Rough vector fields with vanishing Lie brackets

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Abstract

We consider two vector fields $X, Y \in W^{1,p}_{loc}(\mathbb{R}^d) \cap \mathbf{L}^\infty$ with bounded divergence. If $p \geq 2$, this paper shows that $DX\,Y - DY\,X = 0$ almost everywhere if and only if their Regular Lagrangian Flows commute. If instead $1 \leq p < 2$, we present an example of two divergence free locally Sobolev vector fields in $\mathbb{R}^2$ with vanishing Lie brackets whose flows do not commute. In the regime $2 \leq p < \infty$ this paper also yields a quantitative commutation result not depending on the Lipschitz constant for vector fields with $[X, Y] \neq 0$, which is new even in the smooth setting.

1. Introduction

If $X \in W^{1,p}_{loc}(\mathbb{R}^d) \cap \mathbf{L}^\infty$ is a Sobolev vector field with $\operatorname{div} X \in \mathbf{L}^\infty$, DiPerna and Lions [4] (and then Ambrosio [1] in the BV setting) proved that there is a unique natural flow map $\Phi^t_X : \mathbb{R}^d \to \mathbb{R}^d$ such that for all $T > 0$ there is $C > 0$ such that there holds

$$(\Phi^t_X)_\sharp \mathcal{L}^d \leq C\,\mathcal{L}^d \qquad \text{for all } |t| \leq T \tag{1.1}$$

and for almost every x, the map $t \mapsto \Phi^t_X(x)$ is an absolutely continuous solution to

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\Phi^t_X(x) = X(\Phi^t_X(x)) \\ \Phi^0_X(x) = x. \end{cases} \tag{1.2}$$

The map Φ^t_X is called the *Regular Lagrangian flow* of the vector field X. The classical Frobenius theorem states that the (classical) flows Φ^t_X, Φ^s_Y of two smooth vector fields $X, Y : \mathbb{R}^d \to \mathbb{R}^d$ commute

$$\Phi^t_X \circ \Phi^s_Y = \Phi^s_Y \circ \Phi^t_X, \qquad \forall\, t, s \geq 0$$

if and only if their Lie bracket vanishes, i.e.

$$[X, Y] := DX\,Y - DY\,X = 0. \tag{1.3}$$

The aim of this paper is to extend the classical Frobenius theorem to the setting of Regular Lagrangian Flows.

Let $X, Y \in W^{1,p}$ be two bounded Sobolev fields, with bounded divergence. If their Regular Lagrangian Flows commute, it is not difficult (see e.g. [2]) to show that $[X, Y] = 0$ almost everywhere, therefore this paper and the previous literature on the topic only focus on the opposite implication. Assuming that (1.3) is satisfied almost everywhere, the following results are known about the commutation of their Regular Lagrangian flows.

(1) A first paper [10] proves that the flows commute when X, Y are locally Lipschitz.

2020 *Mathematics Subject Classification.* 35F35,37C10,46E35.

Key words and phrases. Sobolev vector fields, Regular Lagrangian flows, Lie brackets.

(2) Later, [2] proved that the flows commute assuming additionally that for every $t \in \mathbb{R}$, the flow map Φ^t_X is weakly Lie differentiable in the direction Y, with locally bounded derivative.
(3) The paper [12] proved the result when one of the vector fields is Lipschitz and the other belongs to $W^{1,p}$ and admits a unique Regular Lagrangian flow.
(4) Finally [11] proved the commutativity of the flows for bounded, continuous Sobolev vector fields with bounded divergence on $\mathbb{R}^2$.

The main result of this paper is as follows. Here d is any dimension $d \geq 2$.

Theorem 1.1. *Let $X, Y \in W^{1,p}_{loc}(\mathbb{R}^d) \cap \mathbf{L}^\infty$ with $\operatorname{div} X$, $\operatorname{div} Y \in \mathbf{L}^\infty$, and assume that $[X,Y] = 0$. If $p \geq 2$, then for all $t, s > 0$,*

$$\Phi^t_X \circ \Phi^s_Y(x) = \Phi^s_Y \circ \Phi^t_X(x) \qquad \text{for a.e. } x. \tag{1.4}$$

If instead $1 \leq p < 2$, there are $X, Y \in W^{1,p}_{loc}(\mathbb{R}^d) \cap \mathbf{L}^\infty$ with zero divergence, such that (1.4) fails for x in an open set with positive measure.

Some comments are in order.

1.1. *Lie differential and commutation.* To understand why a different approach is needed compared to the existing literature, assume that the flow Φ^t_X is Lie-differentiable in the direction of Y, with locally bounded derivative (see [2]). For the purpose of this sketch, assume that there exists a function $\vec{\rho}_t \in \mathbf{L}^\infty([0,T] \times \mathbb{R}^d; \mathbb{R}^d)$ such that

$$\frac{1}{h}\Big(\Phi^t_X \circ \Phi^h_Y - \Phi^t_X\Big) \longrightarrow \vec{\rho}_t \qquad \text{in } \mathbf{L}^1_{loc} \text{ as } h \to 0^+.$$

At least formally $\vec{\rho}_t$ satisfies

$$\partial_t \vec{\rho}_t = DX(\Phi^t_X)\, \vec{\rho}_t, \qquad \vec{\rho}_0 = Y. \tag{1.5}$$

Notice that by applying Gronwall's estimate along almost every trajectory of X, the above equation has a unique solution $\vec{\rho}_t$ in $\mathbf{L}^\infty$. However, due to the assumption $[X,Y] = 0$, it is immediate to check that $Y(\Phi^t_X(x))$ is, in fact, another bounded solution of (1.5), and therefore

$$\vec{\rho}_t = Y(\Phi^t_X).$$

Therefore

$$\frac{\mathrm{d}}{\mathrm{d}s} \Phi^t_X \circ \Phi^s_Y(x) = Y(\Phi^t_X \circ \Phi^s_Y(x))$$

so that $s \mapsto \Phi^t_X \circ \Phi^s_Y(x)$ is a trajectory of the flow of Y for a.e. x, starting at the point $\Phi^t_X(x)$. Since it satisfies (1.1) and (1.2), again by uniqueness of the DiPerna Lions flows one must have that it is equal to $\Phi^s_Y \circ \Phi^t_X(x)$ at a.e. x.

Results about the differentiability of the Regular Lagrangian Flow of Sobolev fields are available in two dimensions under continuity assumptions [7] (and, classically, for Lipschitz vector fields in general dimension), but it is well known, see [7, 9] that the RLF of Sobolev vector fields need not be differentiable or inherit any Sobolev regularity. In particular, there seems not to be an easy way to fix the previous argument in a general Sobolev setting.

1.2. *Quantitative commutation.* For $p > 2$, strictly greater than the critical exponent, the proof also provides a quantitative commutation result for vector fields in $W^{1,p}$, as follows.

Theorem 1.2. *Let $p > 2$ and let $X, Y \in W^{1,p}_{loc}(\mathbb{R}^d) \cap \mathbf{L}^\infty$ with $\operatorname{div} X$, $\operatorname{div} Y \in \mathbf{L}^\infty$. Then*

$$\sup_{|s|,|t| \leq T} \int_{B_r} \Big|\Phi^t_X \circ \Phi^s_Y(x) - \Phi^s_Y \circ \Phi^t_X(x)\Big| \,\mathrm{d}x \lesssim \left[\log\left(1 + \frac{1}{\|[X,Y]\|_{\mathbf{L}^1(B_R)}}\right)\right]^{-p} \tag{1.6}$$

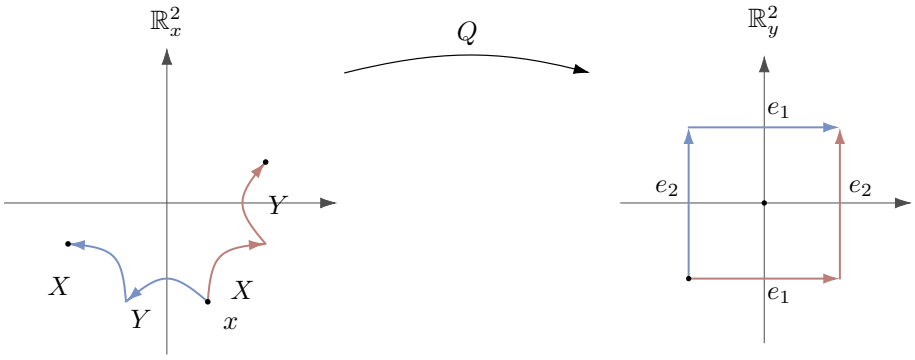


FIGURE 1. Example of two divergence free fields $X, Y \in W^{1,p}_{loc}(\mathbb{R}^2)$, $p < 2$, whose flows do not commute. They are constructed as pullbacks of e_i through a branching map $Q : \mathbb{R}^2_x \to \mathbb{R}^2_y$. For example, in polar coordinates, one can take $Q(\rho, \theta) = (\rho, 2\theta)$.

where

$$R = r + T(\|X\|_\infty + \|Y\|_\infty).$$

The implicit constant depends on d, p, T, r and the norms of X, Y in B_{R+1}.

In the critical case $X, Y \in W^{1,2}$, one readily deduces the existence of an analogous quantitative result, using the compactness and stability properties of the Regular Lagrangian Flows. However, the uniform modulus that we can obtain when $p = 2$ is not as clean as in the case $p > 2$, as it will be clear from the proof below.

Since $X, Y \in \mathbf{L}^\infty$, by a standard localization argument it is easy to see that it is equivalent to prove the commutativity results for vector fields in $W^{1,p}(\mathbb{R}^d)$. To avoid technical issues, we will restrict to this class from now on.

1.3. *The regime $p < 2$.* The example of non-commuting flows in the regime $p < 2$ is surprisingly simple, and it is constructed using a double cover of the plane $Q : \mathbb{R}^2_x \to \mathbb{R}^2_y$ (see Figure 1). The example was essentially already known from [8] (see also [5, 6]), here we just need to check its Sobolev regularity, which is immediate. Say that, in polar coordinates

$$Q(\rho, \theta) = (\rho, 2\theta).$$

Vector fields with vanishing Lie brackets X, Y producing non-commuting flows can be defined as the pullback of $e_1 = (1, 0), e_2 = (0, 1)$ via the map Q:

$$X = (DQ)^{-1}\, e_1, \qquad Y = (DQ)^{-1}\, e_2.$$

It is easy to see that they satisfy all the conditions of Theorem 1.1 if and only if $p < 2$. Their flows do not commute: the reason is that, while clearly e_1, e_2 commute in $\mathbb{R}^2_y$, when they are pulled back via the branching map Q, the points $\Phi^t_X \circ \Phi^s_Y$ and $\Phi^s_Y \circ \Phi^t_X$ may well end up in different branches of the map Q in the original space $\mathbb{R}^2_x$. The exponent $p = 2$ arises as a natural threshold also in [6] for the same geometric reasons.

AI use

We started working on the problem in early 2025, but eventually got stuck and moved away from the topic. In September 2026, we were very surprised when GPT-6 Astra suggested (in a single response) the counterexample in the regime $p < 2$. Moreover, this time after extensive interaction with GPT-6 Astra, in late September

2026 we obtained a first proof, completely different from the one presented here, that worked for $p > 2$ in arbitrary dimension. Finally, in late October 2026, through further interactions with the same model, we obtained the present proof, which we then decided to rework and write up in its current form. The paper was written entirely by the authors.

2. **Short time estimate**

We start with some preliminary results, needed for the following.

2.1. *Maximal functions and truncations.* Recall that if $f \in W^{1,p}(\mathbb{R}^d;\mathbb{R}^m)$ then the *maximal function* of Df is defined as

$$g_f(x) := \mathcal{M}(|Df|)(x) \;=\; \sup_{\rho>0} \frac{1}{|B_\rho(x)|}\int_{B_\rho(x)} |Df(y)|\,\mathrm{d}y.$$

For the following statements, we refer to [13]. If $p > 1$, one has

$$\|g_f\|_{\mathbf{L}^p} \;\le\; C_{d,p}\,\|f\|_{W^{1,p}}.$$

Then there exists a null set N such that for all $x, y \in \mathbb{R}^d \setminus N$, there holds

$$|f(x) - f(y)| \;\le\; C_d\,(g_f(x) + g_f(y))|x - y|.$$

On the set

$$G_\lambda := \Big\{x \in \mathbb{R}^d \mid g_f(x) \le \lambda/(2C_d)\Big\} \setminus N,$$

the function f is λ-Lipschitz, therefore, by the Kirszbraun Lipschitz extension theorem $f_{|G_\lambda}$ admits a Lipschitz extension f_λ, with

$$\|f_\lambda\|_\infty \le \|f\|_\infty, \qquad \mathrm{Lip}\, f_\lambda \le \lambda, \qquad f_\lambda(x) = f(x) \quad \text{for a.e. } x \in G_\lambda.$$

2.2. *Lusin-Lipschitz estimates.* It is well known that the Regular Lagrangian Flow of Sobolev vector fields satisfies Lipschitz estimates outside a small set, see [3]. We recall here a version in our setting.

Lemma 2.1. *Let $p > 1$ and $X \in W^{1,p}(\mathbb{R}^d)$ be a vector field with bounded divergence, and let g_X be its maximal function. Then for every $\lambda > 0$ and $0 \le h \le T$ there is a set $\mathcal{B}_{\lambda,h}$ such that*

$$|\mathcal{B}_{\lambda,h}| \le 2C\frac{h}{\lambda}\int_{\{g_X \ge \lambda/(2h)\}} g_X(x)\,\mathrm{d}x$$

and

$$x \mapsto \Phi^h_X(x) \qquad \textit{is } e^{2C_d\lambda}\textit{-Lipschitz in } \mathbb{R}^d \setminus \mathcal{B}_{\lambda,h}.$$

Proof. Up to a mollification argument, we can assume that X is a smooth vector field. Then we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}|\Phi^t_X(x) - \Phi^t_X(y)| &\le |X(\Phi^t_X(x)) - X(\Phi^t_X(y))|\\ &\le\; C_d\,(g_X(\Phi^t_X(x)) + g_X(\Phi^t_X(y)))|\Phi^t_X(x) - \Phi^t_X(y)|.\end{aligned}$$

By Gronwall's Lemma,

$$|\Phi^h_X(x) - \Phi^h_X(y)| \le |x - y| \exp\left(C_d\int_0^h (g_X(\Phi^t_X(x)) + g_X(\Phi^t_X(y)))\,\mathrm{d}t\right).$$

Define for $x \in \mathbb{R}^d$

$$G_{\lambda,h}(x) := \int_0^h g_X(\Phi^t_X(x))\,\mathrm{d}t$$

and define the set

$$\mathcal{B}_{\lambda,h} := \Big\{x \in \mathbb{R}^d \mid G_{\lambda,h}(x) \ge \lambda\Big\}.$$

Then for a.e. $x, y \in \mathbb{R}^d \setminus \mathcal{B}_{\lambda,h}$, there holds

$$|\Phi_X^h(x) - \Phi_X^h(y)| \;\le\; e^{(2C_d\lambda)}\,|x-y|.$$

Moreover,

$$\begin{aligned}
|\mathcal{B}_{\lambda,h}| &\le \frac{1}{\lambda}\int_{\mathcal{B}_{\lambda,h}} G_{\lambda,h}(x)\,\mathrm{d}x \\
&= \frac{1}{\lambda}\int_{\mathcal{B}_{\lambda,h}}\int_0^h g_X(\Phi_X^t(x))\mathbf{1}_{\{g_X(\Phi_X^t(x))\le\lambda/(2h)\}}(x)\,\mathrm{d}t\,\mathrm{d}x \\
&\qquad + \frac{1}{\lambda}\int_{\mathcal{B}_{\lambda,h}}\int_0^h g_X(\Phi_X^t(x))\mathbf{1}_{\{g_X(\Phi_X^t(x))>\lambda/(2h)\}}(x)\,\mathrm{d}t\,\mathrm{d}x \\
&\le \frac{1}{2}|\mathcal{B}_{\lambda,h}| + \frac{Ch}{\lambda}\int_{\mathbb{R}^d} g_X(x)\mathbf{1}_{\{g_X(x)\ge\lambda/(2h)\}}(x)\,\mathrm{d}x.
\end{aligned}$$

where we have used (1.1) to perform the change of variable in the last inequality. Since the first term in the last line can be absorbed in the left hand side, this concludes the proof. □

2.3. *A short-time estimate.* The main goal of this section is to prove the following preliminary proposition, that is a short time estimate on the commutativity of flows when $t, s \sim h$.

Proposition 2.2. *Let $X, Y \in W^{1,p}(\mathbb{R}^d)\cap\mathbf{L}^\infty$ be vector fields with bounded divergence. Fix $h > 0$ and $r > 0$. Then for all $0 \le t, s \le h$ there holds*

$$\int_{B_r}\left|\Phi_X^t\circ\Phi_Y^s(x) - \Phi_Y^s\circ\Phi_X^t(x)\right|\mathrm{d}x \lesssim h\cdot\left|\left\{g \ge h^{-1}\right\}\cap B_R\right| + h^2\left\|[X,Y]\right\|_{\mathbf{L}^1(B_R)} \tag{2.1}$$

where

$$R \equiv R_{r,h} := r + h(|X|_\infty + |Y|_\infty) \tag{2.2}$$

and $g = g_X + g_Y$.

The following is a preliminary estimate for smooth vector fields.

Lemma 2.3. *Let $X, Y : \mathbb{R}^d \to \mathbb{R}^d$ be bounded, Lipschitz vector fields with Lipschitz constant at most λ. Then there holds, for all $0 \le t, s \le h := 2C_d\lambda^{-1}$,*

$$\int_{B_r}\left|\Phi_X^t\circ\Phi_Y^s(x) - \Phi_Y^s\circ\Phi_X^t(x)\right|\mathrm{d}x \;\lesssim_d\; t\,s\,e^{\lambda(s+t)}\left\|[X,Y]\right\|_{\mathbf{L}^1(B_{R_{r,h}})}. \tag{2.3}$$

where the implicit constant is a dimensional constant.

Proof. Without loss of generality we prove the result for $t, s \ge 0$. By a standard mollification argument, it is sufficient to prove the result for smooth vector fields. First consider the quantity

$$\mathrm{E}^t(z) := D\Phi_X^t(z)Y(z) - Y(\Phi_X^t(z)), \qquad t \ge 0, \quad z \in \mathbb{R}^d$$

Differentiating with respect to t, we obtain

$$\partial_t\mathrm{E}^t(z) = DX(\Phi_X^t(z))\mathrm{E}^t(z) + [X,Y](\Phi_X^t(z)), \qquad \mathrm{E}_0(z) = 0.$$

Therefore, by Gronwall's Lemma we obtain

$$|\mathrm{E}^t(z)| \le e^{\lambda t}\int_0^t \left|\,[X,Y](\Phi_X^p(z))\right|\mathrm{d}p. \tag{2.4}$$

Next, consider the function

$$[0,s] \ni u \mapsto \mathrm{C}^{t,u}(x) := \Phi_X^t\circ\Phi_Y^u(x) - \Phi_Y^u\circ\Phi_X^t(x).$$

Its derivative with respect to u is

$$\begin{aligned}\partial_u \mathrm{C}^{t,u}(x) &= D\Phi_X^t(\Phi_Y^u(x))\, Y(\Phi_Y^u(x)) - Y(\Phi_Y^u \circ \Phi_X^t(x)) \\ &= \Big(D\Phi_X^t(\Phi_Y^u(x))\, Y(\Phi_Y^u(x)) - Y(\Phi_X^t \circ \Phi_Y^u(x))\Big) \\ &\qquad + \Big(Y(\Phi_X^t \circ \Phi_Y^u(x)) - Y(\Phi_Y^u \circ \Phi_X^t(x))\Big).\end{aligned}$$

Therefore, by triangular inequality, using (2.4)

$$\begin{aligned}|\partial_u \mathrm{C}^{t,u}(x)| &\le |\mathrm{E}^t(\Phi_Y^u(x))| + \lambda |\mathrm{C}^{t,u}(x)| \\ &\le e^{\lambda t} \int_0^t |\,[X,Y](\Phi_X^p \circ \Phi_Y^u(x))\,|\,\mathrm{d}p + \lambda |\mathrm{C}^{t,u}(x)|\end{aligned}$$

Therefore again by Gronwall's Lemma we obtain

$$|\Phi_X^t \circ \Phi_Y^s - \Phi_Y^s \circ \Phi_X^t(x)| \le e^{\lambda(s+t)} \int_0^s \int_0^t |\,[X,Y](\Phi_X^p \circ \Phi_Y^u(x))\,|\,\mathrm{d}p\,\mathrm{d}u$$

A final integration in x yields

$$\begin{aligned}\int_{B_r} |\,\Phi_X^t \circ \Phi_Y^s - \Phi_Y^s \circ \Phi_X^t(x)\,|\,\mathrm{d}x &\le e^{\lambda(s+t)} \int_0^s \int_0^t \left(\int_{B_r} |\,[X,Y](\Phi_X^p \circ \Phi_Y^u(x))|\,\mathrm{d}x \right) \mathrm{d}p\,\mathrm{d}u \\ &\le C\, ts\, e^{\lambda(s+t)} \int_{B_R} |\,[X,Y](x)\,|\,\mathrm{d}x\end{aligned}$$

where in the last inequality we use the compression bounds on the flows of X, Y for small times $0 \le t, s \le 2C_d\lambda^{-1}$, which imply

$$(\Phi_X^p)_\sharp \mathcal{L}^d, \quad (\Phi_Y^u)_\sharp \mathcal{L}^d \le C\mathcal{L}^d$$

where C is a dimensional constant. □

The following standard Lemma is an estimate of how much the flow of X can differ from the flow of the truncated field X_λ.

Lemma 2.4. *If $X \in W^{1,p}(\mathbb{R}^d) \cap \mathbf{L}^\infty$ has bounded divergence, and X_λ is its λ-Lipschitz truncation, then there holds for all $0 \le t \le T$*

$$\int_{B_r} \left|\Phi_X^t(x) - \Phi_{X_\lambda}^t(x)\right| \mathrm{d}x \le C\, t\, e^{\lambda t} \left|\{X \ne X_\lambda\} \cap B_{r+t|X|_\infty}\right|.$$

Proof. We have

$$\begin{aligned}\partial_t\Big(\Phi_X^t(x) - \Phi_{X_\lambda}^t(x)\Big) &= \Big(X(\Phi_X^t(x)) - X_\lambda(\Phi_X^t(x))\Big) \\ &\quad + \Big(X_\lambda(\Phi_X^t(x)) - X_\lambda(\Phi_{X_\lambda}^t(x))\Big)\end{aligned}$$

By estimating the second term using the Lipschitz regularity of X_λ, again by Gronwall's Lemma we obtain

$$|\Phi_X^t(x) - \Phi_{X_\lambda}^t(x)| \le e^{\lambda t} \int_0^t |X(\Phi_X^s(x)) - X_\lambda(\Phi_X^s(x))|\,\mathrm{d}s$$

Integrating in x

$$\begin{aligned}\int_{B_r} |\Phi_X^t(x) - \Phi_{X_\lambda}^t(x)|\,\mathrm{d}x &\le C\, t\, e^{\lambda t} \int_{B_R} |X(x) - X_\lambda(x)|\,\mathrm{d}x \\ &\le \|X\|_\infty\, C\, t\, e^{\lambda t} |\{X \ne X_\lambda\} \cap B_R|\end{aligned}$$

where in the first inequality we used the compression bounds (1.1) for the Regular Lagrangian flow of X. □

Proof of Proposition 2.2. We set

$$\lambda := 2C_d/h$$

and we consider the truncated fields X_λ, Y_λ. We can write

$$\begin{aligned}\int_{B_r} \left|\Phi^t_X \circ \Phi^s_Y(x) - \Phi^s_Y \circ \Phi^t_X(x)\right| \mathrm{d}x \le &\int_{B_r} \left|\Phi^t_{X_\lambda} \circ \Phi^s_{Y_\lambda}(x) - \Phi^s_{Y_\lambda} \circ \Phi^t_{X_\lambda}(x)\right| \mathrm{d}x \\ &+ \int_{B_r} \left|\Phi^t_{X_\lambda} \circ \Phi^s_{Y_\lambda}(x) - \Phi^t_X \circ \Phi^s_Y(x)\right| \mathrm{d}x \qquad (2.5)\\ &+ \int_{B_r} \left|\Phi^s_{Y_\lambda} \circ \Phi^t_{X_\lambda}(x) - \Phi^s_Y \circ \Phi^t_X(x)\right| \mathrm{d}x.\end{aligned}$$

Notice that by applying Lemma 2.4 twice, there holds

$$\begin{aligned}\int_{B_r} \left|\Phi^t_{X_\lambda} \circ \Phi^s_{Y_\lambda}(x) - \Phi^t_X \circ \Phi^s_Y(x)\right| \mathrm{d}x &\lesssim \int_{B_r} \left|\Phi^t_{X_\lambda} \circ \Phi^s_{Y_\lambda}(x) - \Phi^t_{X_\lambda} \circ \Phi^s_Y(x)\right| \mathrm{d}x \\ &\quad + \int_{B_r} \left|\Phi^t_{X_\lambda} \circ \Phi^s_Y(x) - \Phi^t_X \circ \Phi^s_Y(x)\right| \mathrm{d}x \\ &\lesssim e^{\lambda t} \int_{B_r} \left|\Phi^s_{Y_\lambda}(x) - \Phi^s_Y(x)\right| \mathrm{d}x \, + \, t e^{\lambda t} \, |\{X \neq X_\lambda\} \cap B_R| \\ &\lesssim s\, e^{\lambda t} e^{\lambda s} |\{Y \neq Y_\lambda\} \cap B_R| \, + \, t\, e^{\lambda t} |\{X \neq X_\lambda\} \cap B_R|\end{aligned}$$

The third term in (2.5) can be estimated in an entirely symmetrical way, while for the first one can use Lemma 2.3. Combining all these estimates, and recalling the properties of the Lipschitz truncation of Section 2.1, one gets

$$\begin{aligned}\int_{B_r} \left|\Phi^t_X \circ \Phi^s_Y(x) - \Phi^s_Y \circ \Phi^t_X(x)\right| \mathrm{d}x &\le h^2 \int_{B_R} |\,[X_\lambda, Y_\lambda](x)\,| \,\mathrm{d}x + h|B_R \cap \{g \ge h^{-1}\}| \\ &\lesssim h^2 \int_{B_R} |\,[X, Y]\,| \,\mathrm{d}x + h|B_R \cap \{g \ge h^{-1}\}|\end{aligned}$$

where we used that $|DX_\lambda|, |DY_\lambda| \le \lambda = 2C_d/h$ to pass from the first to the second line and moreover that, on the good set $\{g \le h^{-1}\}$, by locality of the weak gradient one has $DX = DX_\lambda$ and $DY = DY_\lambda$ almost everywhere.

□

3. A quantitative commutation estimate

In this section we prove the first part of Theorem 1.1, i.e. that the flows commute whenever $p \ge 2$.

The following is the key proposition: the initial commutator is split into 4^k smaller commutators, up to removing a set with controlled measure. Here we recall that g is the sum of the maximal functions of the derivatives of X, Y:

$$g = g_X + g_Y$$

as defined in Section 2.1.

Proposition 3.1. *Let $X, Y \in W^{1,p}(\mathbb{R}^d) \cap \mathbf{L}^\infty$, with $p > 1$ and bounded divergence. Let $0 \le s, t \le T$, and let $\{\lambda_j\}_{j\in\mathbb{N}}$ be a sequence of positive numbers. Then, for every*

$r > 0$ and $k > 0$,

$$\begin{aligned}
&\int_{B_r} \left|\Phi_X^t \circ \Phi_Y^s(x) - \Phi_Y^s \circ \Phi_X^t(x)\right| \mathrm{d}x \\
&\lesssim \sum_{j=1}^{k} \frac{2^j}{\lambda_j} \int_{\{g > 2^j \lambda_j/(2T)\}} g(x)\,\mathrm{d}x \\
&\quad + 4^k \left(\prod_{j=1}^{k} e^{c_d \lambda_j}\right) \int_{B_{R_k}} \left|\Phi_X^{\frac{t}{2^k}} \circ \Phi_Y^{\frac{s}{2^k}}(x) - \Phi_Y^{\frac{s}{2^k}} \circ \Phi_X^{\frac{t}{2^k}}(x)\right| \mathrm{d}x.
\end{aligned} \tag{3.1}$$

where here

$$R_k = r + (1 - 2^{-k})T(|X|_\infty + |Y|_\infty).$$

and c_d is a dimensional constant, and the implicit constant depends on r, T, d.

Proof of Proposition 3.1. **1.** For convenience of notation, let us introduce, for all $j \in \mathbb{N}$,

$$A_j := \Phi_X^{t/2^j}, \qquad B_j := \Phi_Y^{s/2^j},$$
$$\mathbf{\Gamma}_j := A_j \circ B_j - B_j \circ A_j.$$

Notice the recursive relation

$$\begin{aligned}
\mathbf{\Gamma}_j =& A_{j+1} \circ \mathbf{\Gamma}_{j+1} \circ B_{j+1} \\
&+ A_{j+1} \circ B_{j+1} \circ \mathbf{\Gamma}_{j+1} \\
&+ \mathbf{\Gamma}_{j+1} \circ B_{j+1} \circ A_{j+1} \\
&+ B_{j+1} \circ \mathbf{\Gamma}_{j+1} \circ A_{j+1}.
\end{aligned} \tag{3.2}$$

Here we use the convention that for two maps $D_1, D_2 : \mathbb{R}^d \to \mathbb{R}^d$,

$$D_1 \circ \mathbf{\Gamma}_j \circ D_2 \equiv D_1 \circ A_j \circ B_j \circ D_2 - D_1 \circ B_j \circ A_j \circ D_2.$$

Introduce the set of indices $\mathcal{I}^k := \{1,2,3,4\}^k$. An element $\alpha \in \{1,2,3,4\}^k$ is a vector of the form

$$\alpha = (\alpha^1, \dots, \alpha^k), \qquad \alpha^i \in \{1,2,3,4\}.$$

Expanding the first commutator $\mathbf{\Gamma}_0$ recursively for $k \geq 1$ iterations, we therefore obtain for every $k \geq 1$ that

$$\mathbf{\Gamma}_0 = \sum_{\alpha \in \mathcal{I}^k} (\mathcal{L}^1_{\alpha^1} \circ \dots \circ \mathcal{L}^k_{\alpha^k}) \circ \mathbf{\Gamma}_k \circ (\mathcal{R}^k_{\alpha^k} \circ \dots \circ \mathcal{R}^1_{\alpha^1}). \tag{3.3}$$

In particular, at step k, we decomposed $\mathbf{\Gamma}_0$ into the sum of 4^k commutators $\mathbf{\Gamma}_k$, left and right composed with some operators. Here $\mathcal{L}^j_{\alpha^j}$ is one of the four elements appearing in the left compositions in (3.2), more precisely:

$$\mathcal{L}^j_{\alpha^j} \in \Big\{A_j,\ A_j \circ B_j,\ \mathrm{Id},\ B_j\Big\}. \tag{3.4}$$

In an entirely similar way

$$\mathcal{R}^j_{\alpha^j} \in \Big\{B_j,\ \mathrm{Id},\ B_j \circ A_j,\ A_j\Big\}. \tag{3.5}$$

Let us name

$$\widetilde{\mathcal{L}}^k_\alpha := \mathcal{L}^1_{\alpha^1} \circ \dots \circ \mathcal{L}^k_{\alpha^k}, \qquad \widetilde{\mathcal{R}}^k_\alpha := \mathcal{R}^k_{\alpha^k} \circ \dots \circ \mathcal{R}^1_{\alpha^1}.$$

Our aim is to estimate at each step k the commutator $\mathbf{\Gamma}_k$ with the commutator $\mathbf{\Gamma}_{k+1}$, up to removing a small set where the new four left compositions $\mathcal{L}^{k+1}_p$, $p = 1,2,3,4$ are not e^{λ_i}-Lipschitz.

Notice that for every $j \geq 1$, $\mathcal{L}^j_p$ is a composition of the flows of X, Y for a total time of order 2^{-j}, as it immediately follows from (3.5). Recall that $g = g_X + g_Y$ is

the sum of the maximal functions of DX, DY, which are in $\mathbf{L}^p_{loc}$. By Lemma 2.1, for all $j \geq 0$ and $p \in \{1,2,3,4\}$, there exist sets $\mathcal{B}^j_p$ such that

$$|\mathcal{B}^j_p| \lesssim \frac{1}{\lambda_j 2^j} \int_{\{g \geq \lambda_j 2^j/(2T)\}} g(x)\, \mathrm{d}x$$

and

$$\mathcal{L}^j_p \quad \text{is } e^{c_d \lambda_j}\text{-Lipschitz on } \mathbb{R}^d \setminus \mathcal{B}^j_p. \tag{3.6}$$

Define the bad set:

$$\mathcal{B}_j := \bigcup_{\alpha \in \mathcal{I}^j} \bigcup_{p=1}^4 \left\{ x \in \mathbb{R}^d \;\middle|\; A_j \circ B_j \circ \widetilde{\mathcal{R}}^j_\alpha(x) \in \mathcal{B}^j_p \right\} \bigcup_{\alpha \in \mathcal{I}^j} \bigcup_{p=1}^4 \left\{ x \in \mathbb{R}^d \;\middle|\; B_j \circ A_j \circ \widetilde{\mathcal{R}}^j_\alpha(x) \in \mathcal{B}^j_p \right\}.$$

Therefore we estimate

$$|\mathcal{B}_j| \leq 2 \cdot 4 \cdot 4^j \cdot \frac{1}{\lambda_j 2^j} \int_{\{g \geq \lambda_j 2^j/(2T)\}} g(x)\, \mathrm{d}x. \tag{3.7}$$

The *good set* at stage k is then

$$\mathcal{G}_k := \mathbb{R}^d \setminus \bigcup_{j=1}^k \mathcal{B}_j.$$

Therefore, in the good region $\mathcal{G}_k$ we have that every appearing left factor $\mathcal{L}^j_p$ is e^{λ_j} Lipschitz on the points given by the right composition.

2. We want to show that for every $k' \leq k$, there holds

$$\int_{\mathcal{G}_k \cap B_r} |\mathbf{\Gamma}_0(x)|\, \mathrm{d}x \leq \sum_{\alpha \in \mathcal{I}^{k'}} \left(\prod_{j=1}^{k'} e^{c_d \lambda_j} \right) \int_{\mathcal{G}_k \cap B_r} |\mathbf{\Gamma}_{k'} \circ \widetilde{\mathcal{R}}^{k'}_\alpha|\, \mathrm{d}x. \tag{3.8}$$

This can be done, for example, by an easy induction. For $k' = 0$, the result is trivial (here for $k = 0$, we set $\mathcal{R}^0_\alpha = \mathrm{Id}$).

Assume to have proved (3.8) up to $k' \in \mathbb{N}$, where $1 \leq k' < k$. Decomposing $\mathbf{\Gamma}_{k'}$ as in (3.2), we obtain

$$\begin{aligned}
&\sum_{\alpha \in \mathcal{I}^{k'}} \left(\prod_{j=1}^{k'} e^{c_d \lambda_j} \right) \int_{\mathcal{G}_k \cap B_r} |\mathbf{\Gamma}_{k'} \circ \widetilde{\mathcal{R}}^{k'}_\alpha|\, \mathrm{d}x \\
&\quad \leq \sum_{p=1}^4 \sum_{\alpha \in \mathcal{I}^{k'}} \left(\prod_{j=1}^{k'} e^{c_d \lambda_j} \right) \int_{\mathcal{G}_k \cap B_r} |\mathcal{L}^{k'+1}_p \circ \mathbf{\Gamma}_{k'+1} \circ \mathcal{R}^{k'+1}_p \circ \widetilde{\mathcal{R}}^{k'}_\alpha|\, \mathrm{d}x \\
&\quad = \sum_{\alpha \in \mathcal{I}^{k'+1}} \left(\prod_{j=1}^{k'} e^{c_d \lambda_j} \right) \int_{\mathcal{G}_k \cap B_r} |\mathcal{L}^{k'+1}_{\alpha^{k'+1}} \circ \mathbf{\Gamma}_{k'+1} \circ \widetilde{\mathcal{R}}^{k'+1}_\alpha|\, \mathrm{d}x \\
&\quad \leq \sum_{\alpha \in \mathcal{I}^{k'+1}} \left(\prod_{j=1}^{k'+1} e^{c_d \lambda_j} \right) \int_{\mathcal{G}_k \cap B_r} |\mathbf{\Gamma}_{k'+1} \circ \widetilde{\mathcal{R}}^{k'+1}_\alpha|\, \mathrm{d}x.
\end{aligned} \tag{3.9}$$

Here in the equality of the third line we used the fact that if $\alpha \in \mathcal{I}^{k'}$ and $p = \alpha^{k'+1}$, then $\alpha' := (\alpha, p) \in \mathcal{I}^{k'+1}$, and by definition

$$\mathcal{R}^{k'+1}_p \circ \widetilde{\mathcal{R}}^{k'}_\alpha = \widetilde{\mathcal{R}}^{k'+1}_{\alpha'}$$

In the last inequality, we used the fact that $\mathcal{L}^{k'+1}_{\alpha^{k'+1}}$ is calculated only on pair of points where it is $e^{\lambda_{k'+1}}$-Lipschitz thanks to the definition of the good set $\mathcal{G}_k$. This proves the inductive step, so that (3.8) holds in particular for $k' = k$, for every $k \geq 1$.

Next, let $k \in \mathbb{N}$, apply (3.8) with $k' = k$, and write

$$\begin{aligned}\int_{B_r} |\mathbf{\Gamma}_0(x)|\,\mathrm{d}x &\lesssim \sum_{j=1}^{k} |\mathcal{B}_j| + \sum_{\alpha\in\mathcal{I}^k}\left(\prod_{j=1}^{k} e^{c_d\lambda_j}\right)\int_{\mathcal{G}_k\cap B_r} |\mathbf{\Gamma}_k\circ\widetilde{\mathcal{R}}^k_\alpha|\,\mathrm{d}x\\ &\lesssim \sum_{j=1}^{k}\frac{2^j}{\lambda_j}\int_{\{g\geq\lambda_j 2^j/(2T)\}} g(x)\,\mathrm{d}x \\ &\quad + 4^k\left(\prod_{j=1}^{k} e^{c_d\lambda_j}\right)\int_{B_{R_k}} |\mathbf{\Gamma}_k(x)|\,\mathrm{d}x.\end{aligned} \tag{3.10}$$

Here in the last step we used that every right factor $\widetilde{\mathcal{R}}^k_\alpha$ has uniformly bounded compression, since it is a composition of flows of time at most $2T$, and that the points $\widetilde{\mathcal{R}}^k_\alpha$ are contained in the ball B_{R_k}. □

We are now in the position of proving the main theorem for vector fields in $W^{1,2}$.

Theorem 3.2. *Let $X, Y \in W^{1,2}(\mathbb{R}^d)\cap\mathbf{L}^\infty$ with bounded divergence, and assume that $[X,Y]=0$. Then for all s,t, there holds*

$$\Phi^t_X\circ\Phi^s_Y(x) - \Phi^s_Y\circ\Phi^t_X(x) = 0 \qquad \text{for a.e. } x\in\mathbb{R}^d. \tag{3.11}$$

Proof. Let $r>0$ be fixed and take $0\leq s,t\leq T=1$. From Proposition 3.1, we deduce

$$\begin{aligned}&\int_{B_r}\left|\Phi^t_X\circ\Phi^s_Y(x) - \Phi^s_Y\circ\Phi^t_X(x)\right|\mathrm{d}x\\ &\lesssim \sum_{j=1}^{k}\frac{2^j}{\lambda_j}\int_{\{g>2^j\lambda_j/(2T)\}} g(x)\,\mathrm{d}x\\ &\quad + 4^k\left(\prod_{j=1}^{k} e^{c_d\lambda_j}\right)\int_{B_{R_k}}\left|\Phi^{\frac{t}{2^k}}_X\circ\Phi^{\frac{s}{2^k}}_Y(x) - \Phi^{\frac{s}{2^k}}_Y\circ\Phi^{\frac{t}{2^k}}_X(x)\right|\mathrm{d}x.\end{aligned} \tag{3.12}$$

Let $\eta>0$ be fixed but sufficiently small, and $M>0$. Define

$$\lambda_j := \eta + M2^{-j}$$

where η is fixed and sufficiently small to be chosen later. The first term is then bounded by

$$\begin{aligned}\sum_{j=1}^{k}\frac{2^j}{\lambda_j}\int_{\{g>2^j\lambda_j/(2T)\}} g(x)\,\mathrm{d}x &= \int_{\{g\geq M/(2T)\}} g(x)\sum_{j=1}^{k}\frac{2^j}{\lambda_j}\mathbf{1}_{\{g>2^j\lambda_j/(2T)\}}(x)\,\mathrm{d}x\\ &\leq \frac{1}{\eta}\int_{\{g\geq M/(2T)\}} g(x)\sum_{j=1}^{\lfloor\log_2(2Tg/\eta)\rfloor} 2^j\,\mathrm{d}x\\ &\leq \frac{4T}{\eta^2}\int_{\{g\geq M/(2T)\}} g^2(x)\,\mathrm{d}x.\end{aligned} \tag{3.13}$$

For the second term, we use Proposition 2.2, so that we obtain

$$\begin{aligned} 4^k \left(\prod_{j=1}^{k} e^{c_d \lambda_j} \right) \int_{B_{R_k}} \left| \Phi_X^{\frac{t}{2^k}} \circ \Phi_Y^{\frac{s}{2^k}}(x) - \Phi_Y^{\frac{s}{2^k}} \circ \Phi_X^{\frac{t}{2^k}}(x) \right| \mathrm{d}x \\ \lesssim 4^k e^{c_d(k\eta + 2M)} 2^{-k} |\{g \geq 2^k/T\}| \\ \lesssim e^{2c_d M} e^{c_d k \eta} 2^{-k} \int_{\{g \geq 2^k/T\}} g^2(x)\, \mathrm{d}x \end{aligned} \tag{3.14}$$

If η is sufficiently small ($\eta \leq \log(2)/c_d$), the right hand side goes to zero as $k \to +\infty$ and M is fixed. Then, we let M go to infinity, so that the right hand side of (3.13) goes to zero as well. This proves the result. □

For $p > 2$, there is a clean quantitative version using a different choice of the sequence λ_i.

Theorem 3.3. *Let $X, Y \in W^{1,p}(\mathbb{R}^d) \cap \mathbf{L}^\infty$ with $2 < p < \infty$, with bounded divergence. Then for all $0 \leq s, t \leq T$, there holds*

$$\int_{B_r} \left| \Phi_X^t \circ \Phi_Y^s(x) - \Phi_Y^s \circ \Phi_X^t(x) \right| \mathrm{d}x \lesssim \left[\log \left(1 + \frac{1}{\|[X,Y]\|_{\mathbf{L}^1(B_R)}} \right) \right]^{-p}. \tag{3.15}$$

where R is as in Proposition 3.1.

Proof. We apply the same argument as in the proof of the theorem above. This time, choose

$$\lambda_j = \lambda 2^{-\alpha j}, \qquad \alpha = \frac{p-2}{2p}.$$

From (3.12), we can now estimate the first term in the right hand side as

$$\begin{aligned} \sum_{j=1}^{k} \frac{2^j}{\lambda_j} \int_{\{g > 2^j \lambda_j/(2T)\}} g(x)\, \mathrm{d}x &\leq \frac{1}{\lambda^p} \left(\sum_{j=1}^{k} 2^{j\left(1-\frac{p}{2}\right)} \right) \int_{\mathbb{R}^d} g^p(x)\, \mathrm{d}x \\ &\leq \frac{C_p}{\lambda^p} \int_{\mathbb{R}^d} g^p(x)\, \mathrm{d}x \end{aligned} \tag{3.16}$$

where C_p is a constant depending on p. The second term in (3.12) is now estimated by Proposition 2.2 as

$$\begin{aligned} 4^k \left(\prod_{j=1}^{k} e^{c_d \lambda_j} \right) \int_{B_{R_k}} \left| \Phi_X^{\frac{t}{2^k}} \circ \Phi_Y^{\frac{s}{2^k}}(x) - \Phi_Y^{\frac{s}{2^k}} \circ \Phi_X^{\frac{t}{2^k}}(x) \right| \mathrm{d}x \\ \lesssim 4^k e^{\widetilde{C}_p \lambda} \left(2^{-k} |\{g \geq 2^k\}| + 2^{-2k} \|[X,Y]\|_{\mathbf{L}^1(B_R)} \right) \\ \lesssim e^{\widetilde{C}_p \lambda} 2^{-(p-1)k} \int_{\mathbb{R}^d} g^p(x)\, \mathrm{d}x + e^{\widetilde{C}_p \lambda} \|[X,Y]\|_{\mathbf{L}^1(B_R)}. \end{aligned} \tag{3.17}$$

Then, send k to infinity. Since $p > 2$ the first term in (3.17) vanishes, and we get

$$\int_{B_r} \left| \Phi_X^t \circ \Phi_Y^s(x) - \Phi_Y^s \circ \Phi_X^t(x) \right| \mathrm{d}x \leq \frac{C_p}{\lambda^p} \int_{\mathbb{R}^d} g^p(x)\, \mathrm{d}x + e^{\widetilde{C}_p \lambda} \|[X,Y]\|_{\mathbf{L}^1(B_R)}. \tag{3.18}$$

Finally, optimizing in λ we obtain the result. □

4. **Non-commutativity for** $1 \le p < 2$

Proposition 4.1. *Let $p \in [1,2)$. Then there exist divergence free vector fields $X, Y \in W^{1,p}_{loc}(\mathbb{R}^2)$ with $[X,Y] = 0$ such that their regular Lagrangian flows do not commute.*

Proof. The vector fields X, Y are constructed using a double covering of the plane. Consider the map $Q : \mathbb{R}^2 \to \mathbb{R}^2$ which doubles the angle. In polar coordinates,

$$Q(\rho, \theta) := (\rho, 2\theta).$$

The map Q is clearly measure preserving for the Lebesgue measure, since it is in particular measure preserving on every circle $\{|x| = \rho\}$. Consider the vector fields e_1, e_2 in $\mathbb{R}^2$, constant and parallel to the coordinates axis, with unit norm. Clearly, e_1, e_2 are divergence free and $[e_1, e_2] = 0$. We define the pullbacks of the vector fields e_1, e_2 by

$$X(x) := (DQ(x))^{-1} e_1, \qquad Y(x) := (DQ(x))^{-1} e_2.$$

Q is measure preserving and has Jacobian $\det(DQ(x)) = 2$ for all $x \neq 0$, therefore X, Y are divergence free. Moreover, the pullback of vector fields naturally preserves the Lie brackets, therefore

$$[X, Y] = 0.$$

In polar coordinates, one has

$$X = \cos(2\theta) e_r - \tfrac{1}{2} \sin(2\theta) e_\theta, \qquad Y = \sin(2\theta) e_r + \tfrac{1}{2} \cos(2\theta) e_\theta. \tag{4.1}$$

It is therefore not difficult to show that

$$|D\,X(x)| + |D\,Y(x)| \sim \frac{1}{|x|}$$

therefore $X, Y \in W^{1,p}_{loc}(\mathbb{R}^2)$ if and only if $p \in [1,2)$. Consider a small neighborhood U of the point $(-1,-1) \in \mathbb{R}^2_y$ and consider trajectories starting from $y \in U$, obtained by first following e_1 for time 2 and e_2, also for time 2, and conversely. Then the corresponding pulled back trajectories for X, Y in the x-plane do not commute, because they belong to different branches of Q (see Figure 1). To see this, fix $y_0 = (-1,-1)$ and choose any $x_0 \in Q^{-1}(y_0)$. Consider the two polygonal paths in $\mathbb{R}^2_y$:

$$\gamma_1 : (-1,-1) \to (1,-1) \to (1,1),$$
$$\gamma_2 : (-1,-1) \to (-1,1) \to (1,1)$$

each one made of two segments. Since both paths avoid the origin, they can be uniquely lifted through Q, starting from x_0. By construction, the endpoints of these lifts are respectively $\Phi^2_Y \circ \Phi^2_X(x_0)$ and $\Phi^2_X \circ \Phi^2_Y(x_0)$. The closed curve obtained by concatenating γ_1 with the reverse of γ_2 winds once around the origin. Since Q is a two-branch covering of $\mathbb{R}^2$, the two lifts have distinct endpoints. Hence

$$\Phi^2_Y \circ \Phi^2_X(x_0) \neq \Phi^2_X \circ \Phi^2_Y(x_0).$$

The same argument applies to all starting points y_0 in a sufficiently small neighborhood of $(-1,-1)$, since the corresponding rectangles still enclose the origin. Thus the two compositions differ on an open set of positive Lebesgue measure. □

Acknowledgements. The authors are partially supported by the GNAMPA–INDAM Project 2026, "Problemi di controllo e regolarità per evoluzione di insiemi"

Declaration of interests. The author reports there are no competing interests to declare.

Data availability statement. Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

References

[1] L. Ambrosio, *Transport equation and Cauchy problem for BV vector fields*, Invent. Math. **158** (2004), 227–260.

[2] M. Colombo and R. Tione, *On the commutativity of flows of rough vector fields*, J. Math. Pures Appl. (9) **159** (2022), 294–312.

[3] G. Crippa and C. De Lellis, *Estimates and regularity results for the DiPerna–Lions flow*, J. Reine Angew. Math. **616** (2008), 15–46.

[4] R. J. DiPerna and P.-L. Lions, *Ordinary differential equations, transport theory and Sobolev spaces*, Invent. Math. **98** (1989), 511–547.

[5] B. Fuglede, *Conditions for two self-adjoint operators to commute or to satisfy the Weyl relation*, Math. Scand. **51** (1982), 163–178.

[6] J. Heinonen and S. Keith, *Flat forms, bi-Lipschitz parametrizations, and smoothability of manifolds*, Publ. Math. Inst. Hautes Études Sci. **113** (2011), 1–37.

[7] E. Marconi, *Differentiability properties of the flow of 2d autonomous vector fields*, J. Differential Equations **301** (2021), 330–352.

[8] E. Nelson, *Analytic vectors*, Ann. of Math. (2) **70** (1959), 572–615.

[9] P.-E. Jabin, *Critical non-Sobolev regularity for continuity equations with rough velocity fields*, J. Differential Equations **260** (2016), no. 5, 4739–4757.

[10] F. Rampazzo and H. J. Sussmann, *Commutators of flow maps of nonsmooth vector fields*, J. Differential Equations **232** (2007), 134–175.

[11] A. Rebucci and M. Zizza, *Geometric interpretation of the vanishing Lie Bracket for two-dimensional rough vector fields*, J. Funct. Anal. **289** (2025), no. 1, 110919.

[12] C. Rigoni, E. Stepanov, and D. Trevisan, *Lie brackets of nonsmooth vector fields and commutation of their flows*, J. Lond. Math. Soc. (2) **106** (2022), 1232–1256.

[13] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton Mathematical Series, vol. 30, Princeton University Press, Princeton, NJ, 1970.

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