$$s_r(g) = \min_{0\ne z\in L_r} N_g^{(r)}(z) > 0 \qquad (r \in D(M)).$$

In top degree, $H_n(M;\mathbb{Z}) = \mathbb{Z}[M]$ and

$$s_n(g) = N_g^{(n)}([M]) = \mathrm{Vol}(M,g);$$

see [13, arXiv version 5, §3, following (3.2)]. For $E \subset D(M)$ and $\ell \in (0,\infty)^E$, set

$$V_M^E(\ell) = \inf\{\mathrm{Vol}(M,g) \,:\, s_r(g) \ge \ell_r \quad (r \in E)\}.$$

Let $\mathcal{A}_M \subset \mathbb{Z}_{\geq 0}^{D(M)}$ consist of the vectors $a = (a_r)_{r \in D(M)}$ for which there are classes $x_{r,j} \in H^r(M;\mathbb{Q})$, $r \in D(M)$, $1 \leq j \leq a_r$, such that

$$\sum_{r \in D(M)} r a_r = n, \qquad \left\langle \prod_{r \in D(M)} \prod_{j=1}^{a_r} x_{r,j}, [M] \right\rangle \neq 0. \tag{3.1}$$

The factors are ordered first by degree and then by j. Factors of the same degree need not be equal. Since $0 \leq a_r \leq n/r$, $\mathcal{A}_M$ is finite. It contains the vector with $a_n = 1$ and all other entries zero. For $a \in \mathcal{A}_M$, $\ell \in (0,\infty)^{D(M)}$ and $v \in \mathbb{R}^{D(M)}$, put

$$\ell^a = \prod_{r \in D(M)} \ell_r^{a_r}, \qquad P_M(\ell) = \max_{a \in \mathcal{A}_M} \ell^a, \qquad h_M(v) = \max_{a \in \mathcal{A}_M} \sum_{r \in D(M)} v_r a_r.$$

Write $\operatorname{conv} \mathcal{A}_M$ for the convex hull of $\mathcal{A}_M$.

Theorem 3.1. *Let $p \in \mathbb{R}_{\geq 0}^{D(M)}$ satisfy $\sum_{r \in D(M)} r p_r = n$. There is a constant $C = C(M,p) > 0$ such that*

$$\prod_{r \in D(M)} s_r(g)^{p_r} \leq C \operatorname{Vol}(M,g)$$

for every smooth Riemannian metric g on M if and only if $p \in \operatorname{conv} \mathcal{A}_M$.

Let (X,h) be a finite polyhedron with a positive piecewise smooth metric. For $r \geq 1$, write $\operatorname{Mass}_h(c)$ for the mass of a real Lipschitz r-cycle c. The stable norm on $H_r(X;\mathbb{R})$ is

$$N_h^{(r)}(z) = \inf_{[c]=z} \operatorname{Mass}_h(c) \qquad (z \in H_r(X;\mathbb{R})),$$

where the infimum is over such cycles [13, arXiv version 5, Definition 3.1]. Let (Y,k) be another such polyhedron and let $f : (X,h) \to (Y,k)$ be 1-Lipschitz. Write $f_\#$ and f_* for the induced maps on Lipschitz chains and homology. The mass inequality for Lipschitz pushforwards gives

$$\operatorname{Mass}_k(f_\# c) \leq \operatorname{Mass}_h(c), \qquad [f_\# c] = f_*[c].$$

Taking infima over the cycles representing z gives

$$N_k^{(r)}(f_* z) \leq N_h^{(r)}(z) \qquad (z \in H_r(X;\mathbb{R})).$$

3.1. **Metrics from a fixed finite complex.**

Lemma 3.2. *Suppose $n \geq 3$. There are constants $\sigma_r > 0$, $r \in D(M)$, a finite nonempty set $\mathcal{J}$, constants $B_\Delta > 0$, and vectors $a(\Delta) \in \mathcal{A}_M$, $\Delta \in \mathcal{J}$, such that for every $\lambda \in (0,\infty)^{D(M)}$ and every $\eta > 0$ there is a smooth Riemannian metric g on M satisfying*

$$\begin{aligned} s_r(g) &\geq \sigma_r \lambda_r^r \quad (r \in D(M)), \\ \operatorname{Vol}(M,g) &\leq \sum_{\Delta \in \mathcal{J}} B_\Delta \prod_{r \in D(M)} \lambda_r^{r a_r(\Delta)} + \eta. \end{aligned} \tag{3.2}$$

Proof. Let E_r be the sphere and James spaces of Section 1, and put $E_1 = S^1$. Their cells have dimensions $0, r$ for odd r, and jr, $j \geq 0$, for even r. All integral cellular differentials vanish, and $H^*(E_r;\mathbb{Q})$ is the free graded-commutative algebra on a generator of degree r. By [27, Lemma 3.4], for each $r \in D(M)$ with $r \geq 2$ there is a map

$$f_r : M \longrightarrow E_r^{b_r(M)}$$

that is injective on rational homology in degree r. If $1 \in D(M)$, take for f_1 the Jacobi map determined by an integral basis of $H^1(M;\mathbb{Z})$. Its homomorphism on fundamental groups is the abelianization followed by the quotient by torsion. The other factors are simply connected. Hence the product

$$f = (f_r)_{r \in D(M)} : M \longrightarrow \mathcal{E} = \prod_{r \in D(M)} E_r^{b_r(M)}$$

is surjective on fundamental groups. By cellular approximation [39, Theorem 4.8], replace f by a homotopic map into the finite n-skeleton $\mathcal{E}^{(n)}$. It remains surjective on fundamental groups, since $\pi_1(\mathcal{E}^{(n)}) \to \pi_1(\mathcal{E})$ is an isomorphism.

Write $\mathcal{E} = \prod_{i\in I} E_{r_i}$, with a finite ordered index set I, and let $u_i \in H^{r_i}(\mathcal{E};\mathbb{Q})$ be the pullback of the generator of the ith factor. Orient the factor cells and give the product cells their product orientations. A product cell e has factor dimensions $r_i j_i$, with $j_i \in \{0,1\}$ for odd r_i. For an n-cell, put

$$a_r(e) = \sum_{i\,:\,r_i=r} j_i \qquad (r \in D(M)).$$

The class dual to e is a nonzero rational multiple of $\prod_{i\in I} u_i^{j_i}$ [27, Lemmas 3.9–3.10]. Since the cellular differentials of $\mathcal{E}$ vanish, including the one from degree $n+1$, the coefficient m_e of e in $f_*[M]$ is the local degree of f at e. Consequently,

$$m_e \neq 0 \quad\Longrightarrow\quad \left\langle \prod_{i\in I} (f^* u_i)^{j_i}, [M] \right\rangle \neq 0.$$

Since $\sum_{r\in D(M)} r a_r(e) = n$, this gives $a(e) \in \mathcal{A}_M$ whenever $m_e \neq 0$. Let $\mathcal{C}$ be the set of n-cells with $m_e \neq 0$. The cell cancellation theorem [27, Lemma 2.3] applies to $f : M \to \mathcal{E}^{(n)}$: $n > 2$, the source is closed and oriented, and f is surjective on fundamental groups. It gives a homotopic map $f' : M \to \mathcal{E}'$, where

$$\mathcal{E}' = \mathcal{E}^{(n-1)} \cup \bigcup_{e\in\mathcal{C}} \overline{e}.$$

For each factor consider the skeleta $E_{r_i}^{(r_i j)}$ with $r_i j \le n$, taking only $j = 0, 1$ when r_i is odd. Theorem 2C.5 and its proof in [39, pp. 182–184] give a finite simplicial complex K_i, nested subcomplexes $K_{i,j} \subset K_i$ with $\dim K_{i,j} \le r_i j$, and a homotopy equivalence

$$\varphi_i : E_{r_i}^{(n)} \longrightarrow K_i$$

whose restriction to each $E_{r_i}^{(r_i j)}$ is a homotopy equivalence to $K_{i,j}$. Let K' be the union of the products $\prod_{i\in I} K_{i,j_i}$ corresponding to the cells of $\mathcal{E}'$. The product of the φ_i maps $\mathcal{E}'$ into K'. Products corresponding to cells of dimension less than n have dimension less than n; every other product corresponds to $e \in \mathcal{C}$, with

$$\sum_{i\in I} r_i j_i = n, \qquad a(e) \in \mathcal{A}_M.$$

Triangulate K' compatibly with its products of factor simplices. By simplicial approximation [39, Theorem 2C.1], the map $(\prod_i \varphi_i) \circ f'$ is homotopic to a simplicial map $F : M \to K'$ on a finite smooth triangulation of M. Give each K_i a fixed piecewise Euclidean metric h_i. For $r \in D(M)$, let k_r be the product metric on

$$Q_r = \prod_{i\,:\,r_i=r} K_i,$$

and let $F_r : M \to Q_r$ be the projection of F. Composition with the homotopy inverses of the φ_i and the inclusions of the original stages recovers f_r up to homotopy. Thus F_r is injective on $H_r(M;\mathbb{Q})$. Let $\sigma_r > 0$ be the stable r-systole of (Q_r, k_r).

For $\lambda \in (0,\infty)^{D(M)}$, multiply h_i by $\lambda_{r_i}^2$, and let h_λ be the resulting product metric on K' (cf. [27, proof of Lemma 3.8]). The projection

$$(K', h_\lambda) \longrightarrow (Q_r, \lambda_r^2 k_r)$$

is nonexpanding, and the target stable r-systole is $\sigma_r \lambda_r^r$. Let $\mathcal{J}$ be the source n-simplices on whose interiors F has rank n, in simplex coordinates. This set is nonempty, since F_n is nonzero on top

homology. For $\Delta \in \mathcal{J}$, choose $e_\Delta \in \mathcal{C}$ and factor simplices $\tau_i \subset K_{i,j_i(\Delta)}$ such that $F(\Delta) \subset \prod_i \tau_i$, where $r_i j_i(\Delta)$ are the factor dimensions of e_Δ. Writing $d_i = \dim \tau_i$, one has

$$n \le \sum_{i \in I} d_i \le \sum_{i \in I} r_i j_i(\Delta) = n.$$

Hence $d_i = r_i j_i(\Delta)$ for every i, and the derivative of F maps onto the full product tangent space. Put $a(\Delta) = a(e_\Delta)$, let $\mathbf{1} = (1)_{r \in D(M)}$, and set

$$B_\Delta = \int_\Delta dv_{F^* h_1} > 0.$$

The determinant of the product metric gives

$$\int_\Delta dv_{F^* h_\lambda} = B_\Delta \prod_{i \in I} \lambda_{r_i}^{r_i j_i(\Delta)} = B_\Delta \prod_{r \in D(M)} \lambda_r^{r a_r(\Delta)}.$$

Every other source n-simplex has zero pullback volume. The constants σ_r, B_Δ and the vectors $a(\Delta)$ are independent of λ and η.

Lemma 1.7, applied to F and h_λ with error η, gives a smooth Riemannian metric g with the required volume bound and with F nonexpanding. Hence each F_r is nonexpanding. If $0 \neq z \in L_r$, then $(F_r)_* z$ is nonzero in $H_r(Q_r; \mathbb{Z})/$ torsion, by rational homology injectivity. Therefore

$$N_g^{(r)}(z) \ge N_{\lambda_r^2 k_r}^{(r)}((F_r)_* z) \ge \sigma_r \lambda_r^r.$$

Taking the lattice minimum proves (3.2). □

3.2. **Exponents and volume thresholds.**

Proof of Theorem 3.1. For every $a \in \mathcal{A}_M$, Gromov's cup-product inequality [32, §7.4.C] gives a constant $C_a > 0$ such that

$$\prod_{r \in D(M)} s_r(g)^{a_r} \le C_a \operatorname{Vol}(M, g)$$

for every smooth Riemannian metric g on M. If $p = \sum_{a \in \mathcal{A}_M} \theta_a a$, where $\theta_a \ge 0$ and $\sum_{a \in \mathcal{A}_M} \theta_a = 1$, then

$$\prod_{r \in D(M)} s_r(g)^{p_r} = \prod_{a \in \mathcal{A}_M} \left(\prod_{r \in D(M)} s_r(g)^{a_r} \right)^{\theta_a} \le \left(\prod_{a \in \mathcal{A}_M} C_a^{\theta_a} \right) \operatorname{Vol}(M, g).$$

Suppose $n \ge 3$ and $p \notin \operatorname{conv} \mathcal{A}_M$. Since $\operatorname{conv} \mathcal{A}_M$ is nonempty and compact, strict separation [61, Corollary 11.4.2, p. 99] gives $v \in \mathbb{R}^{D(M)}$ such that

$$\sum_{r \in D(M)} v_r p_r > h_M(v).$$

For $t \ge 1$, take $\lambda_r = t^{v_r / r}$ and $\eta = t^{h_M(v)}$ in Lemma 3.2. For each $\Delta \in \mathcal{J}$, $a(\Delta) \in \mathcal{A}_M$, so $\sum_{r \in D(M)} v_r a_r(\Delta) \le h_M(v)$. Put $C_0 = 1 + \sum_{\Delta \in \mathcal{J}} B_\Delta$. The lemma gives a metric g_t with

$$s_r(g_t) \ge \sigma_r t^{v_r}, \qquad \operatorname{Vol}(M, g_t) \le C_0 t^{h_M(v)}. \tag{3.3}$$

Consequently

$$\frac{\prod_{r \in D(M)} s_r(g_t)^{p_r}}{\operatorname{Vol}(M, g_t)} \ge C_0^{-1} \prod_{r \in D(M)} \sigma_r^{p_r} \, t^{\sum_{r \in D(M)} v_r p_r - h_M(v)} \longrightarrow \infty.$$

Thus no constant independent of g exists for p.

For $n = 1$, M is a circle, $D(M) = \{1\}$, and $p_1 = 1$. For $n = 2$, a sphere has $D(M) = \{2\}$ and $p_2 = 1$. An oriented surface of positive genus has $D(M) = \{1, 2\}$ and $\mathcal{A}_M = \{(2, 0), (0, 1)\}$ [39, Example 3.7]. Their convex hull is $\{(p_1, p_2) \in \mathbb{R}^2_{\ge 0} : p_1 + 2p_2 = 2\}$. These cases follow from the sufficiency already proved. □

Corollary 3.3. *There are constants $c_M, C_M > 0$ such that, for every $\ell \in (0,\infty)^{D(M)}$,*

$$c_M P_M(\ell) \le V_M^{D(M)}(\ell) \le C_M P_M(\ell).$$

Proof. With the cup-product constants C_a in the proof of Theorem 3.1, every metric g satisfying $s_r(g) \ge \ell_r$ for $r \in D(M)$ satisfies

$$\mathrm{Vol}(M,g) \ge C_a^{-1} \ell^a \qquad (a \in \mathcal{A}_M).$$

Thus $c_M = \min_{a\in\mathcal{A}_M} C_a^{-1} > 0$ gives the lower bound.

For $n \ge 3$, put

$$C_M = 1 + \sum_{\Delta\in\mathcal{J}} B_\Delta \prod_{r\in D(M)} \sigma_r^{-a_r(\Delta)}.$$

Take $\lambda_r = (\ell_r/\sigma_r)^{1/r}$ and $\eta = P_M(\ell)$ in Lemma 3.2. Then $s_r(g) \ge \ell_r$ for $r \in D(M)$, and

$$\begin{aligned}\mathrm{Vol}(M,g) &\le \sum_{\Delta\in\mathcal{J}} B_\Delta \prod_{r\in D(M)} (\ell_r/\sigma_r)^{a_r(\Delta)} + P_M(\ell)\\ &\le C_M P_M(\ell).\end{aligned}$$

For a surface of positive genus, fix a metric g_0, and put $A_0 = \mathrm{Vol}(M,g_0)$ and $s_0 = s_1(g_0)$. Set

$$q = \max\{\ell_1^2/s_0^2, \ell_2/A_0\}, \qquad g = q g_0.$$

Then $s_1(g) = q^{1/2} s_0 \ge \ell_1$, $s_2(g) = qA_0 \ge \ell_2$, and

$$\mathrm{Vol}(M,g) = qA_0 \le \max\{A_0/s_0^2, 1\}\max\{\ell_1^2, \ell_2\} = \max\{A_0/s_0^2, 1\} P_M(\ell).$$

For a circle or a sphere, $s_n(g) = \mathrm{Vol}(M,g)$ and homothety gives $V_M^{D(M)}(\ell) = P_M(\ell) = \ell_n$. □

For $p \in \mathbb{Z}_{\ge 0}^{D(M)}$ with $\sum_{r\in D(M)} r p_r = n$, the partition containing p_r copies of r is *stable categorical* if the inequality in Theorem 3.1 holds. The stable systolic category $\mathrm{cat}_{\mathrm{stsys}}(M)$ is the maximum of $\sum_{r\in D(M)} p_r$ over the stable categorical partitions [27, Definition 1.2]. If there is no such partition, this maximum is defined to be zero. The rational cup-length $\mathrm{cl}_{\mathbb{Q}}(M)$ is the greatest number of homogeneous positive-degree classes in $H^*(M;\mathbb{Q})$ with nonzero product.

Corollary 3.4. *For every closed connected oriented smooth manifold of dimension $n \ge 1$,*

$$\mathrm{cat}_{\mathrm{stsys}}(M) = \mathrm{cl}_{\mathbb{Q}}(M).$$

Proof. Theorem 3.1 gives

$$\mathrm{cat}_{\mathrm{stsys}}(M) = \max_{p\in \mathrm{conv}\,\mathcal{A}_M \cap \mathbb{Z}^{D(M)}} \sum_{r\in D(M)} p_r = \max_{a\in\mathcal{A}_M} \sum_{r\in D(M)} a_r.$$

The second equality follows because a linear function has the same maximum on a finite set and its convex hull, and $\mathcal{A}_M$ consists of integral vectors.

Let $x_1 \cdots x_m \ne 0$ be a product of maximal length, with $x_i \in H^{r_i}(M;\mathbb{Q})$ and $r_i > 0$, and put $d = \sum_{i=1}^m r_i$. If $d < n$, Poincaré duality [39, Proposition 3.38] gives $y \in H^{n-d}(M;\mathbb{Q})$ such that $x_1 \cdots x_m y \ne 0$, contrary to maximality. Hence $d = n$, and $m = \max_{a\in\mathcal{A}_M} \sum_{r\in D(M)} a_r$. □

Example 3.5. An integral point of $\mathrm{conv}\,\mathcal{A}_M$ need not belong to $\mathcal{A}_M$. Let

$$X = S^2 \times S^2 \times S^6 \times S^6, \qquad Y = (S^4)^4, \qquad M = X \# Y.$$

Let u, v, x, y be the pullbacks to X of rational cohomology generators of the four sphere factors, in degrees $2, 2, 6, 6$, and let $z_1, \dots, z_4 \in H^4(Y;\mathbb{Q})$ be the corresponding classes for Y. By the Künneth formula [39, Theorem 3.15],

$$\begin{aligned}H^*(X;\mathbb{Q}) &= \mathbb{Q}[u,v,x,y]/(u^2, v^2, x^2, y^2),\\ H^*(Y;\mathbb{Q}) &= \mathbb{Q}[z_1,z_2,z_3,z_4]/(z_1^2, z_2^2, z_3^2, z_4^2).\end{aligned}$$

The connected-sum formula [56, Remark 8.9.17, p. 190] identifies these positive-degree classes with classes in $H^*(M;\mathbb{Q})$; mixed products from the two summands vanish, and the orientation classes are identified. Thus $uvxy \neq 0$ and $z_1z_2z_3z_4 \neq 0$ give profiles a, b with

$$(a_2, a_4, a_6) = (2, 0, 2), \qquad (b_2, b_4, b_6) = (0, 4, 0),$$

all other entries being zero. Let $C_a, C_b > 0$ be the constants in Gromov's inequalities for these profiles. Then

$$s_2(g)s_4(g)^2s_6(g) = \left(s_2(g)^2s_6(g)^2\right)^{1/2}\left(s_4(g)^4\right)^{1/2} \leq (C_aC_b)^{1/2}\,\mathrm{Vol}(M, g).$$

The exponent vector is $(a + b)/2$, with entries $(1, 2, 1)$ in degrees $2, 4, 6$.

Every $\alpha \in H^2(M;\mathbb{Q})$ and $\beta \in H^4(M;\mathbb{Q})$ have the form

$$\alpha = \lambda u + \mu v, \qquad \beta = \nu\, uv + \sum_{i=1}^{4} c_i z_i, \qquad \lambda, \mu, \nu, c_i \in \mathbb{Q}.$$

The ring relations give $\alpha\beta = 0$. Hence every cup product of degree list $(2, 4, 4, 6)$ is zero. If $I = \bigoplus_{k>0} H^k(M;\mathbb{Q})$, then $uvxy \in I^4 \setminus \{0\}$ and $I^5 = 0$: each summand ring has four square-zero generators, and mixed positive-degree products vanish. Thus $\mathrm{cl}_{\mathbb{Q}}(M) = 4$, and Corollary 3.4 gives $\mathrm{cat}_{\mathrm{stsys}}(M) = 4$. The stable categorical partition $(2, 4, 4, 6)$ has maximal length and gives a negative answer to [27, Remark 4.10].

4. Cohomological Hodge forms

Let M be a closed connected oriented smooth manifold of dimension $n \geq 3$, with fundamental class $[M] \in H_n(M;\mathbb{Z})$. For each integer r with $0 \leq r \leq n$, put

$$V_r = H^r(M;\mathbb{R}), \qquad \Lambda_r = H^r(M;\mathbb{Z})/\,\mathrm{torsion}, \qquad b_r = \dim V_r.$$

Identify V_r with de Rham cohomology [77, Theorem 5.45], and Λ_r with its image in V_r. Write $\Omega^r(M)$ for the space of smooth real r-forms. For a smooth Riemannian metric g, let $*_g$ be its Hodge star and dv_g its volume form. The Hodge decomposition theorem gives a unique harmonic representative ω_v^g of each $v \in V_r$ [77, Theorems 6.8 and 6.11]. Define the Hodge form by

$$P_{r,g}(v, w) = \int_M \omega_v^g \wedge *_g\, \omega_w^g \qquad (v, w \in V_r).$$

It is symmetric and positive definite. Orthogonality to exact forms gives

$$P_{r,g}(v, v) = \min_{\substack{\alpha\in\Omega^r(M)\\ d\alpha=0,\ [\alpha]=v}} \int_M \alpha \wedge *_g\, \alpha.$$

For bases (e_i) of V_r and (f_j) of V_{n-r}, let

$$(Q_r)_{ij} = \langle e_i \smile f_j, [M]\rangle.$$

Poincaré duality makes Q_r nonsingular [77, Theorem 6.13]. In middle degree the two bases are the same. Bilinear forms and their matrices in the indicated bases are denoted by the same symbols. For symmetric forms A, B on the same vector space, $A \preceq B$ means that $B - A$ is positive semidefinite.

Lemma 4.1. *For $0 \leq r \leq n$, the Hodge forms satisfy*

$$P_{n-r,g} = Q_r^t P_{r,g}^{-1} Q_r. \tag{4.1}$$

If $n = 2m$, $Q = Q_m$, and $\epsilon = (-1)^m$, then

$$P_{m,g} Q^{-1} P_{m,g} = \epsilon Q. \tag{4.2}$$

In the bases $1 \in H^0(M;\mathbb{R})$ and $z \in H^n(M;\mathbb{Z})$ with $\langle z, [M]\rangle = 1$, the matrices in degrees $0, n$ are $\mathrm{Vol}(M, g)$ and $\mathrm{Vol}(M, g)^{-1}$.

Proof. Put $\sigma = (-1)^{r(n-r)}$. The Hodge star preserves harmonic forms and its square in degree r is multiplication by σ [77, §6.1]. Let S be the matrix of

$$V_r \longrightarrow V_{n-r}, \qquad v \longmapsto [*_g\, \omega_v^g].$$

Then $P_{r,g} = Q_r S$. The reverse star has matrix σS^{-1}, and the reverse cup matrix is σQ_r^t. Hence

$$P_{n-r,g} = (\sigma Q_r^t)(\sigma S^{-1}) = Q_r^t S^{-1} = Q_r^t P_{r,g}^{-1} Q_r.$$

For $r = m$, this identity gives $P_{m,g} Q^{-1} P_{m,g} = Q^t = \epsilon Q$. The harmonic representatives of 1 and z are 1 and $dv_g / \operatorname{Vol}(M, g)$, respectively, which gives the last assertion. □

For a finite-dimensional real vector space V, write $\operatorname{Sym}^+(V)$ for its positive definite symmetric bilinear forms. For a nondegenerate bilinear form Q on V with $Q^t = \epsilon Q$, $\epsilon \in \{1, -1\}$, put

$$\mathcal{D}(Q) = \{P \in \operatorname{Sym}^+(V) \,:\, PQ^{-1}P = \epsilon Q\}.$$

The matrix identity is independent of the choice of basis. When $V = 0$, this space is a point. The space of independent Hodge forms of M is

$$\mathscr{H}(M) = \begin{cases} \displaystyle\prod_{1 \le r < n/2} \operatorname{Sym}^+(V_r), & n \text{ odd}, \\ \displaystyle\left(\prod_{1 \le r < n/2} \operatorname{Sym}^+(V_r) \right) \times \mathcal{D}(Q_m), & n = 2m. \end{cases} \tag{4.3}$$

Factors on zero vector spaces are omitted. Let $\operatorname{Met}(M)$ be the space of smooth Riemannian metrics, and define

$$\Pi \,:\, \operatorname{Met}(M) \longrightarrow \mathscr{H}(M), \qquad \Pi(g) = (P_{r,g})_{1 \le r \le \lfloor n/2 \rfloor},$$

with the same zero factors omitted. If μ is a positive smooth volume form, put $\operatorname{Met}_\mu(M) = \{g \,:\, dv_g = \mu\}$. Then

$$T_g \operatorname{Met}_\mu(M) = \{h \in C^\infty(M; \operatorname{Sym}^2 T^*M) \,:\, \operatorname{tr}_g h = 0\}.$$

Theorem 4.2 (Prescribed Hodge forms and volume forms). *Let B be a compact smooth manifold. Let $b \mapsto \mathsf{P}_b \in \mathscr{H}(M)$ and $b \mapsto \mu_b$ be smooth, where each μ_b is a positive volume form on M. There is a smooth family of metrics g_b such that*

$$\Pi(g_b) = \mathsf{P}_b, \qquad dv_{g_b} = \mu_b.$$

There is also a smooth family of linear maps

$$\mathcal{R}_b \,:\, T_{\mathsf{P}_b} \mathscr{H}(M) \longrightarrow T_{g_b} \operatorname{Met}_{\mu_b}(M), \qquad D\Pi_{g_b} \mathcal{R}_b = \mathrm{id}.$$

Fix a background metric on M for the C^k tensor norms, and a smooth bundle metric on $T\mathscr{H}(M)$, inducing the norm $\|\cdot\|_b$ at P_b. For every integer $k \ge 0$ there is a constant C_k such that

$$\|\mathcal{R}_b \xi\|_{C^k} \le C_k \|\xi\|_b \qquad \big(b \in B,\ \xi \in T_{\mathsf{P}_b} \mathscr{H}(M)\big).$$

The same statements hold for compact parameter manifolds with boundary or corners when the data extend smoothly to a neighborhood.

Thus all positive definite symmetric bilinear forms on cohomology may be prescribed subject to (4.1), (4.2), and the degree-0 and degree-n values determined by $\int_M \mu_b$.

Connectedness cannot be omitted. Let $N = (S^2 \times S^3) \sqcup (S^2 \times S^3)$. Choose degree-two classes e_1, e_2 supported on the two respective components and the dual degree-three classes f_1, f_2. These bases have cup matrix I_2, the 2×2 identity matrix, and

$$P_{2,g}(e_1, e_2) = 0$$

for every metric g on N. The positive definite matrices

$$P_2 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \qquad P_3 = P_2^{-1}$$

satisfy (4.1), but cannot be the Hodge forms of a metric on N.

4.1. **Coordinates and parameter estimates.** Write I for the identity matrix or endomorphism of the indicated space. For each integer $0 \le r < n/2$, choose an integral basis of Λ_r and its integral Poincaré dual basis in Λ_{n-r}. The integral cup pairing is unimodular after torsion is factored out [39, Proposition 3.38]; hence $Q_r = I$. If n is even, fix an integral basis of $\Lambda_{n/2}$ as well.

Lemma 4.3. *There is a diffeomorphism $\Theta : \mathscr{H}(M) \longrightarrow \mathbb{R}^d$, where $d = \dim \mathscr{H}(M)$. In a suitable basis, the matrix logarithm gives global coordinates on every nonempty $\mathcal{D}(Q)$.*

Proof. The matrix exponential is a diffeomorphism from the real symmetric matrices onto the positive definite symmetric matrices [28, Corollary 3.10(2)]. If $n = 2m$, the Hodge form of any smooth metric belongs to $\mathcal{D}(Q_m)$ by Lemma 4.1; thus this factor is nonempty. For a nonempty $\mathcal{D}(Q)$, choose $P_* \in \mathcal{D}(Q)$ and a P_*-orthonormal basis. Then $P_* = I$, $Q^2 = \epsilon I$ and $Q^t Q = I$. The defining equation becomes

$$PQP = Q, \qquad \text{or equivalently} \qquad Q^{-1}PQ = P^{-1}.$$

Uniqueness of the symmetric logarithm gives

$$Q^{-1}(\log P)Q = -\log P, \qquad \log P \in \mathfrak{p}_Q := \{H : H^t = H,\ HQ + QH = 0\}.$$

Conversely, $H \in \mathfrak{p}_Q$ gives $Q^{-1}e^H Q = e^{-H}$, hence $e^H \in \mathcal{D}(Q)$. Thus $\log : \mathcal{D}(Q) \to \mathfrak{p}_Q$ and the exponential are inverse diffeomorphisms. Linear coordinates on these spaces and on the symmetric matrix spaces give Θ. The lattices $\Lambda_r \subset V_r$ remain fixed. □

Let B be a compact smooth parameter manifold. A smooth family on a compact parameter set is understood to extend smoothly to a neighborhood. For $R > 0$, write $B_R(0)$ for the open ball of radius R in the indicated Euclidean space. Vector spaces of parameters and values carry fixed Euclidean norms; multilinear maps carry the corresponding operator norms. For $u = (u_1, \dots, u_q) \in \mathbb{R}^q$, put $\partial_a = \partial/\partial u_a$. A smooth metric family $g_{b,u}$ has relative derivatives bounded by $C \ge 0$ through order two if

$$-C g_{b,u} \preceq \partial_a g_{b,u} \preceq C g_{b,u}, \qquad -C g_{b,u} \preceq \partial_c \partial_a g_{b,u} \preceq C g_{b,u} \tag{4.4}$$

for all b, u and $1 \le a, c \le q$.

Lemma 4.4. *The harmonic representative of a fixed cohomology class depends smoothly, in the C^∞ topology, on a smooth family of metrics. Let $(g_{j,b,u})_{j\ge1}$ be smooth metric families satisfying* (4.4) *with one constant C. Fix classes $v_1, \dots, v_s \in V_r$, where $0 \le r \le n$, and put*

$$A_j(b,u) = \left(P_{r,g_{j,b,u}}(v_i, v_\ell)\right)_{1\le i,\ell\le s}.$$

If $\operatorname{tr} A_j$ is uniformly bounded in j, b, u, then the first and second u-derivatives of A_j are uniformly bounded. The same holds for the first and second u-derivatives of the total volumes if those volumes are uniformly bounded.

Proof. Let d_g^* be the $L^2(g)$-adjoint of d, and put $\Delta_g = d d_g^* + d_g^* d$. Choose smooth closed forms $\rho_i \in \Omega^r(M)$ and $\psi_i \in \Omega^{n-r}(M)$, $1 \le i \le b_r$, whose classes are bases satisfying $\int_M \rho_i \wedge \psi_\ell = \delta_{i\ell}$ for $1 \le i, \ell \le b_r$, where $\delta_{i\ell}$ is the Kronecker delta. Define

$$\lambda(\alpha) = \left(\int_M \alpha \wedge \psi_i\right)_{i=1}^{b_r}, \qquad \mathcal{L}_g(\alpha, z) = \left(\Delta_g \alpha + \sum_{i=1}^{b_r} z_i \rho_i, \lambda(\alpha)\right)$$

on $\Omega^r(M) \times \mathbb{R}^{b_r}$, with values in the same space. The harmonic projections of the ρ_i form a basis of $\ker \Delta_g$, and λ restricts to an isomorphism on this kernel. Since the image of Δ_g is the orthogonal complement of its kernel [77, Theorem 6.8], $\mathcal{L}_g$ is bijective. The operator Δ_g is elliptic of order two, with smooth coefficients depending smoothly on g; the finite-dimensional terms in $\mathcal{L}_g$ are fixed. For a fixed background metric g_0, there is a constant C_λ such that $|\lambda(\alpha)| \le C_\lambda \|\alpha\|_{L^2(g_0)}$. The theorem on inverses of augmented elliptic operators [37, Part II, Theorem 3.3.3] gives smooth dependence of $\mathcal{L}_g^{-1}$ in the C^∞ topology. Moreover,

$$\mathcal{L}_g^{-1}(0, c) = \left(\omega^g_{\sum_i c_i[\rho_i]}, 0\right), \qquad c \in \mathbb{R}^{b_r}. \tag{4.5}$$

Fix j, b, and write $g = g_{j,b,u}$, $A = A_j(b,u)$, and $h_i(u) = \omega^g_{v_i}$. Put

$$B_u(\alpha,\beta) = \int_M \alpha\wedge *_g \beta, \qquad \|\alpha\|_u^2 = B_u(\alpha,\alpha).$$

Write $B_{u,a}$ and $B_{u,ca}$ for derivatives of B_u with both arguments fixed. Put $K_a = g^{-1}\partial_a g$ and $K_{ca} = g^{-1}\partial_c\partial_a g$. For an endomorphism K of TM, define its action on r-forms by

$$(K^{[r]}\alpha)(X_1,\dots,X_r) = \sum_{\ell=1}^{r} \alpha(X_1,\dots,KX_\ell,\dots,X_r).$$

In the pointwise operator norms induced by g, $\|K^{[r]}\|_g \le r\|K\|_g$. Set

$$L_a = \tfrac{1}{2}\operatorname{tr}(K_a)I - K_a^{[r]}.$$

Differentiation of the density and inverse metric gives

$$\begin{aligned} B_{u,a}(\alpha,\beta) &= B_u(L_a\alpha,\beta),\\ B_{u,ca}(\alpha,\beta) &= B_u\big((L_cL_a + \partial_c L_a)\alpha,\beta\big),\\ \partial_c K_a &= K_{ca} - K_cK_a. \end{aligned}$$

Let $\kappa = n/2 + r$. By (4.4),

$$\|L_a\|_g \le \kappa C, \qquad \|\partial_c L_a\|_g \le \kappa(C + C^2).$$

Put $C_1 = \kappa C$ and $C_2 = \kappa^2C^2 + \kappa(C + C^2)$. Then

$$\begin{aligned} |B_{u,a}(\alpha,\beta)| &\le C_1\|\alpha\|_u\|\beta\|_u,\\ |B_{u,ca}(\alpha,\beta)| &\le C_2\|\alpha\|_u\|\beta\|_u. \end{aligned} \tag{4.6}$$

The forms $\partial_a h_i$ are exact, since $[h_i] = v_i$ is constant. For a fixed smooth exact form e, differentiation of $B_u(h_i, e) = 0$ gives

$$B_u(\partial_a h_i, e) = -B_{u,a}(h_i, e).$$

At each fixed parameter value, take $e = \partial_a h_i$ and apply (4.6). This gives $\|\partial_a h_i\|_u \le C_1\|h_i\|_u$. For $A_{i\ell} = B_u(h_i, h_\ell)$, orthogonality gives

$$\partial_a A_{i\ell} = B_{u,a}(h_i, h_\ell).$$

Differentiating once more and applying the differentiated orthogonality relation gives

$$\partial_c\partial_a A_{i\ell} = B_{u,ca}(h_i,h_\ell) - B_u(\partial_a h_i, \partial_c h_\ell) - B_u(\partial_c h_i, \partial_a h_\ell).$$

Hence

$$|\partial_a A_{i\ell}| \le C_1\sqrt{A_{ii}A_{\ell\ell}}, \qquad |\partial_c\partial_a A_{i\ell}| \le (C_2 + 2C_1^2)\sqrt{A_{ii}A_{\ell\ell}}. \tag{4.7}$$

Since $\sqrt{A_{ii}A_{\ell\ell}} \le \operatorname{tr} A$, the bounds are uniform in j, b, u. For $r = 0$ and the class 1, the Gram matrix is $(\mathrm{Vol}(M, g))$ by Lemma 4.1; this gives the volume estimates. □

Lemma 4.5. *Let $R > 0$, and let f_j, f be smooth maps from $B \times \overline{B_{2R}(0)}$, with $B_{2R}(0) \subset \mathbb{R}^q$, to a finite-dimensional real vector space. Suppose $f_j \to f$ uniformly and all second u-derivatives of f_j are uniformly bounded. Then $D_u f_j \to D_u f$ uniformly on $B \times \overline{B_R(0)}$.*

Proof. Write $\|\cdot\|_{\infty,\rho}$ for the supremum norm on $B \times \overline{B_\rho(0)}$. Put $e_j = f_j - f$ and $\varepsilon_j = \|e_j\|_{\infty,2R}$. Then

$$\|D_u^2 e_j\|_{\infty,2R} \le C_0 := \sup_j \|D_u^2 f_j\|_{\infty,2R} + \|D_u^2 f\|_{\infty,2R} < \infty.$$

Taylor's formula with integral remainder [37, Part I, Theorem 3.5.6] gives, for $0 < t \le R$,

$$\|D_u e_j\|_{\infty,R} \le \frac{2\varepsilon_j}{t} + \frac{C_0 t}{2}.$$

For $\varepsilon_j > 0$, take $t = \sqrt{\varepsilon_j}$ for all sufficiently large j. If $\varepsilon_j = 0$, the derivative vanishes on the smaller ball. □

Lemma 4.6. *Let $\ell \geq 0$ be an integer, $R > 0$, and $B_R(0) \subset \mathbb{R}^\ell$. Let the maps $(b,u) \mapsto q_b(u)$ from $B \times \overline{B_R(0)}$ to $\mathbb{R}^\ell$ and $b \mapsto y_b$ from B to $\mathbb{R}^\ell$ be smooth. Suppose that, for every $b \in B$,*

$$|q_b(0) - y_b| < R/4, \qquad \sup_{|u|\leq R} \|D_u q_b(u) - I\| < 1/4.$$

There is a unique $u_b \in \overline{B_{R/2}(0)}$ with $q_b(u_b) = y_b$. It is smooth in b, lies in $B_{R/2}(0)$, and $D_u q_b(u_b)$ is invertible. Moreover, $|u_b| \leq \frac{4}{3}|q_b(0) - y_b|$.

Proof. Put $T_b(u) = u - q_b(u) + y_b$. For $|u| \leq R/2$,

$$|T_b(u)| \leq |q_b(0) - y_b| + \tfrac{1}{4}|u| < 3R/8, \qquad \|D_u T_b(u)\| < 1/4.$$

The contraction theorem [37, Part I, Theorem 5.2.1] on the complete ball $\overline{B_{R/2}(0)}$ gives a unique fixed point u_b. Its equation gives

$$|u_b| \leq |q_b(0) - y_b| + \tfrac{1}{4}|u_b|.$$

Moreover, $|D_u q_b(u_b)v| \geq \frac{3}{4}|v|$ for $v \in \mathbb{R}^\ell$, so $D_u q_b(u_b)$ is invertible. The implicit function theorem [77, Theorem 1.37] gives smooth local solutions; uniqueness identifies them on overlaps. □

5. Comparison of marked energy forms

Let $s \geq 0$ be an integer. For an ordered list $(r_i)_{i=1}^s$ of integers with $1 \leq r_i \leq n$, put

$$\mathcal{I} = \left\{ a \in \mathbb{Z}^s_{\geq 0} \,:\, \sum_i r_i a_i = n, \quad a_i \leq 1 \text{ if } r_i \text{ is odd} \right\}.$$

For classes $\alpha_i \in H^{r_i}(P;\mathbb{R})$ on a closed oriented n-manifold P with fundamental class $[P]$, their top moments are $\langle \alpha_1^{a_1} \cdots \alpha_s^{a_s}, [P] \rangle$, $a \in \mathcal{I}$, with products in the indicated order. For $1 \leq r \leq n$, put $J_r = \{i \,:\, r_i = r\}$, and let Id_r be the identity on $\mathbb{R}^{J_r}$. For a metric h on P and forms $\eta_i \in \Omega^{r_i}(P)$, define

$$G^{(r)}(\eta, h) = \left(\int_P \eta_i \wedge *_h \, \eta_j \right)_{i,j \in J_r}.$$

The same notation applies to piecewise smooth forms and metrics. Empty blocks are omitted. Matrix norms are Euclidean operator norms.

Let B be a compact smooth parameter manifold, possibly with boundary or corners, $R > 0$, and $\mathcal{P} = B \times \overline{B_R(0)}$, where $B_R(0) \subset \mathbb{R}^q$. Smooth families on $\mathcal{P}$ are understood to extend smoothly to a neighborhood.

Proposition 5.1. *Let M, N be closed connected oriented smooth n-manifolds, $n \geq 3$, and let $\alpha_i \in H^{r_i}(M;\mathbb{Q})$ and $\alpha_i' \in H^{r_i}(N;\mathbb{Q})$, $1 \leq i \leq s$, have equal top moments. Suppose that J_1 is empty or that $(\alpha_i)_{i \in J_1}$ is an integral basis of $H^1(M;\mathbb{Z})$ and every α_i', $i \in J_1$, is integral. Let $h_{b,u}$ be a smooth metric family on N, parametrized by $\mathcal{P}$, and let $\eta_i(b,u)$ be smooth closed representatives of α_i'.*

For every $\delta > 0$, there are smooth metric families $k_{b,u}$ on M and smooth closed forms $\theta_i(b,u)$ representing α_i, such that, uniformly on $\mathcal{P}$,

$$\begin{aligned} &\mathrm{Vol}(M, k_{b,u}) \leq \mathrm{Vol}(N, h_{b,u}) + \delta, \\ &G^{(r)}(\theta, k_{b,u}) \preceq G^{(r)}(\eta, h_{b,u}) + \delta \mathrm{Id}_r \qquad (J_r \neq \emptyset). \end{aligned} \tag{5.1}$$

If $h_{b,u}$ satisfies (4.4) *with constant C, then $k_{b,u}$ may be chosen to satisfy the same inequalities with the same C. In particular, this constant is independent of δ and of the rational markings and comparison complexes.*

Proof. Let $a = |J_1|$, and take the spaces E_r of Lemma 1.8. Put

$$E = T^a \times \prod_{i \,:\, r_i \geq 2} E_{r_i}.$$

For $i \in J_1$, let u_i be the corresponding coordinate class of the torus, in the order inherited from the marked list; for $r_i \geq 2$, let u_i be the bottom generator of the corresponding factor. All these classes are pulled back to E. The integral degree-one classes determine maps to $S^1 = K(\mathbb{Z}, 1)$ [39, Theorem 4.57] and hence the torus components. Lemma 1.8 supplies the higher-degree components and common positive multipliers. Thus there are maps $f : M \to E$, $j : N \to E$ and positive integers q_i such that

$$f^* u_i = q_i \alpha_i, \qquad j^* u_i = q_i \alpha_i', \qquad q_i = 1 \quad (i \in J_1).$$

When $a > 0$, the torus component of f induces $\pi_1(M) \twoheadrightarrow H_1(M;\mathbb{Z})/\text{ torsion} \simeq \mathbb{Z}^a$. The remaining factors are simply connected; hence f is surjective on fundamental groups, also when $a = 0$. The top monomials in the u_i span $H^n(E;\mathbb{Q})$, and $H_n(E;\mathbb{Z})$ is free by the integral Künneth theorem [39, Theorem 3B.6]. Equality of top moments therefore gives

$$f_*[M] = j_*[N] \quad \text{in } H_n(E;\mathbb{Z}).$$

Apply the finite-skeleton and mapping-cylinder construction in the proof of Proposition 1.9 to these maps. It gives a finite polyhedron X containing N as a subcomplex, maps $c : X \to E$ and $f_X : M \to X$, and the inclusion $\iota : N \hookrightarrow X$, with

$$c \circ f_X \simeq f, \qquad c \circ \iota \simeq j, \qquad (f_X)_*[M] = \iota_*[N].$$

The map c induces an isomorphism on fundamental groups, so $(f_X)_*$ is surjective on them. Brunnbauer's lemma [17, Lemma 3.2] gives an $(n, 1)$-monotone simplicial map

$$F : M \longrightarrow Y = N \cup X^{(n-1)}$$

whose composite with $Y \hookrightarrow X$ is homotopic to f_X. The compatible triangulations and subdivision in the proof of Proposition 1.9 make N full in Y, with smooth triangulations on M, N. Each open n-simplex of N has at most one open n-simplex of M over it, mapped isomorphically onto it. Put

$$v_i = q_i^{-1}(c^* u_i)|_Y \in H^{r_i}(Y;\mathbb{Q}),$$
$$v_i|_N = \alpha_i', \qquad F^* v_i = \alpha_i.$$

All these objects are independent of (b, u).

Fix smooth closed representatives η_i^0 of α_i' and a reference metric on N, with codifferential d_0^* and Green operator G_0. Set

$$\psi_i(b, u) = d_0^* G_0(\eta_i(b, u) - \eta_i^0).$$

Hodge decomposition and commutation with d give $d\psi_i = \eta_i - \eta_i^0$ [77, §6.9 and Proposition 6.10]. The Sobolev estimates for G_0 [30, Theorem A.1] imply that $d_0^* G_0$ is continuous in the C^∞ topology. Hence ψ_i is smooth in all variables.

The construction in the proof of Lemma 1.5 gives a piecewise smooth retraction $\rho : U \to N$ from a neighborhood in Y and fixed closed representatives $\widetilde{\eta}_i^0$ of v_i equal to $\rho^* \eta_i^0$ on U, after shrinking U. Choose fixed piecewise smooth functions $\chi, \zeta : Y \to [0, 1]$, equal to one near N, with

$$\operatorname{supp} \chi \subset U, \qquad \operatorname{supp} \zeta \subset \operatorname{int}\{\chi = 1\}.$$

Extend products supported in U by zero, and put

$$\widetilde{\eta}_i(b, u) = \widetilde{\eta}_i^0 + d(\chi \rho^* \psi_i(b, u)). \tag{5.2}$$

These forms are closed, represent v_i, and equal $\rho^* \eta_i(b, u)$ near N.

Fix a positive piecewise smooth metric h_0 on Y. For $\tau > 0$, define

$$\widehat{h}_{b,u} = \zeta \rho^* h_{b,u} + (\zeta \tau^2 + 1 - \zeta) h_0.$$

It is positive and restricts to $h_{b,u} + \tau^2 h_0|_N$ on N. Every n-simplex of Y belongs to N. Thus, uniformly on $\mathcal{P}$ as $\tau \downarrow 0$,

$$\operatorname{Vol}_n(Y, \widehat{h}_{b,u}) \longrightarrow \operatorname{Vol}(N, h_{b,u}),$$
$$G^{(r)}(\eta, \widehat{h}_{b,u}|_N) \longrightarrow G^{(r)}(\eta, h_{b,u}).$$

Fix τ, choose a smooth metric g_0 on M, and put $\omega_i = F^*\widetilde{\eta}_i$ and

$$q_{b,u} = F^*\widehat{h}_{b,u} + \varepsilon^2 g_0, \qquad \varepsilon > 0. \tag{5.3}$$

The simplexwise estimates in the proof of Proposition 1.9 are uniform on $\mathcal{P}$. Indeed, the pointwise $\widehat{h}_{b,u}$-norms of $\widetilde{\eta}_i$ are bounded uniformly on the finitely many simplices and the compact parameter set. Since F is nonexpanding, there are constants D_r, independent of ε, b, u, such that, for every $z \in \mathbb{R}^{J_r}$ and every n-simplex Δ of rank $l < n$,

$$\int_\Delta \left| \sum_{i \in J_r} z_i \omega_i \right|^2_{q_{b,u}} dv_{q_{b,u}} \le D_r |z|^2 \operatorname{Vol}(\Delta, q_{b,u}) = O(\varepsilon^{n-l})|z|^2.$$

On a simplex of rank n, the positive metrics $q_{b,u}$ converge uniformly to $F^*\widehat{h}_{b,u}$. Change of variables gives the limiting energy on the image simplex in N, uniformly for $|z| \le 1$. Each such image occurs at most once. Consequently, uniformly on $\mathcal{P}$ as $\varepsilon \downarrow 0$,

$$\operatorname{Vol}(M, q_{b,u}) \le \operatorname{Vol}_n(Y, \widehat{h}_{b,u}) + o(1),$$
$$G^{(r)}(\omega, q_{b,u}) \preceq G^{(r)}(\eta, \widehat{h}_{b,u}|_N) + o(1)\mathrm{Id}_r.$$

Fix ε. Let S be the $(n-1)$-skeleton of M and $U_t = \{x : d_{g_0}(x, S) < t\}$, for $t > 0$. Choose a smooth metric H dominating every simplexwise value of $q_{b,u}$, uniformly on $\mathcal{P}$. For each n-simplex Δ of M, choose a smooth function $\chi_{\Delta,t} : M \to [0,1]$ supported in its interior and equal to one on $\Delta \setminus U_t$. These supports are disjoint. Set

$$k_{b,u} = \sum_\Delta \chi_{\Delta,t} q_{b,u} + \left(1 - \sum_\Delta \chi_{\Delta,t}\right) H,$$

where each simplexwise summand is extended by zero. Then $k_{b,u}$ is smooth, $q_{b,u} \preceq k_{b,u} \preceq H$, and $k_{b,u} = q_{b,u}$ outside U_t. The forms ω_i have uniformly bounded pointwise norms for both metrics. Since $\operatorname{Vol}(U_t, H) \to 0$, uniformly on $\mathcal{P}$,

$$\operatorname{Vol}(M, k_{b,u}) - \operatorname{Vol}(M, q_{b,u}) \longrightarrow 0,$$
$$G^{(r)}(\omega, k_{b,u}) - G^{(r)}(\omega, q_{b,u}) \longrightarrow 0 \qquad (t \downarrow 0).$$

Fix t, and take the regularization operators R_σ of Lemma 1.6 for g_0. They are uniformly bounded on closed $L^2(g_0)$ forms and converge strongly to the identity [30, Theorem 12.5(3)–(4)]. For each r, the set

$$\left\{ \sum_{i \in J_r} z_i \omega_i(b,u) : (b,u) \in \mathcal{P},\ |z| \le 1 \right\}$$

is compact in $L^2(g_0)$. The convergence is therefore uniform on this set. The metrics $k_{b,u}$ have uniformly equivalent L^2 norms, so, uniformly on $\mathcal{P}$,

$$G^{(r)}(R_\sigma \omega, k_{b,u}) \longrightarrow G^{(r)}(\omega, k_{b,u}) \qquad (\sigma \downarrow 0).$$

For fixed σ, the regularization is a continuous smoothing operator [30, §§12.1–12.3]. Thus $\theta_i = R_\sigma \omega_i$ is smooth in all variables, closed, and represents α_i. Choose $\tau, \varepsilon, t, \sigma$ successively so that each volume error and each Gram-matrix error above is less than $\delta/4$. The three volume errors and four matrix errors give (5.1).

For these fixed choices, the metric has the form $k_{b,u} = A(h_{b,u}) + H_0$, where A is a positive linear map from symmetric tensors on N to symmetric tensors on M, and H_0 is positive semidefinite; both are independent of (b,u). This follows directly from the displayed formulas for $\widehat{h}, q, k$. If (4.4) holds, then for every u-multi-index ν with $1 \le |\nu| \le 2$,

$$-CA(h_{b,u}) \preceq \partial_u^\nu k_{b,u} = A(\partial_u^\nu h_{b,u}) \preceq CA(h_{b,u}).$$

Since $A(h_{b,u}) \preceq k_{b,u}$, the same C applies for every choice of the fixed maps, background metrics, cutoffs, and positive numbers $\tau, \varepsilon, t, \sigma$. □

Lemma 5.2. *Let $m \geq 1$ be an integer, and let $M_1, \dots, M_m$ be closed connected oriented smooth n-manifolds, $n \geq 3$. On each M_j, let h_j be a smooth metric family parametrized by $\mathcal{P}$, and let $\eta_i^{(j)}$, $1 \leq i \leq s$, be smooth families of closed forms of degree r_i, where $1 \leq r_i < n$. For every $\delta > 0$, the oriented connected sum $M = M_1 \# \cdots \# M_m$ carries a smooth metric family k and smooth closed form families θ_i such that, under the connected-sum identification*

$$H^r(M;\mathbb{R}) \simeq \bigoplus_{j=1}^{m} H^r(M_j;\mathbb{R}) \qquad (0 < r < n),$$

one has $[\theta_i] = ([\eta_i^{(j)}])_{j=1}^m$. Their top moments are the sums of the corresponding moments on the M_j, and, uniformly on $\mathcal{P}$,

$$\left| \mathrm{Vol}(M,k) - \sum_{j=1}^{m} \mathrm{Vol}(M_j,h_j) \right| < \delta,$$

$$\left\| G^{(r)}(\theta,k) - \sum_{j=1}^{m} G^{(r)}(\eta^{(j)},h_j) \right\| < \delta \qquad (J_r \neq \emptyset).$$

If the h_j satisfy (4.4) *with one constant C, then k may be chosen to satisfy the same bounds with the same C.*

Proof. For separate index sets, adjoin zero forms to obtain one ordered degree list. For $m = 1$, take $k = h_1$ and $\theta_i = \eta_i^{(1)}$. Suppose $m \geq 2$. Fix disjoint oriented coordinate balls for a connected-sum chain. In each chart write $D_\rho = \{x \in \mathbb{R}^n : |x| < \rho\}$ and let e be the Euclidean metric; the chart is defined on a neighborhood of $\overline{D_1}$. Fix $\rho_0 \in (0,1)$ and let $0 < \rho < \rho_0$. All auxiliary choices below are independent of $(b,u) \in \mathcal{P}$.

Let K_r be the radial homotopy operator of [77, Lemma 4.18]. For a smooth r-form ω, $r \geq 1$, and vectors $v_1, \dots, v_{r-1} \in \mathbb{R}^n$, it is given by

$$(K_r\omega)_x(v_1,\dots,v_{r-1}) = \int_0^1 t^{r-1}\omega_{tx}(x,v_1,\dots,v_{r-1})\,dt.$$

For closed ω, one has

$$dK_r\omega = \omega, \qquad \|K_r\omega\|_{\infty,D_\rho} \leq \frac{\rho}{r}\|\omega\|_{\infty,D_\rho},$$

where these are supremum Euclidean pointwise norms. Choose a smooth function $\chi : \mathbb{R}^n \to [0,1]$ equal to one on $D_{1/2}$ and supported in $D_{3/4}$, and put $\chi_\rho(x) = \chi(x/\rho)$. On every attachment ball replace $\eta_i^{(j)}$ by

$$\widetilde{\eta}_i^{(j)} = \eta_i^{(j)} - d\big(\chi_\rho K_{r_i}\eta_i^{(j)}\big) = (1-\chi_\rho)\eta_i^{(j)} - d\chi_\rho \wedge K_{r_i}\eta_i^{(j)},$$

extending each correction by zero. These forms are closed, cohomologous to $\eta_i^{(j)}$, zero on $D_{\rho/2}$, and unchanged outside $D_{3\rho/4}$. Since $|d\chi_\rho| \leq \rho^{-1}\sup|d\chi|$, compactness of $\mathcal{P}$ and the preceding estimate give uniform pointwise bounds for the modified forms, independent of ρ.

Choose a smooth function $a : \mathbb{R}^n \to [0,1]$ equal to zero on $D_{1/3}$ and to one outside $D_{5/12}$, and put $a_\rho(x) = a(x/\rho)$. Replace h_j on each attachment ball by

$$a_\rho h_j + (1-a_\rho)e.$$

This metric equals e on $D_{\rho/3}$, and every modified form vanishes wherever the metric changes. Apply the collar and cylinder construction in the proof of Theorem 1.1, with its constant $L = 1$ and its radius parameter equal to $\rho/2$. Thus the collar lies between radii $\rho/8$ and $\rho/4$, and each joining cylinder has length $\rho/2$ and spherical radius $\rho/8$. The outer collar germ is Euclidean and the inner germ is a product. The fixed boundary identifications reverse the induced boundary orientations and are isometries of the round spheres. The resulting metric k_ρ is smooth, and the forms extend by zero

across the collars and cylinders to smooth closed forms $\theta_{i,\rho}$. Radial diffeomorphisms, equal to the identity near the outer chart boundaries and to homotheties near the boundary spheres, extend by the identity off the charts and by linear rescaling along the cylinders. They identify these connected sums with the fixed M constructed at radius ρ_0.

On the attachment charts the metrics h_j are uniformly equivalent to e, uniformly on $\mathcal{P}$. The interpolated metrics have the same property, and the collar and cylinder metrics become ρ^2 times fixed metrics after dilation. Consequently, for a constant C_0 independent of ρ, b, u,

$$\left|\mathrm{Vol}(M, k_\rho) - \sum_j \mathrm{Vol}(M_j, h_j)\right| \le C_0 \rho^n.$$

Since all metric changes occur where the modified forms vanish,

$$G^{(r)}(\theta_\rho, k_\rho) = \sum_j G^{(r)}(\widetilde{\eta}^{(j)}, h_j).$$

For $z \in \mathbb{R}^{J_r}$, put $\eta_z^{(j)} = \sum_{i \in J_r} z_i \eta_i^{(j)}$ and $\widetilde{\eta}_z^{(j)} = \sum_{i \in J_r} z_i \widetilde{\eta}_i^{(j)}$. Their pointwise h_j-norms are bounded by $C_r|z|$, uniformly in ρ, b, u. They agree outside the union of the balls $D_{3\rho/4}$, whose total h_j-volume is $O(\rho^n)$. Hence, for a constant D_r independent of ρ, b, u, z,

$$\left| z^t \left(G^{(r)}(\theta_\rho, k_\rho) - \sum_j G^{(r)}(\eta^{(j)}, h_j) \right) z \right| \le D_r \rho^n |z|^2.$$

Taking the supremum over $|z| = 1$ proves the required matrix estimate. Choose ρ so that $C_0\rho^n < \delta$ and $D_r\rho^n < \delta$ for every nonempty block.

The connected-sum cohomology formula [56, Remark 8.9.17, p. 190] identifies $[\theta_{i,\rho}]$ with $([\eta_i^{(j)}])_j$, since the cutoff changes are exact. It also identifies the orientation classes and makes products of positive-degree classes from distinct summands zero. Thus all top moments add exactly. The radial integral, multiplication by the fixed cutoffs, exterior differentiation and the fixed gluing maps are continuous linear operations on smooth forms. Hence the families are smooth on $\mathcal{P}$.

The metric k_ρ is a positive linear function of the tuple $(h_1, \dots, h_m)$ plus a fixed positive semidefinite tensor. The calculation in the last paragraph of Proposition 5.1 therefore gives the relative derivative bounds with the same C. The identifications with M are parameter independent, so their pullbacks preserve these bounds. □

6. Compact families in middle degree

Throughout this section $n = 2m \ge 4$, $V = V_m$, $\Lambda = \Lambda_m$, $Q = Q_m$, and $\epsilon = (-1)^m$. Put $d = \dim \mathcal{D}(Q)$, write $P_g = P_{m,g}$, and choose logarithm coordinates $\theta : \mathcal{D}(Q) \to \mathbb{R}^d$ as in Lemma 4.3. For a symmetric bilinear form H, write $H[v] = H(v, v)$.

Lemma 6.1. *Let q be a nondegenerate bilinear form on a finite-dimensional real vector space L, with $q^t = \epsilon q$, and let $K \subset L$ be a linear subspace with $q(K, K) = 0$. Put $\bar{L} = K^{\perp_q}/K$, and let $\bar{q}$ be the induced form. For $G \in \mathcal{D}(q)$, the formula*

$$G_K([v], [v]) = \min_{k \in K} G[v + k] \qquad (v \in K^{\perp_q})$$

defines a form $G_K \in \mathcal{D}(\bar{q})$, smooth in G. If $K_1 \subset L$ is a linear subspace with $K \subset K_1$ and $q(K_1, K_1) = 0$, then $(G_K)_{K_1/K} = G_{K_1}$ under the natural identification of the quotient spaces.

Fix a basis of L and its Euclidean norm. For every compact $\mathcal{C} \subset \mathcal{D}(q)$ there is a constant C such that

$$A, P \in \mathcal{D}(q), \quad P \in \mathcal{C}, \quad A \preceq P + \delta I, \quad 0 < \delta \le 1 \quad \Longrightarrow \quad \|A - P\| \le C\delta,$$

where $\|\cdot\|$ is the corresponding operator norm.

Proof. Put $J = q^{-1}G$. Compatibility gives

$$J^2 = \epsilon I, \qquad J^t G J = G, \qquad G(v, Jk) = \epsilon q(v, k).$$

The G-minimal representatives in $K^{\perp_q}$ form

$$W = K^{\perp_G} \cap K^{\perp_q} = (K + JK)^{\perp_G}.$$

If $Jk \in K$ with $k \in K$, then $G[k] = q(k, Jk) = 0$, so $k = 0$. Thus $K \cap JK = 0$. The space $K + JK$ is J-invariant; since J is G-orthogonal, so is W. Moreover, $\dim W = \dim L - 2\dim K = \dim \bar{L}$ and $W \cap K = 0$. Hence $W \to \bar{L}$ is an isomorphism. Under this isomorphism, $G_K = G|_W$ and $\bar{q} = q|_W$. Restricting $G(v, w) = q(v, Jw)$ and $J^2 = \epsilon I$ to W gives $G_K \bar{q}^{-1} G_K = \epsilon \bar{q}$.

For $v \in K_1^{\perp_q}$, successive minimization gives

$$\min_{[k_1] \in K_1/K} \min_{k \in K} G[v + k_1 + k] = \min_{k_1 \in K_1} G[v + k_1].$$

For $K = 0$ the reduction is G itself. Otherwise choose a fixed complement of K in $K^{\perp_q}$ and fixed bases. If the restriction of G has matrix $\left(\begin{smallmatrix} D & B \\ B^t & H \end{smallmatrix}\right)$, with $H = G|_K > 0$, the Schur-complement formula [16, §A.5.5, p. 650] gives $G_K = D - BH^{-1}B^t$. This proves smoothness.

Compatibility gives $A = q^t A^{-1} q$ and $P = q^t P^{-1} q$. The Schur-complement criterion [16, §A.5.5, p. 650], applied to $\left(\begin{smallmatrix} P + \delta I & I \\ I & A^{-1} \end{smallmatrix}\right)$, gives $(P + \delta I)^{-1} \preceq A^{-1}$. Hence

$$q^t (P + \delta I)^{-1} q \preceq A \preceq P + \delta I.$$

For nonempty $\mathcal{C}$ and $L \neq 0$, put $\mu = \min_{P \in \mathcal{C}} \lambda_{\min}(P) > 0$. The resolvent identity gives

$$\begin{aligned} q^t (P + \delta I)^{-1} q - P &= -\delta q^t (P + \delta I)^{-1} P^{-1} q, \\ \| q^t (P + \delta I)^{-1} q - P \| &\leq \delta \|q\|^2 \mu^{-2}. \end{aligned}$$

Thus $-\delta \|q\|^2 \mu^{-2} I \preceq A - P \preceq \delta I$, and one may take $C = \max\{1, \|q\|^2 \mu^{-2}\}$. For $\mathcal{C} = \emptyset$ the assertion is vacuous; for $L = 0$ one has $A - P = 0$. □

Let $l \geq 1$ be an integer, and put

$$\widehat{V} = V \oplus \bigoplus_{i=1}^{l} (\mathbb{R} e_i \oplus \mathbb{R} f_i), \qquad \widehat{\Lambda} = \Lambda \oplus \bigoplus_{i=1}^{l} (\mathbb{Z} e_i \oplus \mathbb{Z} f_i).$$

Extend Q to $\widehat{V}$ by making the added planes orthogonal to V and setting

$$\widehat{Q}(e_i, f_j) = \delta_{ij}, \quad \widehat{Q}(f_i, e_j) = \epsilon \delta_{ij}, \quad \widehat{Q}(e_i, e_j) = \widehat{Q}(f_i, f_j) = 0.$$

Choose $a_1, \dots, a_l \in 2\Lambda$, put $C_{ij} = Q(a_i, a_j)/2$, and define $T : \widehat{V} \to \widehat{V}$ by

$$\begin{aligned} T f_i &= f_i, \\ T e_i &= e_i + a_i - \sum_j C_{ji} f_j, \\ T v &= v - \epsilon \sum_i Q(v, a_i) f_i \quad (v \in V). \end{aligned} \tag{6.1}$$

Since $C^t = \epsilon C$,

$$\widehat{Q}(T e_i, T e_j) = Q(a_i, a_j) - C_{ij} - \epsilon C_{ji} = 0, \qquad \widehat{Q}(Tv, T e_i) = 0.$$

The remaining pairings are unchanged. The entries C_{ij} and $Q(v, a_i)$, for $v \in \Lambda$, are integral. Thus T has an integral unipotent triangular matrix in the block order (f_i, V, e_i) and is an isometry of $\widehat{\Lambda}$.

Put $E = \operatorname{span}\{e_1, \dots, e_l\}$. Then $E^{\perp_{\widehat{Q}}} = V \oplus E$, so T induces an isometry

$$(V, Q) \longrightarrow \left((TE)^{\perp_{\widehat{Q}}} / TE, \overline{\widehat{Q}} \right), \qquad v \longmapsto [Tv].$$

For $R \in \mathcal{D}(Q)$ and $s = (s_1, \dots, s_l) \in (0,\infty)^l$, put

$$\widehat{R}_s = R \oplus \bigoplus_i \operatorname{diag}(s_i^{-1}, s_i),$$

where the ith block is written in (e_i, f_i). Each block is compatible with $\left(\begin{smallmatrix} 0 & 1 \\ \epsilon & 0 \end{smallmatrix}\right)$. Hence $\widehat{R}_s \in \mathcal{D}(\widehat{Q})$. Let $F_R(s) \in \mathcal{D}(Q)$ be its reduction along TE, identified with a form on V by $v \mapsto [Tv]$.

Define

$$S = \operatorname{diag}(s_i), \qquad A : \mathbb{R}^l \to V, \quad Au = \sum_i u_i a_i, \qquad G_A(R) = (R(a_i, a_j))_{ij}, \quad q(v) = (Q(v, a_i))_i.$$

The squared norm of $Tv + \sum_i u_i T e_i$ is

$$R[v + Au] + u^t S^{-1} u + (\epsilon q(v) + Cu)^t S(\epsilon q(v) + Cu).$$

Put

$$z_{R,s}(v) = (R(v, a_i))_i + CSq(v), \qquad D_R(s) = G_A(R) + C^t S C.$$

The linear term in u is $2u^t z_{R,s}(v)$, since $\epsilon C^t = C$, and its quadratic coefficient is $S^{-1} + D_R(s) > 0$. The Schur-complement formula gives

$$F_R(s)[v] = R[v] + q(v)^t S q(v) - z_{R,s}(v)^t (I + SD_R(s))^{-1} S z_{R,s}(v), \tag{6.2}$$

with unique minimizing coefficients

$$u = -(I + SD_R(s))^{-1} S z_{R,s}(v).$$

Here $(I + SD_R(s))^{-1} S = (S^{-1} + D_R(s))^{-1}$.

At $s = 0$ the determinant $\det(I + SD_R(s))$ is 1. Formula (6.2) therefore extends smoothly to a neighborhood of $\mathcal{D}(Q) \times \{0\}$. Multiplication by $\det(I + SD_R(s))^2$ turns $F_R(s) Q^{-1} F_R(s) = \epsilon Q$ into a polynomial identity in s. It holds when all $s_i > 0$, hence identically. Since $F_R(0) = R > 0$, the extension takes values in $\mathcal{D}(Q)$ on a smaller neighborhood. Differentiation gives

$$F_R(0) = R, \qquad \partial_{s_i} F_R(0)[v] = Q(v, a_i)^2 - R(v, a_i)^2. \tag{6.3}$$

For each compact $\mathcal{C} \subset \mathcal{D}(Q)$ there is $\tau > 0$ such that this extension is defined on a neighborhood of $\mathcal{C} \times [-\tau, \tau]^l$. Every fixed mixed derivative in the logarithm coordinates and in s is bounded on this compact set. For an empty list, put $\widehat{V} = V$, $\widehat{\Lambda} = \Lambda$, $\widehat{Q} = Q$, $T = I$, and $F_R : \mathbb{R}^0 \to \mathcal{D}(Q)$ with $F_R(0) = R$.

For $a \in V$, let $X_a(R)$ be the symmetric form

$$X_a(R)[v] = Q(v, a)^2 - R(v, a)^2.$$

A finite family positively spans a real vector space if every vector is a nonnegative linear combination of its members.

Lemma 6.2. *Suppose $d > 0$. For every $R_0 \in \mathcal{D}(Q)$ there is a finite list $a_i \in 2\Lambda$ such that the forms $X_{a_i}(R) = \partial_{s_i} F_R(0)$ positively span $T_R \mathcal{D}(Q)$ throughout a neighborhood of R_0.*

Proof. Put $J = Q^{-1} R_0$. In an R_0-orthonormal basis, the logarithm description in Lemma 4.3 gives

$$T_{R_0} \mathcal{D}(Q) = \{H : H^t = H,\ HJ + JH = 0\}.$$

Equivalently, its forms satisfy $H(Jv, Jw) = -H(v, w)$. If $\epsilon = 1$, write $V = V_+ \oplus V_-$ for the ± 1 eigenspaces of J. Write $a = a_+ + a_-$ and $v = v_+ + v_-$ according to this decomposition. Then

$$X_a(R_0)[v] = -4 R_0(v_+, a_+) R_0(v_-, a_-).$$

For orthonormal basis vectors $u \in V_+$ and $w \in V_-$, the vectors $u + w$ and $u - w$ give opposite mixed coordinate forms. These form a finite positive spanning family.

If $\epsilon = -1$, then

$$X_a(R_0)[v] = R_0(v, Ja)^2 - R_0(v, a)^2, \qquad X_{Ja}(R_0) = -X_a(R_0).$$

The functions $\ell_a(v) = R_0(v, a) + \mathrm{i}R_0(v, Ja)$ are precisely the complex linear functions for J, and $X_a(R_0)[v] = -\operatorname{Re}(\ell_a(v)^2)$. Their squares span the complex symmetric quadratic forms by polarization. A complex coefficient can be absorbed into the linear function by taking a square root. Thus their real parts span all real symmetric forms H satisfying $H(Jv, Jw) = -H(v, w)$; the corresponding complex bilinear form is $H(v, w) - \mathrm{i}H(Jv, w)$. Choose a finite real spanning subfamily and adjoin Ja for every chosen a. The resulting family positively spans $T_{R_0}\mathcal{D}(Q)$.

In a Euclidean space of positive dimension, a nonempty finite family (x_i) positively spans if and only if

$$\min_{|\xi|=1} \max_i \langle \xi, x_i \rangle > 0$$

by the separation theorem for convex cones [61, Corollary 11.7.3, p. 101]. The minimum is over the compact unit sphere, and

$$\left| \max_i \langle \xi, x_i \rangle - \max_i \langle \xi, y_i \rangle \right| \le \max_i |x_i - y_i| \qquad (|\xi| = 1).$$

Thus positive spanning persists under small perturbations of a finite family.

Apply this criterion in the coordinates $D\theta_{R_0}$. Approximate the chosen vectors a_i by vectors in $\Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$, retaining positive spanning. Multiplying each by a positive integer puts it in 2Λ; the identity $X_{ca}(R) = c^2 X_a(R)$ preserves its positive cone. For this fixed integral list, the vectors $D\theta_R X_{a_i}(R) \in \mathbb{R}^d$ depend continuously on R. The same criterion therefore gives positive spanning throughout a neighborhood of R_0. □

Lemma 6.3. *Suppose $d > 0$. For every compact $K \subset \mathcal{D}(Q)$ there are a finite list $a_1, \dots, a_l \in 2\Lambda$, a relatively compact open neighborhood W of K, and numbers $\delta_0, \sigma_-, \sigma_+ > 0$ with the following property. On*

$$\mathcal{U} = \{(R, P) \in W \times \mathcal{D}(Q) : |\theta(P) - \theta(R)| < \delta_0\}$$

there is a smooth map $s : \mathcal{U} \to (0, \infty)^l$ such that

$$F_R(s(R, P)) = P, \qquad \sigma_- \le s_i(R, P) \le \sigma_+ \quad (1 \le i \le l).$$

Proof. For $K = \emptyset$, take $W = \emptyset$, the empty list, and $\delta_0 = \sigma_- = \sigma_+ = 1$. Assume $K \neq \emptyset$. Concatenating finitely many lists from Lemma 6.2 gives one list for which the columns of

$$L_R = D_s(\theta \circ F_R)(0) : \mathbb{R}^l \longrightarrow \mathbb{R}^d$$

positively span $\mathbb{R}^d$ on an open neighborhood U of K. In particular, L_R is onto, and

$$B_R = L_R^t (L_R L_R^t)^{-1}, \qquad L_R B_R = I$$

depends smoothly on $R \in U$.

Fix $R_0 \in U$, and put $\mathbf{1} = (1, \dots, 1) \in \mathbb{R}^l$. Positive spanning gives $\lambda \in \mathbb{R}^l_{\ge 0}$ with $L_{R_0}\lambda = -L_{R_0}\mathbf{1}$. Thus $c_0 = \lambda + \mathbf{1} \in \ker L_{R_0}$ has positive coordinates. The vector

$$c_{R_0}(R) = (I - B_R L_R) c_0$$

belongs to $\ker L_R$ and remains positive near R_0. Choose finitely many such neighborhoods, centered at R_α, covering K, and a subordinate smooth partition of unity (χ_α) on their union [77, Theorem 1.11]. Then $c(R) = \sum_\alpha \chi_\alpha(R) c_{R_\alpha}(R)$ is smooth, belongs to $\ker L_R$, and has positive coordinates. Choose W containing K with compact closure in this union, and put

$$c_* = \min_{R \in \overline{W}, i} c_i(R) > 0, \qquad c^* = \max_{R \in \overline{W}, i} c_i(R), \qquad \beta = \max\left\{1, \sup_{R \in \overline{W}} \|B_R\|\right\}.$$

The signed extension of F_R after (6.2) defines

$$H(R,t,z)=\theta\big(F_R(tc(R)+B_Rz)\big)-\theta(R)$$

for R near $\overline{W}$ and t,z sufficiently small. At $(t,z)=(0,0)$,

$$H(R,0,0)=0,\qquad \partial_tH(R,0,0)=L_Rc(R)=0,\qquad D_zH(R,0,0)=L_RB_R=I.$$

Smoothness and compactness give $r,t_*>0$ and $A\geq 1$ such that H is defined on a neighborhood of $\overline{W}\times[-t_*,t_*]\times\overline{B_r(0)}$ and

$$\|D_zH(R,t,z)-I\|<\tfrac{1}{4},\qquad |H(R,t,0)|\leq At^2$$

there. Choose $t_0\in(0,t_*)$ and then $\delta_0>0$ so that

$$At_0^2+\delta_0<r/4,\qquad \tfrac{4}{3}\beta(At_0^2+\delta_0)<\tfrac{1}{2}t_0c_*.$$

For $(R,P)\in\mathcal{U}$, Lemma 4.6, applied on compact parameter neighborhoods, gives a solution $z(R,P)\in B_{r/2}(0)$ of

$$H(R,t_0,z(R,P))=\theta(P)-\theta(R),\qquad |z(R,P)|\leq\tfrac{4}{3}\big(At_0^2+|\theta(P)-\theta(R)|\big).$$

Uniqueness in $\overline{B_{r/2}(0)}$ identifies the local smooth solutions. Put $s(R,P)=t_0c(R)+B_Rz(R,P)$. Then

$$\tfrac{1}{2}t_0c_*\leq s_i(R,P)\leq t_0c^*+\tfrac{1}{2}t_0c_*.$$

Since θ is injective, the equation for H gives $F_R(s(R,P))=P$. These bounds give σ_- and σ_+. □

Lemma 6.4. *Let B be a compact smooth parameter manifold, possibly with boundary or corners; all families are assumed to extend smoothly to a neighborhood. Let g_b be a smooth metric family on M, and let $R>0$, with $B_{2R}(0)\subset\mathbb{R}^q$. Fix a finite list $a_i\in 2\Lambda$ in (6.1), and let $s:B\times\overline{B_{2R}(0)}\to(0,\infty)^l$ be smooth. There are smooth metric families $k_{j,b,u}$ such that*

$$P_{k_{j,b,u}}\longrightarrow F_{P_{g_b}}(s(b,u))$$

uniformly on $B\times\overline{B_{2R}(0)}$ and with their first u-derivatives on $B\times\overline{B_R(0)}$.

Proof. For the empty list, take $k_{j,b,u}=g_b$. Suppose $l>0$, put $R_b=P_{g_b}$, and take $N=M\#_{i=1}^l(S^m\times S^m)$, with middle pairing $\widehat{Q}$. Let $v_1,\dots,v_{b_m}$ be the fixed integral basis of Λ. On M, take their harmonic representatives for g_b; these depend smoothly on b by Lemma 4.4.

Fix a unit-volume round metric h_* on S^m. On the ith product, take

$$h_i(s_i)=s_i^{1/m}h_*\oplus s_i^{-1/m}h_*$$

and the pullbacks of dv_{h_*} as representatives of e_i,f_i. The factor volumes are $s_i^{1/2},s_i^{-1/2}$, and these forms have Gram matrix $\operatorname{diag}(s_i^{-1},s_i)$. Thus the sum of the Gram forms on the components is

$$\widehat{R}_{b,u}=R_b\oplus\bigoplus_{i=1}^{l}\operatorname{diag}\big(s_i(b,u)^{-1},s_i(b,u)\big).$$

On the compact parameter set, the s_i are bounded away from zero and their derivatives through order two are bounded. The component metrics therefore satisfy (4.4) with one constant C.

In the fixed bases, define

$$\iota_T:V\oplus\mathbb{R}^l\longrightarrow\widehat{V},\qquad \iota_T(v,w)=Tv+\sum_i w_iTe_i.$$

For $\eta>0$, apply Lemma 5.2 with error $\eta/[2(1+\|\iota_T\|^2)]$. Its closed representatives of the full basis give, by the fixed linear combinations defining ι_T, forms β_i,γ_j representing Tv_i,Te_j. For the resulting metric $h_{\eta,b,u}$ on N,

$$G^{(m)}((\beta,\gamma),h_{\eta,b,u})\preceq\iota_T^t\widehat{R}_{b,u}\iota_T+\tfrac{\eta}{2}I.$$

The metrics $h_{\eta,b,u}$ satisfy (4.4) with the same C, independently of η.

The marked classes (Tv_i, Te_j) on N and $(v_i, 0)$ on M have the same cup matrix $\left(\begin{smallmatrix} Q & 0 \\ 0 & 0 \end{smallmatrix}\right)$, since T preserves $\widehat{Q}$. Every marked degree is m, so their top moments are precisely these pairwise products. The classes are rational and $m \geq 2$. Proposition 5.1 therefore applies with empty degree-one list and error $\eta/2$. It gives metrics $k_{\eta,b,u}$ and closed forms θ_i, ζ_j on M, with $[\theta_i] = v_i$, $[\zeta_j] = 0$, such that

$$G^{(m)}((\theta, \zeta), k_{\eta,b,u}) \preceq \iota_T^t \widehat{R}_{b,u} \iota_T + \eta I.$$

The metrics $k_{\eta,b,u}$ again satisfy (4.4) with the same C.

Let $w_{b,u}(v)$ minimize $\widehat{R}_{b,u}[\iota_T(v, w)]$ over $w \in \mathbb{R}^l$. The formula following (6.2) shows that $w_{b,u}$ is linear in v and smooth in (b, u). Hence $|w_{b,u}(v)| \leq C_0|v|$ for one constant C_0. For $v = \sum_i x_i v_i$, put $\theta_v = \sum_i x_i \theta_i$. Evaluating the preceding bound at $(v, w_{b,u}(v))$ and minimizing over representatives of v gives

$$\begin{aligned} P_{k_{\eta,b,u}}[v] &\leq \left\| \theta_v + \sum_i w_{b,u}(v)_i \zeta_i \right\|^2_{L^2(k_{\eta,b,u})} \\ &\leq F_{R_b}(s(b,u))[v] + (1 + C_0^2)\eta|v|^2. \end{aligned}$$

The target forms have compact image in $\mathcal{D}(Q)$, and $P_{k_{\eta,b,u}}$ is compatible with Q. Choose $\eta_j \downarrow 0$ with $(1 + C_0^2)\eta_j \leq 1$. Lemma 6.1 gives uniform convergence of $P_{k_{\eta_j,b,u}}$ on the larger parameter set. The same upper bound gives a uniform bound for their traces. Lemma 4.4, with the relative derivative constant C, therefore bounds their second u-derivatives uniformly in j. Lemma 4.5 gives the required first-derivative convergence on the smaller ball. Set $k_{j,b,u} = k_{\eta_j,b,u}$. □

Proposition 6.5. *Let B be a compact smooth parameter manifold. Every smooth family $P_b \in \mathcal{D}(Q)$ has a smooth family of metrics g_b on M with $P_{g_b} = P_b$. The statement holds with boundary or corners when the family extends to a neighborhood.*

Proof. The assertion is vacuous if $B = \emptyset$. If $d = 0$, Lemma 4.3 makes $\mathcal{D}(Q)$ a point, and any constant metric family suffices. Suppose $B \neq \emptyset$ and $d > 0$. Fix a metric g_0, and put

$$p_0 = \theta(P_{g_0}), \qquad p(b) = \theta(P_b), \qquad K = \{\theta^{-1}((1-t)p_0 + tp(b)) \,:\, b \in B,\ 0 \leq t \leq 1\}.$$

This set is compact. Apply Lemma 6.3 to K, and retain its list, neighborhood W, map s, and number δ_0. Choose an integer $L \geq 1$ such that

$$L^{-1} \max_{b \in B} |p(b) - p_0| < \delta_0/4, \qquad p_j(b) = (1 - j/L)p_0 + (j/L)p(b) \quad (0 \leq j \leq L).$$

Put $\rho = \delta_0/8$. For $1 \leq j \leq L$, $b \in B$, and $u \in \overline{B_{2\rho}(0)} \subset \mathbb{R}^d$, set

$$\sigma_j(b, u) = s\big(\theta^{-1}(p_{j-1}(b)), \theta^{-1}(p_j(b) + u)\big).$$

The first argument lies in $K \subset W$, and

$$|p_j(b) + u - p_{j-1}(b)| < \delta_0/4 + 2\rho = \delta_0/2.$$

Thus these smooth scale families are defined on a neighborhood of the compact parameter set and have common positive lower and finite upper bounds. Moreover,

$$F_{\theta^{-1}(p_{j-1}(b))}(\sigma_j(b, u)) = \theta^{-1}(p_j(b) + u).$$

Start with $g_{0,b} = g_0$. Suppose $P_{g_{j-1,b}} = \theta^{-1}(p_{j-1}(b))$. Lemma 6.4 gives families $k_{\nu,b,u}$ whose Hodge forms converge to $\theta^{-1}(p_j(b) + u)$ uniformly, with first u-derivatives on $B \times \overline{B_\rho(0)}$. The limit has compact image in $\mathcal{D}(Q)$. Smoothness of θ and the chain rule therefore give

$$q_{\nu,b}(u) := \theta(P_{k_{\nu,b,u}}) \longrightarrow p_j(b) + u$$

uniformly with first u-derivatives on this set. Choose one ν, independent of b, such that

$$\sup_b |q_{\nu,b}(0) - p_j(b)| < \rho/4, \qquad \sup_{b,\,|u|\le\rho} \|D_u q_{\nu,b}(u) - I\| < 1/4.$$

Lemma 4.6 gives a smooth $u_b \in B_{\rho/2}(0)$ with $q_{\nu,b}(u_b) = p_j(b)$. Then $g_{j,b} = k_{\nu,b,u_b}$ is smooth and $P_{g_{j,b}} = \theta^{-1}(p_j(b))$. After L steps, $g_b = g_{L,b}$ has $P_{g_b} = P_b$.

If B has boundary or corners, extend $p = \theta(P)$ to the ambient neighborhood in the hypothesis and choose a compact smooth neighborhood B' of B inside it, with $B \subset \operatorname{int} B'$. Carry out the preceding construction over B', with the extended data, and restrict the resulting family to B. All auxiliary lemmas apply there because the data extend to a neighborhood of B'. □

7. Simultaneous realization in all degrees

7.1. **Complementary degrees.** Write Sym_b for the real symmetric $b \times b$ matrices and Sym_b^+ for their positive definite cone.

Lemma 7.1. *Let $K \subset \mathrm{Sym}_b^+$ be compact, $b \ge 1$. On a neighborhood $U \subset \mathrm{Sym}_b^+$ of K there are an integral matrix $A \in \mathbb{Z}^{k\times b}$ of rank b and smooth positive functions $d_i : U \to (0,\infty)$ such that*

$$P = A^t D(P) A, \qquad D(P) = \operatorname{diag}(d_1(P), \dots, d_k(P)). \tag{7.1}$$

Every row of A may be taken nonzero.

Proof. For $K = \emptyset$, take $U = \emptyset$ and $A = I_b$. For $b = 1$, take $U = \mathrm{Sym}_1^+$, $A = (1)$, and $d_1(P) = P_{11}$. Assume $K \ne \emptyset$ and $b > 1$.

Fix $P_0 \in K$, and put $q = b(b+1)/2$. The local rank-one decomposition [24, Lemma 4.3], after congruence by $P_0^{1/2}$, gives nonzero vectors $v_1, \dots, v_q \in \mathbb{R}^b$ such that

$$L_v : \mathbb{R}^q \longrightarrow \mathrm{Sym}_b, \qquad L_v(c) = \sum_{i=1}^{q} c_i v_i v_i^t$$

is an isomorphism and $L_v^{-1}(P_0)$ has positive coordinates. Choose nonzero rational vectors a_i sufficiently close to v_i. Continuity of inversion gives an isomorphism

$$L_a(c) = \sum_{i=1}^{q} c_i a_i a_i^t$$

with $L_a^{-1}(P_0) > 0$ coordinatewise. Thus the linear coefficient functions $L_a^{-1}(P)$ are positive on an open neighborhood of P_0 in Sym_b^+.

Choose finitely many such neighborhoods U_α covering K, and concatenate their rational lists into $a_1, \dots, a_k$. Pad each local coefficient vector with zeros outside its list, and denote the resulting linear functions on Sym_b by λ_i^α. Put $V = \bigcup_\alpha U_\alpha$. A smooth partition of unity (χ_α) on V subordinate to this cover [77, Theorem 1.11] gives

$$c_i(Q) = \sum_\alpha \chi_\alpha(Q)\lambda_i^\alpha(Q) \ge 0, \qquad Q = \sum_i c_i(Q) a_i a_i^t \quad (Q \in V).$$

Put $S_0 = \sum_i a_i a_i^t$. Since K is compact in V, choose $\delta > 0$ with $K - \delta S_0 \subset V$, and set

$$U = \mathrm{Sym}_b^+ \cap (V + \delta S_0).$$

This is an open neighborhood of K. For each i, choose a positive integer m_i with $m_i a_i \in \mathbb{Z}^b$, and put

$$A_{i\cdot} = m_i a_i^t, \qquad d_i(P) = m_i^{-2}\big(\delta + c_i(P - \delta S_0)\big) \quad (P \in U).$$

These functions are smooth and positive, and $A^t D(P) A = \delta S_0 + (P - \delta S_0) = P$. The rows are nonzero. Since P is positive definite, $\ker A = 0$. □

Lemma 7.2. *Let $A \in \mathbb{Q}^{k\times b}$ have rank $b \geq 1$. Choose rational matrices $C \in \mathbb{Q}^{k\times b}$ and $K \in \mathbb{Q}^{k\times(k-b)}$ with $A^tC = I$, $\operatorname{im} K = \ker A^t$, and $\operatorname{rank} K = k - b$. For every positive diagonal matrix $D \in \operatorname{Sym}_k^+$ and $w \in \mathbb{R}^b$,*

$$\min_{z\in\mathbb{R}^{k-b}} (Cw + Kz)^t D^{-1}(Cw + Kz) = w^t(A^tDA)^{-1}w. \tag{7.2}$$

The minimizing z is unique, linear in w, smooth in D, and uniformly bounded as an operator on compact sets of positive diagonal matrices.

Proof. The indicated rational matrices exist: one may take $C = A(A^tA)^{-1}$ and a rational basis of $\ker A^t$ as the columns of K. If $k = b$, then $C = (A^t)^{-1}$ and $C^tD^{-1}C = (A^tDA)^{-1}$; the variable z lies in $\mathbb{R}^0$. Suppose $k > b$. The affine space $Cw + \operatorname{im} K$ is $\{v \in \mathbb{R}^k : A^tv = w\}$. The constrained quadratic minimum [16, §10.1.1, pp. 522–523], with positive definite Hessian $2D^{-1}$ and full-row-rank constraint matrix A^t, has unique minimizer

$$v_* = DA(A^tDA)^{-1}w, \qquad v_*^tD^{-1}v_* = w^t(A^tDA)^{-1}w.$$

Since K is injective, the minimizing coefficient is

$$z_*(D, w) = (K^tK)^{-1}K^t\big(DA(A^tDA)^{-1} - C\big)w.$$

Its coefficient matrix is smooth in D, hence bounded on every compact set of positive diagonal matrices. □

Fix an integer $1 \leq r < n/2$, put $s = n - r$, and suppose $b = b_r > 0$. Let $(\alpha_j)_{j=1}^b$ in Λ_r and $(\beta_j)_{j=1}^b$ in Λ_s be the fixed integral dual bases, with $\langle \alpha_j\beta_l, [M]\rangle = \delta_{jl}$. Let $P : \mathcal{P} \to \operatorname{Sym}_b^+$ be a smooth family on the compact parameter set $\mathcal{P}$ of Section 5. Apply Lemma 7.1 to its image and choose C, K as in Lemma 7.2. Thus A, C, K are fixed and $D = D(P)$ varies smoothly.

Let $N_r = \bigsqcup_{i=1}^k (S^r \times S^s)$, with product orientations. Let x_i, y_i be the integral generators from the two factors of the ith component, extended by zero to the other components, with $\langle x_iy_i, [N_r]\rangle = 1$. Mark the classes

$$\alpha_j' = \sum_i A_{ij}x_i, \qquad \beta_j' = \sum_i C_{ij}y_i, \qquad \zeta_a' = \sum_i K_{ia}y_i \quad (1 \leq j \leq b,\ 1 \leq a \leq k - b). \tag{7.3}$$

They are rational. The classes corresponding to ζ_a' on M are zero and have degree $s \geq 2$. If $r = 1$, the degree-one classes α_j' are integral because A is integral.

Fix unit-volume metrics h_r, h_s on S^r, S^s. For $t > 0$ independent of the parameters, give the ith component the product metric

$$g_{i,t}(P) = t^{2/r}d_i(P)^{-1/r}h_r \oplus t^{2/s}d_i(P)^{1/s}h_s.$$

The factor volumes are $td_i^{-1/2}$ and $td_i^{1/2}$. The fixed pullbacks of dv_{h_r}, dv_{h_s} represent x_i, y_i and have squared L^2 norms d_i, d_i^{-1}, respectively. Take their linear combinations in (7.3) as representatives of the marked classes. Their degree-r Gram matrix is $P = A^tDA$, and the Gram matrix in degree s, ordered by (β', ζ'), is

$$[C\ K]^tD^{-1}[C\ K]. \tag{7.4}$$

Lemma 7.2 gives P^{-1} after minimization over the auxiliary coefficients. Each component has volume t^2, so $\operatorname{Vol}(N_r) = kt^2$.

The marked pairings are

$$\langle \alpha_j'\beta_l', [N_r]\rangle = (A^tC)_{jl} = \delta_{jl}, \qquad \langle \alpha_j'\zeta_a', [N_r]\rangle = (A^tK)_{ja} = 0.$$

On each component the marked classes are multiples of x_i or y_i. Since $x_i^2 = y_i^2 = 0$, every marked product of length at least three vanishes.

On $\mathcal{P}$, the d_i have a common positive lower bound and bounded derivatives through order two. For $\lambda \in \{-1/r, 1/s\}$,

$$\frac{\partial_a(d_i^\lambda)}{d_i^\lambda} = \lambda\frac{\partial_a d_i}{d_i}, \qquad \frac{\partial_c\partial_a(d_i^\lambda)}{d_i^\lambda} = \lambda\frac{\partial_c\partial_a d_i}{d_i} + \lambda(\lambda-1)\frac{\partial_c d_i\,\partial_a d_i}{d_i^2}.$$

Thus the metrics $g_{i,t}$ satisfy (4.4) with one constant independent of t.

7.2. **Higher top products and total volume.**

Lemma 7.3. *Fix an ordered degree list $(r_i)_{i=1}^{s}$ with $1 \le r_i \le n-1$, a top multiindex $a \in \mathcal{I}$ of length $\ell = \sum_i a_i \ge 3$, and $c \in \mathbb{Q}$. If every r_i with $a_i > 0$ is one, suppose $c \in \mathbb{Z}$. There are closed marked forms on flat n-tori whose ath top moment is c and whose other top moments vanish. Their classes are rational, and their degree-one classes are integral. They admit a sequence with these exact moments, volume tending to zero, and every degreewise Gram matrix tending to zero.*

Proof. Give $T^n = \mathbb{R}^n/\mathbb{Z}^n$ its standard orientation and unit-volume flat metric g_0. For $c = 0$, take zero forms and the metrics $k^{-2}g_0$, $k \ge 1$. Suppose $c \ne 0$. Take the ordered coordinate blocks from the proof of Lemma 1.2, with a_i blocks of size r_i for each i, and denote their volume forms by $\vartheta_{i,j}$. That construction requires only positive degrees, $\sum_i r_i a_i = n$, and $a_i \le 1$ for odd r_i; it therefore also applies when $r_i = 1$. The classes of the $\vartheta_{i,j}$ are integral [39, Example 3.16, p. 216]. For rational coefficients $c_{i,j}$, put

$$\omega_i = \sum_{j=1}^{a_i} c_{i,j}\vartheta_{i,j}.$$

The ordered-block calculation in that proof gives

$$\int_{T^n} \prod_i \omega_i^{a_i} = \left(\prod_i a_i!\right) \prod_{i,j} c_{i,j}, \qquad \int_{T^n} \prod_i \omega_i^{b_i} = 0 \quad (b \in \mathcal{I} \setminus \{a\}).$$

Take all coefficients equal to one except one, equal to $c/\prod_i a_i!$. Choose this coefficient in a block of degree at least two if such a block occurs. Otherwise every participating degree is one, all $a_i \le 1$, and $c \in \mathbb{Z}$. Thus all degree-one coefficients are integral.

For an integer $k \ge 1$, set

$$g_k = k^{-2\ell} g_0, \qquad \omega_{i,k} = k^{n-\ell r_i}\omega_i.$$

Since $\sum_i a_i(n - \ell r_i) = 0$, the ath moment remains c; all other top moments remain zero. The degree-r forms have a common multiplier, and metric scaling gives

$$G^{(r)}(\omega_k, g_k) = k^{-n(\ell-2)} G^{(r)}(\omega, g_0), \qquad \mathrm{Vol}(T^n, g_k) = k^{-n\ell}.$$

Both quantities tend to zero because $\ell \ge 3$. All coefficients remain rational. Moreover, $\ell \le \sum_i r_i a_i = n$, so the degree-one multiplier $k^{n-\ell}$ is an integer. □

For each nonzero top moment of length at least three in the fixed integral bases, take one torus from Lemma 7.3. Set all auxiliary markings equal to zero on these tori. There are finitely many such moments, so the total volume and every sum of Gram matrices tend to zero. The marked classes may depend on the sequence index, but are constant on the parameter set $\mathcal{P}$.

If $n = 2m$ and $b_m > 0$, take one further copy of M. Proposition 6.5 supplies a smooth metric family for the prescribed middle Hodge forms. Compactness of $M \times \mathcal{P}$ gives (4.4) for this fixed family. Choose the harmonic representatives of the middle basis, which depend smoothly on the parameters by Lemma 4.4, and set all other markings equal to zero. Its only nonzero top moments are the middle pairings Q_m. Multiplication of the metric by t^2, with $t > 0$ independent of the parameters, multiplies volume by t^n and leaves the middle Gram matrix and the relative derivative bounds unchanged.

Choose sequences of the products of spheres from the preceding subsection, the tori, and, when present, the copy of M, so that the sum of their volumes tends uniformly to zero on $\mathcal{P}$. After adjoining zero forms for absent markings, write $H_j^{(r)}$ for the sum of their degree-r Gram matrices.

Lemma 7.4. *Let $v : \mathcal{P} \to (0,\infty)$ be smooth. The preceding components can be joined to one additional sphere to give a fixed closed connected oriented manifold N, smooth metric families $h_{j,b,u}$, and smooth closed marked forms $\eta_{i,j}(b,u)$ such that*

$$\mathrm{Vol}(N, h_{j,b,u}) = v(b,u), \qquad \left\| G^{(r)}(\eta_j, h_j) - H_j^{(r)} \right\| \longrightarrow 0 \quad (J_r \ne \emptyset),$$

uniformly on $\mathcal{P}$. Their top moments are the sums of the component moments. The marked forms vanish on the added sphere and its neck. The metrics satisfy (4.4) *with one constant independent of j.*

Proof. The case $\mathcal{P} = \emptyset$ is vacuous. Otherwise, put $v_{\min} = \min_{\mathcal{P}} v > 0$ and $v_{\max} = \max_{\mathcal{P}} v$. Fix a metric a on S^n with $\mathrm{Vol}(S^n, a) < v_{\min}/8$, and set every marked form on this sphere equal to zero. Choose a ball for its attachment at one end of the connected-sum chain. Fix an open set W with compact closure disjoint from that ball, and a function $\chi \in C_c^\infty(W)$ with $\chi \geq 0$, positive on a nonempty open set.

Discard finitely many terms and relabel so that the sum of the other component volumes is less than $v_{\min}/4$ uniformly. Choose $0 < \varepsilon_j < \min\{j^{-1}, v_{\min}/8\}$. Apply Lemma 5.2 to the components and the sphere, with error ε_j, and denote the resulting metrics by $h^0_{j,b,u}$ and forms by $\eta_{i,j}$. Then

$$\mathrm{Vol}(N, h^0_{j,b,u}) < v_{\min}/2, \qquad \left\| G^{(r)}(\eta_j, h^0_j) - H^{(r)}_j \right\| < \varepsilon_j.$$

The construction in that proof leaves the metric equal to a on W, and its identifications restrict to the identity on W. Thus W is fixed in N, and the forms vanish on the sphere and its neck. The relative derivative bounds for h^0_j have one constant independent of j.

Put

$$U_j(b,u) = \int_{N \setminus W} dv_{h^0_{j,b,u}}, \qquad R(T) = \int_W e^{nT\chi}\, dv_a.$$

The function R is smooth, and

$$R'(T) = n \int_W \chi e^{nT\chi}\, dv_a > 0.$$

On some open subset of W of positive volume, χ has a positive lower bound; hence $R(T) \to \infty$ as $T \to \infty$. Since $U_j + R(0) < v_{\min}/2 < v$, there is a unique

$$T_j(b,u) = R^{-1}\big(v(b,u) - U_j(b,u)\big) > 0.$$

It is smooth by the inverse function theorem [77, Theorem 1.30]. Define $h_{j,b,u}$ to equal $e^{2T_j(b,u)\chi} a$ on W and $h^0_{j,b,u}$ outside W. Since χ has compact support in W, this is a smooth metric family, and its volume is $U_j + R(T_j) = v$.

As $U_j \geq 0$, $0 < T_j \leq R^{-1}(v_{\max}) =: T_{\max}$. Thus R' has a positive lower bound and R'' is bounded on $[0, T_{\max}]$. The identity $U_j = \mathrm{Vol}(N, h^0_j) - R(0)$ and the volume conclusion of Lemma 4.4 give uniform first and second u-derivatives of U_j. Differentiation of $U_j + R(T_j) = v$ gives

$$\partial_a T_j = \frac{\partial_a v - \partial_a U_j}{R'(T_j)},$$

$$\partial_c \partial_a T_j = \frac{\partial_c \partial_a v - \partial_c \partial_a U_j - R''(T_j) \partial_c T_j\, \partial_a T_j}{R'(T_j)}.$$

These derivatives are uniformly bounded. The first and second relative derivatives of $e^{2T_j\chi}$ are $2\chi\partial_a T_j$ and $2\chi\partial_c\partial_a T_j + 4\chi^2 \partial_c T_j\, \partial_a T_j$. They are uniformly bounded, so h_j satisfies (4.4) with one constant independent of j. The metric changes only where every $\eta_{i,j}$ vanishes; their Gram matrices and top moments are therefore unchanged. □

For each resulting family $h_{b,u} = h_{j,b,u}$, put

$$\eta_{\mathrm{top}}(b,u) = \frac{dv_{h_{b,u}}}{v(b,u)}. \tag{7.5}$$

By Lemma 4.1, this form represents the integral orientation class and has squared L^2 norm $v(b,u)^{-1}$. All other marked degrees are positive, so its only new top moment is its integral, equal to one.

7.3. **Approximation and exact realization.**

Proposition 7.5. *Let $R > 0$, and let $(\mathsf{P}_{b,u}, v(b,u))$ be a smooth map from $B \times \overline{B_{2R}(0)}$ to $\mathscr{H}(M) \times (0,\infty)$. There are smooth metric families $k_{j,b,u}$ such that*

$$\big(\Theta\Pi(k_{j,b,u}), \mathrm{Vol}(M, k_{j,b,u})\big) \longrightarrow \big(\Theta\mathsf{P}_{b,u}, v(b,u)\big)$$

uniformly on $B \times \overline{B_{2R}(0)}$, together with the first u-derivatives on $B \times \overline{B_R(0)}$.

Proof. Put $\mathcal{P} = B \times \overline{B_{2R}(0)}$, and denote the components of $\mathsf{P}_{b,u}$ by $P_{r,b,u}$. For each $1 \le r < n/2$ with $b_r > 0$, choose the fixed matrices A_r, C_r, K_r and the smooth positive diagonal matrix $D_r = D_r(P_{r,b,u})$ of Lemmas 7.1 and 7.2. Define

$$F^{(r)} = P_{r,b,u}, \qquad F^{(n-r)} = [C_r\ K_r]^t D_r^{-1} [C_r\ K_r],$$

where the second matrix is ordered by the original and auxiliary marks in (7.3). If $n = 2m$ and $b_m > 0$, put $F^{(m)} = P_{m,b,u}$.

Apply the component constructions of the preceding two subsections and Lemma 7.4. They give one closed connected oriented manifold N, metric families $h_{j,b,u}$ of volume $v(b,u)$, and closed marked forms $\eta_{i,j}(b,u)$ such that

$$G^{(r)}(\eta_j, h_j) \longrightarrow F^{(r)} \qquad (1 \le r < n,\ J_r \ne \emptyset)$$

uniformly on $\mathcal{P}$. The metrics have a common relative derivative bound. Indeed, the sphere-product Gram matrices are the displayed $F^{(r)}$, the higher-moment tori contribute matrices tending to zero, and the middle component has Gram matrix $P_{m,b,u}$. Lemma 7.4 preserves these limits. Add the orientation representative (7.5); its squared norm is exactly $v(b,u)^{-1}$.

The marked top moments on N equal those on M. The sphere products give the complementary pairings, the middle component gives Q_m, and the tori give the products of length at least three. On the sphere products for degree r, the only possible top product containing an auxiliary mark is its pairing with an original degree-r mark. The matrix of these pairings, summed over those components, is $A_r^t K_r = 0$. Auxiliary marks vanish on all other components. The marked cohomology classes are rational and independent of (b,u) for each j. The degree-one list on M is its fixed integral basis, and the corresponding classes on N are integral; auxiliary degrees are at least two.

Choose $0 < \varepsilon_j \le 1/2$ with $\varepsilon_j \to 0$. Passing to a subsequence and relabelling, arrange

$$\|G^{(r)}(\eta_j, h_j) - F^{(r)}\| < \varepsilon_j \qquad (1 \le r < n,\ J_r \ne \emptyset)$$

uniformly on $\mathcal{P}$. Proposition 5.1, with $\delta = \varepsilon_j$, applies to the full marked list, including the auxiliary zero classes and the orientation class. It gives metrics $k_{j,b,u}$ with one common relative derivative bound and closed representatives $\theta_{i,j}$ on M. Denote the orientation representative by $\theta_{\mathrm{top},j}$. Then

$$\begin{aligned} &G^{(r)}(\theta_j, k_j) \preceq F^{(r)} + 2\varepsilon_j \mathrm{Id}_r \quad (1 \le r < n,\ J_r \ne \emptyset), \\ &\|\theta_{\mathrm{top},j}\|^2_{L^2(k_j)} \le v(b,u)^{-1} + \varepsilon_j, \\ &\mathrm{Vol}(M, k_j) \le v(b,u) + \varepsilon_j, \end{aligned}$$

Write $G_{r,j} = P_{r,k_{j,b,u}}$. For $1 \le r < n/2$ with $b_r > 0$, harmonic minimization gives

$$G_{r,j} \preceq P_{r,b,u} + 2\varepsilon_j I.$$

Put $s = n - r$. A transferred form with coefficients (w,z) in the original and auxiliary degree-s marks represents the class with coefficients w, since the auxiliary forms are exact. The minimizing coefficient z in Lemma 7.2 satisfies $|z| \le L|w|$ with one $L \ge 0$ for all the finitely many compact diagonal families. Consequently,

$$\begin{aligned} G_{s,j}[w] &\le w^t P_{r,b,u}^{-1} w + 2\varepsilon_j(|w|^2 + |z|^2), \\ G_{s,j} &\preceq P_{r,b,u}^{-1} + c\varepsilon_j I, \qquad c = 2(1+L^2). \end{aligned}$$

The bases are dual, so $G_{s,j} = G_{r,j}^{-1}$ by Lemma 4.1. Hence

$$(P_{r,b,u}^{-1} + c\varepsilon_j I)^{-1} \preceq G_{r,j} \preceq P_{r,b,u} + 2\varepsilon_j I. \tag{7.6}$$

The target matrices and their inverses range over compact sets. Thus $\|G_{r,j} - P_{r,b,u}\| = O(\varepsilon_j)$ uniformly on $\mathcal{P}$. If $n = 2m$ and $b_m > 0$, harmonic minimization gives $G_{m,j} \preceq P_{m,b,u} + 2\varepsilon_j I$; both matrices belong to $\mathcal{D}(Q_m)$, so Lemma 6.1 gives the same conclusion.

Since $\int_M \theta_{\mathrm{top},j} = 1$, Cauchy–Schwarz gives

$$\frac{1}{\mathrm{Vol}(M, k_j)} \le \|\theta_{\mathrm{top},j}\|^2_{L^2(k_j)} \le v(b,u)^{-1} + \varepsilon_j.$$

Together with the volume upper bound this yields

$$(v(b,u)^{-1} + \varepsilon_j)^{-1} \le \mathrm{Vol}(M, k_j) \le v(b,u) + \varepsilon_j. \tag{7.7}$$

The independent Hodge matrices and the volumes therefore converge uniformly to their prescribed values. Their traces and volumes are uniformly bounded, and the matrices remain in a fixed compact subset of the positive definite cones. Lemma 4.4 gives uniform first and second u-derivative bounds for the matrices and volumes. The chain rule gives the same bounds after Θ, whose first two derivatives are bounded on that compact set. Lemma 4.5 gives the asserted first-derivative convergence on the smaller ball. □

Proof of Theorem 4.2. For $B = \emptyset$, take the empty families. Assume $B \neq \emptyset$. Put $p(b) = \Theta(\mathsf{P}_b)$ and $v(b) = \int_M \mu_b$. Put $d = \dim \mathscr{H}(M)$, and write $u = (u_E, u_V) \in \mathbb{R}^d \times \mathbb{R}$. Choose $R > 0$ so that $v(b) + u_V > 0$ for $|u| \le 2R$, and prescribe

$$\mathsf{P}_{b,u} = \Theta^{-1}(p(b) + u_E), \qquad v(b,u) = v(b) + u_V.$$

Proposition 7.5 gives a sequence $k_{j,b,u}$. Choose one index, and write $k_{b,u}$ for that family, so that, with

$$q_b(u) = \big(\Theta\Pi(k_{b,u}), \mathrm{Vol}(M, k_{b,u})\big), \qquad y_b = (p(b), v(b)),$$

one has

$$\sup_b |q_b(0) - y_b| < R/4, \qquad \sup_{b,\,|u|\le R} \|D_u q_b(u) - I\| < 1/4.$$

Lemma 4.6 gives a smooth $u_b \in B_{R/2}(0)$ with $q_b(u_b) = y_b$ and $D_u q_b(u_b)$ invertible. Set $k_b = k_{b,u_b}$ and $\nu_b = dv_{k_b}$. Then $\Pi(k_b) = \mathsf{P}_b$ and $\int_M \nu_b = \int_M \mu_b$.

Fix a reference metric on M, with codifferential d_0^* and Green operator G_0 on n-forms, and put $L = d_0^* G_0$. As in the proof of Proposition 5.1, L is continuous in the C^∞ topology. Hodge decomposition gives

$$dL\tau = \tau \qquad \left(\tau \in \Omega^n(M),\ \int_M \tau = 0\right)$$

[77, §6.9 and the proof of Theorem 6.11, equation (3)]. Moser's theorem applies to the positive forms $\nu_{b,t} = (1-t)\nu_b + t\mu_b$, $0 \le t \le 1$, whose integrals are constant in t on the closed connected manifold M [58, Theorem 2, pp. 292–293]. In its construction, take the primitive $L(\mu_b - \nu_b)$. The vector field $X_{b,t}$ determined by

$$\iota_{X_{b,t}} \nu_{b,t} = -L(\mu_b - \nu_b)$$

is smooth for $(b,t,x) \in B \times [0,1] \times M$. Apply the smooth-flow theorem [77, Theorem 1.48(d)] to $(0, 1, X_{b,t})$ on the product of a parameter neighborhood, an open time interval containing $[0,1]$, and M. The flow depends smoothly on the parameters; compactness of M gives its existence through $0 \le t \le 1$. If ϕ_b is the inverse of the time-one flow, Moser's identity gives $\phi_b^* \nu_b = \mu_b$. Thus

$$g_b = \phi_b^* k_b$$

is a smooth metric family with $dv_{g_b} = \mu_b$. Each ϕ_b is isotopic to the identity and hence acts trivially on cohomology. Naturality of harmonic representatives under pullback gives $\Pi(g_b) = \mathsf{P}_b$.

For $\xi \in T_{\mathsf{P}_b}\mathscr{H}(M)$, define

$$w_b(\xi) = (D_u q_b(u_b))^{-1}(D\Theta_{\mathsf{P}_b}\xi, 0),$$
$$h_b(\xi) = \phi_b^*\left(D_u k_{b,u}\big|_{u=u_b}[w_b(\xi)]\right).$$

Differentiating q_b in this direction gives

$$D\Pi_{g_b} h_b(\xi) = \xi, \qquad \int_M \tfrac{1}{2} \operatorname{tr}_{g_b} h_b(\xi)\, \mu_b = 0.$$

Put $f_b(\xi) = \frac{1}{2} \operatorname{tr}_{g_b} h_b(\xi)$, and define the vector field $Y_b(\xi)$ by

$$\iota_{Y_b(\xi)} \mu_b = L(f_b(\xi)\mu_b).$$

Writing $\mathcal{L}_Y$ for Lie derivative, one obtains

$$\mathcal{L}_{Y_b(\xi)} \mu_b = d\big(\iota_{Y_b(\xi)} \mu_b\big) = f_b(\xi)\mu_b,$$
$$\tfrac{1}{2} \operatorname{tr}_{g_b} \mathcal{L}_{Y_b(\xi)} g_b = f_b(\xi).$$

Consequently

$$\mathcal{R}_b(\xi) = h_b(\xi) - \mathcal{L}_{Y_b(\xi)} g_b \tag{7.8}$$

has zero pointwise trace. The flow of $Y_b(\xi)$ is isotopic to the identity, so $D\Pi_{g_b}(\mathcal{L}_{Y_b(\xi)} g_b) = 0$. Hence $D\Pi_{g_b} \mathcal{R}_b(\xi) = \xi$. The map $\mathcal{R}_b$ is smooth in b and linear in ξ. For each fixed k, its operator norm for the prescribed tangent norm and spatial C^k norm is continuous in b, and hence bounded on compact B. All constructions apply on neighborhoods of the compact parameter sets, giving the boundary and corner cases. □

Corollary 7.6. *Let M be closed, connected, and oriented of dimension $2m \geq 4$. Every compact smooth family in $\mathcal{D}(Q_m)$ is induced by a smooth family of metrics with any prescribed smooth positive volume forms. The middle period differentials on the prescribed-volume-form slices have smooth right inverses bounded in each fixed spatial C^k norm. Every compact subset of $\mathcal{D}(Q_m)$ has a neighborhood on which the middle period map has a smooth section with these properties and a fixed prescribed volume form.*

Proof. Choose constant positive definite bilinear forms in the lower-degree factors of (4.3) and apply Theorem 4.2. For a middle tangent vector, set the other tangent components equal to zero in its right inverse. For the section assertion, enclose the compact set in the interior of a closed ball in the global logarithm coordinates, apply the family statement to that ball with the fixed volume form, and restrict to its interior. □

8. Local variation of Hodge periods

Throughout this section, X is a closed connected oriented smooth four-manifold. Put

$$V = H^2(X;\mathbb{R}), \qquad \Lambda = H^2(X;\mathbb{Z})/\text{torsion}, \qquad Q(v,w) = \langle v \smile w, [X] \rangle,$$

and identify Λ with its image in V. The form Q is nondegenerate; let (b_+, b_-) be its signature. A *compatible majorant* is a positive definite symmetric form P on V satisfying

$$PQ^{-1}P = Q.$$

We denote their space by $\mathcal{D}(Q)$; when $V = 0$, this is a singleton. The involution $J = Q^{-1}P$ has $+1$ and -1 eigenspaces which are Q-positive and Q-negative, respectively. Conversely, a maximal positive subspace and its Q-orthogonal complement determine P. Fix one such decomposition $V = V_+ \oplus V_-$. Every maximal positive subspace is the graph of a map $V_+ \to V_-$ of operator norm less than one, for the inner products $Q|_{V_+}$ and $-Q|_{V_-}$. The graph map gives a chart $\theta : \mathcal{D}(Q) \to U$, where U is the open operator unit ball in $\operatorname{Hom}(V_+, V_-)$. Fix a Euclidean norm $|\cdot|$ on this vector space. The set U is convex. If $b_+ b_- = 0$, the domain is a point. For a symmetric form M, write $M[v] = M(v,v)$. For $t \in \mathbb{R}$, put $t_+ = \max(t, 0)$. The notation $\|\cdot\|_2$ denotes the L^2 norm in the current metric. For a smooth form η, $\operatorname{comass}_g(\eta)$ denotes the pointwise function $x \mapsto \operatorname{comass}_g(\eta_x)$.

For a smooth metric g, let ω_v be the harmonic representative of v, and set

$$P_g(v,w) = \int_X \omega_v \wedge *_g\, \omega_w.$$

The harmonic representative minimizes the L^2 norm in its cohomology class; see [9, arXiv version, §7, before (7.3)]. The Hodge star induces the involution $Q^{-1}P_g$; hence P_g is compatible by Lemma 4.1. Taking determinants in the compatibility identity gives

$$\det P = |\det Q| \qquad (P \in \mathcal{D}(Q)) \tag{8.1}$$

in every fixed basis. A metric is *regular* if the derivative of $g \mapsto P_g$ is onto $T_{P_g}\mathcal{D}(Q)$. This condition is vacuous when the target is a point. The cohomological comass $\|\alpha\|_g^*$ is the infimum of the supremum pointwise comass over smooth closed representatives of α.

Theorem 8.1 (Uniform local period variation). *Fix $P_0 \in \mathcal{D}(Q)$, finitely many real classes $\alpha_1, \dots, \alpha_s \in V$, and positive numbers β_i. There are a neighborhood W of P_0 and a constant C such that the following holds. If g is any smooth metric with $M = P_g \in W$ and $\|\alpha_i\|_g^* \le \beta_i$, and if $P \in W$, then for every $\varepsilon > 0$ there is a smooth regular metric h satisfying*

$$P_h = P, \qquad \|\alpha_i\|_h^* \le \beta_i, \qquad \operatorname{Vol}(X,h) \le \operatorname{Vol}(X,g) + C|\theta(P) - \theta(M)| + \varepsilon. \tag{8.2}$$

The constant is independent of the initial metric g. The list may be empty, dependent, or zero. No hypothesis on $\pi_1(X)$ is imposed.

8.1. **Conformal changes.**

Lemma 8.2. *Let $\eta_1, \dots, \eta_s$ be smooth two-forms and let k be a smooth metric on X. Suppose* $\operatorname{comass}_k(\eta_i) \le \beta_i + e_i$, *where $\beta_i > 0$ and the functions e_i are continuous and nonnegative. Put*

$$E = \left(\sum_i e_i^2/\beta_i^2 \right)^{1/2}.$$

For every $\delta > 0$ there is a smooth function $u \ge 1 + E$ such that $\operatorname{comass}_{uk}(\eta_i) \le \beta_i$ *for every i and*

$$\operatorname{Vol}(X, uk) \le \operatorname{Vol}(X,k) + 2\operatorname{Vol}(X,k)^{1/2}\|E\|_{L^2(k)} + \|E\|_{L^2(k)}^2 + \delta. \tag{8.3}$$

The Hodge star on two-forms, their L^2 inner products, the period, and regularity are unchanged.

For a compact smooth parameter manifold B, suppose that k_b and $\eta_{i,b}$ depend smoothly on b and that the functions $e_{i,b}$ are jointly continuous on $B \times X$. Then u_b may be chosen smooth in (b,x), with (8.3) *for every b and one prescribed $\delta > 0$. If $A \subset X$ is closed and $E_b = 0$ on $B \times O$ for one open neighborhood O of A, then u_b may also be chosen equal to one on a fixed smaller neighborhood of A.*

Proof. For a positive function u,

$$\operatorname{comass}_{uk}(\eta_i) = u^{-1}\operatorname{comass}_k(\eta_i), \qquad dv_{uk} = u^2\, dv_k, \qquad *_{uk}\,|_{\Omega^2(X)} = *_k\,|_{\Omega^2(X)}.$$

Since $e_i/\beta_i \le E$, the inequality $u \ge 1 + E$ gives the required comass bounds.

For $a > 0$, take a finite open cover on which the oscillation of E is less than $a/2$. A subordinate smooth partition of unity [77, Theorem 1.11], applied to one value of E on each member of the cover, gives a smooth function f with $|f - E| < a/2$. Put $u = 1 + f + a/2$. Then $0 < u - (1+E) < a$, and

$$\int_X u^2\, dv_k \le \int_X (1+E)^2\, dv_k + a\operatorname{Vol}(X,k)\big(2 + 2\max_X E + a\big).$$

Choose a so that the last term is less than δ. Expanding $(1+E)^2$ and applying Cauchy–Schwarz gives (8.3).

For a compact parameter family, choose an open cover of a neighborhood of $B \times X$ whose intersections with $B \times X$ have E-oscillation less than $a/2$. Apply the same partition-of-unity construction, with values taken on $B \times X$. The bounds $\sup_{b,x} E_b(x) < \infty$ and $\sup_b \operatorname{Vol}(X,k_b) < \infty$ permit one value of a for all b. For the relative assertion, choose a closed neighborhood A' of A contained in O. There is a smooth function $0 \le \chi \le 1$ supported in O and equal to one on A' [77, Corollary to Theorem 1.11]. Replace u_b by $1 + (1-\chi)(u_b - 1)$. On the support of χ one has $E_b = 0$; elsewhere the function is unchanged. Thus the replacement is at least $1 + E_b$, is one near A, and does not increase the volume.

The equality of the stars gives equality of the L^2 inner products on two-forms and of the period maps. For regularity, keep u fixed. The identities $P_{ug_t} = P_{g_t}$ for all smooth metric families g_t show that multiplication of variations by u identifies the images of the two period differentials. □

8.2. **Integral transvections and positive parameters.** Let $a_1, \dots, a_l \in 2\Lambda$. Retain the hyperbolic pairs (e_i, f_i) and the integral isometry T of (6.1), with $\epsilon = 1$. For $M \in \mathcal{D}(Q)$ and $s = (s_1, \dots, s_l) \in (0,\infty)^l$, let $F_M(s)$ be the reduction of $M \oplus \bigoplus_i \operatorname{diag}(s_i^{-1}, s_i)$ along $T\ \operatorname{span}\{e_i\}$, marked by $v \mapsto [Tv]$. Lemma 6.1 and (6.2) give this compatible form. The latter formula extends F_M smoothly to signed s near zero, with first derivatives (6.3).

Lemma 8.3. *Suppose $b_+ b_- > 0$, and fix $P_0 \in \mathcal{D}(Q)$. There are a finite list $a_i \in 2\Lambda$, a neighborhood W of P_0, and constants $C_0, \eta_0 > 0$ with the following property. For $M, P \in W$, put $\delta = |\theta(P) - \theta(M)|$. For every $0 < \eta < \eta_0$, there are positive scales satisfying*

$$F_M(s) = P, \qquad \sum_i s_i \le C_0(\delta + \eta). \tag{8.4}$$

For fixed M, P, η, these scales extend to a smooth positive map $P' \mapsto s(P')$ on a neighborhood of P in W, with $F_M(s(P')) = P'$ and

$$\sum_i s_i(P') \le 2C_0\big(|\theta(P') - \theta(M)| + \eta\big).$$

Proof. By Lemma 6.2, choose a finite list whose derivatives positively span the tangent spaces near P_0, and put $L_M = D_s(\theta \circ F_M)(0)$. The construction in the proof of Lemma 6.3, applied in the present chart, gives smooth right inverses R_M and vectors c_M such that

$$L_M R_M = I, \qquad L_M c_M = 0, \qquad (c_M)_i \ge m_0 > 0$$

on the closure of a sufficiently small neighborhood W_0 of P_0. Choose W_0 with compact closure. The maps R_M and c_M are bounded there.

For $z \in \operatorname{Hom}(V_+, V_-)$, the signed extension of F_M defines

$$H(M, t, z) = \theta\big(F_M(tc_M + R_M z)\big) - \theta(M)$$

near $\overline{W}_0 \times \{0\} \times \{0\}$. One has

$$H(M, 0, 0) = 0, \qquad \partial_t H(M, 0, 0) = 0, \qquad D_z H(M, 0, 0) = I.$$

Smoothness and compactness give $r, t_* > 0$ and $K \ge 1$ such that

$$\|D_z H(M, t, z) - I\| < \tfrac{1}{4}, \qquad |H(M, t, 0)| \le Kt^2$$

for $M \in \overline{W}_0$, $|t| \le t_*$, and $|z| \le r$. After reducing t_* and taking a smaller neighborhood $W \Subset W_0$, assume that $|\theta(P) - \theta(M)| + Kt_*^2 < r/4$ for $M, P \in \overline{W}$. Lemma 4.6, with target $\theta(P) - \theta(M)$, gives a smooth solution $z = z(M, P, t)$ of

$$H(M, t, z) = \theta(P) - \theta(M), \qquad |z| \le \tfrac{4}{3}(\delta + Kt^2).$$

For a constant $C_2 \ge 1$ independent of M, P, t, it follows that $|(R_M z)_i| \le C_2(\delta + t^2)$ for every i.

Choose $A \ge \max\{1, 4C_2/m_0\}$ and put $t = A\delta + \eta$. Reduce W and choose $\eta_0 > 0$ so that $t < t_*$ and $C_2 t \le m_0/4$ whenever $M, P \in W$ and $0 < \eta < \eta_0$. Then $s = tc_M + R_M z$ satisfies

$$s_i \ge m_0 t - C_2\delta - C_2 t^2 \ge m_0 t/2 > 0.$$

The uniform bounds for c_M and R_M give $\sum_i s_i \le C_0(\delta + \eta)$ for a fixed C_0. The equation for H and injectivity of θ give $F_M(s) = P$.

For fixed M, P, η, keep this value of t fixed and put $s(P') = tc_M + R_M z(M, P', t)$. This map is smooth. Positivity persists near P, and continuity of

$$P' \longmapsto \frac{\sum_i s_i(P')}{|\theta(P') - \theta(M)| + \eta}$$

gives the asserted bound after reducing the neighborhood. When $P = M$, one has $t = \eta > 0$, so the same argument applies. □

8.3. **Surgery estimates and parameter derivatives.** Let $S_1, \dots, S_r$ be disjoint oriented framed two-spheres in a closed oriented four-manifold Y. Fix disjoint tubular neighborhoods $\nu_i \simeq S^2 \times D^2$, and put

$$C = Y \setminus \bigcup_i \operatorname{int} \nu_i, \qquad K = \operatorname{span}_{\mathbb{R}}\{\mathrm{PD}[S_i]\} \subset H^2(Y;\mathbb{R}),$$

where PD is Poincaré duality. Write Q_Y for the intersection form. Let X' be the oriented manifold obtained by attaching $D^3 \times S^1$ to each boundary component of C, with the gluing maps determined by the framings. The sphere classes may be dependent or zero.

Lemma 8.4. *Restriction to C gives an identification preserving the intersection form,*

$$H^2(X';\mathbb{R}) = K^{\perp_{Q_Y}}/K. \tag{8.5}$$

Let g_x, $x \in B$, be a smooth metric family on a compact parameter set B. There are smooth metric families $k_{j,x}$ on the fixed marked X' such that, uniformly in x,

$$P_{k_{j,x}} \longrightarrow (P_{g_x})_K, \qquad \operatorname{Vol}(X', k_{j,x}) - \operatorname{Vol}(Y, g_x) \longrightarrow 0.$$

Here $(P_{g_x})_K$ is the reduction of Lemma 6.1. The removed tubes, interpolating annuli and attached caps have volume tending to zero. Any prescribed finite smooth family of closed two-forms whose classes lie in $K^{\perp_{Q_Y}}$ can be cut off and extended by zero to X'. Their classes are given by (8.5)*, and their $L^2(g_x)$ cutoff errors tend uniformly to zero. The cutoffs and identifications with X' are independent of x.*

Proof. Disjointness and the framings give $Q_Y(K,K) = 0$. The normal disk bundles are oriented. Excision and the Thom isomorphism [39, §3.1, Corollary 4D.9 and Theorem 4D.10] give

$$H^2(Y, C;\mathbb{R}) = \mathbb{R}^r, \qquad H^3(Y, C;\mathbb{R}) = 0.$$

The image of the first group in $H^2(Y;\mathbb{R})$ is K. The exact sequence of the pair thus identifies $H^2(C;\mathbb{R})$ with $H^2(Y;\mathbb{R})/K$. The restriction of a lift v to the ith boundary component is determined by $\langle v, [S_i]\rangle = Q_Y(v, \mathrm{PD}[S_i])$. For each cap, the restriction $H^1(D^3 \times S^1;\mathbb{R}) \to H^1(S^2 \times S^1;\mathbb{R})$ is an isomorphism and $H^2(D^3 \times S^1;\mathbb{R}) = 0$. Mayer–Vietoris [39, §3.1, pp. 203–204] therefore identifies $H^2(X';\mathbb{R})$ with the kernel of these boundary restrictions. This gives (8.5).

On a fixed tube $\nu \simeq S^2 \times D^2$, let $\iota : S^2 \to \nu$ be the zero section and $\pi : \nu \to S^2$ the projection. Fix a metric q on S^2, its codifferential d_q^*, and its Green operator G_q. If ω is a closed two-form with $\int_{S^2} \iota^*\omega = 0$, Hodge decomposition gives $d d_q^* G_q \iota^* \omega = \iota^*\omega$ [77, Definition 6.9, Proposition 6.10 and Theorem 6.11]. For the fibre contraction $F(s,p,z) = (p, sz)$, define

$$\boldsymbol{\mathcal{H}}\omega = \int_0^1 \iota_{\partial_s} F^*\omega\big|_{\{s\}\times\nu}\, ds.$$

Cartan's formula [77, Proposition 2.25(d)] gives $d\boldsymbol{\mathcal{H}}\omega = \omega - \pi^*\iota^*\omega$. Hence

$$\psi = \pi^* d_q^* G_q \iota^* \omega + \boldsymbol{\mathcal{H}}\omega, \qquad d\psi = \omega.$$

This is a fixed continuous linear choice of primitive. For a compact smooth family of forms, ψ and $d\psi$ are uniformly bounded.

Write (t, ϑ) for polar coordinates in the normal disk. For $0 < \rho < 1/4$, choose a smooth radial cutoff χ_ρ equal to one for $t \le 2\rho^2$ and to zero for $t \ge \rho$, with

$$|d\chi_\rho| \le \frac{C}{t|\log\rho|}.$$

Such cutoffs are obtained from a fixed smooth function of $\log(t/(2\rho^2))/\log(1/(2\rho))$. The metrics g_x are uniformly equivalent on the fixed tubes to $q + dt^2 + t^2 d\vartheta^2$. With all norms taken in g_x,

$$\|\chi_\rho d\psi\|_2^2 \le C\rho^2, \qquad \|d\chi_\rho \wedge \psi\|_2^2 \le \frac{C}{|\log\rho|^2}\int_{2\rho^2}^{\rho} \frac{dt}{t} \le \frac{C}{|\log\rho|}.$$

Extending the corrections by zero gives globally exact forms, and

$$\|d(\chi_\rho\psi)\|_2 \le C\big(\rho + |\log\rho|^{-1/2}\big). \tag{8.6}$$

The corrected forms $\omega - d(\chi_\rho\psi)$ vanish for $t \le 2\rho^2$.

Put $a = \rho^2$. Choose a fixed metric h on D^3 with a collar coordinate u and boundary germ $du^2 + q$. Fix smooth functions f, A on $[1,2]$ such that $f > 0$, $f = 1$ near 1, $f(s) = s$ for $s \ge 4/3$, $0 \le A \le 1$, $A = 0$ for $s \le 3/2$, and $A = 1$ near 2. On $a \le t \le 2a$ set

$$k_{\rho,x} = A(t/a)g_x + (1 - A(t/a))\big(q + dt^2 + a^2 f(t/a)^2 d\vartheta^2\big).$$

Retain g_x for $t \ge 2a$, remove $t < a$, and attach the cap with metric $h + a^2 d\vartheta^2$. These positive metrics have matching product germs. All changes occur where the corrected forms vanish; the derivative of χ_ρ is supported where g_x is retained. The annular metrics are uniformly equivalent to $q + dt^2 + t^2 d\vartheta^2$. The annuli and removed regions have volume $O(a^2)$, and each cap has volume $2\pi a\,\mathrm{Vol}(D^3, h) = O(a)$. All bounds are uniform in x.

The corrected forms extend smoothly by zero across the caps. Exactness of the corrections preserves their wedge integrals on Y, and their vanishing on the removed regions identifies those integrals with the wedge integrals on X'. Thus (8.5) preserves the intersection form. Radial reparametrizations in the fixed tubes, extended across the caps, identify all radii with one fixed marked X', independently of x.

Fix a basis and a Euclidean norm on $\bar V = K^{\perp_{Q_Y}}/K$. Let $\ell_x : \bar V \to K^{\perp_{Q_Y}}$ assign the P_{g_x}-minimal lift. This linear map is smooth in x by the minimization in Lemma 6.1. Let $\omega_x(v)$ be the harmonic representative of $\ell_x(v)$. Linearity in the cohomology class and Lemma 4.4 give its smooth dependence on x. Apply the preceding cutoffs to a basis of these forms and to the prescribed forms. If $D_{\rho,x}(v)$ is the exact correction of $\omega_x(v)$, linearity and compactness give

$$\|D_{\rho,x}(v)\|^2_{L^2(g_x)} \le C\big(\rho^2 + |\log\rho|^{-1}\big)|v|^2.$$

Write $\widehat{\omega}_{\rho,x}(v)$ for the zero extension of the corrected form to X'. Harmonic orthogonality gives

$$\begin{aligned}\|\widehat{\omega}_{\rho,x}(v)\|^2_{L^2(k_{\rho,x})} &= \|\omega_x(v) - D_{\rho,x}(v)\|^2_{L^2(g_x)}\\ &= (P_{g_x})_K[v] + \|D_{\rho,x}(v)\|^2_{L^2(g_x)}.\end{aligned}$$

Harmonic minimization on X' therefore gives

$$P_{k_{\rho,x}} \preceq (P_{g_x})_K + C\big(\rho^2 + |\log\rho|^{-1}\big)I.$$

The reduced forms range over a compact subset of the compatible forms for the quotient intersection form. Lemma 6.1 gives uniform convergence as $\rho \downarrow 0$. Choose any sequence $\rho_j \downarrow 0$ and put $k_{j,x} = k_{\rho_j,x}$. □

For connected sums of compact smooth metric families $g_{i,x}$, apply Lemma 5.2 to the prescribed forms and harmonic representatives of fixed cohomology bases, each extended by zero on the other summands. Its Gram-matrix estimate and harmonic minimization give, uniformly in x,

$$P_{k_{j,x}} \preceq \bigoplus_i P_{g_{i,x}} + o(1)I.$$

The forms $P_{k_{j,x}}$ and $\bigoplus_i P_{g_{i,x}}$ are compatible for the direct sum intersection form, and the latter range over a compact set. Lemma 6.1 gives uniform convergence to $\bigoplus_i P_{g_{i,x}}$. Every metric change occurs where all the modified forms vanish.

Lemma 8.5. *Let $U \subset \mathbb{R}^d$ be open, and let $k_{j,x}$, $x \in U$, be smooth metric families on a fixed closed oriented four-manifold Z. Suppose that, for every multi-index γ with $1 \le |\gamma| \le 2$,*

$$|\partial_x^\gamma k_{j,x}(w,w)| \le C k_{j,x}(w,w) \qquad (w \in TZ), \tag{8.7}$$

with C independent of j and x. If the period matrices converge uniformly on U to a smooth map, then they converge in C^1 on each compact subset of U. The preceding connected sums and sphere surgeries, and their finite successive compositions, can be chosen to satisfy (8.7) *before conformal restoration.*

Proof. On each relatively compact parameter neighborhood, uniform convergence bounds the harmonic energies of a fixed cohomology basis. Lemma 4.4 gives uniform second parameter derivatives of the period matrices. A finite covering by parameter balls and Lemma 4.5 give C^1 convergence on compact subsets.

A fixed smooth metric family on a compact parameter set has bounded relative derivatives through order two. The same holds for a fixed smooth family of positive product scales: their minimum is positive, and their first and second derivatives are bounded. For a parameter-independent positive tensor a_j and a fixed cutoff $0 \le \chi_j \le 1$, put $\widetilde{k}_{j,x} = \chi_j k_x + (1-\chi_j)a_j$. Then

$$|\partial_x^\gamma \widetilde{k}_{j,x}(w,w)| \le C\chi_j k_x(w,w) \le C\widetilde{k}_{j,x}(w,w).$$

Pullbacks by parameter-independent maps preserve this inequality. The connected-sum metrics of Lemma 5.2 and the annular metrics above have this form, and their cap metrics are parameter independent. Thus finite compositions retain one relative derivative bound, independent of the radii.

After fixing the radii at earlier stages, the next stage is a compact smooth family. For the jth family, choose each successive radius so that its cutoff and volume errors, and its period difference from the corresponding direct sum or reduction of the preceding period, are at most $1/j$, uniformly in the parameter. Direct sums are continuous, and isotropic reduction is smooth by Lemma 6.1; induction on the finite number of stages gives uniform convergence of the periods. The relative derivative bound is independent of all these choices. The first part of the lemma gives C^1 convergence on smaller compact parameter sets. The constants may depend on the fixed metric and scale families. □

Lemma 8.6. *Fix an initial metric and smooth closed two-forms η_i, $1 \le i \le N$, with pointwise comass bounds $\beta_i > 0$. The list may be empty. Fix a compact target parameter ball B on which the product scales $s_1, \dots, s_m$ are smooth and strictly positive, and suppose that the smooth closed forms on the added product pieces obey the same bounds. Consider a fixed finite sequence of the preceding ball, connected-sum and sphere constructions, with embeddings and markings independent of $x \in B$. At each sphere surgery assume that every prescribed class has zero period on each surgery sphere.*

Without intermediate conformal changes, the radii can be chosen in a sequence such that the final metrics $k_{j,x}$ satisfy (8.7) *with one constant. Let $\eta_{i,j,x}$ be the resulting closed forms, and put*

$$E_{j,x} = \left(\sum_i \left(\frac{\operatorname{comass}_{k_{j,x}}(\eta_{i,j,x})}{\beta_i} - 1 \right)_+^2 \right)^{1/2}.$$

Then $\sup_{x\in B} \|E_{j,x}\|_{L^2(k_{j,x})} \to 0$. The volumes converge uniformly to the sum of the piece volumes, and the periods to the forms obtained by the corresponding direct sums and isotropic reductions.

The excess, volume and relative derivative conclusions also hold with preliminary surgeries on framed circles. At a circle surgery, the period comparison is made after reduction by the span of the Poincaré duals of the belt spheres; this reduction converges to the period before surgery. A final smooth conformal change restores all comass bounds, adds volume tending uniformly to zero, and preserves the period map and all its parameter derivatives.

Proof. At each stage all corrected forms vanish on a neighborhood of the metric-change region. The cutoff derivative is supported where the old metric is retained. Let k and k' be the metrics before and after one construction. If $d\psi_i = \eta_i$ on the tube, and $e_i = (\operatorname{comass}_k(\eta_i)/\beta_i - 1)_+$, then on the retained region the new normalized excess satisfies

$$e_i' \le e_i + \beta_i^{-1} |d(\chi\psi_i)|_k.$$

It is zero on the changed region and the cap. Put $E = (\sum_i e_i^2)^{1/2}$ and $E' = (\sum_i (e_i')^2)^{1/2}$. Minkowski's inequality gives

$$\|E'\|_{L^2(k')} \le \|E\|_{L^2(k)} + \left(\sum_i \beta_i^{-2} \|d(\chi\psi_i)\|_{L^2(k)}^2 \right)^{1/2}.$$

The ball estimates in the proof of Lemma 5.2 and (8.6) make the second term arbitrarily small, uniformly on each fixed compact family. There are finitely many stages, and the initial excess on every piece is zero. Successive choices of radii therefore give the asserted L^2 convergence.

In a circle tube $S^1 \times D^3$, write y for the coordinate in D^3. The fibre contraction gives a bounded primitive of every smooth closed two-form, since its restriction to the central circle is zero. For small $r > 0$, choose a smooth radial cutoff equal to one for $|y| \le r$ and to zero for $|y| \ge 2r$, with derivative $O(r^{-1})$. The region $|y| < 2r$ has volume $O(r^3)$, so the squared L^2 norm of the correction is $O(r)$. In the annular formula above, set $a = r/2$, replace q by a fixed circle metric and $d\vartheta^2$ by the unit round metric on S^2. Attach a fixed disk metric with that boundary circle, times the round sphere of radius a. The cap volume is $O(r^2)$ and the annular volume is $O(r^3)$; the relative derivative bounds are unchanged. Let g_x and $k_{r,x}$ denote the metrics before and after this circle surgery, and let K be the span of the Poincaré duals of the belt spheres on the new manifold. The identification (8.5), applied to the reverse surgery, identifies its reduced cohomology with the old cohomology. Zero extensions of a harmonic basis give $(P_{k_{r,x}})_K \preceq P_{g_x} + o(1)I$ by harmonic minimization. Lemma 6.1 gives convergence of this reduction.

The volume and period comparisons preceding Lemma 8.5 do not require comass bounds. They therefore apply before conformal rescaling. The initial data and scale functions are fixed first; the successive radii are then chosen uniformly on B. Lemma 8.5 supplies (8.7). The volumes are uniformly bounded. Apply Lemma 8.2 with an additional volume allowance $\delta_j \downarrow 0$. Its volume increase is at most

$$2\,\mathrm{Vol}(k_{j,x})^{1/2}\|E_{j,x}\|_2 + \|E_{j,x}\|_2^2 + \delta_j,$$

which tends uniformly to zero. The conformal factors can be chosen smooth in the parameter. The period maps before and after this change are identical, so all their parameter derivatives agree. □

8.4. **The simply connected construction.** Fix the data P_0, α_i, and β_i of Theorem 8.1. Assume that X is simply connected and Q is indefinite. Choose $a_1, \dots, a_m \in 2\Lambda$, W, C_0, and η_0 as in Lemma 8.3. The list is nonempty, since $\dim \mathcal{D}(Q) = b_+ b_- > 0$. Put

$$B_j = 1 + \max_i \frac{|Q(\alpha_i, a_j)|}{\beta_i}, \tag{8.8}$$

with $B_j = 1$ for an empty constraint list, and set $C = 2C_0 \max_j B_j^2$.

Let g_0, $M = P_{g_0} \in W$, and $P \in W$ satisfy the hypotheses of Theorem 8.1, and fix $\varepsilon > 0$. Choose $0 < \eta < \eta_0$ with $C\eta < \varepsilon/3$, and choose $\tau > 0$ such that

$$\big((1+\tau)^2 - 1\big)\,\mathrm{Vol}(X, g_0) < \varepsilon/3.$$

By the definition of cohomological comass, there are smooth closed representatives η_i of α_i with $\mathrm{comass}_{g_0}(\eta_i) \le (1+\tau)\beta_i$. For $g = (1+\tau)g_0$, one has

$$\mathrm{comass}_g(\eta_i) \le \beta_i, \qquad P_g = M, \qquad \mathrm{Vol}(X, g) < \mathrm{Vol}(X, g_0) + \varepsilon/3.$$

Put $p = \theta(P)$ and $\delta = |p - \theta(M)|$. Lemma 8.3 gives positive scales at P whose sum is at most $C_0(\delta + \eta)$, and a smooth positive extension to neighboring targets. Choose $r > 0$ such that this extension is defined on a neighborhood of $\overline{B_{2r}(p)} \subset \theta(W)$. Write it as $s(x) = (s_1(x), \dots, s_m(x))$, so that

$$F_M(s(x)) = \theta^{-1}(x).$$

After decreasing r, continuity and $\sum_j s_j(p) > 0$ give

$$\sum_j s_j(x) \le 2 \sum_j s_j(p) \le 2C_0(\delta + \eta) \qquad \big(x \in \overline{B_{2r}(p)}\big).$$

On each copy of $S^2 \times S^2$, fix metrics of area one on the two factors. Let σ_j and ω_j be their area forms, pulled back to the product, and put $e_j = [\sigma_j]$, $f_j = [\omega_j]$. Rescale the factor metrics to areas

$s_j(x)B_j$ and B_j. In the basis (e_j, f_j) the period is $\operatorname{diag}(s_j(x)^{-1}, s_j(x))$; the pointwise comass of ω_j is B_j^{-1}, and the volume is $s_j(x)B_j^2$. For the isometry T of (6.1), the forms

$$\eta_i \quad \text{on } X, \qquad -Q(\alpha_i, a_j)\omega_j \quad \text{on the } j\text{th product} \tag{8.9}$$

represent $T\alpha_i$ under the direct-sum identification. Their pointwise comasses are at most β_i, since $|Q(\alpha_i, a_j)|/B_j \le \beta_i$.

Put $Y = X\#m(S^2 \times S^2)$, with its usual cohomology marking, and write Q_Y for its intersection form. Choose points $p_j \in S^2$ and connected-sum balls disjoint from the standard spheres $S_j^0 = \{p_j\}\times S^2$. Then $\mathrm{PD}[S_j^0] = e_j$. The manifold $Z = X\#(m-1)(S^2 \times S^2)$ is closed, oriented and simply connected, with indefinite intersection form. Wall's theorem [75, Theorem 2, p. 131], applied to $Y = Z\#(S^2 \times S^2)$ and the integral isometry T, gives an orientation-preserving diffeomorphism $\varphi : Y \to Y$ with $\varphi^* = T^{-1}$. The disjoint spheres $S_j = \varphi(S_j^0)$ therefore have $\mathrm{PD}[S_j] = Te_j$. Give them the transported product framings.

With the product framing, surgery on S_j^0 replaces its summand by

$$(D^2 \times S^2) \cup_{S^1\times S^2} (S^1 \times D^3) = \partial(D^2 \times D^3) \cong S^4,$$

after rounding corners. Transport of the framings extends φ over the surgeries. Thus surgery on the S_j returns a manifold diffeomorphic to the original smooth X. Put $K = T\,\mathrm{span}\{e_1, \dots, e_m\}$. The quotient marking is

$$V \longrightarrow K^{\perp_{Q_Y}}/K, \qquad v \longmapsto [Tv].$$

Moreover, $Q_Y(T\alpha_i, Te_j) = Q_Y(\alpha_i, e_j) = 0$ for every i, j.

Apply the connected-sum and sphere constructions of Subsection 8.3 to these fixed embeddings and to (8.9). Lemmas 8.4 and 8.6 give smooth metric families $k_{n,x}$ on the fixed marked X and closed representatives of the classes α_i. Uniformly for $x \in \overline{B_{2r}(p)}$,

$$P_{k_{n,x}} \longrightarrow F_M(s(x)) = \theta^{-1}(x),$$

and, for some $\varepsilon_n \downarrow 0$,

$$\mathrm{Vol}(X, k_{n,x}) \le \mathrm{Vol}(X, g) + \sum_j s_j(x)B_j^2 + \varepsilon_n. \tag{8.10}$$

These are the metrics before conformal restoration. They satisfy (8.7) with one constant independent of n; the metric g and the smooth positive scale functions are fixed. Lemma 8.5 gives, for $q_n(x) = \theta(P_{k_{n,x}})$,

$$q_n \longrightarrow \mathrm{id} \quad \text{in } C^1(\overline{B_r(p)}).$$

For sufficiently large n,

$$|q_n(p) - p| < r/4, \qquad \sup_{x\in\overline{B_r(p)}} \|Dq_n(x) - I\| < 1/4.$$

Apply Lemma 4.6 to $u \mapsto q_n(p + u)$ with target p. It gives $x_n \in B_{r/2}(p)$ with $q_n(x_n) = p$ and $Dq_n(x_n)$ invertible. Since $q_n = \theta \circ (x \mapsto P_{k_{n,x}})$, the chain rule shows that the period differential at k_{n,x_n} is surjective. Thus k_{n,x_n} is regular.

Lemma 8.6 supplies a final conformal metric h_n with $P_{h_n} = P$ and $\|\alpha_i\|_{h_n}^* \le \beta_i$, whose volume exceeds that of k_{n,x_n} by a quantity tending to zero. It remains regular by Lemma 8.2. The choice of the scale functions and (8.10) give

$$\mathrm{Vol}(X, h_n) \le \mathrm{Vol}(X, g_0) + \varepsilon/3 + C(\delta + \eta) + o(1) \qquad (n \to \infty).$$

Since $C\eta < \varepsilon/3$, a sufficiently large n gives (8.2). The constant $C = 2C_0 \max_j B_j^2$ depends only on the fixed cohomological data and the period neighborhood.

8.5. **Arbitrary fundamental group.** Assume first that $b_+ b_- > 0$. Choose $a_1, \dots, a_m \in 2\Lambda$, W, C_0, and η_0 as in Lemma 8.3. Then $m \geq 1$. Define B_j by (8.8), with $B_j = 1$ when the constraint list is empty, and put $C = 2C_0 \max_j B_j^2$.

Choose disjoint framed embedded circles whose conjugacy classes normally generate $\pi_1(X)$. Surgery on these circles gives a closed oriented simply connected manifold Y [57, §3, Lemmas 2–3, p. 44, and p. 46]. Let $S_1, \dots, S_N$ be the belt spheres, with the framings of the inverse surgeries, and let A be the common complement of the circle tubes in X and the attached $D^2 \times S^2$ in Y. Put

$$V_Y = H^2(Y;\mathbb{R}), \qquad \Lambda_Y = \mathrm{im}\big(H^2(Y;\mathbb{Z}) \to V_Y\big), \qquad K = \mathrm{span}\{\mathrm{PD}[S_1], \dots, \mathrm{PD}[S_N]\}.$$

Write Q_Y for the intersection form on V_Y. Lemma 8.4, applied to the inverse surgeries, gives

$$V = K^{\perp_{Q_Y}}/K. \tag{8.11}$$

For $v \in K^{\perp_{Q_Y}}$, write $[v] \in V$ for its image. This identification preserves the intersection forms. No independence or nonvanishing of the belt classes is required.

Excision and the integral Thom isomorphism [39, §3.1, Corollary 4D.9 and Theorem 4D.10] give

$$H^3(Y, A;\mathbb{Z}) \simeq \bigoplus_{\ell=1}^{N} H^1(S^2;\mathbb{Z}) = 0.$$

Thus $H^2(Y;\mathbb{Z}) \to H^2(A;\mathbb{Z})$ is onto. On each original circle tube, restriction induces an isomorphism $H^1(S^1 \times D^3;\mathbb{Z}) \to H^1(S^1 \times S^2;\mathbb{Z})$, and $H^2(S^1 \times D^3;\mathbb{Z}) = 0$. Mayer–Vietoris [39, §3.1, pp. 203–204] therefore gives

$$H^2(X;\mathbb{Z}) \simeq \ker\left(H^2(A;\mathbb{Z}) \longrightarrow \bigoplus_{\ell=1}^{N} H^2(S^1 \times S^2;\mathbb{Z}) \right).$$

Every integral class on X consequently has an integral lift on Y with zero periods on the belt spheres. Choose an integral representative of each $a_j/2$, lift it, and double the lift. Its real image $\widetilde{a}_j$ satisfies

$$\widetilde{a}_j \in 2\Lambda_Y \cap K^{\perp_{Q_Y}}, \qquad [\widetilde{a}_j] = a_j, \qquad Q_Y(u,v) = Q([u],[v]) \quad (u, v \in K^{\perp_{Q_Y}}).$$

Fix the framings and the radial identifications with Y as the circle radii vary. The lifts $\widetilde{a}_j$ are then fixed.

Choose points $p_j \in S^2$ and connected-sum balls disjoint from the belt spheres and from the standard factor spheres $\{p_j\} \times S^2$. Put $\widehat{Y} = Y \# m(S^2 \times S^2)$ and denote its intersection form by $\widehat{Q}$. Retain the hyperbolic pairs (e_j, f_j) of Subsection 8.4. Apply (6.1) on $V_Y \oplus \bigoplus_j (\mathbb{R}e_j \oplus \mathbb{R}f_j)$ with $\widetilde{a}_j$ in place of a_j, and denote the resulting integral isometry by T. Since $\widetilde{a}_j \in K^{\perp_{Q_Y}}$, $T|_K = I$. Put

$$E = \mathrm{span}\{e_1, \dots, e_m\}, \qquad K' = K + TE.$$

The space K' is isotropic. Its marked quotient is

$$V \longrightarrow (K')^{\perp_{\widehat{Q}}}/K', \qquad [v] \longmapsto [Tv] \quad (v \in K^{\perp_{Q_Y}}).$$

The form Q_Y is indefinite because its quotient in (8.11) is Q. Hence $Y\#(m-1)(S^2 \times S^2)$ is closed, oriented and simply connected, with indefinite intersection form. Wall's theorem [75, Theorem 2, p. 131] gives an orientation-preserving diffeomorphism $\varphi : \widehat{Y} \to \widehat{Y}$ with $\varphi^* = T^{-1}$. Transport all these spheres and their framings by φ. Their dual classes span K'. The extension over the surgery caps described in Subsection 8.4 applies to this entire framed collection. The inverse circle surgeries and the product surgeries therefore return the original smooth X, with the displayed quotient marking.

Let g_0, $M = P_{g_0} \in W$, $P \in W$, and $\varepsilon > 0$ be as in Theorem 8.1. The initial rescaling in Subsection 8.4 gives a constant multiple g of g_0 and smooth closed representatives η_i of α_i such that

$$P_g = M, \qquad \mathrm{comass}_g(\eta_i) \leq \beta_i, \qquad \mathrm{Vol}(X, g) < \mathrm{Vol}(X, g_0) + \varepsilon/5.$$

Put $\delta = |\theta(P) - \theta(M)|$, choose $0 < \eta < \eta_0$ with $C\eta < \varepsilon/5$, and set

$$L = \mathrm{Vol}(X, g) + C(\delta + 1 + \eta) + 2.$$

Apply the circle construction of Lemma 8.6 to g and the forms η_i. For a circle radius $\rho > 0$, denote the resulting metric and closed forms on the fixed marked Y by $\widetilde{g}_\rho$ and $\widetilde{\eta}_{i,\rho}$. Their classes lie in $K^{\perp Q_Y}$ and have images α_i in (8.11). Put

$$R_\rho = (P_{\widetilde{g}_\rho})_K, \qquad e_\rho = \left\| \left(\sum_i \left(\frac{\mathrm{comass}_{\widetilde{g}_\rho}(\widetilde{\eta}_{i,\rho})}{\beta_i} - 1 \right)_+^2 \right)^{1/2} \right\|_{L^2(\widetilde{g}_\rho)}.$$

That lemma gives

$$R_\rho \longrightarrow M, \qquad \mathrm{Vol}(Y, \widetilde{g}_\rho) \longrightarrow \mathrm{Vol}(X, g), \qquad e_\rho \longrightarrow 0 \qquad (\rho \downarrow 0).$$

Choose and fix ρ so that, with $R = R_\rho$ and $\kappa = |\theta(R) - \theta(M)|$,

$$R \in W, \qquad \kappa < 1, \qquad C\kappa < \varepsilon/10,$$
$$\mathrm{Vol}(Y, \widetilde{g}_\rho) < \mathrm{Vol}(X, g) + \min\{1, \varepsilon/10\}, \qquad 2L^{1/2} e_\rho + e_\rho^2 < \varepsilon/5.$$

Write $\widetilde{g} = \widetilde{g}_\rho$, $\widetilde{\eta}_i = \widetilde{\eta}_{i,\rho}$, and $\widetilde{\alpha}_i = [\widetilde{\eta}_i] \in V_Y$.

Put $p = \theta(P)$ and $\delta_R = |p - \theta(R)| \le \delta + \kappa$. Lemma 8.3 and the choice of the parameter ball in Subsection 8.4 give smooth positive scales on a neighborhood of $\overline{B_{2r}(p)} \subset \theta(W)$, for some $r > 0$, with

$$F_R(s(x)) = \theta^{-1}(x), \qquad \sum_j s_j(x) \le 2C_0(\delta_R + \eta).$$

Give the added products the metrics and normalized forms ω_j of Subsection 8.4, with factor areas $s_j(x)B_j$ and B_j. The forms

$$\widetilde{\eta}_i \quad \text{on } Y, \qquad -Q(\alpha_i, a_j)\omega_j \quad \text{on the } j\text{th product}$$

represent $T\widetilde{\alpha}_i$, since $Q_Y(\widetilde{\alpha}_i, \widetilde{a}_j) = Q(\alpha_i, a_j)$. They have zero periods on all the transported spheres. The product forms satisfy the bounds β_i, and the jth product has volume $s_j(x)B_j^2$.

The isometry T induces the original transvection on the quotient by K. Associativity of reduction in Lemma 6.1 gives

$$\left(P_{\widetilde{g}} \oplus \bigoplus_j \mathrm{diag}(s_j(x)^{-1}, s_j(x)) \right)_{K'} = F_R(s(x)).$$

Apply the connected-sum and sphere constructions of Subsection 8.3 to this fixed framed collection. They give smooth metrics $k_{n,x}$ on the fixed marked X and closed representatives of α_i. Uniformly on $\overline{B_{2r}(p)}$,

$$P_{k_{n,x}} \longrightarrow \theta^{-1}(x),$$
$$\mathrm{Vol}(X, k_{n,x}) \le \mathrm{Vol}(Y, \widetilde{g}) + C(\delta_R + \eta) + o(1),$$
$$\|E_{n,x}\|_{L^2(k_{n,x})} \le e_\rho + o(1).$$

Here $E_{n,x}$ is the normalized excess defined in Lemma 8.6. The last inequality follows from the successive cutoff inequality in its proof; the initial excess is e_ρ on Y and zero on each product. For all sufficiently large n, the volume bound is less than L.

The fixed metric $\widetilde{g}$ and scale family give (8.7) for $k_{n,x}$, with a constant independent of n. Lemma 8.5 therefore implies that $q_n(x) = \theta(P_{k_{n,x}})$ converges to the identity in $C^1(\overline{B_r(p)})$. Lemma 4.6, applied as in Subsection 8.4, gives $x_n \in B_{r/2}(p)$ with $q_n(x_n) = p$ and $Dq_n(x_n)$ invertible. The corresponding metrics are regular. Lemma 8.2 gives conformal metrics h_n with

$$P_{h_n} = P, \qquad \|\alpha_i\|_{h_n}^* \le \beta_i,$$

still regular, and

$$\begin{aligned}\operatorname{Vol}(X,h_n) &\le \operatorname{Vol}(Y,\widetilde{g}) + C(\delta_R+\eta) + 2L^{1/2}e_\rho + e_\rho^2 + o(1)\\ &< \operatorname{Vol}(X,g_0) + C\delta + 4\varepsilon/5 + o(1).\end{aligned}$$

Taking n sufficiently large proves (8.2). The constant $C = 2C_0 \max_j B_j^2$ depends only on the original cohomological data and the period neighborhood.

If $b_+ b_- = 0$, the period domain is a point. The initial metric is regular and already satisfies (8.2). This proves Theorem 8.1.

9. Finite comass constraints at a prescribed Hodge period

Retain X, V, Q and $\mathcal{D}(Q)$ from Section 8. Fix $P \in \mathcal{D}(Q)$, real classes $\alpha_1,\dots,\alpha_s \in V$, and positive bounds β_i. Put $d = \dim V$. In this section, $V_\pm$ are the ±1 eigenspaces of $Q^{-1}P$. For $v \in V$, write $v = v^+ + v^-$ with $v^\pm \in V_\pm$. Put $\beta = (\beta_1,\dots,\beta_s)$ and write

$$Q_\alpha = (Q(\alpha_i,\alpha_j))_{i,j=1}^s, \qquad P_\alpha = (P(\alpha_i,\alpha_j))_{i,j=1}^s.$$

On an oriented Euclidean four-space, identify the self-dual and anti-self-dual two-forms isometrically with two copies of $\mathbb{R}^3$. Write $|\cdot|$ for their Euclidean norm. Let $e_1,\dots,e_s$ be the standard basis of $\mathbb{R}^s$. Define

$$\boldsymbol{\mathcal{K}}_\beta = \{(A,B) \in \operatorname{Hom}(\mathbb{R}^s,\mathbb{R}^3)^2 \,:\, |Ae_i| + |Be_i| \le \sqrt{2}\,\beta_i \ (1 \le i \le s)\}.$$

This set is compact. For real symmetric $s \times s$ matrices S, T, write $S \succeq T$ if $c^t(S-T)c \ge 0$ for every $c \in \mathbb{R}^s$, and put

$$\begin{aligned}\Gamma(S,T;\beta) = \inf\{\nu(\boldsymbol{\mathcal{K}}_\beta) \,:\, &\nu \ge 0 \text{ is a finite Borel measure on } \boldsymbol{\mathcal{K}}_\beta,\\ &\int (A^tA - B^tB)\,d\nu = S,\\ &\int (A^tA + B^tB)\,d\nu \succeq T\}.\end{aligned} \tag{9.1}$$

Theorem 9.1 (Exact finite moment formula). *For every closed connected oriented smooth four-manifold, every compatible P, and every finite real marked list with positive bounds,*

$$\inf_{\substack{P_g = P\\ \|\alpha_i\|_g^* \le \beta_i \ (1\le i\le s)}} \operatorname{Vol}(X,g) = \Gamma(Q_\alpha, P_\alpha;\beta). \tag{9.2}$$

The infimum is unchanged if the metrics are required to be regular. Dependent and zero classes, and the empty list, are allowed. The value depends on the full prescribed period only through P_α.

There is a minimizing measure supported on at most $1 + s(s+1)$ points. For exactly specified real algebraic entries of $Q_\alpha, P_\alpha, \beta$, the value is a computable real algebraic number.

9.1. **Necessity and finite measures.** Let η be a two-form with Hodge components (a,b). For an oriented orthonormal coframe $(e^1,\dots,e^4)$, put $e^{ij} = e^i \wedge e^j$ and $dv = e^1 \wedge \cdots \wedge e^4$. Choose the coframe so that $\eta = xe^{12} + ye^{34}$. Then

$$\|\eta\|_{\mathrm{comass}} = \frac{|a| + |b|}{\sqrt{2}}, \qquad \eta \wedge \eta = (|a|^2 - |b|^2)\,dv, \qquad |\eta|^2 = |a|^2 + |b|^2. \tag{9.3}$$

Indeed, the comass is $\max(|x|,|y|)$, while the Hodge component norms are $|x+y|/\sqrt{2}$ and $|x-y|/\sqrt{2}$. In particular,

$$|\eta|^2 \le 2\|\eta\|_{\mathrm{comass}}^2. \tag{9.4}$$

Let g satisfy the constraints on the left of (9.2), and put $v = \operatorname{Vol}(X,g)$. For $\delta > 0$, choose smooth closed representatives $\eta_{i,\delta}$ of α_i with pointwise comass at most $(1+\delta)\beta_i$. Let ω_i be the g-harmonic representative of α_i. Hodge decomposition [77, Theorems 6.8 and 6.11, pp. 223–226] gives

$$\left(\int_X \eta_{i,\delta} \wedge *_g\, \eta_{j,\delta}\right)_{ij} = P_\alpha + \left(\langle \eta_{i,\delta} - \omega_i, \eta_{j,\delta} - \omega_j\rangle_{L^2}\right)_{ij} \succeq P_\alpha.$$

Their wedge matrix is Q_α.

Choose Borel orthonormal frames of the self-dual and anti-self-dual two-form bundles by partitioning a finite smooth trivializing cover into Borel sets. In these frames, let the columns of $A_\delta(x)$ and $B_\delta(x)$ be the Hodge components of $\eta_{i,\delta}(x)$. Let ν_δ be the pushforward of dv_g by

$$x \longmapsto \frac{(A_\delta(x), B_\delta(x))}{1+\delta}.$$

Equation (9.3) and polarization give

$$\operatorname{supp} \nu_\delta \subset \mathcal{K}_\beta, \qquad \nu_\delta(\mathcal{K}_\beta) = v,$$
$$\int (A^tA - B^tB)\, d\nu_\delta = (1+\delta)^{-2} Q_\alpha, \qquad \int (A^tA + B^tB)\, d\nu_\delta \succeq (1+\delta)^{-2} P_\alpha.$$

Write $C(\mathcal{K}_\beta)$ for the real continuous functions on $\mathcal{K}_\beta$, with the supremum norm. Positive measures of mass v on the compact metric space $\mathcal{K}_\beta$ are sequentially compact for the weak-$*$ topology of $C(\mathcal{K}_\beta)^*$. Taking $\delta = 1/n$, $n \geq 1$, and passing to a subsequence gives a limit ν. Since 1 and the moment entries are continuous and the positive-semidefinite cone is closed, ν has mass v and satisfies the moment conditions in (9.1) with $S = Q_\alpha$ and $T = P_\alpha$. Thus

$$\Gamma(Q_\alpha, P_\alpha; \beta) \leq \operatorname{Vol}(X, g). \tag{9.5}$$

Lemma 9.2. *For the matrices Q_α, P_α fixed above, the constraints in* (9.1) *are feasible. A minimizing measure exists with at most $1 + s(s+1)$ atoms. For unit bounds,*

$$\Gamma(Q_\alpha, P_\alpha; 1) \leq \tfrac{1}{2} \operatorname{tr} P_\alpha. \tag{9.6}$$

Proof. The matrices $C_\pm = (P_\alpha \pm Q_\alpha)/2$ are positive semidefinite: they are the Gram matrices of the projections $\alpha_i^\pm$ in (V_+, Q) and $(V_-, -Q)$. Write each $C_\pm$ as a finite sum $\sum cc^t$. For a nonzero c, choose $\lambda > 0$ with $|c_i|/\sqrt{\lambda} \leq \sqrt{2}\,\beta_i$ for all i, and take a pure plus or pure minus atom with one row $c^t/\sqrt{\lambda}$. Its weight is λ. These atoms give wedge matrix Q_α and energy matrix P_α exactly. For unit bounds take $\lambda = |c|^2/2$, proving (9.6).

Choose a minimizing sequence of feasible measures with bounded mass. Weak-$*$ compactness in $C(\mathcal{K}_\beta)^*$ gives a convergent subsequence. The moment entries are continuous and the cone of positive-semidefinite matrices is closed, so the limit is feasible. Continuity of the mass makes this limit a minimizing measure. If its mass is zero, the zero measure suffices.

Otherwise, identify each symmetric matrix with the vector of its upper triangular entries and define $\Phi : \mathcal{K}_\beta \to \mathbb{R}^{1+s(s+1)}$ by

$$\Phi(A, B) = (1, A^tA - B^tB, A^tA + B^tB).$$

The set $\Phi(\mathcal{K}_\beta)$ has compact convex hull [61, Theorem 17.2, p. 158]. The normalized integral of Φ belongs to this convex hull; hence its integral belongs to the cone generated by $\Phi(\mathcal{K}_\beta)$. Conical Carathéodory [61, Corollary 17.1.2, p. 156] gives a positive combination of at most $1 + s(s+1)$ such vectors, preserving the mass and both moments. □

9.2. **Independent representatives and energy matrices.** Equip V_+ and V_- with the inner products Q and $-Q$, respectively. Let $T^4 = \mathbb{R}^4/\mathbb{Z}^4$ have its Euclidean quotient metric and standard orientation, and write $e^r = dx^r$ for its parallel orthonormal coframe. Put $e^{pq} = e^p \wedge e^q$ for $p < q$ and $dv = e^1 \wedge \cdots \wedge e^4$. For a finite family of copies of T^4 with positive weights λ_l, integrals mean the sums of the integrals on the components multiplied by λ_l. The total mass is $\sum_l \lambda_l$.

Lemma 9.3. *Let finitely many atoms $(A_l, B_l) \in \mathcal{K}_\beta$, with positive weights λ_l, satisfy*

$$\sum_l \lambda_l (A_l^tA_l - B_l^tB_l) = Q_\alpha, \qquad E := \sum_l \lambda_l (A_l^tA_l + B_l^tB_l) \succeq P_\alpha.$$

Fix a basis $v_1, \ldots, v_d$ of V. For every $\varepsilon > 0$ there is a finite weighted family of copies of T^4, of total mass at most $\sum_l \lambda_l + \varepsilon$, carrying constant two-forms η_i, h_a with the following properties:

(1) $\|\eta_i\|_{\mathrm{comass}} \le \beta_i$ *on each torus;*
(2) the weighted wedge matrix of the full list (η_i, h_a) *is the cup matrix of* (α_i, v_a)*;*
(3) the weighted energy matrix of (h_a) *is exactly* $(P(v_a, v_b))_{a,b=1}^{d}$.
There is no comass bound on the forms h_a.

Proof. On the lth original torus, let η_i have Hodge components $(A_l e_i, B_l e_i)$. Let $\mathcal{H}_+ = \bigoplus_l \mathbb{R}^3$ and $u_i^+ = (\sqrt{\lambda_l} A_l e_i)_l$; define $\mathcal{H}_-$ and u_i^- similarly, with B_l in place of A_l. Then

$$\mathrm{Gram}(u_i^{\pm}) = \tfrac{1}{2}(E \pm Q_\alpha) \succeq \tfrac{1}{2}(P_\alpha \pm Q_\alpha) = \mathrm{Gram}(\alpha_i^{\pm}).$$

Douglas's factorization theorem [26, Theorem 1 and the added remark on p. 415] gives contractions $T_\pm : \mathcal{H}_\pm \to V_\pm$ such that $T_\pm u_i^\pm = \alpha_i^\pm$. Write $I_\pm$ for the identity on $V_\pm$. Define

$$J_\pm : V_\pm \longrightarrow \mathcal{H}_\pm \oplus V_\pm, \qquad J_\pm v = (T_\pm^* v, (I_\pm - T_\pm T_\pm^*)^{1/2} v).$$

Since $J_\pm^* J_\pm = T_\pm T_\pm^* + (I_\pm - T_\pm T_\pm^*) = I_\pm$, these maps are isometric. Moreover,

$$\langle (u_i^\pm, 0), J_\pm v \rangle = \langle \alpha_i^\pm, v \rangle. \tag{9.7}$$

Take $J_+ v_a^+$ and $J_- v_a^-$ as the vectors of weighted Hodge components of h_a. On the original tori, divide their components by $\sqrt{\lambda_l}$. Choose $r \ge 0$ and isometric embeddings $V_\pm \hookrightarrow (\mathbb{R}^3)^r$ with the same r for both signs. Give these r additional tori positive weights of total at most ε, put $\eta_i = 0$ there, and divide each component of h_a by the square root of its torus's weight. These components determine constant two-forms on each copy of T^4.

The isometries $J_\pm$ and (9.7) give

$$\int h_a \wedge h_b = Q(v_a, v_b), \qquad \int h_a \wedge * h_b = P(v_a, v_b), \qquad \int \eta_i \wedge h_a = Q(\alpha_i, v_a).$$

The original tori give the $\eta_i \eta_j$ wedge matrix Q_α. The column bounds in $\mathcal{K}_\beta$ give the required comass bounds by (9.3). □

Writing $\alpha_i = \sum_a c_{ia} v_a$, the weighted wedge pairing of $\eta_i - \sum_a c_{ia} h_a$ with every marked form is zero. Its cohomology class on the disjoint union of the tori need not vanish.

9.3. **Rational forms and transfer of energy**.

Lemma 9.4. *For every rational symmetric* $k \times k$ *matrix* D*, there is a finite family of copies of* T^4 *with positive rational weights and constant two-forms* $\zeta_1, \dots, \zeta_k$*, with rational coefficients in the basis* $(e^{pq})_{p<q}$*, whose weighted wedge matrix is* D *and whose comasses are at most one. The total mass is at most* $\sum_{i \le j} |D_{ij}|$*, and the weighted energy trace is at most twice this number.*

Proof. Omit zero entries. For $D_{ii} \neq 0$, take $\zeta_i = e^{12} + \mathrm{sgn}(D_{ii}) e^{34}$, with weight $|D_{ii}|/2$, and set the other forms to zero. Its wedge square is $2\,\mathrm{sgn}(D_{ii})\, dv$ and its squared norm is two. For $D_{ij} \neq 0$, $i < j$, take $\zeta_i = e^{12}$ and $\zeta_j = \mathrm{sgn}(D_{ij}) e^{34}$, with weight $|D_{ij}|$, and set the other forms to zero. Their wedge squares vanish, their wedge product is $\mathrm{sgn}(D_{ij})\, dv$, and both squared norms are one. The displayed nonzero forms have comass one. The total mass and energy trace are therefore

$$\frac{1}{2} \sum_i |D_{ii}| + \sum_{i<j} |D_{ij}|, \qquad \sum_i |D_{ii}| + 2 \sum_{i<j} |D_{ij}|,$$

respectively. Each energy matrix is positive semidefinite. □

Lemma 9.5. *Assume* $\beta_i = 1$ *and take* (v_a) *to be a basis of* $H^2(X; \mathbb{Q})$*. From the data of Lemma 9.3 one obtains connected closed oriented smooth Riemannian four-manifolds* (N_j, g_j)*, smooth closed forms* $(\eta_i^{(j)}, h_a^{(j)})$ *with rational cohomology classes, and rational classes* $\alpha_i^{(j)} \to \alpha_i$ *on* X*, such that their full wedge matrices are exactly the cup matrices of* $(\alpha_i^{(j)}, v_a)$*, while*

$$\|\eta_i^{(j)}\|_{\infty, g_j} \le 1 + o(1), \qquad \limsup_j \mathrm{Vol}(N_j, g_j) \le \sum_l \lambda_l, \qquad \left(\int_{N_j} h_a^{(j)} \wedge *_{g_j} h_b^{(j)} \right)_{ab} \longrightarrow (P(v_a, v_b))_{ab}.$$

Proof. Choose $\varepsilon_j \downarrow 0$, with $0 < \varepsilon_j < 1$, and apply Lemma 9.3 with additional mass at most ε_j. For each j, the family of tori is finite. Replace each η_i by $(1-\varepsilon_j)\eta_i$. Approximate the positive weights and the coefficients of every form in the coordinate basis $(e^{pq})_{p<q}$ by rational numbers. Also approximate the coordinates of α_i in (v_a) by rationals with error at most ε_j in each coordinate, obtaining $\alpha_i^{(j)}$. Choose these approximations so that all constrained comasses remain at most one and the errors in the total mass, full wedge matrix, and auxiliary energy matrix are at most ε_j, with matrix errors measured in operator norm. This is possible by continuity in the finitely many coefficients and weights, which are fixed before the approximation is chosen. The total masses then have upper limit at most $\sum_l \lambda_l$, the auxiliary energy matrices tend to $(P(v_a, v_b))_{ab}$, and the full wedge matrices tend to the cup matrix of (α_i, v_a).

Let D_j be the cup matrix of $(\alpha_i^{(j)}, v_a)$ minus the wedge matrix of the approximating forms. Its entries are rational and tend to zero. Apply Lemma 9.4 to these $s+d$ marks. Every wedge pairing is then exact. The additional mass is at most $\sum_{\mu\le\nu} |(D_j)_{\mu\nu}|$, and the added energy matrix is positive semidefinite with trace at most twice this sum. It therefore tends to zero in operator norm. All constrained comasses remain at most one. If the family is empty, append a torus with all marked forms zero and positive rational weight at most $1/j$.

For each positive rational weight μ, replace its unit torus by

$$T_\mu = \mathbb{R}^4/(\mu\mathbb{Z} \oplus \mathbb{Z}^3)$$

with the Euclidean quotient metric and the same constant forms, as in the proof of Theorem 1.1. Their periods are rational. Their pointwise norms and comasses are unchanged, while volume and all integrated wedge and energy pairings are multiplied by μ. Thus the disjoint union has the prescribed weighted pairings and energy matrices exactly.

For fixed j, apply Lemma 5.2 to the finite family of tori, with a one-point parameter space. Denote its metric and constrained forms by k_ρ and $\eta_{i,\rho}$, where ρ is the attachment radius. The cohomology classes remain rational, every wedge pairing is unchanged, and the volume and auxiliary Gram matrix tend to those of the disjoint union. Put

$$e_{i,\rho} = (\operatorname{comass}_{k_\rho}(\eta_{i,\rho}) - 1)_+.$$

The cutoff estimates in the proof of that lemma bound the modified forms uniformly on the attachment balls. They are unchanged outside those balls and vanish wherever the metric changes. Hence

$$\int \sum_i e_{i,\rho}^2 \, dv_{k_\rho} = O(\rho^4),$$

with a constant depending on j. Lemma 8.2 gives unit comass bounds with a volume increase tending to zero as $\rho \downarrow 0$ and leaves the auxiliary energy matrix unchanged. For each j, choose the radius and the smoothing error so that the volume and auxiliary Gram errors, including the conformal volume increase, are less than $1/j$. This gives the required sequence. □

Lemma 9.6. *Let (N, h) be a connected closed oriented smooth Riemannian four-manifold carrying smooth closed two-forms (η_i, h_a) with rational cohomology classes. Suppose their full wedge matrix equals the cup matrix of (α_i', v_a) on X, where $\alpha_i' \in H^2(X;\mathbb{Q})$ and (v_a) is a rational basis. Put*

$$E = \left(\int_N h_a \wedge *_h \, h_b \right)_{a,b=1}^{d},$$

and let I_d be the identity matrix of size d. For every $\delta > 0$, there are a smooth metric g on X, smooth closed representatives ξ_i of α_i', and smooth closed representatives θ_a of v_a, such that

$$\|\xi_i\|_{\infty,g} \le \|\eta_i\|_{\infty,h} + \delta, \qquad \operatorname{Vol}(X,g) \le \operatorname{Vol}(N,h) + \delta,$$

and

$$(P_g(v_a, v_b))_{ab} \preceq \left(\int_X \theta_a \wedge *_g \, \theta_b \right)_{ab} \preceq E + \delta I_d.$$

Proof. Apply Proposition 1.9 to the complete rational list, with bounds $\|\eta_i\|_{\infty,h} + \delta$ and $\|h_a\|_{\infty,h} + 1$, and specify (h_a) as the sublist whose Gram matrix is controlled. These bounds are positive. In dimension four, all top products of a degree-two list are wedge pairings, so the proposition applies without an independence hypothesis. It gives the volume and comass bounds and the upper Gram inequality. For every $c \in \mathbb{R}^d$, harmonic minimization gives

$$P_g\Big[\sum_a c_a v_a\Big] \le \Big\|\sum_a c_a \theta_a\Big\|^2_{L^2(g)}.$$

This proves the first matrix inequality. □

9.4. **Proof of the exact formula.** If every α_i is zero, including the empty list, the zero measure gives $\Gamma(Q_\alpha, P_\alpha; \beta) = 0$. Corollary 7.6 gives regular metrics with period P and arbitrarily small positive volume. Thus (9.2) holds in this case.

Assume that some α_i is nonzero. Put $D = \operatorname{diag}(\beta_1, \dots, \beta_s)$ and $\widetilde{\alpha}_i = \alpha_i/\beta_i$. The map $(A, B) \mapsto (AD^{-1}, BD^{-1})$ identifies the feasible measures for $(Q_\alpha, P_\alpha; \beta)$ with those for $(Q_{\widetilde{\alpha}}, P_{\widetilde{\alpha}}; 1)$ and preserves their masses. Since $\|\widetilde{\alpha}_i\|^*_g = \|\alpha_i\|^*_g/\beta_i$, it suffices to assume $\beta_i = 1$.

Fix a basis $v_1, \dots, v_d$ of $H^2(X; \mathbb{Q})$, viewed as a real basis of V. Identify bilinear forms on V with their matrices in this basis; let I_d be the identity matrix and $\|\cdot\|_{\mathrm{op}}$ the corresponding operator norm. By Lemma 9.2, choose a finitely supported minimizing measure, and put $v = \Gamma(Q_\alpha, P_\alpha; 1)$. Lemmas 9.3, 9.5, and 9.6 give smooth metrics g_j, smooth closed forms $\xi_i^{(j)}, h_a^{(j)}$, rational classes $\alpha_i^{(j)} \to \alpha_i$, and numbers $\varepsilon_j > 0$ tending to zero, such that

$$[\xi_i^{(j)}] = \alpha_i^{(j)}, \qquad [h_a^{(j)}] = v_a, \qquad \|\xi_i^{(j)}\|_{\infty,g_j} \le 1 + \varepsilon_j.$$

Put

$$H_j = \left(\int_X h_a^{(j)} \wedge *_{g_j} h_b^{(j)}\right)_{a,b=1}^d.$$

These metrics satisfy $\limsup_j \operatorname{Vol}(X, g_j) \le v$ and $P_{g_j} \preceq H_j \preceq P + \varepsilon_j I_d$. Replace g_j by $(1 + \varepsilon_j)g_j$ and keep the same notation. The conformal identities in Lemma 8.2 leave P_{g_j} and H_j unchanged and multiply the volume by $(1 + \varepsilon_j)^2$. Thus

$$\limsup_j \operatorname{Vol}(X, g_j) \le v, \qquad \|\xi_i^{(j)}\|_{\infty,g_j} \le 1, \qquad P_{g_j} \preceq H_j \preceq P + \varepsilon_j I_d. \tag{9.8}$$

In particular, the matrices H_j are bounded. Since $P_{g_j}, P \in \mathcal{D}(Q)$, Lemma 6.1 gives $P_{g_j} \to P$.

Write

$$\alpha_i - \alpha_i^{(j)} = \sum_a c_{ia}^{(j)} v_a, \qquad c_i^{(j)} = (c_{i1}^{(j)}, \dots, c_{id}^{(j)})^t \longrightarrow 0,$$

and put

$$e_i^{(j)} = \sum_a c_{ia}^{(j)} h_a^{(j)}, \qquad \zeta_i^{(j)} = \xi_i^{(j)} + e_i^{(j)}, \qquad E_j = \left(\sum_i |e_i^{(j)}|^2_{g_j}\right)^{1/2}.$$

The forms $\zeta_i^{(j)}$ are closed representatives of α_i, and $\operatorname{comass}_{g_j}(\zeta_i^{(j)}) \le 1 + |e_i^{(j)}|_{g_j}$. Moreover,

$$\|E_j\|^2_{L^2(g_j)} = \sum_i (c_i^{(j)})^t H_j c_i^{(j)} \le \|H_j\|_{\mathrm{op}} \sum_i |c_i^{(j)}|^2 \longrightarrow 0.$$

Lemma 8.2 gives conformal metrics k_j with $\|\alpha_i\|^*_{k_j} \le 1$, $P_{k_j} = P_{g_j}$, and

$$\operatorname{Vol}(X, k_j) \le \operatorname{Vol}(X, g_j) + 2\operatorname{Vol}(X, g_j)^{1/2}\|E_j\|_{L^2(g_j)} + \|E_j\|^2_{L^2(g_j)} + 1/j.$$

Hence $\limsup_j \operatorname{Vol}(X, k_j) \le v$ and $P_{k_j} \to P$.

Apply Theorem 8.1 at P to the fixed list (α_i) with unit bounds. Let W and C be its neighborhood and constant. For all sufficiently large j, one has $P_{k_j} \in W$, so there is a smooth regular metric $\widehat{g}_j$ with $P_{\widehat{g}_j} = P$, $\|\alpha_i\|^*_{\widehat{g}_j} \le 1$, and

$$\operatorname{Vol}(X, \widehat{g}_j) \le \operatorname{Vol}(X, k_j) + C|\theta(P) - \theta(P_{k_j})| + 1/j.$$

Its upper limit is at most v. Together with (9.5), this proves (9.2), including the restriction to regular metrics. The support bound is Lemma 9.2.

For arbitrary positive bounds β_i, put $\gamma = \Gamma(Q_\alpha, P_\alpha; \beta)$ and $N = 1 + s(s+1)$. Let F be the ordered subfield of $\mathbb{R}$ generated by the entries of Q_α, P_α, and β. For $1 \le l \le N$, introduce weights $\lambda_l \ge 0$, matrices $A_l, B_l \in \mathbb{R}^{3\times s}$, and variables $r^+_{li}, r^-_{li} \ge 0$ satisfying

$$(r^+_{li})^2 = |A_l e_i|^2, \qquad (r^-_{li})^2 = |B_l e_i|^2, \qquad (r^+_{li} + r^-_{li})^2 \le 2\beta_i^2.$$

For each l, these polynomial conditions over F admit a solution in the $r^\pm_{li}$ if and only if $(A_l, B_l) \in \boldsymbol{\mathcal{K}}_\beta$. The wedge equalities are polynomial, and the energy inequality is expressed by nonnegativity of all principal minors of

$$\sum_l \lambda_l (A_l^t A_l + B_l^t B_l) - P_\alpha.$$

Zero weights allow fewer than N atoms. Since the minimum is attained, the existence of these variables with $\sum_l \lambda_l \le t$ is equivalent to $t \ge \gamma$. Quantifier elimination [14, Theorem 2.77] gives a quantifier-free formula $\Phi(t)$ over F defining $[\gamma, \infty)$. The singleton $\{\gamma\}$ is defined by

$$\Phi(t) \ \wedge \ \neg\exists u \left(u < t \ \wedge \ \Phi(u) \right).$$

For real algebraic data, effective quantifier elimination and algebraic sampling compute an exact real-algebraic representation of this point [14, Theorem 14.16 and Algorithm 13.2].

Corollary 9.7. *Every compatible majorant on $H^2(X;\mathbb{R})$ is the period of a smooth regular metric. The volume may be any prescribed positive number. More generally, the metric may have any prescribed smooth positive density on X.*

Proof. Apply Corollary 7.6 with $m = 2$ and a one-point parameter space. Its right inverse makes the resulting metric regular. On the oriented manifold X, a smooth positive density determines a positive volume form. Choosing its integral gives the prescribed total volume. □

Corollary 9.8. *For every $\alpha \in V$ and $\beta > 0$,*

$$\inf_{P_g = P,\ \|\alpha\|^*_g \le \beta} \operatorname{Vol}(X, g) = \frac{P(\alpha,\alpha)}{2\beta^2}. \tag{9.9}$$

Regular metrics give the same infimum. For $\alpha \ne 0$, a metric g with $P_g = P$ attains the infimum if and only if α has a closed representative which is g-self-dual or g-anti-self-dual and has constant g-comass β. In particular, the infimum is not attained when both Hodge components of α are nonzero.

Proof. Put $p = P(\alpha,\alpha)$ and $q = Q(\alpha,\alpha)$. Every admissible measure has energy at least p and at most $2\beta^2$ times its mass. Its mass is therefore at least $p/(2\beta^2)$. Pure plus and pure minus atoms of comass β, with weights

$$\frac{p+q}{4\beta^2}, \qquad \frac{p-q}{4\beta^2},$$

have that mass, wedge moment q, and energy p. The weights are nonnegative since $P \pm Q \succeq 0$; omit any zero weight. Theorem 9.1 gives the formula and its regular version.

Suppose $\alpha \ne 0$ and a metric g attains the infimum. Let ω be the g-harmonic representative of α, and choose smooth closed representatives η_j of α with $\|\eta_j\|_{\infty,g} \le \beta + 1/j$. Hodge orthogonality and (9.4) give

$$\|\eta_j - \omega\|^2_{L^2(g)} = \|\eta_j\|^2_{L^2(g)} - p \le 2(\beta + 1/j)^2 \operatorname{Vol}(X, g) - p \longrightarrow 0.$$

After passing to a subsequence, $\eta_j \to \omega$ almost everywhere. Thus $\operatorname{comass}_g(\omega) \le \beta$ almost everywhere, and hence everywhere by continuity. Since

$$\int_X |\omega|_g^2 \, dv_g = p = 2\beta^2 \operatorname{Vol}(X, g),$$

the continuous nonnegative function $2\beta^2 - |\omega|_g^2$ is zero. Put $\omega^{\pm} = (\omega \pm *_g \omega)/2$. Equation (9.3) gives $|\omega^+|_g |\omega^-|_g = 0$. The continuous function $|\omega^+|_g^2/(2\beta^2)$ therefore takes values in $\{0, 1\}$ and is constant on connected X. Consequently ω is self-dual or anti-self-dual and has constant comass β.

Conversely, a closed self-dual or anti-self-dual form is coclosed, hence harmonic. If its comass is constantly β, then (9.3) gives $p = 2\beta^2 \operatorname{Vol}(X, g)$, proving attainment. Such a form has only one nonzero Hodge component. For $\alpha = 0$, the infimum is zero and no metric attains it. □

Corollary 9.9. *For a fixed finite real marked list and positive bounds, the value in* (9.2) *is finite and locally Lipschitz as a function of the period. Restricting its infimum to regular metrics does not change it.*

Proof. Write $F(P)$ for this value. Theorem 9.1 gives finiteness and the equality of the regular and unrestricted infima. Fix $P_0 \in \mathcal{D}(Q)$, and let W, C be given by Theorem 8.1 for the fixed classes and bounds. For $M, P \in W$ and $\varepsilon > 0$, choose an admissible metric g of period M with $\operatorname{Vol}(X, g) < F(M) + \varepsilon$. Equation (8.2) gives

$$F(P) \le F(M) + C|\theta(P) - \theta(M)| + 2\varepsilon.$$

Let $\varepsilon \downarrow 0$ and interchange M, P. Then

$$|F(P) - F(M)| \le C|\theta(P) - \theta(M)|.$$

□

10. Stable norms at fixed period and volume

Retain X, V, Q and $\mathcal{D}(Q)$ from Section 8. Put $H = H_2(X; \mathbb{R})$ and identify $H^* = V$. Integral homology classes below are taken modulo torsion. Write $N_g = N_g^{(2)}$ for the stable two-norm of Section 1. For a norm N on H, put $B_N = \{h \in H : N(h) \le 1\}$ and

$$C_N = \{\alpha \in V : |\alpha(h)| \le N(h) \text{ for every } h \in H\}.$$

Thus C_N is the dual unit ball. For $P \in \mathcal{D}(Q)$, define

$$G_N(P) = \inf_{\substack{P_g = P \\ N_g \ge N}} \operatorname{Vol}(X, g). \tag{10.1}$$

The set of metrics in this infimum is nonempty: Corollary 9.7 supplies a metric of period P, and constant rescaling makes its stable norm dominate N. If $C_N = \operatorname{conv}\{\pm\alpha_1, \dots, \pm\alpha_s\}$, duality (1.1) gives

$$N_g \ge N \iff C_N \subset C_{N_g} \iff \|\alpha_i\|_g^* \le 1 \quad (1 \le i \le s).$$

Hence Theorem 9.1 gives

$$G_N(P) = \Gamma(Q_\alpha, P_\alpha; 1). \tag{10.2}$$

This value is independent of the finite presentation of C_N. The classes α_i may be real and redundant; they must span V to define a norm.

Equip the space of norms on H with uniform convergence on the unit ball of a fixed auxiliary norm. Convergence $N_j \to N$ is equivalent to

$$(1 - \delta_j)N \le N_j \le (1 + \delta_j)N, \qquad \delta_j \ge 0, \qquad \delta_j \longrightarrow 0.$$

All closures below are taken in the product of this topology and the usual topologies on $\mathcal{D}(Q)$ and $(0, \infty)$. These topologies are metrizable.

Theorem 10.1 (Joint norm–period–volume closure). *Suppose $b_2(X) > 0$. For every norm N on H, every $P \in \mathcal{D}(Q)$, and every $v > 0$, the following are equivalent:*

(1) $v \geq G_N(P)$;

(2) there are smooth regular metrics g_j satisfying

$$P_{g_j} = P, \qquad \mathrm{Vol}(X, g_j) = v, \qquad N_{g_j} \longrightarrow N \tag{10.3}$$

uniformly on bounded subsets of H;

(3) (N, P, v) belongs to the closure of $\{(N_g, P_g, \mathrm{Vol}(X, g)) : g \text{ smooth Riemannian}\}$.

The function $(N, P) \mapsto G_N(P)$ is positive, finite, and continuous. For a polyhedral dual ball it is given by (10.2).

Fix $s \geq \dim V$. For a spanning list $\alpha = (\alpha_1, \dots, \alpha_s) \in V^s$, put $N_\alpha(h) = \max_i |\alpha_i(h)|$. In a fixed rational basis of V, the function $(\alpha, P) \mapsto G_{N_\alpha}(P)$, defined for spanning lists $\alpha \in V^s$ and $P \in \mathcal{D}(Q)$, has a semialgebraic epigraph. If Q_α and P_α have real algebraic entries, then $G_{N_\alpha}(P)$ is computable and real algebraic.

When $b_2(X) = 0$, there is a unique norm on the zero vector space and a unique period. The same exact realization statement holds for every $v > 0$, with $G = 0$.

10.1. **Embedded surfaces and their neighborhoods.**

Lemma 10.2. *Given finitely many nonzero integral classes $h_1, \dots, h_m \in H$, there are closed connected oriented smoothly embedded representatives S_j, pairwise transverse and with no triple intersections. There are tubular neighborhoods U_j of S_j, with projections $\pi_j : U_j \to S_j$. Put $U = \bigcup_{j=1}^m U_j$. Restriction induces an isomorphism*

$$H^2(U; \mathbb{R}) \xrightarrow{\simeq} \bigoplus_{j=1}^m H^2(S_j; \mathbb{R}). \tag{10.4}$$

For each j, let σ_j be a smooth nonnegative two-form on S_j of integral one, vanishing near the crossing points. After the tubes are shrunk, every finite list $\alpha_1, \dots, \alpha_s \in V$ admits smooth closed two-forms κ_i on U such that

$$\kappa_i|_{U_j} = \alpha_i(h_j)\pi_j^* \sigma_j.$$

These forms vanish on pairwise overlaps. Fix an open neighborhood U' of $\bigcup_j S_j$ with $\overline{U'} \subset U$. For every smooth closed representative η_i of α_i on X, there is a smooth one-form ψ_i on U with $\eta_i - \kappa_i = d\psi_i$. Both ψ_i and $d\psi_i$ are bounded on U' in any fixed smooth metric on X.

Proof. Choose an integral lift of each h_j. Thom's realization theorem [73, Théorème II.27, p. 55] gives a closed oriented embedded representative of each lift. Join its components by oriented tubes along arcs whose interiors avoid the surface. Each replacement changes the cycle by the boundary of a three-chain, so the integral class is unchanged. Apply smooth transversality successively to the preceding surfaces and their crossing points [44, Chapter 3, §2, Theorem 2.1]. The perturbations can be chosen small enough to preserve the embeddings and their homology classes. Pairwise intersections are finite; transversality to their points excludes triple intersections, since $2 + 0 < 4$.

Choose disjoint product coordinate neighborhoods of the crossings. Shrink disk-bundle neighborhoods U_j so that each $U_i \cap U_j$ is a finite disjoint union of contractible crossing neighborhoods and there are no triple overlaps. Each U_j retracts to S_j. Put $U^{(k)} = \bigcup_{j=1}^k U_j$ for $1 \leq k \leq m$. For $2 \leq k \leq m$, the intersection $U^{(k-1)} \cap U_k$ has zero first and second cohomology. The cohomological Mayer–Vietoris sequence [39, §3.1, pp. 203–204] therefore makes restriction

$$H^2(U^{(k)}; \mathbb{R}) \longrightarrow H^2(U^{(k-1)}; \mathbb{R}) \oplus H^2(U_k; \mathbb{R})$$

an isomorphism. Induction gives (10.4).

Nonnegative two-forms of integral one supported away from the crossings exist on every S_j. For any such σ_j, shrink the tubes so that $\pi_j^*\sigma_j$ vanishes on every overlap. The prescribed pullbacks are closed and agree there, so they define smooth closed forms κ_i on U. Moreover,

$$\int_{S_j}(\eta_i-\kappa_i)=\alpha_i(h_j)-\alpha_i(h_j)\int_{S_j}\sigma_j=0.$$

Integration identifies $H^2(S_j;\mathbb{R})$ with $\mathbb{R}$. Thus (10.4) gives $[\eta_i-\kappa_i]=0$ on U. Choose a smooth primitive ψ_i there. Its restriction, and that of $d\psi_i$, to the compact set $\overline{U'}$ are bounded. □

Lemma 10.3. *Put $S=\bigcup_j S_j$ and fix a smooth metric g_0 on X. For all sufficiently small $\delta>0$, there is a smooth function $\chi_\delta : X\to[0,1]$ which equals one where $\operatorname{dist}_{g_0}(x,S)\le\delta^2$ and satisfies*

$$\operatorname{supp}\chi_\delta\subset\{x : \operatorname{dist}_{g_0}(x,S)<\delta\},\qquad \|\chi_\delta\|_2=O(\delta),\qquad \|d\chi_\delta\|_2=O(|\log\delta|^{-1/2}).$$

Here the norms are taken with respect to g_0. Let U' be a fixed neighborhood of S, and let ψ be a smooth one-form on U' with ψ and $d\psi$ bounded. Extending $\chi_\delta\psi$ by zero gives

$$\|d(\chi_\delta\psi)\|_2=O(\delta+|\log\delta|^{-1/2}).\tag{10.5}$$

For fixed S and U', the same cutoffs satisfy these estimates uniformly when the norms are taken in a compact smooth family of metrics and $\psi, d\psi$ have uniform bounds.

Proof. On the normal disk bundle of S_j defined by the g_0-exponential map, take the logarithmic profile preceding (8.6) with $\rho=\delta/\sqrt{2}$. In normal polar coordinates (r,ϑ), $r=\operatorname{dist}_{g_0}(\cdot,S_j)$ and the volume density is bounded by $Cr\,dr\,d\vartheta\,dA_{g_0|S_j}$. The same integral therefore gives the stated bounds for the resulting functions $\chi_{j,\delta}$, extended by zero to X. They equal one for $r\le\delta^2$ and have support in $r<\delta$. Put

$$\chi_\delta=1-\prod_j(1-\chi_{j,\delta}).$$

Then

$$0\le\chi_\delta\le\sum_j\chi_{j,\delta},\qquad |d\chi_\delta|\le\sum_j|d\chi_{j,\delta}|,$$

so the same L^2 bounds hold for χ_δ. For small δ, its support is compactly contained in U'; hence the extension of $\chi_\delta\psi$ is smooth. The product rule gives

$$\|d(\chi_\delta\psi)\|_2\le\left(\sup_{U'}|\psi|_{g_0}\right)\|d\chi_\delta\|_2+\left(\sup_{U'}|d\psi|_{g_0}\right)\|\chi_\delta\|_2,$$

which proves (10.5). Uniform equivalence of metrics on a compact smooth family gives the final assertion. □

10.2. **Prescribed surface areas in a conformal class.**

Lemma 10.4. *Let g be a smooth Riemannian metric on X and let $\eta_1,\dots,\eta_s$ be smooth closed two-forms with $\operatorname{comass}_g(\eta_i)\le1$. Put $\alpha_i=[\eta_i]\in V$. Let $h_1,\dots,h_m\in H$ be nonzero integral classes, and let $A_j>0$ satisfy $|\alpha_i(h_j)|\le A_j$ for every i,j. For every $\varepsilon>0$ there are a smooth metric $\widehat{g}$ conformal to g and closed oriented smoothly embedded surfaces S_j representing h_j such that*

$$\|\alpha_i\|_{\widehat{g}}^*\le1,\qquad \operatorname{Vol}(X,\widehat{g})\le\operatorname{Vol}(X,g)+\varepsilon,\qquad \operatorname{Area}_{\widehat{g}}(S_j)\le A_j+\varepsilon.$$

Moreover, $P_{\widehat{g}}=P_g$, and $\widehat{g}$ is regular if and only if g is regular.

Let B be a compact smooth parameter manifold. For smooth families $g_b,\eta_{i,b},A_{j,b}$ satisfying these hypotheses, the metrics $\widehat{g}_b$ may be chosen smooth in b, with the same surfaces S_j and the same ε for every $b\in B$.

Proof. If $m = 0$, take $\widehat{g} = g$. Suppose $m > 0$, and choose the surfaces of Lemma 10.2. Put $S = \bigcup_{j=1}^m S_j$ and $\rho_j = dA_{g|S_j}$. Choose smooth functions $0 \le \zeta_j \le 1$ on S_j which vanish on disks about the crossing points and equal one on a nonempty open set. Choose a nonzero smooth function $f_j \ge 0$ with $\operatorname{supp} f_j \subset \{\zeta_j = 1\}$. Multiplying its pullback in a tube by a nonnegative cutoff gives a smooth extension $F_j \ge 0$ on X whose support is disjoint from every S_ℓ with $\ell \ne j$.

Choose $c > 0$ so that $c \int_{S_j} \rho_j < \varepsilon/8$ for every j. Put $a_j = A_j + \varepsilon/2$ and

$$t_j = \frac{a_j - c\int_{S_j}\rho_j}{\int_{S_j} f_j\rho_j} > 0, \qquad u_j = c + t_j f_j, \qquad u_0 = c + \sum_j t_j F_j.$$

Then $u_0 > 0$, $u_0|_{S_j} = u_j$, and $u_0 = c$ near the crossings. Also $\int_{S_j} u_j\rho_j = a_j$ and

$$D_j := \int_{S_j} \zeta_j u_j \rho_j = a_j - c\int_{S_j}(1-\zeta_j)\rho_j > A_j + 3\varepsilon/8.$$

Apply Lemma 10.2 with $\sigma_j = D_j^{-1}\zeta_j u_j \rho_j$, choosing tubular projections which are g-orthogonal along S_j. Write U for the resulting neighborhood of S and κ_i for the closed forms on U. By [79, arXiv version, Lemma 3.4], the form $\pi_j^*\rho_j$ has g-comass one along S_j. Hence

$$\operatorname{comass}_{u_0 g}(\kappa_i)\big|_{S_j} = \frac{|\alpha_i(h_j)|\zeta_j}{D_j} \le \frac{A_j}{D_j} < 1. \tag{10.6}$$

The forms vanish near the crossings. Since S is compact, there are a neighborhood $W \subset U$ of S and a number $q < 1$ such that $\operatorname{comass}_{u_0 g}(\kappa_i) \le q$ on W for every i.

Choose smooth one-forms ψ_i on U with $d\psi_i = \eta_i - \kappa_i$, and an open neighborhood U' of S with $\overline{U'} \subset U$. Their restrictions and those of $d\psi_i$ are bounded on U'. Fix a smooth background metric g_0, and take the functions χ_δ of Lemma 10.3. For small δ, extend $\chi_\delta\psi_i$ by zero outside U' and put

$$\eta_{i,\delta} = \eta_i - d(\chi_\delta\psi_i).$$

These forms are closed, represent α_i, and equal κ_i on the interior of $\{\chi_\delta = 1\}$. Choose a smooth function $0 \le \vartheta_\delta \le 1$, equal to one near S, with support in this interior, and set

$$w_\delta = 1 + \vartheta_\delta(u_0 - 1), \qquad k_\delta = w_\delta g, \qquad R_\delta = \{w_\delta \ne 1\}.$$

The functions w_δ have positive lower and finite upper bounds independent of δ. Moreover, $R_\delta \subset \{x : \operatorname{dist}_{g_0}(x, S) < \delta\}$. Thus

$$\operatorname{Vol}(X, k_\delta) \longrightarrow \operatorname{Vol}(X, g), \qquad \operatorname{Area}_{k_\delta}(S_j) = a_j, \qquad P_{k_\delta} = P_g.$$

Put

$$E_\delta = \left(\sum_i (\operatorname{comass}_{k_\delta}(\eta_{i,\delta}) - 1)_+^2\right)^{1/2}.$$

On $X \setminus R_\delta$ the metric is g, and

$$(\operatorname{comass}_{k_\delta}(\eta_{i,\delta}) - 1)_+ \le |d(\chi_\delta\psi_i)|_g.$$

On R_δ one has $\eta_{i,\delta} = \kappa_i$. The conformal identity $|\kappa_i|_{k_\delta}^2\, dv_{k_\delta} = |\kappa_i|_g^2\, dv_g$ therefore gives

$$\|E_\delta\|_{L^2(k_\delta)}^2 \le \sum_i \|d(\chi_\delta\psi_i)\|_{L^2(g)}^2 + \sum_i \int_{R_\delta} |\kappa_i|_g^2\, dv_g \longrightarrow 0. \tag{10.7}$$

The first sum tends to zero by (10.5). The second does so because the κ_i are bounded on U' and $\operatorname{Vol}_g(R_\delta) \to 0$. On a neighborhood of S contained in W, one has $k_\delta = u_0 g$ and $\eta_{i,\delta} = \kappa_i$; hence $E_\delta = 0$ there.

The relative assertion of Lemma 8.2, with $\beta_i = 1$ and δ as above, gives a smooth conformal metric $\widehat{g}_\delta$ equal to k_δ near S, with $\mathrm{comass}_{\widehat{g}_\delta}(\eta_{i,\delta}) \le 1$ and

$$\mathrm{Vol}(X, \widehat{g}_\delta) \le \mathrm{Vol}(X, k_\delta) + 2\,\mathrm{Vol}(X, k_\delta)^{1/2}\|E_\delta\|_{L^2(k_\delta)} + \|E_\delta\|^2_{L^2(k_\delta)} + \delta.$$

Its surface areas are $a_j < A_j + \varepsilon$, and $\limsup_{\delta\downarrow 0} \mathrm{Vol}(X, \widehat{g}_\delta) \le \mathrm{Vol}(X, g)$. Thus a sufficiently small δ gives the volume bound. The period and regularity are unchanged by Lemma 8.2.

For the family assertion, keep S_j, ζ_j, f_j, F_j and g_0 fixed. The densities $\rho_{j,b}$ vary smoothly. Compactness of B permits one choice of c and a constant $q < 1$ such that

$$\frac{A_{j,b}}{D_{j,b}} \le q \qquad (b \in B,\ 1 \le j \le m).$$

The orthogonal tubular projections depend smoothly on b and are defined on fixed sufficiently small tubes U_j. Shrink these tubes so that every projection maps their overlaps into the fixed disks where $\zeta_j = 0$. Then the forms $\kappa_{i,b}$ are smooth on $B \times U$, where $U = \bigcup_j U_j$ is independent of b. Equation (10.4) gives $[\eta_{i,b} - \kappa_{i,b}] = 0$ on U.

For the C^∞ topology on compact subsets, the de Rham differential on the open manifold U admits a fixed continuous linear right inverse on exact two-forms which takes smooth families to smooth families [3, Theorem A.1 and Remark A.3(i)]. Applied to $\eta_{i,b} - \kappa_{i,b}$, it gives smooth primitives $\psi_{i,b}$. For fixed $\overline{U'} \subset U$, these primitives and their exterior derivatives are bounded on $B \times U'$. The strict comass bound holds on one neighborhood W of S, by compactness of $B \times S$. The same cutoffs $\chi_\delta, \vartheta_\delta$ give (10.7) and volume convergence uniformly in b. For each sufficiently small δ, $E_{b,\delta} = 0$ on a fixed neighborhood of S for every b. The relative family assertion of Lemma 8.2 now gives smooth metrics $\widehat{g}_{b,\delta}$; one sufficiently small δ gives all the required inequalities. □

10.3. **Polyhedral stable norms.**

Proposition 10.5. *If the unit ball of N is a rational polytope, then for every $\varepsilon > 0$ and every $P \in \mathcal{D}(Q)$ there is a smooth regular metric g with*

$$P_g = P, \qquad N \le N_g \le (1+\varepsilon)N, \qquad \mathrm{Vol}(X, g) < G_N(P) + \varepsilon. \tag{10.8}$$

Moreover, there are finitely many embedded surfaces whose areas give the upper bound.

Proof. If $H = 0$, the norm and the period are unique, and every metric is regular. Constant rescaling gives $G_N(P) = 0$ and a metric of volume less than ε; the family of surfaces is empty. Suppose $H \neq 0$. Write

$$C_N = \mathrm{conv}\{\pm\alpha_1, \dots, \pm\alpha_s\}, \qquad B_N = \mathrm{conv}\{\pm h_1/A_1, \dots, \pm h_m/A_m\}, \qquad A_j = N(h_j) > 0,$$

where each h_j is a nonzero integral class on a vertex ray of B_N. Such classes exist because the vertices are rational with respect to the integral homology lattice. Theorem 9.1 and (10.2) give a smooth regular metric g_1 such that

$$P_{g_1} = P, \qquad \|\alpha_i\|^*_{g_1} \le 1, \qquad v_1 := \mathrm{Vol}(X, g_1) < G_N(P) + \varepsilon/3.$$

Choose $\tau > 0$ with $((1+\tau)^2 - 1)v_1 < \varepsilon/3$. By the definition of cohomological comass, there are smooth closed representatives η_i of α_i satisfying $\mathrm{comass}_{g_1}(\eta_i) \le 1 + \tau$. For $g_2 = (1+\tau)g_1$, these pointwise bounds are one, $P_{g_2} = P$, and g_2 is regular. Also

$$\mathrm{Vol}(X, g_2) = (1+\tau)^2 v_1 < G_N(P) + 2\varepsilon/3.$$

Since $N(h) = \max_i |\alpha_i(h)|$, one has $|\alpha_i(h_j)| \le A_j$. Choose

$$0 < r < \min\{\varepsilon/3,\ \varepsilon \min_j A_j\}.$$

Apply Lemma 10.4 to g_2, the forms η_i, and the classes h_j, with r in place of ε. It gives a smooth regular metric g of period P and surfaces S_j representing h_j, with

$$\|\alpha_i\|_g^* \le 1, \qquad \mathrm{Vol}(X,g) \le \mathrm{Vol}(X,g_2) + r < G_N(P) + \varepsilon.$$

Duality (1.1) gives $N_g(h) \ge \max_i |\alpha_i(h)| = N(h)$ for every $h \in H$. On the other hand,

$$N_g(h_j) \le \mathrm{Area}_g(S_j) \le A_j + r < (1+\varepsilon)A_j.$$

Thus the convex ball $(1+\varepsilon)B_{N_g}$ contains every vertex of B_N, and hence B_N itself. This is the upper bound in (10.8). □

10.4. **Continuity of the volume infimum.** For a real symmetric matrix S, put $|S| = (S^2)^{1/2}$ and $S_+ = (|S| + S)/2$.

Monotonicity and rescaling in (10.1) give

$$N_1 \le N_2 \implies G_{N_1}(P) \le G_{N_2}(P), \qquad G_{cN}(P) = c^2 G_N(P) \quad (c > 0). \tag{10.9}$$

Multiplication of the metric tensor by c multiplies the stable two-norm by c, four-volume by c^2, and leaves the period fixed. Thus, for $0 \le \delta < 1$,

$$(1-\delta)N \le N' \le (1+\delta)N \implies (1-\delta)^2 G_N(P) \le G_{N'}(P) \le (1+\delta)^2 G_N(P). \tag{10.10}$$

Lemma 10.6. *Fix an integer $s \ge 1$. Let S, T, S', T' be real symmetric $s \times s$ matrices, with both pairs (S,T) and (S',T') feasible for unit bounds in* (9.1)*. Then*

$$|\Gamma(S',T';1) - \Gamma(S,T;1)| \le \sum_{1 \le i \le j \le s} |S'_{ij} - S_{ij}| + \tfrac{1}{2}\,\mathrm{tr}\,|T' - T|. \tag{10.11}$$

The function $(N,P) \mapsto G_N(P)$ is finite and jointly continuous; it is positive if $\dim H > 0$.

Proof. The atoms in the proof of Lemma 9.4, with real nonnegative weights, have any prescribed symmetric wedge matrix D and total mass at most $\sum_{i \le j} |D_{ij}|$. Their energy matrix is positive semidefinite. For a positive-semidefinite matrix R, write $R = \sum cc^t$, omitting zero summands. For each c, take two atoms whose only nonzero row is $\sqrt{2}c^t/|c|$, in A and B, respectively, each with weight $|c|^2/4$. Each column has comass at most one. The sum of their wedge matrices is zero, their energy matrix is cc^t, and their mass is $|c|^2/2$. Thus R can be added with zero wedge and mass $\mathrm{tr}\,R/2$.

To any measure feasible for (S,T), add the wedge matrix $S' - S$ and the zero-wedge energy matrix $(T'-T)_+$. Since $T + (T'-T)_+ \succeq T'$, the resulting measure is feasible for (S',T'). Taking infima gives

$$\Gamma(S',T';1) \le \Gamma(S,T;1) + \sum_{i \le j} |S'_{ij} - S_{ij}| + \tfrac{1}{2}\,\mathrm{tr}((T'-T)_+).$$

Interchanging the pairs and using $\mathrm{tr}((T'-T)_+), \mathrm{tr}((T-T')_+) \le \mathrm{tr}\,|T'-T|$ proves (10.11).

If $H = 0$, constant rescaling gives $G = 0$. Suppose $\dim H > 0$. The ball B_N is compact, centrally symmetric, and contains zero in its interior. Polytope approximation [66, Theorem 1.8.19 and Remark 1.8.17] gives, for every $0 < \delta < 1$, a rational polytope K_δ such that

$$(1+\delta)^{-1} B_N \subset K_\delta \subset B_N.$$

Replacing K_δ by $\mathrm{conv}(K_\delta \cup (-K_\delta))$ preserves these inclusions and makes it centrally symmetric. Its norm N_δ therefore satisfies $N \le N_\delta \le (1+\delta)N$.

For a finite set $F = \{\alpha_1, \dots, \alpha_s\} \subset V$, put $Q_F = (Q(\alpha_i,\alpha_j))$ and $P_F = (P(\alpha_i,\alpha_j))$. The value $\Gamma(Q_F, P_F; 1)$ is independent of the ordering of F, by simultaneous permutation of the columns of A and B. We have

$$G_N(P) = \sup_{F \subset C_N \text{ finite}} \Gamma(Q_F, P_F; 1). \tag{10.12}$$

If $F \subset C_N$, duality and Theorem 9.1 give $\Gamma(Q_F, P_F; 1) \le G_N(P)$, whether or not F spans V. Conversely, polytope approximation of C_N gives a finite set $F \subset C_N$ with

$$(1-\delta)C_N \subset \operatorname{conv}(\pm F) \subset C_N.$$

This set spans V, and $N_F(h) = \max_{\alpha\in F} |\alpha(h)|$ is a norm with $(1-\delta)N \le N_F \le N$. Hence

$$\Gamma(Q_F, P_F; 1) = G_{N_F}(P) \ge (1-\delta)^2 G_N(P)$$

by (10.2) and (10.10). Letting $\delta \downarrow 0$ proves (10.12).

For polyhedral norms, continuity in P follows from (10.2) and (10.11). Choose a basis $\beta_1, \dots, \beta_d$ of V, where $d = \dim V$, such that $C_N \subset \operatorname{conv}\{\pm\beta_1, \dots, \pm\beta_d\}$, and put $M(h) = \max_a |\beta_a(h)|$. For a nonempty compact set $K \subset \mathcal{D}(Q)$, set

$$C_K = \tfrac{1}{2} \max_{P\in K} \sum_{a=1}^{d} P(\beta_a, \beta_a).$$

Equations (10.2) and (9.6) give $G_N(P) \le G_M(P) \le C_K$ for $P \in K$. Therefore

$$0 \le G_{N_\delta}(P) - G_N(P) \le (2\delta + \delta^2)C_K \qquad (P \in K).$$

Uniform approximation on compact sets proves continuity in P. For $P, P' \in K$ and $(1-\delta)N \le N' \le (1+\delta)N$, (10.10) gives

$$|G_{N'}(P') - G_N(P)| \le (2\delta + \delta^2)C_K + |G_N(P') - G_N(P)|.$$

This proves joint continuity. Finally, for $0 \ne \alpha \in C_N$, the one-class case of (10.12) and the energy bound (9.4) give

$$G_N(P) \ge \tfrac{1}{2} P(\alpha, \alpha) > 0.$$

Finiteness follows from the nonempty set of metrics in (10.1). □

10.5. **Proof of the closure theorem.** Suppose $(N_{g_j}, P_{g_j}, \operatorname{Vol}(X, g_j)) \to (N, P, v)$. After finitely many terms are omitted, relative norm convergence gives $N_{g_j} \ge (1-\delta_j)N$, with $0 \le \delta_j < 1$ and $\delta_j \to 0$. By (10.1) and (10.9),

$$\operatorname{Vol}(X, g_j) \ge (1-\delta_j)^2 G_N(P_{g_j}).$$

Continuity in P gives $v \ge G_N(P)$.

Conversely, suppose $v \ge G_N(P)$ and $\dim H > 0$. Choose $0 < \delta_j < 1$ with $\delta_j \downarrow 0$ and rational polyhedral norms $N \le N_j \le (1+\delta_j)N$ as above. Proposition 10.5, applied to N_j with error δ_j, gives smooth regular metrics k_j satisfying $P_{k_j} = P$ and $N_j \le N_{k_j} \le (1+\delta_j)N_j$. Moreover,

$$G_N(P) \le \operatorname{Vol}(X, k_j) < G_{N_j}(P) + \delta_j \le (1+\delta_j)^2 G_N(P) + \delta_j.$$

Consequently

$$P_{k_j} = P, \qquad N \le N_{k_j} \le (1+\delta_j)^2 N, \qquad \operatorname{Vol}(X, k_j) \longrightarrow G_N(P). \tag{10.13}$$

The proof of Proposition 10.5 supplies finitely many embedded surfaces whose areas give the upper bound $N_{k_j} \le (1+\delta_j)N_j$.

If $v > G_N(P)$, omit finitely many terms so that $\operatorname{Vol}(X, k_j) < v$. The complement of these finitely many closed surfaces contains an open ball. Choose a nonzero nonnegative smooth function ϕ_j supported in that ball. The metrics $(1+t\phi_j)k_j$, $t \ge 0$, have period P and remain regular. Their volumes are

$$\operatorname{Vol}(X, k_j) + 2t \int_X \phi_j \, dv_{k_j} + t^2 \int_X \phi_j^2 \, dv_{k_j}.$$

Both integrals are positive. There is therefore a unique $t_j > 0$ for which $g_j = (1+t_j\phi_j)k_j$ has volume v. Since $g_j \ge k_j$, one has $N_{g_j} \ge N_{k_j} \ge N$. The surface areas are unchanged, so $N_{g_j} \le (1+\delta_j)N_j \le (1+\delta_j)^2 N$.

If $v = G_N(P) > 0$, set

$$c_j = (v/\operatorname{Vol}(X, k_j))^{1/2}, \qquad g_j = c_j k_j.$$

Then $c_j \to 1$. The metrics g_j remain regular, have period P and volume v, and satisfy $N_{g_j} = c_j N_{k_j} \to N$. Thus (i) implies (ii) in Theorem 10.1, and (ii) implies (iii).

Fix $s \geq d = \dim V$. In a rational basis of V, the spanning condition on $\alpha \in V^s$ is a polynomial rank condition, and $\mathcal{D}(Q) = \{P = P^t > 0 : PQ^{-1}P = Q\}$ is semialgebraic. The finite polynomial system in the proof of Theorem 9.1, with $1 + s(s+1)$ atoms and unit bounds, has coefficients polynomial in (α, P). The minimum is attained by Lemma 9.2; hence the existence of its variables with total mass at most t is equivalent to $t \geq G_{N_\alpha}(P)$ by (10.2). Quantifier elimination [14, Theorem 2.77] gives the semialgebraic epigraph. If Q_α, P_α have real algebraic entries, Theorem 9.1 with unit bounds gives the algebraicity and computability of $G_{N_\alpha}(P)$. When $H = 0$, constant rescaling of any metric gives every positive volume, the period is unique, and every metric is regular. This proves Theorem 10.1.

Corollary 10.7. *Let X, Y be closed connected oriented smooth four-manifolds, with intersection forms Q_X, Q_Y. Let $L : H^2(X;\mathbb{R}) \to H^2(Y;\mathbb{R})$ be a real linear isomorphism preserving these forms. Its dual is $L^* : H_2(Y;\mathbb{R}) \to H_2(X;\mathbb{R})$. For a norm N_X on $H_2(X;\mathbb{R})$ and $P_X \in \mathcal{D}(Q_X)$, define a norm N_Y and a form P_Y by*

$$N_Y(h) = N_X(L^* h), \qquad P_Y(L\alpha, L\beta) = P_X(\alpha, \beta).$$

The map $(N_X, P_X, v) \mapsto (N_Y, P_Y, v)$ identifies the two norm–period–positive-volume closures. Every triple (N, P, v) in either closure admits smooth regular metrics g_j on the corresponding manifold, with period P and volume v for every j, such that $N_{g_j} \to N$.

Proof. If $H^2(X;\mathbb{R}) = 0$, apply the zero-dimensional case of Theorem 10.1. Otherwise, one has $P_Y \in \mathcal{D}(Q_Y)$ and $C_{N_Y} = L(C_{N_X})$. For every finite $F \subset C_{N_X}$, the Q- and P-Gram matrices of F and $L(F)$ agree. Equation (10.12) therefore gives $G_{N_Y}(P_Y) = G_{N_X}(P_X)$. Theorem 10.1 identifies the closures with these epigraphs and supplies the stated sequences. □

11. Smooth attainment of the conformal two-systole maximum

Let X be a closed connected oriented smooth four-manifold with $b = b_2(X) > 0$. Retain the marked intersection space (V, Q, Λ) from Section 8, and let (b_+, b_-) be the signature of Q. The integral form Q is unimodular by Poincaré duality [39, Proposition 3.38, p. 250]. For $P \in \mathcal{D}(Q)$ put

$$\ell(P) = \min_{0 \neq v \in \Lambda} P(v, v)^{1/2}.$$

For a smooth Riemannian metric g and $v \in V$, let ω_v be the harmonic representative of v. The conformal two-systole [64, p. 3] is

$$\operatorname{confsys}_2(X, g) = \min_{0 \neq v \in \Lambda} \left(\int_X \omega_v \wedge *_g \, \omega_v \right)^{1/2} = \ell(P_g). \tag{11.1}$$

It depends only on the conformal class of g.

Proposition 11.1 (Bavard). *The function ℓ attains its maximum on $\mathcal{D}(Q)$. In particular,*

$$T(Q) := \max_{P \in \mathcal{D}(Q)} \ell(P)^2 \tag{11.2}$$

is a positive finite number.

Proof. Fix a $\mathbb{Z}$-basis of Λ, and identify forms with their matrices in this basis. The symmetric matrix Q lies in $\mathrm{GL}_b(\mathbb{Z})$, and $Q(Q^t)^{-1} = I_b$, where I_b is the identity matrix. Equation (8.1) gives $\det P = 1$ for every $P \in \mathcal{D}(Q)$. Thus $\mathcal{D}(Q)$ is the space of Gram matrices denoted by V_Q in [12, equation (2), p. 2]. On this space the Hermite invariant is $\ell(P)^2$. Bavard's compactness theorem [12, Proposition 4.6 and §4.5, pp. 34 and 42] gives its maximum. The domain is nonempty, since $(Q^2)^{1/2} \in \mathcal{D}(Q)$, so this maximum is positive. □

Theorem 11.2. *For every positive smooth density μ on X, there is a smooth metric $g_{\max}$ with $dv_{g_{\max}} = \mu$ such that*

$$\operatorname{confsys}_2(X, g_{\max})^2 = \max_g \operatorname{confsys}_2(X, g)^2 = T(Q). \tag{11.3}$$

Proof. Every metric has a period majorant in $\mathcal{D}(Q)$, so (11.1) gives the upper bound $T(Q)$. Choose $P_{\max}$ by Proposition 11.1. Corollary 9.7 gives a smooth metric $g_{\max}$ with period $P_{\max}$ and density μ. Equation (11.1) gives $\operatorname{confsys}_2(X, g_{\max}) = \ell(P_{\max})$. □

The function ℓ is continuous [47, Remark 9.1, p. 782]. Thus Scaduto's density theorem gives $\sup_g \operatorname{confsys}_2(X, g)^2 = T(Q)$ [64, Theorem 1.1 and Corollary 1.3]. If $b_+ = 1$ or $b_- = 1$, smooth attainment also follows from his surjectivity theorem [64, Theorem 1.2 and p. 2] and Proposition 11.1.

12. Sharp stable systolic constants

Let $p \geq 1$ be an integer. For a closed connected oriented smooth $2p$-manifold X with $b_p(X) > 0$, put

$$\mathrm{SR}_p(X) = \sup_g \frac{\operatorname{stsys}_p(X, g)^2}{\operatorname{Vol}(X, g)},$$

where g ranges over smooth Riemannian metrics. For integers $1 \leq p < n$, define

$$D_{n,p} = \max_{\substack{\omega \in \Lambda^p(\mathbb{R}^n)^* \\ \operatorname{comass}(\omega) \leq 1}} |\omega|^2, \tag{12.1}$$

where $|\omega|$ is the norm induced by the Euclidean metric. The maximum exists because the comass unit ball is compact. Hodge star preserves Euclidean norm and comass, so $D_{n,p} = D_{n,n-p}$ [42, Remark 2.2(1)]. In this squared normalization,

$$D_{4,2} = 2, \qquad D_{6,3} = 4, \qquad D_{8,4} = 14; \tag{12.2}$$

see [42, Remark 2.2(2) and Propositions 3.1–3.2]. Set

$$B_2 = 2, \qquad B_p = \binom{2p}{p} \quad (p \geq 3).$$

For $p \geq 2$, $D_{2p,p} \leq B_p$ by [42, Remark 2.2(2)–(3)].

For a full Euclidean lattice L of positive rank, let L^* be its dual, $\operatorname{covol}(L)$ its covolume, and $\lambda_1(L) = \min_{0 \neq z \in L} |z|$. For $d \geq 1$, the Hermite constant is

$$\gamma_d = \sup_{\operatorname{rank} L = d} \frac{\lambda_1(L)^2}{\operatorname{covol}(L)^{2/d}}.$$

In particular, $\gamma_2 = 2/\sqrt{3}$, attained by the hexagonal lattice [19, Chapter II, §3.1].

Lemma 12.1. *For a Euclidean lattice L of rank $d \geq 1$ and covolume Δ,*

$$\lambda_1(L)^2 \leq d\Delta^{2/d}.$$

Thus $\gamma_d \leq d$.

Proof. In orthonormal coordinates on the span of L, the cube $[-\Delta^{1/d}, \Delta^{1/d}]^d$ is compact, convex, centrally symmetric, and has volume $2^d\Delta$. Minkowski's convex-body theorem [19, Chapter III, §2.2, Theorem II] therefore gives $0 \neq z \in L$ in this cube. Hence $\lambda_1(L)^2 \leq |z|^2 \leq d\Delta^{2/d}$. □

12.1. **Stable and harmonic norms.** Let $V = H^p(X;\mathbb{R})$, $H = H_p(X;\mathbb{R})$, and $L = H_p(X;\mathbb{Z})/\text{torsion} \subset H$. Choose an integral basis of $H^p(X;\mathbb{Z})/\text{torsion}$ and its evaluation-dual basis of L. Put $Q(\alpha,\beta) = \langle \alpha \smile \beta, [X]\rangle$. A positive definite form P on V is compatible with Q if, in the chosen cohomology basis,

$$Q^t P^{-1} Q = P. \tag{12.3}$$

Write

$$|\alpha|_P = P(\alpha,\alpha)^{1/2}, \qquad B_P = \{\alpha \in V : |\alpha|_P \le 1\}, \qquad N_P(z) = \max_{\alpha \in B_P} |\alpha(z)|.$$

Thus $N_P(z)^2 = z^t P^{-1} z$ in the dual homology basis. For a marked list $\alpha = (\alpha_1, \dots, \alpha_s)$ of classes in V, put $Q_\alpha = (Q(\alpha_i,\alpha_j))_{ij}$ and $P_\alpha = (P(\alpha_i,\alpha_j))_{ij}$.

Proposition 12.2. *Let g be a smooth Riemannian metric on X, let P_g be the L^2 form on V defined by harmonic representatives, and put $v = \mathrm{Vol}(X,g)$. Then P_g is compatible with Q, its determinant in the chosen integral basis is one, and*

$$N_g^{(p)}(z) \le \sqrt{D_{2p,p} v}\, N_{P_g}(z) \qquad (z \in H). \tag{12.4}$$

Consequently, for $p \ge 2$ and $d = b_p(X)$,

$$\frac{\mathrm{stsys}_p(X,g)^2}{\mathrm{Vol}(X,g)} \le D_{2p,p}\,\gamma_d \le B_p\,\gamma_d \le B_p\, d. \tag{12.5}$$

Proof. The proof of Lemma 4.1 gives (12.3) for every $p \ge 1$. The integral cup pairing modulo torsion is unimodular [39, Proposition 3.38]. Taking determinants in (12.3) gives $\det P_g = |\det Q| = 1$. Thus L, with norm N_{P_g}, has covolume one.

Hebda's stable–harmonic norm comparison on real homology [40, Corollary 3, p. 343] gives (12.4): take $C_2(p;2p) = \sqrt{D_{2p,p}}$ in his notation; the L^2 norm of the Poincaré dual of z is $N_{P_g}(z)$ by (12.3). Let $0 \ne z_0 \in L$ minimize N_{P_g}. The definition of γ_d gives $N_{P_g}(z_0)^2 \le \gamma_d$. Apply (12.4) to z_0. The bounds $D_{2p,p} \le B_p$ and Lemma 12.1 then give (12.5). □

Lemma 12.3. *Let X be a closed connected oriented smooth $2p$-manifold, with $p \ge 2$, and let P be a positive definite form on $H^p(X;\mathbb{R})$. Let $\mathcal{G}$ be a family of smooth metrics closed under multiplication by positive constants. Suppose $v_0 \ge 0$ has the following property: for every finite set $F \subset B_P \cap H^p(X;\mathbb{Q})$ and every $\varepsilon > 0$, there is $g \in \mathcal{G}$ such that*

$$\mathrm{Vol}(X,g) < v_0 + \varepsilon, \qquad \|\alpha\|_g^* \le 1 \quad (\alpha \in F).$$

Then

$$\inf_{\substack{g \in \mathcal{G} \\ N_g^{(p)} \ge N_P}} \mathrm{Vol}(X,g) \le v_0.$$

Proof. For $V = 0$, take $F = \emptyset$. Suppose $V \ne 0$ and fix $0 < \delta < 1$. Rational polytope approximation [66, Theorem 1.8.19 and Remark 1.8.17], in a rational basis of V, gives a finite symmetric set $F_\delta \subset B_P \cap H^p(X;\mathbb{Q})$ such that

$$(1-\delta)B_P \subset \mathrm{conv}(F_\delta) \subset B_P.$$

For $\varepsilon > 0$, choose $g \in \mathcal{G}$ with $\mathrm{Vol}(X,g) < v_0 + \varepsilon$ and $\|\alpha\|_g^* \le 1$ for $\alpha \in F_\delta$. By (1.1),

$$N_g^{(p)}(z) \ge \max_{\alpha \in F_\delta} |\alpha(z)| \ge (1-\delta) N_P(z) \qquad (z \in H).$$

The metric $g' = (1-\delta)^{-2/p} g$ belongs to $\mathcal{G}$ and satisfies

$$N_{g'}^{(p)} \ge N_P, \qquad \mathrm{Vol}(X,g') < (1-\delta)^{-2}(v_0+\varepsilon).$$

Let $\delta, \varepsilon \downarrow 0$. □

12.2. **One middle-dimensional class.** Let $r \geq 2$ be even, and let dv denote the standard oriented Euclidean volume form on $\mathbb{R}^{2r}$. Define

$$W_r = \max_{\substack{\omega \in \Lambda^r(\mathbb{R}^{2r})^* \\ \operatorname{comass}(\omega) \leq 1}} \left| \frac{\omega \wedge \omega}{dv} \right|. \tag{12.6}$$

The maximum exists by compactness of the comass unit ball. After rescaling, the sum of two complementary unit volume forms has comass at most one and nonzero square. Hence $W_r > 0$.

Theorem 12.4. *Let X be a closed connected oriented smooth $2r$-manifold, with $r \geq 2$ even and $b_r(X) = 1$. Then*

$$\mathrm{SR}_r(X) = W_r. \tag{12.7}$$

Proof. The upper bound is [13, arXiv v5, Proposition 5.4], whose Wirtinger constant is W_r:

$$\mathrm{SR}_r(X) \leq W_r. \tag{12.8}$$

Let α generate $H^r(X;\mathbb{Z})/\text{torsion}$. Unimodularity [39, Proposition 3.38] gives

$$\sigma = \langle \alpha^2, [X] \rangle \in \{-1, 1\}.$$

The free homology and cohomology lattices are dual. Let z be the homology generator with $\alpha(z) = 1$. By (1.1),

$$\mathrm{stsys}_r(X, g) = N_g^{(r)}(z) = \|\alpha\|_g^{*\,-1}. \tag{12.9}$$

For the degree list (r), the sole top index is (2) and its moment is $t_{(2)}(\omega) = \omega^2/dv$. An orientation-reversing orthogonal map preserves comass and changes this moment's sign. Thus the body (1.2) is $K_{2r;r} = [-W_r, W_r]$, and $\gamma_{K_{2r;r}}(T_\alpha) = W_r^{-1}$, since $T_\alpha = \sigma$. Theorem 1.1, applied to the integral class α with comass bound one, gives

$$\inf_{g\colon \|\alpha\|_g^* \leq 1} \mathrm{Vol}(X, g) = W_r^{-1}. \tag{12.10}$$

For metrics approaching this infimum, (12.9) gives $\mathrm{stsys}_r(X, g) \geq 1$. Hence $\mathrm{SR}_r(X) \geq W_r$. □

Corollary 12.5. *The quaternionic projective plane and complex projective four-space satisfy*

$$\mathrm{SR}_4(\mathbb{H}P^2) = \mathrm{SR}_4(\mathbb{C}P^4) = 14. \tag{12.11}$$

These are suprema over smooth metrics.

Proof. Both manifolds have fourth Betti number one [39, Example 3.40]. The identity $W_4 = 14$ is [13, arXiv v5, Proposition 9.1]. Apply Theorem 12.4. □

Bangert–Katz–Shnider–Weinberger proved

$$6 \leq \mathrm{SR}_4(\mathbb{H}P^2) = \mathrm{SR}_4(\mathbb{C}P^4) \leq 14,$$

and equality with 14 under the existence of a compact Joyce manifold with Spin(7) holonomy and $b_4 = 1$ [13, arXiv v5, Theorem 1.2, Proposition 1.4 and Theorem 1.5].

12.3. **One class in each of two complementary degrees.** Let $p, q \geq 1$ be distinct integers, and put $n = p + q$. For a closed connected oriented smooth n-manifold X with $b_p(X), b_q(X) > 0$, define

$$\mathrm{SR}_{p,q}(X) = \sup_g \frac{\mathrm{stsys}_p(X, g)\,\mathrm{stsys}_q(X, g)}{\mathrm{Vol}(X, g)},$$

where g ranges over smooth Riemannian metrics.

Theorem 12.6. *Let $p, q \geq 2$, $p \neq q$, and $n = p + q$. Let X be a closed connected oriented smooth n-manifold such that*

$$H^p(X;\mathbb{Z})/\text{torsion} = \mathbb{Z}x, \qquad H^q(X;\mathbb{Z})/\text{torsion} = \mathbb{Z}y, \qquad \langle xy, [X]\rangle = 1.$$

Assume that every pure power of x or y of degree n vanishes. Then

$$\mathrm{SR}_{p,q}(X) = D_{n,p}. \tag{12.12}$$

In particular this formula holds for $X = S^p \times S^q$.

Proof. The rank-one case of [42, Theorem 5.2], with the normalization (12.1), gives $\mathrm{SR}_{p,q}(X) \leq D_{n,p}$. The free homology lattices have rank one, so (1.1) gives

$$\mathrm{stsys}_p(X, g) = \|x\|_g^{*\,-1}, \qquad \mathrm{stsys}_q(X, g) = \|y\|_g^{*\,-1}.$$

For the ordered degree list (p, q), let $\mathcal{I}$, t, $K = K_{n;p,q}$ and $T = T_{(x,y)}$ be the index set, moment map, body and top moment vector of (1.2)–(1.3).

Let dv be the standard oriented Euclidean volume form on $\mathbb{R}^n$. Choose $\omega \in \Lambda^p(\mathbb{R}^n)^*$ with $\mathrm{comass}(\omega) = 1$ and $|\omega|^2 = D_{n,p}$, and put $\eta = *\,\omega$. Then $\mathrm{comass}(\eta) = 1$ and $\omega \wedge \eta = D_{n,p}\,dv$. Let R be an orientation-reversing orthogonal reflection, and put

$$\tau = (\omega, \eta), \qquad \tau' = \begin{cases} (R^*\omega, -R^*\eta), & p < q, \\ (-R^*\omega, R^*\eta), & p > q. \end{cases}$$

Every component of τ and τ' has comass one. Their mixed moments equal $D_{n,p}$, since

$$(R^*\omega) \wedge (-R^*\eta) = (-R^*\omega) \wedge R^*\eta = -R^*(\omega \wedge \eta) = D_{n,p}\,dv.$$

If $a, b \geq 1$ and $ap + bq = n$, then $a = b = 1$. Put $d = \min(p, q)$. Since $2\max(p, q) > n$, every other index in $\mathcal{I}$, if any, corresponds to a power of the degree-d component, with d even and $d \mid n$. The degree-d component of τ' is the pullback by R of that of τ, with no additional sign. Consequently

$$t_\nu(\tau') = -t_\nu(\tau) \qquad \big(\nu \in \mathcal{I} \smallsetminus \{(1,1)\}\big).$$

These coordinates of T vanish by hypothesis, while $T_{(1,1)} = 1$. It follows that

$$T = \frac{t(\tau) + t(\tau')}{2D_{n,p}} \in D_{n,p}^{-1}K, \qquad \gamma_K(T) \leq D_{n,p}^{-1}.$$

The classes x, y are integral, and the Euclidean forms in (1.2) may have real coefficients. Theorem 1.1 therefore gives, for every $\varepsilon > 0$, a smooth metric g such that

$$\|x\|_g^*, \|y\|_g^* \leq 1, \qquad \mathrm{Vol}(X, g) < D_{n,p}^{-1} + \varepsilon.$$

Both stable systoles are at least one. Hence $\mathrm{SR}_{p,q}(X) \geq (D_{n,p}^{-1} + \varepsilon)^{-1}$. Let $\varepsilon \downarrow 0$.

For $S^p \times S^q$, take x, y to be the pullbacks of the oriented generators of the factors. They satisfy the hypotheses by the Künneth formula [39, Theorem 3.15]. □

The pure-power hypothesis is automatic if $\min(p, q)$ is odd or does not divide n.

12.4. **The symmetric hyperbolic plane.** Put

$$U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Lemma 12.7. *A positive definite symmetric 2×2 matrix P satisfies $UP^{-1}U = P$ if and only if $P = \mathrm{diag}(a, a^{-1})$ for some $a > 0$.*

Proof. The identity is equivalent to $PUP = U$. For $P = \left(\begin{smallmatrix} a & c \\ c & b \end{smallmatrix}\right)$ the diagonal entries of PUP are $2ac$ and $2bc$. Since $a, b > 0$, one has $c = 0$; the off-diagonal entry gives $ab = 1$. Conversely, every such diagonal matrix satisfies the identity. □

Theorem 12.8. *Let $p \geq 2$ be even. If the free integral middle cup form of a closed connected oriented smooth $2p$-manifold X is isometric to U, then*

$$\mathrm{SR}_p(X) = D_{2p,p}.$$

Proof. Choose an integral cohomology basis x, y with cup matrix U, and put $D = D_{2p,p}$. For every smooth metric g, the harmonic energy matrix is $\mathrm{diag}(a, a^{-1})$, $a > 0$, by Proposition 12.2 and Lemma 12.7. The squared first minimum of its dual integral lattice is $\min(a, a^{-1}) \leq 1$. Equation (12.4) gives $\mathrm{stsys}_p(X, g)^2 \leq D\,\mathrm{Vol}(X, g)$.

Choose $\Omega \in \Lambda^p(\mathbb{R}^{2p})^*$ with $\mathrm{comass}(\Omega) = 1$ and $|\Omega|^2 = D$, and an orthogonal reflection R with determinant -1. Put

$$\omega = (\Omega, *\,\Omega), \qquad \omega' = (R^*\Omega, -R^*(*\,\Omega)).$$

Every entry of these tuples has comass one. Write $\Omega \wedge \Omega = c_1\, dv_0$ and $(*\,\Omega) \wedge (*\,\Omega) = c_2\, dv_0$. Since $\Omega \wedge *\,\Omega = D\, dv_0$ and $R^* dv_0 = -dv_0$, the matrices of wedge-product coefficients of ω and ω' are, respectively,

$$\begin{pmatrix} c_1 & D \\ D & c_2 \end{pmatrix}, \qquad \begin{pmatrix} -c_1 & D \\ D & -c_2 \end{pmatrix}.$$

The top products of two degree-p classes are their two squares and their product. Thus, with the notation of (1.2)–(1.3),

$$T_{(x,y)} = \frac{t(\omega) + t(\omega')}{2D} \in D^{-1} K_{2p;p,p}.$$

The classes x, y are rational. Theorem 1.1 therefore gives, for every $\varepsilon > 0$, a smooth metric g with

$$\|x\|_g^*, \|y\|_g^* \leq 1, \qquad \mathrm{Vol}(X, g) < D^{-1} + \varepsilon.$$

Every nonzero integral homology class modulo torsion has a nonzero integer pairing with x or y. Formula (1.1) gives $\mathrm{stsys}_p(X, g) \geq 1$. Consequently,

$$\mathrm{SR}_p(X) \geq \frac{1}{D^{-1} + \varepsilon}.$$

Letting $\varepsilon \downarrow 0$ proves the lower bound. □

Let (X, g) be an oriented Riemannian four-manifold. For any two-form ξ, write $\xi^{\pm} = (\xi \pm *_g\, \xi)/2$. For a smooth two-form θ, define the function $\mathrm{mass}_g(\theta)$ by

$$\mathrm{mass}_g(\theta)(x) = \max_{\substack{\eta \in \Lambda^2 T_x^* X \\ \mathrm{comass}_g(\eta) \leq 1}} \left| \frac{\theta_x \wedge \eta}{dv_g(x)} \right|.$$

At each point,

$$\frac{\theta \wedge \eta}{dv_g} = \langle \theta^+, \eta^+ \rangle_g - \langle \theta^-, \eta^- \rangle_g.$$

Taking the dual norm in (9.3) gives

$$\mathrm{mass}_g(\theta) = \sqrt{2}\max(|\theta^+|_g, |\theta^-|_g) \leq \sqrt{2}\,|\theta|_g. \tag{12.13}$$

Equality in the inequality holds precisely where $\theta^+ = 0$ or $\theta^- = 0$. Suppose that X is closed and $d\theta = 0$, and let $\mathrm{PD} : H^2(X; \mathbb{R}) \to H_2(X; \mathbb{R})$ denote Poincaré duality. For every $\alpha \in H^2(X; \mathbb{R})$ and every smooth closed representative η of α,

$$|\langle \alpha, \mathrm{PD}[\theta] \rangle| = \left| \int_X \theta \wedge \eta \right| \leq \left(\int_X \mathrm{mass}_g(\theta)\, dv_g \right) \|\eta\|_{\infty,g}.$$

Taking the infimum over these representatives and applying (1.1) gives

$$N_g^{(2)}(\mathrm{PD}[\theta]) \leq \int_X \mathrm{mass}_g(\theta)\, dv_g. \tag{12.14}$$

Proposition 12.9. *If the free integral intersection form of a closed connected oriented smooth four-manifold X is isometric to U, then every smooth metric satisfies*

$$\operatorname{stsys}_2(X,g)^2 < 2\operatorname{Vol}(X,g).$$

Proof. Choose an integral cohomology basis with cup matrix U. By Proposition 12.2 and Lemma 12.7, one of its two classes has a harmonic representative h with $0 < \|h\|^2_{L^2(g)} \le 1$. Since $\int_X h\wedge h = 0$,

$$\|h^+\|^2_{L^2(g)} = \|h^-\|^2_{L^2(g)} = \tfrac12\|h\|^2_{L^2(g)} > 0.$$

Both components are smooth, closed and coclosed. The unique continuation theorem [4, §2, Theorem 1, pp. 421–422] therefore applies: the metric and forms are smooth, and their exterior derivatives and codifferentials vanish. Since X is connected and neither component is identically zero, their nonzero sets are open and dense. Their intersection is a nonempty open subset of X.

The class $\mathrm{PD}[h]$ is a nonzero integral homology class modulo torsion. The inequality in (12.13) is strict for h on the open subset above. Thus (12.14) and Cauchy–Schwarz give

$$\begin{aligned} N^{(2)}_g(\mathrm{PD}[h]) &\le \int_X \operatorname{mass}_g(h)\,dv_g < \sqrt{2}\int_X |h|_g\,dv_g \\ &\le \sqrt{2\operatorname{Vol}(X,g)}\,\|h\|_{L^2(g)} \le \sqrt{2\operatorname{Vol}(X,g)}. \end{aligned}$$

The stable systole is at most this value. □

For X as in Proposition 12.9, Theorem 12.8 and (12.2) give $\mathrm{SR}_2(X) = 2$, and no smooth metric attains this supremum. Kumar [51, Theorem A, pp. 3–4] proved

$$\operatorname{stsys}_2(S^2\times S^2, g) \le 2\sqrt{\operatorname{Vol}(S^2\times S^2, g)}.$$

His Theorem B, p. 4, gives the same coefficient for every closed oriented four-manifold with $b_2 = 2$.

12.5. **The alternating plane in dimension six.** Give $\mathbb{R}^6 = \mathbb{C}^3$ its standard Euclidean metric and complex orientation. Write $z_j = x_j + iy_j$, $1 \le j \le 3$, and put

$$dv = dx_1\wedge dy_1\wedge dx_2\wedge dy_2\wedge dx_3\wedge dy_3, \qquad \Omega = (dx_1 + i\,dy_1)\wedge(dx_2 + i\,dy_2)\wedge(dx_3 + i\,dy_3) = A + iB.$$

Expansion in the orthonormal exterior basis gives

$$|A|^2 = |B|^2 = 4, \qquad \langle A, B\rangle = 0, \qquad A\wedge B = 4\,dv.$$

By the special Lagrangian calibration theorem [38, Chapter III, §1, Theorem 1.10 and the remark after (1.5), pp. 87–89], the form $\operatorname{Re}(e^{-it}\Omega)$ has comass one for every $t\in\mathbb{R}$. For $a, b\in\mathbb{R}$, write $a + ib = \rho e^{it}$, with $\rho \ge 0$. Then $aA + bB = \rho\operatorname{Re}(e^{-it}\Omega)$. Hence

$$\operatorname{comass}(aA + bB) = \sqrt{a^2+b^2} \qquad (a, b\in\mathbb{R}). \tag{12.15}$$

Theorem 12.10. *Every closed connected oriented smooth six-manifold X with $b_3(X) = 2$ satisfies*

$$\mathrm{SR}_3(X) = \frac{8}{\sqrt{3}}.$$

Proof. The integral middle cup pairing modulo torsion is unimodular and alternating. Choose an integral basis in which its matrix is

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Fix a positive definite symmetric matrix P with $\det P = 1$; then $J^tP^{-1}J = P$. For a finite list $\alpha = (\alpha_1,\dots,\alpha_s)$ in $B_P\cap H^3(X;\mathbb{Q})$, put

$$u_i = P^{1/2}\alpha_i = (u_{i1}, u_{i2})^t, \qquad \eta_i = u_{i1}A + u_{i2}B,$$

where $P^{1/2}$ is the positive symmetric square root. Equation (12.15) gives $\operatorname{comass}(\eta_i) = |u_i| = |\alpha_i|_P \le 1$. Since $\det P^{1/2} = 1$,

$$\frac{\eta_i \wedge \eta_j}{dv} = 4\det(u_i, u_j) = 4\det(\alpha_i, \alpha_j) = 4Q(\alpha_i, \alpha_j).$$

The top products of degree-three classes on X are the products $\alpha_i\alpha_j$, $i < j$; their squares vanish. Thus

$$T_\alpha = \tfrac{1}{4}t(\eta_1, \dots, \eta_s) \in \tfrac{1}{4}K_{6;3,\dots,3}.$$

Theorem 1.1 gives metrics of volume less than $1/4 + \varepsilon$ with $\|\alpha_i\|_g^* \le 1$, for every such finite list and every $\varepsilon > 0$. The family of all smooth metrics is closed under positive scaling. Lemma 12.3 therefore gives

$$\inf_{N_g^{(3)} \ge N_P} \operatorname{Vol}(X, g) \le \frac{1}{4}.$$

Choose

$$P^{-1} = \frac{2}{\sqrt{3}}\begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}.$$

This is a determinant-one hexagonal Gram matrix, with squared first minimum $\gamma_2 = 2/\sqrt{3}$ [19, Chapter II, §3.1]. For every $\varepsilon > 0$ there is a smooth metric with $N_g^{(3)} \ge N_P$ and $\operatorname{Vol}(X, g) < 1/4 + \varepsilon$. Hence

$$\operatorname{SR}_3(X) \ge \frac{\gamma_2}{1/4 + \varepsilon}.$$

Letting $\varepsilon \downarrow 0$ gives the lower bound $4\gamma_2$. Proposition 12.2 and $D_{6,3} = 4$ give the reverse inequality. □

The middle cup form of $S^p \times S^p$ is U for even p and J for odd p, by the integral Künneth formula [39, Theorem 3.15 and Example 3.16]. Thus

$$\operatorname{SR}_2(S^2 \times S^2) = 2, \qquad \operatorname{SR}_3(S^3 \times S^3) = \frac{8}{\sqrt{3}}, \qquad \operatorname{SR}_4(S^4 \times S^4) = 14.$$

13. Indefinite intersection forms and systolic growth

Let Λ be a free abelian group of rank $d \ge 1$, and let Q be a nondegenerate symmetric integral form on Λ, extended to $\Lambda \otimes_{\mathbb{Z}} \mathbb{R}$. A compatible positive majorant of Q is a positive definite symmetric real form P whose matrix in an integral basis of Λ satisfies

$$PQ^{-1}P = Q.$$

This condition is independent of the basis. Write $\mathcal{D}(Q)$ for the set of these majorants, and put

$$m_\Lambda(P) = \min_{0 \ne z \in \Lambda} P(z, z), \qquad T(Q) = \sup_{P \in \mathcal{D}(Q)} m_\Lambda(P). \tag{13.1}$$

The matrix I_r is the identity of order r. The Hermite constant γ_d has the normalization of Section 12. For a Euclidean lattice L of positive rank, let $\lambda_1(L)$ denote the length of its shortest nonzero vector. By the Conway–Thompson theorem, there are odd positive definite integral unimodular lattices L_d of rank d with $\lambda_1(L_d)^2 \ge k(d)$, where $k(d) \sim d/(2\pi e)$ as $d \to \infty$ [47, Theorem 2.2, p. 775, and §8, p. 779]. Fix such lattices and an absolute constant $c_0 > 0$ such that

$$\lambda_1(L_d)^2 \ge c_0 d \qquad (d \ge 1). \tag{13.2}$$

Such a constant exists by the asymptotic estimate and the positivity of the finitely many remaining minima.

Theorem 13.1. *For every indefinite integral unimodular symmetric form Q of rank d, one has*

$$\frac{c_0 d}{60000} \le T(Q) \le \gamma_d \le d. \tag{13.3}$$

If Q is odd, the lower bound may be replaced by $c_0 d/15000$. In each case a compatible majorant satisfies the stated lower bound.

The linear lower bounds also follow from Siegel's formula for the normalized theta average on the space of majorants modulo integral isometries, with $S = Q$ and $a = 0$ in his notation, for $d > 4$ [71, Chapter IV, §4 and Theorem 12]. The remaining ranks follow from the classification of indefinite unimodular forms [68, Chapter V, §2.2, Theorems 4–6]. For even unimodular forms of signature (r, r), the stronger estimate $T(Q) \geq (1 + o(1))r/(\pi e)$ as $r \to \infty$ follows from [1, §4.5]. The majorants in Lemma 13.3 also give Corollary 13.4.

13.1. **A random subspace estimate.**

Lemma 13.2. *Let a, b be integers with $a \geq 2b \geq 2$, and equip the Grassmannian of b-planes in $\mathbb{R}^a$ with its rotation-invariant probability measure. If U is a random plane, $x \in \mathbb{R}^a \setminus \{0\}$ and $0 < \delta \leq 1/2$, then*

$$\Pr\{\operatorname{dist}(x, U) < \delta|x|\} \leq 2^a \delta^{a-b}. \tag{13.4}$$

Proof. Set $A = (a - b)/2$ and $B = b/2$. Fix a b-plane U_0, and let G have Haar probability distribution on the orthogonal group of $\mathbb{R}^a$. Then GU_0 has the distribution of U, and $G^{-1}x/|x|$ is uniformly distributed on the unit sphere. The orthogonal projection onto $U_0^\perp$ is symmetric and idempotent, of rank $a - b$. Put

$$Z = \operatorname{dist}(x, U)^2/|x|^2, \qquad \mathrm{B}(A, B) = \int_0^1 z^{A-1}(1 - z)^{B-1}\, dz,$$

so that [59, Theorem 1.5.7(ii), pp. 38–39] gives the density

$$\frac{z^{A-1}(1 - z)^{B-1}}{\mathrm{B}(A, B)}\, dz \qquad (0 < z < 1)$$

for Z. If $b \geq 2$, then $A, B \geq 1$ and

$$\mathrm{B}(A, B) \geq \int_{1/4}^{3/4} 4^{-(A-1)} 4^{-(B-1)}\, dz = 2^{3-a}.$$

Consequently,

$$\Pr\{\operatorname{dist}(x, U) < \delta|x|\} \leq \frac{\delta^{2A}}{A\mathrm{B}(A, B)} \leq 2^{a-3}\delta^{a-b}.$$

If $b = 1$, then $A = (a - 1)/2 > 0$ and $\mathrm{B}(A, 1/2) \geq 1/A$. Therefore

$$\Pr\{\operatorname{dist}(x, U) < \delta|x|\} \leq \frac{\delta^{2A}}{A\mathrm{B}(A, 1/2)\sqrt{1 - \delta^2}} \leq 2\delta^{a-1}.$$

□

Lemma 13.3. *Let a, b be integers with $a \geq 2b \geq 2$, and let L_a, L_b satisfy* (13.2)*. The integral form $L_a \oplus (-L_b)$ has a compatible positive majorant P with*

$$\min_{0 \neq (x,y) \in L_a \oplus L_b} P((x, y), (x, y)) \geq \frac{c_0 a}{10^4}. \tag{13.5}$$

Proof. Identify the ambient Euclidean spaces of L_a and L_b with $\mathbb{R}^a$ and $\mathbb{R}^b$. Let $U \subset \mathbb{R}^a$ be a b-plane and let $i : \mathbb{R}^b \longrightarrow U$ be an isometry. For $x = x_\perp + i(v)$, with $x_\perp \perp U$, set

$$e^t = \sqrt{a/b}, \qquad B_{U,i}(x, y) = \big(x_\perp + i(\cosh(t)v + \sinh(t)y), \sinh(t)v + \cosh(t)y\big).$$

This map is an orthogonal sum of the identity on $U^\perp$ and two-dimensional Lorentz transformations; compare [47, §§9–10, pp. 781–784]. In orthonormal coordinates put $S = \operatorname{diag}(I_a, -I_b)$. Then $B_{U,i}^{\mathsf{T}} S B_{U,i} = S$, whence

$$(B_{U,i}^{\mathsf{T}} B_{U,i}) S (B_{U,i}^{\mathsf{T}} B_{U,i}) = S.$$

Thus $P_{U,i}((x,y),(x,y)) = |B_{U,i}(x,y)|^2$ is a compatible positive majorant. The map $B_{U,i}$ is symmetric, with eigenvalues $1, e^t, e^{-t}$; its inverse has Euclidean operator norm e^t. Moreover,

$$P_{U,i}((x,y),(x,y)) = |x_\perp|^2 + \frac{a}{2b}|v+y|^2 + \frac{b}{2a}|v-y|^2. \tag{13.6}$$

Put

$$R = \frac{\sqrt{c_0 a}}{100}, \qquad \lambda = \lambda_1(L_a).$$

If $0 \neq (x,y) \in L_a \oplus L_b$ and $P_{U,i}((x,y),(x,y)) < R^2$, then

$$\operatorname{dist}(x,U) < R, \qquad |x| \le |(x,y)| < \sqrt{a/b}\, R. \tag{13.7}$$

For $x = 0$ and $0 \neq y \in L_b$, however,

$$P_{U,i}((0,y),(0,y)) = \cosh(2t)|y|^2 \ge \frac{a}{2b}\, c_0 b = \frac{c_0 a}{2} > R^2.$$

For each integer $j \ge 0$, disjoint open balls of radius $\lambda/2$ centered at lattice points give

$$\#\{x \in L_a \,:\, 2^j\lambda \le |x| < 2^{j+1}\lambda\} \le (1 + 2^{j+2})^a \le (5\,2^j)^a. \tag{13.8}$$

If $\sqrt{a/b}\, R < \lambda$, every U and i give the required majorant. Otherwise choose an integer $J \ge 0$ such that

$$2^J\lambda \le \sqrt{a/b}\, R < 2^{J+1}\lambda.$$

Since $R/\lambda \le 1/100$, Lemma 13.2 applies in every shell. The probability that some $x \in L_a \setminus \{0\}$ satisfies (13.7) is at most

$$\begin{aligned}\sum_{j=0}^{J} (5\,2^j)^a\, 2^a \left(\frac{R}{2^j\lambda}\right)^{a-b} &= 10^a (R/\lambda)^{a-b} \sum_{j=0}^{J} 2^{jb} \\ &\le 2\,10^a (R/\lambda)^{a-b} (\sqrt{a/b}\, R/\lambda)^b \\ &= 2\,10^a (a/b)^{b/2} (R/\lambda)^a.\end{aligned}$$

For $u = b/a$, the inequality $-u\log u \le 1/e$ gives $u^{-u/2} \le e^{1/(2e)} < 2$. The last expression is therefore at most $2(1/5)^a < 1$. Choose a plane outside this finite union of events, and then any isometry $i : \mathbb{R}^b \longrightarrow U$. The corresponding majorant satisfies (13.5). □

13.2. **Parity and signature.**

Proof of Theorem 13.1. Since $\mathcal{D}(-Q) = \mathcal{D}(Q)$ and $T(-Q) = T(Q)$, we may assume that the signature (a,b) of Q satisfies $a \ge b \ge 1$. Thus $a + b = d$.

Suppose first that Q is odd. By the classification of indefinite integral unimodular forms, it is isometric to $L_a \oplus (-L_b)$ [68, Chapter V, §2.2, Theorem 6]. If $a < 2b$, the direct-sum Euclidean majorant has squared minimum at least

$$c_0 b \ge \frac{c_0(a+b)}{3}.$$

If $a \ge 2b$, Lemma 13.3 gives a compatible majorant with squared minimum at least

$$\frac{c_0 a}{10^4} \ge \frac{c_0(a+b)}{15000}.$$

Suppose now that Q is even. Then $a - b \equiv 0 \pmod 8$ [68, Chapter V, §2.1, Theorem 2, Corollary 1]. On $L = \mathbb{Z}^{a+b}$ set

$$Q_0 = \operatorname{diag}(I_a, -I_b), \qquad L_0 = \{z \in L \,:\, Q_0(z,z) \in 2\mathbb{Z}\}, \qquad w = (1,\dots,1), \qquad L' = L_0 + \mathbb{Z}(w/2).$$

The lattice (L, Q_0) is odd and unimodular, and $Q_0(w,z) \equiv Q_0(z,z) \pmod 2$ for every $z \in L$. Since $a - b \equiv 0 \pmod 8$, the even-neighbor construction [15, §0.2, p. 6] shows that (L', Q_0) is even and

unimodular. Its signature is (a, b), so Theorem 6 of [68, Chapter V, §2.2] identifies it with (Λ, Q). Moreover, $2L' \subset L$, whence every compatible majorant P of Q_0 satisfies

$$m_{L'}(P) \geq \tfrac{1}{4} m_L(P).$$

The odd case provides such a P with $m_L(P) \geq c_0 d/15000$. Thus $m_{L'}(P) \geq c_0 d/60000$.

For every $P \in \mathcal{D}(Q)$, taking determinants in $PQ^{-1}P = Q$ gives $\det P = |\det Q| = 1$. Hence $m_\Lambda(P) \leq \gamma_d \leq d$ by Lemma 12.1. Taking the supremum over P proves the upper bound. □

Corollary 13.4. *For each positive integer r, put $I_{r,1} = \operatorname{diag}(I_r, -1)$. For every $0 < \alpha < 1/(2\pi e)$ and all sufficiently large r, there are a line $U \subset \mathbb{R}^r$ and an isometry $i : \mathbb{R} \longrightarrow U$ such that the majorant $P_{U,i}$ of* (13.6)*, with $a = r$ and $b = 1$, satisfies*

$$m_{L_r \oplus \mathbb{Z}}(P_{U,i}) \geq \frac{\alpha r}{100}.$$

In particular,

$$\liminf_{r \to \infty} \frac{T(I_{r,1})}{r} \geq \frac{1}{200\pi e}.$$

The estimate for $T(I_{r,1})$ also follows from Siegel's theta-average formula [71, Chapter IV, Theorem 12].

Proof. Fix $0 < \alpha < 1/(2\pi e)$. For all sufficiently large integers $r \geq 100$, the Conway–Thompson lattice L_r satisfies $\lambda_1(L_r) \geq d_r := \sqrt{\alpha r}$. Put $R = d_r/10$. Let $U \subset \mathbb{R}^r$ be a random line with the rotation-invariant probability distribution. For an isometry $i : \mathbb{R} \longrightarrow U$, let $P_{U,i}$ be the majorant of Lemma 13.3, with $a = r$, $b = 1$, and $e^t = \sqrt{r}$. For $k \in \mathbb{Z} \setminus \{0\}$, formula (13.6) gives

$$P_{U,i}((0,k),(0,k)) = \frac{r + r^{-1}}{2} k^2 \geq \frac{r}{2} > R^2.$$

By (13.7), every nonzero $(x, k) \in L_r \oplus \mathbb{Z}$ with $P_{U,i}((x,k),(x,k)) < R^2$ therefore has $x \neq 0$ and satisfies

$$|x| < \sqrt{r} R, \qquad \operatorname{dist}(x, U) < R.$$

Write $\mathcal{E}$ for the event that some $x \in L_r \setminus \{0\}$ satisfies these two inequalities. Choose $J \geq 0$ with $2^J d_r \leq \sqrt{r} R < 2^{J+1} d_r$. Distinct points of L_r are at least d_r apart, so (13.8) holds with $a = r$ and $\lambda = d_r$. Since $R/d_r = 1/10$, the bound $2\delta^{r-1}$ in the one-dimensional case of Lemma 13.2 applies in every shell. Consequently,

$$\begin{aligned} \Pr(\mathcal{E}) &\leq 2\, 5^r (R/d_r)^{r-1} \sum_{j=0}^{J} 2^j \\ &\leq 4\, 5^r (R/d_r)^{r-1} \frac{\sqrt{r} R}{d_r} = 4\sqrt{r}\, 2^{-r} < 1. \end{aligned}$$

Choose U outside $\mathcal{E}$ and any isometry $i : \mathbb{R} \longrightarrow U$. Then $m_{L_r \oplus \mathbb{Z}}(P_{U,i}) \geq R^2 = \alpha r/100$. The form $L_r \oplus (-\mathbb{Z})$ is odd, unimodular, and of signature $(r, 1)$; hence it is isometric to $I_{r,1}$ by [68, Chapter V, §2.2, Theorem 6]. Thus $T(I_{r,1})/r \geq \alpha/100$ for all sufficiently large r. Taking the lower limit and then letting $\alpha \uparrow 1/(2\pi e)$ proves the corollary. □

13.3. **Finite comass moments.** Let $p \geq 2$ be an integer. Set

$$c_{\mathrm{K}} = \log(1 + \sqrt{2}), \qquad \kappa = \frac{\pi}{2c_{\mathrm{K}}}, \qquad C_{\mathrm{K}} = \frac{\kappa}{2}.$$

The sine construction is due to Krivine [50, Theorem 2 and its corollary, pp. I–II.10–11].

Lemma 13.5. *Let $\mathbb{R}^{mp} = E_1 \oplus \cdots \oplus E_m$ be an orthogonal sum of oriented p-planes, where $m \geq 2$, and let ξ_r be their volume forms, extended by orthogonal projection. For $a_r \in \mathbb{R}$,*

$$\operatorname{comass}\left(\sum_{r=1}^{m} a_r \xi_r \right) = \max_r |a_r|.$$

Proof. For an orthonormal p-frame $(v_1, \dots, v_p)$ and the orthogonal projection π_r onto E_r, Hadamard's inequality gives

$$|\xi_r(v_1, \dots, v_p)| \le \prod_{j=1}^{p} |\pi_r v_j| \le |\pi_r v_1||\pi_r v_2|.$$

Since $\sum_r |\pi_r v_j|^2 = 1$, Cauchy–Schwarz gives

$$\sum_r |\xi_r(v_1, \dots, v_p)| \le \sum_r |\pi_r v_1||\pi_r v_2| \le 1.$$

Hence the comass is at most $\max_r |a_r|$. Evaluation on an oriented orthonormal basis of E_{r_0}, where $|a_{r_0}| = \max_r |a_r|$, gives the reverse inequality. □

If u, v are unit vectors and Z is a standard Gaussian vector in their span, the Gaussian sign identity [50, proof of Lemma 1, p. I–II.10] is

$$\mathbb{E}\big[\operatorname{sign}\langle Z, u\rangle \operatorname{sign}\langle Z, v\rangle\big] = \frac{2}{\pi} \arcsin\langle u, v\rangle. \tag{13.9}$$

Here $\operatorname{sign}(0) = 1$.

Lemma 13.6. *Let $u_1, \dots, u_s$ be vectors of norm at most one in a finite-dimensional Euclidean space. Let J be orthogonal, with $J^t = \sigma J$, $\sigma \in \{1, -1\}$. Set*

$$T_{ij} = \langle u_i, u_j\rangle, \qquad B_{ij} = \langle u_i, J u_j\rangle.$$

The symbol $\succeq$ denotes positive-semidefinite order on symmetric matrices. There is a finitely supported probability law on pairs $(x, y) \in [-1, 1]^s \times [-1, 1]^s$ such that

$$\mathbb{E}[xy^t] = \kappa^{-1} B, \qquad C_{\mathrm{K}} \mathbb{E}[xx^t + yy^t] \succeq T. \tag{13.10}$$

Proof. If all u_i vanish, take $x = y = 0$. Otherwise, it suffices to treat the nonzero u_i: the omitted coordinates of x and y are set to zero. Thus write $u_i = a_i \widehat{u}_i$, with $0 < a_i \le 1$ and $|\widehat{u}_i| = 1$. The tensor construction in [2, author version, proof of Lemma 5.2, p. 15] gives maps F, G from the unit sphere of the given Euclidean space to the unit sphere of a real Hilbert space such that

$$\langle F(u), G(v)\rangle = \sin(c_{\mathrm{K}} \langle u, v\rangle), \tag{13.11}$$

$$\langle F(u), F(v)\rangle = \langle G(u), G(v)\rangle = \sinh(c_{\mathrm{K}} \langle u, v\rangle). \tag{13.12}$$

Take a standard Gaussian vector Z in the span of the at most $2s$ vectors $F(\widehat{u}_i)$, $G(J\widehat{u}_i)$, and set

$$x_i = a_i \operatorname{sign}\langle Z, F(\widehat{u}_i)\rangle, \qquad y_i = a_i \operatorname{sign}\langle Z, G(J\widehat{u}_i)\rangle.$$

This law has at most 2^{2s} values. Since $c_{\mathrm{K}} < \pi/2$, (13.9) and (13.11) give

$$\mathbb{E}[x_i y_j] = \frac{2 a_i a_j}{\pi} \arcsin\big(\sin(c_{\mathrm{K}} \langle \widehat{u}_i, J\widehat{u}_j\rangle)\big) = \frac{2 c_{\mathrm{K}}}{\pi} B_{ij}.$$

Put $R = (\langle \widehat{u}_i, \widehat{u}_j\rangle)_{ij}$ and $D = \operatorname{diag}(a_i)$, so that $T = DRD$. Orthogonality of J and (13.12) give

$$C_{\mathrm{K}} \mathbb{E}[xx^t + yy^t] = DH[R]D, \qquad H(r) = c_{\mathrm{K}}^{-1} \arcsin(\sinh(c_{\mathrm{K}} r)),$$

where $H[R] = (H(R_{ij}))_{ij}$. On $[-1, 1]$ one has

$$H(r) = r + \sum_{\ell \ge 1} d_\ell r^{2\ell+1}, \qquad d_\ell \ge 0, \qquad \sum_{\ell \ge 1} d_\ell = \kappa - 1. \tag{13.13}$$

Indeed, composition of the power series for $\arcsin z$ and $\sinh(c_{\mathrm{K}} r)$ gives nonnegative coefficients on $0 \le r < 1$; the linear coefficient after division by c_{K} is one. Monotone convergence as $r \uparrow 1$ and $\sinh c_{\mathrm{K}} = 1$ give $\sum_{\ell \ge 1} d_\ell = H(1) - 1 = \kappa - 1$. The series therefore converges absolutely and uniformly

on $[-1,1]$. For each positive integer k, the entrywise power $R^{\circ k} = (R_{ij}^k)_{ij}$ is the Gram matrix of $\widehat{u}_i^{\otimes k}$ and is positive semidefinite. Consequently,

$$DH[R]D - T = \sum_{\ell \geq 1} d_\ell D R^{\circ(2\ell+1)} D \succeq 0,$$

with convergence in matrix norm. □

Let V be a finite-dimensional real vector space with a nondegenerate bilinear form Q satisfying $Q^t = \sigma Q$, where $\sigma = (-1)^p$. A positive-definite form P is compatible with Q if it satisfies (12.3). Let dv denote the standard oriented Euclidean volume form on $\mathbb{R}^{2p}$.

Proposition 13.7. *If P is compatible with Q and $\alpha_1, \dots, \alpha_s \in V$ satisfy $P(\alpha_i, \alpha_i) \leq 1$, put $P_\alpha = (P(\alpha_i, \alpha_j))_{ij}$ and $Q_\alpha = (Q(\alpha_i, \alpha_j))_{ij}$. There is a finitely supported positive measure ν of mass C_{K} on tuples $(\eta_1, \dots, \eta_s) \in (\Lambda^p(\mathbb{R}^{2p})^*)^s$, supported where* $\operatorname{comass}(\eta_i) \leq 1$ *for every i, such that*

$$\int \frac{\eta_i \wedge \eta_j}{dv}\, d\nu = Q(\alpha_i, \alpha_j), \qquad \left(\int \langle \eta_i, \eta_j \rangle \, d\nu \right)_{ij} \succeq P_\alpha.$$

Proof. Choose a basis of V and regard the α_i as coordinate columns. Put $u_i = P^{1/2}\alpha_i$ and $J = P^{-1/2} Q P^{-1/2}$. Then

$$J^t J = P^{-1/2} Q^t P^{-1} Q P^{-1/2} = I, \qquad J^t = \sigma J,$$

and the matrices T, B of Lemma 13.6 are P_α, Q_α. Choose ξ, ζ as in Lemma 13.5, with $m = 2$ and $\xi \wedge \zeta = dv$. For the law in Lemma 13.6, put $\eta_i = x_i \xi + y_i \zeta$ and multiply each probability by C_{K}. The comass bound follows from Lemma 13.5. Moreover,

$$\frac{\eta_i \wedge \eta_j}{dv} = x_i y_j + \sigma y_i x_j, \qquad \langle \eta_i, \eta_j \rangle = x_i x_j + y_i y_j.$$

Since $Q_\alpha^t = \sigma Q_\alpha$ and $2C_{\mathrm{K}} = \kappa$,

$$\int \frac{\eta_i \wedge \eta_j}{dv}\, d\nu = C_{\mathrm{K}} \kappa^{-1} \big((Q_\alpha)_{ij} + \sigma (Q_\alpha)_{ji} \big) = (Q_\alpha)_{ij}.$$

The energy inequality is (13.10). □

13.4. **A lower bound for an entire stable norm.** Let X be a closed connected oriented smooth $2p$-manifold, with $p \geq 2$ and $b_p(X) > 0$. Retain Q, B_P, N_P from Section 12.

Proposition 13.8. *For every compatible P one has*

$$\inf_{g \,:\, N_g^{(p)} \geq N_P} \operatorname{Vol}(X, g) \leq C_{\mathrm{K}}. \tag{13.14}$$

If $p = 2$, the same bound holds with the additional condition $P_g = P$:

$$\inf_{\substack{g \,:\, P_g = P \\ N_g^{(2)} \geq N_P}} \operatorname{Vol}(X, g) \leq C_{\mathrm{K}}. \tag{13.15}$$

Proof. Let $\alpha_1, \dots, \alpha_s \in B_P \cap H^p(X; \mathbb{Q})$. Proposition 13.7 gives a measure of mass C_{K} with their prescribed pairwise cup moments. Since $\dim X = 2p$, these are all the top moments. If T_α and K denote their top cup-moment vector and the moment body of Theorem 1.1, then

$$T_\alpha / C_{\mathrm{K}} \in K, \qquad \gamma_K(T_\alpha) \leq C_{\mathrm{K}}.$$

That theorem gives, for every $\varepsilon > 0$, a metric with $\|\alpha_i\|_g^* \leq 1$ for all i and $\operatorname{Vol}(X, g) < C_{\mathrm{K}} + \varepsilon$. Lemma 12.3, applied to the family of all smooth metrics, gives (13.14).

Suppose now that $p = 2$. For the forms ξ, ζ in the preceding construction, $* \, \xi = \zeta$. Choose orthonormal bases of the self-dual and anti-self-dual two-forms whose first vectors are $(\xi + \zeta)/\sqrt{2}$

and $(\xi - \zeta)/\sqrt{2}$, respectively. Put $e = (1,0,0)^t \in \mathbb{R}^3$. The Hodge components of $\eta_i = x_i\xi + y_i\zeta$ have coordinate columns

$$A_i = 2^{-1/2}(x_i + y_i)e, \qquad B_i = 2^{-1/2}(x_i - y_i)e.$$

By (9.3),

$$|A_i| + |B_i| = \sqrt{2}\,\mathrm{comass}(\eta_i) \le \sqrt{2}.$$

For $A = (A_1, \dots, A_s)$ and $B = (B_1, \dots, B_s)$, the measure of Proposition 13.7 therefore satisfies

$$\int (A^tA - B^tB)\,dv = Q_\alpha, \qquad \int (A^tA + B^tB)\,dv \succeq P_\alpha.$$

Theorem 9.1, with $\beta_i = 1$, gives metrics with $P_g = P$, $\|\alpha_i\|_g^* \le 1$ for all i, and $\mathrm{Vol}(X,g) < C_{\mathrm{K}} + \varepsilon$. The family $\mathcal{G}_P = \{g : P_g = P\}$ of smooth metrics is closed under homotheties, since $P_{t^2 g} = P_g$ for $t > 0$ in dimension four. Lemma 12.3, applied to $\mathcal{G}_P$, gives (13.15). □

13.5. **Systolic consequences.** Let X be a closed connected oriented smooth four-manifold with $b_2(X) > 0$. Write

$$Q_X(\alpha,\beta) = \langle \alpha \smile \beta, [X]\rangle, \qquad \Lambda = H^2(X;\mathbb{Z})/\mathrm{torsion}, \qquad L = H_2(X;\mathbb{Z})/\mathrm{torsion}.$$

For a compatible majorant P of Q_X, put

$$C_2(X) = \sup_g \mathrm{confsys}_2(X,g)^2, \qquad \mathrm{SR}_2(X;P) = \sup_{P_g = P} \frac{\mathrm{stsys}_2(X,g)^2}{\mathrm{Vol}(X,g)},$$

where the suprema are taken over smooth Riemannian metrics on X, with the indicated period condition in the second case. By (11.1),

$$\mathrm{confsys}_2(X,g)^2 = m_\Lambda(P_g).$$

Theorem 13.9. *For every compatible majorant P of Q_X,*

$$\frac{2}{\kappa} m_\Lambda(P) \le \mathrm{SR}_2(X;P) \le 2m_\Lambda(P). \tag{13.16}$$

Consequently,

$$C_2(X) = T(Q_X), \qquad \frac{2}{\kappa} C_2(X) \le \mathrm{SR}_2(X) \le 2C_2(X). \tag{13.17}$$

If Q_X is indefinite and $b = b_2(X)$, then, with $c_* = c_0/60000$,

$$c_* b \le C_2(X) \le b, \qquad \frac{2c_*}{\kappa} b \le \mathrm{SR}_2(X) \le 2b. \tag{13.18}$$

Under this hypothesis, there are a compatible P with $m_\Lambda(P) \ge c_* b$ and smooth metrics g_j of volume one such that

$$P_{g_j} = P, \qquad \mathrm{confsys}_2(X,g_j)^2 = m_\Lambda(P), \qquad \liminf_{j\to\infty} \mathrm{stsys}_2(X,g_j)^2 \ge \frac{2}{\kappa} m_\Lambda(P).$$

Proof. Choose an integral basis of Λ and its evaluation-dual basis of L, and write Q_X, P for the corresponding matrices. Integral Poincaré duality makes Q_X an isomorphism from Λ to L [39, Proposition 3.38]. Since $Q_X^t P^{-1} Q_X = P$,

$$\min_{0 \ne z \in L} N_P(z)^2 = \min_{0 \ne \alpha \in \Lambda} \alpha^t Q_X^t P^{-1} Q_X \alpha = m_\Lambda(P). \tag{13.19}$$

For every $\epsilon > 0$, (13.15) gives a smooth metric g with $P_g = P$, $N_g^{(2)} \ge N_P$, and $\mathrm{Vol}(X,g) < \kappa/2 + \epsilon$. Hence

$$\frac{\mathrm{stsys}_2(X,g)^2}{\mathrm{Vol}(X,g)} \ge \frac{m_\Lambda(P)}{\kappa/2 + \epsilon}.$$

Letting $\epsilon \downarrow 0$ gives the lower bound in (13.16). For every metric in the same period fiber, Proposition 12.2 gives

$$\operatorname{stsys}_2(X,g)^2 \le 2\operatorname{Vol}(X,g) m_\Lambda(P),$$

since $D_{4,2} = 2$.

Equation (11.3) gives $C_2(X) = T(Q_X)$. Every metric has a compatible period, and (13.15) makes each period fiber nonempty. Thus

$$\operatorname{SR}_2(X) = \sup_{P \in \mathcal{D}(Q_X)} \operatorname{SR}_2(X;P).$$

Taking suprema in (13.16) proves (13.17).

Suppose that Q_X is indefinite. Theorem 13.1 gives (13.18) and a compatible P satisfying $m_\Lambda(P) \ge c_* b$. Choose smooth metrics h_j with $P_{h_j} = P$, $N^{(2)}_{h_j} \ge N_P$, and $v_j = \operatorname{Vol}(X,h_j) < \kappa/2 + 1/j$. Put $g_j = v_j^{-1/2} h_j$. Then $P_{g_j} = P$, $\operatorname{Vol}(X,g_j) = 1$, and

$$\operatorname{stsys}_2(X,g_j)^2 = v_j^{-1} \operatorname{stsys}_2(X,h_j)^2 \ge \frac{m_\Lambda(P)}{\kappa/2 + 1/j}.$$

Also $\operatorname{confsys}_2(X,g_j)^2 = m_\Lambda(P)$ by (11.1). Let $j \to \infty$. □

Corollary 13.10. *Let $m \ge 1$ be an integer, and let X^{4m} be a closed connected oriented smooth manifold with indefinite middle intersection form. With the same constant $c_* > 0$,*

$$\frac{2c_*}{\kappa} b_{2m}(X) \le \operatorname{SR}_{2m}(X) \le D_{4m,2m}\, b_{2m}(X).$$

Proof. Put $\Lambda = H^{2m}(X;\mathbb{Z})/\text{torsion}$ and $b = b_{2m}(X)$. The middle cup pairing is symmetric and unimodular. By Theorem 13.1, it has a compatible majorant P with $m_\Lambda(P) \ge c_* b$. The calculation in (13.19) applies in degree $2m$. Proposition 13.8 therefore gives

$$\operatorname{SR}_{2m}(X) \ge \frac{2}{\kappa} m_\Lambda(P) \ge \frac{2c_*}{\kappa} b.$$

Equation (12.5) and Lemma 12.1 give $\operatorname{SR}_{2m}(X) \le D_{4m,2m}\gamma_b \le D_{4m,2m} b$. □

For an integer $r \ge 1$ and a full Euclidean lattice $L \subset \mathbb{R}^r$, let $L^* = \{v \in \mathbb{R}^r : \langle v, L\rangle \subset \mathbb{Z}\}$ be its dual lattice. The Bergé–Martinet constant is

$$\gamma_r' = \sup_{L \subset \mathbb{R}^r} \lambda_1(L)\lambda_1(L^*),$$

where the supremum runs over all full Euclidean lattices. It satisfies

$$\liminf_{r\to\infty} \frac{\gamma_r'}{r} \ge \frac{1}{2\pi e}$$

by [9, §2, equations (2.2)–(2.3)].

Corollary 13.11. *For integers $p \ge 2$ and $r \ge 1$, put $M_{p,r} = \#^r(S^p \times S^p)$. Then*

$$\frac{2\gamma_r'}{\kappa} \le \operatorname{SR}_p(M_{p,r}) \le D_{2p,p}\gamma_{2r}, \qquad \liminf_{r\to\infty} \frac{\operatorname{SR}_p(M_{p,r})}{r} \ge \frac{1}{\pi e \kappa}.$$

Proof. The integral Künneth formula [39, Theorem 3.15 and Example 3.16] and the connected-sum formula [56, Remark 8.9.17, p. 190] give, in the factor classes, the middle cup matrix

$$Q = \begin{pmatrix} 0 & I_r \\ (-1)^p I_r & 0 \end{pmatrix}.$$

Let $L = A\mathbb{Z}^r$, where $A \in \mathrm{GL}_r(\mathbb{R})$, and put $G = A^tA$. The form $P = \operatorname{diag}(G, G^{-1})$ satisfies $Q^tP^{-1}Q = P$. Its dual norm has squared minimum

$$\min_{0 \ne z \in \mathbb{Z}^{2r}} z^t P^{-1} z = \min\{\lambda_1(L)^2, \lambda_1(L^*)^2\}.$$

Put $t = (\lambda_1(L^*)/\lambda_1(L))^{1/2}$. Replacing L by tL makes both squared minima equal to $\lambda_1(L)\lambda_1(L^*)$. Proposition 13.8 gives

$$\mathrm{SR}_p(M_{p,r}) \geq \frac{2}{\kappa}\lambda_1(L)\lambda_1(L^*).$$

Taking the supremum over L proves the lower bound. The upper bound follows from (12.5), since $b_p(M_{p,r}) = 2r$. The asymptotic bound for γ'_r gives the last assertion. □

Corollary 13.12. *Let Y be a closed connected oriented smooth four-manifold whose intersection form has positive index $b_+(Y) > 0$. For integers $r \geq 1$, $C_2(Y\#r\overline{\mathbb{CP}^2})$ and $\mathrm{SR}_2(Y\#r\overline{\mathbb{CP}^2})$ have order r as $r \to \infty$. In particular, for $X_r = \mathbb{CP}^2\#r\overline{\mathbb{CP}^2}$,*

$$\liminf_{r\to\infty} \frac{C_2(X_r)}{r} \geq \frac{1}{200\pi e}, \qquad \liminf_{r\to\infty} \frac{\mathrm{SR}_2(X_r)}{r} \geq \frac{1}{100\kappa\pi e}.$$

Proof. For every integer $r \geq 1$, the first intersection form is indefinite and has rank $b_2(Y) + r$, by the connected-sum formula [56, Remark 8.9.17, p. 190]. Theorem 13.9 therefore gives bounds above and below by positive constant multiples of $b_2(Y) + r$. For X_r, the intersection form is integrally isometric to $-I_{r,1}$. Since $T(-Q) = T(Q)$, Corollary 13.4 gives

$$\liminf_{r\to\infty} \frac{C_2(X_r)}{r} = \liminf_{r\to\infty} \frac{T(I_{r,1})}{r} \geq \frac{1}{200\pi e}.$$

The bound for $\mathrm{SR}_2(X_r)$ follows from (13.17). □

The two bounds for X_r answer Questions 1.4 and 1.5 of Katz [47, p. 774]. Moreover, for every $C > 0$, there is an integer $r \geq 1$ with $\mathrm{SR}_2(X_r) > C$. By the definition of the supremum, there is a smooth metric g on X_r such that

$$\mathrm{stsys}_2(X_r, g)^2 > C\,\mathrm{Vol}(X_r, g).$$

Thus the upper bound in [22, Conjecture 5.1] does not hold.

14. Critical quotient norms and cup products

Let (M, g) be a closed connected oriented Riemannian manifold of dimension $n \geq 2$, and let $\Omega^d(M)$ denote the space of smooth real d-forms. For $1 < r < \infty$ and a measurable d-form ω, write

$$\|\omega\|_{r,*} = \left(\int_M \mathrm{comass}_g(\omega_x)^r \, dv_g(x)\right)^{1/r}.$$

For $1 \leq d < n$, put $r_d = n/d$. The quotient norm on $H^d(M;\mathbb{R})$ is

$$Q_{d,g}(\alpha) = \inf_{\substack{\omega\in\Omega^d(M),\ d\omega=0 \\ [\omega]=\alpha}} \|\omega\|_{r_d,*}, \qquad \alpha \in H^d(M;\mathbb{R}), \tag{14.1}$$

and $Q^*_{d,g}$ denotes its dual norm on $H_d(M;\mathbb{R})$ under evaluation. Regard

$$L^d = H^d(M;\mathbb{Z})/\mathrm{torsion}, \qquad L_d = H_d(M;\mathbb{Z})/\mathrm{torsion}$$

as lattices in real cohomology and homology. They are dual under evaluation [9, Lemma 3.2]. For $b_d(M) > 0$, put

$$\sigma_d(M, g) = \min_{0\neq z\in L_d} Q^*_{d,g}(z).$$

This is the conformal systole with the quotient-comass convention of [10, §2, equation (2.7)]. The norms $Q_{d,g}$ and $Q^*_{d,g}$, and hence σ_d, depend only on the conformal class of g.

For $v = \mathrm{Vol}(M, g)$, the inequality $Q_{d,g}(\alpha) \leq v^{d/n}\|\alpha\|^*_g$ and stable–comass duality give

$$N^{(d)}_g \leq v^{d/n} Q^*_{d,g}, \qquad \mathrm{stsys}_d(M, g) \leq v^{d/n}\sigma_d(M, g), \tag{14.2}$$

where the second inequality assumes $b_d(M) > 0$; see also [10, §2, equation (2.8)]. If $n = 4$ and $b_2(M) > 0$, then

$$\operatorname{confsys}_2(M,g) \le \sigma_2(M,g) \le \sqrt{2}\operatorname{confsys}_2(M,g). \tag{14.3}$$

Indeed, the pointwise inequalities $\operatorname{comass}(\omega) \le |\omega| \le \sqrt{2}\operatorname{comass}(\omega)$ and the minimizing property of harmonic representatives give, for the Hodge norm $|\alpha|_{P_g} = P_g(\alpha,\alpha)^{1/2}$,

$$Q_{2,g} \le |\cdot|_{P_g} \le \sqrt{2}Q_{2,g}.$$

Duality, together with (11.1) and (13.19), gives (14.3).

Theorem 14.1. *Let $p_1,\dots,p_k$ be positive integers, $k \ge 2$, with $p_1+\cdots+p_k = n$. Suppose the cup product*

$$H^{p_1}(M;\mathbb{R}) \otimes \cdots \otimes H^{p_k}(M;\mathbb{R}) \longrightarrow H^n(M;\mathbb{R})$$

is nonzero. There is a constant $C_{p_1,\dots,p_k}$, depending only on these degrees, such that every smooth metric satisfies

$$\prod_{j=1}^{k} \sigma_{p_j}(M,g) \le C_{p_1,\dots,p_k} \min_i \prod_{j\ne i} b_{p_j}(M), \tag{14.4}$$

$$\prod_{j=1}^{k} \operatorname{stsys}_{p_j}(M,g) \le C_{p_1,\dots,p_k} \operatorname{Vol}(M,g) \min_i \prod_{j\ne i} b_{p_j}(M). \tag{14.5}$$

For a full lattice L in a normed vector space (V,N) of dimension b, let

$$\lambda_i(L,N) = \inf\{t > 0 \,:\, \dim \operatorname{span}(L \cap \{N \le t\}) \ge i\}, \qquad 1 \le i \le b.$$

Lemma 14.2. *For $b = b_d(M) > 0$, there is a constant $B_{n,d}$, independent of M, g, b, such that*

$$\lambda_1(L_d, Q_{d,g}^*)\lambda_b(L^d, Q_{d,g}) \le B_{n,d}\, b. \tag{14.6}$$

Proof. Put $r = n/d \in (1,\infty)$, and let $\mathcal{L}_r$ be the Banach space of measurable d-forms of finite norm $\|\cdot\|_{r,*}$, modulo equality almost everywhere. Let Z and E be the closures in $\mathcal{L}_r$ of smooth closed and smooth exact forms. Let $\mathcal{H}_g^d$ be the space of harmonic d-forms, and choose an L^2-orthonormal basis $h_1,\dots,h_b$. The harmonic projection is

$$P\omega = \sum_{a=1}^{b} \left(\int_M \langle \omega, h_a\rangle_g \, dv_g \right) h_a.$$

Since the h_a are smooth, Hölder's inequality makes P a bounded map from $\mathcal{L}_r$ to $\mathcal{H}_g^d$. The Hodge decomposition [77, Theorems 6.8 and 6.11] gives

$$E = \ker(P|_Z), \qquad Z/E \simeq \mathcal{H}_g^d \simeq H^d(M;\mathbb{R}).$$

Indeed, if smooth closed z_j converge to $z \in Z$, the exact forms $z_j - Pz_j$ converge to $z - Pz$. Conversely, P vanishes on E. For $h = Pz$, the forms $z_j + h - Pz_j$ all represent $[h]$ and converge to z. Thus the quotient norm on Z/E is $Q_{d,g}$.

Let $(\Xi, \mathcal{A}, \mathbb{P})$ be a probability space carrying independent standard real Gaussian variables γ_ℓ, $\ell \ge 1$, and let $\mathbb{E}$ denote expectation. For a real Banach space Y, its Gaussian K-convexity constant is

$$K(Y) = \sup_{m\ge 1} \|G_m\|_{L^2(\Xi;Y)\to L^2(\Xi;Y)}, \qquad G_m f = \sum_{\ell=1}^{m} \gamma_\ell \mathbb{E}(\gamma_\ell f);$$

see [74, Definition 10.3]. Put $N = \binom{n}{d}$, $a_{n,d} = \sqrt{N}$, and let $\mathcal{U}_r$ be the space of measurable d-forms with the L^r Euclidean fiber norm. Then

$$\|\omega\|_{r,*} \le \|\omega\|_{\mathcal{U}_r} \le a_{n,d}\|\omega\|_{r,*}, \qquad K(\mathcal{L}_r) \le a_{n,d}K(\mathcal{U}_r).$$

A finite Borel partition subordinate to orthonormal trivializations of $\Lambda^d T^*M$ identifies $\mathcal{U}_r$ isometrically with $L^r(M, dv_g; \mathbb{R}^N)$. Since $1 < r < \infty$ and dv_g is finite, [74, Examples 10.5–10.6] gives $K(\mathcal{U}_r) \le c_r$, where c_r depends only on r. The dual of Z/E is isometric to a closed subspace of Z^*. Subspace monotonicity and self-duality of K [74, Definition 10.3 and Proposition 10.4] give

$$\begin{aligned} K(Z/E) = K((Z/E)^*) &\le K(Z^*) = K(Z) \\ &\le K(\mathcal{L}_r) \le a_{n,d} c_r. \end{aligned}$$

The Gaussian and Rademacher constants differ by at most a numerical factor [8, Remark 5.6]. Banaszczyk's transference theorem [7, Theorem 1, p. 308], applied to $(H^d(M;\mathbb{R}), Q_{d,g})$ and its dual integral lattices, therefore gives (14.6) with $B_{n,d} = C a_{n,d} c_{n/d}$ for a numerical constant C. □

Proof of Theorem 14.1. The cup-product argument is that of [9, proof of Theorem 2.7], with the norms $Q_{d,g}$ and Lemma 14.2. Put $b_j = b_{p_j}(M)$, $Q_j = Q_{p_j,g}$, $\sigma_j = \sigma_{p_j}(M,g)$, and $\lambda_j = \lambda_{b_j}(L^{p_j}, Q_j)$. All b_j are positive by the cup-product hypothesis. For $\epsilon > 0$, choose a real basis of each $H^{p_j}(M;\mathbb{R})$ consisting of integral classes of Q_j-norm less than $\lambda_j + \epsilon$. Multilinearity gives one class u_j from each basis such that $\langle u_1 \cdots u_k, [M]\rangle \neq 0$. Choose smooth closed representatives ω_j with $\|\omega_j\|_{n/p_j,*} < \lambda_j + 2\epsilon$.

For every i, integral Poincaré duality gives a nonzero class

$$h_i = \mathrm{PD}(u_1 \cdots u_{i-1} u_{i+1} \cdots u_k) \in L_{p_i}.$$

It is independent of the integral lifts modulo torsion and pairs with u_i to the nonzero top product, up to sign. Set $W = n!/(p_1! \cdots p_k!)$. For every smooth closed p_i-form β, the alternating-product formula and Hölder's inequality give

$$|\langle [\beta], h_i\rangle| \le W \int_M \mathrm{comass}_g(\beta) \prod_{j\neq i} \mathrm{comass}_g(\omega_j)\, dv_g \le W \|\beta\|_{n/p_i,*} \prod_{j\neq i} (\lambda_j + 2\epsilon).$$

Taking the infimum over representatives and the dual supremum gives $\sigma_i \le Q_i^*(h_i) \le W \prod_{j\neq i}(\lambda_j + 2\epsilon)$. Hence $\sigma_i \le W \prod_{j\neq i} \lambda_j$ as $\epsilon \downarrow 0$. By Lemma 14.2,

$$\prod_{j=1}^{k} \sigma_j \le W \prod_{j\neq i} (\sigma_j \lambda_j) \le W \prod_{j\neq i} (B_{n,p_j} b_j).$$

This holds for every i, so $C_{p_1,\dots,p_k} = W \max_i \prod_{j\neq i} B_{n,p_j}$ is admissible. Multiplication of (14.2) over j gives (14.5). □

Corollary 14.3. *If $n = mp$, $p \ge 1$, $m \ge 2$, and some product of m real classes of degree p is nonzero, then*

$$\sigma_p(M,g)^m \le C_{p,m} b_p(M)^{m-1}, \qquad \mathrm{stsys}_p(M,g)^m \le C_{p,m} b_p(M)^{m-1} \mathrm{Vol}(M,g).$$

Proof. Take $p_1 = \cdots = p_m = p$ in Theorem 14.1. □

For two complementary degrees, the linear Betti-number dependence of the stable estimate is already contained in [9, Theorem 2.2]. For $m = 3$, the stable estimate with the power b_p^2 is [9, Theorem 8.1], with coefficient

$$\binom{3p}{p} \frac{(3p)!}{(p!)^3}.$$

Their Theorem 2.7 gives the equal-degree estimate for arbitrary m with $(b_p(1 + \log b_p))^{m-1}$ in place of b_p^{m-1}.

15. Sharp Betti number growth

Fix integers $p \ge 2$ and $m \ge 3$, and put $n = mp$. For a closed connected oriented smooth n-manifold with $b_p(M) > 0$, put

$$\mathrm{SR}_p(M) = \sup_g \frac{\mathrm{stsys}_p(M,g)^m}{\mathrm{Vol}(M,g)},$$

where g ranges over the smooth Riemannian metrics on M. For each integer $b > 0$, let $A_{p,m}(b)$ be the supremum of $\mathrm{SR}_p(M)$ over these manifolds with $b_p(M) = b$ and a nonzero product of m real classes of degree p. Let $A^{\mathrm{par}}_{p,m}(b)$ be the supremum restricted to $(p-1)$-connected stably parallelizable manifolds, and let $A^{\mathrm{c}}_{p,m}(b)$ be the supremum of $\sigma_p(M,g)^m$ over all manifolds in the definition of $A_{p,m}(b)$ and all their smooth Riemannian metrics. The supremum of an empty family is zero. Stable parallelizability means that $TM \oplus (M \times \mathbb{R}^a)$ is trivial for some integer $a \geq 0$.

Put

$$q = \begin{cases} 1, & p \text{ even}, \\ m, & p \text{ odd}, \end{cases} \qquad D = 5 \cdot 13 \cdot 17 \cdot 29 \cdot 37 \cdot 41 \cdot 53 \cdot 61.$$

Theorem 15.1. *For every integer $b \geq 2q$,*

$$(2q)^{1-m} D^{-m/2} b^{m-1} \leq A^{\mathrm{par}}_{p,m}(b) \leq A_{p,m}(b) \leq A^{\mathrm{c}}_{p,m}(b) \leq C_{p,m} b^{m-1}. \tag{15.1}$$

The lower estimate holds on manifolds with free integral homology in degree p. When p is even, these manifolds have a primitive class $u \in H^p(M;\mathbb{Z})$ with $u^m \neq 0$.

15.1. **Integral top cup forms.**

Lemma 15.2. *Let A be a free abelian group of positive finite rank. If p is even, let $\tau : A^m \to \mathbb{Z}$ be symmetric and multilinear, and set $T = m!\tau$. If p is odd, let $T : A^m \to \mathbb{Z}$ be alternating and multilinear. There is a closed oriented smooth stably parallelizable $(p-1)$-connected mp-manifold M with free integral homology in degree p and an isomorphism $\iota : A \to H^p(M;\mathbb{Z})$ such that, for all $x_1, \dots, x_m \in A$,*

$$\langle \iota(x_1) \cdots \iota(x_m), [M] \rangle = T(x_1, \dots, x_m). \tag{15.2}$$

For $p = 2$ and $m = 3$, this follows from Wall's classification with trilinear form 6τ and zero first Pontryagin class [76, Theorem 5 and pp. 357, 361].

Proof. Choose a basis $a_1, \dots, a_d$ of A. Let $X = K(\mathbb{Z}^d, p)$, and let $u_1, \dots, u_d \in H^p(X;\mathbb{Z})$ be dual to the standard basis of $H_p(X;\mathbb{Z}) \simeq \mathbb{Z}^d$. The space X represents d-tuples of integral degree-p classes [39, Theorem 4.57]. Give $Y = (S^p)^m$ the product orientation, and let $v_r \in H^p(Y;\mathbb{Z})$ be the pullback of the positive generator from its rth sphere factor.

Suppose first that p is even. For $\nu = (\nu_1, \dots, \nu_d) \in \mathbb{Z}^d_{\geq 0}$, write $|\nu| = \sum_i \nu_i$. If $|\nu| = m$, choose a partition $\{1, \dots, m\} = I_1 \sqcup \cdots \sqcup I_d$ with $|I_i| = \nu_i$. There is a map $f_\nu : Y \to X$ such that

$$f_\nu^* u_i = \sum_{r \in I_i} v_r.$$

The classes v_r commute and satisfy $v_r^2 = 0$ and $\langle v_1 \cdots v_m, [Y] \rangle = 1$. Hence, for $\eta \in \mathbb{Z}^d_{\geq 0}$ with $|\eta| = m$,

$$\left\langle f_\nu^*(u_1^{\eta_1} \cdots u_d^{\eta_d}), [Y] \right\rangle = \begin{cases} \prod_i \nu_i!, & \eta = \nu, \\ 0, & \eta \neq \nu. \end{cases}$$

For $\eta = \nu$, each block contributes the factor $\nu_i!$. For $\eta \neq \nu$, some $\eta_i > \nu_i$, and that factor vanishes. Put

$$c_\nu = \frac{m!}{\prod_i \nu_i!} \tau(\underbrace{a_1, \dots, a_1}_{\nu_1}, \dots, \underbrace{a_d, \dots, a_d}_{\nu_d}).$$

Since $m! / \prod_i \nu_i!$ is a multinomial coefficient, c_ν is an integer. Take $|c_\nu|$ copies of f_ν, reversing their orientations when $c_\nu < 0$.

Suppose now that p is odd. For every $I = \{i_1 < \cdots < i_m\} \subset \{1, \dots, d\}$, choose a map $f_I : Y \to X$ with $f_I^* u_{i_r} = v_r$ and all other $f_I^* u_i = 0$. Write δ_{ij} for the Kronecker symbol. For $j_1, \dots, j_m \in \{1, \dots, d\}$, graded commutativity gives

$$\left\langle f_I^*(u_{j_1} \cdots u_{j_m}), [Y] \right\rangle = \det(\delta_{i_r j_s})_{1 \leq r,s \leq m}.$$

Put $t_I = T(a_{i_1}, \dots, a_{i_m})$. Take $|t_I|$ copies of f_I, reversing their orientations when $t_I < 0$.

Adjoin d copies of $S^p \times S^{n-p}$. On the ith copy, take the map $h_i \circ \mathrm{pr}_1$, where $h_i : S^p \to X$ represents the ith standard basis vector of $\pi_p(X) = \mathbb{Z}^d$ and pr_1 is projection to the first sphere. These maps contribute zero to every product of m degree-p classes. Let $h : Q \to X$ be the resulting map from the finite disjoint union. Since $d > 0$, the manifold Q is nonempty. For all $i_1, \dots, i_m \in \{1, \dots, d\}$,

$$\langle h^*(u_{i_1} \cdots u_{i_m}), [Q] \rangle = T(a_{i_1}, \dots, a_{i_m}).$$

Make the maps constant on the joining balls and form the oriented connected sum P. The maps extend over the joining one-handles. Products of spheres are stably parallelizable, and their stable framings extend over these handles [57, Lemma 2′ and §4]. The manifold P is $(p-1)$-connected and stably parallelizable. Its degree-p integral homology is the direct sum of those of the sphere products, and is free. The induced map $f : P \to X$ gives a split surjection

$$f_* : H_p(P; \mathbb{Z}) \twoheadrightarrow H_p(X; \mathbb{Z}) \simeq \mathbb{Z}^d.$$

The first sphere classes of the d added products give a right inverse. Thus $L = \ker f_*$ is a free direct summand of $H_p(P; \mathbb{Z})$.

Choose a basis $\ell_1, \dots, \ell_r$ of L, and put $P_0 = P$, $f_0 = f$. Suppose that $0 \le j < r$ and that P_j is $(p-1)$-connected and stably parallelizable, with a map $f_j : P_j \to X$ and an identification

$$H_p(P_j; \mathbb{Z}) \simeq H_p(P; \mathbb{Z}) / \langle \ell_1, \dots, \ell_j \rangle$$

under which $(f_j)_*$ is induced by f_*. Write $\bar{\ell}_{j+1}$ for the image of ℓ_{j+1} in this quotient. By the Hurewicz theorem [39, Theorem 4.32], it is represented by a class in $\pi_p(P_j)$. Since $n = mp \ge 3p \ge 2p + 2$, Milnor's theorem [57, Theorem 2] gives a framed embedded p-sphere in this class such that surgery preserves stable parallelizability. By naturality of Hurewicz, its image in $\pi_p(X)$ is zero. Thus f_j extends over the surgery trace W_j to a map $F_j : W_j \to X$. Let P_{j+1} be the outgoing boundary, and put $f_{j+1} = F_j|_{P_{j+1}}$.

The trace W_j has one handle of index $p+1$ relative to P_j and one of index $n - p \ge p + 2$ relative to P_{j+1}. The relative integral homology sequences give

$$H_p(P_{j+1}; \mathbb{Z}) \simeq H_p(W_j; \mathbb{Z}) \simeq H_p(P_j; \mathbb{Z}) / \mathbb{Z}\bar{\ell}_{j+1},$$

since the boundary of the incoming handle core represents $\bar{\ell}_{j+1}$. The induced map to $H_p(X; \mathbb{Z})$ is the quotient of $(f_j)_*$. Moreover, P_{j+1} is $(p-1)$-connected [57, Lemma 2]. Induction gives, for $M = P_r$ and $g = f_r$, an isomorphism $g_* : H_p(M; \mathbb{Z}) \to H_p(X; \mathbb{Z})$. In particular, $H_p(M; \mathbb{Z})$ is free.

Since $H_{p-1}(M; \mathbb{Z}) = H_{p-1}(X; \mathbb{Z}) = 0$, the natural universal coefficient isomorphisms [39, Theorem 3.2] identify g^* in degree p with the integral dual of g_*. Hence

$$\iota(a_i) = g^* u_i \qquad (1 \le i \le d)$$

defines an isomorphism $\iota : A \to H^p(M; \mathbb{Z})$. The joining handles and the surgery traces form an oriented bordism over X from (Q, h) to (M, g). Its boundary orientation is $M \sqcup (-Q)$, so

$$g_*[M] = h_*[Q] \quad \text{in } H_n(X; \mathbb{Z}).$$

Evaluation of the products of the u_i gives (15.2) on basis tuples, and multilinearity gives it on A^m. □

15.2. **A totally real unramified tower.** Let $F = \mathbb{Q}(\sqrt{D})$. For a number field K, write $\mathcal{O}_K$ for its ring of integers and $\mathrm{disc}(K)$ for its discriminant. Its codifferent is

$$\mathcal{O}_K^\vee = \{x \in K : \mathrm{Tr}_{K/\mathbb{Q}}(x\mathcal{O}_K) \subset \mathbb{Z}\},$$

and its different is the inverse fractional ideal $\mathfrak{D}_{K/\mathbb{Q}} = (\mathcal{O}_K^\vee)^{-1}$.

Lemma 15.3. *There is a sequence of totally real number fields $F = K_0 \subset K_1 \subset \cdots$, unramified at every finite place over F, such that, for every integer $j \ge 0$,*

$$d_j = [K_j : \mathbb{Q}] = 2^{j+1}, \qquad |\mathrm{disc}(K_j)|^{1/d_j} = \sqrt{D}, \qquad \mathcal{O}_{K_j}^\vee = \frac{1}{\sqrt{D}} \mathcal{O}_{K_j}.$$

Moreover,

$$\min_{0\neq x\in\mathcal{O}_{K_j}^{\vee}} \sum_{\sigma\,:\,K_j\hookrightarrow\mathbb{R}} |\sigma(x)| = \frac{d_j}{\sqrt{D}}. \tag{15.3}$$

Proof. The integer D is squarefree and congruent to one modulo four, so F is real quadratic of discriminant D. Let S be its two real places, and let L/F be the maximal pro-2 extension unramified at finite places in which S splits completely. Put $G = \mathrm{Gal}(L/F)$. By class field theory and the pro-2 Burnside basis theorem, G is finitely generated; denote by $d(G)$ its minimal number of topological generators. The extension $F/\mathbb{Q}$ has eight ramified finite places, and the image of S in $\mathbb{Q}$ consists of one place. Since $\mathbb{Q}$ contains the second roots of unity, the genus bound [35, Theorem 2.5] gives

$$d(G) \geq 8 - 1 - 1 = 6.$$

For L/F, the ramification set is empty, $|S| = 2$, and F contains the second roots of unity. Since $6 > 2 + 2\sqrt{2+1}$, Theorem 2.4 of [35] implies that G is infinite.

Set $H_0 = G$. Every open subgroup of G is infinite, and every nontrivial pro-2 group has an open subgroup of index two. Choose inductively open subgroups $H_{j+1} \subset H_j$ with $[H_j : H_{j+1}] = 2$, and put $K_j = L^{H_j}$. Then $K_0 = F$, $[K_j : F] = [G : H_j] = 2^j$, and $K_j \subset K_{j+1}$. Every K_j is totally real and unramified at finite places over F, since it is a subfield of L.

Fix $j \geq 0$, and put $K = K_j$ and $d = d_j$. Let $\mathfrak{p}$ be a nonzero prime ideal of $\mathcal{O}_K$ above a rational prime ℓ, and let $v_{\mathfrak{p}} : K^{\times} \to \mathbb{Z}$ be its normalized valuation. Multiplicativity of ramification indices gives ramification index two over $\mathbb{Q}$ if $\ell \mid D$, and one otherwise. All ramification is tame, since D is odd. If $\ell \mid D$, then

$$2v_{\mathfrak{p}}(\sqrt{D}) = v_{\mathfrak{p}}(D) = 2;$$

if $\ell \nmid D$, both valuations are zero. The tame different formula [20, Theorem 4.13] therefore gives

$$\mathfrak{D}_{K/\mathbb{Q}} = \sqrt{D}\,\mathcal{O}_K.$$

Every embedding $\sigma : K \hookrightarrow \mathbb{R}$ satisfies $|\sigma(\sqrt{D})| = \sqrt{D}$. The norm formula for the different [20, Theorem 4.8] gives

$$|\operatorname{disc}(K)| = \left|\mathrm{N}_{K/\mathbb{Q}}(\sqrt{D})\right| = D^{d/2}.$$

Inverting the different gives $\mathcal{O}_K^{\vee} = (1/\sqrt{D})\mathcal{O}_K$.

For $0 \neq y \in \mathcal{O}_K$, the field norm is a nonzero integer. The arithmetic–geometric mean inequality gives

$$\begin{aligned}\sum_{\sigma\,:\,K\hookrightarrow\mathbb{R}} \left|\sigma\left(\frac{y}{\sqrt{D}}\right)\right| &= D^{-1/2} \sum_{\sigma\,:\,K\hookrightarrow\mathbb{R}} |\sigma(y)| \\ &\geq D^{-1/2} d\,|\mathrm{N}_{K/\mathbb{Q}}(y)|^{1/d} \geq \frac{d}{\sqrt{D}}.\end{aligned}$$

Equality holds for $y = 1$, proving (15.3). □

15.3. **Trace tensors and comass moments.** Let $\mathbb{R}^{mp} = E_1 \oplus \cdots \oplus E_m$ and ξ_r be as in Lemma 13.5, with $\xi_1 \wedge \cdots \wedge \xi_m = dv$. Let F be a totally real number field of degree b, let $\mathcal{O}_F$ be its ring of integers, and let $\sigma_1, \ldots, \sigma_b : F \to \mathbb{R}$ be its distinct embeddings. Put $F_{\mathbb{R}} = F \otimes_{\mathbb{Q}} \mathbb{R}$, and extend each σ_ν to an $\mathbb{R}$-algebra homomorphism $F_{\mathbb{R}} \to \mathbb{R}$. The map $a \mapsto (\sigma_1(a), \ldots, \sigma_b(a))$ identifies $F_{\mathbb{R}}$ with $\mathbb{R}^b$. Write

$$\mathrm{Tr}(a) = \sum_{\nu=1}^{b} \sigma_\nu(a)$$

for the extended trace. If p is even, set $V = F_{\mathbb{R}}$ and define

$$T(a_1, \ldots, a_m) = m!\,\mathrm{Tr}(a_1 \cdots a_m), \qquad |a|_\infty = \max_\nu |\sigma_\nu(a)|. \tag{15.4}$$

If p is odd, set $V = F_{\mathbb{R}}^m$. For $a_i = (a_{i1}, \ldots, a_{im}) \in V$, define

$$T(a_1, \ldots, a_m) = \mathrm{Tr}\det(a_{ir})_{1\leq i,r\leq m}, \qquad |a_i|_\infty = \max_{\nu,r} |\sigma_\nu(a_{ir})|. \tag{15.5}$$

Lemma 15.4. *Let $a_1, \dots, a_s \in V$ satisfy $|a_i|_\infty \le 1$. There are b tuples $\eta^{(\nu)} = (\eta_1^{(\nu)}, \dots, \eta_s^{(\nu)})$ of real constant p-forms on $\mathbb{R}^{mp}$ such that*

$$\operatorname{comass}(\eta_i^{(\nu)}) \le 1$$

and, for every $1 \le i_1, \dots, i_m \le s$,

$$\sum_{\nu=1}^{b} \frac{\eta_{i_1}^{(\nu)} \wedge \cdots \wedge \eta_{i_m}^{(\nu)}}{dv} = T(a_{i_1}, \dots, a_{i_m}). \tag{15.6}$$

Proof. Suppose first that p is even. Put

$$\omega = \sum_{r=1}^{m} \xi_r, \qquad \eta_i^{(\nu)} = \sigma_\nu(a_i)\omega.$$

Lemma 13.5 gives $\operatorname{comass}(\eta_i^{(\nu)}) = |\sigma_\nu(a_i)| \le 1$. Since the ξ_r commute, $\xi_r^2 = 0$, and $\xi_1 \wedge \cdots \wedge \xi_m = dv$, one has $\omega^m = m!\,dv$. Hence

$$\frac{\eta_{i_1}^{(\nu)} \wedge \cdots \wedge \eta_{i_m}^{(\nu)}}{dv} = m! \prod_{j=1}^{m} \sigma_\nu(a_{i_j}).$$

Summation over ν gives (15.6).

Suppose now that p is odd. Put

$$\eta_i^{(\nu)} = \sum_{r=1}^{m} \sigma_\nu(a_{ir})\xi_r.$$

By Lemma 13.5, $\operatorname{comass}(\eta_i^{(\nu)}) = \max_r |\sigma_\nu(a_{ir})| \le 1$. Let $\mathfrak{S}_m$ be the permutation group of $\{1, \dots, m\}$. The block forms anticommute, so

$$\begin{aligned} \frac{\eta_{i_1}^{(\nu)} \wedge \cdots \wedge \eta_{i_m}^{(\nu)}}{dv} &= \sum_{\tau \in \mathfrak{S}_m} \operatorname{sgn}(\tau) \prod_{j=1}^{m} \sigma_\nu(a_{i_j,\tau(j)}) \\ &= \sigma_\nu\left(\det(a_{i_j,r})_{1 \le j,r \le m}\right). \end{aligned}$$

Summation over ν gives (15.6). If two indices coincide, both the determinant and the wedge product vanish. □

Let M be a closed connected oriented smooth mp-manifold. Suppose an isomorphism of abelian groups

$$H^p(M;\mathbb{Z})/\text{ torsion} \simeq \begin{cases} \mathcal{O}_F, & p \text{ even}, \\ \mathcal{O}_F^m, & p \text{ odd} \end{cases} \tag{15.7}$$

identifies the top cup form with T. Its real extension identifies $H^p(M;\mathbb{R})$ with V. Let N_∞ denote the norm on $H_p(M;\mathbb{R})$ dual to $|\cdot|_\infty$.

Proposition 15.5. *For every $0 < \epsilon < 1$ and $\delta > 0$, there is a smooth metric g on M such that*

$$N_g^{(p)} \ge (1-\epsilon)N_\infty, \qquad \operatorname{Vol}(M,g) < b + \delta. \tag{15.8}$$

Consequently,

$$\inf_{g\,:\,N_g^{(p)} \ge N_\infty} \operatorname{Vol}(M,g) \le b. \tag{15.9}$$

Proof. Put $B = \{a \in V : |a|_\infty \le 1\}$, and let $\mathcal{E}$ be its set of vertices. In the embedding coordinates, B is a cube. Since $H^p(M;\mathbb{Q})$ is dense in V, for each $v \in \mathcal{E}$ choose $a_v \in H^p(M;\mathbb{Q})$ such that

$$|a_v - (1-\epsilon/2)v|_\infty < \epsilon/2.$$

Then $|a_v|_\infty < 1$. Order the finite list $A = (a_v)_{v\in\mathcal{E}}$, and let T_A, K_A and t_A be its top cup-moment vector, moment body and Euclidean moment map, as in Theorem 1.1. All degrees equal p, so every top monomial has m factors. Lemma 15.4 gives tuples $\eta^{(\nu)}$ such that

$$\frac{1}{b}T_A = \frac{1}{b}\sum_{\nu=1}^{b} t_A(\eta^{(\nu)}) \in K_A, \qquad \gamma_{K_A}(T_A) \le b.$$

Theorem 1.1 therefore gives a smooth metric g with $\mathrm{Vol}(M,g) < b+\delta$ and $\|a_v\|_g^* \le 1$ for every $v \in \mathcal{E}$. For $z \in H_p(M;\mathbb{R})$, choose $v \in \mathcal{E}$ such that $\langle v, z\rangle = N_\infty(z)$. By (1.1),

$$\begin{aligned} N_g^{(p)}(z) &\ge |\langle a_v, z\rangle| \\ &\ge (1-\epsilon/2)N_\infty(z) - |a_v - (1-\epsilon/2)v|_\infty N_\infty(z) \\ &\ge (1-\epsilon)N_\infty(z). \end{aligned}$$

This proves (15.8).

The metric $g' = (1-\epsilon)^{-2/p} g$ satisfies

$$N_{g'}^{(p)} \ge N_\infty, \qquad \mathrm{Vol}(M,g') < (1-\epsilon)^{-m}(b+\delta).$$

Letting $\epsilon, \delta \downarrow 0$ proves (15.9). □

Write N for the absolute norm of a nonzero fractional ideal of F. Let $D_F = |\operatorname{disc}(F)|$ and $R_F = D_F^{1/b}$ be its absolute and root discriminants. Its codifferent is

$$\mathfrak{D}_F^{-1} = \{y \in F : \mathrm{Tr}(y\mathcal{O}_F) \subset \mathbb{Z}\},$$

and its inverse $\mathfrak{D}_F$ is the different. Put $\Lambda = H_p(M;\mathbb{Z})/\text{ torsion}$, viewed as a lattice in $H_p(M;\mathbb{R})$. The integral homology and cohomology lattices are dual [9, Lemma 3.2]. Under (15.7), the trace pairing therefore identifies Λ with $\mathfrak{D}_F^{-1}$ when p is even and with $(\mathfrak{D}_F^{-1})^m$ when p is odd. In the odd case, write $a = (a_1,\dots,a_m)$ and $y = (y_1,\dots,y_m)$. The evaluation pairing is

$$\langle a, y\rangle = \begin{cases} \mathrm{Tr}(ay), & p \text{ even}, \\ \displaystyle\sum_{r=1}^{m} \mathrm{Tr}(a_r y_r), & p \text{ odd}. \end{cases}$$

Thus the dual of the coordinate maximum norm is

$$\begin{aligned} N_\infty(y) &= \sum_{\nu=1}^{b} |\sigma_\nu(y)| \qquad (p \text{ even}), \\ N_\infty(y_1,\dots,y_m) &= \sum_{r=1}^{m}\sum_{\nu=1}^{b} |\sigma_\nu(y_r)| \quad (p \text{ odd}). \end{aligned} \tag{15.10}$$

Lemma 15.6. *Every $0 \ne y \in \Lambda$ satisfies*

$$N_\infty(y) \ge b/R_F.$$

Proof. The norm formula for the different [20, Theorem 4.8] gives $\mathrm{N}(\mathfrak{D}_F^{-1}) = D_F^{-1}$. The ℓ^1 case of the fractional-ideal bound [60, Lemma 6.2, p. 15, full version] therefore gives, for $0 \ne y \in \mathfrak{D}_F^{-1}$,

$$\sum_{\nu=1}^{b} |\sigma_\nu(y)| \ge b\,\mathrm{N}(\mathfrak{D}_F^{-1})^{1/b} = b/R_F.$$

If p is odd, at least one component of a nonzero vector in $(\mathfrak{D}_F^{-1})^m$ is nonzero. Apply this bound to that component and then (15.10). □

Corollary 15.7. *Under* (15.7),

$$\sup_g \frac{\mathrm{stsys}_p(M,g)^m}{\mathrm{Vol}(M,g)} \geq \frac{b^{m-1}}{R_F^m}.$$

Proof. For every $\delta > 0$, (15.9) gives a metric g with $N_g^{(p)} \geq N_\infty$ and $\mathrm{Vol}(M,g) < b + \delta$. Lemma 15.6 gives $\mathrm{stsys}_p(M,g) \geq b/R_F$. Hence

$$\sup_g \frac{\mathrm{stsys}_p(M,g)^m}{\mathrm{Vol}(M,g)} \geq \frac{(b/R_F)^m}{b+\delta}.$$

Let $\delta \downarrow 0$. □

15.4. **All sufficiently large Betti numbers.**

Theorem 15.8. *For every integer $d \geq 2q$ there is a closed oriented smooth stably parallelizable $(p-1)$-connected mp-manifold $M_{p,m,d}$, with free integral homology in degree p and $b_p(M_{p,m,d}) = d$, such that some product of m integral degree-p classes is nonzero and*

$$\mathrm{SR}_p(M_{p,m,d}) \geq (2q)^{1-m} D^{-m/2} d^{m-1}. \tag{15.11}$$

For $d = q2^{j+1}$, $j \geq 0$, one can replace the factor $(2q)^{1-m}$ by q^{1-m}. If p is even, a primitive class $u \in H^p(M_{p,m,d};\mathbb{Z})$ may be chosen with $u^m \neq 0$.

Proof. Fix a field K_j from Lemma 15.3, and write $b = d_j$ and $\mathcal{O}_j = \mathcal{O}_{K_j}$. If p is even, apply Lemma 15.2 to $A = \mathcal{O}_j$ and

$$\tau(a_1, \dots, a_m) = \mathrm{Tr}_{K_j/\mathbb{Q}}(a_1 \cdots a_m).$$

If p is odd, take $A = \mathcal{O}_j^m$ and the alternating form

$$T(a_1, \dots, a_m) = \mathrm{Tr}_{K_j/\mathbb{Q}} \det(a_{ir})_{1 \leq i,r \leq m}, \qquad a_i = (a_{i1}, \dots, a_{im}).$$

These forms are integral. Denote the resulting manifold by M and identify $H^p(M;\mathbb{Z})$ with A by the isomorphism in that lemma. Then $b_p(M) = qb$ and the cup form is that of (15.4) or (15.5). Since K_j has root discriminant $\sqrt{D}$, Corollary 15.7 gives

$$\mathrm{SR}_p(M) \geq D^{-m/2} b^{m-1}. \tag{15.12}$$

For $d = qb$, this is the stated estimate with factor q^{1-m}. If p is even, the class u corresponding to $1 \in \mathcal{O}_j$ is primitive, since $\mathcal{O}_j \cap \mathbb{Q} = \mathbb{Z}$, and

$$\langle u^m, [M] \rangle = m! b.$$

If p is odd, let u_r correspond to the element of $\mathcal{O}_j^m$ with 1 in the rth component and 0 elsewhere. Then

$$\langle u_1 \cdots u_m, [M] \rangle = b.$$

For arbitrary $d \geq 2q$, choose a power of two $b \geq 2$ with

$$\frac{d}{2q} < b \leq \frac{d}{q},$$

and put $r = d - qb$. Let M be the manifold constructed above for the field of degree b. If $r = 0$, take $M_{p,m,d} = M$. Suppose $r > 0$, and form

$$\widehat{M} = M \mathbin{\#} \overset{r}{\#} (S^p \times S^{n-p}).$$

The connected sum is $(p-1)$-connected and, with compatible stable framings, stably parallelizable [57, Lemma $2'$ and §4]. The connected-sum homology formula [39, §3.3, Exercise 6(a)] gives

$$H_p(\widehat{M};\mathbb{Z}) \simeq H_p(M;\mathbb{Z}) \oplus \mathbb{Z}^r,$$
$$H^p(\widehat{M};\mathbb{Z}) \simeq A \oplus \mathbb{Z}^r.$$

In particular, $H_p(\widehat{M};\mathbb{Z})$ is free and $b_p(\widehat{M}) = qb + r = d$. Products of positive-degree classes from distinct summands vanish. Since $n - p > p$, the degree-p generator in each added summand is pulled back from S^p and has square zero [39, Examples 3.14 and 3.16]. Thus, for real classes $v_i = (a_i, t_i)$ under the preceding identification,

$$\langle v_1 \cdots v_m, [\widehat{M}]\rangle = \langle a_1 \cdots a_m, [M]\rangle.$$

Put $X = M \vee \bigvee_{\ell=1}^{r} S^p$, and let $i : M \hookrightarrow X$ be the inclusion. The connected-sum pinch map, followed by the identity on M and the first projection on each added summand, gives a map $f : \widehat{M} \to X$ such that

$$f_* : H_p(\widehat{M};\mathbb{Z}) \xrightarrow{\simeq} H_p(X;\mathbb{Z}),$$
$$f_*[\widehat{M}] = i_*[M] \quad \text{in } H_n(X;\mathbb{Z}).$$

Since X is simply connected and $n \geq 6$, Brunnbauer's lemma [17, Lemma 3.2] gives a simplicial map homotopic to f for which the inverse image of each open n-simplex of X is empty or one open n-simplex mapped homeomorphically onto it. Define $\mathrm{SR}_p(X)$ by the same supremum over continuous piecewise smooth Riemannian metrics on X [17, §2]. The space X is obtained from M by attaching r cells of dimension $p < n$, and i_* is a split injection on integral degree-p homology. The comparison and extension lemmas [17, Lemmas 2.2 and 2.3] therefore give

$$\begin{aligned}\mathrm{SR}_p(\widehat{M}) &\geq \mathrm{SR}_p(X) = \mathrm{SR}_p(M)\\ &\geq D^{-m/2} b^{m-1} \geq (2q)^{1-m} D^{-m/2} d^{m-1}.\end{aligned}$$

Take $M_{p,m,d} = \widehat{M}$. The nonzero product on M remains nonzero on $\widehat{M}$. If p is even, $(u, 0)$ remains primitive in $A \oplus \mathbb{Z}^r$. □

Proof of Theorem 15.1. Theorem 15.8 gives the lower estimate with $c_{p,m} = (2q)^{1-m} D^{-m/2}$. The inequality $A^{\mathrm{par}}_{p,m}(b) \leq A_{p,m}(b)$ follows by inclusion of the two classes of manifolds. Equation (14.2) gives $A_{p,m}(b) \leq A^{\mathrm{c}}_{p,m}(b)$, and Corollary 14.3 gives the upper estimate. □

For every $d \geq 2q$, put $M = M_{p,m,d}$ and

$$c'_{p,m} = 2^{-1/m} (2q)^{(1-m)/m} D^{-1/2}.$$

By (15.11), there is a metric g_0 on M with

$$\frac{\mathrm{stsys}_p(M, g_0)^m}{\mathrm{Vol}(M, g_0)} > \frac{1}{2} (2q)^{1-m} D^{-m/2} d^{m-1}.$$

For $g = \mathrm{Vol}(M, g_0)^{-2/n} g_0$, one has

$$\mathrm{Vol}(M, g) = 1, \qquad \mathrm{stsys}_p(M, g) > c'_{p,m} d^{1-1/m}.$$

The rational cup-length of M is exactly m. The integral products above remain nonzero over $\mathbb{Q}$. Conversely, $H^k(M;\mathbb{Q}) = 0$ for $0 < k < p$, so a nonzero product of ℓ positive-degree classes has degree at least ℓp; hence $\ell p \leq n = mp$.

16. The full-rank stable two-systole

Let E be a real vector space of dimension $b \geq 1$, let N be a norm on E, and let $L \subset E$ be a full lattice. For a Lebesgue measure μ on E, denote by $\mathrm{covol}_\mu(L)$ the μ-measure of a fundamental parallelepiped. The Hausdorff–Busemann covolume is

$$\mathrm{covol}_{\mathrm{HB}}(L, N) = \frac{\omega_b \, \mathrm{covol}_\mu(L)}{\mu(\{N \leq 1\})}, \qquad \omega_b = \frac{\pi^{b/2}}{\Gamma(1 + b/2)},$$

where Γ is Euler's gamma function and ω_b is the volume of the Euclidean unit ball. The quotient is independent of μ; see [41, §2]. For a closed oriented smooth Riemannian four-manifold (M, g) with $b = b_2(M) > 0$, put

$$\mathrm{stsys}_{2,b}(M, g) = \mathrm{covol}_{\mathrm{HB}}\big(H_2(M;\mathbb{Z})/\mathrm{torsion}, N_g^{(2)}\big)^{1/b},$$

where the integral lattice is identified with its image in $H_2(M;\mathbb{R})$ and $N_g^{(2)}$ is the stable norm. This is the rank-b case of [41, Definition 4.1]. The ordinary stable two-systole $\mathrm{stsys}_2(M, g)$ is the first minimum of the same lattice. Write $[g] = \{f^2 g : f \in C^\infty(M),\ f > 0\}$ for the smooth conformal class of g.

For integers $p, q \geq 2$, set $b = p + q$ and

$$c_+ = \sqrt{\frac{2p}{b}}, \qquad c_- = \sqrt{\frac{2q}{b}}, \qquad v_{p,q} = \frac{p!q!\,\omega_p\omega_q}{b!\,c_+^p c_-^q}, \qquad C_{p,q} = \left(\frac{\omega_b}{v_{p,q}}\right)^{2/b}. \tag{16.1}$$

Theorem 16.1. *Let M be a closed connected oriented smooth four-manifold with $b_+(M) = p \geq 2$ and $b_-(M) = q \geq 2$. There is a function κ on the smooth conformal classes of M, given by* (16.8)*, with values in $(0, 1)$, such that every smooth Riemannian metric g on M satisfies*

$$\frac{\mathrm{stsys}_{2,p+q}(M, g)^2}{\mathrm{Vol}(M, g)} \leq C_{p,q}\kappa([g])^2 < C_{p,q}. \tag{16.2}$$

If $p, q \in \{2, 3\}$, then

$$\sup_g \frac{\mathrm{stsys}_{2,p+q}(M, g)^2}{\mathrm{Vol}(M, g)} = C_{p,q}, \tag{16.3}$$

where the supremum is over all smooth Riemannian metrics on M. In these signatures, no smooth metric attains this supremum. The supremum over any fixed smooth conformal class is strictly smaller than $C_{p,q}$. The distinct values are

$$C_{2,2} = \sqrt{3}, \qquad C_{2,3} = C_{3,2} = \frac{2^{11/5}3^{3/5}}{5}, \qquad C_{3,3} = \left(\frac{15\pi}{8}\right)^{1/3}.$$

16.1. **Two integral inequalities.** A Gamma variable of integer shape $k \geq 1$ and rate one has density $r^{k-1}e^{-r}/(k-1)!$ for $r > 0$.

Lemma 16.2. *Let R_p, S_q be independent Gamma random variables with integer shapes $p, q \geq 2$ and rate one. For $a, d \geq 0$,*

$$\mathbb{E}\max\left(\frac{aR_p}{p}, \frac{dS_q}{q}\right) \leq \sqrt{a^2 + d^2},$$

with strict inequality if $ad > 0$.

Proof. If $ad = 0$, the assertion follows from $\mathbb{E}R_p = p$ and $\mathbb{E}S_q = q$. Suppose $a, d > 0$. First let $p = q = 2$, and put $s = a + d$ and $t = ad/s^2 \in (0, 1/4]$. The shape-two survival function is $e^{-r}(1 + r)$. Independence and integration give

$$\begin{aligned}\mathbb{E}\min(aR_2, dS_2) &= \int_0^\infty e^{-(1/a+1/d)r}(1 + r/a)(1 + r/d)\,dr \\ &= \frac{2ad}{s} + \frac{2a^2d^2}{s^3}.\end{aligned}$$

Consequently

$$\mathbb{E}\max(aR_2/2, dS_2/2) = s(1 - t - t^2).$$

Since

$$(1-2t)-(1-t-t^2)^2 = t^2(1-2t-t^2) \geq \frac{7}{16}t^2 > 0,$$

this expectation is strictly smaller than $s\sqrt{1-2t} = \sqrt{a^2+d^2}$.

For each integer $k \geq 1$, let R_k have the Gamma law of shape k and rate one. For every continuous convex function $h : \mathbb{R} \to \mathbb{R}$ of at most linear growth, the quantity $\mathbb{E}h(R_k/k)$ is nonincreasing in k; see [21, Corollary A.2]. For each fixed nonnegative value of one variable, $\max(ax, dy)$ is a continuous convex function of the other, of at most linear growth. Applying this inequality successively to the two independent variables gives

$$\mathbb{E}\max(aR_p/p, dS_q/q) \leq \mathbb{E}\max(aR_2/2, dS_2/2) < \sqrt{a^2+d^2}.$$

□

Hereafter $|\cdot|$ denotes the Euclidean norm. Define

$$F_0(u,v) = c_+|u| + c_-|v|, \qquad (u,v) \in \mathbb{R}^p \oplus \mathbb{R}^q.$$

Let (U,V) have probability density $Z_0^{-1}e^{-F_0(u,v)}\,du\,dv$, where

$$Z_0 = \int_{\mathbb{R}^p\oplus\mathbb{R}^q} e^{-F_0(u,v)}\,du\,dv = \frac{p!\omega_p}{c_+^p}\frac{q!\omega_q}{c_-^q} = b!v_{p,q}.$$

Write $\mathbb{E}_0 H = \mathbb{E}[H(U,V)]$ for every function $H : \mathbb{R}^b \to \mathbb{R}$ integrable under this law. If θ and ϕ are uniform on the unit spheres S^{p-1} and S^{q-1}, respectively, and R_p, S_q, θ, ϕ are mutually independent, then

$$(U,V) \overset{\text{law}}{=} \left(\frac{R_p}{c_+}\theta, \frac{S_q}{c_-}\phi\right).$$

In particular, $\mathbb{E}_0 F_0 = p+q = b$. For a linear map A between Euclidean spaces, let A^t be its adjoint and put $\|A\|_{\mathrm{HS}}^2 = \mathrm{tr}(A^tA)$.

Lemma 16.3. *For linear maps $A : \mathbb{R}^p \to \mathbb{R}^3$ and $B : \mathbb{R}^q \to \mathbb{R}^3$,*

$$\mathbb{E}_0\max(|AU|,|BV|) \leq \sqrt{\frac{b}{2}}\left(\|A\|_{\mathrm{HS}}^2 + \|B\|_{\mathrm{HS}}^2\right)^{1/2}. \tag{16.4}$$

The inequality is strict if both maps are nonzero.

Proof. Condition on θ and ϕ. In Lemma 16.2, take $a = p|A\theta|/c_+$ and $d = q|B\phi|/c_-$. Integration over the directions and Cauchy–Schwarz give

$$\begin{aligned}\mathbb{E}_0\max(|AU|,|BV|) &\leq \sqrt{\frac{b}{2}}\,\mathbb{E}_{\theta,\phi}\sqrt{p|A\theta|^2 + q|B\phi|^2} \\ &\leq \sqrt{\frac{b}{2}}\,\sqrt{\|A\|_{\mathrm{HS}}^2 + \|B\|_{\mathrm{HS}}^2}.\end{aligned}$$

Here $\mathbb{E}|A\theta|^2 = \|A\|_{\mathrm{HS}}^2/p$, and the corresponding identity holds for B. If A and B are nonzero, their kernels meet the unit spheres in sets of measure zero. The conditional inequality of Lemma 16.2 is then strict for almost every pair of directions. Its integral is therefore strict. □

Lemma 16.4. *Every norm F on $\mathbb{R}^b$ satisfies*

$$\mathrm{vol}_{\mathbb{R}^b}\{F \leq 1\} \geq v_{p,q}\left(\frac{b}{\mathbb{E}_0 F}\right)^b. \tag{16.5}$$

Proof. Put $K_0 = \{F_0 \le 1\}$ and $K = \{F \le 1\}$, and let $\rho_0 = 1/F_0$ and $\rho = 1/F$ be their radial functions on S^{b-1}. These are positive continuous functions, since K_0 and K are convex bodies containing the origin in their interiors. With $d\sigma$ denoting Euclidean surface measure, put

$$I = \frac{1}{b} \int_{S^{b-1}} \rho_0^{b+1} \rho^{-1} \, d\sigma.$$

Polar integration gives

$$\mathrm{vol}_{\mathbb{R}^b}(K_0) = v_{p,q}, \qquad \mathbb{E}_0 F = \frac{1}{v_{p,q}} \int_{S^{b-1}} \rho_0^{b+1} \rho^{-1} \, d\sigma = \frac{bI}{v_{p,q}}.$$

The quantity I is Lutwak's dual mixed volume $\widetilde{V}_{-1}(K_0, K)$ [54, Definition 2*]. His inequality [54, Theorem 2], with indices $(i, j, k) = (-1, 0, b)$, gives

$$v_{p,q}^{b+1} \le I^b \, \mathrm{vol}_{\mathbb{R}^b}(K).$$

Substituting $I = v_{p,q} \mathbb{E}_0 F / b$ proves (16.5). □

16.2. **The upper bound and the conformal gap.** Let M satisfy the hypotheses of Theorem 16.1. Fix a smooth metric g and put $V_g = \mathrm{Vol}(M, g)$. For $x \in M$, write

$$\Lambda_x^{\pm} = \{\xi \in \Lambda^2 T_x^* M \, : * _g \, \xi = \pm \xi\}.$$

Choose $L^2(g)$-orthonormal bases $\alpha_1, \dots, \alpha_p$ and $\beta_1, \dots, \beta_q$ of the self-dual and anti-self-dual harmonic two-forms, respectively. Define $A_x : \mathbb{R}^p \to \Lambda_x^+$ and $B_x : \mathbb{R}^q \to \Lambda_x^-$ by

$$A_x u = \sum_{i=1}^{p} u_i \alpha_i(x), \qquad B_x v = \sum_{j=1}^{q} v_j \beta_j(x).$$

Give these three-dimensional spaces their pointwise Euclidean norms. With I_p and I_q denoting identity matrices, orthonormality gives

$$\int_M A_x^t A_x \, dv_g = I_p, \qquad \int_M B_x^t B_x \, dv_g = I_q. \tag{16.6}$$

Identify $H_2(M; \mathbb{R})$ with $\mathbb{R}^p \oplus \mathbb{R}^q$ by

$$z \longmapsto (u, v), \qquad u_i = \langle [\alpha_i], z \rangle, \quad v_j = \langle [\beta_j], z \rangle.$$

The integral lattice has Euclidean covolume one, by Proposition 12.2.

Let $\mathrm{PD} : H^2(M; \mathbb{R}) \to H_2(M; \mathbb{R})$ be Poincaré duality. The harmonic representative of $\mathrm{PD}^{-1}(z)$ is

$$\theta_z = \sum_{i=1}^{p} u_i \alpha_i - \sum_{j=1}^{q} v_j \beta_j.$$

Equations (12.13) and (12.14) give

$$N_g^{(2)}(u, v) \le F_g(u, v), \qquad F_g(u, v) = \sqrt{2} \int_M \max(|A_x u|, |B_x v|) \, dv_g. \tag{16.7}$$

The integrand is a seminorm of (u, v), so F_g is a seminorm. Since it dominates the norm $N_g^{(2)}$, it is a norm.

Define $L_g : M \to [0, \infty)$ and $\kappa(g) \ge 0$ by

$$L_g(x) = \mathbb{E}_0 \max(|A_x U|, |B_x V|), \qquad \kappa(g)^2 = \frac{2}{b^2} \int_M L_g(x)^2 \, dv_g. \tag{16.8}$$

The evaluation maps are smooth on the compact manifold M, and $\mathbb{E}_0(|U| + |V|) = p/c_+ + q/c_- < \infty$. Dominated convergence therefore makes L_g finite and continuous. Lemma 16.3 gives

$$L_g(x)^2 \le \frac{b}{2} \big(\|A_x\|_{\mathrm{HS}}^2 + \|B_x\|_{\mathrm{HS}}^2 \big).$$

The nonzero forms α_1 and β_1 are smooth, closed and coclosed. By [4, §2, Theorem 1, pp. 421–422], neither vanishes on a nonempty open set. Thus

$$O = \{x \in M : \alpha_1(x) \neq 0\} \cap \{x \in M : \beta_1(x) \neq 0\}$$

is a nonempty open set. On O, both evaluation maps are nonzero, so the preceding inequality is strict. Moreover, $L_g > 0$ there, since the law of (U, V) has a positive density. The set O has positive volume; hence (16.6) gives

$$0 < \int_M L_g^2 \, dv_g < \frac{b}{2} \int_M \left(\|A_x\|_{\mathrm{HS}}^2 + \|B_x\|_{\mathrm{HS}}^2 \right) dv_g = \frac{b^2}{2}.$$

Consequently $0 < \kappa(g) < 1$.

Tonelli's theorem and Cauchy–Schwarz give

$$\mathbb{E}_0 F_g = \sqrt{2} \int_M L_g \, dv_g \leq b \sqrt{V_g} \, \kappa(g).$$

Put $B_g = \{N_g^{(2)} \leq 1\}$. By (16.7), $\{F_g \leq 1\} \subset B_g$. Lemma 16.4 therefore gives

$$\mathrm{vol}_{\mathbb{R}^b}(B_g) \geq v_{p,q} \left(\frac{b}{\mathbb{E}_0 F_g} \right)^b \geq v_{p,q} \left(\sqrt{V_g} \, \kappa(g) \right)^{-b}.$$

Since the integral lattice has covolume one,

$$\frac{\mathrm{stsys}_{2,b}(M, g)^2}{V_g} = \frac{1}{V_g} \left(\frac{\omega_b}{\mathrm{vol}_{\mathbb{R}^b}(B_g)} \right)^{2/b} \leq C_{p,q} \kappa(g)^2 < C_{p,q}.$$

The density defining $\mathbb{E}_0$ is invariant under orthogonal changes in each factor. Hence L_g and $\kappa(g)$ are independent of the chosen harmonic bases. Let $\widehat{g} = f^2 g$, where $f \in C^\infty(M)$ and $f > 0$. The conformal identities in the proof of Lemma 8.2 give

$$*_{\widehat{g}} \, |_{\Omega^2(M)} = *_g \, |_{\Omega^2(M)}, \qquad dv_{\widehat{g}} = f^4 \, dv_g.$$

The harmonic two-forms, their self-dual decomposition and their L^2 inner product are unchanged. In the same harmonic bases, pointwise two-form norms are multiplied by f^{-2}, so

$$L_{\widehat{g}} = f^{-2} L_g, \qquad \int_M L_{\widehat{g}}^2 \, dv_{\widehat{g}} = \int_M L_g^2 \, dv_g.$$

Thus $\kappa([g]) = \kappa(g)$ is well defined, and

$$\sup_{\widehat{g} \in [g]} \frac{\mathrm{stsys}_{2,b}(M, \widehat{g})^2}{\mathrm{Vol}(M, \widehat{g})} \leq C_{p,q} \kappa([g])^2 < C_{p,q}.$$

16.3. **Sharpness in signatures** $(2,2)$, $(2,3)$ **and** $(3,3)$. Let M satisfy the hypotheses of Theorem 16.1, with $p, q \in \{2, 3\}$. Write Q for its integral intersection form. Suppose first that $Q \simeq I_{p,q}$, where $I_{p,q} = \mathrm{diag}(I_p, -I_q)$. In an integral diagonal cohomology basis, write $\alpha = (x, y) \in \mathbb{Q}^p \oplus \mathbb{Q}^q$. Let g_0 be the unit flat metric on $\mathbb{R}^4/\mathbb{Z}^4$, with its standard orientation and volume form dv_0. Put $dx^{ij} = dx^i \wedge dx^j$ and

$$\Omega_1 = dx^{12} + dx^{34}, \qquad \Omega_2 = dx^{13} - dx^{24}, \qquad \Omega_3 = dx^{14} + dx^{23}.$$

These forms have integral periods and satisfy, for $t \in \mathbb{R}^3$,

$$\mathrm{comass}_{g_0} \left(\sum_{i=1}^3 t_i \Omega_i \right) = |t|, \qquad \Omega_i \wedge \Omega_j = 2\delta_{ij} \, dv_0. \tag{16.9}$$

The comass identity follows from (9.3), since the Ω_i are orthogonal self-dual forms of squared norm two.

Let T_1, T_2 be positively oriented copies of this torus and T_3, T_4 negatively oriented copies. Give the first two the metric $h_+ = \sqrt{p/(2b)}\, g_0$ and the last two the metric $h_- = \sqrt{q/(2b)}\, g_0$, and denote these metrics by h_ℓ. Their volumes are $p/(2b), p/(2b), q/(2b), q/(2b)$, whose sum is one. For each rational class $\alpha = (x, y)$, define the closed forms

$$\eta_\ell(\alpha) = \begin{cases} \frac{1}{2}\sum_{i=1}^{p} x_i \Omega_i, & \ell = 1, 2, \\ \frac{1}{2}\sum_{j=1}^{q} y_j \Omega_j, & \ell = 3, 4 \end{cases}.$$

Their periods are rational. Equation (16.9) gives

$$\sum_{\ell=1}^{4} \int_{T_\ell} \eta_\ell(\alpha) \wedge \eta_\ell(\alpha') = x \cdot x' - y \cdot y' = Q(\alpha, \alpha').$$

Their comasses are $|x|/c_+$ on T_1, T_2 and $|y|/c_-$ on T_3, T_4. Put

$$C_0 = \{(x, y) \in \mathbb{R}^p \oplus \mathbb{R}^q \,:\, |x| \le c_+,\ |y| \le c_-\}.$$

For $\alpha \in C_0 \cap \mathbb{Q}^b$, one has $\|\eta_\ell(\alpha)\|_{\infty, h_\ell} \le 1$ for $1 \le \ell \le 4$. The support function of C_0 in the dual homology coordinates is $F_0(u, v) = c_+|u| + c_-|v|$.

Fix $0 < \varepsilon < 1$. Rational polytope approximation [66, Theorem 1.8.19 and Remark 1.8.17] gives a finite symmetric set $F = \{\alpha_1, \dots, \alpha_s\} \subset C_0 \cap \mathbb{Q}^b$ such that

$$(1 - \varepsilon) C_0 \subset \operatorname{conv}(F) \subset C_0.$$

Let K_F be the moment body in (1.2) for this ordered list, with every degree equal to two, and put

$$T_F = (Q(\alpha_i, \alpha_j))_{1 \le i \le j \le s}.$$

These pairings are all its top moments in dimension four. For $1 \le \ell \le 4$, let τ_ℓ be the top-moment vector of the tuple $(\eta_\ell(\alpha_i))_{i=1}^{s}$ on T_ℓ. Lemma 1.4 gives $\tau_\ell \in \operatorname{Vol}(T_\ell, h_\ell) K_F$. Since $\sum_\ell \tau_\ell = T_F$ and $\sum_\ell \operatorname{Vol}(T_\ell, h_\ell) = 1$, convexity gives $T_F \in K_F$. Theorem 1.1 therefore yields

$$\mathcal{V}_M((\alpha_i)_{i=1}^{s}) = \gamma_{K_F}(T_F) \le 1.$$

For every $\delta > 0$, there is consequently a smooth metric g on M such that

$$\operatorname{Vol}(M, g) < 1 + \delta, \qquad \|\alpha\|_g^* \le 1 \quad (\alpha \in F).$$

By stable–comass duality (1.1), for every $z \in H_2(M; \mathbb{R})$,

$$N_g^{(2)}(z) \ge \max_{\alpha \in F} \langle \alpha, z \rangle \ge (1 - \varepsilon) F_0(z).$$

The integral homology lattice has covolume one in these dual coordinates. Since the unit ball of F_0 has volume $v_{p,q}$,

$$\frac{\operatorname{stsys}_{2,b}(M, g)^2}{\operatorname{Vol}(M, g)} \ge \frac{(1 - \varepsilon)^2}{1 + \delta} C_{p,q}.$$

Letting $\varepsilon, \delta \downarrow 0$ and applying (16.2) proves (16.3) in this case.

The classification of indefinite unimodular integral forms gives $Q \simeq I_{p,q}$ when Q is odd. When Q is even, it gives $p = q$ and $Q \simeq H^{\oplus p}$, where

$$H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix};$$

see [68, Chapter V, §2.1, Corollary 1 to Theorem 2, and §2.2, Theorem 6]. For each hyperbolic pair e, f, the rational vectors $e + \frac{1}{2} f$ and $e - \frac{1}{2} f$ have squares $1, -1$, zero pairing, and change-of-basis determinant -1. Thus the integral homology lattice has covolume one in the dual coordinates of this rational diagonal basis. The preceding argument applies: the classes in $F \subset \mathbb{Q}^b$ are rational cohomology classes, and the covolume is unchanged. This proves (16.3) also when Q is even.

For $M = \mathbb{T}^4$, the intersection form is $H^{\oplus 3}$. Hence

$$\sup_g \frac{\mathrm{stsys}_{2,6}(\mathbb{T}^4, g)^2}{\mathrm{Vol}(\mathbb{T}^4, g)} = \left(\frac{15\pi}{8}\right)^{1/3} > \left(\frac{3\pi}{4}\right)^{1/3}.$$

The last constant is the sharp value for conformally flat metrics [41, Theorem 6.7]. The extension of this bound to all smooth metrics, considered in [41, p. 123], is therefore false.

17. Diagonal intersection forms in dimension eight

Let M be a closed connected oriented smooth eight-manifold, and let $Q(\alpha, \beta) = \langle \alpha\beta, [M]\rangle$ be its middle intersection pairing. Suppose that its matrix on $H^4(M;\mathbb{Z})/\text{ torsion}$ is the identity matrix I_b, where $b \geq 1$. Choose a basis $a_1, \dots, a_b$ with

$$Q(a_i, a_j) = \delta_{ij},$$

and let $z_1, \dots, z_b$ be the evaluation-dual basis of $H_4(M;\mathbb{Z})/\text{ torsion}$. For a smooth Riemannian metric g, write

$$c_g(a) = \|a\|_g^* \quad (a \in H^4(M;\mathbb{R})), \qquad R_4(g) = \frac{\mathrm{stsys}_4(M, g)^2}{\mathrm{Vol}(M, g)}.$$

Let Φ_0 be the Cayley form of [49, arXiv version, (4.11)]. It satisfies

$$*\,\Phi_0 = \Phi_0, \qquad \mathrm{comass}(\Phi_0) = 1, \qquad |\Phi_0|^2 = 14;$$

see [49, arXiv version, Theorem 1.1]. A positive normalized Cayley form on an oriented Euclidean eight-space is a form in the SO(8) orbit of Φ_0. Write dv_0 for Euclidean volume. At $x \in M$, denote this orbit by $\mathcal{C}_g(x)$ and put

$$\mathcal{K}_g(x) = \{t\Phi \,:\, t \in \mathbb{R},\ \Phi \in \mathcal{C}_g(x)\}.$$

Thus $\mathcal{K}_g(x)$ includes zero and both signs of the orbit.

Theorem 17.1. *For every smooth metric on M, one has $R_4(g) \leq 14$, and*

$$\mathrm{SR}_4(M) = 14.$$

Proof. Every harmonic four-form is self-dual: its anti-self-dual part is harmonic and has negative square unless it is zero, whereas Q is positive definite. The L^2 pairing of harmonic representatives is therefore Q. Proposition 12.2, (12.2), and $N_Q(z_i) = 1$ give $N_g^{(4)}(z_i) \leq \sqrt{14\,\mathrm{Vol}(M, g)}$, proving the upper bound.

For each $\varepsilon \in \{1, -1\}^b$, put $\omega_i^\varepsilon = \varepsilon_i \Phi_0$. These forms have comass one and

$$\frac{1}{14}\, 2^{-b} \sum_{\varepsilon \in \{1,-1\}^b} \frac{\omega_i^\varepsilon \wedge \omega_j^\varepsilon}{dv_0} = \delta_{ij}.$$

All classes in the tuple have degree four, so these pairings are all its top moments. With the notation of (1.2) and (1.3), this gives $T_{(a_1,\dots,a_b)} \in \frac{1}{14} K_{8;4,\dots,4}$. Theorem 1.1, applied to this integral tuple, gives, for every $\delta > 0$, a smooth metric with

$$c_g(a_i) \leq 1 \quad (1 \leq i \leq b), \qquad \mathrm{Vol}(M, g) < \frac{1}{14} + \delta.$$

For a nonzero integral class $z = \sum_i m_i z_i$, (1.1) gives $N_g^{(4)}(z) \geq \max_i |m_i| \geq 1$. Hence $R_4(g) \geq (1/14 + \delta)^{-1}$. Let $\delta \downarrow 0$. □

17.1. **The signed equality set.** Fix $x \in M$, and write $\Lambda^4_+ T^*_x M$ for the space of self-dual four-forms at x. For ξ in this space, one has

$$|\xi|^2 \le 14\,\mathrm{comass}(\xi)^2; \tag{17.1}$$

see [49, arXiv version, Theorem 1.1].

Lemma 17.2. *Equality holds in* (17.1) *if and only if* $\xi \in \mathcal{K}_g(x)$. *The orbits* $\mathcal{C}_g(x)$ *and* $-\mathcal{C}_g(x)$ *are disjoint.*

Proof. Suppose first that $\xi \neq 0$, and put $\widehat{\xi} = \xi/\,\mathrm{comass}(\xi)$. By [23, Theorem 3.7], there are $N \ge 1$, numbers $t_j > 0$ with $\sum_{j=1}^N t_j = 1$, and forms $\Phi_j \in \mathcal{C}_g(x) \cup (-\mathcal{C}_g(x))$ such that $\widehat{\xi} = \sum_{j=1}^N t_j \Phi_j$. Since $|\Phi_j|^2 = 14$,

$$14 - |\widehat{\xi}|^2 = \frac{1}{2}\sum_{i,j=1}^{N} t_i t_j |\Phi_i - \Phi_j|^2.$$

Thus equality holds precisely when all the Φ_j coincide. Conversely, every form in either normalized orbit gives equality. Homogeneity gives the assertion for nonzero ξ; it also holds for $\xi = 0$.

For a Euclidean space E, let $\mathrm{Sym}_0(E)$ denote the traceless self-adjoint endomorphisms, with inner product $\langle A, B\rangle = \mathrm{tr}(AB)$. Let Δ^+ be the real eight-dimensional positive half-spin module, with its invariant inner product. Triality gives a Spin(8)-equivariant isometry

$$\psi \,:\, \mathrm{Sym}_0(\Delta^+) \longrightarrow \Lambda^4_+(\mathbb{R}^8)^*, \qquad 4\psi(P_0 - I_8/8) = \Phi_0,$$

where P_0 is a rank-one orthogonal projector; see [49, arXiv version, §3 and (4.12)]. The half-spin representation has image $\mathrm{SO}(\Delta^+)$. Under ψ^{-1}, the two normalized Cayley orbits therefore have the respective eigenvalue lists

$$(7/2, -1/2, \dots, -1/2), \qquad (1/2, \dots, 1/2, -7/2).$$

These lists differ, so the orbits are disjoint. □

On an oriented Riemannian spin eight-manifold X, fix a spin structure and let S^+ be the associated real positive spinor bundle. The isometry ψ induces a parallel isometry $\mathrm{Sym}_0(S^+) \simeq \Lambda^4_+ T^*X$ for the connections induced by the Levi–Civita connection. A positive normalized Cayley form is equivalently of the form

$$\Phi = 4\psi(P - I_8/8), \tag{17.2}$$

where P is a rank-one orthogonal projector on S^+.

17.2. **Harmonic representatives at equality.** For $a \in H^4(M;\mathbb{R})$, let h_a be its harmonic representative and put

$$e_g(a) = \inf_{d\eta = 0,\,[\eta] = a} \left(\int_M \mathrm{comass}_g(\eta)^2 \, dv_g \right)^{1/2}, \tag{17.3}$$

where the representatives are smooth. This is the cohomological L^2 comass norm of [9, arXiv version, §2, before Corollary 2.5]. Write $\|\eta\|_2^2 = \int_M |\eta|_g^2 \, dv_g$, and denote $Q(a,a)^{1/2}$ by $|a|_Q$. Since all harmonic four-forms are self-dual, $\|h_a\|_2^2 = Q(a,a)$.

Lemma 17.3. *For every* $a \in H^4(M;\mathbb{R})$, *one has*

$$e_g(a)^2 \ge \frac{Q(a,a)}{14}. \tag{17.4}$$

If equality holds, then $h_a(x) \in \mathcal{K}_g(x)$ *for every* x. *If* $a \neq 0$ *and*

$$c_g(a)^2\,\mathrm{Vol}(M,g) = \frac{Q(a,a)}{14}, \tag{17.5}$$

then one of $h_a/c_g(a)$ *and* $-h_a/c_g(a)$ *is a positive parallel Cayley form compatible with* g.

Proof. Hodge star preserves comass in middle degree, so the self-dual part $\eta^+ = (\eta + * \eta)/2$ of any representative satisfies $\operatorname{comass}(\eta^+) \le \operatorname{comass}(\eta)$. Since $\eta - h_a$ is exact and h_a is harmonic and self-dual,

$$\langle h_a, \eta^+ \rangle_{L^2} = \langle h_a, \eta \rangle_{L^2} = Q(a, a).$$

Consequently

$$Q(a,a) + \|\eta^+ - h_a\|_2^2 = \|\eta^+\|_2^2 \le 14 \int_M \operatorname{comass}_g(\eta)^2 \, dv_g. \tag{17.6}$$

Taking the infimum gives (17.4).

Suppose equality holds in (17.4), and choose representatives η_j whose squared comass energies tend to $Q(a,a)/14$. Equation (17.6) gives $\eta_j^+ \to h_a$ strongly in L^2. For four-forms α, β at the same point, one has

$$|\operatorname{comass}(\alpha) - \operatorname{comass}(\beta)| \le \operatorname{comass}(\alpha - \beta) \le |\alpha - \beta|.$$

Hence the comass functions of η_j^+ converge to that of h_a in L^2. Thus $\int_M \operatorname{comass}(h_a)^2 \, dv_g \le Q(a,a)/14$. The reverse inequality follows by integrating (17.1). The nonnegative continuous function $14 \operatorname{comass}(h_a)^2 - |h_a|^2$ has integral zero and hence vanishes everywhere. Lemma 17.2 gives the cone condition.

Suppose now that $a \neq 0$ and (17.5) holds. Put $c = c_g(a) > 0$ and choose representatives with $\|\eta_j\|_{\infty,g} \le c + j^{-1}$. Then (17.6) gives

$$\|\eta_j^+ - h_a\|_2^2 \le 14(c + j^{-1})^2 \operatorname{Vol}(M, g) - Q(a,a) \longrightarrow 0.$$

An almost everywhere convergent subsequence of η_j^+ gives $\operatorname{comass}(h_a) \le c$ almost everywhere, hence everywhere by continuity. Therefore

$$|h_a|^2 \le 14c^2 = \frac{Q(a,a)}{\operatorname{Vol}(M,g)}.$$

The integral of $|h_a|^2$ is $Q(a,a)$, so equality holds everywhere. It follows that h_a is nowhere zero and $h_a(x)/c \in C_g(x) \cup (-C_g(x))$ for every x. In every oriented orthonormal trivialization, these two orbits are compact and disjoint. The sign is therefore locally constant, and connectedness makes it constant on M. The resulting positive Cayley form is closed and has associated metric g; it is parallel by [46, arXiv version, §2, after Definition 2.2]. □

17.3. **Zeros of closed forms in the Cayley cone.**

Lemma 17.4. *Let (N^8, g) be a connected oriented smooth Riemannian manifold without boundary. Let ω be a smooth closed four-form with $\omega_x \in \mathcal{K}_g(x)$ for every x. Then either $\omega \equiv 0$ or ω is nowhere zero. In the latter case, one of ω and $-\omega$ is a positive parallel Cayley form for the smooth conformal metric*

$$g_\omega = (|\omega|_g / \sqrt{14})^{1/2} g. \tag{17.7}$$

Proof. Every form in the cone is self-dual, so $d\omega = d_g^* \omega = 0$. Assume $\omega \not\equiv 0$. By [4, §2, Theorem 1], a smooth form satisfying these equations and vanishing to infinite order in 1-mean at one point vanishes identically. A zero of infinite Taylor order has infinite order in 1-mean. Thus every zero of ω has finite Taylor order.

On $U = \{\omega \neq 0\}$, the sign of the normalized Cayley orbit is locally constant. Thus ω or $-\omega$ is a closed positive Cayley form for g_ω on each component of U. This metric is Ricci-flat, by [46, arXiv version, §2, after Definition 2.2].

Suppose x_0 is a zero of order $k \ge 1$. In normal coordinates centred at x_0, let P_k be the nonzero four-form whose coefficients are the homogeneous degree-k terms in the Taylor expansion of ω. Write g_0 for the Euclidean metric in these coordinates and $\delta_t(x) = tx$ for $t > 0$. Taylor's formula gives

$$t^{-k-4} \delta_t^* \omega \longrightarrow P_k, \qquad t^{-2} \delta_t^* g \longrightarrow g_0 \qquad \text{in } C^\infty_{\mathrm{loc}} \quad (t \downarrow 0).$$

Consequently, on every compact subset of $\{P_k \neq 0\}$,

$$t^{-2-k/2}\delta_t^* g_\omega \longrightarrow (|P_k|_{g_0}/\sqrt{14})^{1/2} g_0 \qquad \text{in } C^\infty \quad (t \downarrow 0).$$

On each such compact set the metrics on the left are defined and Ricci-flat for all sufficiently small t. Their limit is Ricci-flat.

Set $v = (|P_k|_{g_0}/\sqrt{14})^{-1/4}$ on $\{P_k \neq 0\}$. For a symmetric two-tensor, a subscript 0 denotes its trace-free part with respect to g_0. The limit metric is $v^{-2}g_0$, so the conformal Ricci formula [52, (2.5)] gives

$$0 = 6v^{-1}(\mathrm{Hess}_{g_0} v)_0.$$

On a small ball in $\{P_k \neq 0\}$, write $\partial_i\partial_j v = \lambda\delta_{ij}$. For each j, choose $i \neq j$ and commute third derivatives to obtain $\partial_j\lambda = \partial_j\partial_i^2 v = \partial_i(\partial_i\partial_j v) = 0$. Hence there are constants $A, c \in \mathbb{R}$ and $\ell \in \mathbb{R}^8$ such that $v = A|x|^2 + \ell\cdot x + c$ on that ball. But v is homogeneous of degree $q = -k/4 < 0$, so

$$(2-q)A|x|^2 + (1-q)\ell\cdot x - qc = 0$$

there. The polynomial vanishes on an open set, so $A = 0$, $\ell = 0$ and $c = 0$, contrary to $v > 0$. Thus ω has no zero. Connectedness makes the Cayley sign constant. □

17.4. **Parallel Cayley forms and rank.**

Lemma 17.5. *Let (N^8, g) be closed, connected and oriented, and let $b \geq 1$. Suppose that its harmonic four-forms have a basis of positive parallel Cayley forms $\Phi_1, \dots, \Phi_b$, compatible with g and satisfying*

$$\langle \Phi_i, \Phi_j \rangle = 14\delta_{ij}.$$

Then $b = 1$.

Proof. The Spin(7) structure defined by Φ_1 gives a spin structure and a global unit parallel positive spinor s_1; see [46, arXiv version, §2, after Definition 2.2]. In this spin structure, (17.2) associates to each Φ_i a global parallel rank-one orthogonal projector P_i, with P_1 the projector onto $\mathbb{R}s_1$. The isometry of triality gives

$$\mathrm{tr}(P_iP_j) = \frac{1}{8} + \frac{1}{16}\langle \Phi_i, \Phi_j \rangle = \begin{cases} 1, & i = j, \\ 1/8, & i \neq j. \end{cases} \tag{17.8}$$

For $i > 1$, the section $P_i s_1$ has squared norm $\mathrm{tr}(P_iP_1) = 1/8$. Hence $s_i = \sqrt{8}P_i s_1$ is a global unit parallel spinor, and P_i is its line projector. All projectors and spinors therefore belong to the same spin structure.

Let W be the subbundle spanned by these parallel spinors, and let $r = \mathrm{rank}\, W$. Their Gram matrix is constant, so W has a global parallel orthonormal frame. Denote by $\mathscr{S}(W)$ the real vector space of parallel self-adjoint endomorphisms of W; it has dimension $r(r+1)/2$. Extend its elements by zero on $W^\perp$. The map

$$\mathscr{S}(W) \ni A \longmapsto 4\psi\big(A - (\mathrm{tr}\, A)I_8/8\big) \tag{17.9}$$

has parallel self-dual four-forms as its image. Such forms are harmonic on N. The image contains all Φ_i and therefore is the full harmonic space.

If $r < 8$, the map is injective: its kernel consists of scalar multiples of I_8 vanishing on $W^\perp$. Thus $b = r(r+1)/2$. Write P_i also for its restriction to W; these restrictions form a basis of $\mathscr{S}(W)$. Put $\mathbf{1} = (1, \dots, 1)^t \in \mathbb{R}^b$. Their Gram matrix for the trace inner product is $G = (7I_b + \mathbf{1}\mathbf{1}^t)/8$, by (17.8). Let I_W denote the identity of W. Since $\mathrm{tr}(I_W P_i) = 1$, expansion of I_W in this basis gives

$$r = \mathrm{tr}(I_W^2) = \mathbf{1}^t G^{-1}\mathbf{1} = \frac{8b}{b+7}.$$

Together with $b = r(r+1)/2$, this implies $(r-1)(r-6) = 0$. Hence $(r, b) = (1, 1)$ or $(6, 21)$. If $r = 8$, the kernel of (17.9) is the scalar line, so $b = \dim \mathrm{Sym}_0(\mathbb{R}^8) = 35$.

Fix $x \in N$. By (17.8),

$$|\langle s_i(x), s_j(x)\rangle| = 1/\sqrt{8} \qquad (i \neq j).$$

Thus the b lines $\mathbb{R}s_i(x) \subset W_x \simeq \mathbb{R}^r$ are distinct and equiangular. Neumann's theorem [11, arXiv version, Theorem 1.2] gives $b \leq 2r$, since $\sqrt{8}$ is not an odd integer. This excludes $(6, 21)$ and $(8, 35)$, leaving $b = 1$. □

Theorem 17.6. *If $R_4(g) = 14$ on M, then $b = 1$ and g admits a compatible positive parallel Cayley form. Conversely, if $b = 1$ and g admits such a form, then $R_4(g) = 14$.*

Proof. Write $V = \mathrm{Vol}(M, g)$ and suppose $R_4(g) = 14$. For each i,

$$\sqrt{14V} = \mathrm{stsys}_4(M, g) \leq N_g^{(4)}(z_i) \leq \sqrt{14V}.$$

The cohomological comass unit ball is compact. Choose u_i in it with $\langle u_i, z_i\rangle = \sqrt{14V}$. By (17.4) and $e_g(a) \leq \sqrt{V}\, c_g(a)$,

$$\sqrt{14V} = Q(u_i, a_i) \leq |u_i|_Q \leq \sqrt{14V}\; c_g(u_i) \leq \sqrt{14V}.$$

Equality in Cauchy–Schwarz gives $u_i = \sqrt{14V}\, a_i$; the last inequalities give $c_g(u_i) = 1$. Hence $c_g(a_i) = (14V)^{-1/2}$.

Lemma 17.3 supplies constant signs $\epsilon_i \in \{1, -1\}$ for which $\Phi_i = \epsilon_i \sqrt{14V}\, h_{a_i}$ are positive parallel Cayley forms compatible with g. Their pointwise inner products are constant, so

$$\langle \Phi_i, \Phi_j\rangle = \frac{1}{V}\int_M \langle \Phi_i, \Phi_j\rangle\, dv_g = 14\epsilon_i\epsilon_j Q(a_i, a_j) = 14\delta_{ij}.$$

They form a basis of the harmonic four-forms. Lemma 17.5 yields $b = 1$.

Conversely, let $b = 1$ and let Φ be a compatible positive parallel Cayley form. Write $[\Phi] = t a_1$. Then $t^2 = \int_M \Phi^2 = 14V > 0$, and Φ/t gives $c_g(a_1) \leq |t|^{-1}$. Rank-one stable–comass duality gives $\mathrm{stsys}_4(M, g) = c_g(a_1)^{-1} \geq |t|$. The upper bound in Theorem 17.1 gives equality. □

17.5. **Conformal classes and uniform limits.** Let e_g^* be the norm on $H_4(M; \mathbb{R})$ dual to e_g under evaluation. In the notation of Section 14, one has $e_g = Q_{4,g}$. Put

$$\Gamma([g]) = \min_{0 \neq z \in H_4(M;\mathbb{Z})/\mathrm{torsion}} e_g^*(z)^2 = \sigma_4(M, g)^2. \tag{17.10}$$

Thus Γ is the square of the conformal four-systole of [9, arXiv version, §2] and depends only on $[g]$. The minimum is positive and attained, since e_g^* is a norm on a finite-dimensional space and the integral lattice is discrete.

Theorem 17.7. *For every smooth conformal class,*

$$\sup_{\widehat{g} \in [g]} R_4(\widehat{g}) \leq \Gamma([g]) \leq 14. \tag{17.11}$$

Equality $\Gamma([g]) = 14$ holds if and only if $b = 1$ and $[g]$ contains a metric with a compatible positive parallel Cayley form. In particular, if $b \geq 2$, then

$$\sup_{\widehat{g} \in [g]} R_4(\widehat{g}) < 14.$$

Proof. The first inequality follows from (14.2) with $n = 8$ and $d = 4$; see [9, arXiv version, (7.5)]. The dual of (17.4) gives $e_g^*(z_i) \leq \sqrt{14}$ for $1 \leq i \leq b$, hence $\Gamma([g]) \leq 14$.

Suppose $\Gamma([g]) = 14$. Then $e_g^*(z_i) = \sqrt{14}$ for every i. Compactness of the e_g unit ball gives a class u_i with $e_g(u_i) \leq 1$ and $\langle u_i, z_i\rangle = \sqrt{14}$. By (17.4),

$$\sqrt{14} = Q(u_i, a_i) \leq |u_i|_Q \leq \sqrt{14}\, e_g(u_i) \leq \sqrt{14}.$$

Equality in Cauchy–Schwarz gives $u_i = \sqrt{14}\, a_i$. Thus $e_g(a_i)^2 = 1/14$.

Put $h_i = h_{a_i}$. By Lemma 17.3, $h_i(x) \in \mathcal{K}_g(x)$ for every x. Since $[h_i] = a_i \neq 0$, Lemma 17.4, applied to $\sqrt{14}\, h_i$, gives constant signs $\epsilon_i \in \{1, -1\}$ such that

$$\Phi_i = \epsilon_i \sqrt{14}\, h_i, \qquad g_i = |h_i|_g^{1/2} g$$

are respectively a positive parallel Cayley form and its compatible smooth metric. Each g_i is Ricci-flat, and

$$dv_{g_i} = |h_i|_g^2\, dv_g, \qquad \mathrm{Vol}(M, g_i) = \|h_i\|_2^2 = Q(a_i, a_i) = 1.$$

All g_i belong to $[g]$. Obata's uniqueness theorem [67, Proposition 1.4], applied to the unit-volume Einstein metric g_1 and the constant-scalar-curvature metrics g_i, gives $g_i = g_1$: the round-sphere exception is excluded by Ricci-flatness. Put $g_* = g_1$.

The forms Φ_i are parallel for g_* and their cohomology classes $\epsilon_i \sqrt{14}\, a_i$ form a basis of $H^4(M; \mathbb{R})$. They therefore form a basis of the g_*-harmonic four-forms. Their pointwise inner products are constant. By self-duality and $\mathrm{Vol}(M, g_*) = 1$,

$$\langle \Phi_i, \Phi_j \rangle_{g_*} = \int_M \Phi_i \wedge \Phi_j = 14 \epsilon_i \epsilon_j Q(a_i, a_j) = 14 \delta_{ij}.$$

Lemma 17.5 gives $b = 1$.

Conversely, suppose $b = 1$ and $\widehat{g} \in [g]$ admits a compatible positive parallel Cayley form. Theorem 17.6 gives $R_4(\widehat{g}) = 14$, and (17.11) gives $\Gamma([g]) = 14$. □

Proposition 17.8. *The quantity Γ is invariant under diffeomorphisms and is continuous under C^0 convergence of smooth metrics to a smooth positive definite metric.*

Proof. For a diffeomorphism $\varphi : M \to M$, pullback of forms and change of variables give

$$e_{\varphi^* g}(\varphi^* a) = e_g(a), \qquad e^*_{\varphi^* g}(z) = e^*_g(\varphi_* z).$$

Since φ_* preserves the integral lattice, Γ is invariant under φ.

Suppose

$$(1 - \delta) g \leq g' \leq (1 + \delta) g, \qquad 0 < \delta < 1.$$

For every four-form η, one has pointwise

$$(1 + \delta)^{-2} \operatorname{comass}_g(\eta) \leq \operatorname{comass}_{g'}(\eta) \leq (1 - \delta)^{-2} \operatorname{comass}_g(\eta),$$
$$(1 - \delta)^4\, dv_g \leq dv_{g'} \leq (1 + \delta)^4\, dv_g.$$

Squaring the first inequalities, integrating with the second, and taking infima over representatives gives

$$C_\delta^{-1} e_g \leq e_{g'} \leq C_\delta e_g, \qquad C_\delta = \left(\frac{1 + \delta}{1 - \delta} \right)^2.$$

Duality gives the same bounds for the dual norms, uniformly on $H_4(M; \mathbb{R})$. Taking squared minima over the nonzero integral lattice gives

$$C_\delta^{-2} \Gamma([g]) \leq \Gamma([g']) \leq C_\delta^2 \Gamma([g]).$$

Since M is compact, C^0 convergence to a positive definite metric gives the preceding metric comparison with $\delta \to 0$. Thus Γ is continuous. □

Corollary 17.9. *Suppose $b \geq 2$ and $R_4(g_j) \to 14$. There are no positive smooth functions f_j and diffeomorphisms φ_j such that a subsequence of $f_j \varphi_j^* g_j$ converges in C^0 to a smooth positive definite metric.*

Proof. Suppose, after passing to a subsequence, that $f_j \varphi_j^* g_j \to g_\infty$ in C^0, where g_∞ is a smooth positive definite metric. By (17.11), $\Gamma([g_j]) \to 14$. Conformal invariance and Proposition 17.8 give

$$\Gamma([g_\infty]) = \lim_j \Gamma([f_j \varphi_j^* g_j]) = \lim_j \Gamma([g_j]) = 14,$$

contrary to Theorem 17.7. □

Appendix A. Nonorientable manifolds

Let M be a closed connected nonorientable smooth manifold of dimension n. Put

$$V_r = H_r(M;\mathbb{R}), \qquad L_r = H_r(M;\mathbb{Z})/\text{ torsion},$$

and identify L_r with its image in V_r. For a smooth Riemannian metric g, let $N_g^{(r)}$ be the stable norm on V_r, and put $s_r(g) = \min_{0\neq z\in L_r} N_g^{(r)}(z)$ when $b_r(M) > 0$. Since $H_n(M;\mathbb{R}) = 0$, put

$$D(M) = \{r \,:\, 1 \le r < n,\ b_r(M) > 0\}, \qquad D_+(M) = D(M) \cap \{2, \dots, n-1\}.$$

For $r \in D_+(M)$, let N_r be a norm on V_r. Inequalities between norms are understood pointwise. If $b_1(M) > 0$, define

$$\sigma_1^{\mathrm{st}}(M) = \inf_g \frac{\mathrm{Vol}(M,g)}{s_1(g)^n},$$

where g ranges over smooth Riemannian metrics on M. Set this number equal to zero when $b_1(M) = 0$.

Theorem A.1. *Suppose $n \ge 3$. If $b_1(M) > 0$, then, for every $a > 0$,*

$$\inf_{\substack{s_1(g)\ge a \\ N_g^{(r)}\ge N_r\ (r\in D_+(M))}} \mathrm{Vol}(M,g) = \sigma_1^{\mathrm{st}}(M) a^n. \tag{A.1}$$

Every strictly larger volume occurs for a smooth metric satisfying these constraints. Without a degree-one constraint, the infimum under all the higher norm constraints is zero, for every value of $b_1(M)$, and every positive volume occurs.

The assertion without a degree-one constraint follows from the construction in [17, proof of Theorem 1.3], by comparison of norms on the spaces V_r.

A.1. **Relative comparison**. For a finite polyhedron X of dimension at most n with a positive piecewise smooth metric h, let $\mathrm{Vol}_n(X,h)$ be the sum of the volumes of its n-simplices. A simplicial map between polyhedra of dimension at most n is $(n,1)$-*monotone* if the inverse image of each open target n-simplex is empty or one open source n-simplex mapped homeomorphically onto it. For such a map $q : M \to X$, the semidefinite pullback satisfies

$$\int_M dv_{q^*h} \le \mathrm{Vol}_n(X,h).$$

Write $\rho_M : \pi_1(M) \to \{\pm 1\}$ for the orientation character. For a map $f : M \to W$ that is surjective on fundamental groups, choose a coefficient system $\mathcal{L}$ on W as follows. If ρ_M factors through f_*, take the resulting local integer system; its pullback to M is the orientation system. Otherwise take $\mathcal{L} = \mathbb{F}_2$. Write $[M]_{\mathcal{L}}$ for a fundamental class in $H_n(M; f^*\mathcal{L})$, with a choice of generator in the local integer case.

Lemma A.2. *Suppose $n \ge 3$. Let M_0 be a copy of M and a subcomplex of a connected finite CW complex W, with inclusion i. Let $f : M \to W$ be surjective on fundamental groups. In the local integer case, suppose $i^*\mathcal{L}$ is the orientation system of M_0. Let $[M_0]_{\mathcal{L}} \in H_n(M_0; i^*\mathcal{L})$ be a fundamental class. If*

$$f_*[M]_{\mathcal{L}} = i_*[M_0]_{\mathcal{L}} \quad \text{in } H_n(W;\mathcal{L}), \tag{A.2}$$

there are a finite polyhedron X obtained from M_0 by attaching cells of dimensions $1, \dots, n-1$, a map $j : X \to W$ extending i, and an $(n,1)$-monotone map $q : M \to X$ such that $j \circ q$ is homotopic to f. The triangulation of X may be chosen so that all its n-simplices belong to M_0.

Proof. Choose a smooth triangulation of M_0. After a homotopy equivalence relative to M_0, the CW structure on W may be taken to extend this triangulation, by cellular approximation of the relative attaching maps [39, Proposition 0.18 and Theorem 4.8]. Let c be the image of $[M_0]_{\mathcal{L}}$ in $H_n(W^{(n)};\mathcal{L})$. By [18, Lemma 2.3] and (A.2), the map f is homotopic to a map into $W^{(n)}$ whose fundamental-class image is c.

The cellular coefficients of c have absolute value one on the n-simplices of M_0 and are zero on the other n-cells. Since $W^{(n)} \hookrightarrow W$ induces an isomorphism on fundamental groups, [18, Lemma 2.2] applies with the coefficient system specified above. It gives a homotopic map

$$f_0 : M \longrightarrow M_0 \cup W^{(n-1)}$$

whose inverse image of each open n-simplex of M_0 is one open ball mapped homeomorphically onto that simplex. The closed balls can be chosen disjoint, with the maps on them equal to characteristic homeomorphisms onto the corresponding closed simplices.

Put $A = W^{(n-1)}$ and $B = M_0^{(n-1)}$. The relative construction in the proof of [39, Theorem 2C.5] gives a finite polyhedral model K of A of dimension at most $n-1$, with B fixed. Since A and B are connected, choose a forest in the one-skeleton of K containing every vertex outside B, with each tree meeting B in exactly one vertex. Collapse each tree to that vertex, and denote the quotient by P.

For one such tree T, choose a subdivision in which every simplex meets T in a face [62, Lemma 3.3(a)]. Let $\lambda : K \to [0,1]$ be affine on each simplex, zero on the vertices of T, and one on the other vertices. Put $C = \{\lambda \geq 1/2\}$ and $D = \{\lambda = 1/2\}$. Barycentric coordinates identify K/T with C glued to the cone on D along D. This is a finite polyhedron of dimension at most $n-1$ [62, Exercise 2.27(2)]; its triangulation is compatible with B, since $T \cap B$ is one vertex. Successive collapses give the same conclusion for P. Its quotient CW structure has no relative zero-cells, and its other relative cells have dimensions $1, \dots, n-1$. Each collapse is a homotopy equivalence relative to B [39, Propositions 0.17 and 0.19]. Thus there are maps

$$u : A \longrightarrow P, \qquad v : P \longrightarrow A, \qquad u|_B = v|_B = \mathrm{id}_B, \qquad vu \simeq \mathrm{id}_A \quad \text{relative to } B.$$

Set $X = M_0 \cup_B P$ and extend u and v by the identity on M_0. Let $j : X \to W$ be the resulting map induced by v. Then $juf_0 \simeq f$. Since $u(A) \subset P$ and $u|_{M_0}$ is the identity, uf_0 has the same inverse images of the open n-simplices of M_0 as f_0.

Triangulate the closed inverse-image balls compatibly with their maps to M_0. Relative simplicial approximation [78, Theorem, p. 40], applied to the complement of their interiors with target P, gives a simplicial map q homotopic to uf_0 relative to these balls. Thus q is $(n,1)$-monotone and $jq \simeq f$. Since $\dim P \leq n-1$, every n-simplex of X belongs to M_0. □

Let $b \geq 1$ be an integer, let X be a finite polyhedron, and let $\varphi : H_1(X;\mathbb{Z})/\,\mathrm{torsion} \to \mathbb{Z}^b$ be surjective. Write $\varphi_{\mathbb{R}} : H_1(X;\mathbb{R}) \to \mathbb{R}^b$ for its real extension. For a positive piecewise smooth metric h on X, define

$$\overline{N}_{h,\varphi}(u) = \inf_{\substack{z \in H_1(X;\mathbb{R}) \\ \varphi_{\mathbb{R}}(z) = u}} N_h^{(1)}(z) \quad (u \in \mathbb{R}^b), \qquad s_\varphi(X,h) = \min_{0 \neq u \in \mathbb{Z}^b} \overline{N}_{h,\varphi}(u).$$

This is the quotient norm of [17, Definition 2.4]. Surjectivity makes it a norm on $\mathbb{R}^b$, with positive lattice minimum.

Lemma A.3. *Let $b = b_1(M) > 0$, and let X be a finite polyhedron obtained from a copy M_0 of M by attaching finitely many cells of dimensions $1, \dots, n-1$. Suppose φ restricts to an integral isomorphism*

$$H_1(M_0;\mathbb{Z})/\,\mathrm{torsion} \;\simeq\; \mathbb{Z}^b.$$

Then

$$\inf_h \frac{\mathrm{Vol}_n(X,h)}{s_\varphi(X,h)^n} = \sigma_1^{\mathrm{st}}(M), \tag{A.3}$$

where h ranges over positive piecewise smooth metrics.

Proof. Let $\psi : H_1(M_0;\mathbb{Z})/\,\mathrm{torsion} \to \mathbb{Z}^b$ be the restriction of φ. The induced homomorphisms from the fundamental groups to $\mathbb{Z}^b$ are surjective. The degree-one relative extension theorem [18,

Definitions 3.2 and 4.3, Lemma 4.4], in the quotient-norm formulation of [17, Definition 2.4 and proof of Lemma 2.3], gives

$$\inf_h \frac{\mathrm{Vol}_n(X,h)}{s_\varphi(X,h)^n} = \inf_{h_0} \frac{\mathrm{Vol}(M_0,h_0)}{s_\psi(M_0,h_0)^n},$$

where h_0 ranges over positive piecewise smooth metrics on M_0. Since ψ is an integral isomorphism, $s_\psi(M_0,h_0) = s_1(M_0,h_0)$. By Lemma 1.7 applied to the identity, a piecewise smooth metric on M_0 admits a smooth majorant with arbitrarily small increase in volume. Its stable systole cannot decrease. The infimum on the right is therefore unchanged if h_0 is required to be smooth, and equals $\sigma_1^{\mathrm{st}}(M)$. □

A.2. **Fundamental-class matching.** Fix a copy M_0 of M, with a chosen identification. Maps and coefficient systems on M are transported to M_0 by this identification. Set $b = b_1(M)$ and let

$$\Phi : M \longrightarrow T^b, \qquad T^b = \mathbb{R}^b/\mathbb{Z}^b,$$

be the Jacobi map associated with the integral quotient $\pi_1(M) \to L_1 \simeq \mathbb{Z}^b$. When $b = 0$, the torus is a point.

For $r \geq 2$ and $m \geq 0$, let $J_m(S^r)$ be the subspace of the James reduced product $J(S^r)$ consisting of words of length at most m. For each $r \in D_+(M)$, put

$$Y_r = \begin{cases} (S^r)^{b_r(M)}, & r \text{ odd}, \\ J_{\lfloor n/r \rfloor}(S^r)^{b_r(M)}, & r \text{ even}. \end{cases}$$

These are finite simply connected CW complexes with free integral homology [39, Proposition 3.22 and Theorem 3B.6]. With E_r as in Section 1, one has $(E_r^{b_r(M)})^{(n)} \subset Y_r$. Thus [17, Lemma 5.1] and cellular approximation [39, Theorem 4.8] give maps

$$f_r : M \longrightarrow Y_r$$

injective on real homology in degree r.

Choose based maps $\alpha_r : S^r \to S^r$ of degree two. Let $d_r : Y_r \to Y_r$ be the product of $b_r(M)$ copies of α_r for odd r, and of $J_{\lfloor n/r \rfloor}(\alpha_r)$ for even r. The latter maps preserve each word-length stage. By the natural finite-stage homology calculation in [39, Proposition 3C.8 and its proof] and the integral Künneth formula,

$$(d_r)_* = 2^j \,\mathrm{id} \quad \text{on } H_{jr}(Y_r;\mathbb{Z}) \quad (j \geq 0).$$

Replace each Y_r by a based finite polyhedron homotopy equivalent to it, and transport f_r and d_r by based homotopy inverses [39, Theorem 2C.5]. Retain the notation Y_r, f_r, d_r. Put

$$Y = \prod_{r \in D_+(M)} Y_r, \qquad d = \prod_{r \in D_+(M)} d_r, \qquad Z = T^b \times Y.$$

An empty product is a point. The space Y is simply connected and has free integral homology. The displayed scalar formula gives

$$d_* H_j(Y;\mathbb{Z}) \subset 2H_j(Y;\mathbb{Z}) \quad (j > 0), \tag{A.4}$$

and d_* is injective on real homology in every degree. The same homological properties occur in [17, §4 and the proof of Theorem 1.3].

Let $* \in Y$ be the product basepoint, let $p : Z \to T^b$ be the projection, and let $s : T^b \to Z$ be given by $s(t) = (t,*)$. Define

$$f = (f_r)_r : M \longrightarrow Y, \qquad F = (\Phi, d \circ f) : M \longrightarrow Z, \qquad F_0 = s \circ \Phi : M_0 \longrightarrow Z.$$

Both F and F_0 induce the Jacobi epimorphism on fundamental groups. Define a coefficient system $\mathcal{K}$ on T^b as follows. If $\ker(\Phi_* : \pi_1(M) \to \mathbb{Z}^b)$ contains an orientation-reversing loop, put $\mathcal{K} = \mathbb{F}_2$. Otherwise let $\omega : \mathbb{Z}^b \to \{\pm 1\}$ be the character such that $\rho_M = \omega \circ \Phi_*$, and put $\mathcal{K} = \mathbb{Z}_\omega$, the local integer system with monodromy ω. Set $\mathcal{L} = p^*\mathcal{K}$, and choose corresponding fundamental classes on M and M_0 under the fixed identification.

Lemma A.4. *One has*

$$F_*[M]_{\mathcal{L}} = (F_0)_*[M_0]_{\mathcal{L}} \quad \text{in } H_n(Z;\mathcal{L}). \tag{A.5}$$

Proof. If $\mathcal{K} = \mathbb{F}_2$, then $2H_i(T^b;\mathcal{K}) = 0$ for every $i \geq 0$. Otherwise ω is nontrivial, since M is nonorientable; in particular, $b \geq 1$. An integral basis change puts its sign monodromy on the first circle only. The cellular complex of that circle, in degrees one and zero, is

$$0 \longrightarrow \mathbb{Z} \xrightarrow{-2} \mathbb{Z} \longrightarrow 0.$$

Tensoring with the cellular complex of T^{b-1} gives

$$H_i(T^b;\mathbb{Z}_\omega) \simeq H_i(T^{b-1};\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}/2, \qquad 2H_i(T^b;\mathbb{Z}_\omega) = 0 \quad (i \geq 0). \tag{A.6}$$

Thus $2H_i(T^b;\mathcal{K}) = 0$ in both cases.

Writing C_* for cellular chains, the pullback definition of $\mathcal{L}$ gives

$$C_*(Z;\mathcal{L}) \simeq C_*(T^b;\mathcal{K}) \otimes_{\mathbb{Z}} C_*(Y;\mathbb{Z}).$$

The second factor and its homology are free over $\mathbb{Z}$. The algebraic Künneth theorem therefore gives a natural cross-product isomorphism [39, Proposition 3B.1, Theorem 3B.5, and §3.H]:

$$H_n(Z;\mathcal{L}) \simeq \bigoplus_{i+j=n} H_i(T^b;\mathcal{K}) \otimes_{\mathbb{Z}} H_j(Y;\mathbb{Z}).$$

Set $\delta = \mathrm{id}_{T^b} \times d$. For $a \in H_i(T^b;\mathcal{K})$ and $z \in H_j(Y;\mathbb{Z})$ with $i + j = n$ and $j > 0$, write $d_*z = 2w$, where $w \in H_j(Y;\mathbb{Z})$, by (A.4). Then

$$(\mathrm{id} \otimes d_*)(a \otimes z) = a \otimes 2w = 2a \otimes w = 0.$$

On the summand with $j = 0$, the map δ_* is the identity. By naturality, $\delta_* = s_*p_*$ on $H_n(Z;\mathcal{L})$. With $G = (\Phi, f) : M \to Z$, one has $F = \delta \circ G$ and $p \circ G = \Phi$. Consequently,

$$F_*[M]_{\mathcal{L}} = s_*p_*G_*[M]_{\mathcal{L}} = s_*\Phi_*[M]_{\mathcal{L}} = (F_0)_*[M_0]_{\mathcal{L}}.$$

□

A.3. **Prescribed higher norms.**

Proof of Theorem A.1. Choose a finite polyhedral model W of the mapping cylinder of F_0, containing M_0 and Z as subcomplexes, and a homotopy equivalence $R : W \to Z$ such that

$$R|_Z = \mathrm{id}_Z, \qquad R|_{M_0} = F_0;$$

see [39, proof of Theorem 2C.5]. Let $i : M_0 \hookrightarrow W$ be the inclusion. Regard F as a map into W, and put $\mathcal{L}_W = R^*\mathcal{L}$. Since R_* is an isomorphism on homology with these coefficients, Lemma A.4 gives

$$F_*[M]_{\mathcal{L}_W} = i_*[M_0]_{\mathcal{L}_W}.$$

Moreover, F is surjective on fundamental groups, and $i^*\mathcal{L}_W = F_0^*\mathcal{L}$ is the orientation system in the integer case. Lemma A.2 therefore gives maps $q : M \to X$ and $j : X \to W$ with $j \circ q \simeq F$.

Write

$$R \circ j = (\Theta, \Psi) : X \longrightarrow T^b \times Y, \qquad \Psi = (\Psi_r)_r.$$

Then $\Theta|_{M_0} = \Phi$ and $\Psi|_{M_0} = *$. Triangulate each Y_r with its basepoint as a vertex, and triangulate Y by subdividing the products of simplices. Since the latter restriction is constant, simplicial approximation may be taken fixed on M_0 [39, Theorem 2C.1 and its proof]. Choose such a representative of Ψ. Subdivide the source of q compatibly with the inverse images of the new target simplices, so that q remains simplicial and $(n, 1)$-monotone. These subdivisions are smooth on M. The homotopies give

$$\Theta \circ q \simeq \Phi, \qquad (\Psi_r \circ q)_* : H_r(M;\mathbb{R}) \longrightarrow H_r(Y_r;\mathbb{R}) \text{ is injective.}$$

When $b > 0$, the homomorphism $\varphi = \Theta_* : H_1(X;\mathbb{Z})/\text{ torsion} \to \mathbb{Z}^b$ restricts to the chosen integral isomorphism on M_0, and is therefore surjective.

Choose positive piecewise smooth metrics k_r on Y_r, and let k be their product metric on Y. For $r \in D_+(M)$, define

$$A_r : V_r \longrightarrow [0,\infty), \qquad A_r(z) = N^{(r)}_{k_r}((\Psi_r \circ q)_* z).$$

The homology injection makes A_r a norm on V_r. Hence

$$c_r = \min_{N_r(z)=1} A_r(z) > 0, \qquad A_r \geq c_r N_r,$$

since the N_r-unit sphere is compact. Fix

$$t = \max\bigl(\{1\} \cup \{c_r^{-1/r} : r \in D_+(M)\}\bigr).$$

For a positive piecewise smooth metric h on X, put

$$h_t = h + t^2 \Psi^* k.$$

Fix $\varepsilon > 0$. Suppose first that $b > 0$, and let $a > 0$. By Lemma A.3, choose $\widehat{h}$ with

$$\frac{\mathrm{Vol}_n(X,\widehat{h})}{s_\varphi(X,\widehat{h})^n} < \sigma_1^{\mathrm{st}}(M) + \frac{\varepsilon}{a^n}.$$

Set $h = (a/s_\varphi(X,\widehat{h}))^2 \widehat{h}$. Then

$$s_\varphi(X,h) = a, \qquad \mathrm{Vol}_n(X,h) < \sigma_1^{\mathrm{st}}(M) a^n + \varepsilon. \tag{A.7}$$

Every n-simplex of X belongs to M_0, where Ψ is constant. Thus $h_t = h$ on these simplices. Since $h_t \geq h$, the quotient norms satisfy $\overline{N}_{h_t,\varphi} \geq \overline{N}_{h,\varphi}$. Consequently,

$$\mathrm{Vol}_n(X,h_t) = \mathrm{Vol}_n(X,h), \qquad s_\varphi(X,h_t) \geq a. \tag{A.8}$$

For $r \in D_+(M)$, the tensor inequality $h_t \geq t^2 \Psi_r^* k_r$ makes $\Psi_r : (X,h_t) \to (Y_r, t^2 k_r)$ nonexpanding. Therefore, for every metric g making $q : (M,g) \to (X,h_t)$ nonexpanding, monotonicity and homogeneity of the stable norm (Section 3) give

$$N_g^{(r)} \geq t^r A_r \geq t^r c_r N_r \geq N_r. \tag{A.9}$$

Apply Lemma 1.7 to q and h_t, with volume error ε. Since q is $(n,1)$-monotone, the resulting smooth metric g makes q nonexpanding and satisfies

$$\mathrm{Vol}(M,g) < \mathrm{Vol}_n(X,h_t) + \varepsilon < \sigma_1^{\mathrm{st}}(M) a^n + 2\varepsilon.$$

For $0 \neq z \in L_1$, the element $u = \varphi(q_* z) = \Phi_* z$ is nonzero in $\mathbb{Z}^b$. Hence

$$N_g^{(1)}(z) \geq N_{h_t}^{(1)}(q_* z) \geq \overline{N}_{h_t,\varphi}(u) \geq a.$$

Thus $s_1(g) \geq a$. Letting $\varepsilon \downarrow 0$ gives the upper bound in (A.1). The lower bound is

$$\mathrm{Vol}(M,g) \geq \sigma_1^{\mathrm{st}}(M) s_1(g)^n \geq \sigma_1^{\mathrm{st}}(M) a^n.$$

If no degree-one constraint is imposed, choose a positive piecewise smooth metric h on X with $\mathrm{Vol}_n(X,h) < \varepsilon$. For the same t and $h_t = h + t^2 \Psi^* k$, the volume identity and (A.9) still hold. Lemma 1.7 gives a smooth metric satisfying all higher constraints and

$$\mathrm{Vol}(M,g) < \mathrm{Vol}_n(X,h_t) + \varepsilon = \mathrm{Vol}_n(X,h) + \varepsilon < 2\varepsilon.$$

This argument also applies when $b = 0$.

Let v be strictly larger than the relevant infimum. Choose an admissible g with $\mathrm{Vol}(M,g) < v$, and put

$$g_v = \left(\frac{v}{\mathrm{Vol}(M,g)}\right)^{2/n} g.$$

Then $\mathrm{Vol}(M,g_v) = v$ and

$$N_{g_v}^{(r)} = \left(\frac{v}{\mathrm{Vol}(M,g)}\right)^{r/n} N_g^{(r)} \geq N_g^{(r)}.$$

All prescribed lower bounds are preserved. □

For $E \subset D(M)$ and $\ell \in (0,\infty)^E$, put

$$V_M^E(\ell) = \inf\{\mathrm{Vol}(M,g) \,:\, s_r(g) \geq \ell_r \quad (r \in E)\},$$

where g ranges over smooth Riemannian metrics on M.

Corollary A.5. *Let $n \geq 3$ and let $\ell_r > 0$ for $r \in D(M)$. Then*

$$V_M^{D(M)}(\ell) = \begin{cases} \sigma_1^{\mathrm{st}}(M)\ell_1^n, & b_1(M) > 0, \\ 0, & b_1(M) = 0. \end{cases}$$

Every strictly larger volume occurs. These conclusions remain valid after any of the higher-degree constraints are omitted.

Proof. For each $r \in D_+(M)$ choose a norm whose minimum on $L_r \setminus \{0\}$ is ℓ_r, and apply Theorem A.1. The definition of $\sigma_1^{\mathrm{st}}(M)$ gives the reverse inequality when $b_1(M) > 0$. For an omitted degree, impose any auxiliary norm in that degree. The assertion about volumes follows from the homothety in the theorem. □

A.4. **Balanced exponents.** For $n \geq 2$, set

$$\epsilon(M) = \begin{cases} 1, & \Phi_*[M]_{\mathbb{F}_2} \neq 0 \text{ in } H_n(T^b;\mathbb{F}_2), \\ 0, & \Phi_*[M]_{\mathbb{F}_2} = 0. \end{cases}$$

For an integral cohomology class u, write $\bar{u}$ for its reduction modulo two.

Lemma A.6. *One has $\sigma_1^{\mathrm{st}}(M) > 0$ if and only if $\epsilon(M) = 1$. Equivalently, there are classes $u_1,\dots,u_n \in H^1(M;\mathbb{Z})$ such that*

$$\bar{u}_1 \cdots \bar{u}_n \neq 0 \quad \text{in } H^n(M;\mathbb{F}_2).$$

Proof. If $b = 0$, then $\sigma_1^{\mathrm{st}}(M) = \epsilon(M) = 0$ and $H^1(M;\mathbb{Z}) = 0$. Suppose $b > 0$. By [17, Corollary 5.2], including its surface case, $\sigma_1^{\mathrm{st}}(M) > 0$ if and only if $\Phi_*[M]_{\mathcal{K}} \neq 0$, with the coefficient system $\mathcal{K}$ defined above. If the orientation character does not descend through Φ_*, then $\mathcal{K} = \mathbb{F}_2$, and this is the required equivalence. Otherwise $\mathcal{K} = \mathbb{Z}_\omega$, with ω nontrivial. The coefficient sequence

$$0 \longrightarrow \mathbb{Z}_\omega \xrightarrow{2} \mathbb{Z}_\omega \longrightarrow \mathbb{F}_2 \longrightarrow 0$$

and (A.6) give

$$\ker\big(H_n(T^b;\mathbb{Z}_\omega) \longrightarrow H_n(T^b;\mathbb{F}_2)\big) = 2H_n(T^b;\mathbb{Z}_\omega) = 0.$$

Reduction sends $\Phi_*[M]_{\mathcal{K}}$ to $\Phi_*[M]_{\mathbb{F}_2}$, so their nonvanishing is equivalent.

The map $\Phi^* : H^1(T^b;\mathbb{Z}) \to H^1(M;\mathbb{Z})$ is an isomorphism, and $H^*(T^b;\mathbb{F}_2)$ is generated by reductions of integral degree-one classes [39, Example 3.16]. For $v \in H^n(T^b;\mathbb{F}_2)$, naturality gives

$$\langle \Phi^* v, [M]_{\mathbb{F}_2} \rangle = \langle v, \Phi_*[M]_{\mathbb{F}_2} \rangle.$$

The evaluation pairings over $\mathbb{F}_2$ are nondegenerate [39, §3.1, pp. 198–199]. Since degree-n products of degree-one classes span $H^n(T^b;\mathbb{F}_2)$, this identity gives the last equivalence. □

Theorem A.7. *Let M be nonorientable of dimension $n \geq 2$. Let $p \in \mathbb{R}_{\geq 0}^{D(M)}$ satisfy $\sum_r r p_r = n$, and put $p_r = 0$ for $r \notin D(M)$. There is a constant $C > 0$ such that*

$$\prod_{r \in D(M)} s_r(g)^{p_r} \leq C \operatorname{Vol}(M, g)$$

for every smooth metric g if and only if

$$\epsilon(M) = 1, \qquad p_1 = n, \qquad p_r = 0 \quad (r \geq 2).$$

Proof. For $p_1 = n$ and $p_r = 0$ when $r \geq 2$, the equivalence follows from Lemma A.6 and the definition of $\sigma_1^{\mathrm{st}}(M)$. If $n = 2$, either $D(M)$ is empty and there is no balanced exponent vector, or $D(M) = \{1\}$ and $p_1 = 2$.

Suppose $n \geq 3$. If $b_1(M) > 0$, then, for each $t \geq 1$, apply Corollary A.5 with $\ell_1 = 1$ and $\ell_r = t$ for $r \in D_+(M)$. There is a smooth metric g_t with

$$s_1(g_t) \geq 1, \qquad s_r(g_t) \geq t \ (r \in D_+(M)), \qquad \operatorname{Vol}(M, g_t) < \sigma_1^{\mathrm{st}}(M) + 1.$$

If $\sum_{r \geq 2} p_r > 0$, then

$$\frac{\prod_{r \in D(M)} s_r(g_t)^{p_r}}{\operatorname{Vol}(M, g_t)} \geq \frac{t^{\sum_{r \geq 2} p_r}}{\sigma_1^{\mathrm{st}}(M) + 1} \longrightarrow \infty.$$

If $b_1(M) = 0$, the same corollary with $\ell_r = 1$ for all $r \in D(M)$ gives metrics g_j, $j \geq 1$, with $s_r(g_j) \geq 1$ and $\operatorname{Vol}(M, g_j) \to 0$. Hence the ratio of every balanced product to $\operatorname{Vol}(M, g_j)$ tends to infinity. □

For nonorientable M, the stable systolic category $\mathrm{cat}_{\mathrm{stsys}}(M)$ is the maximum of $\sum_r p_r$ over $p \in \mathbb{Z}_{\geq 0}^{D(M)}$ satisfying $\sum_r r p_r = n$ and the metric inequality in Theorem A.7 [27, Definition 1.2]. If there is no such vector, set $\mathrm{cat}_{\mathrm{stsys}}(M) = 0$ [63, Definition 1.1].

Corollary A.8. *For a closed connected nonorientable smooth n-manifold, $n \geq 2$,*

$$\mathrm{cat}_{\mathrm{stsys}}(M) = \begin{cases} n, & \epsilon(M) = 1, \\ 0, & \epsilon(M) = 0. \end{cases}$$

Proof. By Theorem A.7, the only possible integral exponent vector has $p_1 = n$ and $p_r = 0$ for $r \geq 2$; it satisfies the inequality precisely when $\epsilon(M) = 1$. □

Example A.9. Let $N_3 = \#_3 \mathbb{RP}^2$. Its standard generators a_1, a_2, a_3 give

$$H_1(N_3; \mathbb{Z}) = \big(\mathbb{Z}a_1 \oplus \mathbb{Z}a_2 \oplus \mathbb{Z}a_3\big)/\langle 2(a_1 + a_2 + a_3)\rangle$$

[39, Example 2.37]. Let $e_1, e_2, e_3 \in H^1(N_3; \mathbb{F}_2)$ be dual to the reductions of a_1, a_2, a_3 in $H_1(N_3; \mathbb{F}_2)$, and let $0 \neq \nu \in H^2(N_3; \mathbb{F}_2)$. Then $e_i e_j = \delta_{ij} \nu$ [39, Example 3.8]. The integral covectors u and v given by

$$(u(a_1), u(a_2), u(a_3)) = (1, -1, 0), \qquad (v(a_1), v(a_2), v(a_3)) = (0, 1, -1)$$

annihilate the relation and hence define $u, v \in H^1(N_3; \mathbb{Z})$ [39, §3.1, p. 198]. Their reductions satisfy

$$\bar{u} = e_1 + e_2, \qquad \bar{v} = e_2 + e_3, \qquad \bar{u}\bar{v} = \nu.$$

For $n \geq 2$, let $x_1, \dots, x_{n-2} \in H^1(T^{n-2}; \mathbb{Z})$ be the generators from the circle factors, with the empty product equal to 1 when $n = 2$. After pullback to $N_3 \times T^{n-2}$,

$$\bar{u}\bar{v}\bar{x}_1 \cdots \bar{x}_{n-2} \neq 0$$

by the Künneth formula [39, Theorem 3.15]. Thus Lemma A.6 and Corollary A.8 give

$$\epsilon(N_3 \times T^{n-2}) = 1, \qquad \mathrm{cat}_{\mathrm{stsys}}(N_3 \times T^{n-2}) = n.$$

For $M = N_3 \times S^2$, one has $n = 4$ and $b_1(M) = 2$, so $H_4(T^2; \mathbb{F}_2) = 0$ and $\epsilon(M) = 0$. By Lemma A.6 and Theorem A.1, the infimum of volume is zero under any prescribed constraint $s_1(g) \geq a > 0$

and arbitrary inequalities $N_g^{(r)} \geq N_r$ for $r \in D_+(M)$. Theorem A.7 therefore excludes an inequality $s_1(g)^2 s_2(g) \leq C\,\mathrm{Vol}(M,g)$ valid for all smooth metrics. If $U = H^1(N_3;\mathbb{Q})$ and z generates $H^2(S^2;\mathbb{Q})$, the Künneth formula gives

$$H^*(M;\mathbb{Q}) \simeq (\mathbb{Q} \oplus U) \otimes \mathbb{Q}[z]/(z^2), \qquad \dim U = 2, \quad U^2 = 0, \quad \deg z = 2.$$

For $0 \neq w \in U$, one has $wz \neq 0$, whereas every product of three positive-degree classes is zero. The rational cup-length is therefore two. Products of this form are considered in [48, arXiv version, Question 8.3] for ordinary systolic category.

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