THE SCALAR CASE OF A DE GIORGI CONJECTURE FOR CARTESIAN AREA

This project, by Giovanni Bellettini, Shokhrukh Kholmatov and Carlo Mantegazza, aims at revising, making clearly readable and possibly simplifying results that were obtained by ChatGPT 5.6 Sol (OpenAI) over a series of chats, without significant human assistance. Not meant for publication. For more details, see the AI statement at the end of the introduction.

ABSTRACT. This note contains a proof of the scalar case of a conjecture of De Giorgi on the Cartesian area functional perturbed by the L^1 distance from a bounded datum. Namely, we show that for every $n \geqslant 1$ there exists a constant $\sigma_n > 0$ such that, for every open ball $B_R \subseteq \mathbb{R}^n$ and every $g \in L^\infty(B_R)$ with $\|g\|_{L^\infty(B_R)} \leqslant \sigma_n$, the functional

$$u \longmapsto \int_{B_R} \sqrt{1+|\nabla u|^2}\, d\mathscr{L}^n + \int_{B_R} |u-g|\, d\mathscr{L}^n$$

attains its minimum over $C^1(B_R)$ at a function of class $C^{1,1/2}_{\mathrm{loc}}(B_R)$. No boundary condition is imposed, and the threshold σ_n depends only on n, not on the radius R.

1. INTRODUCTION

The area of graphs is one of the most classical functionals of the calculus of variations, and one of those in which the direct methods and geometric measure theory interact most closely. In a short contribution on the relaxation of functionals defined on Cartesian manifolds [9], De Giorgi formulated a regularity problem for the Cartesian area of a map u defined on a ball, that is, the area of its graph, perturbed by the L^1 distance of u from a given bounded map g. No boundary condition is imposed, and for large data the relaxed functional may have minimizers which are not of class C^1, and even discontinuous ones, as an explicit one–dimensional example in [4] shows. The question is whether a sufficiently small L^∞ norm of the datum guarantees the existence of a minimizer of class C^1, with a smallness threshold which does not depend on the radius of the ball.

In the formulation recalled by Bellettini and Kholmatov [4], the conjecture reads as follows. Given integers $n, k \geqslant 1$, there should exist a constant $\sigma(n,k) > 0$ such that, for every open ball $B \subseteq \mathbb{R}^n$ and every map $g \in L^\infty(B, \mathbb{R}^k)$ satisfying $\|g\|_\infty \leqslant \sigma(n,k)$, the minimum

$$\min_{u \in C^1(B,\mathbb{R}^k)} \left\{ \int_B \sqrt{1+\sum_i |M_i(\nabla u)|^2}\, d\mathscr{L}^n + \int_B |u-g|\, d\mathscr{L}^n \right\} \tag{1.1}$$

is attained. Here $M_i(\nabla u)$ runs through all minors of positive order of the Jacobian matrix ∇u, so that the first integral is the area of the graph of u. The essential point is that $\sigma(n,k)$ depends only on n and k, and not on the ball B.

In this paper we prove the conjecture in the scalar case $k = 1$, for every $n \geqslant 1$. Since the problem is invariant under translations, it is not restrictive to consider the open ball $B_R = B_R(0) \subseteq \mathbb{R}^n$ of radius R centered at the origin. For $k = 1$ all minors of order at least two vanish, and the functional in equation (1.1) reduces to

$$\mathcal{F}_g(u) = \int_{B_R} \sqrt{1+|\nabla u|^2}\, d\mathscr{L}^n + \int_{B_R} |u-g|\, d\mathscr{L}^n\,. \tag{1.2}$$

Date: November 23, 2026.

2020 *Mathematics Subject Classification.* Primary 49Q20; Secondary 49Q05, 49J45, 28A75.

Key words and phrases. Cartesian area, De Giorgi conjecture, functions of bounded variation, perimeter minimizers, ε–regularity.

The case $n = k = 1$ was treated in [4], where it is shown that every minimizer of the relaxed functional on a bounded interval is of class $C^{1,1}$, provided $\|g\|_\infty$ is smaller than a threshold which depends on the length of the interval. See also [6], and [7] where the threshold is shown to be independent of the interval. Our result gives a threshold independent of the size of the domain, as required by the conjecture, in every dimension $n \geqslant 1$.

Theorem 1.1 (Scalar De Giorgi conjecture). *For every $n \geqslant 1$ there exists $\sigma_n > 0$ with the following property. If $R > 0$ and $g \in L^\infty(B_R)$ satisfy*

$$\|g\|_{L^\infty(B_R)} \leqslant \sigma_n \,,$$

then the minimum of the functional in equation (1.2) over $C^1(B_R)$ is attained. More precisely, there exists a minimizer

$$u \in C^{1,1/2}_{\mathrm{loc}}(B_R) \,.$$

The constant σ_n depends only on n and is independent of R.

Under the assumptions of Theorem 1.1, the proof shows that every minimizer of the relaxed functional in $BV(B_R)$, introduced in Section 2, coincides almost everywhere with a function of class $C^{1,1/2}_{\mathrm{loc}}(B_R)$ which minimizes the functional in equation (1.2) over $C^1(B_R)$.

Remark 1.2 (The smallness assumption is necessary). The assumption on the size of $\|g\|_{L^\infty(B_R)}$ in Theorem 1.1 cannot be dropped. In dimension one, Bellettini and Kholmatov [4] gave an explicit example: on the interval $(-1, 1)$, with $g = -2$ on $(-1, 0)$ and $g = 2$ on $(0, 1)$, the relaxed functional has minimizers which are not of class C^1 at the origin, and some of them are discontinuous there. The following radial example, based on the same mechanism, shows that in every dimension the functional in equation (1.2) may have no minimizer of class C^1, and that Theorem 1.1 cannot hold with $\sigma_n > 1$. Let $M > 2$, let $R \geqslant \max\{6n, 3(n-1)(M-1)/(M-2)\}$ and let $g = M\chi_{B_R \setminus B_{R/2}}$. We show that the functional in equation (1.2) has no minimizer in $C^1(B_R)$. The reason is that the second integral in equation (1.2) bounds by one the mean curvature, that is, the sum of the principal curvatures, of the graph of a minimizer of class C^1, and such a graph, if it has horizontal tangent planes along two concentric spheres of large radii, cannot rise by more than about 2 between them, while a minimizer has to rise from 0 to M near $\partial B_{R/2}$. Since the functional does not change if u and g are replaced by $u+c$ and $g+c$, for every $c \in \mathbb{R}$, it has no minimizer in $C^1(B_R)$ also for the datum $g - M/2$, whose norm in $L^\infty(B_R)$ is $M/2$, and $M/2$ can be any number larger than 1.

We suppose, by contradiction, that $u \in C^1(B_R)$ is a minimizer. Since $C^\infty(B_R) \cap W^{1,1}(B_R)$ is dense in $W^{1,1}(B_R)$ and the functional is continuous in $W^{1,1}(B_R)$, the function u, which belongs to $W^{1,1}(B_R)$, minimizes the functional also in $W^{1,1}(B_R)$, and comparing it with $\min\{\max\{u, 0\}, M\}$, as in Section 2.1, we get $0 \leqslant u \leqslant M$. Since the functional is convex and g is invariant under the orthogonal transformations Q of $\mathbb{R}^n$, the average of the functions $u \circ Q$ with respect to the Haar probability measure of the orthogonal group is a radial minimizer of class C^1, which we still denote by u. Thus $u(x) = v(|x|)$, with $v \in C^1([0, R))$, $v'(0) = 0$ and $0 \leqslant v \leqslant M$, and we set $s = v'/\sqrt{1 + v'^2}$, which is continuous with values in $(-1, 1)$. The first variation of the functional with respect to the radial perturbations $\psi(|x|)$, with $\psi \in C^1_c((0, R))$ supported in an open set where v differs from the radial profile of g, gives

$$(r^{n-1}s)' = r^{n-1} \quad \text{on } \big\{r < R/2 \,|\, v(r) > 0\big\}$$

and

$$(r^{n-1}s)' = -r^{n-1} \quad \text{on } \big\{R/2 < r < R \,|\, v(r) < M\big\} \,.$$

If $v > 0$ on $[0, R/2)$, the first identity, together with $s(0) = 0$, gives $s(r) = r/n$ for $r < R/2$, so that $s(r) \to R/(2n) \geqslant 3$ as $r \to R/2$, which is impossible. If $v < M$ on $[R/2, R)$, the second identity gives $r^{n-1}s(r) \leqslant (R/2)^{n-1} - (r^n - (R/2)^n)/n$ for $R/2 < r < R$, hence $\limsup_{r\to R} s(r) \leqslant 2^{1-n} - 6(1 - 2^{-n}) < -1$, which is impossible. Therefore the numbers

$$\alpha = \max\big\{r \in [0, R/2] \,|\, v(r) = 0\big\} \qquad \text{and} \qquad \beta = \min\big\{r \in [R/2, R) \,|\, v(r) = M\big\}$$

are well defined, and $s(\alpha) = s(\beta) = 0$, because v attains its minimum at α and its maximum at β, and $v'(0) = 0$. If $\alpha = R/2$, the second identity gives $s < 0$ on $(R/2, \beta)$, hence $v < 0$ on this interval, which is impossible, and similarly $\beta \neq R/2$. Then $\alpha < R/2 < \beta$, and the two identities give

$$s(r) = \frac{r^n - \alpha^n}{nr^{n-1}} \quad \text{for } \alpha \leqslant r \leqslant \frac{R}{2}\,, \qquad s(r) = \frac{\beta^n - r^n}{nr^{n-1}} \quad \text{for } \frac{R}{2} \leqslant r \leqslant \beta\,.$$

In particular, $s \geqslant 0$ on $[\alpha, \beta]$, and the inequality $b = s(R/2) < 1$ gives $\alpha^n > (R/2)^n(1 - 2n/R)$, hence $\alpha > R/2 - n \geqslant R/3$. Therefore $s' = 1 - (n-1)s/r \geqslant 1 - 3(n-1)/R = \gamma > 0$ on $(\alpha, R/2)$ and $s' = -1 - (n-1)s/r \leqslant -1$ on $(R/2, \beta)$, and, taking s as a new variable on these two intervals, we get

$$M = v(\beta) - v(\alpha) = \int_\alpha^\beta \frac{s}{\sqrt{1-s^2}}\, dr \leqslant \Big(\frac{1}{\gamma} + 1\Big) \int_0^b \frac{t}{\sqrt{1-t^2}}\, dt < \frac{1}{\gamma} + 1 \leqslant M\,,$$

where the last inequality follows from the choice of R. This contradiction shows that no minimizer of class C^1 exists.

The bound $\sigma_n \leqslant 1$ given by this example is the same in every dimension, and we do not know whether the constant σ_n in Theorem 1.1 can be chosen independently of n. In our proof, σ_n depends on n through the constants of the ε–regularity results recalled in Section 3 and of the estimates of the excess of Section 2.3, and it is not explicit, because the constants of these ε–regularity results are obtained by compactness arguments.

Remark 1.3 (Regularity up to the boundary). The proof also gives the regularity up to the boundary of the ball. Under the assumptions of Theorem 1.1, the function of class $C^{1,1/2}_{\mathrm{loc}}(B_R)$ which coincides almost everywhere with a minimizer of the relaxed functional extends to a function of class $C^{1,1/2}(\overline{B_R})$, whose normal derivative vanishes on ∂B_R; this is the natural boundary condition of the problem. Indeed, near each point of ∂B_R this function is obtained in Section 4 from the function given by the ε–regularity lemma of De Philippis and Maggi at the boundary of a half–space, which is of class $C^{1,1/2}$ up to the boundary of the half–space and satisfies the boundary condition in [10, Lemma 3.4, formula (3.7)]. For the integrand obtained by means of the inversion of Section 3, this condition states that the derivative in the direction normal to the boundary of the half–space vanishes, and the inversion, which preserves angles and maps ∂B_R onto the boundary of the half–space, transforms it into the vanishing of the normal derivative on ∂B_R.

Notation and conventions. Throughout the paper, $B_r(x)$ denotes the open Euclidean ball of radius r centered at x, and we write $B_r = B_r(0)$. The vectors $e_1, \dots, e_{n+1}$ are the vectors of the standard basis of $\mathbb{R}^{n+1}$, and we call e_{n+1} the vertical direction. For every integer $m \geqslant 0$, we write

$$\mathbb{S}^m = \{x \in \mathbb{R}^{m+1} \,|\, |x| = 1\}$$

for the unit sphere in $\mathbb{R}^{m+1}$. We denote by $\mathscr{L}^n$ the n–dimensional Lebesgue measure and by $\mathcal{H}^n$ the n–dimensional Hausdorff measure. The notation $|A|$ denotes the Lebesgue measure of a measurable set A in its ambient Euclidean space, and $\omega_n = |B_1|$. We also write

$$\operatorname{dist}(x, A) = \inf_{y \in A} |x - y|$$

for the Euclidean distance from a point to a set and $\operatorname{diam}(A) = \sup\{|x - y| \,|\, x, y \in A\}$ for the diameter of A.

For an open set $U \subseteq \mathbb{R}^m$, $BV(U)$ denotes the space of functions of bounded variation on U, namely the functions $u \in L^1(U)$ whose distributional derivative is a finite $\mathbb{R}^m$–valued Radon measure, and $BV_{\mathrm{loc}}(U)$ denotes the corresponding local space. If $u \in BV(B_R)$, Du denotes its distributional derivative and

$$Du = \nabla u\, \mathscr{L}^n + D^s u$$

is its Lebesgue decomposition, where ∇u is the density of the absolutely continuous part of Du with respect to $\mathscr{L}^n$ and $D^s u$ is singular with respect to $\mathscr{L}^n$. If μ is a vector–valued Radon measure, $|\mu|$ denotes its total variation measure.

For a measurable set E, χ_E denotes its characteristic function, equal to 1 on E and to 0 outside E. For two sets A and B, we denote by $A \triangle B = (A \setminus B) \cup (B \setminus A)$ their symmetric difference,

while $A \Subset B$ means that the closure of A is a compact subset of the open set B. If $U \subseteq \mathbb{R}^{n+1}$ is open and $E \subseteq \mathbb{R}^{n+1}$ is a set of locally finite perimeter in U, that is, $\chi_E \in BV_{\mathrm{loc}}(U)$, we write

$$P(E,U) = |D\chi_E|(U)$$

for its perimeter in U. We denote by ∂E the topological boundary of E, by $\partial^* E$ its reduced boundary, that is, the set of points at which the measure–theoretic unit normal exists, and by $\nu_E : \partial^* E \to \mathbb{S}^n$ the measure–theoretic outer unit normal, with the convention

$$D\chi_E = -\nu_E \, \mathcal{H}^n \llcorner \partial^* E \, .$$

The same notation is used for every set of locally finite perimeter; for instance, $\partial^* F$ and ν_F denote the reduced boundary and the measure–theoretic outer unit normal of a set F.

If μ is a Radon measure and S is a Borel set, $\mu \llcorner S$ denotes the restriction of μ to S, defined by $(\mu \llcorner S)(A) = \mu(A \cap S)$ for every Borel set A. The support $\operatorname{supp} \mu$ is the closed set of points x such that $|\mu|(B_r(x)) > 0$ for every $r > 0$. If T is a Borel map, $T_{\#}\mu$ denotes the push–forward measure, defined by $(T_{\#}\mu)(A) = \mu(T^{-1}(A))$ for every Borel set A.

For $z = (x_0, t_0) \in \mathbb{R}^n \times \mathbb{R}$ and $r > 0$ we denote by

$$\mathrm{C}_{z,r} = B_r(x_0) \times (t_0 - r, t_0 + r)$$

the open cylinder centered at z with vertical axis, radius r and height $2r$. If E has locally finite perimeter in a neighborhood of $\mathrm{C}_{z,r}$, its *excess* in this cylinder is

$$\mathrm{exc}_{n+1}(E,z,r) = \frac{1}{r^n} \int_{\mathrm{C}_{z,r} \cap \partial^* E} \frac{|\nu_E - e_{n+1}|^2}{2} \, d\mathcal{H}^n \, . \tag{1.3}$$

This is the *cylindrical excess* of [10,19] in the direction e_{n+1}, and it measures how far the outer unit normal ν_E is from e_{n+1} on the part of the reduced boundary of E contained in the cylinder. Finally, C_n denotes a positive constant depending only on the dimension, whose value may change from line to line.

Strategy of the proof. The argument is carried out on the subgraph E of a minimizer u of the relaxed functional in $BV(B_R)$, which satisfies $|u| \leqslant M = \|g\|_{L^\infty(B_R)}$ almost everywhere, and this is where the scalar nature of the problem is essential. By a theorem of Miranda [21], in the form stated by Giusti [12] for functionals with a lower order term and without boundary conditions, the set E minimizes the perimeter in the cylinder $B_R \times \mathbb{R}$ up to a volume term with coefficient one, because the second integral in equation (1.2) is a 1–Lipschitz function of u in $L^1(B_R)$. Comparing u with the functions $(1-\eta)u$, where η is a cut–off function, we bound the excess of E in a cylinder of radius r by a constant times $M/r + M$ and, taking $\eta \equiv 1$, by a constant times MR^n/r^n. Hence the excess of E is small at the scale $\rho = c_0 \min\{R, \sqrt{M}\}$, with c_0 small, as soon as M is small, independently of R: if $R \geqslant \sqrt{M}$ by the first bound, and if $R < \sqrt{M}$ by the second one. This choice of the scale makes the threshold σ_n independent of the radius of the ball.

Far from $\partial B_R \times \mathbb{R}$, the ε–regularity theorem of Tamanini [23,24] for almost–minimizers of the perimeter, applied at the scale ρ, shows that the boundary of E is, near each of its points, the graph of a function of class $C^{1,1/2}$ of the horizontal variables. This theorem does not reach the points whose distance from $\partial B_R \times \mathbb{R}$ is smaller than a multiple of ρ, and the scale cannot be decreased, because both bounds on the excess blow up as $r \to 0$. To extend the result of Tamanini to these points, we flatten $\partial B_R \times \mathbb{R}$ by an inversion in the horizontal variables, which transforms $B_R \times \mathbb{R}$ into a half–space, as De Philippis and Maggi [10] do for curved boundaries by means of diffeomorphisms. The image of E minimizes in the half–space, up to a volume term with coefficient one, an anisotropic perimeter with an explicit integrand, and we apply to it the ε–regularity lemma of De Philippis and Maggi at the boundary of the half–space. We use this quantitative lemma, and not the main theorems of [10], which give the regularity only outside a singular set and do not show that the boundary is a graph over the horizontal directions.

Going back to $B_R \times \mathbb{R}$ by the inverse map, we obtain that, near each point of B_R, the function u coincides almost everywhere with a function of class $C^{1,1/2}$, hence u has a representative in $C^{1,1/2}_{\mathrm{loc}}(B_R)$, which minimizes the functional in equation (1.2) over $C^1(B_R)$. The ε–regularity theory goes back to De Giorgi [8], Almgren [1] and Bombieri [5] in the interior and, for minimizers

of the Euclidean perimeter with no energy on the boundary of the domain, to Grüter and Jost [18] and Grüter [15,16,17] at the boundary. Further references are given in Section 3.

The paper is organized as follows. Section 2 contains the relaxation of the functional, the comparison inequality and the excess estimates. In Section 3 we recall the theorem of Tamanini, which gives the regularity far from $\partial B_R \times \mathbb{R}$, we introduce the inversion and the lemma of De Philippis and Maggi, which give the regularity near $\partial B_R \times \mathbb{R}$, and we verify the assumptions of both results at the scale ρ. In Section 4 we go back to the function u and complete the proof of Theorem 1.1.

AI Statement. *This paper describes results that were obtained by ChatGPT 5.6 Sol (OpenAI) over a series of chats, without significant human assistance. We have checked and reworked the proof, but its fundamental strategy remains close to the original output of the model. The same model, together with Claude Fable 5.2 and Claude Opus 5.5 (Anthropic), was then used to help reorganize the exposition, improve the English and the consistency of the LaTeX source, check the details of the arguments, compare the statement of the problem with the recent literature, and verify the bibliographical data and the references to external regularity results.*

2. Relaxation, perimeter comparison and excess estimates

In this section we collect the facts about the relaxed functional and the subgraph of its minimizers on which the proof is based: the projection identities for subgraphs, the comparison inequality satisfied by the subgraph of a minimizer, and the estimates of its excess. This material is classical, and in each subsection we indicate where it comes from. The comparison inequality follows from a theorem of Miranda, in the form stated by Giusti, while the excess estimates, which we need with explicit constants, are proved in detail. We use standard facts on functions of bounded variation and sets of finite perimeter, for which we refer, for instance, to [3,13,19].

2.1. Relaxation and subgraphs.

The relaxed area of functions of bounded variation, as a function of the measure Du, goes back to Goffman and Serrin [14] and to Miranda [21, Definition 1.7], and the identities relating the perimeter of a subgraph to the relaxed area, which are used below, are proved in [21, Proposition 1.9 and Theorem 1.10]; for these identities and for the relaxation of the area functional we refer also to [13, Chapter 14].

For $u \in BV(B_R)$ we define its *relaxed area measure* by

$$\alpha_u = \sqrt{1+|\nabla u|^2}\,\mathscr{L}^n + |D^s u|$$

and for every Borel set $A \subseteq B_R$, we set

$$\mathcal{A}(u,A) = \alpha_u(A) = \int_A \sqrt{1+|\nabla u|^2}\,d\mathscr{L}^n + |D^s u|(A)\,.$$

The relaxed functional associated to $\mathcal{F}_g$ is given by

$$\overline{\mathcal{F}}_g(u) = \mathcal{A}(u,B_R) + \int_{B_R} |u-g|\,d\mathscr{L}^n\,,$$

which, by the direct method of the calculus of variations, has a minimizer in $BV(B_R)$. We set

$$M = \|g\|_{L^\infty(B_R)}.$$

Every such minimizer u satisfies $|u| \leqslant M$ almost everywhere in B_R. Indeed, the truncation $u_M = \max\{-M, \min\{u, M\}\}$ satisfies $\mathcal{A}(u_M, B_R) \leqslant \mathcal{A}(u, B_R)$ and, since $|g| \leqslant M$ almost everywhere, $|u_M - g| \leqslant |u-g|$ almost everywhere in B_R, with strict inequality on the set $\{|u| > M\}$, which therefore has measure zero, by the minimality of u. In the rest of the paper, u denotes a fixed minimizer of $\overline{\mathcal{F}}_g$ in $BV(B_R)$.

We define the open cylinder $\mathcal{C}_R = B_R \times \mathbb{R}$, whose boundary is $\partial\mathcal{C}_R = \partial B_R \times \mathbb{R}$, and the subgraph of u as

$$E = E_u = \{(x,t) \in \mathcal{C}_R \,|\, t < u(x)\}\,.$$

From now on, all perimeter measures associated with E will be understood relative to the open cylinder $\mathcal{C}_R$; in particular, we set

$$\mu_E = -D\chi_E = \nu_E\,\mathcal{H}^n \llcorner \partial^* E$$

in $\mathcal{C}_R$ and we denote by Γ the support of $|\mu_E|$ in the relative topology of $\mathcal{C}_R$, namely

$$\Gamma = \{z \in \mathcal{C}_R \,|\, |\mu_E|(B_r(z) \cap \mathcal{C}_R) > 0 \text{ for every } r > 0\}\,.$$

Thus Γ is a relatively closed subset of $\mathcal{C}_R$, and it contains no point of $\partial\mathcal{C}_R$. We also write $P(E, A) = |\mu_E|(A)$ for every Borel set $A \subseteq \mathcal{C}_R$.

For the subgraph E of any function $u \in BV(B_R)$, the following projection identities, for which we refer to [21, Theorem 1.10] and to [3,13], hold as equalities between Radon measures on B_R:

$$\pi_{\#}\big(\mathcal{H}^n \llcorner \partial^* E\big) = \alpha_u \qquad \text{and} \qquad \pi_{\#}\big((\nu_E \cdot e_{n+1})\mathcal{H}^n \llcorner \partial^* E\big) = \mathscr{L}^n\,, \tag{2.1}$$

where $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is the horizontal projection $\pi(x,t) = x$. The measure–theoretic outer unit normal ν_E satisfies $\nu_E \cdot e_{n+1} \geqslant 0$ at $\mathcal{H}^n$–almost every point of $\partial^* E$; for the subgraph of a smooth function, it is the upward–pointing normal. The first identity in (2.1) gives

$$P(E, A \times \mathbb{R}) = \mathcal{A}(u, A) \tag{2.2}$$

for every Borel set $A \subseteq B_R$.

2.2. **Perimeter comparison.** The regularity theory of De Philippis and Maggi that we are going to apply in Sections 3 and 4 is a theory for sets rather than for functions. What it requires is the following comparison property: a modification of the set inside a bounded region cannot decrease its perimeter by more than a fixed multiple of the volume of the region where the modification takes place. In this subsection we show that the subgraph E of the minimizer u satisfies such a property with constant one, also for modifications that reach the boundary $\partial\mathcal{C}_R$ of the cylinder.

This property is a consequence of a classical theorem of Miranda, according to which the subgraph of a minimizer of the area functional has least perimeter in the corresponding cylinder; we refer also to [13, Chapter 14]. It rests on two results of [21]: the vertical rearrangement, which replaces every vertical fiber of a set by a half–line with the same signed length, does not increase the perimeter [21, Theorem 2.3], and the perimeter of the subgraph of a function of bounded variation in a cylinder is equal to the relaxed area of the function [21, Theorem 1.10]. For functionals with a lower order term and without boundary conditions, as in our case, the theorem of Miranda is stated by Giusti in [12, Theorem 1.1], where it is attributed to [21], and we recall it in the form in which we use it. In [12] the lower order term is the integral of a primitive, with respect to the vertical variable, of a function which is nondecreasing in this variable. In our case this function is

$$H_g(x,t) = \operatorname{sign}(t - g(x))\,, \qquad x \in B_R\,,\ t \in \mathbb{R}\,,$$

with $\operatorname{sign}(0) = 0$, since $\int_0^s H_g(x,t)\,dt = |s - g(x)| - |g(x)|$ for every $s \in \mathbb{R}$, and there is no boundary term. For $T > 0$ we set $Q_T = B_R \times [-T, T]$ and, for every set $A \subseteq \mathcal{C}_R$ of locally finite perimeter in $\mathcal{C}_R$,

$$\mathcal{G}_T(A) = |D\chi_A|(Q_T) + \int_{Q_T} H_g\,\chi_A\,dx\,dt\,.$$

The functional $\overline{\mathcal{F}}_g$ differs by the constant $\int_{B_R} |g|\,d\mathscr{L}^n$ from the functional [12, (1.1)] with $\Omega = B_R$, with H_g in place of the function denoted there by H, and with no boundary term. Since u minimizes $\overline{\mathcal{F}}_g$ in $BV(B_R)$, [12, Theorem 1.1] shows that E is a local solution in $\mathcal{C}_R$ for the functionals $\mathcal{G}_T$, in the sense of [12, Definitions 1.1 and 1.2]: for every $T > 0$ and every set $S \subseteq Q_T$,

$$\mathcal{G}_T(E) \leqslant \mathcal{G}_T(E \cup S) \qquad \text{and} \qquad \mathcal{G}_T(E) \leqslant \mathcal{G}_T(E \setminus S)\,. \tag{2.3}$$

These two inequalities express the minimality of E with respect to outer and inner variations, respectively, and the set S is not required to stay away from $\partial\mathcal{C}_R$. In the proof below we combine them by means of the submodularity of the perimeter; the coefficient one in front of the volume term comes from the bound $|H_g| \leqslant 1$, which reflects the fact that the function $t \mapsto |t - g(x)|$ is 1–Lipschitz.

Proposition 2.1 (Perimeter comparison). *Let $u \in BV(B_R)$ be the minimizer of $\overline{\mathcal{F}}_g$ fixed in Section 2.1, and let $E = E_u$ be its subgraph. Let $W \subseteq \mathbb{R}^{n+1}$ be an open set and let $F \subseteq \mathcal{C}_R$ be a set of locally finite perimeter in $\mathcal{C}_R$ such that $E \triangle F \Subset W$. Then*

$$P(E, W \cap \mathcal{C}_R) \leqslant P(F, W \cap \mathcal{C}_R) + |E \triangle F|\,. \tag{2.4}$$

Proof. Since the closure of $E\triangle F$ is compact, we can fix $T > M$ such that $\overline{E\triangle F} \subseteq \mathbb{R}^n \times (-T, T)$. The sets E and F coincide in the open set $O = \mathcal{C}_R \setminus \overline{E\triangle F}$, hence $D\chi_E \llcorner O = D\chi_F \llcorner O$, and O contains both $\mathcal{C}_R \setminus W$ and $\mathcal{C}_R \setminus Q_T$. In particular, by identity (2.2),

$$|D\chi_F|(\mathcal{C}_R \setminus W) = |D\chi_E|(\mathcal{C}_R \setminus W) \leqslant P(E, \mathcal{C}_R) = \mathcal{A}(u, B_R) < +\infty\,.$$

If $P(F, W \cap \mathcal{C}_R) = +\infty$, inequality (2.4) holds trivially. Otherwise $P(F, \mathcal{C}_R) < +\infty$, and, by the submodularity of the perimeter [3, Proposition 3.38(d)], [19, Lemma 12.22], the sets $E \cup F$ and $E \cap F$ have finite perimeter in $\mathcal{C}_R$, with

$$P(E \cup F, \mathcal{C}_R) + P(E \cap F, \mathcal{C}_R) \leqslant P(E, \mathcal{C}_R) + P(F, \mathcal{C}_R)\,.$$

The four sets appearing in this inequality coincide with E in O, hence the total variations of their distributional derivatives have the same finite value on the set $\mathcal{C}_R \setminus Q_T$, which is contained in O. Subtracting this value twice from both sides, we see that the same inequality holds with Q_T in place of $\mathcal{C}_R$, and since $\chi_{E\cup F} + \chi_{E\cap F} = \chi_E + \chi_F$, we obtain

$$\mathcal{G}_T(E \cup F) + \mathcal{G}_T(E \cap F) \leqslant \mathcal{G}_T(E) + \mathcal{G}_T(F)\,.$$

On the other hand, the sets $F \setminus E$ and $E \setminus F$ are contained in Q_T, and inequalities (2.3) with $S = F \setminus E$ and with $S = E \setminus F$ give $\mathcal{G}_T(E) \leqslant \mathcal{G}_T(E \cup F)$ and $\mathcal{G}_T(E) \leqslant \mathcal{G}_T(E \cap F)$. Adding these two inequalities and comparing the result with the previous one, we get $\mathcal{G}_T(E) \leqslant \mathcal{G}_T(F)$, because $\mathcal{G}_T(E)$ is finite. As $|H_g| \leqslant 1$ and $E\triangle F \subseteq Q_T$, it follows that

$$|D\chi_E|(Q_T) \leqslant |D\chi_F|(Q_T) + \int_{Q_T} H_g\,(\chi_F - \chi_E)\,dx\,dt \leqslant |D\chi_F|(Q_T) + |E\triangle F|\,.$$

Finally, the measures $|D\chi_E|$ and $|D\chi_F|$ coincide in O, which contains the sets $Q_T \setminus W$ and $(W \cap \mathcal{C}_R) \setminus Q_T$, hence

$$|D\chi_E|(Q_T) - |D\chi_F|(Q_T) = |D\chi_E|(W \cap Q_T) - |D\chi_F|(W \cap Q_T) = P(E, W \cap \mathcal{C}_R) - P(F, W \cap \mathcal{C}_R)\,,$$

and inequality (2.4) follows. □

2.3. **The excess estimate.** Besides the comparison inequality of Proposition 2.1, the regularity theory of De Philippis and Maggi, that we recall in Section 3, requires the excess of E to be small at a suitable scale, that is, the outer normal ν_E has to be close to e_{n+1} on most of the reduced boundary inside the relevant cylinders. We obtain this smallness from the minimality of u, by comparing u with $(1-\eta)u$, where η is a cutoff function. This competitor coincides with u where $\eta = 0$ and vanishes where $\eta = 1$: there it replaces the graph of u by the horizontal hyperplane $\{t = 0\}$, removing the area due to the slope of the graph and to the singular part of Du, while, since $|u| \leqslant M$, the additional cost is controlled by M times the integrals of η and of $|\nabla\eta|$. This is a classical comparison argument: without the term $\int_{B_R} |u-g|\,d\mathscr{L}^n$, it gives the well–known bound of the area of a minimal graph over a ball of radius r by $\omega_n r^n$ plus a multiple of r^{n-1} times the supremum of $|u|$, and comparisons of this kind are standard in the theory of minimal surfaces, for which we refer, for instance, to [13,19].

Lemma 2.2 (Area estimate). *For every Lipschitz function $\eta : B_R \to [0,1]$ we have*

$$\int_{B_R} \eta\,d\alpha_u - \int_{B_R} \eta\,d\mathscr{L}^n \leqslant M \int_{B_R} (|\nabla\eta| + \eta)\,d\mathscr{L}^n\,. \tag{2.5}$$

Consequently, for every $x_0 \in \mathbb{R}^n$ and every $r > 0$,

$$\alpha_u(B_R \cap B_r(x_0)) - \mathscr{L}^n(B_R \cap B_r(x_0)) \leqslant C_n M(r^{n-1} + r^n)\,, \tag{2.6}$$

and

$$\alpha_u(B_R) - \mathscr{L}^n(B_R) \leqslant M|B_R| = \omega_n M R^n\,. \tag{2.7}$$

Proof. Since $u \in BV(B_R)$ is bounded and η is Lipschitz, the function $v = (1-\eta)u$ belongs to $BV(B_R)$, and the product rule gives $Dv = (1-\eta)Du - u\nabla\eta\,\mathscr{L}^n$. The last term is absolutely

continuous with respect to $\mathscr{L}^n$, hence $D^s v = (1-\eta) D^s u$ and $\nabla v = (1-\eta)\nabla u - u\nabla\eta$ almost everywhere. Consequently, almost everywhere in B_R,

$$\begin{aligned}\sqrt{1+|\nabla v|^2} &= \sqrt{1+|(1-\eta)\nabla u - u\nabla\eta|^2} \\ &\leqslant \sqrt{1+(1-\eta)^2|\nabla u|^2} + |u|\,|\nabla\eta| \\ &\leqslant (1-\eta)\sqrt{1+|\nabla u|^2} + \eta + |u|\,|\nabla\eta|\,,\end{aligned}$$

where both inequalities follow from the triangle inequality for the Euclidean norm in $\mathbb{R}^{n+1}$, applied to the sum of the vectors $(1,(1-\eta)\nabla u)$ and $(0,-u\nabla\eta)$ in the first case, and to the sum of the vectors $(1-\eta)(1,\nabla u)$ and $(\eta,0)$ in the second one. Moreover, $|D^s v| = (1-\eta)|D^s u|$, because $0 \leqslant \eta \leqslant 1$; then, integrating over B_R and recalling that $|u| \leqslant M$, we obtain

$$\mathcal{A}(v,B_R) \leqslant \mathcal{A}(u,B_R) - \int_{B_R} \eta\, d\alpha_u + \int_{B_R} \eta\, d\mathscr{L}^n + M\int_{B_R} |\nabla\eta|\, d\mathscr{L}^n\,.$$

On the other hand, since $|v-u| = |u|\,\eta \leqslant M\eta$, we have

$$\int_{B_R} |v-g|\, d\mathscr{L}^n \leqslant \int_{B_R} |u-g|\, d\mathscr{L}^n + M\int_{B_R} \eta\, d\mathscr{L}^n\,.$$

Since no boundary condition is imposed, v is an admissible competitor also when η does not vanish near ∂B_R, so that $\overline{\mathcal{F}}_g(u) \leqslant \overline{\mathcal{F}}_g(v)$; as the quantities $\mathcal{A}(u,B_R)$ and $\int_{B_R}|u-g|\,d\mathscr{L}^n$ are finite, adding these two estimates we obtain inequality (2.5).
To prove inequality (2.6), we choose

$$\eta(x) = \min\{1, \max\{0, 2-|x-x_0|/r\}\}\,,$$

which is equal to 1 on $B_r(x_0)$, vanishes outside $B_{2r}(x_0)$ and satisfies $|\nabla\eta| \leqslant 1/r$. The measure

$$\alpha_u - \mathscr{L}^n = (\sqrt{1+|\nabla u|^2} - 1)\,\mathscr{L}^n + |D^s u|$$

is nonnegative, so its value on $B_R \cap B_r(x_0)$ is at most the left–hand side of inequality (2.5), whose right–hand side is at most $M(r^{-1}+1)|B_{2r}| = 2^n\omega_n M(r^{n-1}+r^n)$. Inequality (2.7) is inequality (2.5) with $\eta \equiv 1$. □

For the subgraph E, the left–hand sides of the inequalities in Lemma 2.2 measure how far the reduced boundary of E is from being horizontal, as the following elementary consequence of the projection identities (2.1) shows.

Lemma 2.3 (Area and excess). *Let $u \in BV(B_R)$. For every Borel set $A \subseteq B_R$ we have*

$$\alpha_u(A) - \mathscr{L}^n(A) = \int_{(A\times\mathbb{R})\cap\partial^* E} (1-\nu_E\cdot e_{n+1})\, d\mathcal{H}^n = \frac{1}{2}\int_{(A\times\mathbb{R})\cap\partial^* E} |\nu_E - e_{n+1}|^2\, d\mathcal{H}^n\,.$$

Proof. By the projection identities (2.1),

$$\alpha_u(A) = \mathcal{H}^n\big((A\times\mathbb{R})\cap\partial^* E\big) \qquad \text{and} \qquad \mathscr{L}^n(A) = \int_{(A\times\mathbb{R})\cap\partial^* E} \nu_E\cdot e_{n+1}\, d\mathcal{H}^n\,.$$

Subtracting these two equalities gives the first identity, and the second follows from $|\nu_E - e_{n+1}|^2 = 2(1-\nu_E\cdot e_{n+1})$, which holds because ν_E and e_{n+1} are unit vectors. □

Combining Lemmas 2.2 and 2.3, we obtain, for every $x_0 \in \mathbb{R}^n$ and every $r>0$,

$$\frac{1}{2}\int_{((B_R\cap B_r(x_0))\times\mathbb{R})\cap\partial^* E} |\nu_E - e_{n+1}|^2\, d\mathcal{H}^n \leqslant C_n M(r^{n-1}+r^n) \tag{2.8}$$

and

$$\frac{1}{2}\int_{(B_R\times\mathbb{R})\cap\partial^* E} |\nu_E - e_{n+1}|^2\, d\mathcal{H}^n \leqslant C_n M R^n\,. \tag{2.9}$$

These two estimates control the excess of E in every cylinder contained in $\mathcal{C}_R$. Let $z=(x_0,t_0) \in \mathcal{C}_R$ and $r>0$ be such that $\mathrm{C}_{z,r} = B_r(x_0)\times(t_0-r,t_0+r) \subseteq \mathcal{C}_R$. Then $B_r(x_0) \subseteq B_R$ and

$\mathrm{C}_{z,r} \cap \partial^* E \subseteq (B_r(x_0) \times \mathbb{R}) \cap \partial^* E$, hence, by the definition of the excess in formula (1.3) and by inequality (2.8), we obtain

$$\begin{aligned} \mathrm{exc}_{n+1}(E,z,r) &= \frac{1}{r^n} \int_{\mathrm{C}_{z,r} \cap \partial^* E} \frac{|\nu_E - e_{n+1}|^2}{2}\, d\mathcal{H}^n \\ &\leqslant \frac{1}{r^n} \int_{(B_r(x_0)\times\mathbb{R}) \cap \partial^* E} \frac{|\nu_E - e_{n+1}|^2}{2}\, d\mathcal{H}^n \leqslant C_n \Big(\frac{M}{r} + M \Big) . \end{aligned} \tag{2.10}$$

In the same way, using inequality (2.9) in place of inequality (2.8), we get

$$\mathrm{exc}_{n+1}(E,z,r) \leqslant \frac{C_n M R^n}{r^n} . \tag{2.11}$$

3. Interior and boundary ε–regularity

By Proposition 2.1, the set E is a $(1,\infty)$–minimizer of the perimeter in $\mathcal{C}_R$, in the sense of Definition 3.1 below. Hence, far from $\partial\mathcal{C}_R$, the regularity of its boundary follows from the ε–regularity theorem of Tamanini [23,24], recalled in Lemma 3.2. This theorem, however, does not reach the points of Γ close to $\partial\mathcal{C}_R$, at the scale at which the excess of E is small by the estimates of Section 2.3.

To extend the result of Tamanini to these points, we transform $\mathcal{C}_R$ into a half–space by an inversion in the horizontal variables. The image of E is then a $(1,\infty)$–minimizer, in the half–space, of an anisotropic perimeter with an explicit integrand (Proposition 3.4), and we apply to it the ε–regularity lemma of De Philippis and Maggi [10] at the boundary of the half–space (Lemma 3.7). Finally, we fix the scale and we verify the assumptions of both results, with constants depending only on n (Proposition 3.8).

We begin with the regularity far from $\partial\mathcal{C}_R$, which is based on the following notion.

Definition 3.1 ((Λ,∞)–minimizers)**.** Let $A \subseteq \mathbb{R}^{n+1}$ be an open set and let $\Lambda \geqslant 0$. A set $F \subseteq \mathbb{R}^{n+1}$ of locally finite perimeter in A is a (Λ,∞)–*minimizer of the perimeter in* A if

$$P(F,W) \leqslant P(G,W) + \Lambda |F \triangle G|$$

for every open set $W \Subset A$ and every set $G \subseteq \mathbb{R}^{n+1}$ of locally finite perimeter in A such that $F \triangle G \Subset W$.

This is the notion of [10, Definition 1.8], in the whole space and with $r_0 = +\infty$, that is, with no restriction on the diameter of the sets W. As observed in [10, Remark 1.9], it is analogous to the notions considered in [2,5,11,24] and in [19, Section 21]. The set E is a $(1,\infty)$–minimizer of the perimeter in $\mathcal{C}_R$: if $W \Subset \mathcal{C}_R$ is open and $G \subseteq \mathbb{R}^{n+1}$ is a set of locally finite perimeter in $\mathcal{C}_R$ such that $E \triangle G \Subset W$, then $G \subseteq \mathcal{C}_R$, and Proposition 2.1 gives $P(E,W) \leqslant P(G,W) + |E \triangle G|$.

The ε–regularity theory for sets which minimize the perimeter, or more generally an elliptic integrand, goes back to De Giorgi [8], Almgren [1,2], Schoen, Simon and Almgren [22] and Bombieri [5]. For sets which minimize the perimeter plus the integral of a bounded function, as E does in the proof of Proposition 2.1, it is due to Massari [20]. Tamanini [23, Theorem 1] showed that the reduced boundary of a set is a hypersurface of class $C^{1,\alpha}$ if, in every ball of sufficiently small radius s, the perimeter of the set exceeds the least perimeter among its modifications compactly contained in the ball by at most a constant times $s^{n+2\alpha}$. A (Λ,∞)–minimizer F of the perimeter in A satisfies this condition with $\alpha = 1/2$, since

$$P(F, B_s(x)) \leqslant P(G, B_s(x)) + \Lambda |B_s(x)|$$

for every ball $B_s(x) \Subset A$ and every set G such that $F \triangle G \Subset B_s(x)$. Hence the reduced boundary of F is a hypersurface of class $C^{1,1/2}$ in A, and the second part of [23, Theorem 1], which concerns sequences of such sets, gives, by a standard compactness and scaling argument, the following ε–regularity statement, which we write in the form in which we use it. We refer also to [24], to Duzaar and Steffen [11] and, for the formulation with the cylindrical excess, to [19, Part III].

Following [10, Lemma 2.16 and Remark 2.17], whenever the topological boundary of a set F is used in an open set A, we choose, within the equivalence class of F modulo sets of Lebesgue measure zero, a representative satisfying

$$A \cap \partial F = A \cap \operatorname{supp} |D\chi_F| \, .$$

Lemma 3.2 (Tamanini). *There exist $\varepsilon_{\rm int} > 0$ and $K_{\rm int} \geqslant 1$, depending only on n, with the following property. Let $A \subseteq \mathbb{R}^{n+1}$ be an open set, let $\Lambda \geqslant 0$ and let $F \subseteq \mathbb{R}^{n+1}$ be a (Λ, ∞)–minimizer of the perimeter in A. We assume that $r > 0$ and $z = (x_0, t_0) \in \partial F$ satisfy $\mathrm{C}_{z, 2K_{\rm int} r} \subseteq A$ and*

$$\mathrm{exc}_{n+1}(F, z, K_{\rm int} r) + \Lambda r < \varepsilon_{\rm int} \, .$$

Then there exists a function $w \in C^{1,1/2}(\overline{B_r(x_0)})$ such that

$$\mathrm{C}_{z,r} \cap \partial F = \{(x, w(x)) \, | \, x \in B_r(x_0)\} \, .$$

Lemma 3.2 applies to E, with $A = \mathcal{C}_R$ and $\Lambda = 1$, at the points of Γ whose distance from $\partial \mathcal{C}_R$ is larger than a fixed multiple of the scale at which the excess of E is small. To reach the other points of Γ, we transform $\mathcal{C}_R$ into a half–space by an inversion, and we use the lemma of De Philippis and Maggi.

We now construct the inversion. We write the points of $\mathbb{R}^{n+1}$ as $y = (y', t) \in \mathbb{R}^n \times \mathbb{R}$, and we denote by

$$H' = \{y' \in \mathbb{R}^n \, | \, y_1 > 0\} \qquad \text{and} \qquad H = H' \times \mathbb{R} = \{y \in \mathbb{R}^{n+1} \, | \, y_1 > 0\}$$

the open half–spaces of $\mathbb{R}^n$ and of $\mathbb{R}^{n+1}$ respectively, whose boundaries are the hyperplanes $\{y_1 = 0\}$; the vertical direction e_{n+1} is tangent to ∂H. As in [10], for a set $S \subseteq \mathbb{R}^{n+1}$ we write $S^+ = S \cap H$.

We fix a point $x_* \in \partial B_R$ and, after a rotation of the coordinates of $\mathbb{R}^n$, we assume $x_* = Re_1$. Let $\widehat{f}$ be the inversion of $\mathbb{R}^n$ with respect to the sphere of center $-x_*$ and radius $2R$, followed by the translation which brings x_* to the origin, and let f act as $\widehat{f}$ on the horizontal variables and as the identity on the vertical one, that is,

$$\widehat{f}(x) = \frac{4R^2(x + Re_1)}{|x + Re_1|^2} - 2Re_1 \, , \qquad f(x,t) = (\widehat{f}(x), t) \, .$$

Since ∂B_R contains the center of the inversion, $\widehat{f}$ maps B_R onto H' and $\partial B_R \setminus \{-x_*\}$ onto $\partial H'$, with $\widehat{f}(x_*) = 0$. Hence the map f is a smooth diffeomorphism of $\mathcal{C}_R$ onto H, and the set

$$\widetilde{E} = f(E)$$

is the subgraph of the function $u \circ \widehat{f}^{-1}$ on H'. The inverse map is

$$\widehat{f}^{-1}(y') = \frac{4R^2(y' + 2Re_1)}{|y' + 2Re_1|^2} - Re_1 \, ,$$

and, as for every inversion, its differential at each point is a multiple of an orthogonal matrix. Precisely, for $y' \neq -2Re_1$ we set

$$h(y') = \frac{4R^2}{|y' + 2Re_1|^2} \, ,$$

and we denote by $\mathrm{O}(y')$ the reflection with respect to the hyperplane orthogonal to $y' + 2Re_1$, which is a symmetric orthogonal matrix with determinant -1; then

$$D\widehat{f}^{-1}(y') = h(y') \, \mathrm{O}(y') \, .$$

In particular, $|D\widehat{f}^{-1}(y')\xi| = h(y')|\xi|$ for every $\xi \in \mathbb{R}^n$, that is, $h(y')$ is the factor by which $\widehat{f}^{-1}$ multiplies the lengths of vectors at the point y'. The map f and the set $\widetilde{E}$ depend on x_*, while all the constants below are independent of it.

We use the map f only near the line $f(\{x_*\} \times \mathbb{R}) = \mathbb{R}e_{n+1}$, in the region $\{|y'| < R/2\}$, where the factor h is controlled by inequalities (3.1) below. For $r > 0$, we denote by

$$D_r = \{y' \in H' \, | \, |y'| < r\} \, , \qquad \mathcal{Q}_r = \{(y', t) \in \mathbb{R}^n \times \mathbb{R} \, | \, |y'| < r\}$$

the open half–ball of radius r in H' and the open cylinder of radius r around the line $\mathbb{R}e_{n+1}$, so that $H \cap \mathcal{Q}_r = D_r \times \mathbb{R}$; a cylinder $\mathrm{C}_{y,r}$ is contained in $\mathcal{Q}_{R/2}$ if and only if $|y'| + r \leqslant R/2$. If $|y'| < R/2$, then $3R/2 < |y' + 2Re_1| < 5R/2$, hence

$$\frac{1}{2} \leqslant h(y') \leqslant 2 \qquad \text{and} \qquad |\nabla h(y')| \leqslant \frac{3}{R} \qquad \text{if } |y'| < R/2\,. \tag{3.1}$$

In particular, $\widehat{f}^{-1}$ is 2–Lipschitz on the ball $B_{R/2}$.

Two further properties of f are used below. First, if $y' \in H'$, then $|y' + 2Re_1| > 2R$, hence $h(y') < 1$, and the Jacobian determinant of $\widehat{f}^{-1}$, which is equal to $-h(y')^n$, has absolute value smaller than one; for this reason, in Proposition 3.4 below, the coefficient of the volume term is still equal to one. Second, if $x \in B_R$ and x_* is a point of ∂B_R nearest to x, then $x = (R-d)e_1$, where $d = \mathrm{dist}(x, \partial B_R)$, and $\widehat{f}(x) = \widetilde{d}\, e_1$ with $\widetilde{d} = 2Rd/(2R-d)$; hence $d \leqslant \widetilde{d} \leqslant 2d$, and the point $f(x,t)$ lies at distance $\widetilde{d}$ both from ∂H and from the line $\mathbb{R}e_{n+1}$.

The map f transforms the perimeter of E into an anisotropic perimeter of $\widetilde{E}$, which we now write explicitly. For $y = (y', t)$ with $y' \neq -2Re_1$ and for $\nu = (\nu', \nu_{n+1}) \in \mathbb{R}^n \times \mathbb{R}$ we set

$$\Phi(y, \nu) = h(y')^{n-1} \big(|\nu'|^2 + h(y')^2 \nu_{n+1}^2 \big)^{1/2}\,, \tag{3.2}$$

and, for a set $F \subseteq H$ of locally finite perimeter in an open set $A \subseteq \mathbb{R}^{n+1}$ and a Borel set $G \subseteq A \cap H$, we call *anisotropic perimeter* of F in G the quantity

$$P_\Phi(F, G) = \int_{G \cap \partial^* F} \Phi(y, \nu_F(y))\, d\mathcal{H}^n(y)\,.$$

Like the map f, the function Φ depends on the point x_*. Only the part of the reduced boundary of F lying in the open half–space H enters this functional, that is, no energy is assigned to ∂H. We have

$$P(E, f^{-1}(B)) = P_\Phi(\widetilde{E}, B) \tag{3.3}$$

for every Borel set $B \subseteq H$. Indeed, by [19, Proposition 17.1], applied to the diffeomorphism f^{-1} and to the set $\widetilde{E}$, as in the proof of [10, Theorem 1.2], the left–hand side is the integral over $B \cap \partial^* \widetilde{E}$ of $|\operatorname{cof} D(f^{-1})(y)\, \nu_{\widetilde{E}}(y)|$, where, for an invertible matrix Y, we denote by Y^{-T} the transpose of its inverse and by $\operatorname{cof} \mathrm{Y} = (\det \mathrm{Y})\, \mathrm{Y}^{-T}$ its cofactor matrix. The matrix $D(f^{-1})(y)$ is block diagonal, with blocks $h(y')\mathrm{O}(y')$ and 1, hence $\operatorname{cof} D(f^{-1})(y)$ is block diagonal, with blocks $-h(y')^{n-1}\mathrm{O}(y')$ and $-h(y')^n$, and therefore $|\operatorname{cof} D(f^{-1})(y)\, \nu| = \Phi(y, \nu)$ for every $\nu \in \mathbb{R}^{n+1}$.

Definition 3.3 (**(Λ, ∞)–minimizers in a half–space**)**.** Let $A \subseteq \mathbb{R}^{n+1}$ be an open set and let $\Lambda \geqslant 0$. A set $F \subseteq H$ of locally finite perimeter in A is a *(Λ, ∞)–minimizer of P_Φ in (A, H)* if

$$P_\Phi(F, H \cap W) \leqslant P_\Phi(G, H \cap W) + \Lambda |F \triangle G|$$

for every open set $W \Subset A$ and every set $G \subseteq H$ of locally finite perimeter in A such that $F \triangle G \Subset W$.

In [10, Definition 1.8] the energy may also contain the integral over $W \cap \partial^* F \cap \partial H$ of a function σ, and the inequality is required only for the sets W with diameter smaller than $2r_0$: Definition 3.3 is the case $\sigma = 0$ and $r_0 = +\infty$.

Proposition 3.4 (Minimality of $\widetilde{E}$)**.** *The set $\widetilde{E}$ is a $(1, \infty)$–minimizer of P_Φ in $(\mathcal{Q}_{R/2}, H)$, that is, for every open set $W \Subset \mathcal{Q}_{R/2}$ and every set $G \subseteq H$ of locally finite perimeter in $\mathcal{Q}_{R/2}$ such that $\widetilde{E} \triangle G \Subset W$, we have*

$$P_\Phi(\widetilde{E}, H \cap W) \leqslant P_\Phi(G, H \cap W) + |\widetilde{E} \triangle G|\,. \tag{3.4}$$

Proof. We argue as in the proof of [10, Theorem 1.2]. Let W and G be as in the statement. The set $\widetilde{G} = f^{-1}(G)$ is contained in $\mathcal{C}_R$ and satisfies $E \triangle \widetilde{G} = f^{-1}(\widetilde{E} \triangle G) \Subset f^{-1}(W)$. Since $f^{-1}(W) \cap \mathcal{C}_R = f^{-1}(H \cap W)$, Proposition 2.1, applied with the open set $f^{-1}(W)$, and equality (3.3), which holds with the same proof for $\widetilde{G}$ in place of E, give

$$P_\Phi(\widetilde{E}, H \cap W) \leqslant P_\Phi(G, H \cap W) + |E \triangle \widetilde{G}|\,.$$

Since $\widetilde{E}\triangle G \subseteq H$ and $|\det D(f^{-1})| = h^n < 1$ in H, we have $|E\triangle\widetilde{G}| \leqslant |\widetilde{E}\triangle G|$, and inequality (3.4) follows. □

The lemma of De Philippis and Maggi requires the integrand Φ to satisfy the conditions of [10, Definition 1.1], which we now check for Φ. We denote by $\nabla\Phi$ and $\nabla^2\Phi$, respectively, the gradient and the Hessian of Φ with respect to the second variable, and by $\|\cdot\|$ the operator norm of a matrix.

Lemma 3.5 (The integrand Φ). *For every $y \in \overline{\mathcal{Q}_{R/2}}$, the function $\Phi(y,\cdot)$ is convex, positively one–homogeneous and smooth in $\mathbb{R}^{n+1}\setminus\{0\}$. Moreover, there exist a constant $\lambda_n \geqslant 1$, depending only on n, and a number $\ell_R \geqslant 0$ with $\ell_R \leqslant C_n/R$, both independent of x_*, such that*

$$\frac{1}{\lambda_n} \leqslant \Phi(y,\nu) \leqslant \lambda_n\,,$$
$$|\Phi(y,\nu)-\Phi(z,\nu)| + |\nabla\Phi(y,\nu)-\nabla\Phi(z,\nu)| \leqslant \ell_R|y-z|\,,$$
$$|\nabla\Phi(y,\nu)| + \|\nabla^2\Phi(y,\nu)\| + \frac{\|\nabla^2\Phi(y,\nu)-\nabla^2\Phi(y,\widetilde{\nu})\|}{|\nu-\widetilde{\nu}|} \leqslant \lambda_n$$

and

$$\nabla^2\Phi(y,\nu)e\cdot e \geqslant \frac{|e-(e\cdot\nu)\nu|^2}{\lambda_n}$$

for every $y,z \in \overline{\mathcal{Q}_{R/2}}$, every $\nu\in\mathbb{S}^n$, every $\widetilde{\nu}\in\mathbb{S}^n\setminus\{\nu\}$ and every $e\in\mathbb{R}^{n+1}$. In addition,

$$\nabla\Phi(y,e_{n+1})\cdot e_1 = 0 \qquad \text{for every } y\in\mathcal{Q}_{R/2}\,. \tag{3.5}$$

Proof. By formula (3.2), $\Phi(y,\nu) = |\mathrm{X}(y')\nu|$, where $\mathrm{X}(y')$ is the diagonal matrix with diagonal entries $h(y')^{n-1}$, repeated n times, and $h(y')^n$. Hence $\Phi(y,\cdot)$ is convex, positively one–homogeneous and smooth outside the origin, and

$$\nabla\Phi(y,\nu) = \frac{\mathrm{X}^2\nu}{|\mathrm{X}\nu|}\,, \qquad \nabla\Phi(y,e_{n+1}) = h(y')^n e_{n+1}\,,$$

which gives identity (3.5). By inequalities (3.1), which also hold for $|y'| = R/2$, by continuity, the matrices $\mathrm{X}(y')$ with $|y'|\leqslant R/2$ belong to a compact set of invertible matrices depending only on n. For every such matrix X and every unit vector ν, the Hessian of the function $\nu\mapsto|\mathrm{X}\nu|$ vanishes on ν, by homogeneity, and it is positive definite on the hyperplane orthogonal to ν, because $\mathrm{X}e$ is not parallel to $\mathrm{X}\nu$ when e is not parallel to ν. By compactness, the first, third and fourth inequalities in the statement hold with a constant λ_n depending only on n, and the second one holds with $\ell_R \leqslant C_n/R$, because X depends on y only through $h(y')$, which is $(3/R)$–Lipschitz on $B_{R/2}$ by inequalities (3.1). □

In the terminology of [10, Definition 1.1], Lemma 3.5 says that Φ is a *regular elliptic integrand* in $\mathcal{Q}_{R/2}\cap H$, with *ellipticity constant* λ_n and *spatial Lipschitz constant* ℓ_R.

Besides the minimality, the lemma at the boundary of a half–space requires the excess relative to the half–space to be small. For a set $F\subseteq H$ of locally finite perimeter in a neighborhood of $\mathrm{C}_{z,r}$, we call

$$\mathrm{exc}^H_{n+1}(F,z,r) = \frac{1}{r^n}\int_{\mathrm{C}^+_{z,r}\cap\partial^*F}\frac{|\nu_F-e_{n+1}|^2}{2}\,d\mathcal{H}^n$$

the excess of F in $\mathrm{C}_{z,r}$ relative to H (where we recall that $\mathrm{C}^+_{z,r} = C_{z,r}\cap H$); it is the quantity denoted by exc^H_n in [10], where the ambient dimension is denoted by n. It differs from the excess defined in formula (1.3) because the part of ∂^*F lying on ∂H is not taken into account.

The excess of $\widetilde{E}$ is controlled by the estimates of Section 2.3.

Lemma 3.6 (Excess of $\widetilde{E}$). *For every cylinder $\mathrm{C}_{z,r}\subseteq\mathcal{Q}_{R/2}$ we have*

$$\mathrm{exc}^H_{n+1}(\widetilde{E},z,r) \leqslant C_n\Big(\frac{M}{r}+M\Big) \tag{3.6}$$

and

$$\mathrm{exc}^H_{n+1}(\widetilde{E},z,r) \leqslant \frac{C_nMR^n}{r^n}\,. \tag{3.7}$$

Proof. By [19, Proposition 17.1], for every Borel set $S \subseteq f^{-1}(H \cap \mathcal{Q}_{R/2})$ we have

$$\int_{f(S)\cap\partial^* \widetilde{E}} \bigl(1 - \nu_{\widetilde{E}} \cdot e_{n+1}\bigr)\, d\mathcal{H}^n = \int_{S\cap\partial^* E} \bigl(1 - \nu_{\widetilde{E}}(f(q)) \cdot e_{n+1}\bigr)\, |\operatorname{cof} Df(q)\, \nu_E(q)|\, d\mathcal{H}^n(q)\,.$$

Let $q \in S \cap \partial^* E$, let $\nu = \nu_E(q) = (\nu', \nu_{n+1}) \in \mathbb{R}^n \times \mathbb{R}$, where $\nu_{n+1} = \nu \cdot e_{n+1}$ is nonnegative for $\mathcal{H}^n$–almost every q, as recalled in Section 2, and let h and O be evaluated at $\widehat{f}(\pi q)$. The matrix $Df(q)$ is block diagonal, with blocks $h^{-1}\mathrm{O}$ and 1, hence the vector $(Df(q))^{-T}\nu = (h\mathrm{O}\nu', \nu_{n+1})$, which is parallel to $\nu_{\widetilde{E}}(f(q))$, has length $s = (h^2(1-\nu_{n+1}^2) + \nu_{n+1}^2)^{1/2}$, and $|\operatorname{cof} Df(q)\,\nu| = h^{-n}s$. Then $\nu_{n+1} \leqslant s$, and, by inequalities (3.1), $1/2 \leqslant s \leqslant 2$ and $h^{-n}s \leqslant 2^{n+1}$, so that

$$1 - \nu_{\widetilde{E}}(f(q)) \cdot e_{n+1} = 1 - \frac{\nu_{n+1}}{s} \leqslant 1 - \frac{\nu_{n+1}^2}{s^2} = \frac{h^2(1-\nu_{n+1}^2)}{s^2} \leqslant 16\,(1-\nu_{n+1}^2) \leqslant 32\,(1 - \nu \cdot e_{n+1})\,.$$

Recalling that $|\nu - e_{n+1}|^2/2 = 1 - \nu \cdot e_{n+1}$ for every unit vector ν, and taking $S = f^{-1}(\mathrm{C}^+_{z,r})$, we obtain

$$\operatorname{exc}^H_{n+1}(\widetilde{E}, z, r) \leqslant \frac{2^{n+6}}{r^n} \int_{f^{-1}(\mathrm{C}^+_{z,r})\cap\partial^* E} \frac{|\nu_E - e_{n+1}|^2}{2}\, d\mathcal{H}^n\,.$$

We write $z = (y_0, t_0) \in \mathbb{R}^n \times \mathbb{R}$ and $x_0 = \widehat{f}^{-1}(y_0)$. Since $\widehat{f}^{-1}$ is 2–Lipschitz on $B_{R/2}$, maps H' onto B_R and f^{-1} does not change the last coordinate, the set $f^{-1}(\mathrm{C}^+_{z,r})$ is contained in $(B_R \cap B_{2r}(x_0)) \times \mathbb{R}$. Inequality (2.8), with $2r$ in place of r, now gives inequality (3.6), and inequality (2.9) gives inequality (3.7). □

For minimizers of the Euclidean perimeter in a domain, with no energy on its boundary, the regularity at the boundary of the domain goes back to Grüter and Jost [18] and Grüter [15,16,17], and it was extended by De Philippis and Maggi [10] to anisotropic integrands and to energies on the boundary. We use the quantitative lemma of [10] at the boundary of the half–space, and not their main theorems, which give the regularity only outside a closed singular set, because we need graphs over the horizontal directions in cylinders of a fixed size, under a smallness condition on the datum independent of R.

For a set $F \subseteq H$, we may have $|D\chi_F|(A \cap \partial H) > 0$, so that $A \cap \partial F$ may also contain points of ∂H which are not limits of points of $H \cap \partial F$; for this reason, the points of ∂H at which the regularity is studied are taken in $\overline{H \cap \partial F} \cap \partial H$, that is, they are limits of boundary points lying in the open half–space. We can now state the result of [10] concerning the points of ∂H.

Lemma 3.7 (De Philippis–Maggi, Lemma 3.4)**.** *There exists $\varepsilon_{\mathrm{reg}} > 0$, depending only on n, with the following property. Let f and Φ be the map and the integrand constructed above at a point $x_* \in \partial B_R$, let $\Lambda \geqslant 0$ and let $F \subseteq H$ be a (Λ, ∞)–minimizer of P_Φ in $(\mathcal{Q}_{R/2}, H)$. We assume that $r > 0$ and $z = (x_0, t_0) \in \overline{H \cap \partial F} \cap \partial H$ satisfy $\mathrm{C}_{z,128r} \subseteq \mathcal{Q}_{R/2}$ and*

$$\operatorname{exc}^H_{n+1}(F, z, 64r) + (\Lambda + \ell_R)r < \varepsilon_{\mathrm{reg}}\,,$$

where ℓ_R is the number given by Lemma 3.5. Then there exists a function $w \in C^{1,1/2}(\overline{D})$, where $D = \{x \in B_r(x_0) \,|\, x_1 > 0\}$, such that

$$\mathrm{C}^+_{z,r} \cap \partial F = \{(x, w(x)) \,|\, x \in D\}\,.$$

This is [10, Lemma 3.4], applied in the open set $\mathcal{Q}_{R/2}$ to the integrand Φ, which satisfies the conditions of [10, Definition 1.1] with ellipticity constant λ_n and spatial Lipschitz constant ℓ_R by Lemma 3.5, with $\sigma = 0$ and $r_0 = +\infty$ in [10, Definition 1.8], after a translation along ∂H sending z to the origin. The threshold $\varepsilon_{\mathrm{reg}}$ is the one given there for the ellipticity constant λ_n, hence it depends only on n, and the term $|\nabla\Phi(z, e_{n+1}) \cdot e_1|$, which also appears in the smallness assumption of [10, Lemma 3.4], vanishes by identity (3.5). In [10] the function w is denoted by u, and the lemma also contains estimates on w and the boundary condition satisfied by w on ∂H. These are not needed for Theorem 1.1, and the boundary condition is used only in Remark 1.3.

The two lemmas apply to our sets with $\Lambda = 1$: the set E is a $(1, \infty)$–minimizer of the perimeter in $\mathcal{C}_R$, as observed after Definition 3.1, and $\widetilde{E}$ is a $(1, \infty)$–minimizer of P_Φ in $(\mathcal{Q}_{R/2}, H)$ by

Proposition 3.4. It remains to choose the scale and to verify the smallness assumptions of the two lemmas.

At a scale r, the excess of E and, by Lemma 3.6, that of $\widetilde{E}$ are at most $C_n(M/r + M)$ and at most $C_n M R^n/r^n$, while the terms $(1+\ell_R)r$ and r in the smallness assumptions of Lemmas 3.7 and 3.2, with $\Lambda = 1$, are at most $(1 + C_n/R)r$, by Lemma 3.5. Hence all these quantities are small at the scale

$$\rho = c_0 \min\{R, \sqrt{M}\}\,,$$

if $c_0 \in (0,1]$ is small and then M is small, depending on c_0: if $R \geqslant \sqrt{M}$ by the first bound on the excess, and if $R < \sqrt{M}$ by the second one. The constant c_0, which depends only on n, is chosen in the proof of Proposition 3.8.

We set

$$K = 2K_{\rm int},$$

where $K_{\rm int}$ is the constant of Lemma 3.2. In Section 4, near a point $x_0 \in B_R$ such that $\mathrm{dist}(x_0, \partial B_R) > K\rho$, we apply Lemma 3.2 to E at a point $z \in \Gamma$ with $\pi z = x_0$, and the cylinder $\mathrm{C}_{z,2K_{\rm int}\rho} = \mathrm{C}_{z,K\rho}$ required by this lemma is contained in $\mathcal{C}_R$. When $\mathrm{dist}(x_0, \partial B_R) \leqslant K\rho$, we use the map f constructed at a point of ∂B_R nearest to x_0; by the second of the two properties of f observed above, the point $\widehat{f}(x_0)$ lies at distance at most $2K\rho$ from the origin. We apply Lemma 3.7 to $\widetilde{E}$ at a point of $(\partial H) \cap H \cap \overline{\partial \widetilde{E}}$ lying on the line $\mathbb{R}e_{n+1}$, with $r = L\rho$, where

$$L = 4K.$$

This choice guarantees that $\widehat{f}(x_0)$ belongs to the half–ball $D_{L\rho}$ over which Lemma 3.7 describes $H \cap \partial\widetilde{E}$ as a graph. Finally, we set

$$\varepsilon_* = \min\bigl\{\varepsilon_{\rm reg},\, \varepsilon_{\rm int}\bigr\} \qquad \text{and} \qquad A_0 = 128L\,,$$

where $\varepsilon_{\rm reg}$ and $\varepsilon_{\rm int}$ are the thresholds of Lemmas 3.7 and 3.2, which depend only on n. All the cylinders used in Section 4 have radius $a\rho$ for some $a \in [1, A_0]$. The constants are chosen in the following order: first the constants K, L and A_0, which depend only on n and determine the cylinders used in Section 4, together with the number ε_*, then c_0, which depends on them, and finally the threshold σ_n of Theorem 1.1, which depends on c_0.

Since we are going to choose $\sigma_n \leqslant 1$, we assume $0 < M \leqslant 1$: if $M = 0$, then $g = 0$ almost everywhere and the constant function $u = 0$ is a minimizer, so that Theorem 1.1 holds trivially. The constant c_0 chosen in the proof of Proposition 3.8 satisfies in particular $c_0 \leqslant 1/(4A_0)$, so that

$$A_0\rho \leqslant \frac{R}{4}\,. \tag{3.8}$$

By the description of the cylinders contained in $\mathcal{Q}_{R/2}$ given above, this inequality implies that the cylinders $\mathrm{C}_{y,A_0\rho}$ with $y' = 0$ are contained in $\mathcal{Q}_{R/2}$, whatever the last coordinate of y is.

Proposition 3.8 (Uniform verification of the smallness assumptions). *There exist constants $c_0 \in (0,1]$ and $\sigma_n \in (0,1]$, depending only on n, with the following property. If $0 < M \leqslant \sigma_n$ and $\rho = c_0 \min\{R, \sqrt{M}\}$, then inequality (3.8) holds and, for every $a \in [1, A_0]$ and every map f constructed as above at a point of ∂B_R,*

$$\mathrm{exc}_{n+1}(E, z, a\rho) < \frac{\varepsilon_*}{2} \qquad \text{and} \qquad \mathrm{exc}^H_{n+1}(\widetilde{E}, y, a\rho) < \frac{\varepsilon_*}{2} \tag{3.9}$$

for every cylinder $\mathrm{C}_{z,a\rho} \subseteq \mathcal{C}_R$ and every cylinder $\mathrm{C}_{y,a\rho} \subseteq \mathcal{Q}_{R/2}$, and

$$(1+\ell_R)a\rho < \frac{\varepsilon_*}{2}\,. \tag{3.10}$$

Consequently, for every such map f, the set $\widetilde{E}$ satisfies all the hypotheses of Lemma 3.7 with $\Lambda = 1$ and $r = L\rho$ at every point $y \in \overline{H \cap \partial\widetilde{E}} \cap \partial H$ such that $y' = 0$. Moreover, the set E satisfies all the hypotheses of Lemma 3.2 with $A = \mathcal{C}_R$, $\Lambda = 1$ and $r = \rho$ at every point $z \in \Gamma$ such that $\mathrm{C}_{z,K\rho} \subseteq \mathcal{C}_R$.

Proof. Let $a \in [1, A_0]$, let $r = a\rho$, and let z and y be as in the statement. If $R \geqslant \sqrt{M}$, then $r \geqslant c_0\sqrt{M}$, and inequalities (2.10) and (3.6) give

$$\max\bigl\{\mathrm{exc}_{n+1}(E, z, r), \mathrm{exc}^H_{n+1}(\widetilde{E}, y, r)\bigr\} \leqslant C_n\Bigl(\frac{M}{r} + M\Bigr) \leqslant C_n\Bigl(\frac{\sqrt{M}}{c_0} + M\Bigr)\,.$$

If $R < \sqrt{M}$, then $r \geqslant c_0 R$, and inequalities (2.11) and (3.7) give

$$\max\bigl\{\mathrm{exc}_{n+1}(E, z, r), \mathrm{exc}^H_{n+1}(\widetilde{E}, y, r)\bigr\} \leqslant \frac{C_n M R^n}{r^n} \leqslant \frac{C_n M}{c_0^n}\,.$$

In both cases $r \leqslant A_0 c_0 \min\{R, \sqrt{M}\}$, hence $r \leqslant A_0 c_0$ and $r/R \leqslant A_0 c_0$, because $M \leqslant 1$. Since $\ell_R \leqslant C_n/R$ by Lemma 3.5, it follows that

$$(1 + \ell_R) r \leqslant C_n c_0\,.$$

Let $C \geqslant 1$ be a constant, depending only on n, such that the last three displayed estimates hold with C in place of C_n. We choose $c_0 \in (0, 1]$, depending only on n, so small that

$$c_0 \leqslant \frac{1}{4A_0} \qquad \text{and} \qquad C c_0 < \frac{\varepsilon_*}{2}\,.$$

Then inequality (3.8) holds, because $\rho \leqslant c_0 R$, and inequality (3.10) follows from the last estimate. Next, we choose $\sigma_n \in (0, 1]$, depending only on n, so small that

$$C\Bigl(\frac{\sqrt{\sigma_n}}{c_0} + \sigma_n + \frac{\sigma_n}{c_0^n}\Bigr) < \frac{\varepsilon_*}{2}\,.$$

If $M \leqslant \sigma_n$, inequalities (3.9) follow from the first two estimates.
We now verify the hypotheses of the two lemmas. Let f be the map constructed as above at a point of ∂B_R. By Proposition 3.4, the set $\widetilde{E}$ is a $(1, \infty)$–minimizer of P_Φ in $(\mathcal{Q}_{R/2}, H)$. Let $y \in \overline{H \cap \partial\widetilde{E}} \cap \partial H$ with $y' = 0$. The cylinder $\mathrm{C}_{y,128L\rho} = \mathrm{C}_{y,A_0\rho}$ is contained in $\mathcal{Q}_{R/2}$, and inequalities (3.9) with $a = 64L$ and (3.10) with $a = L$ give

$$\mathrm{exc}^H_{n+1}(\widetilde{E}, y, 64L\rho) + (1 + \ell_R) L\rho < \varepsilon_* \leqslant \varepsilon_{\mathrm{reg}}\,,$$

which are the remaining hypotheses of Lemma 3.7 with $\Lambda = 1$ and $r = L\rho$. Finally, let $z \in \Gamma$ be such that $\mathrm{C}_{z,K\rho} \subseteq \mathcal{C}_R$. As observed after Definition 3.1, the set E is a $(1, \infty)$–minimizer of the perimeter in $\mathcal{C}_R$, and $\mathcal{C}_R \cap \partial E = \Gamma$ for the representative of E chosen above. Moreover, $\mathrm{C}_{z,2K_{\mathrm{int}}\rho} = \mathrm{C}_{z,K\rho} \subseteq \mathcal{C}_R$, and inequalities (3.9) with $a = K_{\mathrm{int}}$ and (3.10) with $a = 1$ give

$$\mathrm{exc}_{n+1}(E, z, K_{\mathrm{int}}\rho) + \rho \leqslant \mathrm{exc}_{n+1}(E, z, K_{\mathrm{int}}\rho) + (1 + \ell_R)\rho < \varepsilon_* \leqslant \varepsilon_{\mathrm{int}}\,,$$

which are the remaining hypotheses of Lemma 3.2 with $\Lambda = 1$ and $r = \rho$. □

4. Back to the cylinder and proof of the main theorem

By Proposition 3.8, Lemmas 3.2 and 3.7 describe the boundaries of E and of $\widetilde{E}$, in suitable cylinders, as graphs of functions of class $C^{1,1/2}$. Since E and $\widetilde{E}$ are subgraphs, these functions coincide almost everywhere with u and with $u \circ \widehat{f}^{-1}$, respectively, by the following elementary lemma.

Lemma 4.1 (Subgraphs whose boundary is a graph). *Let $U \subseteq \mathbb{R}^n$ be a connected open set and let $w : U \to \mathbb{R}$ be a measurable function whose subgraph*

$$F = \{(x, t) \in U \times \mathbb{R} \,|\, t < w(x)\}$$

has locally finite perimeter in $U \times \mathbb{R}$. If there exist two numbers $a < b$ and a continuous function $v : U \to (a, b)$ such that

$$(U \times (a, b)) \cap \mathrm{supp}\,|D\chi_F| = \{(x, v(x)) \,|\, x \in U\}\,,$$

then $w = v$ almost everywhere in U.

Proof. The graph of v divides the set $V = U \times (a,b)$ into the two connected open sets

$$V^{\pm} = \{(x,t) \in V \mid \pm(t - v(x)) > 0\},$$

on which $|D\chi_F|$ vanishes, hence χ_F is almost everywhere constant on each of them. Since F is a subgraph, if $\chi_F = 1$ almost everywhere on V^+, then $\chi_F = 1$ almost everywhere on V, and if $\chi_F = 0$ almost everywhere on V^-, then $\chi_F = 0$ almost everywhere on V. Both cases are impossible, because V contains the graph of v, which is a nonempty subset of $\operatorname{supp}|D\chi_F|$. Hence $\chi_F = 1$ almost everywhere on V^- and $\chi_F = 0$ almost everywhere on V^+, that is, $w = v$ almost everywhere in U. □

Proof of Theorem 1.1. Let $M = \|g\|_{L^\infty(B_R)} \leqslant \sigma_n$, where σ_n is the constant of Proposition 3.8, and let $u \in BV(B_R)$ be the minimizer of $\overline{\mathcal{F}}_g$ fixed in Section 2, which satisfies $|u| \leqslant M$ almost everywhere. If $M = 0$, Theorem 1.1 holds trivially, as observed in Section 3. Hence we assume $M > 0$ and we set $\rho = c_0 \min\{R, \sqrt{M}\}$. We use the representatives of E and of the sets $\widetilde{E}$ chosen in Section 3, so that $\mathcal{C}_R \cap \partial E = \Gamma$, and we fix a point $x_0 \in B_R$.
If $\operatorname{dist}(x_0, \partial B_R) > K\rho$, we first find a point of Γ above x_0. Since $|u| \leqslant M$ almost everywhere, the function χ_E is equal to 1 almost everywhere on $B_R \times (-\infty, -M)$ and to 0 almost everywhere on $B_R \times (M, +\infty)$. Hence it is not almost everywhere constant on the connected open set $B_\delta(x_0) \times (-M-1, M+1)$, for every $\delta > 0$ such that $B_\delta(x_0) \subseteq B_R$, and Γ meets this set. Since Γ is closed in $\mathcal{C}_R$, it contains a point $z = (x_0, t_0)$ with $|t_0| \leqslant M+1$. Then $\mathrm{C}_{z,K\rho} \subseteq \mathcal{C}_R$ and, by Proposition 3.8, Lemma 3.2 applies to E at z with $r = \rho$. Thus $\mathrm{C}_{z,\rho} \cap \Gamma$ is the graph of a function $v \in C^{1,1/2}(\overline{B_\rho(x_0)})$ with values in $(t_0 - \rho, t_0 + \rho)$, and Lemma 4.1, applied with $U = B_\rho(x_0)$, $a = t_0 - \rho$ and $b = t_0 + \rho$, gives $u = v$ almost everywhere in $B_\rho(x_0)$.
If $\operatorname{dist}(x_0, \partial B_R) \leqslant K\rho$, let f and $\widetilde{E}$ be constructed at a point $x_* \in \partial B_R$ nearest to x_0. As observed in Section 3, we have $\widehat{f}(x_0) = \widetilde{d}\, e_1$ with $\widetilde{d} \leqslant 2\operatorname{dist}(x_0, \partial B_R) \leqslant 2K\rho < L\rho$, hence $\widehat{f}(x_0) \in D_{L\rho}$. The set $\widetilde{E}$ is the subgraph of the function $w = u \circ \widehat{f}^{-1}$, which satisfies $|w| \leqslant M$ almost everywhere in H'. The argument of the previous case, applied to $\widetilde{E}$ in the sets $D_\delta \times (-M-1, M+1)$ with $0 < \delta < R/2$, shows that $H \cap \partial\widetilde{E}$ meets these sets and, letting $\delta \to 0^+$, we find a point $z_b = (0, t_b) \in \overline{H \cap \partial\widetilde{E}} \cap \partial H$. By Proposition 3.8, Lemma 3.7 applies to $\widetilde{E}$ at z_b with $r = L\rho$. Thus $\mathrm{C}^+_{z_b, L\rho} \cap \partial\widetilde{E}$ is the graph of a function $w_b \in C^{1,1/2}(\overline{D_{L\rho}})$ with values in $(t_b - L\rho, t_b + L\rho)$. Since $\mathrm{C}^+_{z_b, L\rho}$ is contained in $H \cap \mathcal{Q}_{R/2}$, where $\partial\widetilde{E}$ coincides with the support of $|D\chi_{\widetilde{E}}|$, Lemma 4.1, applied with $U = D_{L\rho}$, $a = t_b - L\rho$ and $b = t_b + L\rho$, gives $w = w_b$ almost everywhere in $D_{L\rho}$. Hence $u = w_b \circ \widehat{f}$ almost everywhere in the open set $\widehat{f}^{-1}(D_{L\rho}) \subseteq B_R$, which contains x_0, and $w_b \circ \widehat{f}$ is of class $C^{1,1/2}$ in this set, because $\widehat{f}$ is smooth in a neighborhood of its closure.
Therefore every point of B_R has an open neighborhood on which u coincides almost everywhere with a function of class $C^{1,1/2}$. Since two continuous functions which coincide almost everywhere on an open set coincide everywhere on it, these functions define a function $\widetilde{u} \in C^{1,1/2}_{\mathrm{loc}}(B_R)$ such that $u = \widetilde{u}$ almost everywhere in B_R. Then $Du = \nabla\widetilde{u}\,\mathscr{L}^n$, hence $\overline{\mathcal{F}}_g(u) = \mathcal{F}_g(\widetilde{u})$. If $\varphi \in C^1(B_R)$ and $\mathcal{F}_g(\varphi) < +\infty$, then $\varphi \in W^{1,1}(B_R)$, because g is bounded, and

$$\mathcal{F}_g(\varphi) = \overline{\mathcal{F}}_g(\varphi) \geqslant \overline{\mathcal{F}}_g(u) = \mathcal{F}_g(\widetilde{u}).$$

Therefore $\widetilde{u}$ minimizes the functional in equation (1.2) over $C^1(B_R)$. □

References

[1] F. J. Almgren, Jr., *Existence and regularity almost everywhere of solutions to elliptic variational problems among surfaces of varying topological type and singularity structure*, Ann. of Math. (2) **87** (1968), 321–391.
[2] F. J. Almgren, Jr., *Existence and regularity almost everywhere of solutions to elliptic variational problems with constraints*, Mem. Amer. Math. Soc. **4** (1976), no. 165.
[3] L. Ambrosio, N. Fusco and D. Pallara, *Functions of Bounded Variation and Free Discontinuity Problems*, Oxford Mathematical Monographs, Oxford University Press, Oxford, 2000.
[4] G. Bellettini and Sh. Yu. Kholmatov, *A De Giorgi conjecture on the regularity of minimizers of Cartesian area in 1D*, Ann. Univ. Ferrara **72** (2026), Article 115.
[5] E. Bombieri, *Regularity theory for almost minimal currents*, Arch. Ration. Mech. Anal. **78** (1982), 99–130.

[6] R. Continolo, V. Lorenzini, R. Scala and G. Scianna, *Regularity and convergence of critical points of an Ambrosio–Tortorelli functional with linear growth and of its* Γ*–limit*, J. Differential Equations **445** (2025), Article 113654.
[7] R. Continolo, V. Lorenzini, R. Scala and G. Scianna, *Regularity of minimizers of convex functionals with linear growth and small fidelity term*, preprint, 2026, available at https://cvgmt.sns.it/paper/7826/.
[8] E. De Giorgi, *Frontiere orientate di misura minima*, Seminario di Matematica della Scuola Normale Superiore di Pisa, 1960–61, Editrice Tecnico Scientifica, Pisa, 1961.
[9] E. De Giorgi, *On the relaxation of functionals defined on Cartesian manifolds*, in: Developments in Partial Differential Equations and Applications to Mathematical Physics (Ferrara, 1991), Plenum Press, New York, 1992, pp. 33–38.
[10] G. De Philippis and F. Maggi, *Regularity of free boundaries in anisotropic capillarity problems and the validity of Young's law*, Arch. Ration. Mech. Anal. **216** (2015), 473–568.
[11] F. Duzaar and K. Steffen, *Optimal interior and boundary regularity for almost minimizers to elliptic variational integrals*, J. Reine Angew. Math. **546** (2002), 73–138.
[12] E. Giusti, *Generalized solutions for the mean curvature equation*, Pacific J. Math. **88** (1980), 297–321.
[13] E. Giusti, *Minimal Surfaces and Functions of Bounded Variation*, Monographs in Mathematics 80, Birkhäuser, Basel, 1984.
[14] C. Goffman and J. Serrin, *Sublinear functions of measures and variational integrals*, Duke Math. J. **31** (1964), 159–178.
[15] M. Grüter, *Boundary regularity for solutions of a partitioning problem*, Arch. Ration. Mech. Anal. **97** (1987), 261–270.
[16] M. Grüter, *Optimal regularity for codimension one minimal surfaces with a free boundary*, Manuscripta Math. **58** (1987), 295–343.
[17] M. Grüter, *Regularity results for minimizing currents with a free boundary*, J. Reine Angew. Math. **375/376** (1987), 307–325.
[18] M. Grüter and J. Jost, *Allard type regularity results for varifolds with free boundaries*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **13** (1986), 129–169.
[19] F. Maggi, *Sets of Finite Perimeter and Geometric Variational Problems: An Introduction to Geometric Measure Theory*, Cambridge Studies in Advanced Mathematics 135, Cambridge University Press, Cambridge, 2012.
[20] U. Massari, *Esistenza e regolarità delle ipersuperfici di curvatura media assegnata in* $\mathbb{R}^n$, Arch. Ration. Mech. Anal. **55** (1974), 357–382.
[21] M. Miranda, *Superfici cartesiane generalizzate ed insiemi di perimetro localmente finito sui prodotti cartesiani*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (3) **18** (1964), 515–542.
[22] R. Schoen, L. Simon and F. J. Almgren, Jr., *Regularity and singularity estimates on hypersurfaces minimizing parametric elliptic variational integrals. I, II*, Acta Math. **139** (1977), 217–265.
[23] I. Tamanini, *Boundaries of Caccioppoli sets with Hölder-continuous normal vector*, J. Reine Angew. Math. **334** (1982), 27–39.
[24] I. Tamanini, *Regularity Results for Almost Minimal Oriented Hypersurfaces in* $\mathbb{R}^N$, Quaderni del Dipartimento di Matematica dell'Università di Lecce, Lecce, 1984.