COMPLETE CRITICAL METRICS FOR THE QUADRATIC SCALAR CURVATURE FUNCTIONAL: EXAMPLES AND RIGIDITY

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Abstract. We study complete critical metrics for the quadratic scalar curvature functional

$$g \mapsto \int_M S_g^2 \, \mathrm{d}V_g,$$

with criticality understood under compactly supported variations, focusing on rigidity phenomena and nontrivial examples. In dimension three, we prove that every complete critical metric is scalar-flat, without any additional assumption. In dimension four, we construct complete non-Einstein critical metrics with bounded, nonconstant negative scalar curvature. In every dimension $n \geq 5$, we show that either $S_g \equiv 0$ or $S_g < 0$ everywhere, and a new argument proves scalar-flatness whenever $S_g \in L^p$ for some $0 < p \leq P_n$, where $P_n > 2$ is an explicit threshold. Moreover, we construct complete critical metrics with bounded, negative scalar curvature, decaying to zero at infinity. These examples establish the optimality of our rigidity threshold P_n in dimension six and in every dimension $n \geq 10$. To the best of our knowledge, these are the first complete critical metrics for this functional that are neither scalar-flat nor Einstein.

1. Introduction

The aim of this paper is to continue the study of complete, non-compact critical metrics of the quadratic scalar curvature functional

$$\mathfrak{S}^2(g) = \int_M S_g^2 \, \mathrm{d}V_g.$$

On a noncompact manifold, criticality is understood with respect to compactly supported metric variations, without a volume constraint. More precisely, given a smooth family of metrics g_t with $g_{t=0} = g$ and $g_t = g$ outside a fixed compact set, let $U \Subset M$ be a smooth relatively compact domain containing that set in its interior. We require

$$\frac{d}{dt}\bigg|_{t=0} \int_U S_{g_t}^2 \, \mathrm{d}V_{g_t} = 0;$$

this condition is independent of the choice of U and is equivalent to the Euler–Lagrange equation below.

Throughout the paper manifolds are smooth, connected, and without boundary, unless otherwise stated; statements for disconnected manifolds apply to each connected component.

To fix the notation, let M^n, $n \geq 2$, be an n-dimensional smooth manifold without boundary. Given a Riemannian metric g on M^n, we denote by Riem_g, W_g, Ric_g and S_g the Riemann curvature tensor, the Weyl tensor, the Ricci tensor and the scalar curvature, respectively. The *quadratic curvature functionals* are

$$\mathfrak{W}^2 = \int |W_g|^2 dV_g \,, \qquad \mathfrak{r}^2 = \int |\,\mathrm{Ric}_g\,|^2 dV_g \,, \qquad \mathfrak{S}^2 = \int S_g^2 dV_g.$$

2020 *Mathematics Subject Classification.* 53C21, 53C24, 53C25.

Key words and phrases. Quadratic curvature functionals, critical metrics, rigidity, finite energy, scalar curvature.

For the corresponding pointwise quadratic curvature integrands, the space is one-dimensional when $n = 2$ (generated by S_g^2), and two-dimensional when $n = 3$ (generated by S_g^2 and $|\operatorname{Ric}_g|^2$).

From the standard decomposition of the Riemann tensor, for every $n \geq 4$ one has

$$\mathfrak{R}^2 = \int |\operatorname{Riem}_g|^2 dV_g = \int \left(|W_g|^2 + \frac{4}{n-2} |\operatorname{Ric}_g|^2 - \frac{2}{(n-1)(n-2)} S_g^2 \right) dV_g \,.$$

Quadratic curvature functionals have attracted a lot of attention from mathematicians and physicists in recent years (see e.g. [2, 3, 11, 8] and references therein). In [8] we studied $\mathfrak{S}^2$ and the mixed quadratic functionals

$$\mathfrak{F}_t^2(g) = \int_M (|\operatorname{Ric}_g|^2 + t S_g^2) \, \mathrm{d}V_g, \qquad t \in \mathbb{R}.$$

In dimension three, every complete critical metric of $\mathfrak{F}_t^2$ is flat for $t > -1/3$, see [8, Theorem 1.2]. The homogeneous examples of [4] show that this threshold is sharp.

As far as the functional $\mathfrak{S}^2$ is concerned, in [8] we proved the following for complete metrics:

- when $n = 2$ all critical metrics are flat, and thus they are global minima of the functional;
- when $n = 3$, all critical metrics are scalar-flat (and thus they are global minima of the functional) under the additional hypothesis that $S_g \in L^q(M^3)$ for some $q \in (1, \infty)$;
- the dimension $n = 4$ is special, due to the differing behaviour of the quadratic functionals under constant rescaling of the metric when $n = 4$ and $n \neq 4$; note also that, in this case, critical metrics turn out to have harmonic scalar curvature. Thus, if M^4 is compact, then it is scalar flat, or it has constant scalar curvature and it is Einstein; on the other hand, if M^4 is complete, non-compact and $S_g \in L^q(M^4)$ for some $q \in (1, \infty)$, then a classical result of Yau (see [18]) implies that M^4 has constant scalar curvature, and hence it is either scalar-flat or a finite-volume Einstein metric;
- when $n \geq 5$, there exists $q^* > 2$ such that a critical metric of $\mathfrak{S}^2$ having scalar curvature S_g which is *bounded from below* and satisfying $S_g \in L^q(M^n)$ for $q \in (1, q^*)$ must be scalar-flat, and thus it is a global minimum of the functional.

In [9], we removed the boundedness from below of the scalar curvature, assuming $n \geq 10$: the proof uses a conformal reduction to an auxiliary geometric system, in the same spirit as the works [10, 14]. The case of the "missing dimensions" $5 \leq n \leq 9$ has been subsequently settled by Zhang in [19], thus completing the picture for finite energy critical metrics in dimensions greater than or equal to five.

We recall that the Euler–Lagrange equation for a critical metric g of $\mathfrak{S}^2$ reads

$$2 S_g \operatorname{Ric}_g - 2 \operatorname{Hess}(S_g) + 2 \Delta_g S_g \, g = \frac{1}{2} S_g^2 g. \tag{1.1}$$

Equivalently,

$$S_g \operatorname{Ric}_g - \operatorname{Hess}(S_g) = \frac{3}{4(n-1)} S_g^2 g, \tag{1.2}$$

$$\Delta_g S_g = \frac{n-4}{4(n-1)} S_g^2. \tag{1.3}$$

The classical variational formulas may be found in [3, Proposition 4.66]; for critical metrics in the complete setting we refer to [6, 8, 9], and for the wider theory of quadratic curvature energies to [12].

The traced equation separates the compact and noncompact problems: on a compact manifold with $n \neq 4$, integration immediately gives $S_g \equiv 0$; in dimension four, S_g is harmonic, hence constant in the compact case; the tensor equation then gives the alternative between scalar-flatness and the Einstein condition. On a complete noncompact manifold, these integrations require control at infinity. The results below address both the necessity of such control and the range of integrability assumptions that yields rigidity.

In [6], complete critical metrics with nonnegative scalar curvature were classified, and the necessity of the sign assumption was left open. Our first two results answer this question in opposite ways: in dimension three the sign assumption can be removed entirely, whereas in dimension four there exist complete critical metrics with bounded, nonconstant negative scalar curvature.

In dimension three, the following theorem also removes the integrability assumption from [8, Theorem 1.3 and Remark 1.4]:

Theorem 1.1. *Let (M^3, g) be a three-dimensional complete critical metric of $\mathfrak{S}^2$. Then (M^3, g) is scalar-flat, and hence a global minimum of the functional.*

In dimension four, completeness alone no longer gives the same rigidity: indeed, the following examples have bounded, nonconstant negative scalar curvature and are neither scalar-flat nor Einstein. They also show why an additional hypothesis remains necessary in the L^p classification of Proposition 3.1 below: their harmonic scalar curvature belongs to no finite L^p space.

Theorem 1.2. *There exists a smooth, complete, Riemannian metric g_4 on $M^4 = \mathbb{R} \times N$, where N is a closed hyperbolic three-manifold, which is critical for $\mathfrak{S}^2$ and has the following properties:*

(i) *g_4 is neither scalar-flat nor Einstein;*
(ii) *S_{g_4} is negative, bounded, and nonconstant on M^4;*
(iii) *for every $q > 0$,*

$$S_{g_4} \notin L^q(M^4, g_4).$$

In particular, g_4 has infinite energy.

In the second part of the paper we treat all dimensions $n \geq 5$, by a new, unified argument in the original metric (without any conformal change of g). Besides recovering finite-energy rigidity, the theorem establishes rigidity in the range $0 < p \leq n/2$ for $5 \leq n \leq 9$ and $0 < p \leq \frac{7n-10}{2(n-4)}$ for $n \geq 10$, thus including the endpoint in both cases.

Theorem 1.3. *Let (M^n, g), $n \geq 5$, be a complete critical metric of $\mathfrak{S}^2$. Then either $S_g \equiv 0$ or $S_g < 0$ everywhere. Moreover, if $S_g \in L^p(M, g)$ for some $0 < p \leq P_n$, with*

$$P_n = \begin{cases} n/2, & 5 \leq n \leq 9, \\ \frac{7n-10}{2(n-4)}, & n \geq 10, \end{cases} \tag{1.4}$$

then $S_g \equiv 0$. In particular, for $n \geq 5$, every complete critical metric of $\mathfrak{S}^2$ with finite energy is scalar-flat.

The sign alternative in Theorem 1.3 is proved in Lemma 4.1, independently of any integrability assumption. It excludes sign-changing scalar curvature for every complete critical metric in dimension $n \geq 5$. In the negative case, set $u = -S_g > 0$ and $w = u^{-1/2}$. A weighted Bochner argument with respect to the finite measure $u^p \, \mathrm{d}V_g$ first bounds $|\nabla w|^2$ by a constant. A second cutoff test yields the dimensional gradient bound. The scalar equation and a cutoff argument prove rigidity below the upper endpoint; at equality, they force $|\nabla w|$ to be a positive constant, which contradicts completeness and positivity of w.

Finally, in the third part of the paper we construct a family of explicit complete critical metrics on $\mathbb{R}^2 \times \mathbb{T}^{n-2}$, where $\mathbb{T}^{n-2}$ is a flat torus: their scalar curvature is bounded and tends to zero at infinity, but is not identically zero; moreover, the exact curvature integrals also show that the rigidity endpoint is optimal in every dimension $n \geq 10$.

Theorem 1.4. *For every $n \geq 5$, there exists a smooth complete Riemannian metric g_n on*

$$M_n = \mathbb{R}^2 \times \mathbb{T}^{n-2}$$

which is critical for $\mathfrak{S}^2$ and has the following properties:

(i) *g_n is not scalar-flat;*
(ii) *$-1 \leq S_{g_n} < 0$ on M_n, and S_{g_n} tends to zero at infinity;*
(iii) *for every $q > 0$,*

$$S_{g_n} \in L^q(M_n, g_n) \quad \Longleftrightarrow \quad q > q_n = \frac{7n-10}{2(n-4)}.$$

In particular, g_n has infinite energy.

The previous theorem thus implies that completeness, bounded scalar curvature, and even pointwise decay to zero do not force scalar-flatness. Moreover,

$$q_n - 2 = \frac{3(n+2)}{2(n-4)} > 0,$$

so none of these metrics has finite energy. When $n \geq 10$, this highlights sharpness of Theorem 1.3. A second construction in dimension $n = 6$ shows that the rigidity threshold $P_6 = 3$ is optimal.

Theorem 1.5. *Let $(N^5, \bar{h})$ be closed and Einstein, with $\mathrm{Ric}_{\bar{h}} = -\bar{h}/10$. On $M = \mathbb{R} \times N$, set*

$$\widehat{g}_6 = \cosh(t/\sqrt{10})\,(dt^2 + \bar{h}).$$

Then $\widehat{g}_6$ is a smooth complete non-Einstein critical metric, with

$$S_{\widehat{g}_6} = -\operatorname{sech}(t/\sqrt{10}), \qquad S_{\widehat{g}_6} \in L^p(M, \widehat{g}_6) \quad \Longleftrightarrow \quad p > 3.$$

Remark 1.6. Theorem 1.5 shows that the endpoint $P_6 = 3$ in Theorem 1.3 is optimal. The change of variables establishing the exact integrability range is given in the proof of Proposition 5.4.

In dimension six and in dimension $n \geq 10$, the examples also show that the constant in the gradient estimate of Proposition 4.2 is optimal, see Remark 5.5. In dimensions $5, 7, 8, 9$, it remains open whether the rigidity range in Theorem 1.3 can be enlarged.

The paper is organized as follows. Sections 2 and 3 concern dimensions three and four. Section 4 proves rigidity for $n \geq 5$. In Section 5 we construct the family of Theorem 1.4 and the six-dimensional example of Theorem 1.5, and establish the sharpness claims.

Acknowledgements. The first and second authors are members of the GNSAGA, Gruppo Nazionale per le Strutture Algebriche, Geometriche e le loro Applicazioni of INdAM. The third author is a member of GNAMPA, Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni of INdAM.

2. Dimension three: rigidity

In this section we prove Theorem 1.1. Recall that a *nodal domain* of a continuous function f is a connected component of the open set $\{f \neq 0\}$. On each nodal domain, f has a fixed strict sign. Accordingly, a *negative* (respectively, *positive*) nodal domain of f is a connected component of $\{f < 0\}$ (respectively, $\{f > 0\}$).

The critical equations in dimension three, dropping the subscript g for the sake of readability, are

$$S\,\mathrm{Ric} - \mathrm{Hess}(S) = \frac{3}{8}S^2 g, \tag{2.1}$$

$$\Delta S = -\frac{1}{8}S^2. \tag{2.2}$$

The case $S \geq 0$ is covered in [6, Theorem 1.2]; it therefore suffices to exclude a connected component $\Omega \subset \{S < 0\}$. On such a component, $u = -S > 0$ satisfies

$$\mathrm{Hess}(u) = u\,\mathrm{Ric} + \frac{3}{8}u^2 g, \tag{2.3}$$

$$\Delta u = \frac{1}{8}u^2. \tag{2.4}$$

The proof has three steps. First, the boundary of a negative nodal domain is regular and totally geodesic, and its intrinsic completion has minimizing segments that stay in the interior. Second, the tensor equation yields a local weighted distance comparison and cutoffs in the barrier sense. Finally, point selection and the maximum principle applied to a smooth lower support give a contradiction.

Step 1. Since Ω may not be complete, we use the length distance of the restricted metric, namely

$$d_\Omega(x, y) = \inf_c \int_0^1 |\dot{c}(t)|_g \, dt,$$

where the infimum is over piecewise smooth curves $c : [0, 1] \to \Omega$ joining x to y. The curves are required to stay in Ω, so this distance may exceed the ambient distance d_M. By the *intrinsic completion* we mean the ordinary metric completion X of (Ω, d_Ω): its points are equivalence classes of d_Ω-Cauchy sequences, and

$$d_X([(x_j)], [(y_j)]) = \lim_{j\to\infty} d_\Omega(x_j, y_j).$$

We identify Ω with its isometric image in X. A *length space* is a metric space whose distance is the infimum of curve lengths; it is called *geodesic* if this infimum is attained for every pair of points, and proper if every closed bounded ball is compact. For these notions see e.g. [5, Sections 1.5 and 2.1–2.5]. The next lemma proves the required properties of X and identifies it with the ambient closure of Ω. All subsequent differential inequalities are used only in the smooth interior Ω.

Lemma 2.1. *Suppose that $S \not\equiv 0$. Then 0 is a regular value of S, and $\partial\Omega$ is totally geodesic. The intrinsic completion (X, d_X) of (Ω, g) is naturally homeomorphic to $\overline{\Omega} \subset M$ and is proper and geodesic. Every minimizing segment in X whose endpoints belong to Ω is contained in Ω.*

Proof. Suppose that $S(x) = 0$ and $\nabla S(x) = 0$. Along every unit-speed geodesic γ originating from x, the function $v = S \circ \gamma$ satisfies

$$v'' = \mathrm{Ric}(\dot\gamma, \dot\gamma)v - \frac{3}{8}v^2, \qquad v(0) = v'(0) = 0.$$

Standard uniqueness for this ordinary differential equation gives $S = 0$ in a normal neighborhood of x; thus the set $\{S = 0, \nabla S = 0\}$ is both open and closed. Connectedness and $S \not\equiv 0$ imply that this set is empty. At a zero of S, the tensor equation also gives $\mathrm{Hess}(S) = 0$. The second fundamental form of the regular level $\{S = 0\}$ is, up to the choice of the normal, $|\nabla S|^{-1}\,\mathrm{Hess}(S)$ restricted to its tangent bundle; it therefore vanishes.

Let d_Ω denote the intrinsic distance. Since $d_M \le d_\Omega$, every intrinsic Cauchy sequence has an ambient limit in $\overline{\Omega}$, by completeness of M. At an interior limit point, a small coordinate ball identifies intrinsic and ambient convergence. At a boundary point, regularity of the zero level gives a coordinate half-ball on the negative side. This negative side belongs to Ω, since it is connected and meets Ω. Local upper and lower bounds for the metric coefficients show that two sequences approaching the same boundary point have intrinsic distance tending to zero. They also show that every boundary point is the limit of an intrinsic Cauchy sequence. These observations identify X with $\overline{\Omega}$ and establish equality of the two topologies. Under this identification, d_X is also the length distance of the metric on the manifold with boundary $\overline{\Omega}$. Indeed, curves in Ω are admissible in $\overline{\Omega}$. Conversely, a piecewise smooth curve in $\overline{\Omega}$ can be approximated, with converging lengths, by curves in Ω: in boundary half-charts move its boundary portions a distance ε into the interior, using a smooth inward-pointing vector field on a neighborhood of the compact image of the curve. The flow tends to the identity in C^1 as $\varepsilon \to 0$; short interior arcs restore any prescribed interior endpoints. Passing to the completion gives the same equality when an endpoint lies on the boundary. This is an equality with the intrinsic length distance of $\overline{\Omega}$, not with the ambient distance d_M. The half-ball description proves local compactness of X. Moreover, the completion of a length space is a length space. The Hopf–Rinow theorem for length spaces now implies that X is proper, meaning that its closed bounded balls are compact, and that any two points can be joined by a minimizing segment (see [5, Proposition 2.5.22 and Theorem 2.5.23]).

It remains to show that minimizing segments with interior endpoints avoid the boundary. If $\partial\Omega = \varnothing$, the assertion is immediate. Otherwise form the double $\widehat{X}$ by gluing two copies of X along their common boundary. In Fermi coordinates, total geodesicity gives

$$g = dt^2 + g_t, \qquad g_t = g_0 + t^2 A(t, z), \qquad t \ge 0,$$

with A smooth up to $t = 0$. The reflected metric is

$$\widehat{g} = dt^2 + g_0 + t^2 A(|t|, z).$$

It is C^2 across $t = 0$: since the second fundamental form vanishes, the first normal derivative vanishes as well; the second normal derivative tends to $2A(0, z)$ from both sides, and mixed tangential-normal derivatives tend to zero. Tangential derivatives have the same limits as well. Its Christoffel symbols are therefore C^1, hence locally Lipschitz, and the geodesic initial value problem has uniqueness. No smoothness of the doubled metric beyond C^2 is needed.

Folding a curve in $\widehat{X}$ into one copy of X does not increase its length. Inclusion and folding therefore give equality of the distances in X and $\widehat{X}$ for points of the same copy.

An X-minimizing segment is consequently a length-minimizing curve for the C^2 metric on the double. After constant-speed parametrization it satisfies the geodesic equation; this follows from the local first variation of the energy and requires only the stated C^2 regularity.

If it touched $t = 0$ while remaining in $t \ge 0$, its normal velocity would vanish at the contact point. Reflection would produce a geodesic with the same position and velocity. Uniqueness would make the whole segment fixed by reflection, hence contained in the boundary, contradicting the interior endpoints. □

The boundary-avoidance property will also be used in the standard *Calabi trick*: a minimizing segment in X with interior endpoints can be shifted slightly at its starting point to produce a smooth upper support for the intrinsic distance at its endpoint.

Step 2. Extend u continuously to X by zero on $\partial\Omega$, and set on Ω

$$d\mu = u\ \mathrm{d}V, \qquad f = -\log u, \qquad L = \Delta - \langle \nabla f, \nabla \cdot \rangle = u^{-1}\,\mathrm{div}(u\nabla\cdot).$$

We use the convention

$$\mathrm{Ric}_f^m = \mathrm{Ric} + \mathrm{Hess}(f) - \frac{1}{m}\, df \otimes df \qquad (m > 0),$$

where m is the additional dimension parameter: the effective dimension is $n + m$. In the present three-dimensional argument $m = 1$, so the effective dimension is four. This is the convention used in [17, Appendix A]. Since

$$\mathrm{Hess}(f) = -\frac{\mathrm{Hess}(u)}{u} + \frac{du \otimes du}{u^2}, \qquad df \otimes df = \frac{du \otimes du}{u^2},$$

the critical equations imply

$$Lu = \frac{1}{8}u^2 + \frac{|\nabla u|^2}{u} \geq \frac{1}{8}u^2, \qquad \mathrm{Ric}_f^1 = \mathrm{Ric} + \mathrm{Hess}(f) - df \otimes df = -\frac{3}{8}u\, g. \tag{2.5}$$

The weight can vanish at $\partial\Omega$, but only interior pointwise calculations will be used.

Lemma 2.2. *There is a universal constant C with the following property. Suppose that $q \in \Omega$, $r > 0$, and $u \leq \Lambda$ on $B_{8r}^X(q)$. Then there is a Lipschitz function $\eta : X \to [0,1]$, equal to one on $B_r^X(q)$ and supported compactly in $B_{2r}^X(q)$, with the following property: for every $x \in \Omega$ such that $\eta(x) > 0$, there is a smooth function η_x in a neighborhood of x with*

$$\eta_x \leq \eta, \qquad \eta_x(x) = \eta(x) > 0,$$

and

$$\frac{|\nabla \eta_x(x)|^2}{\eta(x)} \leq \frac{C}{r^2}, \qquad L\eta_x(x) \geq -C\left(\frac{1}{r^2} + \frac{\sqrt{\Lambda}}{r}\right). \tag{2.6}$$

Proof. Write $\rho = d_X(q, \cdot)$ and $\kappa = \sqrt{\Lambda/8}$. Every minimizing segment from q to an interior point of $B_{3r}^X(q)$ stays in the smooth interior by Lemma 2.1, and lies in $B_{3r}^X(q)$. Along such a segment, $\mathrm{Ric}_f^1 = -3u\, g/8 \geq -3\kappa^2 g$. At a point where ρ is smooth, put $m_\rho = L\rho$. Bochner's formula and the radial Hessian inequality give

$$m_\rho' = -|\,\mathrm{Hess}\,\rho|^2 - \mathrm{Ric}_f^1(\dot\gamma, \dot\gamma) - \big((f \circ \gamma)'\big)^2 \leq -\frac{m_\rho^2}{3} + 3\kappa^2.$$

Here the inequality uses $|\,\mathrm{Hess}\,\rho|^2 \geq (\Delta\rho)^2/2$ and $(\Delta\rho)^2/2 + (f')^2 \geq (L\rho)^2/3$. Since $m_\rho(t) = 2/t + O(1)$ near the center, Riccati comparison yields the following local form of [17, Theorem A.1]:

$$L\rho \leq 3\kappa \coth(\kappa\rho) \leq \frac{3}{\rho} + 3\kappa \tag{2.7}$$

up to the first cut point, with the interpretation $3/\rho$ when $\kappa = 0$. We now use this estimate at an interior point $x \in B_{2r}^X(q) \setminus \{q\}$, which is the only region needed for the cutoff. Let $\gamma : [0, \ell] \to X$ be an arclength minimizing segment from q to x. It is contained in Ω. For $0 < \delta < \ell$, set $q_\delta = \gamma(\delta)$ and

$$\rho_\delta(y) = \delta + d_X(q_\delta, y).$$

The triangle inequality gives $\rho_\delta \geq \rho$ with $\rho_\delta(x) = \rho(x) = \ell$. Moreover $d_X(q_\delta, \cdot)$ is smooth near x. Indeed, a second minimizing segment from q_δ to x would concatenate with $\gamma|_{[0,\delta]}$ to give a minimizing broken curve from q to x, and therefore must coincide with γ. Nor can x

be conjugate to q_δ along γ. To see this, suppose that a nonzero Jacobi field J vanishes at δ and ℓ, and extend it by zero on $[0,\delta]$. This is a continuous piecewise smooth field with zero endpoints. The index form I of γ is nonnegative, because γ minimizes and its image is a compact subset of Ω. The Jacobi equation gives $I(J,J)=0$. Applying nonnegativity to $J+\tau V$, for every real τ and every smooth endpoint-zero field V, then gives $I(J,V)=0$. Integration by parts on $[\delta,\ell]$ yields

$$I(J,V) = -\langle D_t J(\delta^+), V(\delta)\rangle.$$

Since $V(\delta)$ is arbitrary, $D_tJ(\delta^+)=0$. Together with $J(\delta)=0$, uniqueness for the Jacobi equation would force $J=0$, a contradiction.

Finally, choose a compact neighborhood $K\subset\Omega$ of x. All minimizing segments from q_δ to points of K have uniformly bounded lengths and lie in a fixed compact ball of X. If they were not uniformly separated from $X\setminus\Omega$, a sequence of such segments, parametrized on $[0,1]$ with constant speed, would converge uniformly to a minimizing segment from q_δ to some point of K that touches the boundary. Here one uses continuity of the distance and lower semicontinuity of length; the limit has the distance between its endpoints as its length. Lemma 2.1 excludes this possibility. Thus these segments lie in a common compact subset of Ω.

The exponential map at q_δ is defined on a neighborhood of the initial vector of $\gamma|_{[\delta,\ell]}$, and its differential is invertible there by the absence of conjugacy. Every minimizing segment to an endpoint tending to x has initial vector tending to this vector: otherwise the preceding compactness and the geodesic equation would give a different minimizing segment to x. The local inverse of the exponential map therefore describes all nearby minimizers, and their lengths are a smooth function of the endpoint. This proves that $d_X(q_\delta,\cdot)$ is smooth near x, despite the possible incompleteness of Ω. The segment from q_δ to x is a subsegment of γ, hence lies in $B^X_{2r}(q)$, where $u\le\Lambda$. The same Riccati comparison from q_δ to x therefore gives

$$L\rho_\delta(x) \le \frac{3}{\ell-\delta} + 3\kappa. \tag{2.8}$$

Choose smooth nonincreasing $\zeta:[0,\infty)\to[0,1]$, equal to one on $[0,1]$, compactly supported in $[0,2)$, and satisfying $(\zeta')^2\le C\zeta$ and $|\zeta''|\le C$. Set $\eta=\zeta(\rho/r)$. This function is Lipschitz and has compact support because X is proper. When $\rho(x)<r$, one can take $\eta_x=1$. For $r\le\rho(x)<2r$ and $\eta(x)>0$, take

$$\eta_x(y) = \zeta\big(\rho_\delta(y)/r\big)$$

near x, with $0<\delta<r/2$. Since $\rho_\delta\ge\rho$ and $\zeta'\le 0$, this is a smooth lower support for η at x. At x, $|\nabla\rho_\delta|=1$ and the chain rule, together with (2.8) and $\rho(x)-\delta\ge r/2$, yields

$$\frac{|\nabla\eta_x|^2}{\eta(x)} \le \frac{C}{r^2}, \qquad L\eta_x(x) \ge -C\left(\frac{1}{r^2}+\frac{\kappa}{r}\right).$$

Increasing the universal constant proves the assertion. □

Step 3. Assume that a negative nodal domain Ω exists, and retain the notation above. Fix $x_0\in\Omega$ and $A>1$. We first choose a point $q\in\Omega$ at whose natural scale u is controlled: with $U=u(q)>0$, we require

$$U \ge u(x_0), \qquad u\le 4U \text{ on } B^X_{8A/\sqrt{U}}(q). \tag{2.9}$$

The following point-selection argument is the doubling construction of [16, Lemma 5.1], adapted here to the continuous nonnegative function $\sqrt{u}$ on the complete space X. We include the argument so that its behavior at the nodal boundary is explicit. Start at $q_0=x_0$. If the desired local bound fails at q_j, choose

q_{j+1} within distance $8A/\sqrt{u(q_j)}$ where $\sqrt{u(q_{j+1})} > 2\sqrt{u(q_j)}$. An infinite sequence would satisfy

$$\sqrt{u(q_j)} \geq 2^j \sqrt{u(x_0)}, \qquad \sum_{j=0}^{\infty} d_X(q_j, q_{j+1}) \leq \frac{16A}{\sqrt{u(x_0)}}.$$

It would therefore converge in the complete space X, contradicting continuity of the extension of u because $u(q_j) \to \infty$. The selection must stop, proving (2.9).

Apply Lemma 2.2 with $r = A/\sqrt{U}$ and $\Lambda = 4U$. It supplies a cutoff η whose smooth lower supports satisfy

$$\frac{|\nabla \eta_x(x)|^2}{\eta(x)} \leq \frac{CU}{A^2}, \qquad L\eta_x(x) \geq -CU(A^{-1} + A^{-2}).$$

The function $F = \eta u$, extended by zero to $X \setminus \Omega$, is continuous and compactly supported in X, since X is proper and $u = 0$ on $\partial\Omega$. Therefore it attains its positive maximum at some $x_* \in \Omega$, with $F(x_*) \geq F(q) = U$ and $\eta(x_*) > 0$. Let η_* be the smooth lower support given by the lemma at x_*. Since $u > 0$ nearby, $F_* = u\eta_*$ has a local maximum at x_*, so

$$\nabla F_*(x_*) = 0, \qquad LF_*(x_*) \leq 0.$$

The first identity gives $\nabla u = -u\nabla\eta_*/\eta_*$ at x_*. Hence

$$\begin{aligned} 0 \geq LF_*(x_*) &= \eta_* Lu + uL\eta_* + 2\langle \nabla u, \nabla \eta_* \rangle \\ &\geq u\left(\frac{F(x_*)}{8} - CU(A^{-1} + A^{-2}) - \frac{2CU}{A^2} \right). \end{aligned}$$

Choosing A universally large makes the parentheses strictly positive, because $F(x_*) \geq U$. This contradiction excludes every negative nodal domain. Therefore $S \geq 0$, and [6, Theorem 1.2] gives $S \equiv 0$ and this concludes the proof of Theorem 1.1.

3. Dimension four: complete non-Einstein examples

In dimension four, the critical equations become

$$S\,\mathrm{Ric} - \mathrm{Hess}(S) = \frac{1}{4}S^2 g, \qquad \Delta S = 0. \tag{3.1}$$

As already recalled in the introduction, harmonicity settles the L^p case. The construction below shows that bounded scalar curvature alone does not imply rigidity.

Proposition 3.1. *Let (M^4, g) be complete, connected, without boundary, and critical for $\mathfrak{G}^2$. If M is compact, or if $S \in L^p(M, g)$ for some $1 < p < \infty$, then S is constant and g is scalar-flat or Einstein. In the noncompact case, a nonzero constant scalar curvature under this L^p assumption is necessarily negative and $\mathrm{Vol}_g(M) < \infty$.*

Proof. On a compact manifold, harmonicity forces S to be constant; in the complete case, the same result follows by the classical L^p Liouville argument of Yau [18].

If $S = c \neq 0$, (3.1) gives $\mathrm{Ric} = (c/4)g$. In the noncompact case the L^p assumption implies $\mathrm{Vol}_g(M) < \infty$, and the Bonnet–Myers theorem forces $c < 0$. □

The Einstein alternative is necessary: a complete finite-volume hyperbolic four-manifold has $S = -12 \in L^p$ for every $p > 0$. Also, a complete noncompact critical metric with $S \geq 0$ is scalar-flat by [6, Theorem 1.3] and the Bonnet–Myers theorem. The following family has bounded, strictly negative, nonconstant scalar curvature with infinite energy.

Proposition 3.2 (Complete non-Einstein examples)**.** *Let $(N^3, \bar{h})$ be a closed connected hyperbolic manifold with $\mathrm{Ric}_{\bar{h}} = -2\bar{h}$. For $0 < Q < 16/\sqrt{3}$, set*

$$V_Q(r) = -1 + \frac{r^2}{12} + \frac{Q^2}{4r^4}, \qquad r > 0,$$

and let b_0 be the larger positive zero of V_Q. The solution of

$$b'' = \frac{b}{12} - \frac{Q^2}{2b^5}, \qquad b(0) = b_0, \qquad b'(0) = 0 \tag{3.2}$$

is positive, even, and defined on $\mathbb{R}$. Define

$$w(s) = Q\int_0^s b(t)^{-3}\,dt, \qquad W_Q = Q\int_0^\infty b(s)^{-3}\,ds.$$

Then $0 < W_Q < \infty$, and

$$g_Q = e^{-w(s)}(ds^2 + b(s)^2\bar{h}) \tag{3.3}$$

is a smooth complete critical metric of $\mathfrak{S}^2$ on $\mathbb{R}\times N$. It is not Einstein, and its scalar curvature satisfies

$$S_{g_Q} = -e^{w(s)}, \qquad \lim_{s\to\pm\infty} S_{g_Q}(s) = -e^{\pm W_Q}.$$

In particular, S_{g_Q} is bounded and belongs to no $L^p(M, g_Q)$ with $0 < p < \infty$.

Proof. For $h = ds^2 + b^2\bar{h}$ and $w = w(s)$, consider the auxiliary system

$$\mathrm{Ric}_h = -\frac{1}{4}h + \frac{3}{2}\,dw\otimes dw, \qquad \Delta_h w = 0. \tag{3.4}$$

We will verify below that this system makes $e^{-w}h$ a critical metric. The warped-product formulas are

$$\mathrm{Ric}_h(\partial_s,\partial_s) = -3\frac{b''}{b}, \qquad \mathrm{Ric}_h|_{TN} = \frac{-2 - bb'' - 2(b')^2}{b^2}\,h|_{TN}\,, \qquad \Delta_h w = \frac{(b^3w')'}{b^3}.$$

Thus, the harmonicity of w gives $b^3w' = Q$, with Q constant. The radial Ricci equation becomes

$$-3\frac{b''}{b} = -\frac{1}{4} + \frac{3Q^2}{2b^6}, \qquad \text{hence} \qquad b'' = \frac{b}{12} - \frac{Q^2}{2b^5}.$$

The tangential equation is $-2 - bb'' - 2(b')^2 = -b^2/4$. Substituting the expression for b'' yields

$$(b')^2 = -1 + \frac{b^2}{12} + \frac{Q^2}{4b^4} = V_Q(b).$$

This derives both the ODE and its first integral from the auxiliary system. Conversely, these two equations together with $w' = Qb^{-3}$ imply (3.4).

The function V_Q tends to $+\infty$ as $r\to 0^+$ and as $r\to\infty$, and has a unique critical point, at which

$$r_* = (6Q^2)^{1/6}, \qquad V_Q(r_*) = -1 + \frac{(6Q^2)^{1/3}}{8} < 0.$$

Thus, it has two simple positive zeros. Since $V_Q(\sqrt{8}) < 0 < V_Q(\sqrt{12})$, the larger zero satisfies $\sqrt{8} < b_0 < \sqrt{12}$ and $V_Q'(b_0) > 0$. The initial value problem and its first integral give

$$(b')^2 = V_Q(b), \qquad b''(0) = \tfrac{1}{2}V_Q'(b_0) > 0. \tag{3.5}$$

Uniqueness makes b even. In its positive-time interval of existence, $b' > 0$ and the first integral give $b' = \sqrt{V_Q(b)}$. Separating variables yields

$$s = \int_{b_0}^{b(s)} \frac{dr}{\sqrt{V_Q(r)}}.$$

This integral is finite at its lower endpoint because b_0 is a simple zero and $V_Q'(b_0) > 0$. At infinity its integrand is asymptotic to $\sqrt{12}/r$, so

$$\int_{b_0}^{\infty} \frac{dr}{\sqrt{V_Q(r)}} = \infty.$$

Thus b cannot tend to infinity at a finite value of s. It also cannot approach a finite limit larger than b_0, since $b' = \sqrt{V_Q(b)} > 0$ there. The ODE continuation theorem now gives existence for all $s \geq 0$ and $b(s) \to \infty$; evenness gives the same conclusion at the other end.

The functions just constructed define h, w on $M = \mathbb{R} \times N$ and they satisfy (3.4) on M. Let $u = e^w$, so that $h = ug_Q$. Since $h = e^w g_Q$, the four-dimensional conformal formulas give

$$\Delta_h w = e^{-w}(\Delta_{g_Q} w + |\nabla w|^2_{g_Q}).$$

Thus $\Delta_h w = 0$ implies both $\Delta_{g_Q} w + |\nabla w|^2_{g_Q} = 0$ and $\Delta_{g_Q} u = u(\Delta_{g_Q} w + |\nabla w|^2_{g_Q}) = 0$. The transformation law for the Ricci tensor is (see e.g. [7])

$$\mathrm{Ric}_h = \mathrm{Ric}_{g_Q} - \mathrm{Hess}_{g_Q}(w) + \tfrac{1}{2} dw \otimes dw - \tfrac{1}{2}(\Delta_{g_Q} w + |\nabla w|^2_{g_Q}) g_Q.$$

The last term vanishes by the identity just proved.

Comparing with (3.4), we obtain

$$\mathrm{Hess}_{g_Q}(u) = u\,\mathrm{Ric}_{g_Q} + \frac{1}{4} u^2 g_Q.$$

Its trace gives $S_{g_Q} = -u$, and consequently (3.1) holds.

Finally,

$$\int_0^{\infty} b(s)^{-3}\, ds = \int_{b_0}^{\infty} \frac{dr}{r^3 \sqrt{V_Q(r)}} < \infty.$$

Integrability follows from the simple zero at b_0 and decay of order r^{-4} at infinity. Hence w is odd and strictly increasing, with finite limits $\pm W_Q$, and bounded. Since $b \geq b_0 > 0$ and N is closed, h is complete, and

$$e^{-W_Q} h \leq g_Q \leq e^{W_Q} h$$

proves completeness of g_Q. The function $S_{g_Q} = -e^w$ is nonconstant, so g_Q is not Einstein. For every $p > 0$,

$$\int_M |S_{g_Q}|^p \, \mathrm{d}V_{g_Q} = \mathrm{Vol}_{\bar{h}}(N) \int_{\mathbb{R}} e^{(p-2)w(s)} b(s)^3 \, ds = \infty,$$

because w is bounded and $b \geq b_0 > 0$. □

4. Higher-dimensional rigidity

In this section we first settle the sign of the scalar curvature without any integrability assumption; the proof uses the geometry of nodal domains established in Section 2.

Lemma 4.1. *Let (M^n, g), $n \geq 5$, be a complete, connected Riemannian manifold without boundary, critical for $\mathfrak{S}^2$. Then either $S \equiv 0$ or $S < 0$ everywhere.*

Proof. Suppose that a connected component Ω of $\{S > 0\}$ exists, and set

$$u = S|_\Omega > 0, \qquad a_n = \frac{n-4}{4(n-1)} > 0, \qquad b_n = \frac{3}{4(n-1)}.$$

The critical equations give

$$\mathrm{Hess}(u) = u\,\mathrm{Ric} - b_n u^2 g, \qquad \Delta u = a_n u^2.$$

The proof of Lemma 2.1 applies with b_n in place of $3/8$ and with the positive side of the nodal set in place of the negative side. Indeed, the geodesic uniqueness argument uses only $\mathrm{Hess}(S) = S\,\mathrm{Ric} - b_n S^2 g$; the subsequent completion and boundary-avoidance arguments use regularity and total geodesicity of $\{S = 0\}$. Thus the intrinsic completion X of Ω is a proper geodesic space, naturally identified with $\overline{\Omega}$, and minimizing segments with interior endpoints stay in Ω. The function u extends continuously to X, with value zero on $\partial\Omega$.

As in the three-dimensional proof, put

$$f = -\log u, \qquad d\mu = u\ \mathrm{d}V, \qquad L = \Delta - \langle \nabla f, \nabla \cdot \rangle.$$

This time the identities corresponding to (2.5) are

$$Lu = a_n u^2 + \frac{|\nabla u|^2}{u} \geq a_n u^2, \qquad \mathrm{Ric}^1_f = b_n u\, g \geq 0. \tag{4.1}$$

Fix $q \in \Omega$. The distance comparison and Calabi argument in the proof of Lemma 2.2 apply in any dimension. This time $\mathrm{Ric}^1_f = b_n u g \geq 0$, and the corresponding radial Riccati inequality is

$$(L\rho)' \leq -\frac{(L\rho)^2}{n}, \qquad \rho = d_X(q, \cdot),$$

because the effective dimension is $n+1$. Thus the comparison from any slightly shifted basepoint along an interior minimizing segment gives $L\rho_\delta(x) \leq n/(\rho(x) - \delta)$. Using the same nonincreasing function ζ as before, for any $r > 0$ the function $\eta = \zeta(\rho/r)$ has compact support in $B^X_{2r}(q)$, equals one on $B^X_r(q)$, and at every point $x \in \Omega$ with $\eta(x) > 0$ admits a smooth lower support η_x satisfying

$$\frac{|\nabla \eta_x(x)|^2}{\eta(x)} \leq \frac{C_n}{r^2}, \qquad L\eta_x(x) \geq -\frac{C_n}{r^2}. \tag{4.2}$$

No local bound for u is needed.

Set $U = u(q) > 0$ and $F = \eta u$ on Ω, extended by zero to $X \setminus \Omega$. This function is continuous and has compact support; it reaches its maximum at some $x_* \in \Omega$, where $F(x_*) \geq U$ and $\eta(x_*) > 0$. Let η_* be the smooth lower support at x_*. Then $F_* = u\eta_*$ has a local maximum at x_*, whence $\nabla F_*(x_*) = 0$ and $LF_*(x_*) \leq 0$. Using $Lu \geq a_n u^2$, we obtain at x_*

$$\begin{aligned} 0 \geq LF_* &= \eta_* Lu + uL\eta_* + 2\langle \nabla u, \nabla \eta_* \rangle \\ &\geq u\left(a_n F(x_*) - \frac{3C_n}{r^2}\right) \geq u\left(a_n U - \frac{3C_n}{r^2}\right). \end{aligned}$$

Taking $r^2 > 3C_n/(a_n U)$ gives a contradiction. Hence $S \leq 0$ on M. Finally, $\Delta S = a_n S^2 \geq 0$, so the strong maximum principle and connectedness imply either $S \equiv 0$ or $S < 0$ everywhere. □

We may now assume $S < 0$ and set $u = -S > 0$. From this point onward we impose $S \in L^p(M, g)$ for some $0 < p < \infty$. Its first role is to make $u^p\ \mathrm{d}V$ a finite measure. We use this measure to bound $|\nabla u^{-1/2}|$ directly by a dimensional constant. The scalar equation will turn the latter estimate into rigidity in the stated range of exponents, including the upper endpoints.

4.1. **The gradient estimate.** Set $w = u^{-1/2}$ and $H = |\nabla w|$, so that $H^2 = |\nabla u|^2/(4u^3)$ is invariant under constant rescaling of the metric. We first obtain a preliminary bound by testing a normalized excess against the finite measure $u^p \, \mathrm{d}V$. A second test, localized near the supremum of H^2, gives the dimensional constant in the following proposition.

Proposition 4.2. *Let (M^n, g), $n \geq 5$, be a complete critical metric of $\mathfrak{S}^2$, and assume that $S < 0$ and $S \in L^p(M, g)$ for some $0 < p < \infty$. Then, setting $u = -S$,*

$$\frac{|\nabla u|^2}{u^3} \leq 4K_n, \qquad K_n = \max\left\{\frac{1}{8(n-1)}, \frac{(n-4)^2}{24(n-1)(n+2)}\right\}. \tag{4.3}$$

Remark 4.3. Equivalently,

$$\frac{|\nabla u|^2}{u^3} \leq \begin{cases} 1/[2(n-1)], & 5 \leq n \leq 9, \\ (n-4)^2/[6(n-1)(n+2)], & n \geq 10. \end{cases}$$

Proof. Set

$$u = -S > 0, \qquad w = u^{-1/2}, \qquad H = |\nabla w|.$$

The critical system is

$$\mathrm{Ric} = \frac{\mathrm{Hess}(u)}{u} - \frac{3}{4(n-1)} u g, \qquad \Delta u = -\frac{n-4}{4(n-1)} u^2. \tag{4.4}$$

Throughout the proof set

$$c_n = \frac{n-4}{8(n-1)}, \qquad d_n = \frac{n+2}{8(n-1)}.$$

The critical equations and $u = w^{-2}$ give

$$w\Delta w = 3H^2 + c_n, \qquad \mathrm{Ric} = -\frac{2}{w}\,\mathrm{Hess}(w) + \frac{6}{w^2} dw \otimes dw - \frac{3}{4(n-1)w^2} g. \tag{4.5}$$

We use the symmetric diffusion operator

$$L_p = \Delta - \frac{2p}{w}\langle \nabla w, \nabla \cdot \rangle = w^{2p}\,\mathrm{div}(w^{-2p}\nabla \cdot), \qquad d\nu_p = w^{-2p}\, dV_g = u^p\, dV_g.$$

Thus $\nu_p(M) = \int_M d\nu_p < \infty$.

Completeness provides compactly supported Lipschitz cutoffs η_r, with

$$0 \leq \eta_r \leq 1, \qquad \eta_r = 1 \text{ on } B_r(o), \qquad \mathrm{supp}\,\eta_r \subset B_{2r}(o), \qquad |\nabla \eta_r| \leq C/r. \tag{4.6}$$

Bochner's formula, with (4.5), yields

$$L_p(H^2) = 2|\,\mathrm{Hess}(w)|^2 + \frac{6H^4 - 2d_n H^2}{w^2} + \frac{4-2p}{w}\langle \nabla w, \nabla H^2 \rangle, \tag{4.7}$$

where we used

$$\nabla \Delta w = \frac{3}{w}\nabla H^2 - \frac{3H^2 + c_n}{w^2}\nabla w, \qquad 2\,\mathrm{Hess}(w)(\nabla w, \nabla w) = \langle \nabla w, \nabla H^2 \rangle.$$

Step 1. Let

$$G = \frac{H^2 - 1/12}{w^2};$$

the value $1/12$ is chosen in order to factorize the nonlinear term in H^2. Indeed,

$$L_p(w^{-2}) = \frac{4pH^2 - 2c_n}{w^4}, \qquad \nabla(w^{-2}) = -2w^{-3}\nabla w.$$

Using these two identities, the product rule in (4.7) gives, for any constant K,

$$L_p\left(\frac{H^2-K}{w^2}\right) = \frac{2}{w^2}|\operatorname{Hess}(w)|^2 - \frac{2p}{w}\left\langle \nabla w, \nabla\left(\frac{H^2-K}{w^2}\right)\right\rangle + \frac{6H^4 - \frac{1}{2}H^2 + 2c_nK}{w^4}. \tag{4.8}$$

Here we used $2(c_n+d_n)=1/2$. For $K=1/12$, $6H^4-H^2/2 = 6H^2(H^2-1/12)$, so

$$L_pG = \frac{2}{w^2}|\operatorname{Hess}(w)|^2 - \frac{2p}{w}\langle \nabla w, \nabla G\rangle + 6\frac{H^2}{w^2}G + \frac{c_n}{6w^4}. \tag{4.9}$$

Let $G_+ = \max\{G,0\}$. Choose an integer $k\geq 4$ with $k-2\geq p^2$ and put $\eta=\eta_r$. We multiply (4.9) directly by $\eta^{2k}G_+^{k-2}$ and integrate with respect to $d\nu_p$. This is an admissible compactly supported Lipschitz test: since $k\geq 4$, the function $t\mapsto (t_+)^{k-2}$ is continuously differentiable and

$$\nabla(G_+^{k-2}) = (k-2)G_+^{k-3}\nabla G.$$

The test vanishes where $G\leq 0$, while $G=G_+$ where it is positive; note that the terms involving $|\operatorname{Hess}(w)|^2$ and c_n in (4.9) are nonnegative and may be discarded. Let

$$A=\int_M \eta^{2k}\frac{H^2}{w^2}G_+^{k-1}\,d\nu_p, \qquad B=\int_M \eta^{2k}G_+^{k-3}|\nabla G_+|^2\,d\nu_p;$$

integration by parts, using $|\nabla w|=H$ and the Cauchy–Schwarz inequality, gives

$$6A+(k-2)B \leq 2k\int_M \eta^{2k-1}G_+^{k-2}|\nabla\eta|\,|\nabla G_+|\,d\nu_p + 2p\int_M \eta^{2k}\frac{H}{w}G_+^{k-2}|\nabla G_+|\,d\nu_p.$$

Young's inequality bounds the first term by

$$\frac{k-2}{2}B + \frac{2k^2}{k-2}\int_M \eta^{2k-2}G_+^{k-1}|\nabla\eta|^2\,d\nu_p$$

and the second by

$$\frac{k-2}{2}B + \frac{2p^2}{k-2}A \leq \frac{k-2}{2}B + 2A.$$

Consequently,

$$4A \leq \frac{2k^2}{k-2}\int_M \eta^{2k-2}G_+^{k-1}|\nabla\eta|^2\,d\nu_p. \tag{4.10}$$

Set $I_r=\int_M \eta_r^{2k}G_+^k\,d\nu_p$. Since $\frac{H^2}{w^2}\geq G_+$, we have $I_r\leq A$. Hölder's inequality with conjugate exponents $k/(k-1)$ and k yields

$$I_r \leq C_k I_r^{(k-1)/k}\left(\int_M |\nabla\eta_r|^{2k}\,d\nu_p\right)^{1/k}.$$

If $I_r>0$ we divide and raise to the power k; if $I_r=0$ the resulting inequality is automatic. Thus

$$I_r \leq C_{k,p}\,r^{-2k}\nu_p(M)\longrightarrow 0. \tag{4.11}$$

On any fixed ball $\eta_r=1$ for all sufficiently large r; it follows that $G_+=0$ there, and hence everywhere. We have proved

$$H^2\leq \frac{1}{12} \quad \text{on } M. \tag{4.12}$$

Step 2. At a point with $H > 0$, choose an orthonormal frame with $e_1 = \nabla w/H$. The sum of the remaining diagonal Hessian entries is $\Delta w - \mathrm{Hess}(w)(e_1, e_1)$. Cauchy–Schwarz on these $n-1$ entries, together with nonnegativity of the omitted squares, gives

$$\begin{aligned} |\mathrm{Hess}(w)|^2 &\geq \mathrm{Hess}(w)(e_1,e_1)^2 + \frac{\big(\Delta w - \mathrm{Hess}(w)(e_1,e_1)\big)^2}{n-1} \\ &= \frac{(\Delta w)^2 - 2\Delta w\ \mathrm{Hess}(w)(e_1,e_1)}{n-1} + \frac{n}{n-1}\,\mathrm{Hess}(w)(e_1,e_1)^2 \\ &\geq \frac{(\Delta w)^2 - 2\Delta w\ \mathrm{Hess}(w)(e_1,e_1)}{n-1}. \end{aligned}$$

Since

$$\Delta w = \frac{3H^2 + c_n}{w}, \qquad 2H^2\,\mathrm{Hess}(w)(e_1,e_1) = \langle \nabla w, \nabla H^2\rangle,$$

(4.7) implies

$$L_p(H^2) \geq \frac{F_n(H^2)}{w^2} + \frac{b_p(H^2)}{w}\langle \nabla w, \nabla H^2\rangle, \tag{4.13}$$

where

$$\begin{aligned} F_n(s) &= \frac{2(3s+c_n)^2}{n-1} + 6s^2 - 2d_n s = \frac{6(n+2)}{n-1}(s-\kappa_1)(s-\kappa_2), \\ b_p(s) &= 4 - 2p - \frac{2(3 + c_n/s)}{n-1}, \qquad \kappa_1 = \frac{1}{8(n-1)}, \quad \kappa_2 = \frac{(n-4)^2}{24(n-1)(n+2)}. \end{aligned} \tag{4.14}$$

In particular, $F_n(s) > 0$ for $s > K_n$.

By (4.12), the number $M_0 = \sup_M H^2$ is finite. Suppose that $M_0 > K_n$ and put $m_0 = (M_0 + K_n)/2$. On the compact interval $[m_0, M_0]$, choose constants

$$a_0 = \min_{[m_0,M_0]} F_n > 0, \qquad b_0 = \max_{[m_0,M_0]} |b_p(s)|\sqrt{s} < \infty.$$

Take $0 < \varepsilon < M_0 - m_0$ so small that $b_0^2\varepsilon \leq a_0/2$, and define

$$K = M_0 - \varepsilon, \qquad Z = (H^2 - K)_+.$$

Then $0 \leq Z \leq \varepsilon$ and Z is positive somewhere, by the definition of the supremum. On $\{Z > 0\}$, inequality (4.13) gives

$$L_p(H^2) \geq \frac{a_0}{w^2} - \frac{b_0}{w}|\nabla(H^2)|.$$

We test this inequality directly with $\eta_r^2 Z$.

The support of this Lipschitz test lies in the compact set $\operatorname{supp}\eta_r \cap \{H^2 \geq K\} \subset \{H > 0\}$, because $K > 0$. Thus the coefficient $b_p(H^2)$ is bounded near its support.

The first-order chain rule gives, almost everywhere,

$$\langle \nabla(H^2), \nabla Z\rangle = |\nabla Z|^2, \qquad Z\nabla(H^2) = Z\nabla Z.$$

Integration by parts applied to the smooth function H^2 therefore gives, writing $\eta = \eta_r$,

$$\begin{aligned} \int_M \eta^2 |\nabla Z|^2\, d\nu_p + a_0 \int_M \eta^2 \frac{Z}{w^2}\, d\nu_p \leq 2\int_M \eta Z |\nabla\eta|\,|\nabla Z|\, d\nu_p \\ + b_0 \int_M \eta^2 \frac{Z}{w}|\nabla Z|\, d\nu_p. \end{aligned}$$

Using again Young's inequality, the first term on the right is at most $\frac{1}{4}\int \eta^2|\nabla Z|^2\,d\nu_p + 4\int Z^2|\nabla\eta|^2\,d\nu_p$, while the second is at most $\frac{1}{4}\int \eta^2|\nabla Z|^2\,d\nu_p + b_0^2\int \eta^2 Z^2/w^2\,d\nu_p$. Using $Z^2 \le \varepsilon Z$ and the choice of ε, we obtain

$$\frac{1}{2}\int_M \eta_r^2|\nabla Z|^2\,d\nu_p + \frac{a_0}{2}\int_M \eta_r^2\frac{Z}{w^2}\,d\nu_p \le 4\int_M Z^2|\nabla\eta_r|^2\,d\nu_p \le \frac{C\varepsilon^2}{r^2}\nu_p(M). \tag{4.15}$$

Let $r\to\infty$ while keeping a fixed ball on the left. We conclude that Z/w^2 has zero integral on that ball. Since $w>0$ and Z is continuous, $Z=0$ everywhere, contradicting its positivity somewhere. Thus $M_0 \le K_n$, proving (4.3). □

Remark 4.4. The constant K_n in Proposition 4.2 is optimal in dimension $n=6$ and in every dimension $n\ge 10$. Indeed, the complete examples constructed in Section 5 have scalar curvature in L^p for suitable $p>1$ and satisfy

$$\sup_M |\nabla(-S)^{-1/2}|^2 = K_n;$$

see Remark 5.5 for the explicit formulas. In these examples the supremum is approached at infinity, not attained at a finite point.

4.2. **Rigidity.** The gradient estimate is valid for every integrability exponent $p>0$. The restriction $p\le P_n$ enters in the following argument, where the scalar equation is integrated against powers of u.

Proof of Theorem 1.3. The compact case follows immediately by integrating

$$\Delta S = \frac{n-4}{4(n-1)}S^2.$$

Assume therefore that M is noncompact. By Lemma 4.1, either $S\equiv 0$ or $S<0$ everywhere. In the latter case set $u=-S>0$ and argue by contradiction.

Proposition 4.2 gives

$$\frac{|\nabla u|^2}{u^3} \le 4K_n, \qquad K_n = \max\left\{\frac{1}{8(n-1)}, \frac{(n-4)^2}{24(n-1)(n+2)}\right\}. \tag{4.16}$$

Since $w = u^{-1/2}$,

$$|\nabla w| \le \sqrt{K_n}.$$

Thus, fixing $o\in M$ and integrating along minimizing geodesics,

$$u(x)^{-1/2} \le u(o)^{-1/2} + \sqrt{K_n}\,d(x,o). \tag{4.17}$$

Let η_s be a standard radial cutoff with $0\le\eta_s\le 1$, equal to one on $B_s(o)$, supported in $B_{2s}(o)$, and with $|\nabla\eta_s|\le C/s$, as in (4.6).

Multiplying

$$-\Delta u = \frac{n-4}{4(n-1)}u^2$$

by $u^{p-2}\eta_s^2$ and integrating by parts gives

$$\begin{aligned}\frac{n-4}{4(n-1)}\int_M u^p\eta_s^2\,\mathrm{d}V &= (p-2)\int_M \frac{|\nabla u|^2}{u^3}u^p\eta_s^2\,\mathrm{d}V \\ &\quad + 2\int_M u^{p-2}\eta_s\langle\nabla u,\nabla\eta_s\rangle\,\mathrm{d}V.\end{aligned} \tag{4.18}$$

The last term in (4.18) tends to zero. Indeed, (4.16) gives

$$|\nabla u| \le 2\sqrt{K_n}\,u^{3/2},$$

while (4.17) gives

$$u^{-1/2} \leq Cs$$

on $B_{2s}(o) \setminus B_s(o)$ for all sufficiently large s. Consequently,

$$\begin{aligned}\left|2\int_M u^{p-2}\eta_s\langle\nabla u, \nabla\eta_s\rangle \, \mathrm{d}V\right| &\leq \frac{C}{s}\int_{B_{2s}(o)\setminus B_s(o)} u^{p-\frac{1}{2}} \, \mathrm{d}V \\ &\leq C\int_{B_{2s}(o)\setminus B_s(o)} u^p \, \mathrm{d}V \longrightarrow 0.\end{aligned}$$

Here $u^{p-1/2} = wu^p$, so the estimate uses exactly the assumed integrability of u^p, together with the linear growth bound for w. No volume growth condition or inclusion between different global L^p spaces is required.

If $0 < p \leq 2$, the first term on the right-hand side of (4.18) is nonpositive. Letting $s \to \infty$ therefore gives

$$\frac{n-4}{4(n-1)}\int_M u^p \, \mathrm{d}V \leq 0,$$

which is impossible. The global integrability needed to pass to the limit follows from the gradient estimate:

$$0 \leq u^{p-3}|\nabla u|^2 \leq 4K_n u^p \in L^1(M).$$

This holds for the original exponent p, including $0 < p < 2$.

Suppose now that $p > 2$. By (4.16),

$$\left[\frac{n-4}{4(n-1)} - 4(p-2)K_n\right]\int_M u^p\eta_s^2 \, \mathrm{d}V \leq C\int_{B_{2s}(o)\setminus B_s(o)} u^p \, \mathrm{d}V.$$

The threshold is determined by the balance between the two coefficients:

$$\frac{n-4}{4(n-1)} - 4(p-2)K_n = 4K_n(P_n - p), \qquad P_n = 2 + \frac{n-4}{16(n-1)K_n}.$$

Substitution of the two branches of K_n yields

$$P_n = \begin{cases} 2 + (n-4)/2 = n/2, & 5 \leq n \leq 9, \\ 2 + 3(n+2)/[2(n-4)] = (7n-10)/[2(n-4)], & n \geq 10. \end{cases}$$

For $p < P_n$, the coefficient is positive, and letting $s \to \infty$ contradicts $u > 0$ and $\int_M u^p \, \mathrm{d}V < \infty$.

It remains to consider the endpoint. There

$$\frac{n-4}{4(n-1)} = 4(p-2)K_n.$$

Letting $s \to \infty$ in the exact identity (4.18), using the vanishing of the

boundary term and dominated convergence, gives the following identity. The domination follows from $0 \leq |\nabla u|^2/u^3 \leq 4K_n$ and $u^p \in L^1$:

$$\int_M \left(4K_n - \frac{|\nabla u|^2}{u^3}\right) u^p \, \mathrm{d}V = 0.$$

The integrand is continuous and nonnegative, hence

$$\frac{|\nabla u|^2}{u^3} \equiv 4K_n.$$

Equivalently,

$$|\nabla w|^2 \equiv K_n > 0.$$

The vector field

$$Z = -\frac{\nabla w}{\sqrt{K_n}}$$

is smooth and has unit length. Its integral curves exist for all positive time. Indeed, on any finite time interval they stay in a closed bounded ball, which is compact by completeness; the smooth ordinary differential equation can therefore be extended beyond any finite endpoint of its interval of existence. Along such a curve γ,

$$\frac{\mathrm{d}}{\mathrm{d}t} w(\gamma(t)) = -|\nabla w| = -\sqrt{K_n}.$$

Therefore

$$w(\gamma(t)) = w(\gamma(0)) - \sqrt{K_n}\, t,$$

so w reaches zero in finite positive time. This contradicts $w = u^{-1/2} > 0$.

Hence the alternative $S < 0$ is impossible, and therefore $S \equiv 0$. □

Remark 4.5. Note that, since

$$\frac{(n-4)^2}{24(n-1)(n+2)} - \frac{1}{8(n-1)} = \frac{n-10}{24(n+2)},$$

at $n = 10$ both gradient constants agree and the endpoint is $P_{10} = 5$. The examples in Theorem 5.1 below prove that P_n is optimal for every $n \geq 10$: their scalar curvature belongs to L^p exactly for $p > P_n$.

The sign lemma also yields the following finite-volume rigidity statement.

Corollary 4.6. *Let (M^n, g) be a complete, connected, critical metric for $\mathfrak{S}^2$, with $n \geq 5$. If $\mathrm{Vol}_g(M) < \infty$, then $S_g \equiv 0$.*

Proof. By Lemma 4.1, it suffices to exclude $S < 0$. Set $u = -S > 0$ and $a_n = \frac{n-4}{4(n-1)}$. Testing $-\Delta u = a_n u^2$ with η^2/u, where η is a compactly supported cutoff, gives

$$a_n \int_M \eta^2 u \, \mathrm{d}V = -\int_M \eta^2 |\nabla \log u|^2 \, \mathrm{d}V + 2\int_M \eta \langle \nabla \log u, \nabla \eta \rangle \, \mathrm{d}V \leq \int_M |\nabla \eta|^2 \, \mathrm{d}V.$$

Choosing standard cutoffs η_R with $\eta_R = 1$ on B_R, $\eta_R = 0$ on $M \setminus B_{2R}$ and $|\nabla \eta_R| \leq C/R$, the right-hand side tends to zero because $\mathrm{Vol}(M) < \infty$.

For every ball B_{R_0} and $R \geq R_0$ we obtain

$$a_n \int_{B_{R_0}} u \, \mathrm{d}V \leq C R^{-2} \, \mathrm{Vol}(M).$$

Letting $R \to \infty$ gives $u \equiv 0$ on that ball and then on M, a contradiction.

□

5. Explicit complete examples in higher dimensions

In this section we construct an explicit family showing that completeness, bounded scalar curvature, and its pointwise decay to zero at infinity do not imply scalar-flatness. The metrics are defined on the n-dimensional product $\mathbb{R}^2 \times \mathbb{T}^{n-2}$, where $\mathbb{T}^{n-2}$ is a flat torus of dimension $n-2$ and $n \geq 5$.

For the sake of completeness, we first recall a few basic facts about moving frames (see for instance [1] or [7]).

Fix a local orthonormal coframe $\{\theta^i\}$ for g, with $i = 1, \dots, n$, and let $\{e_i\}$ be its dual frame. We follow the Einstein summation convention: repeated indices in formulas are

summed from 1 to n, unless otherwise specified. The corresponding *Levi-Civita connection forms* $\left\{\theta^i_j\right\}$ are the 1-forms uniquely defined by the requirements

$$d\theta^i = -\theta^i_j \wedge \theta^j \quad \text{(first structure equations)}, \tag{5.1}$$

$$\theta^i_j + \theta^j_i = 0. \tag{5.2}$$

The *curvature forms* $\left\{\Theta^i_j\right\}$ associated to the connection are the 2-forms defined via the *second structure equations*

$$d\theta^i_j = -\theta^i_k \wedge \theta^k_j + \Theta^i_j. \tag{5.3}$$

They are skew-symmetric (i.e. $\Theta^i_j + \Theta^j_i = 0$) and they can be written as

$$\Theta^i_j = \frac{1}{2} R^i_{jkt} \theta^k \wedge \theta^t = \sum_{k<t} R^i_{jkt} \theta^k \wedge \theta^t, \tag{5.4}$$

where R^i_{jkt} are the components of the Riemann curvature tensor in its $(1,3)$ version. Since we are working on an orthonormal coframe, the coefficients of the $(0,4)$ version of the Riemann tensor are simply given by

$$R_{ijkt} = R^i_{jkt},$$

while the coefficients of the Ricci tensor are obtained by tracing:

$$R_{ik} = \sum_{t=1}^{n} R_{itkt}.$$

Similarly, the scalar curvature S is the trace of the Ricci tensor, that is

$$S = \sum_{t=1}^{n} R_{tt}.$$

We use commas for covariant derivatives of tensor components, e.g. $R_{ij,k}$. For a scalar function we instead write subscripts without commas: in particular, S_i and S_{ij} denote the components of dS and of the Hessian of S, respectively. The covariant derivative of a vector field $X \in \mathfrak{X}(M)$ is defined by

$$dX^i + X^j \theta^i_j = X^i_{,k} \theta^k,$$

while the covariant derivative of a 1-form ω is defined by

$$d\omega_i - \omega_j \theta^j_i = \omega_{i,k} \theta^k.$$

The covariant derivative of X defines an endomorphism of TM whose matrix in the orthonormal frame is $(X^i_{,j})$. Its trace is the divergence:

$$\operatorname{div} X = X^i_{,i}. \tag{5.5}$$

For a smooth function f we can write

$$df = f_i \theta^i, \tag{5.6}$$

for smooth coefficients f_i defined on the coframe domain. The Hessian of f, $\operatorname{Hess}(f)$, is the $(0,2)$-tensor defined as

$$\operatorname{Hess}(f) = f_{ij} \theta^j \otimes \theta^i, \tag{5.7}$$

with

$$f_{ij} \theta^j = df_i - f_t \theta^t_i. \tag{5.8}$$

Note that

$$f_{ij} = f_{ji}.$$

The Laplacian of f, Δf, is the trace of the Hessian, in other words

$$\Delta f = \mathrm{tr}(\mathrm{Hess}(f)) = f_{ii}.$$

We are now ready to state the following theorem, which immediately implies Theorem 1.4 in the Introduction.

Theorem 5.1. *Let $n \geq 5$, let g_0 be a flat metric on $\mathbb{T}^{n-2}$, and set*

$$a = -\frac{n+14}{6(n+2)}, \qquad b = -\frac{n-2}{2(n+2)}, \qquad c = \frac{2}{n+2}, \qquad \lambda^2 = \frac{32(n-1)}{3(n+2)}, \quad \lambda > 0. \tag{5.9}$$

In standard polar coordinates (r,θ) on $\mathbb{R}^2$, with $\theta \in \mathbb{R}/(2\pi\mathbb{Z})$, define

$$\begin{aligned} g_n &= A(r)^2\, dr^2 + B(r)^2\, d\theta^2 + C(r)^2 g_0, \\ A(r) &= \lambda(1+r^2)^a, \qquad B(r) = \lambda r(1+r^2)^b, \qquad C(r) = (1+r^2)^c. \end{aligned} \tag{5.10}$$

Then g_n is a smooth complete critical metric of $\mathfrak{S}^2$ on $\mathbb{R}^2 \times \mathbb{T}^{n-2}$, with

$$S_{g_n} = -(1+r^2)^{-(2a+1)} = -(1+r^2)^{-\frac{2(n-4)}{3(n+2)}}. \tag{5.11}$$

In particular, g_n is neither scalar-flat nor Einstein, its scalar curvature is bounded, and $S_{g_n} \to 0$ at infinity. Moreover for every $p > 0$

$$S_{g_n} \in L^p(M, g_n) \quad \Longleftrightarrow \quad p > q_n, \tag{5.12}$$

where

$$q_n = \frac{7n-10}{2(n-4)}.$$

Writing $V_0 = \mathrm{Vol}_{g_0}(\mathbb{T}^{n-2})$, one has, for $p > q_n$,

$$\int_M |S_{g_n}|^p \, \mathrm{d}V_{g_n} = \frac{16\pi(n-1)V_0}{(n-4)(p-q_n)}. \tag{5.13}$$

Proof. Set $\Sigma = \{0\} \times \mathbb{T}^{n-2}$. Throughout the proof, Greek indices $\alpha, \beta, \gamma \in \{3, \dots, n\}$ denote directions tangent to the torus.

Step 1. First we need to choose a suitable frame, and then to compute all the relevant geometric quantities. For $r > 0$, we choose a local, parallel g_0-orthonormal coframe $\{\eta^\alpha\}_{\alpha=3}^n$ on the torus (note that such a choice is possible because g_0 is flat, and $d\eta^\alpha = 0$); thus, an orthonormal coframe for g_n is given by

$$\theta^1 = A\, dr, \qquad \theta^2 = B\, d\theta, \qquad \theta^\alpha = C\eta^\alpha \quad (\alpha = 3, \dots, n).$$

For the moment, A, B, C are arbitrary positive smooth functions on $r > 0$; primes denote derivatives with respect to r. The Levi-Civita connection forms are given by

$$\begin{aligned} \theta_1^2 &= -\theta_2^1 = \frac{B'}{AB}\theta^2, \\ \theta_1^\alpha &= -\theta_\alpha^1 = \frac{C'}{AC}\theta^\alpha, \\ \theta_2^\alpha &= -\theta_\alpha^2 = 0, \\ \theta_\beta^\alpha &= -\theta_\alpha^\beta = 0 \ (\alpha, \beta = 3, \dots, n). \end{aligned}$$

The independent nonzero components of the Riemann tensor are

$$\begin{aligned} R_{1212} &= \frac{1}{A^2 B}\left(\frac{A'}{A}B' - B''\right), \\ R_{1\alpha 1\alpha} &= \frac{1}{A^2 C}\left(\frac{A'}{A}C' - C''\right), \\ R_{2\alpha 2\alpha} &= -\frac{1}{A^2}\frac{B'C'}{BC}, \\ R_{\alpha\beta\alpha\beta} &= -\frac{(C')^2}{A^2C^2} \qquad (\alpha \neq \beta). \end{aligned}$$

All other components vanish, apart from those determined by the curvature symmetries. There is no summation over α or β in these component formulas or in the diagonal formulas below. Tracing, we find that the Ricci tensor is diagonal, with components

$$\begin{aligned} R_{11} &= -\frac{1}{A^2}\left[\frac{B''}{B} - \frac{A'B'}{AB} + (n-2)\left(\frac{C''}{C} - \frac{A'C'}{AC}\right)\right], \\ R_{22} &= -\frac{1}{A^2}\left[\frac{B''}{B} - \frac{A'B'}{AB} + (n-2)\frac{B'C'}{BC}\right], \\ R_{\alpha\alpha} &= -\frac{1}{A^2}\left[\frac{C''}{C} - \frac{A'C'}{AC} + \frac{B'C'}{BC} + (n-3)\left(\frac{C'}{C}\right)^2\right]. \end{aligned}$$

Finally, tracing again we obtain the expression for the scalar curvature:

$$S = \frac{2}{A^2 B}\left(\frac{A'}{A}B' - B''\right) + \frac{2(n-2)}{A^2 C}\left(\frac{A'}{A}C' - C''\right) - \frac{2(n-2)}{A^2}\frac{B'C'}{BC} - \frac{(n-2)(n-3)}{(AC)^2}(C')^2.$$

Since S depends only on r, equation (5.8) shows that its Hessian is also diagonal, with

$$\begin{aligned} S_{11} &= \frac{1}{A^2}\left(S'' - \frac{A'}{A}S'\right), \\ S_{22} &= \frac{1}{A^2}\frac{B'}{B}S', \\ S_{\alpha\alpha} &= \frac{1}{A^2}\frac{C'}{C}S'. \end{aligned}$$

Step 2. We now specialize the choice of A, B and C, proving the smoothness of the corresponding metric and then imposing the critical equation. We look for a critical metric within the family

$$A = \lambda(1+r^2)^a, \qquad B = \lambda r(1+r^2)^b, \qquad C = (1+r^2)^c,$$

where $a, b, c \in \mathbb{R}$ and $\lambda > 0$ are initially unknown.
We prove that the metric extends smoothly across Σ for every $a, b, c \in \mathbb{R}$ and $\lambda > 0$. Indeed, factoring out $A(r)^2$, we can write

$$g_n = \lambda^2(1+r^2)^{2a}\Big[dr^2 + r^2(1+r^2)^{2(b-a)}d\theta^2 + \lambda^{-2}(1+r^2)^{2(c-a)}g_0\Big].$$

The two-dimensional part of the metric in brackets is $dr^2 + \varphi(r)^2 d\theta^2$, where

$$\varphi(r) = r(1+r^2)^{b-a}.$$

This function is smooth and odd, with $\varphi(0) = 0$ and $\varphi'(0) = 1$. Since θ has period 2π, the standard endpoint smoothness criterion implies that this two-dimensional metric extends smoothly across the origin; see [15, Chapter 1, Sections 3–4]. Adding the torus term and

multiplying by the conformal factor therefore gives a smooth Riemannian metric on the whole $\mathbb{R}^2 \times \mathbb{T}^{n-2}$.

We now impose the critical equation and the normalization $S|_\Sigma = -1$; these conditions will determine all four parameters. Put $x = r^2$. The logarithmic derivatives are

$$\frac{A'}{A} = \frac{2ar}{1+x}, \qquad \frac{B'}{B} = \frac{1}{r} + \frac{2br}{1+x}, \qquad \frac{C'}{C} = \frac{2cr}{1+x}.$$

To simplify the writing, define the following coefficients, all independent of r:

$$\begin{aligned} U &= 3b - a + (n-2)c, \\ V_1 &= U + 2b(b-a-1) + 2(n-2)c(c-a-1), \\ V_2 &= U + 2b(b-a-1) + 2(n-2)bc, \\ J &= b - a + (n-2)c. \end{aligned}$$

The Ricci components therefore become

$$\begin{aligned} R_{11} &= -\frac{2(U+V_1 x)}{\lambda^2(1+x)^{2a+2}}, \qquad R_{22} = -\frac{2(U+V_2 x)}{\lambda^2(1+x)^{2a+2}}, \\ R_{\alpha\alpha} &= -\frac{4c(1+Jx)}{\lambda^2(1+x)^{2a+2}}. \end{aligned} \tag{5.14}$$

Taking their trace gives

$$S = \frac{4}{\lambda^2}(1+x)^{-2a-2}(P_0 + P_2 x), \tag{5.15}$$

where

$$\begin{aligned} P_0 &= a - 3b - 2(n-2)c, \\ P_2 &= (a-b)\big(1 + 2b + 2(n-2)c\big) - (n-2)(n-1)c^2. \end{aligned}$$

In particular, the prescribed value at Σ requires

$$P_0 < 0, \qquad \lambda^2 = -4P_0. \tag{5.16}$$

The first derivative of the scalar curvature can be written as

$$S' = \frac{8r}{\lambda^2}(1+x)^{-2a-3}\big(P_1 - (2a+1)P_2 x\big), \qquad \text{where } \; P_1 = P_2 - 2(a+1)P_0.$$

We first use the "torus component" of the critical equation, that is

$$SR_{\alpha\alpha} - S_{\alpha\alpha} = \frac{3}{4(n-1)}S^2.$$

The Hessian formula above gives

$$S_{\alpha\alpha} = \frac{16cx}{\lambda^4(1+x)^{4a+4}}\big(P_1 - (2a+1)P_2 x\big).$$

If $c = 0$, both $R_{\alpha\alpha}$ and $S_{\alpha\alpha}$ vanish, so the critical equation would force $S = 0$, contrary to the normalization; hence, necessarily, $c \neq 0$. Cancelling the common nonzero factor, the torus equation gives the polynomial identity

$$\begin{aligned} 0 = {} & (P_0 + P_2 x)(1 + Jx) + x\big(P_1 - (2a+1)P_2 x\big) \\ & + \frac{3}{4c(n-1)}(P_0 + P_2 x)^2. \end{aligned} \tag{5.17}$$

Its constant coefficient, since $P_0 \neq 0$, gives

$$P_0 = -\frac{4}{3}(n-1)c;$$

in particular, $c > 0$. Using this relation, the coefficient of x becomes $P_0 J + P_1 - P_2 = 0$, hence $J = 2a + 2$; the coefficient of x^2 then becomes $P_2(1 - P_2/P_0) = 0$. We have obtained the three necessary conditions

$$P_0 = -\frac{4}{3}(n-1)c, \qquad b - 3a - 2 + (n-2)c = 0, \qquad P_2 = 0 \text{ or } P_2 = P_0. \tag{5.18}$$

Conversely, these conditions make every coefficient in (5.17) vanish.

The first two conditions are linear in a, b and give

$$a = \frac{7n-10}{24}c - \frac{3}{4}, \qquad b = \frac{6-n}{8}c - \frac{1}{4}. \tag{5.19}$$

Substitution in the expression for P_2 yields

$$48P_2 = -(13n-22)(n+2)c^2 - 32(n-1)c - 12 < 0.$$

Here $n \geq 5$ and $c > 0$. The branch $P_2 = 0$ is therefore impossible, and necessarily

$$P_2 = P_0, \qquad P_1 = -(2a+1)P_0.$$

The remaining relation $P_2 = P_0$ is equivalent to

$$\big((n+2)c - 2\big)\big((13n-22)c - 6\big) = 0. \tag{5.20}$$

Thus the torus component alone leaves two possible values:

$$c = \frac{2}{n+2} \qquad \text{or} \qquad c = \frac{6}{13n-22}.$$

The radial component will distinguish between these two possibilities.

In particular, the scalar curvature has simplified to a single power:

$$S = \frac{4P_0}{\lambda^2}(1+x)^{-k}, \qquad k = 2a+1.$$

We next evaluate the radial component of the critical equation along Σ. Although the polar orthonormal frame is not defined at $r = 0$, the smoothness established above shows that Ric and Hess(S) are smooth there. By rotational invariance, their radial components have limits independent of the direction of approach. Writing $R_{11}(0)$ and $S_{11}(0)$ for these common normal-directional limits, the formulas above give

$$R_{11}(0) = -\frac{2U}{\lambda^2}, \qquad S_{11}(0) = \frac{8P_1}{\lambda^4}.$$

Thus $SR_{11} - S_{11} = 3S^2/[4(n-1)]$ at $r = 0$ is equivalent to

$$P_0 U + P_1 + \frac{3P_0^2}{2(n-1)} = 0,$$

or, after the torus relations,

$$0 = U - k - 2c = 3b - 3a - 1 + (n-4)c = \frac{1}{2} - \frac{n+2}{4}c.$$

Thus $c = 2/(n+2)$. Equations (5.19) and (5.16) now give exactly (5.9). Thus these constants are forced within the chosen family by criticality and $S|_\Sigma = -1$.

Step 3. We now verify the validity of the full critical equation. So far, the parameters have been determined using the normalization $S|_\Sigma = -1$, the torus component of the critical equation for $r > 0$, and its radial component at $r = 0$. These relations are necessary, but they do not by themselves ensure that the tensor equation is satisfied for every r. To establish

that the resulting metric is indeed critical, we now check all its independent components on $r > 0$. Substitution of the parameters just found gives

$$P_0 = P_2 = -\frac{8(n-1)}{3(n+2)}, \qquad J = 2a+2, \qquad P_1 = -(2a+1)P_0.$$

Thus all three relations in (5.18) hold, so the torus equation is satisfied for every $r > 0$. To check the two remaining components, direct differentiation gives

$$S_{11} = -\frac{8kP_0}{\lambda^4(1+x)^{4a+3}}(1-3kx),$$
$$S_{22} = -\frac{8kP_0}{\lambda^4(1+x)^{4a+3}}(1+(1+2b)x).$$

The constant coefficients of both critical equations are $U - k - 2c$, which is already zero. Their coefficients of x are, respectively, $V_1 + 3k^2 - 2c$ and $V_2 - k(1+2b) - 2c$. To evaluate these, write $D = n+2$. The parameters just found give

$$c = \frac{2}{D}, \quad k = \frac{2(D-6)}{3D}, \quad U = \frac{2}{3}, \quad b - a - 1 = -\frac{4}{3} + \frac{4}{D}, \quad c - a - 1 = -\frac{5}{6} + \frac{4}{D},$$
$$V_1 = \frac{4}{D} - \frac{4(D-6)^2}{3D^2} = 2c - 3k^2, \qquad V_2 = \frac{4}{D} + \frac{8(D-6)}{3D^2} = 2c + k(1+2b).$$

All coefficients therefore vanish. The torus component was already settled by (5.18), and all mixed components vanish. We have proved

$$S_{g_n}\,\mathrm{Ric}_{g_n} - \mathrm{Hess}_{g_n}(S_{g_n}) = \frac{3}{4(n-1)} S_{g_n}^2 g_n \tag{5.21}$$

on $r > 0$. Moreover, $4P_0/\lambda^2 = -1$ gives (5.11). Since g_n is smooth across Σ, as proved above, both sides of (5.21) are smooth on the whole manifold; thus the identity extends from $r > 0$ to Σ by continuity. Tracing (5.21) yields the scalar equation (1.3), so the full critical system holds.

Step 4. We now check the completeness of g_n: we prove that every closed metric ball is compact, and then apply the Hopf–Rinow theorem. The key point is that reaching large values of the Euclidean radius r requires arbitrarily large g_n-length, even for curves that also move in the angular or torus directions.

Define

$$\rho(r) = \int_0^r A(t)\,dt = \lambda \int_0^r (1+t^2)^a\,dt. \tag{5.22}$$

This is the length of the radial segment from Σ to radius r, with fixed angular and torus coordinates. Since $A > 0$, ρ is strictly increasing and $\rho(0) = 0$. Moreover,

$$2a+1 = \frac{2(n-4)}{3(n+2)} > 0, \qquad \rho(r) \sim \frac{\lambda}{2a+1} r^{2a+1} \longrightarrow \infty.$$

We next compare this radial length with the length of an arbitrary piecewise smooth curve $c : [t_0, t_1] \to M$. The Euclidean radius r is locally Lipschitz also at Σ, so $t \mapsto \rho(r(c(t)))$ is absolutely continuous. From the radial part $A(r)^2 dr^2$ of g_n we obtain, almost everywhere,

$$\left|\frac{d}{dt}\rho(r(c(t)))\right| = A(r(c(t)))\left|\frac{d}{dt} r(c(t))\right| \le |\dot c(t)|_{g_n}.$$

Integrating, for any such curve from p to q we find

$$|\rho(r(q)) - \rho(r(p))| \le \int_{t_0}^{t_1} \left|\frac{d}{dt}\rho(r(c(t)))\right| dt \le \ell_{g_n}(c).$$

Taking the infimum over curves gives

$$|\rho(r(q)) - \rho(r(p))| \le d_{g_n}(p,q). \tag{5.23}$$

Since $\rho = 0$ on Σ and a radial segment realizes this lower bound, we also obtain

$$d_{g_n}(p,\Sigma) = \rho(r(p)).$$

For each $L \ge 0$, let r_L be the unique radius with $\rho(r_L) = L$. The sublevel set of the radial length is

$$\{p : \rho(r(p)) \le L\} = \{(X,Y) : X^2 + Y^2 \le r_L^2\} \times \mathbb{T}^{n-2},$$

which is compact because both the closed Euclidean disk and the torus are compact. Now fix a center $o \in M$ and $R > 0$. By (5.23), every point of the closed metric ball $\overline{B}_R^{g_n}(o)$ satisfies

$$\rho(r(p)) \le \rho(r(o)) + R.$$

Thus this ball is a closed subset of a compact sublevel set, and is compact; the Hopf–Rinow theorem ensures that (M, g_n) is complete.

Finally, S_{g_n} is nonconstant, so g_n cannot be Einstein.

Step 5. Finally, we compute the integral of $|S_{g_n}|^p$. The volume form is

$$\mathrm{d}V_{g_n} = ABC^{n-2}\, dr\, d\theta\, \mathrm{d}V_{g_0} = \lambda^2 r(1+r^2)^{a+b+(n-2)c}\, dr\, d\theta\, \mathrm{d}V_{g_0}. \tag{5.24}$$

The exponents satisfy

$$a + b + (n-2)c = 2(2a+1), \qquad q_n = 2 + \frac{1}{2a+1}.$$

Consequently, the change of variable $y = 1 + r^2$ gives

$$\begin{aligned}\int_M |S_{g_n}|^p\, \mathrm{d}V_{g_n} &= 2\pi\lambda^2 V_0 \int_0^\infty r(1+r^2)^{(2-p)(2a+1)}\, dr \\ &= \pi\lambda^2 V_0 \int_1^\infty y^{(2a+1)(q_n-p)-1}\, dy.\end{aligned}$$

Since $2a + 1 > 0$, this integral converges precisely when $p > q_n$. In that case it equals

$$\frac{\pi\lambda^2 V_0}{(2a+1)(p-q_n)} = \frac{16\pi(n-1)V_0}{(n-4)(p-q_n)},$$

as claimed. Note that, in particular,

$$|S_{g_n}|^2\, \mathrm{d}V_{g_n} = \lambda^2 r\, dr\, d\theta\, \mathrm{d}V_{g_0},$$

so the energy is infinite.

□

Remark 5.2. When $n = 10$, $a = b = -1/3$, $c = 1/6$, and $\lambda^2 = 8$; hence

$$g_{10} = (1+r^2)^{1/3}\left(8\frac{dr^2 + r^2 d\theta^2}{1+r^2} + g_0\right).$$

Note that the two-dimensional metric in the fraction is *Hamilton's cigar* (see [13]), and thus the metric is conformal to a gradient steady Ricci soliton.

Remark 5.3. The same metric formula makes sense at $n = 4$, where $a = -1/2$, $b = -1/6$, $c = 1/3$, and $\lambda^2 = 16/3$. It gives

$$g_4 = \frac{16}{3}\frac{dr^2}{1+r^2} + \frac{16}{3}r^2(1+r^2)^{-1/3}d\theta^2 + (1+r^2)^{2/3}g_0 \quad \text{on } \mathbb{R}^2 \times \mathbb{T}^2.$$

The radial length is

$$\rho(r) = \frac{4}{\sqrt{3}}\operatorname{arsinh} r \longrightarrow \infty,$$

so the argument in the previous construction proves completeness. Substitution in (5.14) gives $\mathrm{Ric}_{g_4} = -g_4/4$ and $S_{g_4} = -1$, i.e. g_4 is Einstein and, in particular, it is critical. Its volume form is $\frac{16}{3} r\, dr\, d\theta\, \mathrm{d}V_{g_0}$, so its volume is infinite. The non-Einstein examples of Proposition 3.2 require a different construction.

5.1. **A six-dimensional example.** The torus family does not establish the sharpness of the rigidity endpoint in dimension six, while a different example does so.

Proposition 5.4. *Let $(N^5, \bar{h})$ be a closed Einstein manifold with $\mathrm{Ric}_{\bar{h}} = -\bar{h}/10$. Define on $\mathbb{R} \times N$*

$$v(t) = \cosh(t/\sqrt{10}), \qquad \widehat{g}_6 = v(t)(dt^2 + \bar{h}). \tag{5.25}$$

Then $\widehat{g}_6$ is smooth, complete, non-Einstein, and critical for $\mathfrak{S}^2$, with

$$S_{\widehat{g}_6} = -\operatorname{sech}(t/\sqrt{10}). \tag{5.26}$$

Moreover, for every $p > 0$,

$$S_{\widehat{g}_6} \in L^p(\mathbb{R} \times N, \widehat{g}_6) \quad \Longleftrightarrow \quad p > 3. \tag{5.27}$$

In particular, the endpoint $P_6 = 3$ of Theorem 1.3 is optimal.

Proof. Set $h = dt^2 + \bar{h}$, $u = v^{-1}$, and $a = v'/v$. Since $v'' = v/10$ and $\mathrm{Ric}_h = -\bar{h}/10$, the conformal Ricci and Hessian formulas for $\widehat{g}_6 = vh$ yield

$$\mathrm{Ric}_{\widehat{g}_6} - u^{-1} \mathrm{Hess}_{\widehat{g}_6}(u) = -\frac{3}{20}\, u \widehat{g}_6, \qquad \Delta_{\widehat{g}_6} u = -\frac{1}{10} u^2.$$

Tracing also gives $S_{\widehat{g}_6} = -u$, as asserted. As $v \geq 1$ and the product metric h is complete, so is $\widehat{g}_6$. Its scalar curvature is not constant, hence it is not Einstein.

Put $V_N = \mathrm{Vol}_{\bar{h}}(N)$. Since $dV_{\widehat{g}_6} = v^3\, dt\, dV_{\bar{h}}$, we obtain

$$\int_{\mathbb{R}\times N} |S_{\widehat{g}_6}|^p\, dV_{\widehat{g}_6} = \sqrt{10}\, V_N \int_{\mathbb{R}} \cosh^{3-p} s\, ds.$$

Using $y = \tanh s$ and then $x = y^2$, the last integral becomes

$$\int_{-1}^{1} (1 - y^2)^{(p-5)/2}\, dy = \int_0^1 x^{-1/2} (1 - x)^{(p-5)/2}\, dx. \tag{5.28}$$

At $x = 0$, the power $x^{-1/2}$ is integrable; at $x = 1$, integrability is equivalent to $(p-5)/2 > -1$, that is, $p > 3$. At $p = 3$, the integral over $[-T, T] \times N$ is exactly $2V_N T$. The distance from $\{0\} \times N$ is

$$\rho(t) = \int_0^{|t|} \sqrt{v(\tau)}\, d\tau \sim \sqrt{20}\, e^{|t|/(2\sqrt{10})}.$$

Consequently the integral over the tube $\{\rho \leq R\}$ is $4\sqrt{10} V_N \log R + O(1)$ as $R \to \infty$. This specifies the distance with respect to which the endpoint divergence is logarithmic. For example, a closed hyperbolic five-manifold, rescaled to sectional curvature $-1/40$, gives an admissible N. □

Remark 5.5. The preceding examples show that the dimensional constant in (4.3) cannot be decreased in dimension six or in any dimension $n \geq 10$. Indeed, for the torus family of Theorem 5.1, set $u = -S_{g_n}$; from the radial component of (5.10) one computes

$$H^2 = |\nabla u^{-1/2}|^2 = \frac{(n-4)^2}{24(n-1)(n+2)} \frac{r^2}{1 + r^2}.$$

For $n \geq 10$, the coefficient equals K_n; hence $\sup_{M_n} H^2 = K_n$. In dimension six, the product example (5.25) satisfies

$$H^2 = |\nabla u^{-1/2}|^2 = \frac{1}{40}\tanh^2(t/\sqrt{10}), \qquad \sup_{\mathbb{R}\times N} H^2 = \frac{1}{40} = K_6.$$

All these metrics have $S \in L^p$ for some $p > 1$, so they fall under Proposition 4.2. In both constructions the sharp constant is approached at an end; pointwise equality at a finite point is not asserted.

Use of artificial intelligence

The authors used ChatGPT (OpenAI) during the preparation of this manuscript as a supporting tool for mathematical exploration, critical discussion, and editorial revision. The authors take full responsibility for the mathematical content and the final manuscript.

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