SHARP HÖLDER ESTIMATES FOR ELLIPTIC EQUATIONS WITH MEASURE DATA WHEN $p > n$

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Abstract. We show that local energy solutions of scalar equations of p-Laplacian type, with measurable coefficients and signed Radon measure data, are locally Hölder continuous for $p > n \geq 2$, with the same exponent as the fundamental solution of the p-Laplacian; this exponent is therefore optimal. We also show that the gradient need not have sufficient integrability to recover this exponent via the Sobolev–Morrey embedding.

1. Introduction and results

We consider the scalar equation

$$-\operatorname{div} A(x, Du) = \mu, \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is a bounded domain, μ is a signed Radon measure on Ω, and where the differential operator is modeled upon the classic p-Laplacian:

$$\Delta_p u = \operatorname{div}\big[|Du|^{p-2}Du\big].$$

The nonlinearity $A\colon \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ in (1.1) is indeed supposed to be a Carathéodory vector field satisfying for $0 < \nu \leq L$

$$|A(x,\xi)| \leq L|\xi|^{p-1}, \qquad \langle A(x,\xi), \xi\rangle \geq \nu|\xi|^p \tag{1.2}$$

for almost every $x \in \Omega$ and every $\xi \in \mathbb{R}^n$; throughout the paper, we assume $p > n$ unless stated otherwise.

When $1 < p \leq n$, signed Radon measures need not belong to the dual of the natural energy space $W^{1,p}_0(B)$, even on balls $B \Subset \Omega$. Distributional solutions to (1.1) can be found on each such ball, under the usual strict monotonicity assumptions, by approximation arguments (with Du understood as the generalized gradient defined by truncation when $p \leq 2 - 1/n$) and, due to the low integrability of such very weak solutions, uniqueness may fail in the larger class of distributional solutions, see [21, 5]. When $p > n$, μ acts continuously on $W^{1,p}_0(B)$ for every ball $B \Subset \Omega$ and under the same strict monotonicity assumptions, the usual monotonicity methods then give a unique energy solution in B for fixed Dirichlet data.

The basic model for properties of solutions to the equation in (1.1) is the so-called fundamental solution of the p-Laplacian $\mathcal{G}_p$, which satisfies $-\Delta_p \mathcal{G}_p = \delta_0$ in the sense of distributions for every $p > 1$, where δ_0 is the Dirac delta measure centered at the origin; see [18, p. 53]. Setting $c_{n,p} := (n\omega_n)^{-1/(p-1)}$ with $\omega_n = |B_1|$, it is given, for every $p > 1$ and $x \neq 0$ with continuous extension at the origin when $p > n$, by

$$\mathcal{G}_p(x) = -c_{n,p}\begin{cases} \dfrac{1}{\gamma}\,|x|^{\gamma}, & \text{if } p > 1,\ p \neq n \\ \log|x|, & \text{if } p = n \end{cases}, \qquad \gamma := \frac{p-n}{p-1}; \tag{1.3}$$

for $2 - 1/n < p \leq n$, the function $\mathcal{G}_p - \mathcal{G}_p(e_1)$ is the unique solution obtained as a limit of approximations (SOLA) to $-\Delta_p u = \delta_0$ in B_1 with zero boundary values, see [13, Theorem 2.1] and [14], and the same fact is evident, upon replacing SOLA with energy solution, when $p > n$: $\delta_0 \in W^{-1,p'}(B_1)$ by Morrey's embedding and uniqueness follows from the strict monotonicity of $\xi \mapsto |\xi|^{p-2}\xi$. $\mathcal{G}_p$ indeed locally belongs to $W^{1,p}(\mathbb{R}^n)$ if and only if $p > n$, as it is easily seen by computing for every $p > 1$

$$D\mathcal{G}_p(x) = -c_{n,p}|x|^{\gamma-1}\frac{x}{|x|}, \qquad |D\mathcal{G}_p(x)| = c_{n,p}|x|^{\gamma-1}, \qquad x \neq 0.$$

Since $\gamma - 1 = -(n-1)/(p-1)$, we also stress that $\mathcal{G}_p$ locally belongs to $W^{1,1}(\mathbb{R}^n)$ if and only if $p > 2 - 1/n$; for $1 < p \leq 2 - 1/n$, the flux $|D\mathcal{G}_p|^{p-2}D\mathcal{G}_p$ is still locally integrable and its negative divergence equals δ_0 in the sense of distributions, as mentioned above.

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The main point of this paper is the relation between the regularity of the fundamental solution of the p-Laplacian and that of solutions to equations with general structure as in (1.2) and general measure data. It has been known for more than thirty years, at least in the case $p < n$, that solutions to (1.1) in a suitable class of weak solutions, depending on the framework, and their gradients satisfy the same endpoint regularity as the fundamental solution and its gradient, respectively. We refer to Bénilan, Boccardo, Gallouët, Gariepy, Pierre and Vázquez [4] for $p < n$ and to Dolzmann, Hungerbühler and Müller [8], also for a suitable class of systems, for the more challenging case $p = n$. These global estimates hold for $|\mu|(\Omega) < \infty$, under the structural and domain assumptions of the cited works. For $p < n$, both bounds are expressed in terms of Marcinkiewicz spaces; for $p = n$, the solution belongs to BMO and its gradient to $\mathcal{M}^n$. Recalling that

$$f \in \mathcal{M}^q(U) \qquad \Longleftrightarrow \qquad \sup_{\lambda>0} \lambda^q \big|\{x \in U : |f(x)| > \lambda\}\big| < \infty,$$

the gradient estimate for $p \le n$ reads

$$Du \in \mathcal{M}^{\frac{n}{1-\gamma}}(\Omega)$$

for solutions obtained by approximation in the Dirichlet problem with zero boundary data. We also mention the paper [7] by Dal Maso, Murat, Orsina and Prignet for the treatment of general measures, and the local gradient estimate of Mingione in the borderline case $p = n$, see [20, Theorem 1.8].

In this paper we show that the parallel continues for $p > n$, where the solution satisfies a quantitative Hölder estimate with the same exponent as the fundamental solution.

Theorem 1.1. *Let $p > n \ge 2$ and let $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ be a local energy solution to (1.1), under the assumptions in (1.2). Then $u \in C^{0,\gamma}_{\mathrm{loc}}(\Omega)$ with $\gamma = (p-n)/(p-1)$, and for every ball $B_{2R}(x_0) \Subset \Omega$ we have*

$$[u]_{C^{0,\gamma}(B_{R/2}(x_0))} \le c\left[\frac{1}{\nu R}\int_{B_{2R}(x_0)\setminus B_R(x_0)} |A(x,Du)|\,dx + \frac{|\mu|(B_{2R}(x_0))}{\nu}\right]^{\frac{1}{p-1}} \tag{1.4}$$

and, for every $0 < \rho \le R$,

$$\rho^{-\gamma}\int_{B_\rho(x_0)} |Du|^p\,dx \le c\left[R^{-\gamma}\int_{B_{2R}(x_0)} |Du|^p\,dx + \left[\frac{|\mu|(B_{2R}(x_0))}{\nu}\right]^{p'}\right], \tag{1.5}$$

where $c = c(n,p)$ in (1.4) and $c = c(n,p,L/\nu)$ in (1.5).

The fundamental solution shows that both exponents in Theorem 1.1 are sharp: $\mathcal{G}_p$ has Hölder exponent γ at the origin, and radial integration gives $\int_{B_\rho(0)} |D\mathcal{G}_p|^p\,dx = c(n,p)\rho^\gamma$. For the p-Laplacian with nonnegative measure data, the Hölder exponent γ follows from [15, Theorem 1.14], since $n-p+\gamma(p-1) = 0$. For signed measure data, the same exponent follows by approximation from [17, Theorem 5.1, estimate (5.2)], taking $\alpha = \gamma$ so that $p - \alpha(p-1) = n$. For operators with measurable coefficients, both the result of Kilpeläinen and Zhong and the signed measure estimate in [17, Theorem 1.1] require the desired exponent to lie strictly below a Hölder exponent for solutions of the corresponding homogeneous equation. Under (1.2), we obtain γ directly for arbitrary signed measure data.

The following proposition gives a bound for the Hölder seminorm of u in terms of $|\mu|(B_{2R})$ and the tangential component of Du on $B_{2R} \setminus B_R$, together with a decay estimate for the tangential energy. (1.4) follows from (1.6) with $c = c(n,p,L/\nu)$; the Appendix improves this to $c = c(n,p)$.

Proposition 1.2. *Under the assumptions of Theorem 1.1, denote by $D_\tau u$ the tangential gradient of u with respect to spheres centered at x_0. Then*

$$[u]_{C^{0,\gamma}(B_{R/2}(x_0))} \le c\left[\frac{1}{R}\int_{B_{2R}(x_0)\setminus B_R(x_0)} |D_\tau u|^{p-1}\,dx + \frac{|\mu|(B_{2R}(x_0))}{\nu}\right]^{\frac{1}{p-1}} \tag{1.6}$$

and, for every $0 < \rho \le R$,

$$\rho^{-\gamma}\int_{B_\rho(x_0)} |D_\tau u|^p\,dx \le c\left[R^{-\gamma}\int_{B_{2R}(x_0)} |D_\tau u|^p\,dx + \left[\frac{|\mu|(B_{2R}(x_0))}{\nu}\right]^{p'}\right], \tag{1.7}$$

with $c = c(n,p,L/\nu)$ in both estimates.

We recall that $B_r(x_0)$ denotes the open ball of radius r and center $x_0 \in \Omega$, $|\mu|$ is the total variation of μ and

$$[u]_{C^{0,\gamma}(B_{R/2})} = \sup_{\substack{x,y\in B_{R/2},\\ x\neq y}} \frac{|u(x)-u(y)|}{|x-y|^\gamma}.$$

We also recall that under the structural assumptions in (1.2), for $p > 1$, a local energy solution to (1.1) is a function $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ satisfying the weak formulation

$$\int_\Omega \langle A(x, Du), D\varphi \rangle \, dx = \int_\Omega \varphi \, d\mu \qquad \text{for all } \varphi \in C^\infty_c(\Omega). \tag{1.8}$$

The quantitative Hölder exponent γ in (1.3) improves on the exponent $1-n/p$ given by Morrey-Sobolev's embedding theorem; the same embedding gives a continuous representative of u, a fact that we shall use below. At this point it is natural to wonder whether also the improved Hölder regularity in Theorem 1.1 follows from embedding, and a quick computation ($1 - n/q_* = \gamma$ for $q_* = n(p-1)/(n-1)$) shows that the corresponding integrability exponent for Du, guaranteeing by embedding $u \in C^{0,\gamma}$, is precisely the critical one for $D\mathcal{G}_p$. The fact that the gain in gradient integrability can be arbitrarily small when the ellipticity ratio is large appears to be well known to specialists; however, we have been unable to find a precise reference for this assertion for general p even for the weighted p-Laplacian with a scalar measurable coefficient. For completeness, we give a short proof based on the rank-one constructions of Astala, Faraco and Székelyhidi [2] and Faraco [9].

Proposition 1.3. *Let $n \geq 2$, $1 < p < \infty$ and $\delta > 0$. There exist a measurable coefficient $a : B_1(0) \to [\nu, L]$ with $0 < \nu \leq L$, a finite signed Radon measure μ on $B_1(0)$ and an energy solution $u \in W^{1,p}(B_1(0))$ to*

$$-\operatorname{div}\big(a(x)|Du|^{p-2}Du\big) = \mu \qquad \text{in } B_1(0) \tag{1.9}$$

such that $u \in C^{0,\gamma}_{\mathrm{loc}}(B_1(0))$ if $p > n$, with γ as in (1.3), but

$$Du \notin L^{p(1+\delta)}(B_r(0)) \qquad \text{for every } 0 < r < 1. \tag{1.10}$$

For $0 < \delta \leq 1$, the coefficient can be chosen with $L/\nu \leq c(p)\delta^{-p}$.

In the planar linear case, Faraco [9, Theorem 1.1] established the sharp threshold by constructing a sequence of solutions, and Astala, Faraco and Székelyhidi Jr. [2, Theorem 1.2] attained the same threshold with a single solution. We stress that, for $p > n$, the largest Hölder exponent independent of L/ν that can be deduced from gradient integrability by the Sobolev–Morrey embedding is therefore $1 - n/p$. Indeed, for every $s > p$, choosing $0 < \delta < s/p - 1$ in Proposition 1.3 rules out even $Du \in \mathcal{M}^s_{\mathrm{loc}}$, since $\mathcal{M}^s(B_r) \subset L^{p(1+\delta)}(B_r)$. In particular, this applies to $s = q_* = n(p-1)/(n-1)$, for which the Sobolev–Lorentz embedding would give precisely $1 - n/q_* = \gamma$; see [6, Theorem 5.6(ii)].

Remark 1.4. Even when $1 < p \leq n$, the measure μ in our example belongs to $(W^{1,p}_0(B_1))^*$, since $u \in W^{1,p}(B_1)$; whence $\mu = \operatorname{div} F$ for some $F \in L^{p'}(B_1; \mathbb{R}^n)$. The construction however shows that F can in fact be chosen in $L^\infty(B_1; \mathbb{R}^n)$.

1.1. **Ideas of the proof.** The proof of Theorem 1.1 is done in two steps, and the first one consists in proving the same results for the homogeneous equation, when $\mu \equiv 0$. In this case, the result follows (as in the classic De Giorgi approach for the regularity of equations with measurable coefficients) from the combination of a suitable Sobolev-type inequality and an energy-type inequality following, after an appropriate testing of its weak formulation, from the equation. In our case the Sobolev-type inequality is provided, as $p > n$, by the Sobolev-Morrey inequality on spheres, while the energy estimate is a Stampacchia-Caccioppoli estimate that retains the tangential gradient information. The two estimates are merged using the maximum principle, which ensures that a truncation separating two values in a ball has the same oscillation on each enclosing sphere. Our use of the spherical inequality was inspired by the work of Bella and Schäffner [3] and Hirsch and Schäffner [11] on functionals with (p, q)-growth. To prove the result for the measure data equation (1.1), we use the well-established strategy of comparing u with the homogeneous solution in its Dirichlet class on a fixed ball, and this initially requires imposing on $A(\,\cdot\,)$ the extra assumption of uniform p-monotonicity; removing this assumption to prove the result solely under (1.2) entails some additional work, the construction of a uniformly p-monotone field agreeing with $A(\,\cdot\,)$ at Du. Finally, Proposition 1.3 provides a counterexample, inspired by the theory of laminates (see in particular [2, 5]), showing that higher Lebesgue integrability of Du is not sufficient to recover γ via the Sobolev–Morrey embedding.

During the final stages of this work, we learned of two related contributions: Koskela, Manfredi and Villamor [16] rely on the same technique we use here, namely the maximum principle combined with Sobolev embedding on spheres, to prove Harnack's inequality for nonnegative $\mathcal{A}$-harmonic functions when $p > n - 1$. For measure data of one sign, the exponent γ also follows by taking $a \equiv 0$ and $q = p$ in [10, Proposition 5.1]. Our results were obtained independently of the latter work.

2. The proofs

We start with the standard Caccioppoli estimate for solutions to (1.1). For concentric balls $B_r \subset B_s \Subset \Omega$ with $r < s$ and any $k \in \mathbb{R}$, testing (1.8) with $(u-k)\eta^p$, η a standard cut-off, and applying Young's inequality gives

$$\int_{B_r} |Du|^p \, dx \leq \frac{c}{(s-r)^p} \int_{B_s} |u-k|^p \, dx + \frac{c}{\nu} \int_{B_s} |u-k| \, d|\mu|, \tag{2.1}$$

with $c \equiv c(n,p,L/\nu)$.

2.1. **The homogeneous equation.** For $B_R \equiv B_R(x_0) \Subset B_{2R} \equiv B_{2R}(x_0) \Subset \Omega$, let $v \in W^{1,p}(B_{2R})$ be a local energy solution to (1.1) with $\mu \equiv 0$ in B_{2R} under the assumptions in (1.2); by Morrey-Sobolev embedding we can assume that v is continuous. As mentioned, the first ingredient in the proof of Theorem 1.1 is an appropriate Caccioppoli-Stampacchia estimate and therefore we need to introduce the classic Stampacchia truncation operators [22]: for $h \in \mathbb{R}$ and $k > 0$, we set

$$T_{h,k} \colon \mathbb{R} \to [0,k], \qquad T_{h,k}(t) = \min\{k, (t-h)_+\}. \tag{2.2}$$

Lemma 2.1 (Caccioppoli inequality for truncations)**.** *If $v \in W^{1,p}(B_{2R})$ is a local energy solution to the homogeneous version of (1.1), then we have for every $h \in \mathbb{R}, k > 0$ and $\vartheta \in (1,2]$*

$$\nu \int_{B_R} |DT_{h,k}(v)|^p \, dx \leq \frac{k}{(\vartheta-1)R} \int_{B_{\vartheta R} \setminus B_R} \big|A(x,Dv)\big| \, dx. \tag{2.3}$$

Proof. Let η be a radial Lipschitz function, equal to one on B_R, zero outside $B_{\vartheta R}$, and decreasing linearly in the radial variable otherwise. Testing the weak formulation with $T_{h,k}(v)\eta \in W^{1,p}_0(B_{2R})$ gives

$$\nu \int_{B_R} |DT_{h,k}(v)|^p \, dx \leq - \int_{B_{\vartheta R} \setminus B_R} \big\langle T_{h,k}(v) A(x,Dv), D\eta \big\rangle \, dx \leq \frac{k}{(\vartheta-1)R} \int_{B_{\vartheta R} \setminus B_R} |A(x,Dv)| \, dx.$$

□

Proposition 2.2. *Every $v \in W^{1,p}(B_{2R}(x_0))$ as in Lemma 2.1 satisfies*

$$[v]_{C^{0,\gamma}(B_{R/2})} \leq c(n,p) \left(\frac{1}{\nu R} \int_{B_{2R}(x_0) \setminus B_R(x_0)} |A(x,Dv)| \, dx \right)^{\frac{1}{p-1}}. \tag{2.4}$$

Proof. Fix $x, y \in B_{R/2}$ and first suppose that $0 < r := |x-y| \leq R/4$. Choose in (2.3) $\vartheta = 2$ and levels

$$h = \min\{v(x), v(y)\}, \qquad k = |v(x) - v(y)| \qquad \text{so that} \qquad h + k = \max\{v(x), v(y)\}; \tag{2.5}$$

we can assume without loss of generality that $k > 0$. For every $r < s < 2r$, both x and y lie in $B_s(x)$, and again the maximum principle gives

$$\min_{\partial B_s(x)} T_{h,k}(v) = 0, \qquad \max_{\partial B_s(x)} T_{h,k}(v) = \operatorname*{osc}_{\partial B_s(x)} T_{h,k}(v) = |v(x) - v(y)|;$$

for almost every such s, the restriction of $T_{h,k}(v)$ to $\partial B_s(x)$ belongs to $W^{1,p}(\partial B_s(x))$ and its tangential gradient $D_\tau T_{h,k}(v)$ is the tangential projection of $DT_{h,k}(v)$. Since $p - 1 > n - 1$, Sobolev-Morrey's inequality on the connected sphere $\partial B_s(x)$, see [19, Lemma 2.10], followed by Hölder's inequality, gives

$$\begin{aligned} k = |v(x) - v(y)| &\leq c\, s^{1-\frac{n-1}{p-1}} \left(\int_{\partial B_s(x)} |D_\tau T_{h,k}(v)|^{p-1} \, d\mathcal{H}^{n-1} \right)^{\frac{1}{p-1}} \\ &\leq c\, s^{1-\frac{n-1}{p}} \left(\int_{\partial B_s(x)} |D_\tau T_{h,k}(v)|^{p} \, d\mathcal{H}^{n-1} \right)^{\frac{1}{p}}; \end{aligned}$$

taking p-th powers and integrating in s over $[r, 2r]$, we find

$$|v(x) - v(y)|^p \leq c\, r^{p-n} \int_{B_{2r}(x) \setminus B_r(x)} |D_\tau T_{h,k}(v)|^p \, dx \leq c\, r^{p-n} \int_{B_R(x_0)} |DT_{h,k}(v)|^p \, dx.$$

We apply (2.3) on the fixed pair B_R, B_{2R} and cancel $|v(x) - v(y)|$ on both sides, obtaining

$$|v(x) - v(y)|^{p-1} \leq c\, r^{p-n} \frac{1}{\nu R} \int_{B_{2R}(x_0) \setminus B_R(x_0)} \big|A(x,Dv)\big| \, dx.$$

For arbitrary $x, y \in B_{R/2}$, divide the line segment between them into four equal parts and add the four estimates after taking $(p-1)$-th roots. Since $r = |x-y|$ is now arbitrary, recalling the expression for γ proves (2.4). □

Remark 2.3. Using (2.3) with $\vartheta = 5/4$ in the proof of Proposition 2.2, we obtain the first inequality in

$$[v]_{C^{0,\gamma}(B_{R/2})} \le c(n,p)\left[\frac{1}{\nu R}\int_{B_{5R/4}\setminus B_R} |A(x,Dv)|\,dx\right]^{\frac{1}{p-1}} \le c\,R^{-\gamma}\operatorname*{osc}_{B_{3R/2}} v,$$

while the second inequality follows from the growth bound and the Caccioppoli inequality (2.1) applied to v with $\mu = 0$ and $k = \inf_{B_{3R/2}} v$. To estimate the last term, we observe that for almost every $s \in (3R/2, 2R)$, the maximum principle and Sobolev-Morrey's inequality on $\partial B_s(x_0)$ give

$$\operatorname*{osc}_{B_{3R/2}} v \le \operatorname*{osc}_{\partial B_s(x_0)} v \le c\,R^{\gamma}\left(\int_{\partial B_s(x_0)} |D_\tau v|^{p-1}\,d\mathcal{H}^{n-1}\right)^{\frac{1}{p-1}},$$

where D_τ denotes the tangential gradient with respect to spheres centred at x_0. Averaging in s we find

$$[v]_{C^{0,\gamma}(B_{R/2})} \le c\left[\frac{1}{R}\int_{B_{2R}\setminus B_R} |D_\tau v|^{p-1}\,dx\right]^{\frac{1}{p-1}}, \tag{2.6}$$

where $c = c(n,p,L/\nu)$.

2.2. **Comparison in the presence of a measure.** If $B_s \Subset \Omega$, each $\varphi \in W^{1,p}_0(B_s)$ has a continuous representative vanishing on ∂B_s, and

$$\|\varphi\|_{L^\infty(B_s)} \le c(n,p)s^{1-\frac{n}{p}}\|D\varphi\|_{L^p(B_s)}. \tag{2.7}$$

Since $p > n$, convergence in $W^{1,p}_0(B_s)$ implies uniform convergence. By density, (1.8) therefore holds for every test function $\varphi \in W^{1,p}_0(B_s)$ and in the integral against μ, we use the continuous representative of φ.

Lemma 2.4 (Comparison on a ball)**.** *Assume that, in addition to satisfying (1.2), $A(\,\cdot\,)$ also satisfies, for a parameter $\kappa > 0$,*

$$\big\langle A(x,\xi_1) - A(x,\xi_2), \xi_1 - \xi_2\big\rangle \ge \kappa|\xi_1 - \xi_2|^p. \tag{2.8}$$

Let $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ be an energy solution to (1.1) and $v \in u + W^{1,p}_0(B_R)$ be the homogeneous solution in B_R, for a ball $B_R \equiv B_R(x_0) \Subset \Omega$. Then we have

$$\|u-v\|_{L^\infty(B_R)} \le c\,R^{\gamma}\left[\frac{|\mu|(B_R)}{\kappa}\right]^{\frac{1}{p-1}}, \qquad \int_{B_R}|Du-Dv|^p\,dx \le c\,R^{\gamma}\left[\frac{|\mu|(B_R)}{\kappa}\right]^{p'}, \tag{2.9}$$

for a constant c depending only on n and p. Moreover, the zero extension $\widetilde{w}$ of $w = u - v$ to $\mathbb{R}^n$ satisfies

$$[\widetilde{w}]_{C^{0,\gamma}(\mathbb{R}^n)} \le c\left[\frac{|\mu|(B_R)}{\kappa}\right]^{\frac{1}{p-1}}. \tag{2.10}$$

Proof. It is standard to see that

$$\kappa\int_{B_R}|Dw|^p\,dx \le \int_{B_R} w\,d\mu \le |\mu|(B_R)\|w\|_{L^\infty(B_R)}$$

testing the difference of the two equations with w and using assumption (2.8). Taking the power p of both members of the inequality (2.7), inserting the estimate in the last display and simplifying $\|w\|_\infty$ (when it is non-zero) yields the first estimate in (2.9). Substituting this bound into the preceding display gives the second one.

To prove (2.10), we fix $h \in \mathbb{R}$ and $k > 0$, and we consider the truncated function $\psi = T_{h,k}(w) - T_{h,k}(0)$: note that ψ has zero boundary trace and satisfies $|\psi| \le k$, even in the case where $h < 0 < h + k$. Using ψ as a test function in the difference of the equations leads to

$$\int_{\mathbb{R}^n}|D\widetilde{\psi}|^p\,dx = \int_{B_R}|D\psi|^p\,dx = \int_{\{h<w<h+k\}\cap B_R}|Dw|^p\,dx \le \frac{1}{\kappa}\int_{B_R}|\psi|\,d|\mu| \le k\,\frac{|\mu|(B_R)}{\kappa}, \tag{2.11}$$

where $\widetilde{\psi}$ and $\widetilde{w}$ respectively denote the zero extensions of ψ and w to all of $\mathbb{R}^n$. We fix two points $x, y \in \mathbb{R}^n$ such that $\widetilde{w}(x) \ne \widetilde{w}(y)$, we choose the same levels as in (2.5) with $\widetilde{w}$ in place of v and we apply the Morrey inequality on $\mathbb{R}^n$ to $\widetilde{\psi}$: this gives

$$\big|\widetilde{\psi}(x) - \widetilde{\psi}(y)\big|^p = k^p \le c\,|x-y|^{p-n}\int_{\mathbb{R}^n}|D\widetilde{\psi}|^p\,dx \tag{2.12}$$

because from $T_{h,k}(h) = 0$ and $T_{h,k}(h+k) = k$, it follows that

$$\big|\widetilde{\psi}(x) - \widetilde{\psi}(y)\big| = \big|T_{h,k}(\widetilde{w}(x)) - T_{h,k}(\widetilde{w}(y))\big| = k = |\widetilde{w}(x) - \widetilde{w}(y)|.$$

Inserting now (2.11) into (2.12), dividing both sides by k and taking the $(p-1)$-th root of both sides establishes (2.10). □

If $A(\,\cdot\,)$ additionally satisfies (2.8), the proof of the Hölder regularity of Theorem 1.1 in the general case is now immediate.

Proof of (1.6) with κ in place of ν under the additional assumption (2.8). Let $v \in u + W^{1,p}_0(B_{2R})$ be the solution in B_{2R} to the homogeneous problem, as in Lemma 2.4, for some ball $B_{2R} \equiv B_{2R}(x_0) \Subset \Omega$. We use the triangle inequality and (2.10) and (2.6) to estimate

$$[u]_{C^{0,\gamma}(B_{R/2})} \le [u-v]_{C^{0,\gamma}(B_{R/2})} + [v]_{C^{0,\gamma}(B_{R/2})} \le c\left[\frac{|\mu|(B_{2R})}{\kappa} + \frac{1}{R}\int_{B_{2R}\setminus B_R} |D_\tau v|^{p-1}\,dx\right]^{\frac{1}{p-1}},$$

$c \equiv c(n,p,L/\nu)$. To replace the dependence on $D_\tau v$ with that on $D_\tau u$ on the right-hand side, we employ the triangle inequality and we observe that $|D_\tau v - D_\tau u| \le |Dv - Du|$ and the comparison (2.9) and Hölder's inequality give

$$\int_{B_{2R}} |Du-Dv|^{p-1}\,dx \le c\,R^{\frac{n}{p}}\left(\int_{B_{2R}} |Du-Dv|^p\,dx\right)^{\frac{1}{p'}} \le c\,R^{\frac{n}{p}+\frac{\gamma}{p'}}\frac{|\mu|(B_{2R})}{\kappa} = c\,R\frac{|\mu|(B_{2R})}{\kappa}; \tag{2.13}$$

thus

$$\frac{1}{R}\int_{B_{2R}\setminus B_R} |D_\tau v|^{p-1}\,dx \le \frac{c}{R}\int_{B_{2R}\setminus B_R} |D_\tau u|^{p-1}\,dx + c\,\frac{|\mu|(B_{2R})}{\kappa}.$$

□

To prove (1.6) under the sole assumptions (1.2), it suffices to construct a uniformly p-monotone field $\mathcal{A}$ such that $\mathcal{A}(x,Du) = A(x,Du)$ almost everywhere, since the weak formulation only involves the values of $A(\,\cdot\,)$ at Du. We do this in the following Lemma, inspired by [12, Section 3], perturbing the p-Laplacian vector field.

Lemma 2.5. *Let $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ and let $A(\,\cdot\,)$ satisfy (1.2). There exists a Carathéodory field $\mathcal{A}(\,\cdot\,)$ such that $\mathcal{A}(x,Du(x)) = A(x,Du(x))$ almost everywhere in Ω and*

$$|\mathcal{A}(x,\xi)| \le \Lambda|\xi|^{p-1}, \qquad \langle \mathcal{A}(x,\xi),\xi\rangle \ge \frac{\nu}{2}|\xi|^p, \tag{2.14}$$

where Λ/ν depends only on p and L/ν. Moreover, $\mathcal{A}(\,\cdot\,)$ satisfies (2.8) with $\kappa = 2^{1-p}\nu$.

Proof. Fix a representative in the class of Du and on the set $\{x \in \Omega : Du(x) \neq 0\}$ let

$$b(x) = \frac{2A(x,Du(x))}{\nu|Du(x)|^{p-1}} - e(x) \qquad \text{with} \qquad e(x) = \frac{Du(x)}{|Du(x)|},$$

and set $e(\,\cdot\,) = b(\,\cdot\,) = e_1$ on $\{x \in \Omega : Du(x) = 0\}$; we have $\langle b(x), e(x)\rangle \ge 1$ and $|b(x)| \le 1 + 2L/\nu$ for almost all $x \in \Omega$. For $x \in \Omega$ and $\xi \in \mathbb{R}^n$ we define the vector field, measurable in x and continuous in ξ,

$$\mathcal{A}(x,\xi) = \frac{\nu}{2}\left[|\xi|^{p-2}\xi + \frac{|\langle b(x),\xi\rangle|^{p-2}\langle b(x),\xi\rangle}{\langle b(x),e(x)\rangle^{p-1}}\,b(x)\right]. \tag{2.15}$$

From the bounds on b and $\langle b,e\rangle$ we immediately deduce the growth condition in (2.14) with $\Lambda = \nu[1+(1+2L/\nu)^p]/2$, whereas taking the inner product with ξ establishes the coercivity. To prove the equality, we observe that on the set $\{Du \neq 0\}$ we have $\langle b(x), Du(x)\rangle = |Du(x)|\langle b(x),e(x)\rangle$, and therefore

$$\mathcal{A}(x,Du(x)) = \frac{\nu|Du(x)|^{p-1}}{2}(e(x)+b(x)) = A(x,Du(x)),$$

while the same identity holds trivially on $\{Du = 0\}$. Finally, the contribution of the second term in brackets in (2.15) is monotone, since $t \mapsto |t|^{p-2}t$ is increasing: writing in short $s_k = \langle b(x),\xi_k\rangle$ for $k = 1,2$, it holds

$$\frac{(|s_1|^{p-2}s_1 - |s_2|^{p-2}s_2)(s_1-s_2)}{\langle b(x),e(x)\rangle^{p-1}} \ge 0.$$

Hence, for all $\xi_1,\xi_2 \in \mathbb{R}^n$, the uniform monotonicity for $\mathcal{A}(\,\cdot\,)$ follows from the standard uniform monotonicity of the p-Laplacian for $p \ge 2$:

$$\big\langle \mathcal{A}(x,\xi_1) - \mathcal{A}(x,\xi_2), \xi_1-\xi_2\big\rangle \ge \frac{\nu}{2}\big\langle |\xi_1|^{p-2}\xi_1 - |\xi_2|^{p-2}\xi_2, \xi_1-\xi_2\big\rangle \ge 2^{1-p}\nu|\xi_1-\xi_2|^p.$$

□

Proof of Proposition 1.2. Since $\mathcal{A}(x,Du)=A(x,Du)$ almost everywhere, where $\mathcal{A}$ is the field given by Lemma 2.5, u is a local energy solution of $-\operatorname{div}\mathcal{A}(x,Du)=\mu$ in Ω too. Applying the tangential estimate just proved to $\mathcal{A}$, with $\kappa=2^{1-p}\nu$, gives (1.6) where $c=c(n,p,L/\nu)$ by Lemma 2.5. To prove (1.7), let consider radii $0<\rho\le R/4$. Since $B_{2\rho}\subseteq B_{R/2}$, we have $\operatorname{osc}_{B_{2\rho}}u\le(4\rho)^{\gamma}[u]_{C^{0,\gamma}(B_{R/2})}$. Applying (2.1) on $B_\rho\subset B_{2\rho}$ with $k=\inf_{B_{2\rho}}u$ and using Young's inequality gives

$$\rho^{-\gamma}\int_{B_\rho}|Du|^p\,dx\le c[u]^p_{C^{0,\gamma}(B_{R/2})}+c\left[\frac{|\mu|(B_{2\rho})}{\nu}\right]^{p'}.$$

By (1.6) and Hölder's inequality on $B_{2R}\setminus B_R$, it follows that

$$\rho^{-\gamma}\int_{B_\rho}|Du|^p\,dx\le cR^{-\gamma}\int_{B_{2R}\setminus B_R}|D_\tau u|^p\,dx+c\left[\frac{|\mu|(B_{2R})}{\nu}\right]^{p'}. \tag{2.16}$$

Since $|D_\tau u|\le|Du|$ on B_ρ, this gives (1.7). Estimate (2.16), together with $|D_\tau u|\le|Du|$, also proves (1.5) for $0<\rho\le R/4$; for $R/4<\rho\le R$, both estimates follow from elementary estimates. □

Proof of Proposition 1.3. We first construct a sequence of coefficients b_k and corresponding solutions w_k to

$$-\operatorname{div}(b_k|Dw_k|^{p-2}Dw_k)=0\qquad\text{in } B_1\equiv B_1(0), \tag{2.17}$$

with $w_k=x_1$ on ∂B_1. The coefficients $b_k(\,\cdot\,)$ will have common positive lower and upper bounds, while

$$\sup_k\int_{B_1}|Dw_k|^p\,dx<\infty\qquad\text{and}\qquad\int_{B_{1/2}}|Dw_k|^{p(1+\delta)}\,dx\xrightarrow[k\to+\infty]{}\infty.$$

In a second stage, in order to obtain a single solution, we adapt the gluing argument used by Faraco in the proof of [9, Corollary 1.2] and place rescaled copies of the solutions w_k in pairwise disjoint balls accumulating at the origin.

Construction of the approximate solutions. For $t,s>0$, denote by $M(t,s)$ the $2\times n$ matrix with first row te_1 and second row se_2. Starting from $\delta_{M(1,1)}$, we construct a sequence of discrete probability measures by two successive rank-one splittings at each iteration. Given $\delta>0$, set $h:=1-2^{-p\delta/2}$ and define

$$\lambda_1:=\frac{1-h}{2-h},\qquad\lambda_2:=\frac{1-h}{2^{p-1}-h},\qquad\theta:=\lambda_1\lambda_2.$$

Since $(1-h)^2=2^{-p\delta}$, this choice gives

$$2^{p(1+\delta)}\theta=\frac{2^p}{(2-h)(2^{p-1}-h)}>1. \tag{2.18}$$

Notice that the weights satisfy $(1-\lambda_1)h+2\lambda_1=1$ and $(1-\lambda_2)h+2^{p-1}\lambda_2=1$.

The iterative step. Suppose that the matrix $M(t,t^{p-1})$ carries mass m, that is, consider $m\delta_{M(t,t^{p-1})}$.

Step 1) Keeping the second row fixed, write $M(t,t^{p-1})$ as a convex combination of two matrices that differ only in the first row:

$$M(t,t^{p-1})=(1-\lambda_1)M(ht,t^{p-1})+\lambda_1M(2t,t^{p-1}),$$

accordingly, we split the Dirac mass as $m\,\delta_{M(t,t^{p-1})}\longmapsto(1-\lambda_1)m\,\delta_{M(ht,t^{p-1})}+\lambda_1m\,\delta_{M(2t,t^{p-1})}$. The first Dirac is not modified again; the second is split further in the next Step 2.

Step 2) Keeping the first row fixed, we split the second matrix obtained in *Step 1* as

$$M(2t,t^{p-1})=(1-\lambda_2)M(2t,ht^{p-1})+\lambda_2M(2t,(2t)^{p-1}).$$

We decompose the mass λ_1m putting the weight $\lambda_1(1-\lambda_2)m$ at $M(2t,ht^{p-1})$ and the remaining mass $\lambda_1\lambda_2m=\theta m$ at $M(2t,(2t)^{p-1})$: $\lambda_1m\,\delta_{M(2t,t^{p-1})}\longmapsto\lambda_1(1-\lambda_2)m\,\delta_{M(2t,ht^{p-1})}+\lambda_1\lambda_2m\,\delta_{M(2t,(2t)^{p-1})}$. Combining the two steps, one iteration replaces the mass $m\,\delta_{M(t,t^{p-1})}$ by

$$(1-\lambda_1)m\,\delta_{M(ht,t^{p-1})}+\lambda_1(1-\lambda_2)m\,\delta_{M(2t,ht^{p-1})}+\theta m\,\delta_{M(2t,(2t)^{p-1})}.$$

This replacement has the following properties:

- the total mass of the measure is preserved, since $(1-\lambda_1)+\lambda_1(1-\lambda_2)+\theta=1$;
- the barycenter is also preserved. Indeed, if $M=(1-\lambda)A+\lambda B$, then replacing $m\,\delta_M$ by $(1-\lambda)m\,\delta_A+\lambda m\,\delta_B$ does not change the total mass nor the weighted sum of the matrices, since $(1-\lambda)mA+\lambda mB=mM$; therefore, their ratio remains unchanged. Since at every step, each of the two splittings we make is of this form, the three masses keep the barycenter in $M(t,t^{p-1})$;

- in both splittings (the first row in Step 1, the second in Step 2), we replace a matrix by two matrices that differ in a single row; consequently, their difference has rank one.

We repeat this construction only on the third atom, which has mass θm and lies at $M(2t,(2t)^{p-1})$, while the first two Dirac masses, which carry a total mass of $(1-\theta)m$, are left untouched.

The iteration. We start with mass 1 at $M(1,1)$, $\delta_{M(1,1)}$, and recursively apply the two steps described above. At the j-th iteration, $j=0,\dots,k-1$, this matrix is $M(2^j,2^{j(p-1)})$ (that is, $t=2^j$) and it carries mass θ^j, since each iteration multiplies the mass of the surviving matrix by θ. Iterating only on the last atom, after k steps we obtain

$$\nu_k := \sum_{j=0}^{k-1} \theta^j \Big[(1-\lambda_1)\delta_{M(h2^j,\,2^{j(p-1)})} + \lambda_1(1-\lambda_2)\delta_{M(2^{j+1},\,h2^{j(p-1)})}\Big] + \theta^k \delta_{M(2^k,\,2^{k(p-1)})}, \tag{2.19}$$

with barycenter $M(1,1)$. Write $\nu_k = \sum_{i=1}^{2k+1} m_i\,\delta_{M_i}$, where $M_i = M(t_i,s_i)$ and $m_i>0$ is the mass of the atom M_i. At each step, the atoms at $M(ht,t^{p-1})$ and $M(2t,ht^{p-1})$ have ratios s_i/t_i^{p-1} equal to h^{1-p} and $2^{1-p}h$, respectively; the ratio is 1 at the last atom $M(2^k,2^{k(p-1)})$. Thus, for every i,

$$\nu := 2^{1-p}h \le \frac{s_i}{t_i^{p-1}} \le h^{1-p} =: L. \tag{2.20}$$

The ellipticity ratio $L/\nu = 2^{p-1}h^{-p}$ is independent of k. For $0<\delta\le 1$, the concavity of $t\mapsto 1-2^{-pt/2}$ on $[0,1]$ gives $h = 1-2^{-p\delta/2} \ge (1-2^{-p/2})\delta$, whence

$$\frac{L}{\nu} \le \frac{2^{p-1}}{(1-2^{-p/2})^p}\,\delta^{-p};$$

moreover, $\lambda_1<1/2$ and $\lambda_2<2^{1-p}$, so $\theta<2^{-p}$. Combining this with (2.18) and the preceding bound yields

$$0 < -\log_2\theta - p < p\delta \le c(p)(L/\nu)^{-1/p}, \qquad 0<\delta\le 1.$$

The homogeneous solutions constructed below therefore show that the gain over p in a uniform local higher integrability estimate for the gradient can be arbitrarily small as L/ν increases.

For any fixed $k\in\mathbb{N}$, enumerate the atoms as after (2.19). Since ν_k is obtained from a Dirac mass by finitely many rank-one splittings, we may apply [2, Proposition 2.3]: for every sufficiently small $0<\varepsilon<\frac12\min_{i\neq j}|M_i-M_j|$, there exists a Lipschitz map $(v_\varepsilon,z_\varepsilon): B_{1/2}\to\mathbb{R}^2$ with $v_\varepsilon = x_1$ and $z_\varepsilon = x_2$ on $\partial B_{1/2}$ such that, using the notation

$$E_i^\varepsilon := \{x\in B_{1/2} : |D(v_\varepsilon,z_\varepsilon)(x) - M_i| < \varepsilon\} = \{x\in B_{1/2} : D(v_\varepsilon,z_\varepsilon)(x)\in B_\varepsilon(M_i)\},$$

where $B_\varepsilon(A)$ is the ball with center the matrix A with respect to the Euclidean norm $|\cdot|$ on $\mathbb{R}^{2\times n}$, these sets are pairwise disjoint, they cover $B_{1/2}$ up to a set of measure zero and we have

$$|E_i^\varepsilon| = m_i|B_{1/2}| \quad \text{for every } i=1,2,\dots,2k+1.$$

The boundary condition allows us to extend $(v_\varepsilon,z_\varepsilon)$ to a Lipschitz map on B_1 by setting $v_\varepsilon = x_1$ and $z_\varepsilon = x_2$ on $B_1\setminus B_{1/2}$.

Passage to exact solutions. Using the affine extension $(v_\varepsilon,z_\varepsilon)(x) = (x_1,x_2)$ on $B_1\setminus B_{1/2}$, we consider the divergence-free field on B_1

$$F_\varepsilon = (\partial_2 z_\varepsilon, -\partial_1 z_\varepsilon, 0,\dots,0).$$

We also set $b_\varepsilon = s_i/t_i^{p-1}$ on each E_i^ε and $b_\varepsilon = 1$ elsewhere in B_1 and define in short $P_\varepsilon := b_\varepsilon|Dv_\varepsilon|^{p-2}Dv_\varepsilon$.

On the set E_i^ε, the two fields P_ε and F_ε are both close to s_ie_1: for the second one, we have $|F_\varepsilon - s_ie_1| \le |Dz_\varepsilon - s_ie_2| < \varepsilon$. Since $b_\varepsilon t_i^{p-1} = s_i$ there and $|Dv_\varepsilon - t_ie_1|<\varepsilon$, the local Lipschitz regularity of $\xi\mapsto b_\varepsilon|\xi|^{p-2}\xi$ near $t_ie_1\neq 0$ gives

$$\big|b_\varepsilon|Dv_\varepsilon|^{p-2}Dv_\varepsilon - b_\varepsilon t_i^{p-1}|e_1|^{p-2}e_1\big| = |P_\varepsilon - s_ie_1| \le c_k\varepsilon \qquad \text{a.e. on } E_i^\varepsilon,$$

where c_k is independent of i and ε, since k is fixed and there are only finitely many atoms. Since both fields equal e_1 on $B_1\setminus B_{1/2}$, the triangle inequality gives

$$\big\|F_\varepsilon - P_\varepsilon\big\|_{L^{p'}(B_1)} \le (c_k+1)\varepsilon|B_{1/2}|^{\frac{1}{p'}} \xrightarrow[\varepsilon\searrow 0]{} 0.$$

Let now $w_\varepsilon \in x_1 + W^{1,p}_0(B_1)$ solve $-\operatorname{div}(b_\varepsilon |Dw_\varepsilon|^{p-2}Dw_\varepsilon) = 0$ in B_1; we test the weak formulation with $w_\varepsilon - v_\varepsilon \in W^{1,p}_0(B_1)$ using that F_ε is divergence free, and this quite easily implies that $Dw_\varepsilon - Dv_\varepsilon \to 0$ in L^p. For $p \geq 2$, indeed, uniform monotonicity and Hölder's inequality delivers that for each fixed k

$$\|Dw_\varepsilon - Dv_\varepsilon\|^{p-1}_{L^p(B_1)} \leq c(p,\delta)\big\|F_\varepsilon - P_\varepsilon\big\|_{L^{p'}(B_1)} \xrightarrow[\varepsilon \searrow 0]{} 0.$$

For $1 < p < 2$, the weighted monotonicity and Hölder's inequality provide

$$\|Dw_\varepsilon - Dv_\varepsilon\|_{L^p(B_1)} \leq \frac{c(p)}{\nu}\big(\|Dw_\varepsilon\|_{L^p(B_1)} + \|Dv_\varepsilon\|_{L^p(B_1)}\big)^{2-p}\|F_\varepsilon - P_\varepsilon\|_{L^{p'}(B_1)};$$

for fixed k the gradients Dv_ε are uniformly bounded since they stay in an ε-neighborhood of a fixed finite set on $B_{1/2}$ and equal e_1 elsewhere, and comparing the energy of w_ε with that of x_1 also gives a uniform bound on $\|Dw_\varepsilon\|_{L^p(B_1)}$. Thus $Dw_\varepsilon - Dv_\varepsilon \to 0$ in $L^p(B_1)$ in this case as well. Now consider the atom $M_i = M(2^k, 2^{k(p-1)})$ in (2.19): its mass is $m_i = \theta^k$, so $|E^\varepsilon_i| = \theta^k |B_{1/2}|$. Since $|Dv_\varepsilon - 2^k e_1| < \varepsilon$ on E^ε_i, for $\varepsilon < 2^{k-1}$ we obtain by set inclusion

$$\big|\{x \in B_{1/2} : |Dv_\varepsilon(x)| > 2^{k-1}\}\big| \geq \theta^k |B_{1/2}|.$$

For each k, the convergence of $Dw_\varepsilon - Dv_\varepsilon$ to zero in measure allows us to choose $0 < \varepsilon_k < 1/k$ such that

$$\big|\{x \in B_{1/2} : |Dw_{\varepsilon_k} - Dv_{\varepsilon_k}| > 2^{k-2}\}\big| \leq \frac{1}{2}\theta^k |B_{1/2}| \implies \big|\{x \in B_{1/2} : |Dw_{\varepsilon_k}| > 2^{k-2}\}\big| \geq \frac{1}{2}\theta^k |B_{1/2}|.$$

Relabeling $w_k := w_{\varepsilon_k}$ and $b_k := b_{\varepsilon_k}$ defines the functions sought in (2.17); indeed, recalling that $2^{p(1+\delta)}\theta > 1$ by (2.18), the preceding super-level estimate gives

$$\int_{B_{1/2}} |Dw_k|^{p(1+\delta)}\,dx \geq 2^{(k-2)p(1+\delta)}\big|\{x \in B_{1/2} : |Dw_k(x)| > 2^{k-2}\}\big| \geq \frac{c(n)}{4^{p(1+\delta)}}(2^{p(1+\delta)}\theta)^k \xrightarrow[k \to +\infty]{} +\infty.$$

On the other hand, w_k minimizes the weighted p-energy with boundary values x_1, and using the same function as a competitor and recalling the bound in (2.20), we have

$$\int_{B_1} |Dw_k|^p\,dx \leq c(n,p,\delta). \tag{2.21}$$

Construction of a single solution. Extend w_k by x_1 and b_k by 1 outside B_1. By the maximum principle, $|w_k| \leq 1$ in B_1; moreover, w_k is p-harmonic in $B_1 \setminus \overline{B}_{1/2}$ and equals x_1 on ∂B_1. The boundary estimate in [1, Theorem 1.2], together with the uniform p-energy bound above, therefore gives

$$\sup_{\partial B_1} |Dw_k| \leq c(n,p,h).$$

$b_k|Dw_k|^{p-2}Dw_k$ is divergence free on either side of ∂B_1, so its negative divergence is the signed measure

$$\big\langle |Dw_k|^{p-2}Dw_k - e_1, n\big\rangle\, \mathcal{H}^{n-1} \llcorner\, \partial B_1, \tag{2.22}$$

where n is the outer unit normal and, with a little abuse of notation, we denote with Dw_k the interior trace of the gradient on ∂B_1.

Finally, for $j \geq 1$ let $x_j = 2^{-j}e_1$ and $r_j = 2^{-j-2}$, so that the balls $B_{r_j}(x_j)$ are pairwise disjoint, lie in B_1, and accumulate at the origin. For each j, we can choose k_j large enough so that

$$r_j^n \int_{B_{1/2}} |Dw_{k_j}|^{p(1+\delta)}\,dx \geq 1.$$

Set $\varphi_k(y) = w_k(y) - y_1$ for $y \in \mathbb{R}^n$: $\varphi_k \in W^{1,p}(\mathbb{R}^n)$ and $\varphi_k = 0$ outside B_1. For $x \in B_1$, we define

$$u(x) = x_1 + \sum_{j=1}^{\infty} r_j \varphi_{k_j}\Big(\frac{x - x_j}{r_j}\Big), \qquad a(x) = 1 + \sum_{j=1}^{\infty}\Big[b_{k_j}\Big(\frac{x - x_j}{r_j}\Big) - 1\Big].$$

Since $\varphi_k = 0$ and $b_k = 1$ outside B_1, and the balls $B_{r_j}(x_j)$ are pairwise disjoint, each sum has at most one nonzero term at any given point x. Let u_m, a_m be the partial sums of the series defining u, a: $u_m - x_1 \in W^{1,p}_0(B_1)$ and, for $\ell > m$, (2.21) gives

$$\int_{B_1} |Du_\ell - Du_m|^p\,dx = \sum_{j=m+1}^{\ell} r_j^n \int_{B_1} |Dw_{k_j} - e_1|^p\,dx \leq c(n,p,\delta) \sum_{j=m+1}^{\infty} r_j^n < +\infty.$$

Thus, by Poincaré's inequality, u_m converges in $W^{1,p}(B_1)$ to its pointwise limit u and for each $B_{r_j}(x_j)$,

$$Du(x) = Dw_{k_j}\Big(\frac{x - x_j}{r_j}\Big) \qquad \text{for almost every } x \in B_{r_j}(x_j).$$

Let σ_k denote the measure in (2.22) and set $\mu_j := r_j^{n-1}(T_j)_\# \sigma_{k_j}$, where $T_j(y) = x_j + r_j y$. Then

$$|\mu_j|(\partial B_{r_j}(x_j)) = r_j^{n-1} \int_{\partial B_1} \left|\langle |Dw_{k_j}(y)|^{p-2} Dw_{k_j}(y) - e_1, y\rangle\right| d\mathcal{H}^{n-1}(y) \le c(n,p,\delta) r_j^{n-1},$$

whence the series $\mu = \sum_j \mu_j$ converges in total variation. Since $Du_m \to Du$ in $L^p(B_1)$, $a_m \to a$ almost everywhere and $a_m \le L$, then $a_m |Du_m|^{p-2} Du_m$ converges to $a|Du|^{p-2}Du$ in $L^{p'}(B_1)$, and (1.9) is a consequence of passing to the limit in $-\operatorname{div}(a_m|Du_m|^{p-2}Du_m) = \sum_{j=1}^m \mu_j$. Theorem 1.1 now yields $u \in C^{0,\gamma}_{\mathrm{loc}}(B_1)$ if $p > n$, but at the same time, the choice of k_j gives

$$\int_{B_{r_j/2}(x_j)} |Du|^{p(1+\delta)}\, dx = r_j^n \int_{B_{1/2}} |Dw_{k_j}|^{p(1+\delta)}\, dx \ge 1.$$

For every $0 < r < 1$, all sufficiently large j satisfy $B_{r_j/2}(x_j) \subset B_r(0)$ and summing over these disjoint balls gives (1.10). □

We conclude with the proof of Remark 1.4: letting $B_j = B_{r_j}(x_j)$ be the balls used in the construction and g_j the density of the measure concentrated on ∂B_j, as in (2.22), write

$$\mu = \sum_j g_j\, \mathcal{H}^{n-1} \llcorner \partial B_j, \qquad \sup_j \|g_j\|_{L^\infty(\partial B_j)} < \infty.$$

The field $a|Du|^{p-2}Du - e_1$ is divergence free in B_j, and its interior normal trace on ∂B_j is g_j, whence $\int_{\partial B_j} g_j\, d\mathcal{H}^{n-1} = 0$. For every $\varphi \in C_c^\infty(B_1(0))$, if we denote by $(\varphi)_{B_j}$ its average over B_j, we thus have

$$\int_{\partial B_j} \varphi g_j\, d\mathcal{H}^{n-1} = \int_{\partial B_j} (\varphi - (\varphi)_{B_j}) g_j\, d\mathcal{H}^{n-1}.$$

Since the g_j are uniformly bounded and the balls B_j are pairwise disjoint, using the trace inequality on every ball B_j and Poincaré's inequality, it follows that

$$\left|\int_{B_1} \varphi\, d\mu\right| = \left|\sum_j \int_{\partial B_j} (\varphi - (\varphi)_{B_j}) g_j\, d\mathcal{H}^{n-1}\right| \le c \sum_j \int_{B_j} |D\varphi|\, dx \le c \int_{B_1} |D\varphi|\, dx.$$

We have shown that $D\varphi \mapsto -\int_{B_1} \varphi\, d\mu$ is a bounded linear functional on the subspace of $L^1(B_1;\mathbb{R}^n)$ formed by gradients of test functions; by the Hahn–Banach theorem, it extends to all of $L^1(B_1;\mathbb{R}^n)$ and by duality there exists $G \in L^\infty(B_1;\mathbb{R}^n)$ such that $\operatorname{div} G = \mu$ in B_1.

Appendix A. An alternative proof

We here report an alternative proof of the Hölder estimate of Theorem 1.1, whose origin is described in the Statement on the use of AI tools below. The proof is shorter than the one in Paragraphs 2.1 and 2.2 and provides estimate (1.4) with a constant only depending on n and p, but it does not separate the tangential contribution of the gradient; it is similar to the proof for weak subsolutions in [10, Proposition 5.1], which the author became aware of only after obtaining it.

An alternative proof of (1.4). Fix $x, y \in B_{R/2}(x_0)$, take $T_{h,k}$ as in (2.2) with levels h, k as in (2.5) with u in place of v (but still with $k > 0$ without loss of generality) and η a radial cutoff function as in the proof of Lemma 2.1 for $\vartheta = 2$. Testing (1.8) with $T_{h,k}(u)\eta$ gives

$$\nu \int_{B_R(x_0)} |D(T_{h,k}(u))|^p\, dx \le k\left[\frac{1}{R}\int_{B_{2R}(x_0)\setminus B_R(x_0)} |A(x,Du)|\, dx + |\mu|(B_{2R}(x_0))\right];$$

since $|T_{h,k}(u(x)) - T_{h,k}(u(y))| = k$, Morrey's inequality on $B_R(x_0)$ yields

$$k^p \le c(n,p)|x-y|^{p-n} \int_{B_R(x_0)} |D(T_{h,k}(u))|^p\, dx.$$

Combining the two estimates and dividing by k gives (1.4). □

Statement on the use of AI tools. The author used ChatGPT (Astra 6 and Astra 6.2) for language editing, literature searches, and LaTeX formatting. In addition, the author acknowledges its direct contribution to the proofs of Lemma 2.5 (which removes the necessity of the strong monotonicity assumption (2.8) for the main result) and Proposition 1.3, for which it found the construction given in the proof following the author's suggestion to argue by analogy with earlier constructions in the literature; we mention in particular [2, 9]. Finally, when tested by the author on the day of its release to assess its

performance, Astra 6 produced the short proof given in the Appendix, which we reproduced with the sole change of adapting the notation to that of the present paper.

The author developed the remaining mathematical arguments and results: he independently verified all the arguments in the article and takes sole responsibility for its content and any remaining errors.

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