ON THE REGULARITY OF THE SOLUTION AND THE FREE BOUNDARY IN A WEIGHTED $p-$LAPLACIAN PROBLEM

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ABSTRACT. In this paper, we study the regularity of the minimizer in the Alt-Caffarelli type minimum problem for the "weighted" $p-$Laplace operator $(1 < p < \infty)$ with free boundary:

$$\min\left\{\int_\Omega (w|\nabla u|^p + \psi\,\chi_{\{u>0\}}) \,:\, u \in W^{1,p}(\Omega),\ u \geq 0,\ u = g \ \text{ on } \ \partial\Omega\right\},$$

where w and ψ are two given nonnegative functions on Ω and g is a nonnegative boundary datum. More precisely, under the assumptions that w is a C^2 function with $w \geq w_{\min} > 0$ and ψ belongs to $L^q_{loc}(\Omega)$ for some $q > \frac{N}{p}$, we will show that a minimizer u is locally $\alpha-$Hölderian with $\alpha = 1 - \frac{N}{pq}$. If ψ belongs to $L^\infty_{loc}(\Omega)$ and is bounded away from zero then, thanks to the Lipschitz regularity of u, we will be able also to prove that the free boundary $\partial\{u > 0\}$ is locally of finite perimeter.

1. INTRODUCTION

Given a bounded Lipschitz domain $\Omega \subset \mathbb{R}^N$ and two functions $w,\ \psi : \Omega \mapsto \mathbb{R}^+$, we study the problem of minimizing the energy functional

$$u \mapsto \int_\Omega w|\nabla u|^p + \psi\,\chi_{\{u>0\}}$$

among all functions $u \in W^{1,p}(\Omega)$ with boundary condition $u = g$ on $\partial\Omega$, where χ_A denotes the characteristic function of set A. It is not difficult to see that any minimizer u solves the following $p-$Laplace equation:

$$\Delta_p u = 0 \qquad \text{in} \quad \{u > 0\}$$

where $\Delta_p u := \nabla \cdot [|\nabla u|^{p-2}\nabla u]$. This problem is referred to as Bernoulli-type free boundary problem and is well studied in the literature. The case $p = 2$ has been studied first by H. W. Alt and L. A. Caffarelli in [1] and later, for any $1 < p < \infty$ in [3].

In this paper, we are interested in studying the Hölder regularity of a minimizer u as well as the regularity of the free boundary $\Gamma = \partial\{u > 0\} \cap \Omega$. The main difference in [3] from what was done in [1, 2] (where the authors consider the Laplacian case $p = 2$) is that the $p-$Laplacian (for $p \neq 2$) is not uniformly elliptic (degenerate for $p > 2$ and singular for $1 < p < 2$). On the other side, the authors of [3] consider Problem (2.1) but in the case when $w = 1$ and ψ is also constant. The presence of non-uniform functions w and ψ in (2.1) makes the problem somehow more complicated. In [3, 1], the Lipschitz regularity of a minimizer u has been proven. Now, suppose that ψ is not constant or even unbounded, what kind of regularity can we prove on the minimizer u? Can we still prove Lipschitz (or perhaps Hölder) regularity? Do we also need some regularity assumption on the weight w? The answers to these questions do not seem trivial and this is what motivated us to study the problem (2.1) in a much more general setting and to write this article. To be more precise, we will show that the regularity of a minimizer u is related to the L^q-summability of ψ as well as the regularity of the weight w. More precisely, we will show as soon as w is of class C^2 and bounded away from zero that the following statements hold:

$$\psi \in L^q_{loc}(\Omega) \Rightarrow u \in C^{0,\alpha}_{loc}(\Omega), \qquad \alpha = 1 - \frac{N}{pq} > 0,$$

$$\psi \in L^{\infty}_{loc}(\Omega) \Rightarrow u \in \mathrm{Lip}_{loc}(\Omega).$$

In addition, for any compact set $K \subset \Omega' \subset\subset \Omega$, there is a constant C depending on p, N, $\min_{\Omega'} w$, $||\psi||_{L^q(\Omega')}$, $||w||_{L^\infty(\Omega')}$, $||\nabla w||_{L^\infty(\Omega')}$, $||D^2 w||_{L^\infty(\Omega')}$, $||u||_{L^\infty(\Omega')}$ and $\mathrm{dist}(K, \partial\Omega')$ such that

$$||u||_{C^{0,1-\frac{N}{pq}}(K)} \leq C.$$

We note that in [3, Section 3] the proof of Lipschitz regularity of u (in the case where ψ is constant) is more complicated. First, the authors show $\alpha-$Hölder regularity on u for some $0 < \alpha < 1$ sufficiently small (depensing on p and N). Then, they prove in [3, Lemma 3.2] a uniform bound on the minimizers near to their free boundaries. In this way, they obtain uniform (in u) Lipschitz estimates on the minimizers u. However, it seems that in their proof all what they can show is that if x_0 is a "common" point on the free boundaries of a family of minimizers u, then in a neighborhood of x_0 these minimizers u are uniformly bounded. Anyway, we are not interested here in proving uniform (in u) Lipschitz estimates on the minimizers.

In the case when $\psi \in L^{\infty}_{loc}(\Omega)$ and $\psi \geq \psi_{\min} > 0$, we will show that the free boundary has zero Lebesgue measure. Moreover, the characteristic function $\chi_{\{u>0\}}$ belongs to $BV_{loc}(\Omega)$. This means that the free boundary $\partial\{u > 0\}$ has locally a finite perimeter. The proofs of these results are based on a nondegeneracy property (see [2, Section 2]). Roughly speaking, there is a uniform constant $c > 0$ such that the following statement holds:

$$\text{If} \quad ||u||_{L^\infty(B_r)} \leq cr \quad \text{then} \quad u = 0 \ \text{ on } \ B_{\frac{r}{2}}.$$

In [3, Theorem 4.4], the authors extend the proof of [1, Lemma 3.7] to the case $p \neq 2$, so they show that there exists a constant $c > 0$ such that for any point x on the free boundary we have

$$c \leq \frac{\mathcal{L}^N(B(x,r) \cap \{u > 0\})}{\mathcal{L}^N(B(x,r))} \leq 1 - c.$$

Yet, according to their proof, it is not clear why this constant c can be taken uniform in x. We note that the approach used in [1] is different and based on some estimates on $u - v$, where v is the harmonic replacement of u. In Section 2 below, we will use the same argument as in [1] (the comparison with the "weighted" $p-$Laplacian replacement of u) to prove our Hölder regularity result on the minimizers.

To our knowledge, the Alt-Caffarelli minimization problem for the "weighted" $p-$Laplacian with free boundary has not been previously studied. Moreover, the dependence of the Hölder regularity of the minimizer u on the L^q summability of the density ψ as well as the regularity of the free boundary in the "weighted" $p-$Laplace version of the Alt-Caffarelli minimum problem seems to be new in the literature and it has not been written anywhere.

The paper is organized as follows. In Section 2, we show existence of a minimizer u to Problem (2.1). Moreover, we study the $\alpha-$Hölder regularity of u. Section 3 is devoted to prove that the free boundary has zero Lebesgue measure and that it has locally a finite perimeter.

2. Existence and Hölder regularity of minimizers

Let w and ψ be two nonnegative functions over an open bounded Lipschitz domain $\Omega \subset \mathbb{R}^N$ and g be a nonnegative function on $\partial\Omega$. Then, we consider the following minimization problem:

$$\min\left\{ \int_{\Omega} (w|\nabla u|^p + \psi\, \chi_{\{u>0\}}) \ : \ u \in W^{1,p}(\Omega), \ u = g \ \text{ on } \partial\Omega \right\}. \tag{2.1}$$

It is clear that we can restrict (2.1) to the set of nonnegative functions u since if $u^+ := \max\{u, 0\}$ then we have $u^+ = g$ on $\partial\Omega$ and

$$\int_\Omega w|\nabla u^+|^p + \int_\Omega \psi\,\chi_{\{u^+>0\}} \le \int_\Omega w|\nabla u|^p + \int_\Omega \psi\,\chi_{\{u>0\}}.$$

Throughout this paper, we assume that $w \in L^\infty(\Omega)$, $\psi \in L^1(\Omega)$ and there is a function $\tilde{g} \in W^{1,p}(\Omega)$ such that $\tilde{g} = g$ on $\partial\Omega$ (so, we have $\inf(2.1) < \infty$). First of all, we start by the following result that guarantees the existence of a minimizer for Problem (2.1).

Proposition 2.1. *Assume $w \ge w_{\min} > 0$. Then, there exists a minimizer u for Problem (2.1). In addition, every minimizer u is weighted $p-$subharmonic in the sense that*

$$\int_\Omega w\,|\nabla u|^{p-2}\nabla u\cdot\nabla\varphi \le 0, \qquad \text{for all } \varphi \in C_0^\infty(\Omega) \text{ with } \varphi \ge 0.$$

If u is continuous, then u is also weighted $p-$harmonic inside the positivety set $\{u > 0\}$, i.e. for any function $\varphi \in C_0^\infty(\Omega)$ such that $\mathrm{spt}(\varphi) \subset \{u > 0\}$, we have

$$\int_\Omega w\,|\nabla u|^{p-2}\nabla u\cdot\nabla\varphi = 0.$$

Proof. Let $\{u_n\}_n$ be a minimizing sequence in Problem (2.1). Then, there is a uniform constant $C < \infty$ such that for all $n \in \mathbb{N}$, we have

$$\int_\Omega (w|\nabla u_n|^p + \psi\,\chi_{\{u_n>0\}}) \le C.$$

Since $w \ge w_{\min}$ and $\psi \ge 0$, then one has

$$\int_\Omega |\nabla u_n|^p \le \frac{C}{w_{\min}}.$$

But, $u_n = g$ on $\partial\Omega$. So, this implies that $u_n - \tilde{g} \in W_0^{1,p}(\Omega)$; we recall that $\tilde{g}$ is a $W^{1,p}-$extension of g to Ω. By the Poincaré inequality, this yields that

$$||u_n||_{L^p(\Omega)} \le ||u_n - \tilde{g}||_{L^p(\Omega)} + ||\tilde{g}||_{L^p(\Omega)} \le C||\nabla u_n - \nabla\tilde{g}||_{L^p(\Omega)} + ||\tilde{g}||_{L^p(\Omega)} \le C.$$

Hence, the sequence $\{u_n\}_n$ is bounded in $W^{1,p}(\Omega)$ and so, up to a subsequence, $u_n \rightharpoonup u$ weakly in $W^{1,p}(\Omega)$, for some function $u \in W^{1,p}(\Omega)$ with $u = g$ on $\partial\Omega$ and $u \ge 0$. Thanks to the lower semicontinuity of the L^p-norm, one has

$$\int_\Omega w|\nabla u|^p \le \liminf_n \left[\int_\Omega w|\nabla u_n|^p\right]. \tag{2.2}$$

On the other hand, $u_n \to u$ in $L^p(\Omega)$ and so, $u_n(x) \to u(x)$ at almost everywhere point $x \in \Omega$. Hence, if $x \in \{u > 0\}$ then for n large enough, $u_n(x) > 0$ and so,

$$\chi_{\{u>0\}}(x) = 1 = \liminf_n \chi_{\{u_n>0\}}(x).$$

If $x \in \{u = 0\}$, then we always have

$$\chi_{\{u>0\}}(x) = 0 \le \liminf_n \chi_{\{u_n>0\}}(x).$$

By Fatou's Lemma, we get

$$\int_\Omega \psi\,\chi_{\{u>0\}} \le \int_\Omega \liminf_n[\psi\,\chi_{\{u_n>0\}}] \le \liminf_n \left[\int_\Omega \psi\,\chi_{\{u_n>0\}}\right]. \tag{2.3}$$

Combining (2.2) & (2.3), we infer that

$$\int_\Omega w|\nabla u|^p + \int_\Omega \psi\,\chi_{\{u>0\}} \le \liminf_n \left[\int_\Omega w|\nabla u_n|^p + \int_\Omega \psi\,\chi_{\{u_n>0\}}\right].$$

This implies that u minimizes Problem (2.1).

Fix $\varphi \in C_0^\infty(\Omega)$ with $\varphi \geq 0$. Hence, $u - \varepsilon\varphi \in W^{1,p}(\Omega)$ with $u - \varepsilon\varphi = g$ on $\partial\Omega$ and $u - \varepsilon\varphi \leq u$, for all $\varepsilon > 0$. In particular, $u - \varepsilon\varphi$ is admissible in Problem (2.1). From the minimality of u, we have

$$\int_\Omega w|\nabla u|^p + \int_\Omega \psi\,\chi_{\{u>0\}} \leq \int_\Omega w|\nabla u - \varepsilon\nabla\varphi|^p + \int_\Omega \psi\,\chi_{\{u-\varepsilon\varphi>0\}} \leq \int_\Omega w|\nabla u - \varepsilon\nabla\varphi|^p + \int_\Omega \psi\,\chi_{\{u>0\}}.$$

Therefore, we get

$$\int_\Omega w|\nabla u|^p \leq \int_\Omega w|\nabla u - \varepsilon\nabla\varphi|^p, \qquad \text{for all } \varepsilon > 0.$$

Hence,

$$\int_\Omega w|\nabla u|^{p-2}\nabla u \cdot \nabla\varphi \leq 0.$$

Finally, assume that u is continuous (we will show later that this assumption is always satisfied; see Proposition 2.3). Let $\varphi \in C_0^\infty(\Omega)$ be such that $\mathrm{spt}(\varphi) \subset \{u > 0\}$. Since u is continuous, $\mathrm{spt}(\varphi)$ is compact and $u > 0$ on $\mathrm{spt}(\varphi)$, then we have $\{u + \varepsilon\varphi > 0\} = \{u > 0\}$, for any $\varepsilon > 0$ small enough. Yet, $u + \varepsilon\varphi = g$ on $\partial\Omega$. So again, by the minimality of u in Problem (2.1), we get that

$$\int_\Omega w|\nabla u|^p + \int_\Omega \psi\,\chi_{\{u>0\}} \leq \int_\Omega w|\nabla u + \varepsilon\nabla\varphi|^p + \int_\Omega \psi\,\chi_{\{u+\varepsilon\varphi>0\}}.$$

Thus,

$$\int_\Omega w|\nabla u|^p \leq \int_\Omega w|\nabla u + \varepsilon\nabla\varphi|^p, \qquad \text{for all } \varepsilon > 0.$$

This yields that

$$\int_\Omega w|\nabla u|^{p-2}\nabla u \cdot \nabla\varphi \geq 0. \quad \square$$

For any ball $B \subset \Omega$, we will denote by $v = v_B$ the unique solution of the following weighted $p-$Laplacian problem:

$$\begin{cases} \nabla \cdot [w|\nabla v|^{p-2}\nabla v] = 0 & \text{in } B \\ v = u & \text{on } \partial B. \end{cases}$$

Notice that, by the comparison principle (see [4, Lemma 3.18]), $u \leq v$ on B. In the sequel, we will call this function v the weighted $p-$harmonic replacement of u in B. We recall that for any function $\phi \in W^{1,p}(B)$ such that $v = u$ on ∂B, we have

$$\int_B w|\nabla v|^p \leq \int_B w|\nabla\phi|^p. \tag{2.4}$$

Our aim is to show that a minimizer u in Problem (2.1) is locally $\alpha-$Hölderian, for some $\alpha > 0$ that depends on N, p and q. In order to prove this local Hölder regularity on u, we need the following crucial lemma that we use often in the rest of the paper.

Lemma 2.2. *Assume $\psi \in L^q_{loc}(\Omega)$, for some $q > \frac{N}{p}$. Let u be a minimizer of Problem (2.1) and v be the weighted $p-$harmonic replacement of u in $B \subset \Omega$. Then, there is a universal constant $C = C(p)$ such that*

$$\begin{cases} \int_B w|\nabla u - \nabla v|^p \leq C||\psi||_{L^q(B)}|B \cap \{u = 0\}|^{1-\frac{1}{q}} & \text{if } p \geq 2, \\ \int_B w|\nabla u - \nabla v|^p \leq C\Big(\int_B w + \int_B |\nabla u|^p\Big)^{1-\frac{p}{2}}[||\psi||_{L^q(B)}|B \cap \{u = 0\}|^{1-\frac{1}{q}}]^{\frac{p}{2}} & \text{if } 1 < p < 2. \end{cases}$$

Proof. First of all, let us extend the weighted $p-$harmonic replacement v to a function $\tilde{v}$ on Ω by setting

$$\tilde{v} = \begin{cases} v & \text{in } B, \\ u & \text{on } \Omega\backslash B. \end{cases}$$

Clearly, $\tilde{v} \in W^{1,p}(\Omega)$ with $\tilde{v} = u$ on $\partial\Omega$ (so, $\tilde{v}$ is admissible in (2.1)). From the minimality of u in Problem (2.1), we have

$$\int_\Omega w|\nabla u|^p + \int_\Omega \psi\,\chi_{\{u>0\}} \le \int_\Omega w|\nabla \tilde{v}|^p + \int_\Omega \psi\,\chi_{\{\tilde{v}>0\}}.$$

Thus,

$$\begin{aligned} &\int_B w|\nabla u|^p + \int_{\Omega\backslash B} w|\nabla u|^p + \int_B \psi\,\chi_{\{u>0\}} + \int_{\Omega\backslash B} \psi\,\chi_{\{u>0\}} \\ &\le \int_B w|\nabla v|^p + \int_{\Omega\backslash B} w|\nabla u|^p + \int_B \psi\,\chi_{\{v>0\}} + \int_{\Omega\backslash B} \psi\,\chi_{\{u>0\}}. \end{aligned}$$

We infer that

$$\int_B w|\nabla u|^p + \int_B \psi\,\chi_{\{u>0\}} \le \int_B w|\nabla v|^p + \int_B \psi\,\chi_{\{v>0\}}.$$

Hence,

$$\int_B w[|\nabla u|^p - |\nabla v|^p] \le \int_B \psi\,\chi_{\{v>0\}} - \int_B \psi\,\chi_{\{u>0\}} \le \int_B \psi\,\chi_{\{u=0\}} \le ||\psi||_{L^q(B)}|B\cap\{u=0\}|^{1-\frac{1}{q}}. \tag{2.5}$$

Now, set

$$u_t = (1-t)v + tu, \qquad \text{for every } \ 0 \le t \le 1.$$

We have

$$\int_B w[|\nabla u|^p - |\nabla v|^p] = \int_B w[|\nabla u_1|^p - |\nabla u_0|^p] = \int_B w\left[\int_0^1 \frac{d}{dt}|\nabla u_t|^p\right] = p\int_0^1 \int_B w|\nabla u_t|^{p-2}\nabla u_t\cdot[\nabla u - \nabla v].$$

Yet,

$$\int_B w|\nabla v|^{p-2}\nabla v\cdot[\nabla u - \nabla v] = 0 \qquad \text{and} \qquad \nabla u_t - \nabla v = t[\nabla u - \nabla v].$$

Then, we get that

$$\int_B w[|\nabla u|^p - |\nabla v|^p] = p\int_0^1 t^{-1}\int_B w[|\nabla u_t|^{p-2}\nabla u_t - |\nabla v|^{p-2}\nabla v]\cdot[\nabla u_t - \nabla v]. \tag{2.6}$$

From [6], we have the following inequalities:

$$\begin{cases} |a-b|^2(1+|a|^2+|b|^2)^{\frac{p-2}{2}} \le \frac{1}{p-1}[|a|^{p-2}a - |b|^{p-2}b]\cdot[a-b] & \text{if } \ 1<p<2, \\ |a-b|^p \le 2^{p-2}[|a|^{p-2}a - |b|^{p-2}b]\cdot[a-b] & \text{if } \ p\ge 2. \end{cases}$$

Recalling (2.6), we get that

$$\begin{cases} \int_B w[|\nabla u|^p - |\nabla v|^p] \ge p(p-1)\int_0^1 t^{-1}\int_B w|\nabla u_t - \nabla v|^2(1+|\nabla u_t|^2+|\nabla v|^2)^{\frac{p-2}{2}}, & \text{if } \ 1<p<2, \\ \int_B w[|\nabla u|^p - |\nabla v|^p] \ge p2^{2-p}\int_0^1 t^{-1}\int_B w|\nabla u_t - \nabla v|^p, & \text{if } \ p\ge 2. \end{cases}$$

Assume $p \ge 2$. Then,

$$\int_B w[|\nabla u|^p - |\nabla v|^p] \ge p2^{2-p}\int_0^1 t^{p-1}\int_B w|\nabla u - \nabla v|^p = 2^{2-p}\int_B w|\nabla u - \nabla v|^p.$$

By (2.5), we infer that

$$\int_B w|\nabla u - \nabla v|^p \le 2^{p-2}||\psi||_{L^q(B)}|B\cap\{u=0\}|^{1-\frac{1}{q}}.$$

Finally, assume $1<p<2$. Then, we have

$$||\psi||_{L^q(B)}|B\cap\{u=0\}|^{1-\frac{1}{q}}\geq\int_B w[|\nabla u|^p-|\nabla v|^p]\geq C(p)\int_B w|\nabla u-\nabla v|^2(1+|\nabla u|^2+|\nabla v|^2)^{\frac{p-2}{2}}.$$

By Hölder inequality, one has

$$\int_B w|\nabla u-\nabla v|^p\leq\left(\int_B w|\nabla u-\nabla v|^2(1+|\nabla u|^2+|\nabla v|^2)^{\frac{p-2}{2}}\right)^{\frac{p}{2}}\left(\int_B w(1+|\nabla u|^2+|\nabla v|^2)^{\frac{p}{2}}\right)^{1-\frac{p}{2}}.$$

Then, we get

$$\begin{aligned}\int_B w|\nabla u-\nabla v|^p&\leq\left[C(p)\,||\psi||_{L^q(B)}|B\cap\{u=0\}|^{1-\frac{1}{q}}\right]^{\frac{p}{2}}\left(\int_B w(1+|\nabla u|^2+|\nabla v|^2)^{\frac{p}{2}}\right)^{1-\frac{p}{2}}\\&\leq C(p)\,[||\psi||_{L^q(B)}|B\cap\{u=0\}|^{1-\frac{1}{q}}]^{\frac{p}{2}}\left(\int_B w(1+|\nabla u|^p+|\nabla v|^p)\right)^{1-\frac{p}{2}}\\&\leq C(p)\left(\int_B w+\int_B w|\nabla u|^p)\right)^{1-\frac{p}{2}}[||\psi||_{L^q(B)}|B\cap\{u=0\}|^{1-\frac{1}{q}}]^{\frac{p}{2}},\end{aligned}$$

where the last inequality follows from (2.4). □

Since $u\geq 0$ is weighted $p-$subharmonic, then thanks to [4, Theorem 3.41], u is locally bounded. In the next proposition, we show that u is locally $\alpha-$Hölderian.

Proposition 2.3. *Assume $w\in C^2(\Omega)$ and $\psi\in L^q_{loc}(\Omega)$ with $q>\frac{N}{p}$. Let u be a minimizer in Problem (2.1). For any open subset $\Omega'\subset\subset\Omega$ and a compact set $K\subset\Omega'$, there exists a constant C depending on p, N, $w_{\min}$, $||\psi||_{L^q(\Omega')}$, $||w||_{L^\infty(\Omega')}$, $||\nabla w||_{L^\infty(\Omega')}$, $||D^2w||_{L^\infty(\Omega')}$, $||u||_{L^\infty(\Omega')}$ and, $dist(K,\partial\Omega')$ such that*

$$||u||_{C^{0,1-\frac{N}{pq}}(K)}\leq C.$$

Proof. Fix $x\in K$ and $0<2r<R<\operatorname{dist}(K,\partial\Omega')$ (so, we have $B(x,r)\subset B(x,\frac{R}{2})\subset B(x,R)\subset\Omega'$). Let v be the weighted $p-$harmonic replacement of u on $B=B(x,r)$. Then, we clearly have

$$\left(\int_{B(x,r)}w|\nabla u|^p\right)^{\frac{1}{p}}\leq\left(\int_{B(x,r)}w|\nabla u-\nabla v|^p\right)^{\frac{1}{p}}+\left(\int_{B(x,r)}w|\nabla v|^p\right)^{\frac{1}{p}}.\tag{2.7}$$

But, thanks to Lemma 2.2, one has

(2.8)

$$\begin{cases}\left(\int_{B(x,r)}w|\nabla u-\nabla v|^p\right)^{\frac{1}{p}}\leq C(p,N)\,||\psi||^{\frac{1}{p}}_{L^q(B)}\,r^{\frac{N}{p}(1-\frac{1}{q})}, & \text{if}\quad p\geq 2,\\ \left(\int_{B(x,r)}w|\nabla u-\nabla v|^p\right)^{\frac{1}{p}}\leq C(p,N)\left(\int_B w+\int_B|\nabla u|^p\right)^{\frac{1}{p}-\frac{1}{2}}||\psi||^{\frac{1}{2}}_{L^q(B)}\,r^{\frac{N}{2}(1-\frac{1}{q})}, & \text{if}\quad 1<p<2.\end{cases}$$

On the other hand, we have

$$\int_{B(x,r)}w|\nabla v|^p\leq||w||_\infty\,|B(x,r)|\,||\nabla v||^p_{L^\infty(B(x,r))}.\tag{2.9}$$

By [9, Theorem 1.1],

$$||\nabla v||_{L^\infty(B(x,r))}\leq C(p,N)\left(\frac{1+\sqrt{\kappa}R}{R}\right)||v||_{L^\infty(B(x,\frac{R}{2}))},$$

where $0 \leq \kappa < \infty$ (depends on N, $w_{\min}$, $||\nabla w||_{L^\infty(B(x,\frac{R}{2}))}$, $||D^2 w||_{L^\infty(B(x,\frac{R}{2}))}$) is such that the Ricci tensor $\mathrm{Ric}(w)$ of the Riemannian conformal metric $(\mathbb{R}^N, w)$ satisfies $\mathrm{Ric}(w) \geq -(N-1)\kappa$; it is well known that the Ricci tensor in this case is given by (see, for instance, [8])

$$\mathrm{Ric}(w) = -\frac{N-2}{2}\left[\frac{D^2 w}{w} - \frac{3}{2}\frac{\nabla w \otimes \nabla w}{w^2}\right] - \frac{1}{2}\left[\frac{\Delta w}{w} - \frac{(N-4)}{2}\frac{|\nabla w|^2}{w^2}\right].$$

Recalling (2.9), we get that

$$\int_{B(x,r)} w|\nabla v|^p \leq C(p,N)\,||w||_\infty\, r^N \left(\frac{1+\sqrt{\kappa}R}{R}\right)^p ||v||^p_{L^\infty(B(x,\frac{R}{2}))}. \tag{2.10}$$

Since $u = v$ on ∂B, then thanks to the comparison principle [4, Lemma 3.18], we have the following estimate:

$$||v||_{L^\infty(B(x,\frac{R}{2}))} \leq ||u||_{L^\infty(B(x,R))}.$$

For $0 < \delta < 1$, set $R_\delta = r^{1-\delta}$. Now, choose $r > 0$ small enough so that $2r < R_\delta < \mathrm{dist}(K, \partial\Omega')$. This yields that

$$\int_{B(x,r)} w|\nabla v|^p \leq C\, r^N \left(\frac{1+\sqrt{\kappa}\, r^{1-\delta}}{r^{1-\delta}}\right)^p ||u||^p_{L^\infty(B(x,\frac{R_\delta}{2}))} \leq C r^{N-p+\delta p}.$$

Assume $p \geq 2$. Then, by (2.7) & (2.8), we get

$$\left(\int_{B(x,r)} |\nabla u|^p\right)^{\frac{1}{p}} \leq \frac{1}{w_{\min}^{\frac{1}{p}}}\left(\int_{B(x,r)} w|\nabla u|^p\right)^{\frac{1}{p}} \leq C r^{\frac{N}{p}(1-\frac{1}{q})} + C r^{\frac{N}{p}-1+\delta} \leq C r^{\frac{N}{p}(1-\frac{1}{q})}$$

as soon as $\delta \geq 1 - \frac{N}{pq} > 0$. Thanks to Morrey's Lemma, we conclude that $u \in C^{0,1-\frac{N}{pq}}(K)$. Moreover, we have

$$||u||_{C^{0,1-\frac{N}{pq}}(K)} \leq C(p, N, w_{\min}, ||\psi||_{L^q(\Omega')}, ||w||_{C^2(\Omega')}, ||u||_{L^\infty(\Omega')}, \mathrm{dist}(K, \partial\Omega')).$$

Finally, assume that $1 < p < 2$. So, by (2.8), we recall that

$$\begin{aligned}\int_\Omega w|\nabla u - \nabla v|^p &\leq C\left(|B| + \int_B |\nabla u|^p\right)^{1-\frac{p}{2}} \left(||\psi||_{L^q(B)} r^{N(1-\frac{1}{q})}\right)^{\frac{p}{2}} \\ &\leq C r^{N(1-\frac{p}{2q})} + C r^{\frac{Np}{2}(1-\frac{1}{q})}\left(\int_B |\nabla u|^p\right)^{1-\frac{p}{2}}.\end{aligned}$$

Hence,

$$\begin{aligned}\left(\int_{B(x,r)} w|\nabla u|^p\right)^{\frac{1}{p}} &\leq C r^{\frac{N}{p}(1-\frac{p}{2q})} + C r^{\frac{N}{2}(1-\frac{1}{q})}\left(\int_B |\nabla u|^p\right)^{\frac{1}{p}-\frac{1}{2}} + C r^{\frac{N}{p}-1+\delta} \\ &\leq C r^{\frac{N}{p}(1-\frac{p}{2q})} + \frac{C}{w_{\min}^{\frac{1}{p}-\frac{1}{2}}} r^{\frac{N}{2}(1-\frac{1}{q})}\left(\int_B w|\nabla u|^p\right)^{\frac{1}{p}-\frac{1}{2}}.\end{aligned}$$

Using Young's inequality, we get

$$\left(\int_{B(x,r)} w|\nabla u|^p\right)^{\frac{1}{p}} \leq C r^{\frac{N}{p}(1-\frac{1}{q})}.$$

Consequently, u is locally $\alpha-$Hölder in Ω with $\alpha = 1 - \frac{N}{pq}$. In addition, we also have the following estimate:

$$||u||_{C^{0,\alpha}(K)} \leq C(p, N, w_{\min}, ||\psi||_{L^q(\Omega')}, ||w||_{C^2(\Omega')}, ||u||_{L^\infty(\Omega')}, \mathrm{dist}(K, \partial\Omega')).$$

□

3. Regularity of the free boundary

In this section, we prove under the assumptions $q=\infty$ and $\psi \geq \psi_{\min} > 0$ that the free boundary $\partial\{u>0\}$ of any minimizer u in Problem (2.1) has locally a finite perimeter. First, we start by the following nondegeneracy result:

Lemma 3.1. *Assume $\psi \geq \psi_{\min} > 0$. For any $\kappa \in (0, \frac{1}{2})$, there exists a constant $c>0$ depending only on p, N, $w_{\min}$, $\psi_{\min}$, $||\psi||_{L^\infty(\Omega')}$ and $||w||_{C^2(\Omega')}$ such that the following statement holds*

$$||u||_{L^\infty(B_r)} < cr \qquad \text{implies that} \qquad u=0 \text{ in } B_{\kappa r}$$

for any ball $B_r \subset \Omega' \subset\subset \Omega$.

Proof. Fix $\kappa' \in (k,1)$. Let ϕ be the weighted $p-$harmonic function in $B_{\kappa' r\setminus \kappa r}$ with $\phi=1$ on $\partial B_{\kappa' r}$ and $\phi=0$ on $\partial B_{\kappa r}$. From [5, 7], this function ϕ is of class $C^{1,\beta}$ on $\overline{B_{\kappa' r\setminus \kappa r}}$. Let us extend ϕ by zero in $B_{\kappa r}$. Moreover, one has $0\leq \phi \leq 1$.

Set $m=||u||_{L^\infty(B_{k' r})}$. Then, we define $\Phi=\min\{u, m\phi\}$. Since $m\,\phi = m \geq u$ on $\partial B_{k' r}$, then $\Phi = u$ on $\partial B_{k' r}$. Moreover, $\Phi=0$ in $B_{\kappa r}$. Thanks to the minimality of u in Problem (2.1), we have

$$\int_{B_{k' r}} (w|\nabla u|^p + \psi\chi_{\{u>0\}}) \leq \int_{B_{k' r}} (w|\nabla \Phi|^p + \psi\chi_{\{\Phi>0\}}) = \int_{B_{k' r}\setminus B_{\kappa r}} (w|\nabla \Phi|^p + \psi\chi_{\{\Phi>0\}}).$$

Hence,

$$\int_{B_{kr}} (w|\nabla u|^p + \psi\chi_{\{u>0\}}) + \int_{B_{k' r}\setminus B_{\kappa r}} (w|\nabla u|^p + \psi\chi_{\{u>0\}}) \leq \int_{B_{k' r}\setminus B_{\kappa r}} (w|\nabla \Phi|^p + \psi\chi_{\{\Phi>0\}}).$$

But, $\{\Phi>0\}\subset\{u>0\}$. Then, we get that

$$\int_{B_{kr}} (w|\nabla u|^p + \psi\chi_{\{u>0\}}) \leq \int_{B_{k' r}\setminus B_{\kappa r}} (w[|\nabla \Phi|^p - |\nabla u|^p] + \psi[\chi_{\{\Phi>0\}} - \chi_{\{u>0\}}]) \leq \int_{B_{k' r}\setminus B_{\kappa r}} w[|\nabla \Phi|^p - |\nabla u|^p].$$

Since the map $\xi \mapsto |\xi|^p$ is convex, one has

$$|\nabla u|^p - |\nabla \Phi|^p \geq p\,|\nabla \Phi|^{p-2}\nabla\Phi\cdot[\nabla u - \nabla \Phi].$$

Hence,

$$\begin{aligned}
\int_{B_{kr}} (w|\nabla u|^p + \psi\chi_{\{u>0\}}) &\leq p\int_{B_{k' r}\setminus B_{\kappa r}} w|\nabla \Phi|^{p-2}\nabla\Phi\cdot[\nabla\Phi - \nabla u] \\
&= pm^{p-1}\int_{B_{k' r}\setminus B_{\kappa r}} w|\nabla \phi|^{p-2}\nabla\phi\cdot[\nabla\Phi - \nabla u] \\
&= -pm^{p-1}\int_{B_{k' r}\setminus B_{\kappa r}} \nabla\cdot[w|\nabla\phi|^{p-2}\nabla\phi][\Phi-u] + pm^{p-1}\int_{\partial B_{\kappa r}} w|\nabla\phi|^{p-2}\nabla\phi\cdot\mathbf{n}[\Phi-u] \\
&= pm^{p-1}\int_{\partial B_{\kappa r}} [\Phi-u]w|\nabla\phi|^{p-2}\nabla\phi\cdot\mathbf{n} \\
&\leq pm^{p-1}||\nabla\phi||_\infty^{p-1}\,||w||_\infty \int_{\partial B_{\kappa r}} u.
\end{aligned}$$

By [9, Theorem 1.1], we have

$$|\nabla\phi| \leq \frac{C}{r}.$$

Therefore, we get

$$\min\{w_{\min},\psi_{\min}\}\left(\int_{B_{kr}} |\nabla u|^p + |\{u>0\}|\right) \leq w_{\min}\int_{B_{kr}} |\nabla u|^p + \psi_{\min}|\{u>0\}| \leq \int_{B_{kr}} (w|\nabla u|^p + \psi\chi_{\{u>0\}})$$

$$\leq C\,\frac{m^{p-1}}{r^{p-1}}\int_{\partial B_{\kappa r}} u.$$

Thanks to the $W^{1,1}$ trace inequality, we have

$$\int_{\partial B_{\kappa r}} u \leq C\Big(\int_{B_{\kappa r}} u + \int_{B_{\kappa r}} |\nabla u|\Big).$$

Hence,

$$\int_{B_{kr}} |\nabla u|^p \,+\, |\{u>0\}| \leq C\,\frac{m^{p-1}}{r^{p-1}}\Big(\int_{B_{\kappa r}} u + \int_{B_{\kappa r}} |\nabla u|\Big).$$

This implies

$$\begin{aligned}\int_{B_{kr}} |\nabla u|^p \,+\, |\{u>0\}| &\leq C\frac{m^{p-1}}{r^{p-1}}\Big(|\{u>0\}| + |\{u>0\}|^{\frac{p-1}{p}}\Big(\int_{B_{\kappa r}} |\nabla u|^p\Big)^{\frac{1}{p}}\Big)\\ &\leq C\frac{m^{p-1}}{r^{p-1}}\Big(\int_{B_{\kappa r}} |\nabla u|^p + |\{u>0\}|\Big).\end{aligned}$$

Hence,

$$\Big[1 - C\,\frac{m^{p-1}}{r^{p-1}}\Big]\Big(\int_{B_{kr}} |\nabla u|^p \,+\, |\{u>0\}|\Big) \leq 0$$

which implies that $u=0$ on $B_{\kappa r}$ as soon as

$$m \leq \Big(\frac{1}{C}\Big)^{\frac{1}{p-1}} r.$$

□

Proposition 3.2. *For $K \subset \Omega' \subset\subset \Omega$, there is a constant $c \in (0,1)$ (depending on p, N, $w_{\min}$, $\psi_{\min}$, $||\psi||_{L^\infty(\Omega')}$, $||w||_{C^2(\Omega')}$, $||u||_{L^\infty(\Omega')}$ and, dist$(K, \partial\Omega')$) such that for every $x_0 \in K \cap \partial\{u>0\}$ and $r>0$ with $B(x_0,r) \subset K$, we have*

$$c \leq \frac{\mathcal{L}^N(B(x_0,r)\cap\{u>0\})}{\mathcal{L}^N(B(x_0,r))} \leq 1-c.$$

Proof. Again thanks to Lemma 3.1, we see that there exist a constant $c>0$ and a point $x \in B(x_0, \frac{r}{2})$ such that $u(x) \geq c\,\frac{r}{2}$. Yet, we claim that there is a uniform constant $\delta \in (0,1)$ such that $B(x,\delta r) \subset B(x_0,r)\cap\{u>0\}$. Indeed, if $y \in B(x,\delta r)$ then one has

$$|y-x_0| \leq |y-x| + |x-x_0| \leq \Big(\frac{1}{2}+\delta\Big)r < r.$$

Thanks to the Lipschitz regularity of u, we also have

$$u(y) \geq u(x) - C|y-x| \geq \Big(\frac{c}{2} - C\delta\Big)r > 0$$

as soon as $\delta < \frac{c}{2C}$. Hence, we get

$$\frac{\mathcal{L}^N(B(x_0,r)\cap\{u>0\})}{\mathcal{L}^N(B(x_0,r))} \geq \frac{\mathcal{L}^N(B(x,\delta r))}{\mathcal{L}^N(B(x_0,r))} \geq c\,\delta^N.$$

For the upper bound: assume that there is a sequence of points $x_{0,n}$ in $K\cap\partial\{u>0\}$ and $r_n>0$ with $B(x_{0,n},r_n)\subset K$ such that

$$\frac{\mathcal{L}^N(B(x_{0,n},r_n)\cap\{u=0\})}{\mathcal{L}^N(B(x_{0,n},r_n))} \leq \frac{1}{n} \to 0.$$

Up to a subsequence, $x_{0,n} \to x_0 \in K\cap\partial\{u>0\}$ and $r_n \to r_0$. Assume $r_0>0$. Let v_n be the

weighted $p-$harmonic replacement of u in $B_n := B(x_{0,n}, r_n)$. For all n large enough, it is clear that $B := B(x_0, \frac{r_0}{4}) \subset B(x_{0,n}, \frac{r_n}{2})$. Thanks to Lemma 2.2, we have

$$\begin{cases} \int_B w|\nabla u - \nabla v_n|^p \leq C||\psi||_{L^\infty(\Omega)}\,|B_n \cap \{u = 0\}| & \text{if} \quad p \geq 2, \\ \int_B w|\nabla u - \nabla v_n|^p \leq C\Big(|B_n| + \int_{B_n} w|\nabla u|^p\Big)^{1-\frac{p}{2}} [||\psi||_{L^\infty(\Omega)}|B_n \cap \{u = 0\}|]^{\frac{p}{2}} & \text{if} \quad 1 < p < 2. \end{cases}$$

Hence,

$$\int_B w|\nabla u - \nabla v_n|^p \to 0.$$

Yet, $v_n \to v$ uniformly in $\overline{B}$. In particular, we must have $u = v + C$. But, v is weighted $p-$harmonic in B and so, u is also weighted $p-$harmonic there. However, we have $u(x_0) = 0$. By the maximum principle, we infer that $u = 0$ in B. But, this is a contradiction as $x_0 \in \partial\{u > 0\}$.

Finally, assume that $r_0 = 0$. For every n, we define $u_n(x) = \frac{u(x_{0,n}+r_n x)}{r_n}$ for all $x \in B_1$. It is not difficult to check that u_n minimizes

$$\min\left\{ \int_{B_1} (w_n|\nabla u|^p + \psi_n\,\chi_{\{u>0\}}) \,:\, u \in W^{1,p}(B_1),\ u \geq 0,\ u = u_n \ \text{ on } \ \partial B_1 \right\},$$

where

$$w_n(x) = w(x_{0,n} + r_n x) \qquad \text{and} \qquad \psi_n(x) = \psi(x_{0,n} + r_n x).$$

Now, let v_n be the weighted $p-$harmonic replacement of u_n in B_1. Again, by Lemma 2.2, one has the following:

$$\begin{cases} \int_{B_1} w_n|\nabla u_n - \nabla v_n|^p \leq C||\psi||_{L^\infty(\Omega)}\,|B_1 \cap \{u_n = 0\}| & \text{if} \quad p \geq 2, \\ \int_{B_1} w_n|\nabla u_n - \nabla v_n|^p \leq C\Big(|B_1| + \int_{B_1} w_n|\nabla u_n|^p\Big)^{1-\frac{p}{2}} [||\psi||_{L^\infty(\Omega)}|B_1 \cap \{u_n = 0\}|]^{\frac{p}{2}} & \text{if} \quad 1 < p < 2. \end{cases}$$

But, $w_n \geq \frac{w(x_0)}{2} > 0$ on B_1 for n large enough. Moreover, we have

$$\mathcal{L}^N(B_1 \cap \{u_n = 0\}) = \frac{\mathcal{L}^N(B(x_{0,n}, r_n) \cap \{u = 0\})}{\mathcal{L}^N(B(x_{0,n}, r_n))}\,\mathcal{L}^N(B_1) \to 0.$$

Thus,

$$\int_{B_1} |\nabla u_n - \nabla v_n|^p \to 0.$$

We note that since u is Lipschitz, then u_n are clearly uniformly Lipschitz. In addition, one has $0 \leq u_n \leq C$ for all n since

$$u(x_{0,n} + r_n x) = u(x_{0,n} + r_n x) - u(x_{0,n}) \leq C r_n.$$

So, up to a subsequence, $u_n \to u^\star$ and $v_n \to v$ uniformly in $\overline{B_1}$. And, we have $u^\star = v + C$. However, v is weighted $p-$harmonic in B_1. Hence, $u^\star$ is weighted $p-$harmonic. But, we have $u^\star(0) = 0$ (since $u_n(0) = 0$ for every n). Thanks again to the maximum principle, we get that $u^\star = 0$ in B_1. From Lemma 3.1, we know that there is a uniform constant $c > 0$ such that

$$||u||_{L^\infty(B(x_{0,n}, r_n))} \geq c r_n.$$

Then,

$$||u_n||_{L^\infty(B_1)} \geq c.$$

But, this yields again to a contradiction. □

In particular, we obtain the following:

Theorem 3.3. *The free boundary has zero Lebesgue meausre.*

Proof. This follows from Proposition 3.2 as well as the Lebesgue density theorem. □

Now, let us define the measure Λ as follows:

$$\int_\Omega \varphi \, \mathrm{d}\Lambda = -\int_\Omega w|\nabla u|^{p-2}\nabla u \cdot \nabla \varphi, \qquad \text{for all } \varphi \in C_0^\infty(\Omega).$$

Since u is $p-$subharmonic, then Λ is a nonnegative Borel measure. Moreover, u is $p-$harmonic in the positivety set $\{u > 0\}$ and so, for every smooth function φ such that $\mathrm{spt}(\varphi) \subset \{u > 0\}$ we have

$$\int_\Omega \varphi \, \mathrm{d}\Lambda = 0.$$

Therefore, $\mathrm{spt}(\Lambda) \subset \partial\{u > 0\}$.

Proposition 3.4. *For any compact set $K \subset \Omega' \subset\subset \Omega$, there exist two constants $0 < c < C < \infty$ (depending on p, N, $w_{\min}$, $\psi_{\min}$, $||\psi||_{L^\infty(\Omega')}$, $||w||_{C^2(\Omega')}$, $||u||_{L^\infty(\Omega')}$ and, dist$(K, \partial\Omega'))$ such that for every $x_0 \in K \cap \partial\{u > 0\}$, we have*

$$cr^{N-1} \leq \Lambda(B(x_0, r)) \leq Cr^{N-1}, \qquad \text{for all } \ 0 < r < d(x_0, \partial K).$$

Proof. Fix $\delta > 0$. Let $0 \leq \varphi_\delta \in C_0^\infty(B(x_0, r+\delta))$ be a cutoff function such that $\varphi_\delta = 1$ on $B(x_0, r)$ and $|\nabla\varphi| \leq \frac{C}{\delta}$. Then, we have

$$\Lambda(B(x_0, r)) \leq \int_\Omega \varphi_\delta \, \mathrm{d}\Lambda = -\int_\Omega w|\nabla u|^{p-2}\nabla u \cdot \nabla \varphi_\delta \leq ||w||_\infty ||\nabla u||_\infty^{p-1} \frac{C}{\delta} |B(x_0, r+\delta)\backslash B(x_0, r)|,$$

for any $\delta > 0$. Letting $\delta \to 0^+$, we get

$$\Lambda(B(x_0, r)) \leq Cr^{N-1}.$$

Let us prove the lower estimate. Assume that there is a sequence of points $x_{0,n}$ in $K \cap \partial\{u > 0\}$ and $0 < r_n < d(x_{0,n}, \partial K)$ such that

$$\Lambda(B(x_{0,n}, r_n)) \leq \frac{r_n^{N-1}}{n}.$$

Up to a subsequence, $x_{0,n} \to x_0$ and $r_n \to r_0$. Assume $r_0 > 0$. For every $\varphi \in C_0^\infty(B(x_0, \frac{r_0}{2}))$, we have

$$-\int_\Omega w|\nabla u|^{p-2}\nabla u \cdot \nabla \varphi = \int_{B(x_{0,n}, r_n)} \varphi \, \mathrm{d}\Lambda \leq ||\varphi||_\infty \Lambda(B(x_{0,n}, r_n)) \to 0.$$

Substituting φ with $-\varphi$, we get

$$-\int_\Omega w|\nabla u|^{p-2}\nabla u \cdot \nabla \varphi = 0.$$

Hence, u is weighted $p-$harmonic inside $B(x_0, \frac{r_0}{2})$. But, we have $u(x_0) = \lim_n u(x_{0,n}) = 0$. By the maximum principle, this implies that $u = 0$ on $B(x_0, \frac{r_0}{2})$. Yet, this is a contradiction.

Finally, assume that $r_0 = 0$. For every n, set $u_n(x) = \frac{u(x_{0,n} + r_n x)}{r_n}$ for all $x \in B_1$. Then, we define

$$\int_{B_1} \varphi \, \mathrm{d}\Lambda_n = -\int_{B_1} w_n|\nabla u_n|^{p-2}\nabla u_n \cdot \nabla \varphi, \qquad \text{for all } \varphi \in C_0^\infty(B_1),$$

where

$$w_n(x) = w(x_{0,n} + r_n x).$$

Fix $\varphi \in C_0^\infty(B_1)$. We have

$$\int_{B_1} \varphi \, \mathrm{d}\Lambda_n = -\int_{B_1} w(x_{0,n} + r_n x)|\nabla u(x_{0,n} + r_n x)|^{p-2}\nabla u(x_{0,n} + r_n x) \cdot \nabla \varphi(x) \, \mathrm{d}x$$

$$= -\frac{1}{r_n^N}\int_{B(x_{0,n},r_n)} w(z)|\nabla u(z)|^{p-2}\nabla u(z)\cdot\nabla\varphi\left(\frac{z-x_{0,n}}{r_n}\right)\mathrm{d}z$$
$$= -\frac{1}{r_n^{N-1}}\int_{B(x_{0,n},r_n)} w(z)|\nabla u(z)|^{p-2}\nabla u(z)\cdot\nabla\varphi_n(z)\,\mathrm{d}z$$
$$= \frac{1}{r_n^{N-1}}\int_\Omega \varphi_n\,\mathrm{d}\Lambda$$

where

$$\varphi_n(z) = \varphi\left(\frac{z-x_{0,n}}{r_n}\right).$$

Yet, we see that u_n is uniformly bounded and Lipschitz continuous on B_1. Hence, up to a subsequence, $u_n \to u^\star$ uniformly in $\overline{B_1}$. On the other hand, one has $w_n|\nabla u_n|^{p-2}\nabla u_n \rightharpoonup^\star h$ in $L^\infty(B_1)$. We claim that $h = w(x_0)|\nabla u^\star|^{p-2}\nabla u^\star$. If this is the case, so we get that for every $\varphi \in C_0^\infty(B_1)$,

$$-\int_{B_1} w(x_0)|\nabla u^\star|^{p-2}\nabla u^\star\cdot\nabla\varphi$$
$$= \lim_{n\to\infty} -\int_\Omega w_n|\nabla u_n|^{p-2}\nabla u_n\cdot\nabla\varphi = \lim_{n\to\infty}\int_{B_1}\varphi\,\mathrm{d}\Lambda_n \le ||\varphi||_\infty\Lambda_n(B_1) \le C\,\frac{\Lambda(B(x_{0,n},r_n))}{r_n^{N-1}} \to 0.$$

Hence, $u^\star$ is weighted $p-$harmonic inside B_1. But, we have $u^\star(0) = \lim_n u_n(0) = 0$. By the maximum principle, this implies that $u^\star = 0$ in B_1. However, by Lemma 3.1, there is a constant $c > 0$ (uniform in n) such that

$$||u_n||_{L^\infty(B_1)} \ge c.$$

Passing to the limit when $n \to \infty$, we get

$$||u^\star||_{L^\infty(B_1)} \ge c > 0,$$

which is a contradiction.

It remains to prove the claim. Fix $x \in B_1$. First, assume that $u^\star(x) > 0$. Let $r' > 0$ be sufficiently small so that $\overline{B(x,r')} \subset \{u^\star > 0\}$. Since $u_n \to u^\star$ uniformly in B_1, then for n large enough we must have $\overline{B(x,r')} \subset \{u_n > 0\}$ and so, u_n is weighted $p-$harmonic in $B(x,r')$. Hence, thanks to [7, Theorem 13.1], u_n is bounded in $C^{1,\alpha}(\overline{B(x,r')})$. Then, up to a subsequence, $w_n|\nabla u_n|^{p-2}\nabla u_n \to w(x_0)|\nabla u^\star|^{p-2}\nabla u^\star$ uniformly and so, $h = w|\nabla u^\star|^{p-2}\nabla u^\star$ on $B(x,r')$. Now, assume that there is a small $r' > 0$ such that $B(x,r') \subset \{u^\star = 0\}$. Thanks to Lemma 3.1, for n large enough, $u_n = 0$ on $B(x,\frac{r'}{2})$. In particular, we get that $h = w(x_0)|\nabla u^\star|^{p-2}\nabla u^\star = 0$ in $B(x,\frac{r'}{2})$.

To conclude the proof, we just need to check that $\mathcal{L}^N(\partial\{u^\star > 0\}) = 0$. Fix $x \in \partial\{u^\star > 0\}$. Hence, it is not difficult to see that there is a constant $c > 0$ such that for every $r > 0$ small one has for n large enough

$$||u_n||_{L^\infty(B(x,r))} \ge cr.$$

Passing to the limit when $n \to \infty$, we infer that

$$||u^\star||_{L^\infty(B(x,r))} \ge cr.$$

In particular, there exists a point $x^\star \in B(x,r)$ such that $u^\star(x^\star) \ge cr$. Moreover, thanks to the Lipschitz regularity of $u^\star$, it is easy to check that $B(x^\star,\delta r) \subset B(x,2r)\cap\{u^\star > 0\}$, for $\delta \in (0,1)$ small. Hence, we get

$$\frac{\mathcal{L}^N(B(x,2r)\cap\{u^\star > 0\})}{\mathcal{L}^N(B(x,2r))} \ge \frac{\mathcal{L}^N(B(x^\star,\delta r))}{\mathcal{L}^N(B(x,2r))} \ge c.$$

□

Finally, we arrive to the following regularity result on the free boundary; $\partial\{u>0\}$ is locally of finite perimeter:

$$\text{“}\chi_{\{u>0\}} \in BV_{loc}(\Omega)\text{”}.$$

Theorem 3.5. *Let u be a minimizer for Problem (2.1). For any compact set $K \subset \Omega$, we have $\mathcal{H}^{N-1}(\partial\{u>0\} \cap K) < \infty$.*

Proof. Thanks to Proposition 3.4, we infer that

$$c\,\mathcal{H}^{N-1}(\{\partial\{u>0\} \cap K\}) \leq \Lambda(K) \leq C\,\mathcal{H}^{N-1}(\{\partial\{u>0\} \cap K\}).$$

Yet, $\Lambda(K) < \infty$. Hence,

$$\mathcal{H}^{N-1}(\{\partial\{u>0\} \cap K\}) < \infty. \quad \square$$

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