Fractional minimality of the Simons cone for s close to 1

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Abstract

Let $\Sigma := \{(X,Y) \in \mathbb{R}^4 \times \mathbb{R}^4 : |X| = |Y|\}$ be the Simons cone and let $C := \{|X| \le |Y|\}$. We prove that C minimizes the fractional perimeter P_s, under compact perturbations, whenever s is sufficiently close to 1. The proof constructs two homogeneous families of strict one-sided barriers and converts them into nonlocal sub-calibrations.

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1 Introduction

The Simons cone is the singular hypersurface

$$\Sigma = \{(X,Y) \in \mathbb{R}^4 \times \mathbb{R}^4 : |X| = |Y|\} \subset \mathbb{R}^8. \tag{1.1}$$

We denote by

$$C := \{(X,Y) \in \mathbb{R}^4 \times \mathbb{R}^4 : |X| \le |Y|\} \tag{1.2}$$

one of the two closed cones bounded by Σ. Simons proved the relevant stability and low-dimensional rigidity results for minimal cones [13]; Bombieri, De Giorgi and Giusti subsequently proved that the cone is area minimizing, equivalently that C is locally perimeter minimizing, [2]. Calibration and sub-calibration proofs were later given, among others, by Davini [8] and by De Philippis and Paolini [9].

We study the analogous minimizing property for the fractional perimeter introduced by Caffarelli, Roquejoffre and Savin [5]. For an open set $\Omega \subset \mathbb{R}^n$, $s \in (0,1)$ and a measurable set $E \subset \mathbb{R}^n$, we use the normalization

$$\begin{aligned} P_s(E,\Omega) := \frac{1}{2}\int_\Omega \int_\Omega |\chi_E(x) - \chi_E(y)||x-y|^{n+s}\,dx\,dy \\ + \int_\Omega \int_{\mathbb{R}^n\setminus\Omega} |\chi_E(x) - \chi_E(y)||x-y|^{n+s}\,dx\,dy. \end{aligned} \tag{1.3}$$

At a smooth boundary point $p \in \partial E$, its Euler–Lagrange operator is the fractional mean curvature

$$H_s^E(p) := \mathrm{PV}\int_{\mathbb{R}^n} \chi_{E^c}(q) - \chi_E(q)|q-p|^{n+s}\,dq. \tag{1.4}$$

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The fractional perimeter converges, after the usual normalization, to the classical perimeter as $s \uparrow 1$ [4]; the corresponding curvatures converge to the classical mean curvature on smooth families [1, 6].

Nonlocal minimal cones play a central role in the regularity theory. In the plane, halfplanes are the only minimizing cones [12]. Dávila, del Pino and Wei constructed nonlocal Lawson cones and studied their stability [7]. In the symmetric $4+4$ case, the exchange map $(X, Y) \mapsto (Y, X)$ shows directly that the Simons cone is stationary for every $s \in (0, 1)$. Stationarity, however, does not imply minimality; the purpose of this paper is to establish the latter in the perturbative regime $s \uparrow 1$.

The proof is based on nonlocal calibrations. Cabré developed calibrations and null-Lagrangians associated with foliations by solutions of the nonlocal mean-curvature equation [3]. Pagliari introduced an antisymmetric scalar calibration for nonlocal total-variation energies and used it to prove the minimality and uniqueness of halfspaces [11]. Here the leaves are strict sub- and supersolutions rather than exact solutions, so the appropriate object is a one-sided nonlocal sub-calibration.

The main difficulty is uniformity at infinity. Set

$$E_1^+ := \{|X|^4 \le |Y|^4 + 1\}.$$

On compact portions of ∂E_1^+, positivity of $H_s^{E_1^+}$ follows from the local limit $s \uparrow 1$ and positivity of the classical mean curvature. This argument alone is not uniform at infinity, because the classical mean curvature decays like y^{-13} along the boundary. Rescaling by $\eta = y^{-4}$ transforms the remote boundary point into

$$P_\eta = ((1+\eta)^{1/4} e_1, e_1)$$

and the set into

$$E_\eta = \{|X|^4 \le |Y|^4 + \eta\}.$$

The key estimate is

$$H_s^{E_\eta}(P_\eta) \ge A_s \eta - C\eta^2, \qquad A_s \ge c > 0, \tag{1.5}$$

uniformly for s in a one-sided neighborhood of 1. A Laplace transform turns the curvature into a weighted integral of Gaussian sublevel probabilities. The first variation is computed explicitly using modified Bessel functions. The remainder requires two regimes: an exact noncentral chi-square representation for bounded Gaussian scale, and a quantitative normal-graph expansion with two η-derivatives for large Gaussian scale. The latter isolates a nonnegative local mean-curvature term and leaves an integrable $O(\eta^2 t^{-3/2})$ error.

The estimate (1.5) yields strict mean-convexity of all positive homogeneous leaves of $\Phi(X, Y) = |X|^4 - |Y|^4$ and, by symmetry, of the corresponding leaves of $-\Phi$. The two resulting one-sided calibrations give outward and inward minimality of C; submodularity then produces minimality against arbitrary compact perturbations.

We now state the result. Throughout the paper,

$$\Phi(X, Y) := |X|^4 - |Y|^4, \qquad C := \{\Phi \le 0\}. \tag{1.6}$$

Sets are identified up to Lebesgue-null modifications.

Theorem 1.1. *There exists $s_0 \in (0, 1)$ such that, for every $s \in (s_0, 1)$, the set C is minimizing for the fractional perimeter under compact perturbations. More precisely, for every bounded open set $\Omega \subset \mathbb{R}^8$ and every measurable set $F \subset \mathbb{R}^8$ satisfying $F \,\Delta\, C \Subset \Omega$,*

$$P_s(C, \Omega) \le P_s(F, \Omega).$$

The analytical core is the following barrier theorem.

Theorem 1.2. *There exists $s_0 \in (0,1)$ such that, for every $s \in (s_0, 1)$,*

$$H_s^{E_1^+}(p) > 0 \qquad \text{for every } p \in \partial E_1^+, \tag{1.7}$$

where

$$E_1^+ := \{|X|^4 \le |Y|^4 + 1\}.$$

By the symmetry $(X,Y) \mapsto (Y,X)$, the analogous statement holds for

$$E_1^- := \{|Y|^4 \le |X|^4 + 1\}.$$

Remark 1.3 (Sign convention)**.** *We use*

$$H_s^E(p) := \mathrm{PV} \int_{\mathbb{R}^8} \chi_{E^c}(q) - \chi_E(q)|q-p|^{8+s}\, dq. \tag{1.8}$$

With this convention Euclidean balls have positive fractional mean curvature.

Remark 1.4 (Stationarity of the cone)**.** *For every $s \in (0,1)$,*

$$H_s^C(p) = 0 \qquad \text{for } p \in \Sigma \setminus \{0\}.$$

Indeed, after applying an element of $O(4) \times O(4)$, one may assume $p = (re_1, re_1)$. The exchange isometry $(X,Y) \mapsto (Y,X)$ fixes p, swaps C and C^c, and preserves the kernel in (1.8)*; the principal value therefore cancels. Theorem 1.1 is thus a genuine minimizing result, not merely an Euler–Lagrange statement.*

The paper is organized as follows. Section 2 establishes the one-sided calibration principle. Section 3 proves the strict barrier theorem, including the Gaussian expansion at infinity. Section 4 combines the two one-sided calibrations and submodularity to prove Theorem 1.1.

2 Calibrations and preliminary results

For measurable sets $A, B \subset \mathbb{R}^n$, define

$$L_s(A,B) := \int_A \int_B \frac{1}{|x-y|^{n+s}}\, dy\, dx.$$

For an open set $\Omega \subset \mathbb{R}^n$, the localized fractional perimeter is

$$P_s(E,\Omega) := L_s(E\cap\Omega, E^c\cap\Omega) + L_s(E\cap\Omega, E^c\setminus\Omega) + L_s(E\setminus\Omega, E^c\cap\Omega). \tag{2.1}$$

Equivalently, with

$$\mathcal{Q}_\Omega := \mathbb{R}^{2n} \setminus (\Omega^c \times \Omega^c),$$

one has

$$P_s(E,\Omega) = \frac{1}{2}\int_{\mathcal{Q}_\Omega} \frac{|\chi_E(x) - \chi_E(y)|}{|x-y|^{n+s}}\, dx\, dy. \tag{2.2}$$

Lemma 2.1 (Scaling)**.** *For every $\lambda > 0$,*

$$P_s(\lambda E, \lambda\Omega) = \lambda^{n-s} P_s(E,\Omega), \qquad H_s^{\lambda E}(\lambda p) = \lambda^{-s} H_s^E(p).$$

Proof. Both identities follow by the change of variables $x = \lambda x'$ and $y = \lambda y'$. □

Lemma 2.2 (Complement and submodularity)**.** *For all measurable $A, B \subset \mathbb{R}^n$ and every open Ω,*

$$P_s(A^c, \Omega) = P_s(A, \Omega),$$

and

$$P_s(A \cup B, \Omega) + P_s(A \cap B, \Omega) \leq P_s(A, \Omega) + P_s(B, \Omega). \tag{2.3}$$

Proof. The complement identity follows immediately from (2.2). For (2.3), apply the pointwise lattice inequality

$$|(a \vee b) - (a' \vee b')| + |(a \wedge b) - (a' \wedge b')| \leq |a - a'| + |b - b'|$$

with $a = \chi_A(x)$, $b = \chi_B(x)$, $a' = \chi_A(y)$, $b' = \chi_B(y)$, multiply by the kernel and integrate over $\mathcal{Q}_\Omega$. □

Definition 2.3 (One-sided localized minimality)**.** *Let $E \subset \mathbb{R}^n$ be measurable and let $0 < s < 1$. We say that E is* outward minimizing for P_s *if, for every bounded open set $\Omega \subset \mathbb{R}^n$ and every measurable competitor $F \subset \mathbb{R}^n$ such that*

$$E \subset F, \qquad F \setminus E \Subset \Omega,$$

one has

$$P_s(E, \Omega) \leq P_s(F, \Omega).$$

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We shall use the following one-sided calibration criterion. The formulation with truncated divergences is convenient because it records explicitly the passage to the principal value.

Proposition 2.4 (One-sided nonlocal calibration)**.** *Let $E \subset \mathbb{R}^n$ have locally finite fractional perimeter and let $\zeta : \mathbb{R}^n \times \mathbb{R}^n \to [-1, 1]$ be antisymmetric. For $\rho > 0$ set*

$$\mathcal{D}_{s,\rho}\zeta(x) := \int_{|x-y|>\rho} \zeta(x, y)|x - y|^{-n-s}\, dy.$$

Assume that

(i)
$$\zeta(x, y)(\chi_E(y) - \chi_E(x)) = |\chi_E(y) - \chi_E(x)|$$
for a.e. pair (x, y);

(ii) *there exists $\mathcal{D}_s\zeta \in L^1_{\mathrm{loc}}(E^c)$ such that*

$$\mathcal{D}_{s,\rho}\zeta \longrightarrow \mathcal{D}_s\zeta \quad \text{in } L^1_{\mathrm{loc}}(E^c) \quad (\rho \downarrow 0), \qquad \mathcal{D}_s\zeta \leq 0 \quad \text{a.e. in } E^c.$$

Then E is outward minimizing for P_s.

Proof. Let $F \supset E$ with $F \setminus E \Subset \Omega$. If $P_s(F,\Omega) = +\infty$ there is nothing to prove, so assume it is finite and put $u := \chi_F - \chi_E$. Then $u \geq 0$ and $\operatorname{spt} u \Subset \Omega$. For $\rho > 0$ denote by $P_{s,\rho}$ the expression in (2.2) with the additional restriction $|x-y| > \rho$. Since $|\zeta| \leq 1$, the saturation identity gives

$$P_{s,\rho}(F,\Omega) - P_{s,\rho}(E,\Omega) \geq \frac{1}{2} \int_{\mathcal{Q}_\Omega \cap \{|x-y|>\rho\}} \zeta(x,y)(u(y)-u(x))|x-y|^{-n-s}\,dx\,dy.$$

Because u vanishes outside a compact subset of Ω, the last integral is unchanged if $\mathcal{Q}_\Omega$ is replaced by $\mathbb{R}^{2n}$. It is absolutely convergent for fixed ρ, and antisymmetry yields

$$\frac{1}{2} \iint_{|x-y|>\rho} \zeta(x,y)(u(y)-u(x))|x-y|^{-n-s}\,dx\,dy = -\int_{E^c} u(x)\mathcal{D}_{s,\rho}\zeta(x)\,dx.$$

Letting $\rho \downarrow 0$, monotone convergence gives $P_{s,\rho}(A,\Omega) \uparrow P_s(A,\Omega)$ for $A = E, F$, while assumption (ii) gives convergence of the right-hand side. Consequently

$$P_s(F,\Omega) - P_s(E,\Omega) \geq -\int_{E^c} u(x)\mathcal{D}_s\zeta(x)\,dx \geq 0.$$

□

Corollary 2.5 (Foliation criterion)**.** *Let $\Psi : \mathbb{R}^n \to \mathbb{R}$ be smooth away from a closed null set and set $E_a = \{\Psi \leq a\}$. Assume that E_0 has locally finite fractional perimeter, that ∂E_a is smooth for every $a > 0$,*

$$H_s^{E_a} > 0 \qquad \text{on } \partial E_a, \qquad a > 0,$$

and, for

$$\zeta(x,y) := \operatorname{sgn}(\Psi(x) - \Psi(y)),$$

the truncated divergences $\mathcal{D}_{s,\rho}\zeta$ converge in $L^1_{\mathrm{loc}}(\{\Psi > 0\})$. Then $E_0 = \{\Psi \leq 0\}$ is outward minimizing.

Proof. The saturation condition is immediate. If $x \in E_0^c$ and $a = \Psi(x) > 0$, then, for a.e. y,

$$\zeta(x,y) = \chi_{E_a}(y) - \chi_{E_a^c}(y).$$

Therefore

$$\mathcal{D}_s\zeta(x) = \mathrm{PV} \int_{\mathbb{R}^n} \frac{\chi_{E_a}(y) - \chi_{E_a^c}(y)}{|x-y|^{n+s}}\,dy = -H_s^{E_a}(x) < 0.$$

The conclusion follows from Proposition 2.4. □

The final application of this criterion to the level sets of Φ and $-\Phi$ is postponed to Section 4, after the strict barrier theorem has been proved.

3 Fractional mean curvature of the perturbed cone

We now prove Theorem 1.2. Since E_1^+ is invariant under $O(4) \times O(4)$, every boundary point is equivalent to

$$p(y) := (x(y)e_1, ye_1), \qquad x(y) := (1+y^4)^{1/4}, \qquad y \geq 0. \tag{3.1}$$

The classical mean curvature with the sign convention (1.8) is

$$\mathcal{H}(y) = \frac{3\big(x(y)^2 - y^2\big)^3}{\sqrt{x(y)^2+y^2}\,\big(x(y)^4 - x(y)^2y^2 + y^4\big)^{3/2}}. \tag{3.2}$$

Hence

$$\mathcal{H}(y) > 0 \quad \forall y \geq 0, \qquad \mathcal{H}(y) = \frac{3}{8\sqrt{2}} y^{-13} + O(y^{-17}) \quad (y \to \infty). \tag{3.3}$$

Indeed, $x(y)^4 - y^4 = 1$, and therefore

$$x(y)^2 - y^2 = \frac{x(y)^4 - y^4}{x(y)^2 + y^2} = \frac{1}{x(y)^2 + y^2} > 0.$$

The denominator in (3.2) is strictly positive for every $y \geq 0$, including $y = 0$.

Lemma 3.1 (Compact part)**.** *For every $Y < \infty$ there is $s_Y \in (0,1)$ such that*

$$H_s^{E_1^+}(p(y)) > 0 \qquad \forall y \in [0, Y], \quad s \in (s_Y, 1).$$

Proof. On compact families of uniformly C^2 hypersurfaces,

$$(1-s) H_s^E(p) \to \kappa_n H_{\mathrm{cl}}^E(p)$$

uniformly as $s \uparrow 1$, with $\kappa_n > 0$; see, for instance, [1, 6]. Since (3.3) gives a positive minimum of the classical mean curvature on $[0, Y]$, the claim follows. □

It remains to handle $y \to \infty$. Set

$$\eta := y^{-4}, \qquad E_\eta := \{|X|^4 \leq |Y|^4 + \eta\}, \qquad P_\eta := (a_\eta e_1, e_1), \qquad a_\eta := (1+\eta)^{1/4}. \tag{3.4}$$

Then

$$H_s^{E_1^+}(p(y)) = y^{-s} H_s^{E_\eta}(P_\eta). \tag{3.5}$$

Proposition 3.2 (Uniform expansion at the cone)**.** *There exist $s_* \in (0,1)$, $\eta_0 > 0$, $c_* > 0$ and $C_* < \infty$ such that, for $s \in [s_*, 1)$ and $0 < \eta < \eta_0$,*

$$H_s^{E_\eta}(P_\eta) \geq A_s \eta - C_* \eta^2, \qquad A_s \geq c_*. \tag{3.6}$$

In particular, after decreasing η_0 if necessary,

$$H_s^{E_\eta}(P_\eta) > 0 \qquad \forall s \in [s_*, 1), \quad 0 < \eta < \eta_0.$$

The proof occupies the rest of this section.

3.1 Laplace–Gaussian representation

Let Z_t, W_t be independent Gaussian variables in $\mathbb{R}^4$ with density

$$\gamma_t(z) = \left(\frac{t}{\pi}\right)^2 e^{-t|z|^2}.$$

Define

$$P_t(\eta) := \mathbb{P}\left(|a_\eta e_1 + Z_t|^4 - a_\eta^4 > |e_1 + W_t|^4 - 1\right). \tag{3.7}$$

Lemma 3.3 (Gaussian representation)**.** *For $0 < s < 1$ and $\eta \geq 0$,*

$$H_s^{E_\eta}(P_\eta) = \frac{\pi^4}{\Gamma(4 + s/2)} \int_0^\infty t^{-1+s/2} (2P_t(\eta) - 1)\, dt. \tag{3.8}$$

Moreover $P_t(0) = 1/2$ for every $t > 0$.

Proof. Let

$$\sigma_\eta(q) := \chi_{E_\eta^c}(P_\eta + q) - \chi_{E_\eta}(P_\eta + q).$$

Choose the unit normal ν_η at P_η with the orientation for which $\sigma_\eta(q) = \operatorname{sgn}(q \cdot \nu_\eta)$ to first order, and put $\sigma_\eta^0(q) := \operatorname{sgn}(q \cdot \nu_\eta)$. Since ∂E_η is C^2 near P_η, there exist $r_0, C > 0$ such that

$$\sigma_\eta(q) \neq \sigma_\eta^0(q), \quad |q| < r_0, \qquad \Longrightarrow \qquad |q \cdot \nu_\eta| \leq C|q - (q \cdot \nu_\eta)\nu_\eta|^2.$$

Consequently the set where the two signs differ occupies, in a spherical shell of radius r, a proportion $O(r)$. Hence

$$\int_{B_{r_0}} |\sigma_\eta(q) - \sigma_\eta^0(q)||q|^{-8-s}\, dq < \infty$$

precisely because $s < 1$. The same difference is integrable at infinity. The principal value of the halfspace term is zero, and therefore

$$H_s^{E_\eta}(P_\eta) = \int_{\mathbb{R}^8} (\sigma_\eta(q) - \sigma_\eta^0(q))|q|^{-8-s}\, dq, \tag{3.9}$$

where the right-hand side is absolutely convergent.

Using the Laplace identity

$$|q|^{-8-s} = \frac{1}{\Gamma(4 + s/2)} \int_0^\infty t^{3+s/2} e^{-t|q|^2}\, dt$$

and Fubini's theorem in (3.9), justified by the absolute convergence of the iterated integral, we obtain

$$H_s^{E_\eta}(P_\eta) = \frac{1}{\Gamma(4 + s/2)} \int_0^\infty t^{3+s/2} \int_{\mathbb{R}^8} (\sigma_\eta(q) - \sigma_\eta^0(q)) e^{-t|q|^2}\, dq\, dt.$$

The centered Gaussian integral of σ_η^0 vanishes by reflection in $\nu_\eta^\perp$. Since the density of (Z_t, W_t) is $(t/\pi)^4 e^{-t|q|^2}$, the remaining inner integral equals $\pi^4 t^{-4}(2P_t(\eta) - 1)$, proving (3.8). When $\eta = 0$, the random variables $|e_1 + Z_t|^4 - 1$ and $|e_1 + W_t|^4 - 1$ are independent and have the same continuous law, so $P_t(0) = 1/2$. □

Lemma 3.4 (First derivative at the cone)**.** *For every $t > 0$,*

$$\partial_\eta P_t(0) = \frac{e^{-t}}{16} \left(2tI_0(t) - (2t + 1)I_1(t)\right), \tag{3.10}$$

where I_0 and I_1 are modified Bessel functions of the first kind.

Proof. Write

$$U_a(z) := |ae_1 + z|^4 - a^4, \qquad a_\eta = (1 + \eta)^{1/4}.$$

For $\delta > 0$, first restrict the coarea computation to the region $|e_1 + z| \geq \delta$ and then differentiate the resulting fixed-boundary integral. The omitted part is $O(\delta^4)$, uniformly for η near zero, while its differentiated integrand is $O(\delta^3)$ after the radial Jacobian is included. Letting $\delta \downarrow 0$ therefore justifies differentiation and gives

$$\partial_\eta P_t(0) = \int_{\mathbb{R}^4} \int_{\{U_1(z) = U_1(w)\}} \frac{G(z)}{|\nabla U_1(z)|} \gamma_t(z)\gamma_t(w)\, d\mathcal{H}^3(z)\, dw, \tag{3.11}$$

where

$$G(z) := \partial_\eta U_{a_\eta}(z)\big|_{\eta=0} = |e_1 + z|^2(1 + z_1) - 1.$$

Indeed $a_\eta'(0) = 1/4$ and $\partial_a U_a(z) = 4|ae_1 + z|^2(a + z_1) - 4a^3$.

Use polar coordinates centered at $-e_1$:

$$e_1 + z = r\omega, \qquad e_1 + w = r\nu, \qquad r > 0, \qquad \omega, \nu \in \mathbb{S}^3.$$

Then $U_1(z) = r^4 - 1$, $|\nabla U_1(z)| = 4r^3$, and

$$\gamma_t(z)\gamma_t(w) = \left(\frac{t}{\pi}\right)^4 e^{-2t(r^2+1)} e^{2tr(\omega_1+\nu_1)}.$$

Moreover $1 + z_1 = r\omega_1$ and $|e_1 + z| = r$, whence the corrected identity

$$G(z) = r^3\omega_1 - 1. \tag{3.12}$$

Thus

$$\partial_\eta P_t(0) = \frac{1}{4}\left(\frac{t}{\pi}\right)^4 \int_0^\infty r^3 e^{-2t(r^2+1)} (r^3 A_1(2tr) - A_0(2tr)) A_0(2tr)\, dr, \tag{3.13}$$

where

$$A_0(\rho) := \int_{\mathbb{S}^3} e^{\rho\omega_1}\, d\sigma(\omega), \qquad A_1(\rho) := A_0'(\rho).$$

The standard integral representation of the modified Bessel functions, see [10, Section 10.32], gives

$$A_0(\rho) = \frac{4\pi^2 I_1(\rho)}{\rho}, \qquad A_1(\rho) = 4\pi^2\left(\frac{I_0(\rho)}{\rho} - \frac{2I_1(\rho)}{\rho^2}\right).$$

Substitution in (3.13) gives, with all Bessel functions in the following line evaluated at $2tr$,

$$\partial_\eta P_t(0) = e^{-2t} \int_0^\infty e^{-2tr^2} \left(t^2 r^4 I_0 I_1 - t r^3 I_1^2 - t^2 r I_1^2\right) dr. \tag{3.14}$$

To evaluate the integral, set

$$W(\alpha, \beta) := \int_0^\infty r e^{-\alpha r^2} I_1(\beta r)^2\, dr = \frac{1}{2\alpha} e^{\beta^2/(2\alpha)} I_1\left(\frac{\beta^2}{2\alpha}\right).$$

Using $I_1'(x) = I_0(x) - I_1(x)/x$, differentiation under the integral gives

$$\int_0^\infty e^{-\alpha r^2} \left(t^2 r^4 I_0(\beta r) I_1(\beta r) - t r^3 I_1(\beta r)^2 - t^2 r I_1(\beta r)^2\right) dr$$

$$= -\frac{t^2}{2} W_{\alpha\beta}(\alpha, \beta) + \frac{t}{2} W_\alpha(\alpha, \beta) - t^2 W(\alpha, \beta) \quad \text{when } \beta = 2t.$$

Evaluating at $\alpha = \beta = 2t$ and differentiating the explicit Weber formula yields

$$-\frac{t^2}{2} W_{\alpha\beta} + \frac{t}{2} W_\alpha - t^2 W = \frac{e^t}{16}(2tI_0(t) - (2t+1)I_1(t)).$$

Together with (3.14), this proves (3.10). All differentiations are justified by the Gaussian factor and standard local bounds for I_0, I_1. □

Lemma 3.5 (Positivity of the Bessel coefficient)**.** *For every $t > 0$,*

$$2tI_0(t) - (2t+1)I_1(t) > 0. \tag{3.15}$$

Consequently

$$A_s := \frac{\pi^4}{8\Gamma(4 + s/2)} \int_0^\infty t^{-1+s/2} e^{-t} \left(2tI_0(t) - (2t+1)I_1(t)\right) dt \tag{3.16}$$

is continuous and strictly positive on every compact subinterval of $(0, 1]$. In particular, for $s_* < 1$ sufficiently close to 1, there exists $c_* > 0$ such that

$$A_s \geq c_* \qquad \forall s \in [s_*, 1].$$

Proof. Let $R(t) := I_1(t)/I_0(t)$ and $b(t) := 2t/(2t+1)$. The identities $I_0' = I_1$ and $I_1' = I_0 - I_1/t$ give

$$R' = 1 - R^2 - \frac{R}{t}.$$

Since $R(t) = t/2 + O(t^3)$ while $b(t) = 2t + O(t^2)$, $R < b$ for small positive t. If $R - b$ had a first zero $t_0 > 0$, then $(R-b)'(t_0) \geq 0$. But at $R = b$,

$$(R-b)' = 1 - b^2 - \frac{b}{t} - b' = -\frac{3}{(2t+1)^2} < 0,$$

a contradiction. Hence $R < b$ for all $t > 0$, which is exactly (3.15).

At zero,

$$e^{-t}\big(2tI_0(t) - (2t+1)I_1(t)\big) = \frac{3}{2}t + O(t^2).$$

At infinity, the standard large-argument expansions of I_0 and I_1 [10, Section 10.40] give

$$e^{-t}\big(2tI_0(t) - (2t+1)I_1(t)\big) = O(t^{-3/2}).$$

These two bounds imply uniform integrability of the defining integral for A_s on compact intervals of $s \leq 1$, and continuity follows by dominated convergence. The integrand is positive, hence $A_s > 0$. □

3.2 A one-sided Taylor estimate

The bounded and the large Gaussian scales require different coordinates. At bounded scales we use the exact one-dimensional density of a noncentral chi-square variable. At large scales we use a normal graph, but keep the Taylor remainder in η throughout the argument.

Lemma 3.6 (Bounded Gaussian scales)**.** *There exist $\eta_0 > 0$ and $C < \infty$ such that*

$$|\partial_\eta^2 P_t(\eta)| \leq Ct, \qquad 0 < t \leq 1, \quad 0 \leq \eta \leq \eta_0. \tag{3.17}$$

In particular,

$$2P_t(\eta) - 1 \geq 2\eta\partial_\eta P_t(0) - C\eta^2 t.$$

Proof. Let $\Xi, \Omega \in \mathbb{R}^4$ be independent with density $\pi^{-2}e^{-|\xi|^2}$. After writing $Z_t = t^{-1/2}\Xi$ and $W_t = t^{-1/2}\Omega$, set

$$X_\eta := |\sqrt{t}\, a_\eta e_1 + \Xi|^2, \qquad Y := |\sqrt{t}\, e_1 + \Omega|^2.$$

The event in (3.7) is exactly

$$X_\eta^2 > Y^2 + t^2\eta, \qquad \text{or equivalently} \qquad X_\eta > \sqrt{Y^2 + t^2\eta}.$$

If $\lambda \geq 0$, the variable $|\sqrt{\lambda}e_1 + \Xi|^2$ has density

$$f_\lambda(x) = e^{-(x+\lambda)}\sqrt{\frac{x}{\lambda}}I_1(2\sqrt{\lambda x}) = xh_\lambda(x), \qquad x > 0, \tag{3.18}$$

with the value at $\lambda = 0$ understood by continuity, and

$$h_\lambda(x) = e^{-(x+\lambda)}\sum_{k=0}^{\infty}\frac{(\lambda x)^k}{k!(k+1)!}. \tag{3.19}$$

Let

$$Q(\lambda, x) := \int_x^\infty f_\lambda(r)\, dr, \qquad \lambda_\eta := ta_\eta^2 = t\sqrt{1+\eta}, \qquad x_\eta(y) := \sqrt{y^2 + t^2\eta}.$$

Conditioning on Y gives the fixed-domain formula

$$P_t(\eta) = \int_0^\infty Q(\lambda_\eta, x_\eta(y)) f_t(y)\, dy. \tag{3.20}$$

Formula (3.19), or equivalently the standard bounds for I_1, shows that for $0 \le \lambda \le 2$ the functions h_λ and their λ-derivatives up to order two and x-derivatives up to order one are bounded by $C(1+x)^M e^{-cx}$. The same type of bound holds for Q_λ and $Q_{\lambda\lambda}$. For $\eta > 0$ these bounds justify two differentiations in (3.20). To include $\eta = 0$, differentiate first on $[\delta, \infty)$ in the y-variable, combine the two terms containing x''_η before taking the limit, and then let $\delta \downarrow 0$. The combined expression below is dominated by an integrable function independent of δ and η. Thus the second derivative extends continuously from $\eta > 0$ to the right endpoint $\eta = 0$.

We record the cancellation at $y = 0$, which is the point missed by an argument based on a C^2 density for the difference variable. Since $Q_x = -xh_\lambda(x)$ and $Q_{xx} = -h_\lambda(x) - xh'_\lambda(x)$,

$$Q_{xx}(x'_\eta)^2 + Q_x x''_\eta = -\frac{t^4}{4x_\eta} h'_{\lambda_\eta}(x_\eta),$$

where

$$x'_\eta = \frac{t^2}{2x_\eta}, \qquad x''_\eta = -\frac{t^4}{4x_\eta^3}.$$

Similarly,

$$2Q_{\lambda x}\lambda'_\eta x'_\eta = -\lambda'_\eta t^2 \partial_\lambda h_{\lambda_\eta}(x_\eta).$$

Thus

$$\partial_\eta^2 P_t(\eta) = \int_0^\infty \Big[Q_{\lambda\lambda}(\lambda_\eta, x_\eta)(\lambda'_\eta)^2 + Q_\lambda(\lambda_\eta, x_\eta)\lambda''_\eta \tag{3.21}$$

$$- \lambda'_\eta t^2 \partial_\lambda h_{\lambda_\eta}(x_\eta) - \frac{t^4}{4x_\eta} h'_{\lambda_\eta}(x_\eta) \Big] f_t(y)\, dy. \tag{3.22}$$

Here and below $x_\eta = x_\eta(y)$. Since $x_\eta \ge y$ and $f_t(y) = yh_t(y)$, the last term is integrable uniformly even when $\eta = 0$. Moreover $|\lambda'_\eta| \le Ct$ and $|\lambda''_\eta| \le Ct$. Applying the bounds following (3.19) in (3.21) gives

$$|\partial_\eta^2 P_t(\eta)| \le C(t + t^2 + t^3 + t^4) \le Ct.$$

Taylor's formula and $P_t(0) = 1/2$ finish the proof. □

Lemma 3.7 (Large Gaussian scales)**.** *There exist $\eta_0 > 0$, $\varepsilon_0 > 0$ and $C < \infty$ such that, for $0 < \varepsilon \le \varepsilon_0$ and $0 \le \eta \le \eta_0$, one can write*

$$P_{\varepsilon^{-2}}(\eta) - \frac{1}{2} = B_\varepsilon(\eta) + \varepsilon^3 \Psi_\varepsilon(\eta), \tag{3.23}$$

where

$$B_\varepsilon(\eta) = \frac{\varepsilon}{4\sqrt{\pi}} \frac{3(a_\eta^2 - 1)^3}{\sqrt{a_\eta^2 + 1}\,(a_\eta^4 - a_\eta^2 + 1)^{3/2}} \ge 0, \tag{3.24}$$

$B_\varepsilon(0) = B'_\varepsilon(0) = 0$, *and*

$$\sup_{0<\varepsilon\le\varepsilon_0} \sup_{0\le\eta\le\eta_0} |\Psi''_\varepsilon(\eta)| \le C. \tag{3.25}$$

Consequently

$$2P_t(\eta) - 1 \ge 2\eta \partial_\eta P_t(0) - C\eta^2 t^{-3/2} \qquad (t \ge \varepsilon_0^{-2}). \tag{3.26}$$

Proof. Put $z = \varepsilon\xi$, $w = \varepsilon\omega$ and $F_{\eta,\varepsilon}(\xi,\omega) := F_\eta(\varepsilon\xi, \varepsilon\omega)$. Then

$$P_{\varepsilon^{-2}}(\eta) = \pi^{-4} \int_{\mathbb{R}^8} \mathbf{1}_{\{F_{\eta,\varepsilon}>0\}} e^{-|\xi|^2-|\omega|^2}\, d\xi\, d\omega,$$

and the following expansion is exact:

$$\begin{aligned} F_{\eta,\varepsilon} = 4\varepsilon(a_\eta^3\xi_1 - \omega_1) + \varepsilon^2\{2a_\eta^2|\xi|^2 + 4a_\eta^2\xi_1^2 - 2|\omega|^2 - 4\omega_1^2\} \\ + 4\varepsilon^3\{a_\eta\xi_1|\xi|^2 - \omega_1|\omega|^2\} + \varepsilon^4\{|\xi|^4 - |\omega|^4\}. \end{aligned} \tag{3.27}$$

Introduce the orthogonal coordinates

$$N_\eta = (a_\eta^6 + 1)^{1/2}, \qquad \alpha = \frac{a_\eta^3\xi_1 - \omega_1}{N_\eta}, \qquad \beta = \frac{\xi_1 + a_\eta^3\omega_1}{N_\eta},$$

so that

$$\xi_1 = \frac{a_\eta^3\alpha + \beta}{N_\eta}, \qquad \omega_1 = \frac{-\alpha + a_\eta^3\beta}{N_\eta}.$$

Write $\tau = (\beta, \xi', \omega') \in \mathbb{R}^7$, $u = |\xi'|^2$ and $v = |\omega'|^2$. In these variables

$$F_{\eta,\varepsilon} = 4\varepsilon N_\eta\alpha + \varepsilon^2 R_{2,\eta}(\alpha,\tau) + \varepsilon^3 R_{3,\eta}(\alpha,\tau) + \varepsilon^4 R_{4,\eta}(\alpha,\tau), \tag{3.28}$$

where the $R_{j,\eta}$ are polynomials with coefficients smooth in η. At $\alpha = 0$,

$$R_{2,\eta}(0,\tau) = 2a_\eta^2 u - 2v - \frac{6a_\eta^2\eta}{N_\eta^2}\beta^2. \tag{3.29}$$

We give the localization details needed for uniformity. Set $L_\varepsilon := \varepsilon^{-1/8}$. On $\{|\alpha|+|\tau| \le 2L_\varepsilon\}$, differentiation of (3.28) gives

$$\partial_\alpha F_{\eta,\varepsilon} = 4\varepsilon N_\eta + O(\varepsilon^2(1+L_\varepsilon)^3) \ge 2\varepsilon N_\eta$$

for ε_0 sufficiently small. The signs at $\alpha = \pm L_\varepsilon$ are opposite, since the linear term has size $\varepsilon L_\varepsilon$ whereas all remaining terms are $o(\varepsilon L_\varepsilon)$. Hence the zero set in this cylinder is a unique normal graph

$$\alpha = h_{\eta,\varepsilon}(\tau).$$

Write $h_{\eta,\varepsilon} = \varepsilon k_{\eta,\varepsilon}$ in the implicit equation $F_{\eta,\varepsilon}(h_{\eta,\varepsilon},\tau) = 0$. Since the derivative with respect to k is $4N_\eta + O(\varepsilon(1+|\tau|)^3)$, the quantitative implicit-function theorem applies uniformly on the cylinder. Expanding the equation through order ε^2 and differentiating it twice with respect to η gives

$$h_{\eta,\varepsilon}(\tau) = \varepsilon H_{1,\eta}(\tau) + \varepsilon^2 H_{2,\eta}(\tau) + \varepsilon^3 K_{\eta,\varepsilon}(\tau), \tag{3.30}$$

where, for $j = 0, 1, 2$,

$$|\partial_\eta^j H_{1,\eta}| \le C(1+|\tau|)^2, \quad |\partial_\eta^j H_{2,\eta}| \le C(1+|\tau|)^3, \quad |\partial_\eta^j K_{\eta,\varepsilon}| \le C(1+|\tau|)^M. \tag{3.31}$$

The first coefficient is

$$H_{1,\eta} = \frac{v - a_\eta^2 u}{2N_\eta} + \frac{3a_\eta^2\eta}{2N_\eta^3}\beta^2. \tag{3.32}$$

The coefficient equation at order ε^2 is

$$H_{2,\eta} = -\frac{\partial_\alpha R_{2,\eta}(0,\tau)H_{1,\eta} + R_{3,\eta}(0,\tau)}{4N_\eta}. \tag{3.33}$$

Furthermore, $R_{2,\eta}(0,\tau)$ and $H_{1,\eta}$ are even in β, whereas $\partial_\alpha R_{2,\eta}(0,\tau)$ and $R_{3,\eta}(0,\tau)$ are odd in β. Formula (3.33) therefore gives

$$H_{2,\eta}(-\beta,\xi',\omega') = -H_{2,\eta}(\beta,\xi',\omega'). \tag{3.34}$$

For completeness, we also record how the Gaussian tails are handled. A smooth cutoff equal to one on $|\alpha|+|\tau|\le L_\varepsilon$ and zero outside $|\alpha|+|\tau|\le 2L_\varepsilon$ is inserted. On its support the preceding graph description is valid. The gradient in the original scaled variables is

$$\nabla_\xi F_{\eta,\varepsilon} = 4\varepsilon|a_\eta e_1+\varepsilon\xi|^2(a_\eta e_1+\varepsilon\xi), \qquad \nabla_\omega F_{\eta,\varepsilon} = -4\varepsilon|e_1+\varepsilon\omega|^2(e_1+\varepsilon\omega).$$

Thus the only common critical point is $(-a_\eta e_1/\varepsilon, -e_1/\varepsilon)$, and its value is $-\eta$. It does not belong to the zero set when $\eta>0$; when $\eta=0$ it lies at Gaussian distance $O(\varepsilon^{-1})$. Away from a fixed neighborhood of that point, the coarea formulas for the first two η-derivatives have a nonvanishing gradient denominator and produce only polynomial factors. In the remaining neighborhood use

$$p = a_\eta e_1+\varepsilon\xi, \qquad q = e_1+\varepsilon\omega, \qquad u=|p|^4, \qquad v=|q|^4.$$

Since in four dimensions $r^3\,dr = \frac14\,d(r^4)$, the radial Jacobian in (u,v) is constant and the event is the half-plane $u-v>\eta$. After the angular integrations, each radial density has an expansion $g(u)=g(0)+O(u^{1/2})$ and satisfies $g'(u)=O(u^{-1/2})$ at the origin, with all coefficients bounded by a polynomial in ε^{-1} times e^{-c/ε^2}. For $d\ge 0$ the density of $u-v$ is $k(d)=\int_0^\infty g_1(v+d)g_2(v)\,dv$; hence $k'(0+)=\int_0^\infty g_1'(v)g_2(v)\,dv$ is finite. This gives the two right η-derivatives required at $\eta=0$, with the same exponentially small bound. Combining the two regions, the complementary contribution $\mathcal{R}_\varepsilon(\eta)$ is C^2 and satisfies

$$\sup_{0\le\eta\le\eta_0}|\mathcal{R}_\varepsilon''(\eta)| \le Ce^{-cL_\varepsilon^2}. \tag{3.35}$$

Possible additional roots of the quartic in α lie outside the cutoff cylinder; they are included in $\mathcal{R}_\varepsilon$. This is the reason for localizing before replacing the superlevel set by a single half-line. Since $e^{-cL_\varepsilon^2} = e^{-c\varepsilon^{-1/4}} = O(\varepsilon^N)$ for every N, the remainder in (3.35) may be absorbed at order ε^3 without losing two η-derivatives.

Let

$$q(r) := \pi^{-1/2}\int_r^\infty e^{-\alpha^2}\,d\alpha, \qquad d\gamma_7(\tau) := \pi^{-7/2}e^{-|\tau|^2}\,d\tau.$$

After extending the localized graph smoothly outside the central region, the preceding discussion gives

$$P_{\varepsilon^{-2}}(\eta) = \int_{\mathbb{R}^7} q(h_{\eta,\varepsilon}(\tau))\,d\gamma_7(\tau) + \mathcal{R}_\varepsilon(\eta). \tag{3.36}$$

On the central region one has $|h_{\eta,\varepsilon}|\le C\varepsilon^{3/4}$ for ε_0 sufficiently small. Hence

$$q(r) = \frac12 - \frac{r}{\sqrt{\pi}} + r^3\rho(r),$$

where ρ is smooth. Substitution of (3.30) into (3.36), together with (3.31) and the finiteness of all Gaussian moments, yields

$$P_{\varepsilon^{-2}}(\eta) - \frac12 = -\frac{\varepsilon}{\sqrt{\pi}}\int H_{1,\eta}\,d\gamma_7 - \frac{\varepsilon^2}{\sqrt{\pi}}\int H_{2,\eta}\,d\gamma_7 + \varepsilon^3\Psi_\varepsilon(\eta), \tag{3.37}$$

with (3.25). Notice that the C^2 estimate for Ψ_ε follows from the η-derivative bounds in (3.31); a bare $O(\varepsilon^5)$ remainder would not suffice for this conclusion.

The second integral in (3.37) vanishes by (3.34). From (3.32) and the Gaussian moments $\int \beta^2\, d\gamma_7 = 1/2$ and $\int u\, d\gamma_7 = \int v\, d\gamma_7 = 3/2$, one obtains

$$\int H_{1,\eta}\, d\gamma_7 = -\frac{1}{4}\,\frac{3(a_\eta^2-1)^3}{\sqrt{a_\eta^2+1}\,(a_\eta^4-a_\eta^2+1)^{3/2}}. \tag{3.38}$$

Equations (3.37) and (3.38) prove (3.23)–(3.24). Finally Taylor's theorem, $B_\varepsilon \geq 0$, and $B_\varepsilon(0) = B_\varepsilon'(0) = 0$ give

$$P_{\varepsilon^{-2}}(\eta) - P_{\varepsilon^{-2}}(0) - \eta\partial_\eta P_{\varepsilon^{-2}}(0) \geq -C\eta^2\varepsilon^3,$$

which is (3.26). □

Lemma 3.8 (One-sided Gaussian Taylor estimate)**.** *There are $\eta_0 > 0$ and $C < \infty$ such that, for every $0 < \eta < \eta_0$ and every $t > 0$,*

$$2P_t(\eta) - 1 \geq 2\eta\partial_\eta P_t(0) - C\eta^2 m(t), \tag{3.39}$$

where

$$m(t) = \begin{cases} t, & 0 < t \leq 1, \\ t^{-3/2}, & t \geq 1. \end{cases} \tag{3.40}$$

Proof. The range $0 < t \leq 1$ is Lemma 3.6. The large-scale lemma covers $t \geq \varepsilon_0^{-2}$. On the remaining compact interval $1 \leq t \leq \varepsilon_0^{-2}$, the fixed-domain representation (3.20) and the same differentiation used in (3.21) give $|\partial_\eta^2 P_t(\eta)| \leq C$; here the density parameters remain in a fixed compact interval depending only on ε_0. Since $t^{-3/2}$ is bounded below by a positive constant on this compact interval, Taylor's theorem gives (3.39) after increasing C. □

Remark 3.9. *The delicate point is not differentiability of $P_t(\eta)$ for fixed t. The point is the absence of a negative $\eta^2 t^{-1/2}$ contribution as $t \to \infty$. In the normal form above, the potentially dangerous local term is exactly the classical mean-curvature contribution* (3.24)*; it has nonnegative sign. The remaining negative part starts only at order $t^{-3/2}$, which is integrable against $t^{-1+s/2}$ uniformly for $s \leq 1$.*

3.3 Proof of the uniform expansion

Proof of Proposition 3.2. By Lemmas 3.3 and 3.8,

$$\begin{aligned} H_s^{E_\eta}(P_\eta) &\geq \frac{\pi^4}{\Gamma(4+s/2)} \int_0^\infty t^{-1+s/2} \left(2\eta\partial_\eta P_t(0) - C\eta^2 m(t)\right) dt \\ &= A_s\eta - C\eta^2 \frac{\pi^4}{\Gamma(4+s/2)} \int_0^\infty t^{-1+s/2} m(t)\, dt. \end{aligned}$$

The last integral is uniformly bounded for $s \in [s_*, 1]$, because

$$\int_0^1 t^{-1+s/2} t\, dt = \frac{1}{1+s/2},$$

and

$$\int_1^\infty t^{-1+s/2} t^{-3/2}\, dt = \frac{1}{3/2 - s/2}.$$

The factor $\Gamma(4+s/2)^{-1}$ is uniformly bounded on $[s_*, 1]$. Hence

$$H_s^{E_\eta}(P_\eta) \geq A_s\eta - C_*\eta^2.$$

By Lemma 3.5, choosing $s_* < 1$ sufficiently close to 1 gives $A_s \geq c_* > 0$. Finally decrease η_0 so that $C_*\eta_0 \leq c_*/2$. □

Corollary 3.10 (Uniform positivity at infinity)**.** *There exist $Y < \infty$, $s_* < 1$ and $c > 0$ such that*

$$H_s^{E_1^+}(p(y)) \geq cy^{-s-4} > 0 \qquad \forall y \geq Y, \quad s \in [s_*, 1).$$

Proof. Take $\eta = y^{-4}$ in (3.5). For y large enough, Proposition 3.2 gives

$$H_s^{E_1^+}(p(y)) = y^{-s} H_s^{E_\eta}(P_\eta) \geq y^{-s} \frac{c_*}{2} \eta = \frac{c_*}{2} y^{-s-4}.$$

□

Proof of Theorem 1.2. Use Corollary 3.10 for $y \geq Y$ and Lemma 3.1 on $[0, Y]$. The statement for E_1^- follows from the isometry $(X, Y) \mapsto (Y, X)$. □

4 Proof of the main result

We now turn the strict barriers into two one-sided calibrations and conclude by submodularity.

Proposition 4.1 (From the asymptotic estimate to the global barriers)**.** *There exists $s_b \in (0, 1)$ such that, for every $s \in (s_b, 1)$ and every $a > 0$,*

$$H_s^{E_a^+}(p) > 0 \qquad \text{for every } p \in \partial E_a^+, \qquad E_a^+ := \{\Phi \leq a\}, \tag{4.1}$$

and

$$H_s^{E_a^-}(p) > 0 \qquad \text{for every } p \in \partial E_a^-, \qquad E_a^- := \{-\Phi \leq a\}. \tag{4.2}$$

Proof. By Corollary 3.10 there exist $Y < \infty$, $s_\infty \in (0, 1)$ and $c_\infty > 0$ such that

$$H_s^{E_1^+}(p(y)) \geq c_\infty y^{-s-4} > 0 \qquad \forall y \geq Y, \quad s \in [s_\infty, 1).$$

For the same Y, Lemma 3.1 gives $s_Y \in (0, 1)$ such that

$$H_s^{E_1^+}(p(y)) > 0 \qquad \forall y \in [0, Y], \quad s \in (s_Y, 1).$$

Set

$$s_b := \max\{s_\infty, s_Y\}.$$

Since every point of ∂E_1^+ is, by the $O(4) \times O(4)$ symmetry, of the form $p(y)$ for some $y \geq 0$, the two estimates imply

$$H_s^{E_1^+}(p) > 0 \qquad \forall p \in \partial E_1^+, \quad s \in (s_b, 1).$$

Now fix $a > 0$. Because Φ is homogeneous of degree 4,

$$E_a^+ = a^{1/4} E_1^+.$$

If $p \in \partial E_a^+$ and $q = a^{-1/4} p$, then $q \in \partial E_1^+$. The scaling identity in Lemma 2.1 gives

$$H_s^{E_a^+}(p) = H_s^{a^{1/4} E_1^+}(a^{1/4} q) = a^{-s/4} H_s^{E_1^+}(q) > 0.$$

This proves (4.1). Finally, the exchange map

$$T(X, Y) := (Y, X)$$

is an isometry of $\mathbb{R}^8$ and satisfies $T(E_a^+) = E_a^-$. Fractional mean curvature is invariant under isometries, hence (4.2) follows from (4.1). □

Lemma 4.2 (Integrability of the homogeneous calibration)**.** *Let*

$$\zeta(x,y) := \operatorname{sgn}(\Phi(x) - \Phi(y)).$$

Then, for every $0 < s < 1$, the truncated divergences $\mathcal{D}_{s,\rho}\zeta$ converge in $L^1_{\mathrm{loc}}(C^c)$ as $\rho \downarrow 0$. Their limit satisfies

$$\mathcal{D}_s\zeta(x) = -H_s^{E^+_{\Phi(x)}}(x), \qquad x \in C^c, \tag{4.3}$$

and

$$|\mathcal{D}_{s,\rho}\zeta(x)| + |\mathcal{D}_s\zeta(x)| \le C_s|x|^{-s} \tag{4.4}$$

locally uniformly with respect to $\rho > 0$.

Proof. For $x \in C^c$ put $a = \Phi(x) > 0$. For a.e. y,

$$\zeta(x,y) = \chi_{E_a^+}(y) - \chi_{(E_a^+)^c}(y),$$

which gives (4.3) once the principal value is shown to exist. We verify this uniformly after normalization.

The kernel is homogeneous and Φ has degree four, hence

$$\zeta(\lambda x, \lambda y) = \zeta(x,y), \qquad \mathcal{D}_{s,\rho}\zeta(\lambda x) = \lambda^{-s}\mathcal{D}_{s,\rho/\lambda}\zeta(x).$$

Consider $x = \theta \in \mathbb{S}^7 \cap C^c$. The level hypersurface $\{\Phi = \Phi(\theta)\}$ through θ has uniformly bounded C^2 geometry as θ ranges in the closure of $\mathbb{S}^7 \cap C^c$: on the limiting cone $\{\Phi = 0\} \cap \mathbb{S}^7$ one still has $\nabla\Phi \neq 0$. Subtracting the sign of the tangent halfspace as in the proof of Lemma 3.3, the contribution from a ball of radius r is bounded uniformly by

$$C\int_0^r \rho^{-s}\, d\rho = Cr^{1-s}.$$

The contribution outside a fixed ball is uniformly bounded by the integrable tail of $|x-y|^{-8-s}$. It follows that $\mathcal{D}_{s,\delta}\zeta(\theta)$ converges uniformly in $\theta \in \mathbb{S}^7 \cap C^c$ and is uniformly bounded there.

Writing $x = r\theta$ gives (4.4). Since $|x|^{-s} \in L^1_{\mathrm{loc}}(\mathbb{R}^8)$, pointwise convergence away from the origin and dominated convergence imply convergence in $L^1_{\mathrm{loc}}(C^c)$. □

Proposition 4.3 (Exterior calibration of the cone)**.** *For every $s \in (s_b, 1)$, the cone*

$$C = \{\Phi \le 0\}$$

is outward minimizing for P_s.

Proof. Apply Corollary 2.5 with

$$\Psi = \Phi, \qquad E_a = \{\Psi \le a\} = E_a^+.$$

For $a > 0$, the boundary ∂E_a^+ is smooth because $\nabla\Phi \neq 0$ on $\{\Phi = a\}$. Indeed, $\nabla\Phi(X,Y) = 0$ only when $X = Y = 0$, whereas $0 \notin \{\Phi = a\}$ for $a > 0$. The cone has locally finite classical perimeter; the standard translation estimate for characteristic functions therefore implies locally finite fractional perimeter for every $0 < s < 1$. The positivity assumption in Corollary 2.5 is precisely (4.1), and the required L^1_{loc} convergence of the truncated divergences is Lemma 4.2. Therefore $E_0 = \{\Phi \le 0\} = C$ is outward minimizing.

□

Proposition 4.4. *For every $s \in (s_b, 1)$, the cone C is inward minimizing for P_s.*

Proof. We apply the previous argument to $-\Phi$. The positive leaves are

$$E_a^- = \{-\Phi \le a\} = \{|Y|^4 \le |X|^4 + a\}, \qquad a > 0,$$

and their strict mean-convexity is (4.2). Hence

$$\{-\Phi \le 0\} = \{\Phi \ge 0\}$$

is outward minimizing. Since $\{\Phi \ge 0\}$ differs from $C^c = \{\Phi > 0\}$ only by the boundary $\{\Phi = 0\}$, a Lebesgue null set, the same outward minimality holds for C^c.

Let now $F \subset C$ and $C \setminus F \Subset \Omega$. Then

$$F^c \supset C^c, \qquad F^c \setminus C^c = C \setminus F \Subset \Omega.$$

Outward minimality of C^c gives

$$P_s(C^c, \Omega) \le P_s(F^c, \Omega).$$

Using complement invariance from Lemma 2.2,

$$P_s(C, \Omega) = P_s(C^c, \Omega) \le P_s(F^c, \Omega) = P_s(F, \Omega).$$

This is exactly inward minimality of C. □

Theorem 4.5 (Compact minimality of the cone)**.** *Let $s_0 := s_b$. For every $s \in (s_0, 1)$, every bounded open set $\Omega \subset \mathbb{R}^8$, and every measurable competitor $F \subset \mathbb{R}^8$ satisfying*

$$F \Delta C \Subset \Omega,$$

one has

$$P_s(C, \Omega) \le P_s(F, \Omega).$$

In particular, Theorem 1.1 follows.

Proof. Define the two one-sided competitors

$$F^+ := F \cup C, \qquad F^- := F \cap C.$$

Then

$$F^+ \supset C, \qquad F^+ \setminus C = F \setminus C \subset F \Delta C \Subset \Omega,$$

and

$$F^- \subset C, \qquad C \setminus F^- = C \setminus F \subset F \Delta C \Subset \Omega.$$

By Propositions 4.3 and 4.4,

$$P_s(C, \Omega) \le P_s(F^+, \Omega), \qquad P_s(C, \Omega) \le P_s(F^-, \Omega).$$

Adding the two inequalities yields

$$2P_s(C, \Omega) \le P_s(F \cup C, \Omega) + P_s(F \cap C, \Omega). \tag{4.5}$$

The submodularity inequality (2.3), applied with $A = F$ and $B = C$, gives

$$P_s(F \cup C, \Omega) + P_s(F \cap C, \Omega) \le P_s(F, \Omega) + P_s(C, \Omega). \tag{4.6}$$

Combining (4.5) and (4.6) gives

$$2P_s(C, \Omega) \le P_s(F, \Omega) + P_s(C, \Omega),$$

hence

$$P_s(C, \Omega) \le P_s(F, \Omega).$$

□

References

[1] N. Abatangelo and E. Valdinoci, *A notion of nonlocal curvature*, Numer. Funct. Anal. Optim. **35** (2014), no. 7–9, 793–815. doi:10.1080/01630563.2014.901837.

[2] E. Bombieri, E. De Giorgi and E. Giusti, *Minimal cones and the Bernstein problem*, Invent. Math. **7** (1969), 243–268. doi:10.1007/BF01404309.

[3] X. Cabré, *Calibrations and null-Lagrangians for nonlocal perimeters and an application to the viscosity theory*, Ann. Mat. Pura Appl. (4) **199** (2020), 1979–1995. doi:10.1007/s10231-020-00952-z.

[4] L. Caffarelli and E. Valdinoci, *Uniform estimates and limiting arguments for nonlocal minimal surfaces*, Calc. Var. Partial Differential Equations **41** (2011), no. 1–2, 203–240. doi:10.1007/s00526-010-0359-6.

[5] L. Caffarelli, J.-M. Roquejoffre and O. Savin, *Nonlocal minimal surfaces*, Comm. Pure Appl. Math. **63** (2010), no. 9, 1111–1144. doi:10.1002/cpa.20331.

[6] A. Cesaroni, L. De Luca, M. Novaga and M. Ponsiglione, *Stability results for nonlocal geometric evolutions and limit cases for fractional mean curvature flows*, Comm. Partial Differential Equations **46** (2021), no. 7, 1344–1371. doi:10.1080/03605302.2021.1875485.

[7] J. Dávila, M. del Pino and J. Wei, *Nonlocal s-minimal surfaces and Lawson cones*, J. Differential Geom. **109** (2018), no. 1, 111–175. doi:10.4310/jdg/1525399218.

[8] A. Davini, *On calibrations for Lawson's cones*, Rend. Semin. Mat. Univ. Padova **111** (2004), 55–70.

[9] G. De Philippis and E. Paolini, *A short proof of the minimality of Simons cone*, Rend. Semin. Mat. Univ. Padova **121** (2009), 233–241. doi:10.4171/RSMUP/121-14.

[10] F. W. J. Olver, D. W. Lozier, R. F. Boisvert and C. W. Clark (eds.), *NIST Handbook of Mathematical Functions*, Cambridge University Press, Cambridge, 2010.

[11] V. Pagliari, *Halfspaces minimise nonlocal perimeter: a proof via calibrations*, Ann. Mat. Pura Appl. (4) **199** (2020), 1685–1696. doi:10.1007/s10231-019-00937-7.

[12] O. Savin and E. Valdinoci, *Regularity of nonlocal minimal cones in dimension* 2, Calc. Var. Partial Differential Equations **48** (2013), no. 1–2, 33–39. doi:10.1007/s00526-012-0539-7.

[13] J. Simons, *Minimal varieties in Riemannian manifolds*, Ann. of Math. (2) **88** (1968), no. 1, 62–105. doi:10.2307/1970556.