QUANTITATIVE STABILITY OF THE CUBE WULFF INEQUALITY WITH OPTIMAL DIMENSIONAL DEPENDENCE

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Abstract. We prove a quantitative Wulff inequality for the cube with optimal dimensional dependence. For every set $E \subset \mathbb{R}^n$ of finite perimeter and unit volume, the anisotropic perimeter deficit controls the square of its Fraenkel asymmetry with constant $1/(72\,n)$. Rectangular boxes show that the order $1/n$ is optimal. The proof is by induction on the dimension and rests on an exact decomposition of the deficit along parallel sections, together with a gluing estimate for the asymmetry with coefficient one. We deduce a quantitative Brunn–Minkowski inequality of optimal dimensional order n^{-2} for a cube and an arbitrary measurable summand, valid at every volume ratio.

1. Introduction

The quantitative isoperimetric inequality of Fusco, Maggi and Pratelli [9] states that the isoperimetric deficit of a set controls the square of its distance from a ball, measured by the Fraenkel asymmetry. By mass transportation, Figalli, Maggi and Pratelli [5] proved the analogous statement for the anisotropic perimeter P_K associated with a general open, bounded and convex set K: for every set $E \subset \mathbb{R}^n$ of finite perimeter with $0 < |E| < \infty$,

$$\frac{P_K(E)}{n\,|E|^{(n-1)/n}|K|^{1/n}} - 1 \geq \frac{A_K(E)^2}{C(n)}, \qquad A_K(E) := \inf_{x\in\mathbb{R}^n} \frac{|E\triangle(x+\lambda K)|}{|E|}, \tag{1}$$

where $\lambda := (|E|/|K|)^{1/n}$ and $C(n)$ depends only on the dimension. The quantity A_K is called *Fraenkel asymmetry*. Here and below we denote by $\delta_K(E)$ the left-hand side of (1), so that (1) reads $A_K(E)^2 \leq C(n)\,\delta_K(E)$. More precisely, the estimate obtained in [5, Theorem 1.1 and eq. (1.12)] gives $C(n) \leq Cn^{17}$, with C an absolute constant.

For convex sets E, Segal [15] obtained an estimate $C(n) \leq Cn^7$. Kolesnikov and Milman also established $C(n) \leq Cn^{11/2}$ for every convex set E and every convex Wulff shape K; see [13, Section 12.2 and Theorem 12.9].

Harutyunyan [12, Conjecture 1.4] conjectured that, for convex sets E, the optimal constant in (1), uniform with respect to the Wulff shape K, is of the form Cn^2. By testing the Brunn–Minkowski inequality on pairs of boxes, he also observed that the exponent 2 cannot be lowered. We obtain the conjectured order n^2 in the Wulff inequality when K is a cube and E is an arbitrary set of finite perimeter, and in the Brunn–Minkowski inequality when one summand is a cube and the other is measurable.

In order to describe the main result more precisely, we introduce our setting. Throughout the paper we assume $n \geq 2$ and we denote by $Q_n := (-1/2, 1/2)^n$ the open unit cube. Given a set $E \subset \mathbb{R}^n$ of locally finite perimeter, its *anisotropic perimeter* with respect to Q_n is

$$P_n(E) := \int_{\partial^* E} \|\nu_E\|_*\, d\mathcal{H}^{n-1}, \qquad \|\nu\|_* := \sup_{x\in Q_n} \langle x, \nu\rangle = \frac{1}{2}\sum_{i=1}^n |\nu_i|,$$

where $\partial^* E$ is the reduced boundary of E and ν_E is the measure-theoretic outer normal (see [14] for an introduction to sets of finite perimeter). Since $D\chi_E = -\nu_E\,\mathcal{H}^{n-1}\llcorner\partial^* E$ by De Giorgi's

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structure theorem, we can equivalently write

$$P_n(E) = \frac{1}{2}\sum_{i=1}^{n} |D_i\chi_E|(\mathbb{R}^n), \tag{2}$$

where $|D_i\chi_E|$ is the total variation of the distributional derivative of χ_E in the direction e_i. We use formula (2) throughout the proofs. In particular, $P_n(E) < \infty$ if and only if E has finite perimeter, $P_n(Q_n) = n$, and $P_n(\lambda E) = \lambda^{n-1} P_n(E)$ for every $\lambda > 0$. The Wulff inequality for the cube states that $P_n(E) \geq n$ for every set E with $|E| = 1$, with equality only for translates of Q_n (see [5] and the references therein). We recover this inequality in Lemmas 3 and 6. When $|E| = 1$ the Fraenkel asymmetry $A_{Q_n}(E)$ becomes

$$\alpha_n(E) := \inf_{x\in\mathbb{R}^n} |E\triangle(x + Q_n)| \tag{3}$$

which satisfies $0 \leq \alpha_n(E) \leq 2$. Both $P_n(E) - n$ and $\alpha_n(E)$ are invariant under translations.

Our main result reads as follows.

Theorem 1. *Let $n \geq 2$. Every set $E \subset \mathbb{R}^n$ of finite perimeter with $|E| = 1$ satisfies*

$$P_n(E) - n \geq \frac{1}{72\,n}\,\alpha_n^2(E). \tag{4}$$

Moreover, any constant replacing $1/(72\,n)$ in (4) is at most $n/(8\lfloor n^2/4\rfloor)$.

By scaling, (4) is equivalent to the following scale-invariant statement: for every set E of finite perimeter with $0 < |E| < \infty$,

$$\frac{P_n(E)}{n\,|E|^{(n-1)/n}} - 1 \geq \frac{A_{Q_n}(E)^2}{72\,n^2}. \tag{5}$$

Thus (1) holds for $K = Q_n$ with $C(n) = 72\,n^2$.

The result is sharp in the sense that the order n^2 cannot be improved. The example is elementary and of the same kind as the one in [12, Section 3]. For even n and $\varepsilon > 0$, let R_ε be the box with $n/2$ sides of length e^{ε} and $n/2$ sides of length $e^{-\varepsilon}$. Then $P_n(R_\varepsilon) - n = n(\cosh\varepsilon - 1)$ and $\alpha_n(R_\varepsilon) = 2(1 - e^{-n\varepsilon/2})$. Therefore $\alpha_n^2(R_\varepsilon)/(P_n(R_\varepsilon) - n) \to 2n$ as $\varepsilon \downarrow 0$. For odd $n = 2k + 1$, we take instead k sides of length $e^{(k+1)\varepsilon}$ and $k + 1$ sides of length $e^{-k\varepsilon}$. This volume-preserving deformation gives the limit $2(n^2 - 1)/n$; both calculations are given at the end of Section 4.

Establishing the optimal dimensional dependence in (1) for general convex Wulff shapes remains open. In particular, for the Euclidean ball B, the conjectured dimensional scale in (1) is $A_B(E)^2 \leq Cn\,\delta_B(E)$, with C independent of n (see [11]). This scale agrees with the second variation at the ball and with Fuglede's estimate for small enough nearly spherical perturbations [8]. The corresponding problem in Gauss space is settled: by [1, Corollary 1], the Gaussian isoperimetric deficit controls the square of the Fraenkel asymmetry with respect to half-spaces in a dimension-free way.

Our second result concerns the Euclidean concentration inequality for the cube, namely the Brunn–Minkowski inequality for the Minkowski sum of a set with a cube. Since Q_n is open, $E + rQ_n$ is an open set for every $r > 0$. Its measure is understood for the given representative of E, and the estimates below hold for every representative. Recall that $|E + rQ_n| \geq (1 + r)^n$ for every set E with $|E| = 1$. In our previous work [2, Theorem 2] we proved that for every convex K and every measurable E with $|K| = |E| = 1$, and for every $r > 0$, the concentration gap controls $c_n \max\{r^{n-1}, r\}\,\alpha^2(E)$. In [2, Remark 2] we described the coefficient supplied by that proof as decaying at most like 9^{-n}. For the cube, Theorem 1 and the union estimate in the proof of [2, Lemma 4] give the optimal dimensional order at every radius.

Theorem 2. *Let $n \geq 2$ and let $E \subset \mathbb{R}^n$ be measurable with $|E| = 1$. Then, for every $0 < t < 1$,*

$$|(1 - t)E + tQ_n|^{1/n} - 1 \geq \frac{t(1 - t)}{1152\,n^2}\,\alpha_n^2(E). \tag{6}$$

Since $x^n - 1 \geq n(x-1)$ for $x \geq 1$, (6) also gives $|(1-t)E + tQ_n| - 1 \geq t(1-t)\alpha_n^2(E)/(1152\,n)$. With $t = r/(1+r)$ this volume estimate reads

$$|E + rQ_n| - (1+r)^n \geq \frac{r(1+r)^{n-2}}{1152\,n}\,\alpha_n^2(E) \qquad \text{for every } r > 0. \tag{7}$$

The boxes R_ε introduced above show that the quadratic power of the asymmetry and the order n^{-2} in (6) are sharp. Indeed, for even n,

$$|(1-t)R_\varepsilon + tQ_n|^{1/n} = \sqrt{\big((1-t)e^{\varepsilon} + t\big)\big((1-t)e^{-\varepsilon} + t\big)} = \sqrt{1 + 2t(1-t)(\cosh\varepsilon - 1)}.$$

Since $\alpha_n(R_\varepsilon) \sim n\varepsilon$, for each fixed $0 < t < 1$ we have

$$\lim_{\varepsilon \downarrow 0} \frac{|(1-t)R_\varepsilon + tQ_n|^{1/n} - 1}{t(1-t)\,\alpha_n^2(R_\varepsilon)} = \frac{1}{2n^2}.$$

In the normalization of [4, 12], (6) gives the dimensional order predicted by [12, Conjecture 1.4] at every volume ratio when one summand is a cube. See Remark 3.

The main idea of the proof of Theorem 1 is inspired by the proof of the Brunn–Minkowski inequality due to Hadwiger and Ohmann, as presented in Gardner's survey (see the proof of [10, Theorem 4.1]). There the inequality is first established for finite unions of boxes, which are cut by a coordinate hyperplane dividing the two sets in the same volume ratio, and then extended by approximation. Here we argue by induction on the dimension for finite unions of boxes. Such a set E is a finite stack of horizontal layers $R_j \times (a_{j-1}, a_j)$. To every layer we associate the prism $\mathcal{P}_j := R_j \times [0, 1/|R_j|]$ of unit volume. In Lemma 1 we show that the deficit of E splits as

$$P_n(E) - n = \sum_j \mu_j \big(P_n(\mathcal{P}_j) - n\big) + V,$$

where μ_j is the volume of the j-th layer and $V \geq 0$ measures the mismatch between consecutive layers in the vertical direction. The key point is the gluing estimate of Lemma 4: if the largest section has measure at least $1/2$, then

$$\alpha_n^2(E) \leq \sum_j \mu_j\, \alpha_n^2(\mathcal{P}_j) + 144\, V.$$

The coefficient in front of the sum is one, which prevents the loss of a multiplicative constant at every step of the induction. We then control the asymmetry of each prism through the inductive hypothesis applied to its base (Lemma 5). Finally, in Lemma 6 we pass from finite unions of boxes to general sets of finite perimeter by an averaged thresholding of cell averages. This approach avoids a preliminary reduction to a dimension-dependent neighborhood of the cube and preserves the constant under strict approximation.

With respect to the cut of Hadwiger and Ohmann, here we have the decomposition of Lemma 1. The new ingredient is the gluing estimate. The difficulty of making the translations of different pieces consistent appears also in the general stability theory of Figalli, van Hintum and Tiba [6, 7]. Here a section of maximal measure provides an anchor, and the connectivity of the vertical fibers produces a single interval to which all prisms are compared.

The paper is organized as follows. In Section 2 we introduce our notation for stacks of boxes and prove the deficit decomposition. In Section 3 we establish the gluing lemma. Section 4 contains the prism estimate, the approximation argument, and the proof of Theorem 1. In Section 5 we deduce Theorem 2.

2. Sections and prisms

In this section we fix the notation for finite unions of boxes and prove some preliminary facts. A *box* is a bounded open rectangle with sides parallel to the coordinate axes. Throughout Sections 2, 3 and 4 boundaries are immaterial and sets are identified up to null sets. We write $|\cdot|$

for the Lebesgue measure in the appropriate dimension, and χ_F for the characteristic function of a set F. For every set F of finite perimeter with $|F| = 1$, we define its *deficit* as

$$D_n(F) := P_n(F) - n, \tag{8}$$

so that (4) reads $\alpha_n^2(E) \le 72\, n\, D_n(E)$. For sets E and F of measure one, the triangle inequality gives

$$|\alpha_n(E) - \alpha_n(F)| \le |E \triangle F|. \tag{9}$$

Let $n = m + 1$ and let $E \subset \mathbb{R}^n$ be a finite union of boxes with $|E| = 1$. We choose $a_0 < a_1 < \cdots < a_N$ such that

$$E = \bigcup_{j=1}^{N} R_j \times (a_{j-1}, a_j), \tag{10}$$

where every $R_j \subset \mathbb{R}^m$ is either a finite union of boxes or the empty set. We set

$$h_j := a_j - a_{j-1}, \qquad p_j := |R_j|, \qquad \mu_j := h_j p_j, \qquad M := \max_j p_j. \tag{11}$$

Thus $\sum_j \mu_j = 1$ and $M > 0$. Empty layers are allowed and have $p_j = \mu_j = 0$. By (2) the perimeter of E splits as

$$P_n(E) = \sum_j h_j P_m(R_j) + P_v, \qquad P_v := \frac{1}{2}|D_n \chi_E|(\mathbb{R}^n). \tag{12}$$

Here $|D_n\chi_E|$ is, as in (2), the total variation of the derivative of χ_E in the direction e_n, and it should not be confused with the deficit D_n in (8). By (10), we have

$$P_v = \frac{1}{2}\Big(p_1 + p_N + \sum_{j=1}^{N-1} |R_j \triangle R_{j+1}|\Big). \tag{13}$$

The vertical part P_v has a useful description in terms of the fibers of E. For every $x \in \mathbb{R}^m$ let $k(x)$ be the number of maximal consecutive runs of indices j for which $x \in R_j$. Equivalently, $k(x)$ is the number of intervals forming the fiber $\{y : (x, y) \in E\}$, where consecutive layers (a_{j-1}, a_j) and (a_j, a_{j+1}) both met by the fiber are counted as the single interval (a_{j-1}, a_{j+1}). Then

$$P_v = \int_{\mathbb{R}^m} k(x)\, dx. \tag{14}$$

Setting $\mathcal{U} := \bigcup_j R_j$, from (14) we get

$$P_v \ge |\mathcal{U}| \ge M. \tag{15}$$

To every layer with $p_j > 0$ we associate the prism of unit volume

$$\mathcal{P}_j := R_j \times [0, 1/p_j], \qquad \alpha_j := \alpha_n(\mathcal{P}_j), \qquad H := \sum_{p_j > 0} \mu_j \alpha_j^2, \tag{16}$$

so that $0 \le H \le 4$. We also define the *weighted interface mismatch*

$$V := P_v - \sum_j \mu_j p_j, \tag{17}$$

whose name is justified by (19). Since $\sum_j \mu_j p_j \le M \sum_j \mu_j = M$, by (15) we have $V \ge 0$.

We start with the following decomposition lemma, which shows that the deficit of E is the average deficit of the prisms plus V.

Lemma 1. *Let E be as in (10). Then*

$$D_n(E) = \sum_{p_j>0} \mu_j D_n(\mathcal{P}_j) + V. \tag{18}$$

Moreover, with $s_j := \sum_{i\le j} \mu_i$,

$$V = \sum_{j=1}^{N-1} \Big[s_j \, |R_{j+1} \setminus R_j| + (1-s_j)\, |R_j \setminus R_{j+1}| \Big], \tag{19}$$

i.e., V is a weighted sum of the changes between consecutive sections. Mass that appears going up is weighted by the volume fraction s_j below the interface, mass that disappears by the fraction $1-s_j$ above it.

Proof. By (2) the perimeter of $\mathcal{P}_j$ is

$$P_n(\mathcal{P}_j) = \frac{P_m(R_j)}{p_j} + p_j.$$

Multiplying by $\mu_j = h_j p_j$, summing over j, and using (12) and (17), we get

$$\sum_{p_j>0} \mu_j P_n(\mathcal{P}_j) = \sum_j h_j P_m(R_j) + \sum_j \mu_j p_j = P_n(E) - V.$$

Since $\sum_j \mu_j = 1$, (18) follows. To prove (19), we note that

$$s_j \, |R_{j+1} \setminus R_j| + (1-s_j)\, |R_j \setminus R_{j+1}| = \frac{1}{2}|R_j \triangle R_{j+1}| + \Big(s_j - \frac{1}{2}\Big)(p_{j+1} - p_j).$$

Since $s_j - s_{j-1} = \mu_j$, $s_1 = \mu_1$ and $s_{N-1} = 1 - \mu_N$, summation by parts gives

$$\sum_{j=1}^{N-1} s_j \, (p_{j+1} - p_j) = p_N - \sum_{j=1}^{N} \mu_j p_j.$$

Summing over j and using (13) and (17), we obtain

$$\sum_{j=1}^{N-1} \Big[s_j \, |R_{j+1} \setminus R_j| + (1-s_j)\, |R_j \setminus R_{j+1}| \Big] = \frac{1}{2}\Big(p_1 + p_N + \sum_{j=1}^{N-1} |R_j \triangle R_{j+1}|\Big) - \sum_j \mu_j p_j = V,$$

and (19) follows. □

For every $m \ge 1$ and every $p > 0$, we set

$$g_m(p) := m\, p^{-1/m} + p - m - 1, \quad \beta(p) := 2(1 - \min\{p, p^{-1}\}), \quad c_0 := \log 2 - \tfrac{1}{2}. \tag{20}$$

$g_m(p)$ is the deficit of a unit-volume prism whose base is a cube of measure p, and $\beta(p)$ is the asymmetry of the same prism (see Lemma 5). We collect in the next lemma the elementary scalar inequalities used in the induction.

Lemma 2. *Let $m \ge 1$ and $p > 0$. Then $g_m(p) \ge p - 1 - \log p \ge 0$ and*

$$\beta^2(p) \le 8(p - 1 - \log p) \le 8\, g_m(p). \tag{21}$$

Moreover, if $p \notin [1/2, 2]$, then

$$g_m(p) \ge p - 1 - \log p \ge c_0 \ge \tfrac{1}{6} > 0. \tag{22}$$

Finally, if $m \ge 2$, then

$$m + 1 - m\, 2^{1/m} \ge 3 - 2\sqrt{2} > 0.17. \tag{23}$$

Proof. Set $\psi(p) := p - 1 - \log p$. Since $e^x \geq 1 + x$, the bound $p^{-1/m} \geq 1 - \frac{\log p}{m}$ yields $g_m(p) \geq \psi(p) \geq 0$.

We first prove (21). If $0 < p \leq 1$, then

$$\psi(p) = \int_p^1 \frac{1-t}{t}\,dt \geq \int_p^1 (1-t)\,dt = \frac{(1-p)^2}{2} = \frac{\beta^2(p)}{8}.$$

If $p \geq 1$, then

$$\psi(p) = \int_1^p \frac{t-1}{t}\,dt \geq \frac{1}{p}\int_1^p (t-1)\,dt = \frac{(p-1)^2}{2p} \geq \frac{(1-p^{-1})^2}{2} = \frac{\beta^2(p)}{8}.$$

The function ψ is decreasing on $(0,1)$ and increasing on $(1,\infty)$, with $\psi(1/2) = c_0$ and $\psi(2) = 1 - \log 2 > c_0$. Convexity of $t \mapsto 1/t$ gives

$$\frac{2}{3} \leq \int_1^2 \frac{dt}{t} = \log 2 < \frac{1}{2}\left(1 + \frac{1}{2}\right) = \frac{3}{4}.$$

Thus $c_0 \geq 1/6$, and (22) follows.

Finally, concavity of $x \mapsto x^{2/m}$ for $m \geq 2$ gives

$$2^{1/m} = (\sqrt{2})^{2/m} \leq 1 + \frac{2}{m}(\sqrt{2} - 1),$$

and (23) follows. □

The next lemma provides the Wulff inequality for finite unions of boxes, together with a lower bound on the deficit in terms of M.

Lemma 3. *For every $k \geq 1$ and every finite union of boxes $F \subset \mathbb{R}^k$ with $|F| > 0$,*

$$P_k(F) \geq k\,|F|^{(k-1)/k}. \tag{24}$$

Moreover, if E is as in (10) and $m = n - 1$, then

$$D_n(E) \geq g_m(M). \tag{25}$$

Proof. We argue by induction on k. If $k = 1$, then $P_1(F)$ is the number of essential interval components of F, and hence $P_1(F) \geq 1$. Assume now that (24) holds in dimension m, and let E be as in (10). By (12), (15) and the bound $p_j \leq M$, we have

$$P_n(E) \geq m \sum_{p_j > 0} h_j p_j^{1-1/m} + M \geq m\,M^{-1/m} \sum_j h_j p_j + M = m\,M^{-1/m} + M,$$

which gives (25). Since $g_m \geq 0$ by Lemma 2, we obtain $P_n(E) \geq n$, and (24) in dimension n follows by scaling. □

3. The gluing lemma

We now prove the gluing lemma, which is the core of our inductive argument. It shows that the asymmetry of the stack E is controlled by the average asymmetry of its prisms plus the variation V, with coefficient one in front of the average.

Lemma 4. *Let E be as in (10), and let M, H and V be as in (11), (16) and (17). Then*

$$\alpha_n(E) \leq \sqrt{H} + \frac{18\,V}{M}. \tag{26}$$

Consequently,

$$\alpha_n^2(E) \leq H + \frac{72\,V}{M}, \tag{27}$$

and if $M \geq 1/2$, then

$$\alpha_n^2(E) \leq H + 144\,V. \tag{28}$$

Proof. The main idea of the proof is to use a section of maximal measure R_* to construct a single unit-volume prism $P_* = R_* \times I$ close to E, and then to compare every prism $\mathcal{P}_j$ with P_*.

Let R_* be a section with $|R_*| = M$, and let us split V as

$$V = \sigma + \eta, \qquad \sigma := P_v - M \geq 0, \qquad \eta := M - \sum_j \mu_j p_j \geq 0. \tag{29}$$

By (14) we have $\int_{\mathbb{R}^m} (k-1)_+ \, dx = P_v - |\mathcal{U}| \leq \sigma$, and therefore

$$|\mathcal{U} \setminus R_*| \leq \sigma, \qquad |\{x \in \mathbb{R}^m : k(x) \geq 2\}| \leq \sigma. \tag{30}$$

Let $p : \mathbb{R} \to [0, M]$ be the function that equals p_j on (a_{j-1}, a_j) and vanishes outside (a_0, a_N). Then

$$\int_{\mathbb{R}} p(y)\, dy = 1, \qquad \eta = \int_{\mathbb{R}} p(y)(M - p(y))\, dy. \tag{31}$$

Step 1: construction of the anchor prism P_*. We show that there exists an interval I of length $1/M$ such that the unit-volume prism $P_* := R_* \times I$ satisfies

$$|E \triangle P_*| \leq \frac{16\, V}{M}. \tag{32}$$

If $\sigma \geq M/8$, then (32) is immediate because $|E \triangle P_*| \leq 2 \leq 16\sigma/M$ for every interval I of length $1/M$. Assume therefore that $\sigma < M/8$, and let J be the smallest interval containing $\{y : p(y) > 3M/4\}$, see Figure 1. Since this set contains the layer of R_*, the interval J has positive length. Let $a < b$ be two points of $\{p > 3M/4\}$ that are not layer boundaries, and denote by R_y the section of E at height y. Since $|R_a \cup R_b| \leq |\mathcal{U}| \leq M + \sigma$,

$$|R_a \cap R_b| \geq p(a) + p(b) - |\mathcal{U}| > \frac{M}{2} - \sigma.$$

By (30), outside a set of measure at most σ, every point $x \in R_a \cap R_b$ has a single occupied run of layers, and its fiber contains (a, b) up to layer boundaries. Therefore $p(y) \geq M/2 - 2\sigma$ for every $y \in (a, b)$ that is not a layer boundary. Letting a and b approach the endpoints of J, we obtain

$$p(y) \geq \frac{M}{2} - 2\sigma \geq \frac{M}{4} \qquad \text{for almost every } y \in J. \tag{33}$$

Since $p \leq 3M/4$ outside J, from (31) and (33) we get

$$|J| \leq \frac{4}{M}, \qquad \int_{\mathbb{R}\setminus J} p\, dy \leq \frac{4\eta}{M}, \qquad \int_J (M - p)\, dy \leq \frac{4\eta}{M}. \tag{34}$$

Moreover, by (30), for almost every $y \in \mathbb{R}$,

$$|R_y \triangle R_*| = M - p(y) + 2|R_y \setminus R_*| \leq M - p(y) + 2\sigma.$$

Integrating in y and using (34), we obtain

$$|E \triangle (R_* \times J)| = \int_J |R_y \triangle R_*|\, dy + \int_{\mathbb{R}\setminus J} p\, dy \leq \frac{4\eta}{M} + 2\sigma|J| + \frac{4\eta}{M} \leq \frac{8V}{M}.$$

Let I be the interval of length $1/M$ with the same center as J. Since I and J are nested,

$$|(R_* \times J) \triangle (R_* \times I)| = \big|M|J| - 1\big| = \big||R_* \times J| - |E|\big| \leq |E \triangle (R_* \times J)|,$$

and (32) follows by the triangle inequality.

Step 2: comparison of the prisms $\mathcal{P}_j$ with P_*. For every j with $p_j > 0$, we translate $\mathcal{P}_j$ vertically so that its interval $[0, 1/p_j]$ starts at the left endpoint of I. Since $1/p_j \geq 1/M = |I|$, the translated interval contains I. Moreover, by (30) we have $|R_j \cap R_*| \geq p_j - |\mathcal{U} \setminus R_*| \geq p_j - \sigma$, and therefore

$$|\mathcal{P}_j \triangle P_*| = 2 - 2\,\frac{|R_j \cap R_*|}{M} \leq \frac{2\,(M - p_j + \sigma)}{M}.$$

By (9) we obtain $\alpha_n(P_*) \le \alpha_j + 2(M - p_j + \sigma)/M$. Averaging with the weights μ_j and using the Cauchy–Schwarz inequality, we get

$$\alpha_n(P_*) \le \sum_{p_j>0} \mu_j \alpha_j + \frac{2\,(\eta+\sigma)}{M} \le \sqrt{H} + \frac{2V}{M}.$$

Combining this bound with (9) and (32), we obtain (26).

Step 3: proof of (27) and (28). Since both $\alpha_n(E)$ and $\sqrt{H}$ belong to $[0,2]$, by (26) we have

$$\alpha_n^2(E) - H = (\alpha_n(E) - \sqrt{H})(\alpha_n(E) + \sqrt{H}) \le 4(\alpha_n(E) - \sqrt{H})_+ \le \frac{72V}{M},$$

which gives (27). Inequality (28) follows when $M \ge 1/2$. □

The proof of Lemma 4 uses the sections only through the measures of their finite unions and intersections, the fiber identity (14), and the finiteness of the number of layers. It therefore applies to arbitrary measurable sections of finite measure, with P_v defined by (13).

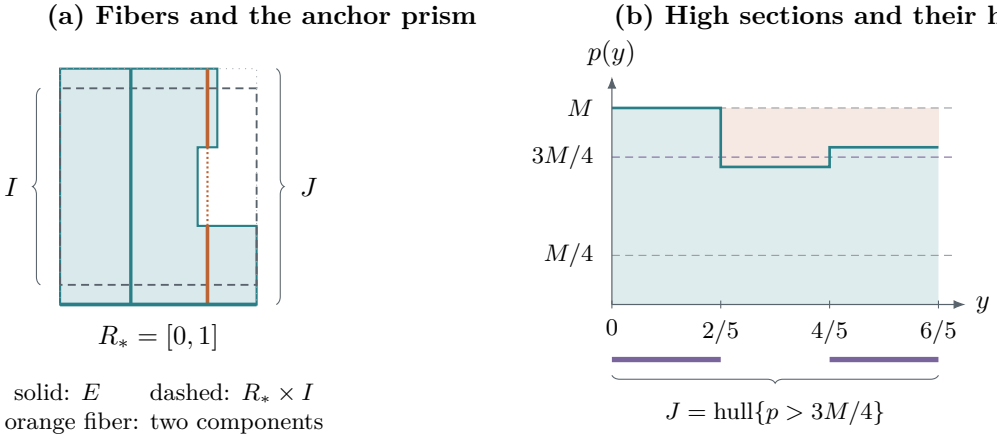

FIGURE 1. Construction of the anchor prism. In (a), the stack E is teal, $R_* \times J$ is dotted, and $P_* = R_* \times I$ is dashed; the orange fiber has two components. In (b), the function $p(y)$ is teal, the defect $M - p(y)$ is orange, and the high set $\{p > 3M/4\}$ is purple. Here the section lengths are $1, 7/10, 4/5$ and the layer heights $2/5$, so that $M = 1$, $\sigma = 1/10 < M/8$, $\eta = 37/250$, $J = (0, 6/5)$ and $I = (1/10, 11/10)$. Fibers with one essential component crossing the high sections force a lower bound on p throughout their interval hull J. The interval I is then chosen so that P_* has volume one. Intervals are used for visibility only: the gluing lemma permits arbitrary measurable sections.

4. Proof of Theorem 1

In this section we prove Theorem 1. Throughout the section we set $C := 72$. We begin with the prism estimate, which is the only step where the dimensional constant is updated.

Lemma 5. *Let $m \ge 2$ and assume that every finite union of boxes $F \subset \mathbb{R}^m$ with $|F| = 1$ satisfies*

$$\alpha_m^2(F) \le C\, m\, D_m(F). \tag{35}$$

Let $R \subset \mathbb{R}^m$ be a finite union of boxes with $|R| = p > 0$ and let $\mathcal{P} := R \times [0, 1/p]$. Then

$$\alpha_{m+1}^2(\mathcal{P}) \le C\,(m+1)\, D_{m+1}(\mathcal{P}). \tag{36}$$

Moreover, (36) holds for $m = 1$ without assuming (35).

Proof. Let $F := p^{-1/m}R$, so that $|F| = 1$, and set $a := \alpha_m(F)$. As in the proof of Lemma 1, we have $P_{m+1}(\mathcal{P}) = P_m(R)/p + p$. Since $P_m(R) = p^{(m-1)/m}P_m(F)$, we obtain

$$D_{m+1}(\mathcal{P}) = \frac{P_m(R)}{p} + p - m - 1 = p^{-1/m}D_m(F) + g_m(p). \tag{37}$$

We next estimate $\alpha_{m+1}(\mathcal{P})$. For every $x \in \mathbb{R}^m$, the box $B_x := (x + p^{1/m}Q_m) \times [0, 1/p]$ has unit volume and satisfies

$$|\mathcal{P} \triangle B_x| = p^{-1}|R \triangle (x + p^{1/m}Q_m)| = |F \triangle (p^{-1/m}x + Q_m)|,$$

so that $\inf_x |\mathcal{P} \triangle B_x| = a$. On the other hand, B_x has m sides of length $p^{1/m}$ and one side of length $1/p$. Thus its largest intersection with a translate of Q_{m+1} has measure $\min\{p, p^{-1}\}$, and $\alpha_{m+1}(B_x) = \beta(p)$. By the triangle inequality, see Figure 2, we obtain

$$\alpha_{m+1}(\mathcal{P}) \le a + \beta(p). \tag{38}$$

Assume first $m \ge 2$ and $p \in [1/2, 2]$. By the Cauchy–Schwarz inequality, (35), (21) and (37), we have

$$\begin{aligned} \alpha_{m+1}^2(\mathcal{P}) &\le (Cm\,p^{1/m} + 8)\Big(\frac{a^2}{Cm\,p^{1/m}} + \frac{\beta^2(p)}{8}\Big) \\ &\le (Cm\,2^{1/m} + 8)\,(p^{-1/m}D_m(F) + g_m(p)) \le C(m+1)\,D_{m+1}(\mathcal{P}), \end{aligned} \tag{39}$$

where the last inequality follows from (23) and $C(3 - 2\sqrt{2}) > 8$.

If $p \notin [1/2, 2]$, then (37), Lemma 3 and (22) give $D_{m+1}(\mathcal{P}) \ge c_0$. Since $\alpha_{m+1}^2(\mathcal{P}) \le 4 \le C(m+1)c_0$, we obtain (36).

Finally, let $m = 1$. If $F \subset \mathbb{R}$ is a finite union of intervals with $|F| = 1$, then $P_1(F)$ is the number of its essential interval components. If F agrees with an interval up to a null set, then $\alpha_1(F) = 0$. Otherwise $D_1(F) \ge 1$. Thus in both cases

$$\alpha_1^2(F) \le 4\,D_1(F). \tag{40}$$

If $p \in [1/2, 2]$, we repeat (39), using (40) in place of (35), that is, with 4 in place of Cm, obtaining the factor $4p + 8 \le 16 \le 2C$. The case $p \notin [1/2, 2]$ is treated as before. □

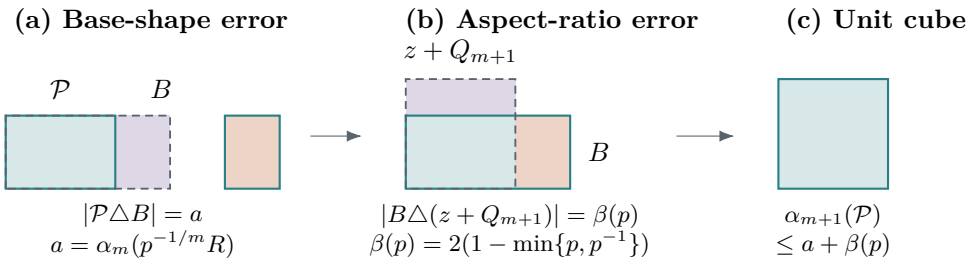

FIGURE 2. The two errors in the prism comparison. Here $R = [0, 1] \cup [2, 5/2]$ and $\mathcal{P} = R \times [0, 2/3]$. Panel (a) compares $\mathcal{P}$ with $B = [0, 3/2] \times [0, 2/3]$, isolating the error in the shape of the base. Panel (b) compares B with the reference square $z + Q_2 = [0, 1]^2$, shown in (c), isolating the error in the aspect ratio. Orange and purple mark the two parts of each symmetric difference. Together the comparisons yield the triangle inequality (38).

By combining Lemma 1, Lemma 4 and Lemma 5, we now prove Theorem 1 for finite unions of boxes.

Proposition 1. *For every $n \ge 2$ and every finite union of boxes $E \subset \mathbb{R}^n$ with $|E| = 1$,*

$$\alpha_n^2(E) \le C\,n\,D_n(E). \tag{41}$$

Proof. We argue by induction on n. Let E be as in (10), with $m = n-1$. Note that when $n = 2$ the sections R_j are finite unions of intervals, so Lemma 5 applies without any inductive assumption.

If $M \notin [1/2, 2]$, then by Lemma 3 and Lemma 2 we have $D_n(E) \geq g_m(M) \geq c_0$. Since $\alpha_n^2(E) \leq 4 \leq Cnc_0$, (41) holds.

Assume now that $M \in [1/2, 2]$. Applying Lemma 5 to every section with $p_j > 0$, and using the inductive hypothesis when $m \geq 2$, we get

$$H = \sum_{p_j>0} \mu_j \alpha_j^2 \leq C\, n \sum_{p_j>0} \mu_j D_n(\mathcal{P}_j).$$

Since $M \geq 1/2$ and $144 \leq Cn$, by Lemma 4 and Lemma 1 we obtain

$$\alpha_n^2(E) \leq H + 144\, V \leq C\, n \sum_{p_j>0} \mu_j D_n(\mathcal{P}_j) + C\, n\, V = C\, n\, D_n(E),$$

and (41) follows. □

To pass from finite unions of boxes to general sets of finite perimeter, we use the following approximation lemma.

Lemma 6. *Let $E \subset \mathbb{R}^n$ be a set of finite perimeter with $|E| < \infty$. Then there exists a sequence of finite unions of cubes $E_k \subset \mathbb{R}^n$ such that*

$$\lim_{k\to\infty} |E_k \triangle E| = 0, \qquad \lim_{k\to\infty} P_n(E_k) = P_n(E). \tag{42}$$

Moreover, if $|E| = 1$, then the sets E_k can be chosen with $|E_k| = 1$.

Proof. For every $h > 0$, let f_h be the function that equals the average of χ_E on each half-open cube of the grid $h([0,1)^n + \mathbb{Z}^n)$. Then $0 \leq f_h \leq 1$ and $f_h \to \chi_E$ in $L^1(\mathbb{R}^n)$ as $h \to 0$. Since translation by he_i maps the grid onto itself, it commutes with the averaging operator, which is a contraction in $L^1(\mathbb{R}^n)$. Consequently,

$$\|f_h(\cdot + he_i) - f_h\|_{L^1} \leq \|\chi_E(\cdot + he_i) - \chi_E\|_{L^1} \leq h\, |D_i \chi_E|(\mathbb{R}^n). \tag{43}$$

For every $\theta \in (0,1)$, set $E_{h,\theta} := \{f_h > \theta\}$. Since every grid cube contained in $E_{h,\theta}$ contains more than θh^n of the mass of E, the set $E_{h,\theta}$ is a finite union of cubes. Moreover, for every finite union G of grid cubes,

$$P_n(G) = \frac{1}{2h} \sum_{i=1}^{n} \|\chi_G(\cdot + he_i) - \chi_G\|_{L^1},$$

because $\|\chi_G(\cdot + he_i) - \chi_G\|_{L^1}$ equals h times the area of the faces of G orthogonal to e_i. Integrating in θ and using the layer-cake formula together with (43), we obtain

$$\int_0^1 |E_{h,\theta} \triangle E|\, d\theta = \|f_h - \chi_E\|_{L^1} =: \tau_h \to 0, \tag{44}$$

$$\int_0^1 P_n(E_{h,\theta})\, d\theta = \frac{1}{2h} \sum_{i=1}^{n} \|f_h(\cdot + he_i) - f_h\|_{L^1} \leq P_n(E). \tag{45}$$

Choose $\lambda_h \to \infty$ with $\lambda_h \tau_h \to 0$, and select $\theta_h \in (0,1)$ such that the value of $P_n(E_{h,\theta}) + \lambda_h |E_{h,\theta} \triangle E|$ at $\theta = \theta_h$ does not exceed its average over $(0,1)$. By (44) and (45) we get

$$|E_{h,\theta_h} \triangle E| \leq \frac{P_n(E) + \lambda_h \tau_h}{\lambda_h} \to 0, \qquad \limsup_{h\to 0} P_n(E_{h,\theta_h}) \leq P_n(E).$$

By the lower semicontinuity of $|D_i \cdot|(\mathbb{R}^n)$ with respect to L^1 convergence, we obtain (42) along any sequence $h_k \to 0$.

If $|E| = 1$, the approximants have positive measure for small h. Dilating E_{h,θ_h} by the factor $|E_{h,\theta_h}|^{-1/n}$ gives volume one and preserves the perimeter convergence, since these factors tend

to one. Continuity of dilations in L^1, together with the triangle inequality, preserves the L^1 convergence as well. □

Proof of Theorem 1. Let $E \subset \mathbb{R}^n$ be a set of finite perimeter with $|E| = 1$, and let E_k be the sequence of finite unions of cubes given by Lemma 6, with $|E_k| = 1$. Proposition 1 gives $\alpha_n^2(E_k) \le 72\, n\, D_n(E_k)$ for every k. By (9) and (42), $\alpha_n(E_k) \to \alpha_n(E)$ and $D_n(E_k) \to D_n(E)$, proving (4).

To prove the upper bound on the constant, let n be even, let $\varepsilon > 0$, and let R_ε be the centered box with $n/2$ sides of length e^{ε} and $n/2$ sides of length $e^{-\varepsilon}$. Then $|R_\varepsilon| = 1$. Since the perimeter of a unit-volume box with sides $\ell_1, \dots, \ell_n$ is $\sum_i 1/\ell_i$, we have

$$P_n(R_\varepsilon) - n = \frac{n}{2}(e^{\varepsilon} + e^{-\varepsilon}) - n = n(\cosh\varepsilon - 1).$$

On the other hand, the largest intersection of R_ε with a translate of Q_n has measure $e^{-n\varepsilon/2}$. Therefore

$$\alpha_n(R_\varepsilon) = 2\big(1 - e^{-n\varepsilon/2}\big), \qquad \lim_{\varepsilon \downarrow 0} \frac{\alpha_n^2(R_\varepsilon)}{P_n(R_\varepsilon) - n} = \lim_{\varepsilon \downarrow 0} \frac{4\big(1 - e^{-n\varepsilon/2}\big)^2}{n(\cosh\varepsilon - 1)} = 2n,$$

so the constant in (4) cannot exceed $1/(2n)$.

Similarly, for odd $n = 2k + 1$, let R_ε have k sides of length $e^{(k+1)\varepsilon}$ and $k + 1$ sides of length $e^{-k\varepsilon}$. Again $|R_\varepsilon| = 1$, and the perimeter and overlap formulas give

$$\begin{aligned} P_n(R_\varepsilon) - n &= k e^{-(k+1)\varepsilon} + (k+1)e^{k\varepsilon} - (2k+1) \\ &= \frac{k(k+1)(2k+1)}{2}\,\varepsilon^2 + O(\varepsilon^3), \\ \alpha_n(R_\varepsilon) &= 2\big(1 - e^{-k(k+1)\varepsilon}\big). \end{aligned}$$

Consequently,

$$\lim_{\varepsilon \downarrow 0} \frac{\alpha_n^2(R_\varepsilon)}{P_n(R_\varepsilon) - n} = \frac{8k(k+1)}{2k+1} = \frac{2(n^2-1)}{n},$$

and the constant in (4) is at most $n/(8\lfloor n^2/4 \rfloor)$. □

Remark 1. The constant C enters the proof of Proposition 1 only through the four inequalities

$$C(3 - 2\sqrt{2}) \ge 8, \qquad 16 \le 2C, \qquad 144 \le Cn, \qquad 4 \le Cnc_0.$$

The first two arise in Lemma 5, the third in the gluing step of Proposition 1, and the fourth when M or p lies outside $[1/2, 2]$. Since $3 - 2\sqrt{2} > 0.17$, $c_0 \ge 1/6$ and $n \ge 2$, all four inequalities hold for $C = 72$, where the active constraint is $144 \le Cn$ at $n = 2$.

5. Euclidean concentration

We express the union estimate from the proof of [2, Lemma 4] in terms of the volume radius.

Lemma 7. *Let $E \subset \mathbb{R}^n$ be measurable with $|E| = 1$. For $\rho > 0$, set $F := E + \rho Q_n$ and suppose that $a := |F|^{1/n} < \infty$. Then*

$$a\Big(1 + \frac{\alpha_n(a^{-1}F)}{2}\Big)^{1/n} \ge \rho + \Big(1 + \frac{\alpha_n(E)}{2}\Big)^{1/n}. \tag{46}$$

Proof. By Brunn–Minkowski, $a \ge 1 + \rho$. Hence, for every $x \in \mathbb{R}^n$,

$$(E \cup (x + Q_n)) + \rho Q_n = F \cup (x + (1+\rho)Q_n) \subset F \cup (x + aQ_n).$$

Brunn–Minkowski gives $|F \cup (x + aQ_n)|^{1/n} \ge |E \cup (x + Q_n)|^{1/n} + \rho$. For sets G, B of equal finite measure, $|G \cup B| = |G| + \frac{1}{2}|G \triangle B|$. Taking the infimum over x and scaling F by a^{-1} yields (46). □

Proof of Theorem 2. Write $\alpha := \alpha_n(E)$ and $\kappa := 1/(72\,n^2)$. Fix $r > 0$, put $E_0 := E$ and $E_\rho := E + \rho Q_n$ for $0 < \rho \le r$, and assume $|E_r| < \infty$ since the case of infinite volume is immediate. Define

$$a_\rho := |E_\rho|^{1/n}, \qquad h_\rho := a_\rho - 1 - \rho, \qquad \alpha_\rho := \alpha_n(a_\rho^{-1} E_\rho).$$

For $t = r/(1+r)$ the left-hand side of (6) is $(1-t)h_r$. Thus it suffices to show that

$$h_r \ge \frac{\kappa \alpha^2}{16}\,\frac{r}{1+r}. \tag{47}$$

For $0 \le u < v \le r$, the identity $E_v = E_u + (v-u)Q_n$ and Brunn–Minkowski give $a_v \ge a_u + v - u$. Thus h_ρ is nondecreasing and $h_0 = 0$.

Fix a differentiability point $\rho \in (0,r)$ of a. For every $X \in C^1_c(\mathbb{R}^n;\mathbb{R}^n)$ with values in Q_n, the map $\Phi_s(x) := x + sX(x)$ is injective for all sufficiently small $s > 0$ and sends E_ρ into $E_{\rho+s}$. By change of variables,

$$a_{\rho+s}^n \ge |\Phi_s(E_\rho)| = a_\rho^n + s\int_{E_\rho} \operatorname{div} X\,dx + O_X(s^2).$$

Passing to the right derivative, we obtain $n a_\rho^{n-1} a'_\rho \ge \int_{E_\rho} \operatorname{div} X\,dx$. The constraint $X(x) \in Q_n$ reads $|X_i| < 1/2$ for every i, so the components of X can be chosen independently, and the supremum over X of the right-hand side is

$$\frac{1}{2}\sum_{i=1}^n \sup\Big\{ \int_{E_\rho} \partial_i \varphi\,dx :\ \varphi \in C^1_c(\mathbb{R}^n),\ |\varphi| \le 1\Big\}.$$

Since $a'_\rho < \infty$, each of these suprema is finite. Hence E_ρ has finite perimeter, the i-th supremum equals $|D_i \chi_{E_\rho}|(\mathbb{R}^n)$, and $n a_\rho^{n-1} a'_\rho \ge P_n(E_\rho)$ by (2). Since almost every $\rho \in (0,r)$ is a differentiability point of a, at these radii Theorem 1 in the form (5) gives

$$h'_\rho \ge \frac{P_n(E_\rho)}{n a_\rho^{n-1}} - 1 \ge \kappa \alpha_\rho^2. \tag{48}$$

Set $\delta := (1+\alpha/2)^{1/n} - 1$. Since $1 - q^n \le n(1-q)$ for $0 \le q \le 1$ and $\alpha \le 2$, we have

$$\frac{n\delta}{1+\delta} \ge 1 - (1+\delta)^{-n} = \frac{\alpha}{2+\alpha} \ge \frac{\alpha}{4}. \tag{49}$$

If $h_r \ge \delta/2$, then $h_r \ge \alpha/(8n) \ge \kappa\alpha^2/16$, which proves (47). Otherwise $h_\rho \le h_r < \delta/2$ and $a_\rho \le (1+\rho)(1+\delta)$ for $0 < \rho \le r$. By Lemma 7 and $(1+s)^{1/n} \le 1 + s/n$ for $s \ge 0$,

$$1 + \rho + \delta \le a_\rho\Big(1 + \frac{\alpha_\rho}{2n}\Big), \qquad \alpha_\rho \ge \frac{2n(\delta - h_\rho)}{a_\rho} \ge \frac{\alpha}{4(1+\rho)},$$

where the last inequality follows from (49). Since h is nondecreasing, its increment dominates the integral of its derivative even if $h_{0+} > 0$. Thus (48) gives

$$h_r \ge \int_0^r h'_\rho\,d\rho \ge \frac{\kappa\alpha^2}{16}\int_0^r \frac{d\rho}{(1+\rho)^2} = \frac{\kappa\alpha^2}{16}\,\frac{r}{1+r}. \qquad \square$$

Remark 2. The argument transfers any admissible coefficient κ of $A_{Q_n}(E)^2$ in (5) to the coefficient $\kappa/16$ of $t(1-t)\alpha_n^2(E)$ in (6). The case $h_r \ge \delta/2$ uses $\kappa \le 1/n$, which holds since Theorem 1 gives $\kappa \le 1/(8\lfloor n^2/4\rfloor)$. Conversely, a bound of the form (7) at small radii, namely $|E + rQ_n| - (1+r)^n \ge c_n r\,\alpha_n^2(E)$ for all sufficiently small $r > 0$, implies $P_n(E) - n \ge c_n \alpha_n^2(E)$ for bounded open sets E of unit volume with Lipschitz boundary. Indeed, the anisotropic Minkowski-content identity gives

$$P_n(E) = \lim_{r \downarrow 0} \frac{|E + rQ_n| - |E|}{r}.$$

Bounded Lipschitz domains satisfy the classical outer Minkowski-content identity, and [3, Theorem 3.4 and Remark 3.6] transfers it to the closed cube $\overline{Q_n}$. The limit is the same for the open cube, since $E + rQ_n \subset E + r\overline{Q_n} \subset E + r'Q_n$ whenever $0 < r < r'$. For general sets of finite perimeter, this passage requires regular approximants because null modifications can change $|E + rQ_n|$.

Remark 3. In this remark, following [4, 12], the letters β and σ denote a Brunn–Minkowski deficit and a volume ratio; they are unrelated to the function β in (20) and to the quantity σ in (29). Moreover, contrary to (1), here K denotes the arbitrary summand and L the convex one. Let $0 < t < 1$, $K := (1-t)E$ and $L := tQ_n$, so that $|K|^{1/n} + |L|^{1/n} = 1$. Write

$$\beta(K, L) := |K + L|^{1/n} - 1, \qquad \sigma(K, L) := \frac{|K|}{|L|}.$$

Then the relative asymmetry $A(K, L)$, which is $A_L(K)$ in the notation of (1), equals $\alpha_n(E)$, and $s := \sigma(K, L)^{1/n} = (1-t)/t$. Since $t(1-t) = s/(1+s)^2 \geq \frac{1}{4}\min\{s, s^{-1}\}$, (6) gives

$$\beta(K, L) \geq \frac{A(K, L)^2}{1152\, n^2} \frac{s}{(1+s)^2} \geq \frac{A(K, L)^2}{4608\, n^2 \max\{\sigma(K, L), \sigma(K, L)^{-1}\}^{1/n}}.$$

By homogeneity, this gives the form of [12, Theorem 1.3] with the conjectured order n^2 for any cube and any measurable summand of finite positive measure.

Origin of the results and use of artificial intelligence

This project began several years ago, after Vesa Julin and I completed our work on quantitative concentration [2]. Motivated by the dimension-free Gaussian estimate in [1], I wanted to understand the optimal dimensional dependence of quantitative isoperimetric estimates in Euclidean space. My idea was to make the Hadwiger–Ohmann proof of the Brunn–Minkowski inequality quantitative. The product structure of the cube made its anisotropic perimeter a natural choice. I returned to the problem several times over the years, but the gluing estimates I obtained introduced a multiplicative loss at each step of the induction on the dimension.

When I took up the problem again in September 2026, ChatGPT 6 Astra provided the decisive argument in Lemma 4 that yields coefficient one in front of the weighted average of the squared asymmetries of the prisms. This allowed me to close the induction with the required dimensional dependence and complete the proof. I also used ChatGPT 6 Astra to track the constants in the remaining estimates and Claude Fable 5.2 to improve the exposition.

Acknowledgments

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