DIRAC SPECTRA AND THE TOPOLOGY OF SPACES OF METRICS

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Abstract. Let M be a closed connected spin manifold of dimension $n \geq 3$. We prescribe arbitrary admissible finite Dirac spectra in bounded open symmetric intervals, including multiplicities and nullity. Kernel dimensions are unbounded on each M. Nonzero eigenvalues generically have the least multiplicity allowed by the spinor structures, also at fixed density.

Compatible multiplication perturbations of Dirac operators are approximated in norm resolvent by metric Dirac operators, including first parameter derivatives for compact smooth families. The metrics are arbitrarily C^0-close, preserve density, and agree with the original metric outside a prescribed neighborhood of the support. In dimensions four and five, such changes in any nonempty open set realize the first admissible nullity above the index minimum.

If invertible metrics exist, ordinary real index difference identifies their components at fixed density with KO_{n+1}; for $n \geq 5$, each contains a positive-scalar-curvature metric whenever one exists. With density and the metric outside any nonempty open set fixed, the Dirac-minimal space has weak homotopy type $\mathrm{Sp} / \mathrm{Sp}(a)$ for $n \equiv 4 \pmod 8$, and $\Omega \mathrm{Sp}$ for $n \equiv 5 \pmod 8$, where $a = |\operatorname{ind}_{\mathbb{H}} D^+|$ and Sp is the stable compact symplectic group. On compact connected spin n-manifolds, invertible boundary metrics admit invertible cylindrical extensions exactly when the relative real index vanishes.

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1. Introduction

Let D_g be the untwisted spin Dirac operator of a Riemannian spin manifold with metric g. Charge conjugation and, in even dimension, chirality constrain its spectrum; the real spin index constrains its nullity. On a closed connected spin manifold of dimension at least three, these restrictions suffice to prescribe the spectrum in a bounded open interval symmetric about zero, including the multiplicity at zero (Theorem P).

Theorem R gives norm-resolvent approximation of a Dirac operator with a compatible multiplication potential by Dirac operators of Riemannian metrics. The metric may be kept fixed outside a neighborhood of the support of the potential, with volume density fixed everywhere. For invertible operators the relative approximation identifies the components of the space of metrics with those of the space of pairs consisting of a metric and a compatible potential. Ordinary real index difference classifies these components (Theorem C). In dimensions congruent to four or five modulo eight, the corresponding spaces with minimal kernel also admit a description up to weak homotopy equivalence (Theorem T).

The manifolds carrying Dirac operators are smooth and oriented, with a fixed spin structure. A connected manifold is understood to be nonempty. Write Σ_g for the complex spinor bundle, dv_g for the volume density, and $\mathrm{Spec}(D_g)$ for the spectrum of D_g. For real covectors ξ, η at the same point, Clifford multiplication c satisfies

$$c(\xi)c(\eta) + c(\eta)c(\xi) = -2g^{-1}(\xi,\eta)I, \qquad c(\xi)^* = -c(\xi), \tag{1.1}$$

where I is the identity of the spinor fibre. Metrics on a compact spin manifold with boundary are products near the boundary. Invertibility then refers to the Dirac operator on the complete manifold obtained by attaching the product half-cylinders. Spaces of smooth metrics and multiplication potentials carry the C^∞ topology. Write $KO_k = KO_k(\mathrm{pt}) = KO^{-k}(\mathrm{pt})$ for the real K-homology of a point and $\alpha(M) \in KO_{\dim M}$ for the real spin index of a closed spin manifold M.

1.1. **Norm-resolvent approximation**. Let X be closed or the cylindrical completion of a compact spin manifold whose boundary Dirac operator is invertible. Fix a reference metric g_{ref} on X, equal to the fixed product metric on each end. On a cylindrical completion all metrics in a family agree with g_{ref} off one compact set. Put

$$\mathscr{D} = H^1(\Sigma_{g_{\mathrm{ref}}}) \subset \mathscr{H} = L^2(\Sigma_{g_{\mathrm{ref}}}). \tag{1.2}$$

We identify the spinor spaces for metrics g, h by the spin lift of the positive map of orthonormal frames, multiplied by $(dv_h/dv_g)^{-1/2}$; see (2.6). For $n = \dim X$, let J be spinor charge conjugation and, if n is even, let Γ be chirality. Put $\epsilon_n = -1$ for $n \equiv 1, 5 \pmod 8$ and $\epsilon_n = 1$ otherwise. A complex-linear multiplication endomorphism V is *compatible* if $V^* = V$, $JV = \epsilon_n VJ$ and, for even n, $\Gamma V = -V\Gamma$.

Write $\mathcal{L}(\mathscr{H})$ for the bounded operators on $\mathscr{H}$. For a family of metrics $b \mapsto g_b$ on X and an operator family on their spinor spaces, the C^1_b norm on a compact set of pairs (b, z) is the supremum norm of the family and its first parameter derivatives at fixed z, after transport to g_{ref}. The derivatives are taken on neighborhoods of the compact set in finitely many parameter charts. Spatial C^0 norms of metrics are measured with g_{ref}.

Theorem R (Norm-resolvent realization). *Suppose $n \geq 3$. Let B be a compact smooth manifold, possibly with boundary, and $B_0 \subset B$ a closed subset. Let g_b be a smooth family of metrics on X and V_b a smooth family of compatible endomorphisms of Σ_{g_b}. Suppose that all V_b have support in one compact subset of an interior open set $U \subset X$ and that $V_b = 0$ for b near B_0. Put $L_b = D_{g_b} + V_b$, and let $\rho(L_b)$ denote its resolvent set.*

For every compact $\mathcal{K} \subset \{(b, z) \in B \times \mathbb{C} : z \in \rho(L_b)\}$ and every $\varepsilon > 0$ there is a smooth family of metrics $\widehat{g}_b$ such that

$$\widehat{g}_b = g_b \text{ on } X \setminus U, \qquad dv_{\widehat{g}_b} = dv_{g_b}, \tag{1.3}$$
$$\widehat{g}_b = g_b \text{ for } b \text{ near } B_0, \qquad \left\|\widehat{g} - g\right\|_{C^1(B;C^0)} < \varepsilon.$$

Let $\widehat{D}_b$ be $D_{\widehat{g}_b}$ transported directly to the g_b-spinor space. For every $(b,z)\in\mathcal{K}$ one has $z\in\rho(\widehat{D}_b)$, and

$$\left\|(\widehat{D}_b-z)^{-1}-(L_b-z)^{-1}\right\|_{C^1_b(\mathcal{K};\mathcal{L}(\mathscr{H}))}<\varepsilon. \tag{1.4}$$

1.2. **Generic nonzero eigenvalues.** Let M^n be closed and connected. Denote its space of smooth Riemannian metrics by $\mathcal{R}(M)$ and, for a positive smooth density μ, set $\mathcal{R}_\mu(M)=\{g\in\mathcal{R}(M)\,:\,dv_g=\mu\}$. Put

$$d_n=\begin{cases}2, & n\equiv 2,3,4 \pmod 8,\\ 1, & \text{otherwise.}\end{cases} \tag{1.5}$$

A nonzero eigenvalue of D_g is *simple* if its complex multiplicity is d_n [61, Definition 6.2.4].

Theorem G (Generic nonzero spectrum). *Suppose $n\geq 3$. For each integer $a\geq 2$ put*

$$K_a=[-a,-a^{-1}]\cup[a^{-1},a],\qquad \mathcal{G}_a=\{g\in\mathcal{R}(M)\,:\,\text{every eigenvalue of } D_g \text{ in } K_a \text{ is simple}\}. \tag{1.6}$$

The set $\mathcal{G}_a$ is C^1-open and C^∞-dense in $\mathcal{R}(M)$; its intersection with $\mathcal{R}_\mu(M)$ is C^1-open and C^∞-dense there. In both spaces the metrics with simple nonzero spectrum thus form a dense countable intersection of open sets.

Write $\mathcal{S}(M)=\bigcap_{a\geq 2}\mathcal{G}_a$. The metrics in $\mathcal{S}(M)$ whose kernel also has the least dimension allowed by the real index form a residual dense subset of $\mathcal{R}(M)$; their intersection with $\mathcal{R}_\mu(M)$ is residual and dense there (Corollary 7.6; [4, Theorem 1.1 and Corollary 1.2]).

1.3. **Path components.** Let M^n be closed and connected, let μ be a positive smooth density, and put

$$\mathcal{R}^{\mathrm{inv}}_\mu(M)=\{g\in\mathcal{R}(M)\,:\,dv_g=\mu,\ \ker D_g=0\}.$$

For $g_*, g\in\mathcal{R}^{\mathrm{inv}}_\mu(M)$, let $\delta_{g_*}(g)\in KO_{n+1}$ be the relative real family index of a metric path from g_* to g, with the sign convention of Section 8.1. Only the endpoint operators are required to be invertible; the space of all metrics of density μ is contractible.

Theorem C (Path components). *If $n\geq 3$ and $g_*\in\mathcal{R}^{\mathrm{inv}}_\mu(M)$, then*

$$\delta_{g_*}\,:\,\pi_0\mathcal{R}^{\mathrm{inv}}_\mu(M)\longrightarrow KO_{n+1}$$

is a bijection of pointed sets, with $[g_*]$ corresponding to zero.

By [4, Theorem 1.1] and Lemma 8.1, for $n\geq 3$ an invertible metric of density μ exists if and only if $\alpha(M)=0$. In that case the numbers of path components, in residues $n\equiv 0,1,\dots,7 \pmod 8$, are

$$2,\quad 2,\quad 1,\quad \aleph_0,\quad 1,\quad 1,\quad 1,\quad \aleph_0.$$

Corollary 1.1. *If $n\geq 5$ and M admits a metric of positive scalar curvature, every path component of $\mathcal{R}^{\mathrm{inv}}_\mu(M)$ contains a metric of positive scalar curvature.*

For $n\geq 6$ this follows from Theorem C and the integral index range of Botvinnik–Ebert–Randal-Williams [26, Theorems C, D and 5.3.1]; for $n=5$, $KO_6=0$. Constant rescaling followed by an isotopy gives the prescribed density (Lemma 14.1). For simply connected M of dimension at least five, this answers Dahl's question on surjectivity on path components [39, p. 453].

1.4. **Boundary metrics and the real index.** Let W^n be a compact spin manifold and h a metric on ∂W with invertible Dirac operator. A product extension of h is a metric on W equal to dt^2+h in a collar of ∂W. Write $\alpha(W,h)\in KO_n$ for the real Dirac index of any such extension on its cylindrical completion [40, §5.4]. For $\partial W=\emptyset$ this is the closed spin index.

Theorem I (Invertible extensions). *Suppose W is connected and $n \geq 3$. Then h has a product extension with invertible Dirac operator on the cylindrical completion if and only if $\alpha(W, h) = 0$.*

If $\alpha(W, h) = 0$, let g be any product extension and $U \Subset \operatorname{int} W$ any nonempty open set. There is a smooth trace-free g-self-adjoint endomorphism H of TW, compactly supported in U, such that $g_t = g(e^{tH}\cdot, \cdot)$ has invertible Dirac operator at $t = 1$ and for all sufficiently small $t > 0$. In particular, $\widehat{g} = g_1$ satisfies

$$\widehat{g} = g \text{ on } W \setminus U, \qquad dv_{\widehat{g}} = dv_g. \tag{1.7}$$

Let R_n^{inv} be the bordism group of Dahl–Große [40, §5.2]. Its index homomorphism sends $[W, h]$ to $\alpha(W, h)$. By Corollary 15.1 in degrees $n \geq 3$ and Appendix A in degrees $0, 1, 2$, it is an isomorphism:

$$\theta_n : R_n^{\mathrm{inv}} \xrightarrow{\cong} KO_n, \qquad n \geq 0. \tag{1.8}$$

1.5. **Finite spectra.** Let M^n be closed and connected. A finite multiset of nonzero real numbers is *admissible* if the multiplicities of λ and $-\lambda$ agree unless $n \equiv 3, 7 \pmod 8$, and every multiplicity is divisible by d_n in (1.5). In dimensions divisible by four write D^+ for the Dirac operator from positive to negative spinors and $\widehat{A}(M)$ for its complex index. In dimensions $1, 2 \pmod 8$ regard the mod-two index $\alpha(M)$ as an integer in $\{0, 1\}$. The allowed complex kernel dimensions h are

$n \bmod 8$	$h \quad (r \in \mathbb{Z}_{\geq 0})$
0	$\lvert\widehat{A}(M)\rvert + 2r$
1	$\alpha(M) + 2r$
2	$2\alpha(M) + 4r$
3	$2r$
4	$\lvert\widehat{A}(M)\rvert + 4r$
5	$2r$
6	$2r$
7	r.

(1.9)

A metric whose nullity is the value at $r = 0$ is called *Dirac-minimal*.

Theorem P (Finite spectral prescription). *Suppose $n \geq 3$. Fix $\Lambda > 0$, a finite admissible multiset Ξ in $(-\Lambda, \Lambda) \setminus \{0\}$, and h as in* (1.9)*. There is a smooth metric g such that, counting complex multiplicities,*

$$\operatorname{Spec}(D_g) \cap \big((-\Lambda, \Lambda) \setminus \{0\}\big) = \Xi, \qquad \dim_{\mathbb{C}} \ker D_g = h. \tag{1.10}$$

For a nonempty disjoint union $M = M_1 \sqcup \cdots \sqcup M_s$ of closed connected spin n-manifolds, $n \geq 3$, the admissible nonzero multisets are the same, and h is possible precisely when $h = \sum_{i=1}^{s} h_i$ with each h_i allowed by (1.9) on M_i. Indeed, D_g is the direct sum of the component operators. Conversely, apply Theorem P with the multiset Ξ on M_1 and the empty multiset on every other component, with the prescribed h_i and the same interval on all components. This gives the prescription of [38, Conjecture 9] in dimension at least three. In particular $\sup_{g \in \mathcal{R}(M)} \dim_{\mathbb{C}} \ker D_g = \infty$ for every nonempty closed spin manifold of dimension at least three, as conjectured by Bär [18, §8]; see also [5, Conjecture 2.2] and Corollary 19.2.

1.6. **Localized harmonic spinors.** In dimension four, let $a = |\operatorname{ind}_{\mathbb{H}} D^+| = \frac{1}{2}|\widehat{A}(M)|$ and choose the chiral block A_g of quaternionic index $-a$; when $a = 0$ take $A_g = D_g^+$. The metric is Dirac-minimal precisely when A_g is injective. In dimension five this condition is $\ker D_g = 0$. Let $\mathfrak{sp}(1)$ denote the skew-adjoint quaternionic endomorphisms of $\mathbb{H}$.

Theorem L (Localized harmonic spinors). *Let M be closed and connected of dimension four or five. For every $g \in \mathcal{R}(M)$, every nonempty open set $U \subset M$, and every $\varepsilon > 0$, there exists a smooth metric h such that*

$$h = g \text{ on } M \setminus U, \qquad dv_h = dv_g, \qquad \|h - g\|_{C^0_g} < \varepsilon. \tag{1.11}$$

In dimension four,

$$\dim_{\mathbb{H}} \ker A_h = 1, \qquad \dim_{\mathbb{H}} \ker A_h^* = a + 1, \qquad \dim_{\mathbb{C}} \ker D_h = 2a + 4. \tag{1.12}$$

In dimension five,

$$\dim_{\mathbb{C}} \ker D_h = 2. \tag{1.13}$$

More precisely, let $E = \mathrm{Hom}_{\mathbb{H}}(\mathbb{H}, \mathbb{H}^{a+1})$ *in dimension four and* $E = \mathfrak{sp}(1)$ *in dimension five. There is a smooth family of metrics* h_q*, for* q *in a neighborhood of* 0 *in* E*, with* $h_0 = h$ *and satisfying* (1.11) *with* h_q *in place of* h*. For some* $\delta > 0$*, the spectral subspaces of* D_{h_q} *for* $(-\delta, \delta)$ *admit smooth quaternionic orthonormal frames in which the chiral block of* D_{h_q} *in dimension four, respectively the restriction of* iD_{h_q} *in dimension five, has matrix* q*.*

1.7. **Weak homotopy type of the minimal locus.** Fix a metric g_* on a closed connected spin manifold M^n, a nonempty open set $U \subset M$, and the density $\mu = dv_{g_*}$. Put

$$\mathcal{R}_\mu(U) = \{g \in \mathcal{R}(M) \,:\, dv_g = \mu, \quad g = g_* \text{ on } M \setminus U\}.$$

For $n \equiv 4 \pmod 8$, let $a = |\operatorname{ind}_{\mathbb{H}} D^+|$ and let A_g be the chiral operator of quaternionic index $-a$ (take D_g^+ if $a = 0$). Write $\mathcal{R}_{\min,\mu}(U)$ for the subspace on which A_g is injective, and $\mathcal{R}_{\mathrm{inv},\mu}(U)$ for the subspace on which D_g is invertible. The same notation for the invertible subspace applies in residue five. Let Sp and U_{st} denote the direct limits of the compact symplectic and unitary groups under the standard inclusions; set $\mathrm{Sp}(0) = \{1\}$. For a topological group G write BG for its classifying space, and ΩY for the based loop space of a pointed space Y.

Theorem T (Weak homotopy type of the minimal locus). *The following metric spaces are nonempty. For any base metric* g_b *in the indicated space there is a based weak homotopy equivalence*

$$\kappa_a \,:\, \mathcal{R}_{\min,\mu}(U) \longrightarrow \mathrm{Sp} \,/\, \mathrm{Sp}(a), \qquad n \equiv 4 \pmod 8, \tag{1.14}$$

$$\kappa_5 \,:\, \mathcal{R}_{\mathrm{inv},\mu}(U) \longrightarrow \Omega\,\mathrm{Sp} \simeq \mathrm{Sp}\,/\mathrm{U}_{\mathrm{st}}, \qquad n \equiv 5 \pmod 8. \tag{1.15}$$

In (1.14) *the target is a model for the homotopy fibre of the stabilization* $B\,\mathrm{Sp}(a) \to B\,\mathrm{Sp}$ *of classifying spaces. The map records* $\operatorname{coker} A_g$ *together with a nullhomotopy of its reduced stable class* $[\operatorname{coker} A_g] - [\mathbb{H}^a]$*, obtained from the ambient path* (22.2)*. For* $a = 0$ *in* (1.14)*, and in* (1.15)*, the map represents ordinary real index difference.*

If $U_0 \subset U_1$ *are nonempty open subsets of* M*, the inclusion between the corresponding metric spaces in* (1.14) *or* (1.15)*, with the same* g_* *and* μ*, is a weak homotopy equivalence.*

In residue four, $\mathcal{R}_{\min,\mu}(U)$ is $(4a+2)$-connected and $\pi_{4a+3}(\mathcal{R}_{\min,\mu}(U), g_b) \cong \mathbb{Z}$. In residue five, $\mathcal{R}_{\mathrm{inv},\mu}(U)$ is simply connected and $\pi_2(\mathcal{R}_{\mathrm{inv},\mu}(U), g_b) \cong \mathbb{Z}$. If $n \equiv 4 \pmod 8$ and $a = 0$, or if $n \equiv 5 \pmod 8$, ordinary real index difference induces

$$\pi_k(\mathcal{R}_{\mathrm{inv},\mu}(U), g_b) \xrightarrow{\cong} KO_{n+k+1}, \qquad k \geq 1. \tag{1.16}$$

In dimensions four and five, the homomorphisms (22.1), given by the signed counts of kernel lines in transverse disk fillings with supported compatible potentials, are isomorphisms onto $\mathbb{Z}$. Every C^0 neighborhood of every metric in $\mathcal{R}_\mu(U)$ contains the image of a sphere which, based at one of its points, represents a generator of the corresponding homotopy group of the full locus (Proposition 22.4).

1.8. **Earlier results.** The spinor identification for two metrics and the variation formula for the Dirac operator are due to Bourguignon–Gauduchon [27, Propositions 1 and 5; Theorems 20–21]. A rotating coframe already gives a quadratic scalar term on the three-torus [43, §8]. A metric connection with skew torsion T has Dirac operator $D_g + \frac{3}{4} c_g(T)$ [1, §2]. In Theorem R the connections are Levi-Civita and the metrics have the prescribed volume density. The Riesz continuity theorem of Bandara–McIntosh–Rosén assumes, in particular, a bound on $\|\nabla^g h\|_\infty$ for the second metric h [17, Theorem 3.1 and Remark 3.2]; the first spatial derivatives of the metrics in Section 4 can grow like

$N^{1/2}$ with the frequency N. For norm-resolvent approximation of a planar Dirac operator by varying its mass, see [68].

Dahl proved generic simplicity in dimension three, and generic simplicity of the nonzero eigenvalues in each conformal class [37, Theorem 1.2 and Remark 1.3]. Theorem G gives the nonzero part of his higher-dimensional conjecture, also at fixed density. Canzani's conformal theorem assumes the absence of rigid eigenspaces [31, Definition 2.2 and Theorem 2.3]. For a nonscalar-compression criterion for bounded potential perturbations of semibounded operators, see [33]; for connection Laplacians, see [29].

Colin de Verdière prescribed finite spectra with multiplicities for the scalar Laplacian in dimension at least three [34, Theorem 1.2]. For local sections through a given scalar Schrödinger operator or Laplace cluster, see [76, Theorems A and 5.3]. For the Dirac operator, Dahl prescribed the simple nonzero eigenvalues in a finite interval [38, Theorem 3], and Nowaczyk obtained an eigenvalue of complex multiplicity at least two in dimensions $0, 6, 7 \pmod 8$ by changing the metric in a prescribed open set [77, Main Theorem 1]. Unbounded nullity was known for the Berger three-sphere [65, 5] and, by a construction of Seeger, for S^{4k}, $k \geq 1$; see [21, Introduction, after Conjecture A].

Maier proved generic minimality of the kernel in dimensions two, three, and four [71, Theorems 1.1, 1.2(i), and 1.3(i)]; Ammann–Dahl–Humbert proved it in all dimensions $n \geq 2$ and obtained minimality by a metric change in any nonempty open set [3, Theorem 1.1], [4, Theorem 1.1]. In the opposite direction, Bär constructed metrics with harmonic spinors in dimensions $3 \pmod 4$, for each spin structure [18, Theorem A and §4]. For the existence question of Bär–Dahl [21, Conjecture A], see also Hitchin [65, Theorem 4.5], Dahl for S^n, $n \geq 5$ [39, Corollary 4.2], and Crowley–Schick–Steimle for every closed spin manifold of dimension at least six [35, Theorem 1.4]. Farinelli states existence in every dimension at least four [48, Theorem 1.1]; Lockman treats the general four- and five-dimensional cases as open [70, §1]. In these two dimensions Theorem L prescribes the first nullity above the index minimum and a transverse family through the metric, in any prescribed open set, with fixed density and arbitrarily small C^0 change. For the construction of divergence-free framings from eigenspinors in dimension three, see Lin [69].

In dimensions $n \geq 7$, $n \equiv 3 \pmod 4$, Dahl distinguished infinitely many components of the invertible locus whenever it is nonempty; in dimensions $8k$ and $8k+1$, $k \geq 1$, he obtained at least two components [39, Theorem 3.3]. Dahl–Große obtained the dimension-three case by handle attachment [40, Proposition 4.3]; Waterstraat gave a spectral-flow proof for all $n \equiv 3 \pmod 4$ [86]. Theorem C shows that two invertible metrics with the same ordinary index difference from a fixed base metric are joined by a path of invertible metrics on the given manifold. Dahl–Große's classification in [40, Theorem 5.3(2)] concerns concordance, which requires invertibility on a cylinder. For block Dirac operators and positive-scalar-curvature concordances, see [62, Theorems B and C]. Dahl–Große proved surjectivity of $\theta_n : R_n^{\mathrm{inv}} \to KO_n$ and conjectured its injectivity [40, §5.4, Conjecture 5.12]. In degree five, connectedness of the invertible locus on S^4 [2, Theorem 1.1], isotopy-to-concordance [39, Corollary 2.4], and [40, Proposition 5.8] already imply $R_5^{\mathrm{inv}} = 0$, hence the extension assertion in that degree by [40, Theorem 5.3(1)]. Theorem I gives a modification in any prescribed interior open set and preserves the volume density.

Ammann–Dahl proved connectedness of the Dirac-minimal locus in dimensions two and four and gave codimension estimates for its complement [2, Theorems 1.1 and 1.3]. For the product of two closed surfaces of genera at least three, with any spin structure, Lockman proved that at least one of π_1, π_2, π_3 of the invertible locus is nonzero [70, Theorem B]. For every closed connected spin four-manifold of index zero, Theorem T with $U = M$ and Lemma 8.1 give $\pi_1 = \pi_2 = 0$ and $\pi_3 \cong \mathbb{Z}$ for that locus. In particular, it is two-connected, as conjectured by Ammann and Dahl [70, Remark 1.6]. Große–Pederzani state a spin-bordism invariance theorem for invertible metric spaces in dimension at least three [56, Theorem 1]; Lockman discusses this invariance as an open question [70, Remark 1.4]. For the corresponding surgery maps see also [2, Remark 1.2(3)], [79].

The stable Fredholm spaces and their Bott equivalences are classical [15, 25]. Ebert's path model fixes the principal symbol and allows all compatible pseudodifferential perturbations of order zero [45,

Definition 4.2 and Proposition 4.3]. The comparison of Theorem T is for metrics, with their density and exterior fixed. Take $M = S^1 \times S^{11}$ with either spin structure on the circle. Write $\mathcal{R}^+(M)$ for the space of metrics of positive scalar curvature and $\mathcal{R}^{\mathrm{inv}}(M)$ for the space of metrics with invertible Dirac operator, both without a density constraint. For any metric g_0 of positive scalar curvature and every $j \geq 1$, the group π_{4j} of the homotopy fibre over g_0 of $\mathcal{R}^+(M) \hookrightarrow \mathcal{R}^{\mathrm{inv}}(M)$ has an infinite cyclic direct summand (Proposition 22.26). Ebert–Randal-Williams give rational surjectivity for the higher index difference on $\mathcal{R}^+(M)$ [46, Theorem C]. The index difference of Theorem T on $\mathcal{R}^{\mathrm{inv}}(M)$ has ordinary real coefficients.

1.9. **From multiplication potentials to metrics.** A compatible potential is a sum of Clifford products of real forms of degree 3 mod 4 (Lemma 2.2). A positive congruence replaces an elementary term of degree $\ell \geq 7$ by terms of degrees 3 and $\ell-4$ (Proposition 5.1); an oscillating metric of unchanged density eliminates each elementary three-form (Proposition 4.2). The resolvent errors add after conjugation by the successive spinor identifications (5.8). The estimate (5.9) passes from their composite to the direct comparison in Theorem R.

For a finite spectral cluster separated by a contour in the resolvent set, integration on the contour gives the spectral projection and the first spectral moment (6.1). In compatible orthonormal frames the compressed operators converge together with their first parameter derivatives (Lemma 6.1). If the potentials have a common compact support in the prescribed open set and their compressed matrix at the zero cluster is the parameter itself, Proposition 6.3 gives a smooth metric family with the same matrices on a smaller parameter neighborhood. Its metric at zero need not be the original metric. For Theorem G, the splitting of a multiple nonzero eigenvalue is followed by analytic perturbation on a fixed spectral window (Proposition 7.5 and Section 7). In Theorem P the comparison under surgery is uniform in the parameters (Proposition 18.1). The mod-two intersection in Lemma 17.1 then yields the prescribed matrices, without differentiation in the surgery parameters (Section 19). The local families of Theorem L are constructed in Sections 20 and 21.

For Theorem C the inclusion of metrics as pairs with zero potential preserves the ordinary index difference and induces a bijection on components (Proposition 8.2 and Section 14). For Theorem I, the zero relative index permits cancellation of the kernel by an interior multiplication potential (Proposition 9.1); Theorem R at the resolvent parameter $z = 0$ then gives an invertible metric. Section 15 proves the assertion about the ray. In Theorem T the relative fillings give weak equivalences of the supported pair spaces with the operator models (Proposition 22.24). Proposition 8.2 and the based comparisons of Section 22 give (1.14) and (1.15) for the metric spaces.

2. Spinor structures and fixed domains

2.1. **Charge conjugation.** Let (X, g) be a Riemannian spin manifold of dimension n, and write $\Sigma X = \Sigma_g X$. Its complex rank is $s_n = 2^{\lfloor n/2 \rfloor}$. Clifford multiplication on covectors extends to differential forms by antisymmetrization. In an oriented orthonormal coframe $e^1, \dots, e^n$, this convention gives

$$c(e^{i_1} \wedge \cdots \wedge e^{i_\ell}) = c(e^{i_1}) \cdots c(e^{i_\ell}), \qquad i_1 < \cdots < i_\ell.$$

For $n = 2m > 0$, put

$$\Gamma = i^m c(e^1) \cdots c(e^{2m}).$$

For $n = 2m + 1$, choose the irreducible complex Clifford representation with

$$i^{m+1} c(e^1) \cdots c(e^{2m+1}) = -I.$$

Both formulas are independent of the oriented orthonormal coframe. In dimension zero, the spinor fibre is a complex line with scalar grading Γ, and J is complex conjugation. In every even dimension, set $\Sigma^{\pm} X = \ker(\Gamma \mp I)$.

The spinor bundle admits a parallel antiunitary endomorphism J [61, Theorem 2.2.2]. Define the signs ϵ_n and, in even dimension, τ_n by

$$J c(\xi) = \epsilon_n c(\xi) J, \qquad J\Gamma = \tau_n \Gamma J \quad (n \text{ even}).$$

Their values are

n mod 8	J^2	ϵ_n	τ_n
0	$+1$	$+1$	$+1$
1	$+1$	-1	$--$
2	-1	$+1$	-1
3	-1	$+1$	$--$
4	-1	$+1$	$+1$
5	-1	-1	$--$
6	$+1$	$+1$	-1
7	$+1$	$+1$	$--$.

(2.1)

Since J is parallel, $JD = \epsilon_n DJ$ for $D = D_g$. In even dimension,

$$\Gamma^* = \Gamma, \qquad \Gamma^2 = I, \qquad \Gamma D = -D\Gamma.$$

In even dimension, write $D^{\pm} : C^\infty(\Sigma^{\pm}X) \to C^\infty(\Sigma^{\mp}X)$ for the chiral restrictions of D. When $J^2 = I$, its fixed space is the real form of the spinor fibre. When $J^2 = -I$, the right actions $u \cdot i = iu$ and $u \cdot j = Ju$ give the fibre a quaternionic structure.

Definition 2.1. The real vector bundle $\mathscr{A}_n \subset \mathrm{End}_{\mathbb{C}}(\Sigma X)$ consists of the endomorphisms V satisfying

$$V^* = V, \qquad JV = \epsilon_n VJ, \qquad \Gamma V = -V\Gamma \quad (n \text{ even}). \tag{2.2}$$

Let E be a finite-dimensional Hermitian space with an antiunitary endomorphism J and, for even n, a self-adjoint involution Γ, with J^2 as in (2.1) and $J\Gamma = \tau_n \Gamma J$ in even dimension. Such a space is called a *zero module*. Write $\mathcal{A}_n(E) \subset \mathrm{End}_{\mathbb{C}}(E)$ for the real vector space of endomorphisms satisfying (2.2). The module is *balanced* if $\mathcal{A}_n(E)$ contains an invertible endomorphism.

For $r \in \mathbb{Z}_{\geq 0}$ and $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, let $\mathrm{Mat}_r(\mathbb{F})$ denote the $r \times r$ matrices over $\mathbb{F}$, and let A^* denote the conjugate transpose. For $\mathbb{F} = \mathbb{R}, \mathbb{C}$, put

$$\mathrm{Sym}_r(\mathbb{F}) = \{A \in \mathrm{Mat}_r(\mathbb{F}) : A^T = A\}, \qquad \mathrm{Skew}_r(\mathbb{F}) = \{A \in \mathrm{Mat}_r(\mathbb{F}) : A^T = -A\}.$$

Also put

$$\mathrm{Herm}_r(\mathbb{H}) = \{A \in \mathrm{Mat}_r(\mathbb{H}) : A^* = A\}, \quad \mathfrak{so}(r) = \mathrm{Skew}_r(\mathbb{R}), \quad \mathfrak{sp}(r) = \{A \in \mathrm{Mat}_r(\mathbb{H}) : A^* = -A\}.$$

In even dimension, set $E^{\pm} = \ker(\Gamma \mp I)$. Every element of $\mathcal{A}_n(E)$ has the form

$$\mathcal{D}(Z) = \begin{pmatrix} 0 & Z^* \\ Z & 0 \end{pmatrix}, \qquad Z : E^+ \longrightarrow E^-. \tag{2.3}$$

In residues zero and four, the block Z is respectively real or quaternionic. In residues two and six, choose orthonormal chiral coordinates in which

$$J(x, y) = (\varepsilon \bar{y}, \bar{x}), \qquad \varepsilon = J^2 \in \{-1, 1\}.$$

Then $J\mathcal{D}(Z) = \mathcal{D}(Z)J$ is equivalent to $Z^T = \varepsilon Z$.

For a balanced module the possible independent matrices and complex dimensions are

n mod 8	independent matrix	$\dim_{\mathbb{C}} E$
0	$\mathrm{Mat}_r(\mathbb{R})$	$2r$
1	$i\mathfrak{so}(2r)$	$2r$
2	$\mathrm{Skew}_{2r}(\mathbb{C})$	$4r$
3	$\mathrm{Herm}_r(\mathbb{H})$	$2r$
4	$\mathrm{Mat}_r(\mathbb{H})$	$4r$
5	$i\mathfrak{sp}(r)$	$2r$
6	$\mathrm{Sym}_r(\mathbb{C})$	$2r$
7	$\mathrm{Sym}_r(\mathbb{R})$	r,

(2.4)

Here the quaternionic matrix spaces act in their standard complex representations. In residue five, multiplication by i is complex scalar multiplication in $\mathrm{End}_{\mathbb{C}}(\mathbb{C}^{2r})$. Indeed, in residue one $V = iA$ with A real skew-symmetric; in residues three and seven V is respectively quaternionic Hermitian or real symmetric; and in residue five $-iV$ commutes with J and is skew-adjoint. In residues zero and four balance requires $\dim_{\mathbb{F}} E^+ = \dim_{\mathbb{F}} E^-$, with $\mathbb{F} = \mathbb{R}, \mathbb{H}$, respectively. In residues one and two balance requires an even matrix size. Each of the other matrix spaces contains invertible elements at every size.

Lemma 2.2. *Clifford multiplication induces a real vector-bundle isomorphism*

$$\bigoplus_{\substack{3\le\ell\le n\\ \ell\equiv 3\ (4)}} \Lambda^\ell T^*X \xrightarrow{\ c\ } \mathscr{A}_n. \tag{2.5}$$

Proof. For $n = 0$, both bundles vanish. Suppose $n > 0$. By the complex Clifford algebra classification [12, §1, (1.4), Table 1 and Proposition 5.4], the odd Clifford monomials form a complex basis of the endomorphisms anticommuting with Γ when n is even, and of all spinor endomorphisms when n is odd. In the odd case, the even Clifford algebra acts as the full endomorphism algebra; left multiplication by a unit covector identifies the even and odd subspaces of the complex Clifford algebra.

Thus every compatible endomorphism has a unique expansion

$$V = \sum_{|I|\ \mathrm{odd}} z_I c_I, \qquad c_I = c(e^{i_1}) \cdots c(e^{i_\ell}), \quad I = (i_1 < \cdots < i_\ell).$$

For odd $\ell = |I|$,

$$J c_I = \epsilon_n c_I J, \qquad c_I^* = (-1)^{\ell(\ell+1)/2} c_I.$$

The first identity and $JV = \epsilon_n VJ$ imply $z_I \in \mathbb{R}$. The second identity and $V^* = V$ then imply $z_I = 0$ unless $|I| \equiv 3 \pmod 4$. Conversely, these two conditions on the coefficients imply (2.2). The construction is $\mathrm{Spin}(n)$-equivariant and gives the stated bundle isomorphism. □

2.2. **Domains and local compactness.** Let X be closed or be the cylindrical completion of a compact spin manifold with invertible boundary Dirac operator. Fix a reference metric g_0. In a parameter family, the metrics equal g_0 outside one compact set and the potentials have a common compact support. Write Σ_h for the spinor bundle of a metric h, and put

$$\mathscr{H} = L^2(\Sigma_{g_0}), \qquad \mathscr{D} = H^1(\Sigma_{g_0}).$$

The Sobolev norm is defined by g_0 and its spin connection.

Write $h = g_0(S\cdot, \cdot)$, where S is positive and g_0-self-adjoint. The map $S^{-1/2}$ takes g_0-orthonormal frames to h-orthonormal frames. Let $\beta^h_{g_0} : \Sigma_{g_0} \to \Sigma_h$ be the bundle isomorphism induced by its spin lift, continued from the identity along $S_t^{-1/2}$, where $S_t = \exp(t \log S)$, $0 \le t \le 1$. With $\rho = dv_h / dv_{g_0}$, the map

$$U_h = \rho^{-1/2} \beta^h_{g_0} : L^2(\Sigma_{g_0}) \longrightarrow L^2(\Sigma_h) \tag{2.6}$$

is unitary [61, Lemmas 2.3.3–2.3.5]. It intertwines charge conjugation and, in even dimension, chirality. Both U_h and U_h^{-1} are bounded between the corresponding H^1 spaces, uniformly on compact smooth parameter families. The local formula [61, Theorem 2.3.6] shows that

$$h \longmapsto U_h^{-1} D_h U_h \in \mathcal{L}(\mathscr{D}, \mathscr{H})$$

is smooth on each finite-dimensional smooth family: the first-order coefficients depend smoothly on h, and the zeroth-order coefficients depend smoothly on h and its first spatial derivatives.

The metrics are complete, have positive injectivity radius, and satisfy $\| \mathrm{Ric}_h \|_\infty + \| \nabla^h \mathrm{Ric}_h \|_\infty < \infty$. The hypotheses of [17, Theorem 3.1] therefore hold with both metrics equal to h and the comparison map equal to the identity. By [17, Propositions 3.4 and 3.6], D_h is essentially self-adjoint on $C_c^\infty(\Sigma_h)$, its closure has domain $H^1(\Sigma_h)$, and its graph norm is equivalent to the H^1 norm. A bounded self-adjoint

multiplication endomorphism V has D_h-relative bound zero. Thus $D_h + V$ is self-adjoint on the same domain [67, Chapter V, Theorem 4.3]. For

$$L_{h,V} = U_h^{-1}(D_h + V)U_h \,:\, \mathscr{D} \longrightarrow \mathscr{H}$$

one has

$$\|u\|_{H^1} \le C\big(\|L_{h,V}u\|_{L^2} + \|u\|_{L^2}\big), \qquad u \in \mathscr{D}. \tag{2.7}$$

On a fixed compact smooth parameter family, the injectivity radii have a common positive lower bound, and the geometric bounds, H^1 comparison bounds, and $\|V\|_\infty$ are uniform. The constant C can consequently be chosen uniformly on that family.

If V is bounded and compactly supported, choose $\chi \in C_c^\infty(X)$ equal to one on its support. Multiplication by χ is compact from H^1 to L^2 by the Rellich–Kondrachov theorem [44, Theorem 16.8], applied in finitely many bundle charts. Since $V = V\chi$, multiplication by V is compact from H^1 to L^2, hence relatively compact with respect to D_h. For $V = V^*$, the essential spectra of $D_h + V$ and D_h agree [67, Chapter IV, Theorem 5.35]. On a closed X, these operators have compact resolvent. On a cylindrical completion, if $\lambda_0 > 0$ and the boundary operator has no spectrum in $(-\lambda_0, \lambda_0)$, then neither operator has essential spectrum there [40, Proposition 2.1(1)]. Every compact subinterval of $(-\lambda_0, \lambda_0)$ therefore contains only finitely many eigenvalues, all isolated and of finite multiplicity.

Lemma 2.3. *Let E, F be smooth vector bundles of equal finite rank over a manifold of positive dimension, equipped with a positive smooth density. Let $A : C^\infty(E) \to C^\infty(F)$ be a first-order differential operator with smooth coefficients, whose principal symbol at x_0 is invertible at every nonzero real covector. If an E-valued distribution T satisfies*

$$\operatorname{supp} T \subset \{x_0\}, \qquad AT = \delta_{x_0} v, \qquad v \in F_{x_0},$$

then $T = 0$ and $v = 0$. Here δ_{x_0} denotes the Dirac mass with respect to the fixed density.

Proof. In local coordinates centered at x_0 and local bundle frames, the point-support theorem [44, Theorem 4.19] gives

$$T = \sum_{|\alpha| \le m} t_\alpha \partial^\alpha \delta_0. \tag{2.8}$$

If $T \neq 0$, let $q = \max\{|\alpha| \,:\, t_\alpha \neq 0\}$. Write $A = \sum_j a^j(x)\partial_j + a^0(x)$ and set

$$p_q(\xi) = \sum_{|\alpha| = q} t_\alpha \xi^\alpha, \qquad a(0,\xi) = \sum_j a^j(0)\xi_j.$$

The derivatives of δ_0 of order $q+1$ in AT have coefficient polynomial $a(0,\xi)p_q(\xi)$; derivatives of the coefficients of A contribute only lower orders. Since $AT = \delta_{x_0} v$, this polynomial vanishes. For every real $\xi \neq 0$, invertibility of $a(0,\xi)$ gives $p_q(\xi) = 0$. Hence $p_q = 0$, contrary to the choice of q. Thus $T = 0$, and then $v = 0$. □

Lemma 2.4. *Let $n \ge 3$, $0 < m < n$, and let $b_p(x, y, \xi)$ be a smooth matrix-valued amplitude on a coordinate ball $\Omega \subset \mathbb{R}^n$, with p in a compact smooth parameter manifold. Suppose that in parameter charts, for all multi-indices $\alpha, \beta, \gamma, \delta$ with $|\alpha| \le 1$,*

$$|\partial_p^\alpha \partial_x^\beta \partial_y^\gamma \partial_\xi^\delta b_p(x, y, \xi)| \le C_{\alpha\beta\gamma\delta}(1 + |\xi|)^{-m-|\delta|}.$$

The oscillatory integral

$$K_p(x, y) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\langle x-y,\xi\rangle} b_p(x, y, \xi)\, d\xi$$

defines a distribution, smooth in (x, y) off the diagonal and C^1 in p there. On every smaller coordinate ball it satisfies

$$|\partial_p^\alpha K_p(x, y)| \le C|x - y|^{m-n}, \qquad 0 < |x - y| < 1, \quad |\alpha| \le 1. \tag{2.9}$$

The constant depends on finitely many uniform symbol seminorms. The same conclusions hold after addition of a smooth kernel with bounded $C_p^1 C_{x,y}^0$ norm.

Proof. Apply [53, Proposition 2.2] to each entry of $\partial_p^\alpha b_p$, $|\alpha| \le 1$, with order $r = -m$ and an integer $N > n - m$. The variables (p, x, y) are fixed parameters in this inverse-Fourier estimate. Since $-m > -n$, the estimate is (2.9), with a constant uniform in these parameters.

Off the diagonal, put

$$L = \frac{(x-y)\cdot \partial_\xi}{i|x-y|^2}, \qquad L e^{i\langle x-y,\xi\rangle} = e^{i\langle x-y,\xi\rangle}.$$

After N integrations by parts, an integrand differentiated k times in (x, y) and at most once in p is bounded on compact sets off the diagonal by $C(1+|\xi|)^{-m-N+k}$. Taking $N > n - m + k$ permits differentiation under the integral. Finally, a smooth remainder is bounded by the right-hand side of (2.9), since $|x-y|^{m-n} \ge 1$ for $0 < |x-y| < 1$. □

2.3. **The real index.** Put $KO_n = KO^{-n}(\mathrm{pt})$. Its values, for $n \equiv 0, \dots, 7 \pmod 8$, are

$$\mathbb{Z}, \quad \mathbb{Z}/2, \quad \mathbb{Z}/2, \quad 0, \quad \mathbb{Z}, \quad 0, \quad 0, \quad 0;$$

see [12, Table 2 and Theorem 11.5]. For a closed spin manifold M^n, let $\alpha(M) \in KO_n$ be its real Dirac index. For a compact spin manifold W^n with invertible boundary metric h, let $\alpha(W, h) \in KO_n$ be the index of the Dirac operator on its cylindrical completion, with the convention of [40, Section 5.4]. All kernels on a cylindrical completion are taken in L^2.

Set $X = M$ or $X = W_\infty$, respectively, and write α for either index. Put $\operatorname{Fix} J = \{u \in L^2(\Sigma X) : Ju = u\}$. In even dimension, D^+ has domain $H^1(\Sigma^+ X)$ and takes values in $L^2(\Sigma^- X)$. With the structures in (2.1),

$n \bmod 8$	α
0	$\operatorname{ind}_{\mathbb{R}} D^+$
1	$\dim_{\mathbb{R}}(\ker D \cap \operatorname{Fix} J) \pmod 2$
2	$\dim_{\mathbb{C}} \ker D^+ \pmod 2$
4	$\operatorname{ind}_{\mathbb{H}} D^+$.

In residue zero, D^+ restricts to the real fixed spaces of J; in residue four, it is quaternionic-linear. In residue one, iD is real skew-adjoint on $\operatorname{Fix} J$, with domain $H^1 \cap \operatorname{Fix} J$. In residue two, J exchanges the chiral kernels. The two parity formulas are [16, Theorems 3.1 and 3.3]. In the other residues, $\alpha = 0$.

For a closed manifold with $n \equiv 0, 4 \pmod 8$, the complex index theorem [14, Theorem 5.3] gives

$$\operatorname{ind}_{\mathbb{C}} D^+ = \widehat{A}(M) = \begin{cases} \alpha(M), & n \equiv 0 \pmod 8, \\ 2\alpha(M), & n \equiv 4 \pmod 8. \end{cases}$$

The factors 1 and 2 agree with the complexification homomorphisms in [12, Section 6, formula (6.11) and the preceding paragraph].

For $V \in C_c^\infty(X; \mathscr{A}_n)$, the operators

$$D + tV \,:\, H^1(\Sigma X) \longrightarrow L^2(\Sigma X), \qquad 0 \le t \le 1,$$

form a continuous Fredholm path. In the closed case this follows from the compact resolvent; in the cylindrical case it follows from the essential spectral gap. The identities (2.2) hold along the path. The KO_n-valued index is constant along this path.

3. Resolvents under positive congruence

Let $\mathscr{D}$ and $\mathscr{H}$ be complex Hilbert spaces with a fixed continuous dense inclusion $\mathscr{D} \hookrightarrow \mathscr{H}$. The unbounded operators on $\mathscr{H}$ considered below have domain $\mathscr{D}$. Write $\mathcal{L}(E, F)$ for the space of bounded linear maps between Banach spaces E, F, put $\mathcal{L}(E) = \mathcal{L}(E, E)$, and denote the resolvent set of a closed operator T by $\rho(T)$.

Let B be a compact smooth manifold, possibly with boundary, and fix a Riemannian metric on B. Differentiability at its boundary is understood by local extension. For an operator family A defined near a compact set $\mathcal{K} \subset B \times \mathbb{C}$, put

$$\|A\|_{C^1_b(\mathcal{K})} = \sup_{(b,z)\in\mathcal{K}} \|A_b(z)\| + \sup_{(b,z)\in\mathcal{K}} \|d_b A_b(z)\|.$$

The second norm is the operator norm on maps from T_bB, and no derivative in z occurs. For a family independent of z, the corresponding norm on B is denoted by $\|\cdot\|_{C^1(B)}$.

Lemma 3.1. *Let $b \mapsto T_b$ be C^1 with values in $\mathcal{L}(\mathscr{D}, \mathscr{H})$, and let $\mathcal{K} \subset B \times \mathbb{C}$ be compact. Suppose $T_b - z : \mathscr{D} \to \mathscr{H}$ is bijective for every $(b, z) \in \mathcal{K}$. Then*

$$R_b(z) = (T_b - z)^{-1} \in \mathcal{L}(\mathscr{H}, \mathscr{D})$$

is defined on an open neighborhood of $\mathcal{K}$, is C^1 in b, and has finite $C^1_b(\mathcal{K}; \mathcal{L}(\mathscr{H}, \mathscr{D}))$ norm. For every parameter coordinate,

$$\partial_b R_b = -R_b(\partial_b T_b)R_b. \tag{3.1}$$

Proof. The bounded inverse theorem and smooth inversion between Banach spaces [50, Remark 11.6 and Theorems 11.7, 11.17] apply to $(b, z) \mapsto T_b - z \in \mathcal{L}(\mathscr{D}, \mathscr{H})$. The derivative of the inverse is (3.1); its norm is bounded on $\mathcal{K}$ by compactness. □

Lemma 3.2. *Let T_b and $T_{j,b}$ be self-adjoint on $\mathscr{D}$, with $b \mapsto T_b$ of class C^1 in $\mathcal{L}(\mathscr{D}, \mathscr{H})$. Let $F_j \in C^1(B; \mathcal{L}(\mathscr{H}))$ take positive invertible values, and suppose $F_{j,b}^{\pm 1}$ restrict to bounded operators on $\mathscr{D}$ for every j, b. Suppose*

$$F_j T_j F_j = T + E_j \tag{3.2}$$

as operators $\mathscr{D} \to \mathscr{H}$, where $E_j \in C^1(B; \mathcal{L}(\mathscr{D}, \mathscr{H}))$ and

$$\|F_j - I\|_{C^1(B;\mathcal{L}(\mathscr{H}))} \longrightarrow 0, \qquad \|E_j\|_{C^1(B;\mathcal{L}(\mathscr{D},\mathscr{H}))} \longrightarrow 0. \tag{3.3}$$

For every compact set

$$\mathcal{K} \subset \{(b, z) \in B \times \mathbb{C} : z \in \rho(T_b)\},$$

one has $z \in \rho(T_{j,b})$ on $\mathcal{K}$ for all sufficiently large j, and

$$(T_{j,b} - z)^{-1} \longrightarrow (T_b - z)^{-1} \quad \text{in } C^1_b(\mathcal{K}; \mathcal{L}(\mathscr{H})).$$

Proof. Domain invariance gives

$$T_j - z = F_j^{-1}(T + E_j - zF_j^2)F_j^{-1} : \mathscr{D} \longrightarrow \mathscr{H}. \tag{3.4}$$

Set

$$R = (T - z)^{-1}, \qquad K_j = E_j - z(F_j^2 - I), \qquad B_j = K_j R,$$

and put

$$\eta_j = \|E_j\|_{C^1(B;\mathcal{L}(\mathscr{D},\mathscr{H}))} + \|F_j - I\|_{C^1(B;\mathcal{L}(\mathscr{H}))}.$$

For $\eta_j \le 1$, the product rule gives $\|F_j^2 - I\|_{C^1(B;\mathcal{L}(\mathscr{H}))} \le 3\eta_j$. Lemma 3.1 and the inclusion $\mathscr{D} \hookrightarrow \mathscr{H}$ therefore imply

$$\|B_j\|_{C^1_b(\mathcal{K};\mathcal{L}(\mathscr{H}))} \le C\eta_j.$$

For $C\eta_j < 1/2$, set $S_j = (I + B_j)^{-1}$. Then

$$\|S_j\|_{C^0(\mathcal{K};\mathcal{L}(\mathscr{H}))} \le 2, \qquad S_j - I = -S_j B_j, \qquad \partial_b S_j = -S_j(\partial_b B_j)S_j,$$

and hence

$$\|S_j - I\|_{C^1_b(\mathcal{K};\mathcal{L}(\mathscr{H}))} \le 4C\eta_j.$$

Since $T + E_j - zF_j^2 = (I + B_j)(T - z)$, (3.4) yields

$$(T_j - z)^{-1} = F_j R S_j F_j. \tag{3.5}$$

The right-hand side maps $\mathscr{H}$ into $\mathscr{D}$ for each j, b, and all four factors are C^1 as operators on $\mathscr{H}$. From

$$F_j R S_j F_j - R = (F_j - I)RS_jF_j + R(S_j - I)F_j + R(F_j - I)$$

and the product rule, one obtains

$$\left\|(T_j - z)^{-1} - (T - z)^{-1}\right\|_{C_b^1(\mathcal{K};\mathcal{L}(\mathscr{H}))} \le C\eta_j. \tag{3.6}$$

The constant depends only on $\mathcal{K}$, T, the fixed parameter norm, and the inclusion $\mathscr{D} \hookrightarrow \mathscr{H}$; it is independent of the norms of $F_j^{\pm 1}|_{\mathscr{D}}$. □

Remark 3.3. Positive congruence preserves invertibility, but need not preserve nonzero eigenvalues. For example,

$$\operatorname{diag}(2,1)\operatorname{diag}(1,-1)\operatorname{diag}(2,1) = \operatorname{diag}(4,-1).$$

Let $U \subset \mathbb{R}^d$ be open. In the next two lemmas, matrix fields take values in $\mathrm{Mat}_q(\mathbb{C})$, $q \ge 1$, with the operator norm of the standard Hermitian product. Write $[H, A] = HA - AH$ and $\mathrm{ad}_H(A) = [H, A]$. Parameter derivatives are taken in a fixed finite coordinate cover of B, restricted to relatively compact subcharts.

Lemma 3.4. *Let H, A, Ω_b be smooth matrix fields on $B \times U$, and let X_b be a smooth family of spatial vector fields. Put $Y(A) = X_b(A) + [\Omega_b, A]$; thus $Y(AC) = Y(A)C + AY(C)$ for matrix fields A, C. For each $r > 0$, on $\|H\| \le r$,*

$$\begin{aligned} e^{-H}Y(e^H) &= YH - \tfrac{1}{2}[H, YH] + \mathcal{E}_-(H, YH), \\ Y(e^H)e^{-H} &= YH + \tfrac{1}{2}[H, YH] + \mathcal{E}_+(H, YH), \end{aligned} \tag{3.7}$$

where $\|\mathcal{E}_\pm(H, YH)\| \le C_r \|H\|^2 \|YH\|$. For every first parameter derivative,

$$\|\partial_b \mathcal{E}_\pm(H, YH)\| \le C_r \left(\|H\| \|\partial_b H\| \|YH\| + \|H\|^2 \|\partial_b(YH)\|\right). \tag{3.8}$$

Also,

$$e^{-H}Ae^H = A - [H, A] + C(H, A),$$

where C satisfies the same bounds with YH replaced by A and $\partial_b(YH)$ by $\partial_b A$.

Proof. The differential formula for the matrix exponential [58, Theorem 5.4 and formula (5.13), p. 115], applied to curves tangent to X_b and to $t \mapsto e^{t\Omega_b} H e^{-t\Omega_b}$, gives

$$e^{-H}Y(e^H) = \int_0^1 e^{-tH}(YH)e^{tH}\,dt = \sum_{j=0}^{\infty} \frac{(-1)^j}{(j+1)!}\,\mathrm{ad}_H^j(YH).$$

The right logarithmic derivative has the series

$$Y(e^H)e^{-H} = \sum_{j=0}^{\infty} \frac{1}{(j+1)!}\,\mathrm{ad}_H^j(YH).$$

By [58, Proposition 3.35, p. 64], $e^{-H}Ae^H = e^{-\mathrm{ad}_H}A$. Thus the remainders are

$$\mathcal{E}_\pm(H, A) = \sum_{j=2}^{\infty} \frac{(\pm 1)^j}{(j+1)!}\,\mathrm{ad}_H^j(A), \qquad C(H, A) = \sum_{j=2}^{\infty} \frac{(-1)^j}{j!}\,\mathrm{ad}_H^j(A).$$

For $j \geq 2$, the Leibniz rule gives

$$\left\| \mathrm{ad}_H^j(A) \right\| \leq 2^j \|H\|^j \|A\|,$$
$$\left\| \partial_b(\mathrm{ad}_H^j(A)) \right\| \leq 2^j \left(j \|H\|^{j-1} \|\partial_b H\| \|A\| + \|H\|^j \|\partial_b A\| \right).$$

Since $\sum_{j\geq 2} 2^j(1+j)r^{j-2}/j! < \infty$, these bounds imply the stated estimates and local uniform convergence of the differentiated series. □

Lemma 3.5. *Let $\mathcal{K}_0 \subset B \times U$ be compact, let P_b, Q_b be smooth matrix fields on a neighborhood of $\mathcal{K}_0$, and let s be a spatial coordinate independent of b. Write $\|F\|_\infty = \sup_{(b,x)\in\mathcal{K}_0} \|F_b(x)\|$. For a coordinate derivative, the supremum is taken over the corresponding part of $\mathcal{K}_0$. For $N \geq 1$, put*

$$Z_{b,N} = N^{-1/2}(P_b \cos Ns + Q_b \sin Ns).$$

Let k be a nonnegative integer. Let $v_b, Y_{1,b}, \dots, Y_{k,b}$ be smooth spatial vector fields on the same neighborhood, independent of N, such that

$$v_b s = 1, \qquad Y_{\nu,b} s = 0 \quad (1 \leq \nu \leq k).$$

These identities hold throughout the neighborhood of $\mathcal{K}_0$. Put $\mathcal{Y}_b = Y_{1,b} \cdots Y_{k,b}$, with $\mathcal{Y}_b = I$ for $k = 0$. For every parameter multi-index α with $|\alpha| \leq 1$,

$$\left\| \partial_b^\alpha(\mathcal{Y}_b Z_{b,N}) \right\|_\infty \leq C_k N^{-1/2}, \tag{3.9}$$

and

$$\left\| \partial_b^\alpha(\mathcal{Y}_b v_b Z_{b,N}) \right\|_\infty \leq C_k N^{1/2}, \tag{3.10}$$
$$\left\| \partial_b^\alpha\left(\mathcal{Y}_b([Z_{b,N}, v_b Z_{b,N}] - [P_b, Q_b])\right) \right\|_\infty \leq C_k N^{-1}.$$

For $\mathcal{R} \in \{\mathcal{E}_-, \mathcal{E}_+, \mathcal{C}\}$,

$$\left\| \partial_b^\alpha \mathcal{Y}_b \mathcal{R}(Z_{b,N}, v_b Z_{b,N}) \right\|_\infty \leq C_k N^{-1/2}.$$

The constants depend on $\mathcal{K}_0$ and finitely many derivatives of the fixed matrix fields and vector-field coefficients, but not on N.

Proof. Put $\epsilon = N^{-1/2}$ and $\vartheta = Ns$. Since $Y_{\nu,b} s = 0$ and $\partial_b s = 0$,

$$\mathcal{Y}_b Z_{b,N} = \epsilon\left((\mathcal{Y}_b P_b)\cos\vartheta + (\mathcal{Y}_b Q_b)\sin\vartheta\right),$$

and $\partial_b \vartheta = 0$. This gives (3.9). Moreover,

$$v_b Z_{b,N} = \epsilon N(-P_b \sin\vartheta + Q_b \cos\vartheta) + \epsilon\left((v_b P_b)\cos\vartheta + (v_b Q_b)\sin\vartheta\right),$$

and, since $\epsilon^2 N = 1$,

$$[Z_{b,N}, v_b Z_{b,N}] - [P_b, Q_b] = \epsilon^2\big[P_b \cos\vartheta + Q_b \sin\vartheta, \\ (v_b P_b)\cos\vartheta + (v_b Q_b)\sin\vartheta\big].$$

Applying $\partial_b^\alpha \mathcal{Y}_b$ to these identities gives (3.10).

Set $A_N = v_b Z_{b,N}$ and $D = \partial_b^\alpha \mathcal{Y}_b$. If W is obtained from D by deleting any of its derivative factors, without changing their order, the preceding identities give

$$\|W Z_{b,N}\|_\infty \leq c_k \epsilon, \qquad \|W A_N\|_\infty \leq c_k \epsilon N,$$

where $c_k \geq 1$ is independent of N; the empty composition is included. Each of the 2^j terms in $\mathrm{ad}^j_{Z_{b,N}}(A_N)$ is a product of j factors $Z_{b,N}$ and one factor A_N. The Leibniz rule therefore gives

$$\left\| D\,\mathrm{ad}^j_{Z_{b,N}}(A_N) \right\|_\infty \leq 2^j (j+1)^{k+1} c_k^{j+1} \epsilon^{j+1} N.$$

Since $0 < \epsilon \le 1$,

$$\sum_{j=2}^{\infty} \frac{\left\| D \operatorname{ad}^j_{Z_{b,N}}(A_N) \right\|_\infty}{j!} \le c_k \epsilon^3 N \sum_{j=2}^{\infty} \frac{(2c_k)^j (j+1)^{k+1}}{j!} \le C_k \epsilon.$$

The same estimates hold on a compact neighborhood of $\mathcal{K}_0$. The series defining the three remainders thus converge uniformly with the indicated derivatives and satisfy the last bound. □

4. Realization of three-form perturbations

Let X be as in Section 2, with $n = \dim X \ge 3$.

4.1. **A local connection formula.** Let g be a smooth Riemannian metric on X, write $\nabla = \nabla^g$, and let s be a smooth function without critical points on a coordinate neighborhood. Put

$$\kappa = |ds|_g, \qquad \nu = \frac{\nabla^g s}{\kappa}, \qquad v = \frac{\nu}{\kappa}.$$

Then $vs = 1$. Flow coordinates $(s, y^1, \dots, y^{n-1})$ for v satisfy $v = \partial_s$ and

$$g = a^2 ds^2 + q_s, \qquad a = \kappa^{-1}, \tag{4.1}$$

where q_s is the metric on the level hypersurface of s. Put $e_0 = \nu$ and choose a frame $e = (e_1, \dots, e_{n-1})$ tangent to these hypersurfaces such that (e_0, e) is an oriented g-orthonormal frame. Denote its dual coframe by $(e^0, \dots, e^{n-1})$; thus $e^0 = a\, ds$. Define L by

$$[v, e_a] = \sum_d e_d L_{da}.$$

The brackets are tangential since $vs = 1$ and $e_a s = 0$. If $e_a = \sum_d E_{da} \partial_{y^d}$, then $L = E^{-1} \partial_s E$. Set

$$(\Omega_0)_{da} = g(\nabla_\nu e_a, e_d), \qquad S_{da} = g(\nabla_{e_a} \nu, e_d).$$

The matrix S is symmetric, and the tangential part of the torsion-free identity gives $\Omega_0 = \kappa L + S$. Since Ω_0 is skew-symmetric,

$$S = -\frac{\kappa}{2}(L + L^T), \qquad \Omega_0 = \frac{\kappa}{2}(L - L^T). \tag{4.2}$$

Let H be a smooth real symmetric $(n-1) \times (n-1)$ matrix field in the frame e. Regard it as a tangential endomorphism, extended by zero on $\mathbb{R}\nu$, and put $\Lambda_H = e^{-H/2}$. Define

$$g_H = a^2 ds^2 + q_s(e^H \cdot, \cdot). \tag{4.3}$$

The fields ν, v and the function κ are unchanged. With $f_a = (e\Lambda_H)_a$, the frame $(\nu, f_1, \dots, f_{n-1})$ is oriented and g_H-orthonormal. Define

$$(\Omega_H)_{da} = g_H(\nabla^{g_H}_\nu f_a, f_d).$$

A prime on a matrix field denotes entrywise differentiation along v.

Lemma 4.1. *One has*

$$\Omega_H = \frac{\kappa}{2}(\Lambda_H^{-1}\Lambda_H' - \Lambda_H'\Lambda_H^{-1}) + \frac{\kappa}{2}(\Lambda_H^{-1} L \Lambda_H - \Lambda_H L^T \Lambda_H^{-1}). \tag{4.4}$$

Proof. Since $[v, e\Lambda_H] = e(L\Lambda_H + \Lambda_H')$, the bracket matrix of $e\Lambda_H$ is

$$L_H = \Lambda_H^{-1} L \Lambda_H + \Lambda_H^{-1} \Lambda_H'.$$

Equation (4.2) for g_H gives $\Omega_H = \frac{\kappa}{2}(L_H - L_H^T)$. Both Λ_H and Λ_H' are symmetric, which gives (4.4). □

For a vector field w, write $c(w) = c_g(w^\flat)$, and put $C_0 = c(\nu)$. In a local spin frame lifting (e_0, e), vector fields act on the spinor components. The local spin connection formula [22, equation (4)] gives

$$D_g = \sum_{i=0}^{n-1} c(e_i)e_i + \frac{1}{4}\sum_{i,j,k=0}^{n-1} g(\nabla_{e_i}e_j, e_k)c(e_i)c(e_j)c(e_k). \tag{4.5}$$

The tangential part of the connection in the ν-direction has the Clifford contraction

$$-\frac{1}{4}C_0\sum_{a,d=1}^{n-1}(\Omega_0)_{ad}c(e_a)c(e_d). \tag{4.6}$$

Here $(\Omega_0)_{da} = -(\Omega_0)_{ad}$. Let ∇^{q_s} denote the Levi-Civita connection of q_s. On these ambient spinor components, put

$$D^\top = \sum_{a=1}^{n-1} c(e_a)e_a + \frac{1}{4}\sum_{a,d,f=1}^{n-1} q_s(\nabla^{q_s}_{e_a}e_d, e_f)c(e_a)c(e_d)c(e_f).$$

The spinorial Gauss formula [22, equations (10)–(11)], with Weingarten matrix $-S$, and (4.5) give

$$D_g = C_0\nu + D^\top + \frac{1}{2}\operatorname{tr}(S)C_0 - \frac{1}{4}C_0\sum_{a,d=1}^{n-1}(\Omega_0)_{ad}c(e_a)c(e_d) - \frac{1}{2}c(\nabla_\nu \nu). \tag{4.7}$$

In the flow coordinates of (4.1), the normal acceleration is

$$\nabla_\nu \nu = -\operatorname{grad}_{q_s}(\log a). \tag{4.8}$$

The Koszul formula gives $g(\nabla_v v, \partial_{y^d}) = -a\,\partial_{y^d}a$. Since $\nu = a^{-1}v$, this gives (4.8); its normal component vanishes because $g(\nu, \nu) = 1$.

4.2. **Parameter-dependent estimates.**

Proposition 4.2. *Let B be a compact smooth manifold, possibly with boundary, and let g_b, $b \in B$, be a smooth family in the setting of Section 2. Let $\mathcal{U} \subset B \times X$ be a product coordinate neighborhood, with a spatial coordinate s independent of b. Let $\mathcal{T}_b = ds \wedge \beta_b$ be a smooth real three-form family with compact support in $\mathcal{U}$. For each $N \geq 1$, there is a smooth family of metrics $g_{b,N}$, equal to g_b outside $\mathcal{U}$, such that*

$$\begin{aligned} dv_{g_{b,N}} &= dv_{g_b}, \qquad \|g_N - g\|_{C^1(B;C^0)} \leq CN^{-1/2}, \\ U_{b,N}^{-1} D_{g_{b,N}} U_{b,N} &= D_{g_b} + c_{g_b}(\mathcal{T}_b) + E_{b,N}, \\ \|E_N\|_{C^1(B;\mathcal{L}(H^1,L^2))} &\leq CN^{-1/2}. \end{aligned} \tag{4.9}$$

Here $U_{b,N} : L^2(\Sigma_{g_b}) \to L^2(\Sigma_{g_{b,N}})$ is the positive spinor comparison of (2.6) *for the pair $(g_b, g_{b,N})$. The metric norm is computed relative to a fixed reference metric. The last norm is the $C^1(B;\mathcal{L}(\mathscr{D},\mathscr{H}))$ norm of $b \mapsto U_{g_b}^{-1}E_{b,N}U_{g_b}$, with $\mathscr{D}, \mathscr{H}$ and U_{g_b} as in Section 2. The constant C is independent of N. If $\mathcal{V} \subset B$ is open and its closure is disjoint from the parameter projection of* $\operatorname{supp}\mathcal{T}$*, the metrics may also be chosen equal to g_b for $b \in \mathcal{V}$.*

Proof. Put

$$\kappa_b = |ds|_{g_b}, \quad \nu_b = \kappa_b^{-1}\nabla^{g_b}s, \quad v_b = \kappa_b^{-1}\nu_b, \quad C_{0,b} = \kappa_b^{-1}c_{g_b}(ds).$$

Replace β_b by $\iota_{v_b}\mathcal{T}_b$. Since $ds(v_b) = 1$ and $ds \wedge \mathcal{T}_b = 0$, this gives

$$ds \wedge \beta_b = \mathcal{T}_b, \qquad \iota_{v_b}\beta_b = 0, \qquad \operatorname{supp}\beta = \operatorname{supp}\mathcal{T}.$$

Orthonormalize the coordinate fields tangent to s to obtain $e_{1,b}, \dots, e_{n-1,b}$. Choose their signs so that $(\nu_b, e_{1,b}, \dots, e_{n-1,b})$ is oriented. Let e_b^a be the dual tangential covectors, extended by zero on ν_b, and put

$$\beta_{ad,b} = \beta_b(e_{a,b}, e_{d,b}), \qquad \beta_b = \tfrac{1}{2}\sum_{a,d}\beta_{ad,b}\,e_b^a \wedge e_b^d, \qquad K_{ad,b} = 16\beta_{ad,b}.$$

Choose a constant diagonal trace-free real matrix A of size $n-1$ with distinct entries, and set

$$B_{ad,b} = \frac{K_{ad,b}}{A_{aa} - A_{dd}} \quad (a \neq d), \qquad B_{aa,b} = 0. \tag{4.10}$$

The matrix K_b is skew-symmetric; hence $B_b^T = B_b$, $\operatorname{tr} B_b = 0$, and $[A, B_b] = K_b$.

Fix $\mathcal{V}$ as in the statement, allowing $\mathcal{V} = \emptyset$. Choose $\chi \in C_c^\infty(\mathcal{U}; \mathbb{R})$ equal to one near $\operatorname{supp} \mathcal{T}$ and zero on $\mathcal{V} \times X$. With $\epsilon = N^{-1/2}$, put

$$H_{b,N} = \epsilon(\chi A \cos Ns + B_b \sin Ns). \tag{4.11}$$

The tangential endomorphism represented by $H_{b,N}$ extends smoothly by zero on $\mathbb{R}v_b$ and outside $\mathcal{U}$. Define $g_{b,N}$ by (4.3). Since $\operatorname{tr} H_{b,N} = 0$,

$$\frac{dv_{g_{b,N}}}{dv_{g_b}} = e^{\frac{1}{2}\operatorname{tr} H_{b,N}} = 1.$$

Put $H_N = H_{b,N}$ and $\Lambda_N = e^{-H_N/2}$. In local spin frames lifting the paired oriented orthonormal frames (v_b, e_b) and $(v_b, e_b\Lambda_N)$, the map $U_{b,N}$ identifies the Clifford matrices. The density factor in (2.6) is one, and the normal principal term $C_{0,b}\, v_b$ is unchanged.

All estimates below hold in the given product coordinates, uniformly with one parameter derivative. Lemma 3.5, with $v_b\, s = 1$ and $e_{a,b}\, s = 0$, gives

$$H_N = O(\epsilon), \qquad e_{a,b}H_N = O(\epsilon), \qquad v_b H_N = O(\epsilon N).$$

The same bounds hold after any fixed number of tangential derivatives. The exponential formula gives

$$\Lambda_N^{\pm 1} - I = O(\epsilon), \qquad e_{a,b}(\Lambda_N^{\pm 1}) = O(\epsilon),$$

and the metric estimate of (4.9) follows.

Write $g = g_b$, $g_N = g_{b,N}$, $e = e_b$, $\beta = \beta_b$, $B = B_b$, $K = K_b$, and $C_0 = C_{0,b}$; the fields v, ν, κ and Clifford multiplication refer to g. For $H_N' = vH_N$, Lemma 3.5 gives

$$\begin{aligned}[H_N, H_N'] &= [\chi A \cos Ns + B \sin Ns, -\chi A \sin Ns + B \cos Ns] + O(N^{-1}) \\ &= K + O(N^{-1}),\end{aligned} \tag{4.12}$$

because $\chi B = B$ and $[A, B] = K$. Lemmas 3.4 and 3.5, applied to $-H_N/2$, give

$$\Lambda_N^{-1}\Lambda_N' - \Lambda_N'\Lambda_N^{-1} = -\tfrac{1}{4}[H_N, H_N'] + O(\epsilon^3 N).$$

Moreover,

$$\Lambda_N^{-1} L \Lambda_N - \Lambda_N L^T \Lambda_N^{-1} - (L - L^T) = O(\epsilon).$$

Since $\epsilon^3 N = \epsilon$, (4.4) yields

$$\Omega_{H_N} - \Omega_0 = -\frac{\kappa}{8}[H_N, H_N'] + O(N^{-1/2}) = -\frac{\kappa}{8}K + O(N^{-1/2}). \tag{4.13}$$

Equation (4.6) therefore gives the contribution

$$\frac{\kappa}{32}\sum_{a,d} K_{ad} C_0\, c(e_a)c(e_d) = \frac{\kappa}{2}\sum_{a,d} \beta_{ad} C_0\, c(e_a)c(e_d) = c(ds \wedge \beta). \tag{4.14}$$

Put $f_a = (e\Lambda_N)_a$ and $q_{s,N} = g_N|_{\ker ds}$. The tangential principal coefficients differ by $O(\epsilon)$ because $\Lambda_N - I = O(\epsilon)$. For the intrinsic connection, define the structure coefficients by

$$[e_i, e_j] = \sum_\ell c_{ij}^\ell e_\ell, \qquad [f_a, f_d] = \sum_r \widetilde{c}_{ad}^{\,r} f_r,$$

where all indices in these two identities range from 1 to $n-1$. The Leibniz rule gives

$$\widetilde{c}_{ad}^{\,r} = (\Lambda_N^{-1})_{r\ell}\big((\Lambda_N)_{ia}(\Lambda_N)_{jd}c_{ij}^\ell + (\Lambda_N)_{ia}e_i((\Lambda_N)_{\ell d}) - (\Lambda_N)_{id}e_i((\Lambda_N)_{\ell a})\big),$$

with repeated indices summed. Thus $\widetilde{c}^{r}_{ad} - c^{r}_{ad} = O(\epsilon)$. The Koszul formula in the orthonormal frame f is

$$2g_N(\nabla^{q_{s,N}}_{f_a} f_d, f_r) = \widetilde{c}^{r}_{ad} - \widetilde{c}^{a}_{dr} + \widetilde{c}^{d}_{ra}.$$

It follows that the intrinsic spin-connection coefficients differ by $O(\epsilon)$, with one parameter derivative.

Define $(S_N)_{da} = g_N(\nabla^{g_N}_{f_a} \nu, f_d)$. The bracket identity in the proof of Lemma 4.1 gives

$$\operatorname{tr} S_N - \operatorname{tr} S = -\kappa \operatorname{tr}(\Lambda_N^{-1}\Lambda_N') = -\kappa v(\log\det\Lambda_N) = \frac{\kappa}{2} v(\operatorname{tr} H_N) = 0.$$

The lapse $a = \kappa^{-1}$ is the same for g and g_N. By (4.8),

$$g_N(\nabla^{g_N}_{\nu} \nu, f_a) = -f_a(\log a).$$

Since $f_a - e_a = O(\epsilon)$, the acceleration coefficients in the paired frames differ by $O(\epsilon)$, with one parameter derivative. Together with (4.14), these estimates account for every term of (4.7).

In a fixed spatial chart $x = (x^1, \dots, x^n)$ and a local g_b-spin frame, put $\partial_j = \partial/\partial x^j$. The difference $E_{b,N}$ has the form

$$E_{b,N} = \sum_j a_{j,b,N}\partial_j + a_{0,b,N}, \qquad \sum_j \|a_{j,b,N}\|_{C^1_b L^\infty_x} + \|a_{0,b,N}\|_{C^1_b L^\infty_x} \le C\epsilon.$$

Here j ranges over the spatial coordinates. The error is supported in a fixed compact subset of $\mathcal{U}$. In a local reference spin frame, write Q_b for the matrix of U_{g_b}. Then

$$Q_b^{-1} E_{b,N} Q_b = \sum_j Q_b^{-1} a_{j,b,N} Q_b \partial_j + Q_b^{-1} a_{0,b,N} Q_b + \sum_j Q_b^{-1} a_{j,b,N} (\partial_j Q_b).$$

The matrices $Q_b^{\pm 1}$ and their first spatial derivatives, each with at most one parameter derivative, are bounded on the compact support independently of N. Thus all coefficients in this expression have $C^1_b L^\infty_x$ norm at most $C\epsilon$. For bounded matrix coefficients A_j, A_0 and $u \in H^1$, the estimate

$$\left\| \sum_j A_j \partial_j u + A_0 u \right\|_{L^2} \le C \left(\sum_j \|A_j\|_\infty + \|A_0\|_\infty \right) \|u\|_{H^1},$$

applied in a fixed finite trivializing cover, also applies to the parameter derivatives of the coefficients. It proves the last estimate of (4.9). Finally, $H_{b,N} = 0$ outside $\mathcal{U}$ and for $b \in \mathcal{V}$, so the corresponding metrics equal g_b there. □

Remark 4.3. The three-torus metric in [43, arXiv version 2, equation (8.1)] has the symmetric coframe given in [43, equation (8.2)]. On spinors depending only on the distinguished coordinate, its Dirac operator has the scalar term, quadratic to leading order, displayed in [43, equation (8.5)].

5. Realization of compatible potentials

Let $n = \dim X \ge 3$. Let B be a compact smooth manifold, possibly with boundary, and let g_b, $b \in B$, be a smooth metric family in the setting of Section 2. Parameter norms for spinor endomorphisms and operators are taken after conjugation by the identifications U_{g_b} of (2.6).

5.1. **Elementary forms.** A smooth family of real differential forms of positive degree on X, with compact support in $B \times X$, admits a finite decomposition subordinate to product coordinate neighborhoods. Expanding each summand in the spatial coordinate coframe gives terms $ds \wedge \gamma_b$, where s is a spatial coordinate independent of b. Put $v_b = \nabla^{g_b} s / |ds|^2_{g_b}$. Replacing γ_b by $\iota_{v_b}(ds \wedge \gamma_b)$ preserves the term and makes $\iota_{v_b}\gamma_b = 0$.

For a term of degree $\ell \ge 7$, choose a smooth orthonormal coframe of the covectors annihilating v_b and expand γ_b in its wedge basis. Each summand has the form

$$\mathcal{T}_b = f_b \, ds \wedge \alpha_b \wedge \beta_b, \tag{5.1}$$

where α_b is a wedge of three covectors of this coframe and β_b is a wedge of $\ell-4$ distinct further covectors. The coefficient f is real, smooth, and compactly supported in the product coordinate neighborhood $\mathcal{U} \subset B \times X$.

Proposition 5.1. *Let $7 \le \ell \le n$, $\ell \equiv 3 \pmod 4$, and let $\mathcal{T}_b$ be as in* (5.1)*. Fix a smooth family of compatible potentials $V_{0,b}$ in the setting of Section 2. For each $N \ge 1$, there are smooth positive invertible multipliers $F_{b,N}$ and compatible potentials $V_{1,b,N}$, sums of Clifford forms of degrees 3 and $\ell-4$, such that*

$$F_{b,N}(D_{g_b} + V_{0,b} + V_{1,b,N})F_{b,N} = D_{g_b} + V_{0,b} + c_{g_b}(\mathcal{T}_b) + E_{b,N}. \tag{5.2}$$

The identity holds as operators $H^1(\Sigma_{g_b}) \to L^2(\Sigma_{g_b})$. There is a constant C, depending on the fixed family but independent of N, such that

$$\|F_N - I\|_{C^1_b L^\infty_x} \le CN^{-1/2}, \qquad \|E_N\|_{C^1_b \mathcal{L}(H^1,L^2)} \le CN^{-1/2}. \tag{5.3}$$

The supports of $F_N - I$, $V_{1,N}$, and E_N lie in one compact subset of $\mathcal{U}$, independent of N. If $\mathcal{V} \subset B$ is open and its closure is disjoint from the parameter projection of supp f*, the multipliers and potentials may be chosen so that $F_{b,N} = I$ and $V_{1,b,N} = 0$ for $b \in \mathcal{V}$.*

Proof. Suppress the parameter b. Write ∇ for the spin connection of g and its induced connection on endomorphisms. Put

$$\kappa = |ds|_g, \qquad \nu = \kappa^{-1}\nabla^g s, \qquad C_0 = \kappa^{-1} c_g(ds), \qquad P = c_g(\alpha), \qquad Q = c_g(\beta).$$

The degrees of P and Q are 3 (mod 4). By Lemma 2.2, they are compatible; in particular, $P^* = P$ and $Q^* = Q$. Their covector factors and ds are mutually orthogonal, so

$$PC_0 = -C_0P, \qquad QC_0 = -C_0Q, \qquad PQ = -QP.$$

Set $S = C_0P$ and $T = C_0Q$. Since $C_0^* = -C_0$ and $C_0^2 = -I$,

$$S^* = S, \quad T^* = T, \quad SC_0 = -C_0S, \quad TC_0 = -C_0T.$$

Moreover, $JS = SJ$ and $JT = TJ$ by (2.1); in even dimension, S and T commute with Γ. Finally, $ST = PQ$ and $TS = QP$, whence

$$[S, T] = 2PQ. \tag{5.4}$$

Fix $\mathcal{V}$ as in the statement, allowing $\mathcal{V} = \emptyset$. Choose $\chi \in C^\infty_c(\mathcal{U}; \mathbb{R})$ equal to one near supp f and zero on $\mathcal{V} \times X$. Define

$$\begin{aligned} H_N &= N^{-1/2}(\chi S \cos Ns + fT \sin Ns), & F_N &= e^{H_N}, \\ V_{1,N} &= \kappa\sqrt{N}(-\chi P \sin Ns + fQ \cos Ns). \end{aligned} \tag{5.5}$$

The fields H_N and $V_{1,N}$ extend smoothly by zero outside $\mathcal{U}$. The multiplier F_N is positive, invertible, and commutes with J and, in even dimension, Γ. Lemma 2.2 gives the compatibility of $V_{1,N}$. Since $H_NC_0 = -C_0H_N$,

$$F_NC_0F_N = C_0. \tag{5.6}$$

Put

$$\dot H_N = \kappa\sqrt{N}(-\chi S \sin Ns + fT \cos Ns), \qquad G_N = \nabla_\nu H_N - \dot H_N.$$

Then $V_{1,N} = -C_0\dot H_N$ and

$$G_N = N^{-1/2}\big(\nabla_\nu(\chi S)\cos Ns + \nabla_\nu(fT)\sin Ns\big).$$

All estimates below include one parameter derivative in the fixed product coordinates. Lemma 3.5, with $v = \nu/\kappa$, and the fixed smooth spin-connection coefficients give

$$H_N = O(N^{-1/2}), \qquad G_N = O(N^{-1/2}), \qquad \dot H_N = O(N^{1/2}).$$

Since $\chi f = f$, direct expansion gives

$$[H_N, \dot{H}_N] = \kappa f [S, T].$$

By Lemma 3.4,

$$\begin{aligned} F_N^{-1}\nabla_\nu F_N - F_N^{-1}\dot{H}_N F_N = G_N + \tfrac{1}{2}[H_N, \dot{H}_N] - \tfrac{1}{2}[H_N, G_N] \\ + \mathcal{E}_-(H_N, \nabla_\nu H_N) - C(H_N, \dot{H}_N). \end{aligned}$$

The two remainders are $O(N^{-1/2})$, also after one parameter derivative, by (3.8). Indeed, each bound has two factors of order $N^{-1/2}$ and one of order $N^{1/2}$. Thus

$$\begin{aligned} F_N C_0 \nabla_\nu F_N + F_N V_{1,N} F_N &= C_0\big(F_N^{-1}\nabla_\nu F_N - F_N^{-1}\dot{H}_N F_N\big) \\ &= \tfrac{1}{2} C_0 [H_N, \dot{H}_N] + O(N^{-1/2}) \\ &= \tfrac{\kappa}{2} f C_0 [S, T] + O(N^{-1/2}) \\ &= c_g(f\, ds \wedge \alpha \wedge \beta) + O(N^{-1/2}). \end{aligned} \tag{5.7}$$

Let $e_1, \dots, e_{n-1}$ be the orthonormal frame of $\ker ds$ dual to the chosen tangential coframe. Then $e_a s = 0$, and Lemmas 3.4 and 3.5 give $\nabla_{e_a} F_N = O(N^{-1/2})$. Also,

$$F_N c_g(e_a^\flat) F_N - c_g(e_a^\flat) = O(N^{-1/2}), \qquad F_N V_0 F_N - V_0 = O(N^{-1/2}).$$

The second bound depends only on the fixed $C_b^1 L_x^\infty$ norm of V_0 on the common support. The Leibniz rule and (5.6) give the normal principal term $C_0\nabla_\nu$. The remaining first-order error is

$$\sum_{a=1}^{n-1} \big(F_N c_g(e_a^\flat) F_N - c_g(e_a^\flat)\big)\nabla_{e_a}.$$

The remaining zeroth-order terms are the remainder in (5.7), $\sum_a F_N c_g(e_a^\flat)\nabla_{e_a} F_N$, and $F_N V_0 F_N - V_0$.

The reference comparisons and their first spatial derivatives, each with at most one parameter derivative, are bounded on the common compact support independently of N. In local reference spinor trivializations and fixed spatial coordinates $x^1, \dots, x^n$, the preceding estimates therefore give

$$U_{g_b}^{-1} E_{b,N} U_{g_b} = \sum_{j=1}^{n} a_{j,b,N}\partial_{x^j} + a_{0,b,N},$$

$$\sum_{j=1}^{n} \|a_{j,b,N}\|_{C_b^1 L_x^\infty} + \|a_{0,b,N}\|_{C_b^1 L_x^\infty} \le C N^{-1/2}.$$

The H^1-to-L^2 estimate at the end of the proof of Proposition 4.2 therefore applies and gives the second bound in (5.3). Smoothness of the exponential and the bound for H_N give the first.

For each finite N, $F_N^{\pm 1}$ are smooth multipliers equal to I outside a fixed compact set. The Leibniz rule makes them bounded isomorphisms of H^1 as well as of L^2. Lemma 3.2 requires no uniform bound for their H^1 operator norms. Finally, $\chi = f = 0$ outside the fixed compact support and on $\mathcal{V} \times X$, which gives the asserted support and parameter equalities. □

Proof of Theorem R. If $V = 0$, take $\widehat{g} = g$. Suppose $V \neq 0$. Let $\pi_B : B \times X \to B$ be the parameter projection. Choose an open neighborhood $\mathcal{V}$ of B_0 such that $\overline{\mathcal{V}} \cap \pi_B(\operatorname{supp} V) = \emptyset$. Cover $\operatorname{supp} V$ by finitely many relatively compact product coordinate neighborhoods whose closures lie in $B \times U$ and are disjoint from $\overline{\mathcal{V}} \times X$. Take all decompositions and cutoffs in these neighborhoods. Their spatial projections have closures in a fixed compact subset of U.

By Lemma 2.2, write V_b as a finite sum of real Clifford forms of degrees $3 \pmod 4$. If a degree greater than three occurs, let ℓ be the largest such degree; then $\ell \ge 7$. Decompose its forms into terms (5.1). For one such term, let V_0 be the sum of the other terms and apply Proposition 5.1. Both operators are self-adjoint on H^1 by Section 2, since the new multiplication potential is bounded for

each finite frequency. Lemma 3.2 replaces the chosen term by forms of degrees 3 and $\ell - 4$, with arbitrarily small C^1_b resolvent error on $\mathcal{K}$. Each degree requires only finitely many replacements, and a replacement creates no term of that degree or a larger one. Descending induction leaves only three-forms. Before choosing a frequency, fix all preceding frequencies and the resulting smooth families, including their spatial derivatives and spinor comparisons.

Decompose the remaining three-form into finitely many terms $ds \wedge \beta_b$ in the fixed spatial coordinates. For one term $\mathcal{T}_b = ds \wedge \beta_b$, apply Proposition 4.2. Denote the current metric by h_b, the resulting metric by $h_{b,N}$, and their direct comparison by $U_{b,N}$. Let $\omega_{\mathrm{rem},b}$ be the sum of the retained differential forms. For each retained form ω_b,

$$U_{b,N}^{-1} c_{h_{b,N}}(\omega_b) U_{b,N} - c_{h_b}(\omega_b) = O_{C^1_b L^\infty_x}(N^{-1/2}).$$

Indeed, under the positive comparison, Clifford multiplication depends smoothly and algebraically on the metric and the form. The form and the current metric are fixed as N varies, and Proposition 4.2 bounds the metric difference in $C^1(B; C^0)$. Set

$$\widetilde{L}_{b,N} = U_{b,N}^{-1}\big(D_{h_{b,N}} + c_{h_{b,N}}(\omega_{\mathrm{rem},b})\big) U_{b,N}.$$

Then

$$\widetilde{L}_{b,N} - \big(D_{h_b} + c_{h_b}(\mathcal{T}_b + \omega_{\mathrm{rem},b})\big) = O_{C^1_b \mathcal{L}(H^1,L^2)}(N^{-1/2}).$$

For each finite N, $U_{b,N}$ is unitary on L^2 and identifies the H^1 domains. Lemma 3.2, with $F_N = I$, gives arbitrarily small resolvent error and keeps $\mathcal{K}$ in the parameter-resolvent set. Repetition removes the finitely many three-form terms.

All metrics equal the original family on $X \setminus U$ and for $b \in \mathcal{V}$. Each metric change preserves the volume density by Proposition 4.2; the higher-degree replacements leave the metric unchanged.

Index all replacements by $j = 1, \dots, m$, and write

$$L_b^{(j)} = D_{g_b^{(j)}} + V_b^{(j)}, \qquad g^{(0)} = g, \qquad L_b^{(0)} = L_b, \qquad V_b^{(m)} = 0.$$

Put

$$\mathscr{H}_{j,b} = L^2(\Sigma_{g_b^{(j)}}), \qquad \mathscr{D}_{j,b} = H^1(\Sigma_{g_b^{(j)}}), \qquad R_b^{(j)}(z) = (L_b^{(j)} - z)^{-1}.$$

Every norm below on these varying spaces is computed through the identifications $U_{g_b^{(j)}}$ with $\mathscr{H}, \mathscr{D}$ from Section 2. Let $U_{j,b} : \mathscr{H}_{j-1,b} \to \mathscr{H}_{j,b}$ be the direct positive comparison of the successive metrics, with $U_{j,b} = I$ when the metric is unchanged. Suppressing b, put

$$C_j = U_j \cdots U_1, \qquad C_0 = I, \qquad \Delta_j = U_j^{-1} R^{(j)} U_j - R^{(j-1)}.$$

Then

$$C_m^{-1} R^{(m)} C_m - R^{(0)} = \sum_{j=1}^{m} C_{j-1}^{-1} \Delta_j C_{j-1}. \tag{5.8}$$

Fix positive numbers θ, η, to be specified below. Let M_j bound conjugation by C_{j-1} in the indicated C^1_b operator norms. The constant M_j is finite and depends only on frequencies chosen before the jth step. Choose the jth frequency so that

$$\|\Delta_j\|_{C^1_b(\mathcal{K};\mathcal{L}(\mathscr{H}_{j-1,b}))} < \frac{2^{-j}\theta}{1 + M_j}, \qquad \delta_j := \left\| g^{(j)} - g^{(j-1)} \right\|_{C^1(B;C^0)} < 2^{-j}\eta.$$

For a higher-degree replacement, $\delta_j = 0$. For a metric change, both inequalities follow from Proposition 4.2 and Lemma 3.2. The degree induction is finite, and these inequalities do not require its number of steps to be fixed in advance. Thus (5.8) gives

$$\left\| C_m^{-1} R^{(m)} C_m - R^{(0)} \right\|_{C^1_b(\mathcal{K};\mathcal{L}(\mathscr{H}_{0,b}))} < \theta, \qquad \sigma := \sum_{j=1}^{m} \delta_j < \eta.$$

Let $P_{j,b} : \mathscr{H}_{0,b} \to \mathscr{H}_{j,b}$ be the direct comparison from the original metric g_b to $g_b^{(j)}$, with $P_{0,b} = I$. In the original spinor bundle, put

$$W_j = P_j^{-1} U_j P_{j-1}, \qquad W = W_m \cdots W_1.$$

Then $C_m = P_m W$. The positive frame maps in (2.6) depend smoothly on the metric matrices. Their spin lifts are continued from the identity; consequently [61, Lemmas 2.3.3–2.3.5] gives a pointwise matrix for W_j which is a smooth function of $g, g^{(j-1)}, g^{(j)}$ and equals I when its last two arguments agree.

All these metrics agree outside a fixed compact subset of U. Choose $s_0 > 0$ so that $\sigma \le s_0$ keeps their values and first parameter derivatives, in a fixed finite cover of that compact subset, in a fixed bounded set of finite-dimensional matrix spaces, with all metric matrices uniformly positive. The closure of this set is compact. The mean-value formula, also after one parameter derivative, gives

$$\left\| W_j - I \right\|_{C_b^1 L_x^\infty} \le C\delta_j.$$

The constant depends only on this compact set and the original family. The product rule and the telescoping expansion of $W - I$ therefore give

$$\| W - I \|_{C_b^1 L_x^\infty} \le C e^{C\sigma} \sigma. \tag{5.9}$$

Indeed, the norms of the intervening products are bounded by $\prod_j (1 + C\delta_j) \le e^{C\sigma}$, and each summand has one factor bounded by $C\delta_j$.

Set

$$R_{\mathrm{comp}} = C_m^{-1} R^{(m)} C_m, \qquad R_{\mathrm{dir}} = P_m^{-1} R^{(m)} P_m.$$

Thus $R_{\mathrm{dir}} = W R_{\mathrm{comp}} W^{-1}$. For a C^1 family R of bounded operators on $\mathscr{H}_{0,b}$,

$$WRW^{-1} - R = (W - I) R W^{-1} + R(W^{-1} - I).$$

The product rule and $d(W^{-1}) = -W^{-1}(dW)W^{-1}$ imply, for $\sigma \le s_0$,

$$\left\| WRW^{-1} - R \right\|_{C_b^1 \mathcal{L}(\mathscr{H}_{0,b})} \le C \left\| W - I \right\|_{C_b^1 L_x^\infty} \left\| R \right\|_{C_b^1 \mathcal{L}(\mathscr{H}_{0,b})}.$$

By Lemma 3.1,

$$M := 1 + \left\| R^{(0)} \right\|_{C_b^1(\mathcal{K}; \mathcal{L}(\mathscr{H}_{0,b}))} < \infty.$$

Choose $0 < \eta < \min\{s_0, \varepsilon\}$ so that (5.9) makes the conjugation error less than $\varepsilon/2$ whenever $\sigma < \eta$ and $\| R \|_{C_b^1(\mathcal{K}; \mathcal{L}(\mathscr{H}_{0,b}))} \le M$. Choose $0 < \theta < \min\{1, \varepsilon/2\}$. The preceding frequency choices give $\left\| R_{\mathrm{comp}} \right\|_{C_b^1(\mathcal{K}; \mathcal{L}(\mathscr{H}_{0,b}))} < M$. In the stated C_b^1 norm,

$$\left\| R_{\mathrm{dir}} - R^{(0)} \right\| \le \left\| R_{\mathrm{dir}} - R_{\mathrm{comp}} \right\| + \left\| R_{\mathrm{comp}} - R^{(0)} \right\| < \varepsilon/2 + \theta < \varepsilon.$$

This proves (1.4). Finally,

$$\left\| g^{(m)} - g \right\|_{C^1(B; C^0)} \le \sigma < \varepsilon$$

proves the metric estimate in (1.3). □

Corollary 5.2. *In the setting of Theorem R, if $D_g + V$ is invertible and V is compatible and compactly supported in U, there is a metric $\widehat{g}$ with $D_{\widehat{g}}$ invertible, $\widehat{g} = g$ outside U, and $dv_{\widehat{g}} = dv_g$.*

Proof. Take a one-point parameter space and apply Theorem R to $\mathcal{K} = \{0\} \subset \rho(D_g + V)$. □

6. Spectral subspaces and local sections

Let B be a compact smooth parameter manifold, possibly with boundary, and let $b \mapsto L_b \in \mathcal{L}(\mathscr{D}, \mathscr{H})$ be a smooth family of self-adjoint operators with common domain $\mathscr{D}$, endowed with a fixed graph norm. Let $\Omega \subset \mathbb{C}$ be a bounded open set with smooth boundary $\gamma = \partial\Omega$, positively oriented. Suppose that $\gamma \subset \rho(L_b)$ and that $\operatorname{Spec}(L_b) \cap \Omega$ consists of eigenvalues of finite total multiplicity for every b. Define

$$P_b = \frac{1}{2\pi i}\int_\gamma (z - L_b)^{-1}\, dz, \qquad Q_b = \frac{1}{2\pi i}\int_\gamma z(z - L_b)^{-1}\, dz. \tag{6.1}$$

The spectral theorem identifies P_b with the orthogonal projection onto the cluster, and spectral decomposition gives $Q_b = L_b P_b \in \mathcal{L}(\mathscr{H})$ [67, Chapter III, Theorem 6.17 and equation (6.24)]. Lemma 3.1 and contour integration give C^1 dependence of P_b in $\mathcal{L}(\mathscr{H}, \mathscr{D})$ and of Q_b in $\mathcal{L}(\mathscr{H})$. For an isometric frame $U_b : E \to \operatorname{im} P_b$, the matrix of $L_b|_{\operatorname{im} P_b}$ is $U_b^* Q_b U_b$.

Suppose that $L_b = D_{g_b} + V_b$ satisfies the hypotheses of Theorem R. For a metric family $\widehat{g}_b$ furnished by that theorem on $B \times \gamma$, write $\widehat{D}_b = D_{\widehat{g}_b}$ in the direct positive spinor comparison, and define $\widehat{P}_b, \widehat{Q}_b$ by (6.1) with $L_b = \widehat{D}_b$. Then

$$\begin{aligned}&\left\|\widehat{P} - P\right\|_{C^1(B;\mathcal{L}(\mathscr{H}))} + \left\|\widehat{Q} - Q\right\|_{C^1(B;\mathcal{L}(\mathscr{H}))}\\ &\qquad \le \frac{\operatorname{length}(\gamma)}{2\pi}\left(1 + \sup_{z\in\gamma}|z|\right)\sup_{z\in\gamma}\left\|(z - \widehat{D}_b)^{-1} - (z - L_b)^{-1}\right\|_{C^1(B;\mathcal{L}(\mathscr{H}))}.\end{aligned}$$

Thus both moments converge in $C^1(B; \mathcal{L}(\mathscr{H}))$ as the resolvent error tends to zero. If $\left\|\widehat{P}_b - P_b\right\| < 1$, the map $P_b : \operatorname{im}\widehat{P}_b \to \operatorname{im} P_b$ is injective; hence $\widehat{P}_b$ has finite rank.

Lemma 6.1. *Let $P_b, \widehat{P}_{j,b}$ be C^1 families of finite-rank orthogonal projections, and let $Q_b, \widehat{Q}_{j,b}$ be bounded self-adjoint operators satisfying*

$$Q_b = P_b Q_b P_b, \qquad \widehat{Q}_{j,b} = \widehat{P}_{j,b}\widehat{Q}_{j,b}\widehat{P}_{j,b}.$$

Suppose that

$$\widehat{P}_j \longrightarrow P, \qquad \widehat{Q}_j \longrightarrow Q \quad \text{in } C^1(B; \mathcal{L}(\mathscr{H})).$$

If $\sup_{b\in B}\left\|\widehat{P}_{j,b} - P_b\right\| < 1$, the polar part of $\widehat{P}_{j,b}P_b : \operatorname{im} P_b \to \operatorname{im}\widehat{P}_{j,b}$ is an isometry onto $\operatorname{im}\widehat{P}_{j,b}$. It intertwines every unitary or antiunitary spinor structure commuting with both projections. Omitting j, its expression in a local C^1 isometric frame $U_b : E \to \operatorname{im} P_b$ is

$$\widehat{U}_b = \widehat{P}_b U_b (U_b^* \widehat{P}_b U_b)^{-1/2}. \tag{6.2}$$

On every compact subset of the frame domain,

$$\widehat{U}_b^* \widehat{Q}_b \widehat{U}_b - U_b^* Q_b U_b \longrightarrow 0 \quad \text{in } C^1_b \operatorname{End}(E). \tag{6.3}$$

Proof. Put $q = \sup_b \left\|\widehat{P}_b - P_b\right\| < 1$ and $G_b = U_b^* \widehat{P}_b U_b$. The close-projection theorem [67, Chapter I, Theorem 6.32], applied on $\operatorname{im} P_b + \operatorname{im}\widehat{P}_b$, gives equal ranks. For $v \in E$,

$$\langle G_b v, v\rangle = \left\|\widehat{P}_b U_b v\right\|^2 \ge (1 - q)^2 \|v\|^2.$$

Thus (6.2) is the polar isometry onto $\operatorname{im}\widehat{P}_b$. Since $G \to I$ in C^1, holomorphic functional calculus on a fixed contour in the domain of $z^{-1/2}$ gives $G^{-1/2} \to I$ in C^1 [67, Chapter I, Section 5.6]. Hence $\widehat{U} \to U$ in $C^1\mathcal{L}(E, \mathscr{H})$, and the product rule gives (6.3).

The polar map is intrinsically $\widehat{P}_b P_b (P_b \widehat{P}_b P_b|_{\operatorname{im} P_b})^{-1/2}$. The positive inverse square root is a uniform limit of polynomials with real coefficients on a positive spectral interval. It therefore commutes with

every unitary or antiunitary structure commuting with P_b and $\widehat{P}_b$. Under $U_b \mapsto U_b R_b$, for a unitary R_b, formula (6.2) gives $\widehat{U}_b \mapsto \widehat{U}_b R_b$. The local maps consequently define a bundle isometry with the stated intertwining properties. □

Definition 6.2. Let E be a zero module. A *full potential section* at zero is a smooth family $L_A = D_g + V(A)$, for A in a neighborhood of zero in $\mathcal{A}_n(E)$, where $V(A)$ is a compatible multiplication endomorphism, with the following properties. There is a positively oriented circle γ centered at zero in $\rho(L_A)$ for every A, and there is a smooth family $U_A \in \mathcal{L}(E, \mathscr{D})$ of isometries, for the $\mathscr{H}$-inner product, from E onto $\operatorname{im} P_A$, where P_A is given by (6.1). These isometries intertwine J and, when present, Γ, and satisfy

$$\operatorname{im} P_0 = \ker L_0, \qquad U_A^* L_A U_A = A.$$

A *full metric section* is defined by taking $L_A = D_{g(A)}$ in the direct positive spinor comparison.

Proposition 6.3. *Assume $n \geq 3$, and let X, g, U satisfy the geometric hypotheses of Theorem R. Every full potential section whose potentials are supported in one compact subset of U yields a full metric section. The metric section equals g outside U and has volume density dv_g. Its reference metric at zero need not equal g.*

Proof. Equip the real vector space $\mathcal{A}_n(E)$ with a Euclidean norm. Choose $r > 0$ so that the closed ball $B = \{A : \|A\| \leq r\}$ lies in the section domain. Let γ and U_A be as in Definition 6.2. A structure commuting with the operator preserves each eigenspace; one anticommuting with it maps the eigenspace at λ onto that at $-\lambda$. Since γ is centered at zero, the corresponding spectral projections commute with J and Γ.

Apply Theorem R with $\mathcal{K} = B \times \gamma$ and $B_0 = \emptyset$. The same spectral symmetry holds for $\widehat{D}_A$. Lemma 6.1 gives an isometric frame $\widehat{U}_A$ intertwining these structures. Therefore

$$F(A) = \widehat{U}_A^* \widehat{Q}_A \widehat{U}_A = \widehat{U}_A^* \widehat{D}_A \widehat{U}_A \in \mathcal{A}_n(E).$$

For a sufficiently small resolvent error,

$$\sup_{\|A\| \leq r} \|F(A) - A\| < r/4, \qquad \sup_{\|A\| \leq r} \|DF(A) - I\| < 1/2. \tag{6.4}$$

Smooth inversion gives smooth dependence of $\widehat{P}_A$ in $\mathcal{L}(\mathscr{H}, \mathscr{D})$. Formula (6.2) then gives smooth dependence of $\widehat{U}_A$ in $\mathcal{L}(E, \mathscr{D})$, and hence of F. For $\|Y\| < r/4$, the map $\Phi_Y(A) = Y + A - F(A)$ satisfies

$$\|\Phi_Y(A)\| \leq \|Y\| + \|A - F(A)\| < r/2, \qquad \|D\Phi_Y(A)\| < 1/2 \quad (\|A\| \leq r/2).$$

The closed ball of radius $r/2$ is complete and convex. The contraction theorem [50, Theorem 10.38] gives a unique $A(Y)$ in its interior with $F(A(Y)) = Y$. By (6.4), DF is invertible there. The inverse function theorem [50, Theorem 11.22 and Corollary 11.35] and uniqueness give a smooth map $Y \mapsto A(Y)$ with

$$DA(Y) = DF(A(Y))^{-1}, \qquad \|DA(Y)\| \leq 2, \qquad \|A(Y) - A(Z)\| \leq 2\,\|Y - Z\|.$$

Put $h_Y = \widehat{g}_{A(Y)}$ and take the frame $\widehat{U}_{A(Y)}$. Its restriction matrix is Y. Since $F(A(0)) = 0$, the spectral subspace is invariant, and 0 lies inside γ,

$$\ker D_{h_0} = \operatorname{im} \widehat{P}_{A(0)}.$$

Thus h_Y is a full metric section, and (1.3) gives $h_Y = g$ on $X \setminus U$ and $dv_{h_Y} = dv_g$. □

Lemma 6.4. *Let $T_t : \mathscr{D} \to \mathscr{H}$, for t near zero, be a C^1 family in $\mathcal{L}(\mathscr{D}, \mathscr{H})$ of self-adjoint operators with common domain $\mathscr{D} = H^1$, whose graph norms are locally uniformly equivalent. Suppose $\lambda \in \mathbb{R}$ is an isolated eigenvalue of finite multiplicity of T_0. Put $E = \ker(T_0 - \lambda)$ and let P_E be the orthogonal*

projection onto E. Let $U_t : E \to \mathscr{D}$ be the polar frame for the corresponding spectral subspace, based at the inclusion $U_0 : E \hookrightarrow \mathscr{H}$. Then

$$\left.\frac{d}{dt}\right|_{t=0} U_t^* T_t U_t = P_E \dot{T}_0|_E.$$

In particular, for a bounded compatible potential V, $T_t = D_g + tV$, and $E = \ker D_g$, the derivative is $P_E V|_E$.

Proof. Lemma 3.1, contour integration, and (6.2) give C^1 dependence of U_t in $\mathcal{L}(E, \mathscr{D})$. Since $T_0 U_0 = \lambda U_0$, self-adjointness gives

$$\left.\frac{d}{dt}\right|_{t=0} U_t^* T_t U_t = U_0^* \dot{T}_0 U_0 + \lambda(\dot{U}_0^* U_0 + U_0^* \dot{U}_0) = P_E \dot{T}_0|_E,$$

where the last equality follows from $U_t^* U_t = I$. □

Lemma 6.5. *Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$, and let S, T be Euclidean, Hermitian, or quaternionic Hermitian bundles over a smooth manifold X, according as $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$. Fix a smooth positive density dv on X. Let $F \subset C^\infty(S) \cap L^2(S)$ and $E \subset C^\infty(T) \cap L^2(T)$ be finite-dimensional $\mathbb{F}$-subspaces, with their L^2 inner products. Denote the evaluation maps by*

$$e_F(x) : F \longrightarrow S_x, \qquad e_E(x) : E \longrightarrow T_x.$$

Let $\mathscr{V} \subset \operatorname{Hom}_{\mathbb{F}}(S, T)$ be a smooth real subbundle and $U \subset X$ a nonempty open set. The real-linear map

$$C : C_c^\infty(U; \mathscr{V}) \longrightarrow \operatorname{Hom}_{\mathbb{F}}(F, E), \qquad C(V) = \int_X e_E(x)^* V(x) e_F(x)\, dv(x),$$

satisfies

$$\operatorname{im} C = \operatorname{span}_{\mathbb{R}}\{e_E(x)^* Q e_F(x) : x \in U,\ Q \in \mathscr{V}_x\}. \tag{6.5}$$

Proof. Regard $W = \operatorname{Hom}_{\mathbb{F}}(F, E)$ as a real vector space, and denote its real dual by W^*. For $\ell \in W^*$, the fundamental lemma for test functions [44, Theorem 1.17, pp. 23–24], applied in local coordinates and a local frame of $\mathscr{V}$, gives

$$\ell \circ C = 0 \quad \Longleftrightarrow \quad \ell\big(e_E(x)^* Q e_F(x)\big) = 0 \quad (x \in U,\ Q \in \mathscr{V}_x).$$

In local coordinates $dv = \rho\, dx$ with $\rho > 0$; the scalar coefficients are smooth, so their almost-everywhere vanishing implies pointwise vanishing. The two subspaces in (6.5) therefore have the same annihilator in W^*. Since W is finite-dimensional, they are equal. □

Lemma 6.6. *Let E_1, E_2, F_1, F_2 be Hilbert spaces, and equip their direct sums with the Hilbert direct-sum norms. Let*

$$T_0 = \operatorname{diag}(A, C) : E_1 \oplus E_2 \longrightarrow F_1 \oplus F_2$$

be bounded. Suppose C is bijective and $\|C^{-1}\|_{F_2 \to E_2} \le c^{-1}$, where $c > 0$. Let $T = T_0 + R$, where $R = (R_{ij})$ and $\|R\|_{E_1 \oplus E_2 \to F_1 \oplus F_2} \le \delta < c/2$. Then $C' = C + R_{22}$ is bijective. Its Schur complement

$$S = A + R_{11} - R_{12}(C')^{-1} R_{21} : E_1 \longrightarrow F_1$$

satisfies

$$\|S - A\|_{E_1 \to F_1} \le \delta + 2\delta^2/c. \tag{6.6}$$

Moreover,

$$\ker T \simeq \ker S, \qquad (F_1 \oplus F_2)/\operatorname{im} T \simeq F_1/\operatorname{im} S.$$

The range of T is closed if and only if the range of S is closed; in that case the quotients are their Hilbert cokernels.

Let J_E and J_F be linear or antilinear maps on $E_1 \oplus E_2$ and $F_1 \oplus F_2$, respectively, preserving the summands. If $J_F T = \epsilon T J_E$, where $\epsilon \in \{1, -1\}$, then $J_{F_1} S = \epsilon S J_{E_1}$, where the subscripts denote restrictions to the summands.

Suppose further that $F_1 \oplus F_2 = \mathscr{H}$ and $E_1 \oplus E_2 = \mathscr{D} \subset \mathscr{H}$, where $\mathscr{D}$ is dense and continuously included in $\mathscr{H}$, with a fixed graph norm. Assume that the orthogonal projections of $\mathscr{H}$ onto F_i preserve $\mathscr{D}$ and that $E_i = \mathscr{D} \cap F_i$. Let $\dagger$ denote the ambient adjoint. On a complex Hilbert space it may instead denote the transpose $T^{\mathrm{t}} = \kappa T^* \kappa$, where κ is a conjugation preserving F_1, F_2 and $\mathscr{D}$. If $T^\dagger = \epsilon T$, with equality of domains, then $S^\dagger = \epsilon S$ and $\operatorname{Dom} S^\dagger = E_1$.

Proof. Bounded inverse stability [67, Chapter IV, Theorem 1.16], with $C : E_2 \to F_2$ and $R_{22} : E_2 \to F_2$, gives

$$\left\|(C')^{-1}\right\|_{F_2 \to E_2} \leq (c - \delta)^{-1} \leq 2/c.$$

The block norms are at most δ, so (6.6) follows. Block multiplication gives

$$\begin{pmatrix} I & -R_{12}(C')^{-1} \\ 0 & I \end{pmatrix} T \begin{pmatrix} I & 0 \\ -(C')^{-1} R_{21} & I \end{pmatrix} = \begin{pmatrix} S & 0 \\ 0 & C' \end{pmatrix}. \tag{6.7}$$

Both triangular factors are bounded and invertible on the respective Hilbert direct sums. The kernel isomorphism is

$$x \longmapsto \big(x, -(C')^{-1} R_{21} x\big).$$

The left factor sends $\operatorname{im} T$ onto $\operatorname{im} S \oplus F_2$, which gives the quotient isomorphism and the equivalence of closed ranges.

The block identities for $J_F T = \epsilon T J_E$ give

$$J_{E_2}(C')^{-1} = \epsilon (C')^{-1} J_{F_2}, \qquad J_{F_1} R_{12}(C')^{-1} R_{21} = \epsilon R_{12}(C')^{-1} R_{21} J_{E_1}.$$

Together with the identity for $A + R_{11}$, these imply $J_{F_1} S = \epsilon S J_{E_1}$.

For the last assertion, E_i is dense in F_i, since $\mathscr{D}$ is dense and the orthogonal projections onto F_i preserve $\mathscr{D}$. Put $B = (C')^{-1}$ and $K = R_{12} B$. The graph-norm bounds make $B : F_2 \to F_2$ and $K : F_2 \to F_1$ bounded in the ambient norms. Write $(u, v)_\dagger = \langle u, v \rangle$ for the adjoint, and $(u, v)_\dagger = \langle \kappa u, v \rangle$ for the transpose. For $y = C'u$ and $z = C'v$, with $u, v \in E_2$,

$$(By, z)_\dagger = (u, C'v)_\dagger = \epsilon (C'u, v)_\dagger = \epsilon (y, Bz)_\dagger.$$

Surjectivity of C' gives $B^\dagger = \epsilon B$. For $x \in E_1$ and $y \in F_2$, the off-diagonal identity gives

$$(Ky, x)_\dagger = \epsilon (By, R_{21} x)_\dagger = (y, B R_{21} x)_\dagger.$$

Hence $K^\dagger|_{E_1} = B R_{21}$. In particular, $K^\dagger E_1 \subset E_2$. Set

$$L = \begin{pmatrix} I & -K \\ 0 & I \end{pmatrix}.$$

The operators L and L^{-1} are bounded on $\mathscr{H}$, and $(L^\dagger)^{\pm 1} \mathscr{D} = \mathscr{D}$. Thus (6.7) becomes

$$L T L^\dagger = \operatorname{diag}(S, C') \quad \text{on } \mathscr{D}.$$

Since L is bounded and invertible,

$$(L T L^\dagger)^\dagger = L T^\dagger L^\dagger, \qquad \operatorname{Dom}(L T L^\dagger)^\dagger = (L^\dagger)^{-1} \operatorname{Dom} T^\dagger = \mathscr{D}.$$

The two diagonal blocks therefore satisfy $S^\dagger = \epsilon S$ on E_1 and $(C')^\dagger = \epsilon C'$ on E_2, with equality of domains. □

7. Generic nonzero eigenvalues

Throughout this section M^n is closed and connected, $n \geq 3$, and its spin structure is fixed. For a metric g, write $\Sigma = \Sigma_g M$. Multiplicities are counted according to (1.5). The spaces of compatible pointwise endomorphisms are given in Section 2. On a nonzero eigenspace the scalar field is

$$\mathbb{F}_n = \begin{cases} \mathbb{H}, & n \equiv 2,3,4 \pmod 8, \\ \mathbb{C}, & n \equiv 1,5 \pmod 8, \\ \mathbb{R}, & n \equiv 0,6,7 \pmod 8. \end{cases} \tag{7.1}$$

When $J^2 = I$, put $\Sigma_{\mathbb{R}} = \operatorname{Fix} J$. In a real case the eigenspace means its real form in $C^\infty(\Sigma_{\mathbb{R}})$; its real dimension equals the complex dimension of the ordinary eigenspace. In a quaternionic case the ordinary complex dimension is twice its quaternionic dimension. For $n \equiv 1,5 \pmod 8$, charge conjugation exchanges the nonzero eigenspaces of opposite signs and imposes no additional multiplicity on either. Denote by P_E the L^2-orthogonal projection corresponding to an eigenspace E, restricted to the real form when appropriate, and write $M_f \psi = f\psi$ for scalar multiplication. Fibre Hermitian products are conjugate linear in the first variable; the associated real product is their real part.

7.1. Splitting a nonzero eigenspace.

Lemma 7.1. *Let $L = D_g + W$, where W is a smooth compatible multiplication endomorphism. Suppose $\lambda \in \mathbb{R} \setminus \{0\}$ and $E = \ker(L - \lambda)$ has $\mathbb{F}_n$-dimension at least two. For every nonempty open set $U \subset M$ there are $f \in C^\infty_c(U;\mathbb{R})$ and $V \in C^\infty_c(U;\mathscr{A}_n)$ such that*

$$P_E V|_E - \lambda P_E M_f|_E \notin \mathbb{R} I_E. \tag{7.2}$$

If $n \not\equiv 2 \pmod 8$, one may take $f = 0$.

Proof. Elliptic regularity for $L - \lambda$ gives $E \subset C^\infty(\Sigma)$ [44, Theorem 16.1 and Remark 16.2]. The square $(L-\lambda)^2$ has the form $\nabla^*\nabla + P_1 + P_0$, where P_j is a differential operator of order at most j. Hence unique continuation [61, Theorem 6.2.8] implies that every nonzero eigenspinor is nonzero on an open dense set.

Let $e_x : E \to \Sigma_x$ denote evaluation, over $\mathbb{F}_n$. Suppose first that $P_E V|_E \in \mathbb{R} I_E$ for every $V \in C^\infty_c(U;\mathscr{A}_n)$. Lemma 6.5, applied to the compatible endomorphism bundle, gives

$$e_x^* Q e_x \in \mathbb{R} I_E \qquad (x \in U,\ Q \in \mathscr{A}_{n,x}).$$

In the real cases the bundle is $\Sigma_{\mathbb{R}}$; in the quaternionic cases it carries the quaternionic structure of Section 2. For the even-dimensional cases, write $V_A = \mathcal{D}(A)$ as in (2.3).

The cases $n \equiv 3,7 \pmod 8$. Choose an orthonormal pair $\psi, \phi \in E$ over $\mathbb{F}_n$. At a point $x \in U$ where $u = \psi(x) \neq 0$, the compatible fibre map $Q = uu^*$ has a compression to this pair of $\mathbb{F}_n$-rank at most one. Its first diagonal entry is $|u|^4 > 0$, whereas a scalar two-by-two matrix with that entry has $\mathbb{F}_n$-rank two. This is impossible.

The case $n \equiv 1 \pmod 8$. Here $J^2 = I$ and $Jc(\xi) = -c(\xi)J$. In a local J-real frame, compatible potentials are iQ, where Q is real skew-adjoint. Let $\wedge$ denote the complex bilinear exterior product in these fibre coordinates. For complex orthonormal $\psi_1, \psi_2 \in E$, vanishing of the pointwise off-diagonal entries and equality of the diagonal entries give

$$\overline{\psi_1} \wedge \psi_2 = 0, \qquad \overline{\psi_1} \wedge \psi_1 = \overline{\psi_2} \wedge \psi_2. \tag{7.3}$$

On a connected ball in U where $\psi_1 \neq 0$, the first equation gives $\psi_2 = b\overline{\psi_1}$, with b smooth and complex-valued. Substitution into the second gives

$$(1 + |b|^2)\overline{\psi_1} \wedge \psi_1 = 0.$$

Thus ψ_1 is pointwise a complex multiple of a nonzero real spinor, and $\psi_2 = a\psi_1$, where $a = \psi_1^*\psi_2/|\psi_1|^2$ is smooth and complex-valued. The product rule gives $c(da)\psi_1 = 0$. At each point write $\psi_1 = zw$, with $z \neq 0$ and $Jw = w$, and put $a = a_1 + ia_2$. Then

$$c(da_1)w + i\, c(da_2)w = 0.$$

The two terms belong to $i\Sigma_{\mathbb{R}}$ and $\Sigma_{\mathbb{R}}$, respectively. They therefore vanish separately. Since $c(\xi)^2 = -|\xi|^2 I$ for real ξ, one has $da_1 = da_2 = 0$. Thus a is constant on the ball. Unique continuation for $\psi_2 - a\psi_1$ contradicts orthonormality.

The case $n \equiv 5 \pmod 8$. Here $J^2 = -I$, $JL = -LJ$, and compatible potentials are $V = -iQ$, where Q is quaternionic-linear and skew-adjoint. For a fibre vector u, let $u^{\perp_{\mathbb{R}}}$ denote its orthogonal complement for the real fibre product. If $u \neq 0$, then

$$\{Qu \,:\, Q^* = -Q,\ QJ = JQ\} = u^{\perp_{\mathbb{R}}}, \qquad \{Vu \,:\, V \in \mathscr{A}_{n,x}\} = (iu)^{\perp_{\mathbb{R}}}. \tag{7.4}$$

Indeed, skew-adjointness gives $\mathrm{Re}\langle u, Qu\rangle = 0$. In a quaternionic orthonormal basis with $u = |u|e_1$, write $w/|u| = (a, v)$ for $w \in u^{\perp_{\mathbb{R}}}$. Then

$$\mathrm{Re}\, a = 0, \qquad Q = \begin{pmatrix} a & -v^* \\ v & 0 \end{pmatrix}, \qquad Q^* = -Q, \qquad Qu = w.$$

Multiplication by $-i$ gives the second equality in (7.4).

Choose complex orthonormal $\psi, \phi \in E$. Pointwise scalarity gives $\langle \phi(x), V\psi(x)\rangle_{\mathbb{C}} = 0$ for every $V \in \mathscr{A}_{n,x}$. Where $\psi(x) \neq 0$, the real and imaginary parts and (7.4) give

$$\phi(x),\ i\phi(x) \in \mathbb{R}\, i\psi(x).$$

A real line contains no nonzero complex line. Hence $\phi = 0$ on a nonempty open set, contrary to unique continuation.

The cases $n \equiv 0, 4 \pmod 8$. Put $\mathbb{F} = \mathbb{R}$ or $\mathbb{H}$, respectively, and write $L^{\pm}$ for the chiral restrictions of L. For $\psi = (u, v) \in \ker(L - \lambda)$, chirality gives $L^+u = \lambda v$ and $L^-v = \lambda u$. Since $\lambda \neq 0$, vanishing of either component on an open set forces vanishing of the other there. Both components of a nonzero eigenspinor are therefore nonzero on open dense sets.

Choose $\mathbb{F}$-orthonormal $\psi_i = (u_i, v_i)$, $i = 1, 2$. For $A \in \mathrm{Hom}_{\mathbb{F}}(\Sigma^+, \Sigma^-)$, put

$$b_{12}(A) = u_1^* A^* v_2 + v_1^* A u_2.$$

For every $q \in \mathbb{F}$,

$$\mathrm{Re}\big(\overline{q}\, b_{12}(A)\big) = \mathrm{Re}\,\mathrm{tr}\big(A^*(v_1 q u_2^* + v_2 \overline{q}\, u_1^*)\big)\,.$$

The real trace pairing is nondegenerate. The vanishing off-diagonal entries and the equality of the diagonal entries give

$$v_1 u_1^* = v_2 u_2^*, \qquad v_1 q u_2^* + v_2 \overline{q}\, u_1^* = 0 \quad (q \in \mathbb{F}). \tag{7.5}$$

At a point of U where $u_1, v_1 \neq 0$, the first equation is an equality of nonzero rank-one maps. Hence

$$u_2 = u_1 a, \qquad v_2 = v_1(\overline{a})^{-1} \quad \text{for some } a \in \mathbb{F} \setminus \{0\}.$$

Taking $q = a$ in the second equation gives $(|a|^2 + 1)v_1 u_1^* = 0$, a contradiction.

The case $n \equiv 6 \pmod 8$. In charge coordinates $J(u, v) = (\overline{v}, \overline{u})$, a real orthonormal pair has the form $\psi = (u, \overline{u})$ and $\phi = (a, \overline{a})$. Compatible blocks satisfy $A^T = A$, and

$$\langle \psi, V_A \phi\rangle_{\mathbb{R}} = 2\,\mathrm{Re}(u^T A a).$$

Since both A and iA are allowed, pointwise scalarity gives $u^T A a = 0$ for every complex symmetric A. Thus

$$u_i a_i = 0, \qquad u_i a_j + u_j a_i = 0 \quad (i \neq j).$$

If $u_i \neq 0$, these equations give $a_j = 0$ for every j. Thus $\phi = 0$ wherever $\psi \neq 0$, contrary to unique continuation.

The case $n \equiv 2 \pmod 8$. Suppose every expression in (7.2) is scalar. Taking $V = 0$ and applying Lemma 6.5 to the real line bundle $\mathbb{R}I_\Sigma$, one obtains, since $\lambda \neq 0$, a smooth nonnegative function ρ such that

$$e_x^* e_x = \rho(x) I_E \qquad (x \in U). \tag{7.6}$$

Unique continuation implies that $\rho > 0$ somewhere. The adjoint is quaternionic; the same identity holds in the complex realization.

Choose quaternionic orthonormal $\psi = (u, v)$ and $\phi = (a, b)$. In charge coordinates,

$$J(u, v) = (-\overline{v}, \overline{u}), \qquad A^T = -A.$$

Taking $f = 0$ gives vanishing pairings for (ψ, ϕ) and $(J\psi, \phi)$. In particular,

$$\langle \psi, V_A \phi \rangle_{\mathbb{C}} = v^* A a + \overline{b^* A u}. \tag{7.7}$$

The equations for A and iA separate the two terms. Applying them also with $J\psi$ gives

$$\overline{v} \wedge a = 0, \qquad \overline{b} \wedge u = 0, \qquad u \wedge a = 0, \qquad v \wedge b = 0. \tag{7.8}$$

At a point where $\rho > 0$, if $u, \overline{v}$ are independent, these equations imply $a = b = 0$. Otherwise write $u = \alpha z$, $\overline{v} = \beta z$, where $z \neq 0$ and $(\alpha, \beta) \neq (0, 0)$. The same equations give $a \in \mathbb{C}z$ and $b \in \mathbb{C}\overline{z}$, including when one coefficient vanishes. The vectors

$$\psi = (\alpha z, \overline{\beta}\,\overline{z}), \qquad J\psi = (-\beta z, \overline{\alpha}\,\overline{z})$$

span this complex plane, since their coefficient determinant is $|\alpha|^2 + |\beta|^2 > 0$. Hence $\phi(x)$ belongs to the quaternionic line of $\psi(x)$. Both alternatives contradict (7.6), which makes their values nonzero and quaternionically orthogonal. This proves (7.2). □

Lemma 7.2. *With the notation of Lemma 7.1, the endomorphism in* (7.2) *is the derivative at zero of the λ-cluster restriction matrix in the polar frame along a real analytic path of compatible pairs based at (g, W) and equal to (g, W) outside U.*

Proof. Set $g_s = e^{2sf} g$ and let $\beta_s : \Sigma_g \to \Sigma_{g_s}$ be the conformal spin isometry. Put

$$W_s = \beta_s e^{-sf} (W + sV) \beta_s^{-1}, \qquad U_s = \beta_s e^{-nsf/2}. \tag{7.9}$$

Since $dv_{g_s} = e^{nsf}\, dv_g$, the map $U_s : L^2(\Sigma_g) \to L^2(\Sigma_{g_s})$ is unitary. Conformal covariance [37, Section 2] gives

$$U_s^{-1} (D_{g_s} + W_s) U_s = e^{-sf/2} (L + sV) e^{-sf/2}. \tag{7.10}$$

The right-hand side is analytic in $\mathcal{L}(H^1, L^2)$ and self-adjoint on the common domain H^1 for real s. Its derivative at zero is $V - (M_f L + L M_f)/2$. Since $L P_E = \lambda P_E$,

$$P_E \left(V - \frac{1}{2} (M_f L + L M_f) \right) \Big|_E = P_E V|_E - \lambda P_E M_f|_E.$$

Lemma 6.4 identifies this compression with the derivative of the restriction matrix. The spinor comparison and real scalar factors preserve the charge and chirality identities. Since $f = V = 0$ outside U, one has $(g_s, W_s) = (g, W)$ there. □

Lemma 7.3. *Let $\Gamma_1, \dots, \Gamma_s$ be positively oriented smooth Jordan curves in $\rho(D_g)$ with pairwise disjoint interiors. Suppose their interiors meet $\mathbb{R}$ in bounded intervals whose closures avoid zero. Every sufficiently small C^∞ neighborhood of $(g, 0)$ contains a smooth path of compatible pairs (g_t, V_t), $0 \le t \le 1$, equal to $(g, 0)$ near $t = 0$, such that every Γ_j remains in the resolvent and every enclosed eigenvalue at $t = 1$ is simple. The variations may be supported in any prescribed nonempty open set $U \subset M$.*

Proof. In the comparison (2.6), the map $(h, W) \mapsto D_h + W \in \mathcal{L}(H^1, L^2)$ is continuous for the spatial C^1 topology on metrics and the C^0 topology on potentials. Lemma 3.1 and contour integration give a neighborhood on which the contours remain in the resolvent and their projection ranks are constant. Write P_{Γ_j} for these projections, and put

$$m = \sum_{j=1}^{s} \operatorname{rank}_{\mathbb{C}} P_{\Gamma_j} / d_n.$$

Every enclosed eigenvalue has complex multiplicity at least d_n. Thus the number of distinct enclosed eigenvalues is at most m.

At a current compatible pair, surround its distinct enclosed eigenvalues by disjoint circles contained in the interiors of the original contours. Their ranks persist under sufficiently small deformations. If an enclosed eigenvalue λ is multiple, Lemmas 7.1 and 7.2 give a restriction matrix

$$A(\tau) = \lambda I + \tau B + o(\tau), \qquad B^* = B, \quad B \notin \mathbb{R}I.$$

Choose disjoint intervals about two distinct eigenvalues of B. Since $(A(\tau)-\lambda I)/\tau \to B$, both intervals contain eigenvalues of this matrix for all sufficiently small $\tau > 0$. The selected circle therefore encloses at least two distinct eigenvalues, while every other circle still encloses an eigenvalue. The total number of distinct enclosed eigenvalues increases.

Let r_0 be its initial value. If $r_0 = m$, the constant path suffices. Otherwise, whenever the current number is less than m, one eigenvalue has $\mathbb{F}_n$-multiplicity greater than one. The preceding deformation applies at that pair. At most $m - r_0$ such deformations give simplicity of every enclosed eigenvalue. Each parameter interval may be chosen so small that its whole path lies in the given neighborhood. Reparameterize the finitely many paths by smooth functions constant near both endpoints and concatenate them. Their supports have a common compact union contained in U. □

7.2. **Analytic metric paths.**

Lemma 7.4. *Let $\widetilde{g}_t$, $0 \le t \le 1$, be a smooth path of metrics, with $\widetilde{g}_0 = g$. Suppose finitely many positively oriented smooth Jordan curves Γ_j lie in $\rho(D_{\widetilde{g}_t})$ for every t. There is a path h_t, real analytic in t with smooth spatial coefficients, such that*

$$h_0 = g, \qquad h_1 = \widetilde{g}_1,$$

and every contour remains in $\rho(D_{h_t})$ with the same rank. Moreover $h_t \to g$ in C^∞ as $t \to 0$.

Proof. Put $H(t) = \log(g^{-1}\widetilde{g}_t)$, a smooth family of g-self-adjoint endomorphism fields. For $N = 1, 2, \ldots$, define

$$B_N(t) = \sum_{j=0}^{N} \binom{N}{j} t^j (1-t)^{N-j} H(j/N), \qquad h_t^{(N)} = g(e^{B_N(t)}\cdot, \cdot). \tag{7.11}$$

Let X be the real Banach space of C^1 g-self-adjoint endomorphism fields, with a fixed C^1 norm. Then $H \in C([0,1]; X)$. For $\delta > 0$, put

$$\omega_1(\delta) = \sup_{|t-u|\le\delta} \|H(t) - H(u)\|_{C^1}, \qquad t, u \in [0,1].$$

The scalar Bernstein operators are positive, preserve constants, and have absolute centered first moment at most $(2\sqrt{N})^{-1}$. The vector-valued operators have the same nonnegative weights; they satisfy the norm domination and constant-vector identities of [6, Theorem 3.10]. Hence [6, Theorem 3.12, p. 219] gives

$$\sup_t \|B_N(t) - H(t)\|_{C^1} \le 2\omega_1\big((2\sqrt{N})^{-1}\big) \longrightarrow 0. \tag{7.12}$$

Consequently $h_t^{(N)} \to \widetilde{g}_t$ uniformly in spatial C^1, and both endpoints agree. In the fixed comparison (2.6), set

$$T_t = D_{\widetilde{g}_t}, \qquad \Delta_{N,t} = D_{h_t^{(N)}} - T_t.$$

These operators have common domain H^1, and the coefficient formula gives $\sup_t \|\Delta_{N,t}\|_{H^1\to L^2} \to 0$.

If a contour is present, let C_{gr} be a uniform constant in (2.7) for T_t, and put

$$C_2 = \sup_{\substack{0\le t\le 1 \\ z\in\bigcup_j \Gamma_j}} \big\|(T_t - z)^{-1}\big\|_{L^2\to L^2}, \qquad Z_\Gamma = \max_{z\in\cup_j\Gamma_j} |z|.$$

Lemma 3.1 and (2.7) give

$$\sup_{\substack{0\le t\le 1\\ z\in\bigcup_j \Gamma_j}} \left\|(T_t - z)^{-1}\right\|_{L^2\to H^1} \le C_{\mathrm{gr}}\big(1 + (1 + Z_\Gamma)C_2\big) =: C_\Gamma < \infty. \tag{7.13}$$

Choose N with $\sup_t \|\Delta_{N,t}\|_{H^1\to L^2} < (2C_\Gamma)^{-1}$. Then

$$\left\|s\Delta_{N,t}(T_t - z)^{-1}\right\|_{L^2\to L^2} < \frac{1}{2} \quad (0 \le s, t \le 1,\ z \in \textstyle\bigcup_j \Gamma_j).$$

Bounded inverse stability [67, Chapter IV, Theorem 1.16] keeps every contour in the resolvent of $T_t + s\Delta_{N,t}$. Its Riesz projections depend continuously on (t, s) and retain their finite ranks. With no contours, there is no restriction on N.

For this fixed N, the polynomial $B_N(t)$ has smooth spatial coefficients and is g-self-adjoint for every real t. Thus $h_t^{(N)}$ is a metric for every real t and is analytic in every spatial C^r norm. The direct spinor comparison gives a locally holomorphic Dirac family on the common domain H^1 [61, Definition A.0.15 and Lemma A.0.16, pp. 88–90]. Take $h_t = h_t^{(N)}$. Since $B_N(0) = 0$, $h_t \to g$ in every smooth seminorm as $t \to 0$. □

Proposition 7.5. *For g and the contours of Lemma 7.3, there is an analytic metric path h_t, $0 \le t \le 1$, with $h_0 = g$, such that the contours remain in the resolvent and every enclosed eigenvalue is simple for all sufficiently small $t > 0$.*

Proof. If every contour has rank zero, take $h_t = g$. Otherwise fix a nonempty open set $U \subset M$ and apply Lemma 7.3. Around each distinct terminal eigenvalue choose a small positively oriented circle $\gamma_{j\ell}$ of rank d_n. Choose them with pairwise disjoint closed disks contained in the corresponding original interiors. Their ranks sum to the original ranks. Apply Theorem R with $B = [0, 1]$, $B_0 = \{0\}$, and

$$\mathcal{K} = \big([0, 1] \times \textstyle\bigcup_j \Gamma_j\big) \ \cup\ \big(\{1\} \times \bigcup_{j,\ell} \gamma_{j\ell}\big).$$

This is a compact subset of the parameter-resolvent set. The potential vanishes near B_0, and its supports lie in one compact subset of U. The realized metric equals g near $t = 0$. Write $P_\gamma(t)$ and $\widehat{P}_\gamma(t)$ for the original and realized contour projections. The contour estimate in Section 6 permits a realization for which

$$\begin{aligned} &\sup_{0\le t\le 1} \left\|\widehat{P}_{\Gamma_j}(t) - P_{\Gamma_j}(t)\right\| < \tfrac{1}{2} \quad \text{for every } j, \\ &\left\|\widehat{P}_{\gamma_{j\ell}}(1) - P_{\gamma_{j\ell}}(1)\right\| < \tfrac{1}{2} \qquad \text{for every } j, \ell. \end{aligned} \tag{7.14}$$

The rank assertion of Lemma 6.1 gives equality of the corresponding ranks. Each terminal circle has rank d_n and its real interval avoids zero. Since every nonzero eigenvalue has complex multiplicity at least d_n, that circle encloses exactly one simple eigenvalue. The subcontour ranks exhaust the original ranks, so no other eigenvalue lies inside an original contour.

Let $\widetilde{g}_t$ be the realized smooth path. Lemma 7.4 gives an analytic metric path with the same endpoints and original contour ranks. By compactness and openness of invertibility, the contours remain in the resolvent on a connected open real interval containing $[0, 1]$. In the fixed comparison with g, put

$$\Pi_j(t) = \frac{1}{2\pi i}\int_{\Gamma_j} (z - D_{h_t})^{-1}\, dz, \qquad Q_j(t) = \frac{1}{2\pi i}\int_{\Gamma_j} z(z - D_{h_t})^{-1}\, dz. \tag{7.15}$$

As in Section 6, these finite-rank operators are locally analytic, and $Q_j(t) = D_{h_t}\Pi_j(t) = \Pi_j(t)Q_j(t)$. Set $m_j = \operatorname{rank}_{\mathbb{C}} \Pi_j / d_n$ and define the power sums

$$p_{j,k}(t) = d_n^{-1} \operatorname{tr}_{\mathbb{C}}(Q_j(t)^k), \qquad 1 \le k \le m_j.$$

The projections admit local analytic frames [67, Chapter VII, Theorem 1.7 and equation (1.4), pp. 368–369]. In these frames the power sums are traces of powers of finite analytic matrices, and hence are analytic. Their definitions are independent of the frame. Define

$$e_{j,0} = 1, \qquad re_{j,r} = \sum_{k=1}^{r} (-1)^{k-1} e_{j,r-k} p_{j,k} \quad (1 \le r \le m_j), \qquad q_j(t,z) = \sum_{r=0}^{m_j} (-1)^r e_{j,r}(t) z^{m_j - r}. \tag{7.16}$$

For real t, Newton's identities identify the roots of this monic polynomial with the enclosed eigenvalues, repeated according to their $\mathbb{F}_n$-multiplicities: the complex trace repeats each such root d_n times.

For a monic polynomial q with roots $\xi_1, \dots, \xi_m$, repeated with multiplicity, define $\operatorname{disc}_z q = \prod_{r<s} (\xi_r - \xi_s)^2$, with empty product one. Set

$$\Delta(t) = \prod_j \operatorname{disc}_z q_j(t,z). \tag{7.17}$$

The discriminant is a polynomial in the coefficients, so this function is real analytic on a connected neighborhood of $[0,1]$. Moreover $\Delta(1) \neq 0$. The identity theorem gives

$$\Delta(t) = t^{\nu} u(t), \qquad \nu \in \mathbb{Z}_{\ge 0}, \quad u(0) \neq 0, \tag{7.18}$$

with u real analytic near zero. Thus $\Delta(t) \neq 0$ for all sufficiently small $t > 0$. Every $q_j(t,\cdot)$ has distinct roots, so every enclosed eigenvalue has complex multiplicity d_n. □

7.3. **Genericity and prescribed volume density.**

Proof of Theorem G. Fix an integer $a \ge 2$ and $g \in \mathcal{G}_a$. By discreteness, there is a bounded open neighborhood N of K_a in $\mathbb{R} \setminus \{0\}$ such that $\operatorname{Spec}(D_g) \cap N = \operatorname{Spec}(D_g) \cap K_a$. Choose disjoint small open intervals about these eigenvalues, with closures contained in N, and surround them by circles of rank d_n. The complement in K_a of the union of these intervals is compact and disjoint from the spectrum; it is covered by the interiors of finitely many circles whose projections have rank zero. All the ranks persist under sufficiently small spatial C^1 changes of metric. Every eigenvalue in K_a is then enclosed by a circle of rank d_n, and hence is simple. This also includes an eigenvalue at an endpoint of K_a. Thus $\mathcal{G}_a$ is C^1-open.

For an arbitrary metric g, enclose the two components of K_a in larger bounded intervals whose closures avoid zero and whose endpoints avoid $\operatorname{Spec}(D_g)$. Proposition 7.5 gives metrics in $\mathcal{G}_a$ converging to g in C^∞. The space of smooth metrics is an open subset of the Fréchet space of smooth symmetric two-tensors, hence a Baire space. Since the K_a exhaust $\mathbb{R} \setminus \{0\}$,

$$\mathcal{S}(M) = \bigcap_{a \ge 2} \mathcal{G}_a$$

is residual and dense in $\mathcal{R}(M)$.

Fix a positive smooth density μ and $g \in \mathcal{R}_\mu(M)$. Choose $h_j \in \mathcal{S}(M)$ with $h_j \to g$ in C^∞, and put

$$c_j = \left(\frac{\int_M \mu}{\operatorname{Vol}(M, h_j)} \right)^{1/n}, \qquad h_j' = c_j^2 h_j, \qquad \nu_j = dv_{h_j'}. \tag{7.19}$$

Then $c_j \to 1$, $\nu_j \to \mu$, and $\int_M (\nu_j - \mu) = 0$. Regard these densities as top forms via the fixed orientation. Fix a reference metric, with codifferential d^* and Green operator G_n for its Hodge Laplacian on top forms, zero on harmonic forms. Set

$$\alpha_j = d^* G_n (\nu_j - \mu), \qquad d\alpha_j = \nu_j - \mu.$$

The second equality follows from the zero integral and Hodge decomposition [44, Theorem 18.15]. Elliptic regularity and the Fredholm inverse on the orthogonal complement of harmonic forms make

d^*G_n continuous on smooth forms [44, Remark 16.2 and Theorems 16.7 and 16.13]. Thus $\alpha_j \to 0$ in C^∞. Define $X_{j,s}$, for $0 \le s \le 1$, by

$$\iota_{X_{j,s}}\big(\mu + s(\nu_j - \mu)\big) = -\alpha_j. \tag{7.20}$$

The forms in parentheses are positive and have the same total volume. Moser's theorem, with this primitive and vector field, gives an isotopy $\phi_{j,s}$ satisfying $\phi_{j,0} = \mathrm{Id}$ and $\phi_{j,1}^*\nu_j = \mu$ [74, Theorem 2, Section 4, pp. 292–293]. The forms remain uniformly positive, and $X_{j,s} \to 0$ in C^∞ uniformly in s. Smooth dependence of flows on a compact manifold gives $\phi_{j,1} \to \mathrm{Id}$ in C^∞.

Constant rescaling multiplies each Dirac eigenvalue by c_j^{-1}. The isotopy lifts to the fixed spin structure, so spin pullback preserves the spectrum. Hence $\phi_{j,1}^* h_j' \in \mathcal{S}(M) \cap \mathcal{R}_\mu(M)$ and converges smoothly to g. Each $\mathcal{G}_a \cap \mathcal{R}_\mu(M)$ is therefore open and dense. For $h = g(e^H\cdot,\cdot)$ with H g-self-adjoint, $dv_h = e^{\operatorname{tr} H/2}\mu$. Logarithmic coordinates identify $\mathcal{R}_\mu(M)$ with the Fréchet space of smooth trace-free g-self-adjoint endomorphism fields. This space is Baire; the countable intersection of the restricted sets gives the conclusion. □

A metric has *index-minimal kernel*, or is *Dirac-minimal*, if its complex nullity is the value corresponding to $r = 0$ in (1.9).

Corollary 7.6. *A residual dense set of smooth metrics has index-minimal kernel and simple nonzero spectrum. The same holds among metrics with any prescribed positive smooth density.*

Proof. The Dirac-minimal locus is C^1-open and dense in every C^r topology, for integers $r \ge 1$, by [4, Theorem 1.1 and Corollary 1.2]. A C^∞ neighborhood contains a neighborhood determined by finitely many seminorms, so this locus is also C^∞-dense. Intersect it with the open dense sets $\mathcal{G}_a$ and apply the Baire property. For a prescribed density, apply (7.19)–(7.20) to a sequence in this intersection. Scaling and spin pullback preserve the kernel dimension and the nonzero multiplicities. Openness and the Baire property on $\mathcal{R}_\mu(M)$ give the conclusion. □

8. Invertible pairs and the real index

Let M^n be a closed connected spin manifold, $n \ge 3$, with a fixed orientation and spin structure, and let μ be a positive smooth density. Write $\mathcal{R}(M)$ for the space of smooth Riemannian metrics on M. All spaces of metrics and sections carry the C^∞ topology. Put

$$\mathcal{R}_\mu(M) = \{g \in \mathcal{R}(M) \,:\, dv_g = \mu\}, \qquad \mathcal{R}^{\mathrm{inv}}(M) = \{g \in \mathcal{R}(M) \,:\, \ker D_g = 0\}.$$

The operator D_g is the ordinary untwisted spin Dirac operator. Denote its bundle of compatible endomorphisms (Definition 2.1) by $\mathscr{A}_n(g)$, and set

$$\mathcal{P}(M) = \{(g, V) \,:\, g \in \mathcal{R}(M),\ V \in C^\infty(M; \mathscr{A}_n(g))\},$$
$$\mathcal{P}_\mu = \{(g, V) \in \mathcal{P}(M) \,:\, dv_g = \mu\}.$$

A superscript inv denotes the locus where $D_g + V$ is invertible; for metrics, it denotes the locus where D_g is invertible. Thus $\mathcal{R}_\mu^{\mathrm{inv}}(M) = \mathcal{R}_\mu(M) \cap \mathcal{R}^{\mathrm{inv}}(M)$. The independent finite-dimensional blocks are those of (2.4). With the real-index normalization below, the nonzero groups KO_{n+1} have the following representatives, where $-1 \le t \le 1$:

n mod 8	independent block	KO_{n+1}
0	$tI_{\mathbb{R}}$	$\mathbb{Z}/2$
1	$t\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$	$\mathbb{Z}/2$
3	$tI_{\mathbb{H}}$	$\mathbb{Z}$
7	$tI_{\mathbb{R}}$	$\mathbb{Z}$.

(8.1)

The first row is chiral; the second is real skew-adjoint. In the remaining residues, $KO_{n+1} = 0$.

Fix $g_* \in \mathcal{R}_\mu(M)$. The comparison (2.6) identifies the L^2 spaces and H^1 domains and intertwines charge conjugation and chirality. Its conjugation identifies $\mathscr{A}_n(g)$ with $\mathscr{A}_n(g_*)$. Write $H^{*_{g_*}}$ for the

adjoint of a tangent-bundle endomorphism with respect to g_*. Every $g \in \mathcal{R}_\mu(M)$ has a unique expression

$$g = g_*(e^H \cdot, \cdot), \qquad H \in C^\infty(M; \mathrm{End}(TM)), \quad H^{*_{g_*}} = H, \quad \operatorname{tr} H = 0.$$

Together with the transported potential, H gives a global smooth chart from $\mathcal{P}_\mu$ onto a product of two real Fréchet vector spaces. For $\mathcal{P}(M)$ the same chart has no trace condition. Both pair spaces are therefore contractible. Multiplication perturbations preserve the principal symbol and the real index, as in Section 2.

For an open set $U \subset M$, put

$$\begin{aligned} \mathcal{R}_\mu(U) &= \{g \in \mathcal{R}_\mu(M) \,:\, g = g_* \text{ on } M \setminus U\}, \\ \mathcal{P}_\mu(U) &= \{(g, V) \in \mathcal{P}_\mu \,:\, g \in \mathcal{R}_\mu(U), \quad V = 0 \text{ on } M \setminus U\}. \end{aligned} \tag{8.2}$$

In the preceding chart, the exterior conditions are $H = 0$ and $V = 0$ on $M \setminus U$. They define closed linear subspaces; hence both spaces are contractible. Write $\mathcal{R}_{\mathrm{inv},\mu}(U) = \mathcal{R}^{\mathrm{inv}}_\mu(U)$ and $\mathcal{P}_{\mathrm{inv},\mu}(U) = \mathcal{P}^{\mathrm{inv}}_\mu(U)$.

Suppose $n \equiv 4 \pmod 8$, and set $a = |\operatorname{ind}_{\mathbb{H}} D^+|$. If $\operatorname{ind}_{\mathbb{H}} D^+ \le 0$, let $A_{g,V} = (D_g + V)^+$; otherwise let $A_{g,V} = (D_g + V)^-$. These chiral restrictions have domain H^1 in the indicated source bundle, and their quaternionic index is $-a$. At index zero the grading is that of Section 2. Write $A_g = A_{g,0}$, and put

$$\begin{aligned} \mathcal{P}_{\min,\mu}(U) &= \{(g, V) \in \mathcal{P}_\mu(U) \,:\, \ker A_{g,V} = 0\}, \\ \mathcal{R}_{\min,\mu}(U) &= \{g \in \mathcal{R}_\mu(U) \,:\, \ker A_g = 0\}. \end{aligned}$$

On these loci, $\dim_{\mathbb{H}} \ker A^*_{g,V} = a$. Theorem R applies on M with the open set U; the choice $U = M$ is the one in Theorem C.

8.1. **The geometric real index.** Let $C_{p,q}$ be the real graded Clifford algebra on p generators of square $+1$ and q generators of square -1, with its standard involution. Thus the former generators are self-adjoint and the latter are skew-adjoint. This is the convention of [28, §2.1]; the algebra is denoted by $\mathrm{Cl}^{q,p}$ in [45, Definition 2.1]. Put $\mathrm{Cl}_n = C_{0,n}$, and denote its standard generators by $c_1, \dots, c_n$. For a metric g, let $P_{\mathrm{Spin},g}(M)$ be the spin double cover of its oriented orthonormal frame bundle. The real graded spin bundle is

$$\mathscr{S}_{\mathrm{fr}} = P_{\mathrm{Spin},g}(M) \times_{\mathrm{Spin}(n)} \mathrm{Cl}_n,$$

where $\mathrm{Spin}(n)$ acts by left multiplication. Right multiplication $R_a(s) = sa$, $a \in \mathrm{Cl}_n$, is therefore globally defined; it commutes with the Dirac operator and satisfies $R_a R_b = R_{ba}$. These are the real spin bundle and operator of [65, §4.2, Proposition 4.2].

Write $n = 8k + r$, where $k \ge 0$ and $0 \le r \le 7$, and put $\omega_n = c_1 \cdots c_n$. Choose a primitive self-adjoint idempotent e in Cl^0_n. For $r = 0, 4$, choose its simple factor so that

$$\omega_n e = (-1)^{n/4} e.$$

The grading induced on $\mathrm{Cl}_n e$ then agrees with $\Gamma = i^{n/2}\omega_n$ on the complex spinors. Put

$$\mathsf{C}_r = e\mathrm{Cl}_n e, \qquad \mathscr{S}_{\mathrm{red}} = \mathscr{S}_{\mathrm{fr}} e.$$

All Clifford algebras, corners, and tensor products in this subsection are real. The quaternion algebra $\mathbb{H}$ is purely even. Let $E^\pm_{\mathbb{F}}$ denote the chiral spinor bundles with the scalar structures of Section 2. Thus $E^\pm_{\mathbb{R}} = \ker(J - I)|_{\Sigma^\pm_g}$ in residue zero, $E^\pm_{\mathbb{H}} = \Sigma^\pm_g$ as quaternionic bundles in residue four, and $E^\pm_{\mathbb{C}} = \Sigma^\pm_g$ in residues two and six. In odd dimensions put

$$W_{\mathbb{R}} = \ker(J - I) \subset \Sigma_g \quad (r = 1, 7), \qquad W_{\mathbb{H}} = \Sigma_g \quad (r = 3, 5),$$

with the quaternionic structure defined by J in the latter case. A subscript $\mathbb{R}$ outside parentheses denotes the underlying real bundle.

For $L = D_g + V$, write $A = L^+$ in even dimension. In odd dimension put $K = iL$ in residues one and five, and $T = L$ in residues three and seven, restricted to $W_{\mathbb{R}}$ or $W_{\mathbb{H}}$ as appropriate. In residues two and six, let $J_\pm = J|_{\Sigma^\pm_g} : \Sigma^\pm_g \to \Sigma^\mp_g$ and define the formal transpose by

$$A^T = J_+ A^* J_+.$$

Here A and A^* have domains $H^1(\Sigma_g^+)$ and $H^1(\Sigma_g^-)$, respectively, as in Section 2. The identity $JL = LJ$ gives $J_-A = A^*J_+$, hence $A^T = J^2A$. On an L-invariant finite-dimensional subspace preserved by J and Γ, it is the transpose in (2.4). The subscripts on A, K, T indicate their scalar fields. The reduced bundles and independent operators are

r	C_r	$\mathscr{S}_{\mathrm{red}}$	independent block	(KO_n, KO_{n+1})
0	$\mathbb{R}$	$E^+_{\mathbb{R}} \oplus E^-_{\mathbb{R}}$	$A_{\mathbb{R}}$	$(\mathbb{Z}, \mathbb{Z}/2)$
1	$C_{0,1}$	$W_{\mathbb{R}} \oplus W_{\mathbb{R}}$	$K^*_{\mathbb{R}} = -K_{\mathbb{R}}$	$(\mathbb{Z}/2, \mathbb{Z}/2)$
2	$C_{0,2}$	$(E^+_{\mathbb{C}} \oplus E^-_{\mathbb{C}})_{\mathbb{R}}$	$A^T_{\mathbb{C}} = -A_{\mathbb{C}}$	$(\mathbb{Z}/2, 0)$
3	$\mathbb{H} \otimes C_{1,0}$	$W_{\mathbb{H}} \oplus W_{\mathbb{H}}$	$T^*_{\mathbb{H}} = T_{\mathbb{H}}$	$(0, \mathbb{Z})$
4	$\mathbb{H}$	$E^+_{\mathbb{H}} \oplus E^-_{\mathbb{H}}$	$A_{\mathbb{H}}$	$(\mathbb{Z}, 0)$
5	$\mathbb{H} \otimes C_{0,1}$	$W_{\mathbb{H}} \oplus W_{\mathbb{H}}$	$K^*_{\mathbb{H}} = -K_{\mathbb{H}}$	$(0, 0)$
6	$C_{2,0}$	$(E^+_{\mathbb{C}} \oplus E^-_{\mathbb{C}})_{\mathbb{R}}$	$A^T_{\mathbb{C}} = A_{\mathbb{C}}$	$(0, 0)$
7	$C_{1,0}$	$W_{\mathbb{R}} \oplus W_{\mathbb{R}}$	$T^*_{\mathbb{R}} = T_{\mathbb{R}}$	$(0, \mathbb{Z})$.

(8.3)

In the even rows the reduced operator is $\mathcal{D}(A)$ from (2.3); in the odd rows it is

$$F_-(K) = \begin{pmatrix} 0 & -K \\ K & 0 \end{pmatrix} \quad (r = 1, 5), \qquad F_+(T) = \begin{pmatrix} 0 & T \\ T & 0 \end{pmatrix} \quad (r = 3, 7).$$

The grading on a doubled odd-dimensional bundle is $\operatorname{diag}(I, -I)$.

The corner algebras follow from the Clifford algebra table and $\mathrm{Cl}^0_n \cong \mathrm{Cl}_{n-1}$ [12, Table 1 and Proposition 5.4]. Their actions are determined by (2.1). Write $I_{\mathbb{C}}$ for multiplication by i. In residues two and six, the odd generators are J and $I_{\mathbb{C}}J$. They anticommute and have squares $-I$ in residue two and $+I$ in residue six. In residue four, $I_{\mathbb{C}}$ and J generate the even quaternionic action. In residues three and five this action is diagonal on the doubled bundle; the remaining odd generator is, respectively,

$$U_+ = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \qquad U_- = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}.$$

By the odd-dimensional convention in Section 2, the action of ω_n on the complex spinors is $iI, I, -iI, -I$ in residues $1, 3, 5, 7$, respectively. On the real doubled bundle, left Clifford multiplication by a covector ξ is $F_-(ic(\xi))$ in residues $1, 5$ and $F_+(c(\xi))$ in residues $3, 7$. Since ω_n is central in odd dimension, its left and right actions agree. Hence

$n \bmod 8$	1	3	5	7
R_{ω_n}	$-U_-$	U_+	U_-	$-U_+$.

Ordinary commutation with these Clifford actions, together with oddness and self-adjointness, gives the operators in (8.3).

The idempotent e is full in Cl_n. In residues three and seven its projection to each of the two simple factors is nonzero. These are the primitive reductions associated with the Clifford algebra decompositions in [12, Table 1 and §5]. In the odd rows the two summands form one module for the residual Clifford algebra.

Let c_{fr} denote left Clifford multiplication on $\mathscr{S}_{\mathrm{fr}}$, and let D_{fr} be its Dirac operator. By Lemma 2.2, every compatible potential has a unique expression

$$V = c(v), \qquad v \in \bigoplus_{\substack{3 \le \ell \le n \\ \ell \equiv 3 \ (4)}} C^\infty(M; \Lambda^\ell T^*M).$$

The form v is real. Define

$$V_{\mathrm{fr}} = c_{\mathrm{fr}}(v), \qquad L_{\mathrm{fr}} = D_{\mathrm{fr}} + V_{\mathrm{fr}}.$$

Then V_{fr} is odd, self-adjoint, and commutes with right Clifford multiplication. Spin-equivariance makes this a global construction compatible with positive spinor comparison. Its restriction to $\mathscr{S}_{\mathrm{red}}$ is the reduced operator associated to V in (8.3).

The Hilbert space $\mathscr{H} = L^2(\mathscr{S}_{\mathrm{fr}})$ is ample in the sense of [45, Definition 2.3]: over a coordinate ball the fibre is the regular graded Clifford module, and every irreducible graded module occurs with infinite multiplicity in its space of sections. Let ι be the grading of $\mathscr{H}$, and put

$$E_j = R_{c_j}, \qquad \omega = \iota E_1 \cdots E_n, \qquad f(t) = t(1+t^2)^{-1/2}, \qquad F = f(L_{\mathrm{fr}}).$$

The operators E_j satisfy Ebert's $\mathrm{Cl}^{n,0}$ relations. For $r = 3, 7$, order reversal in right multiplication gives $E_1 \cdots E_n = -R_{\omega_n}$, and oddness gives $\omega F \iota = (E_1 \cdots E_n)F$. The corresponding primitive reductions of $\omega F \iota$ are, respectively,

$$-\operatorname{diag}(f(T), f(T)), \qquad +\operatorname{diag}(f(T), f(T)).$$

On a coordinate ball choose a real coordinate function s. The endomorphism $ic(ds)$ on the complexified bundle has both eigenvalues $\pm|ds|$: multiplication by any nonzero covector orthogonal to ds interchanges the two eigenspaces. Choose nonzero compactly supported smooth sections $a_\pm$ in these eigenbundles. Then, as $N \to +\infty$,

$$\left\langle T(e^{iNs}a_\pm), e^{iNs}a_\pm \right\rangle_{L^2} = \pm N \int_M |ds|\,|a_\pm|^2\, dv_g + O(1).$$

Thus T has spectrum unbounded in both directions. Its resolvent is compact, so ± 1 belong to the essential spectrum of $f(T)$. The projection R_e commutes with L_{fr} by right Clifford linearity, and with the grading because e is even. It also commutes with R_{ω_n}, since ω_n is central. Thus $L^2(\mathscr{S}_{\mathrm{red}})$ is a reducing subspace for $\omega F \iota$, and these essential spectral values also occur for the full operator. It is therefore neither essentially positive nor essentially negative, as required by [45, Definition 2.6].

Consequently the full operators define $\mathrm{Cl}^{n,0}$-Fredholm families of degree KO^{-n}. Let $I = [-1, 1]$. A smooth coefficient path on I, with invertible endpoints, can be reparametrized to be constant on endpoint collars and extended constantly to $\mathbb{R}$. Its operators are odd, symmetric, Clifford-linear, and of Dirac type on the closed manifold M. They have common domain H^1 after positive comparison. Their bounded transforms are norm-continuous by [45, Proposition 3.7]. The representing theorem and the contractibility of the invertible subspace [45, Theorem 2.7, Lemma 2.8, and Theorem 2.21] give a relative class in

$$KO^{-n}(I, \partial I) = KO^{-(n+1)}(\mathrm{pt}) = KO_{n+1}.$$

More generally, let (B, B_0) be a compact CW pair. Suppose that $(g_{b,t}, V_{b,t})$ depends continuously on b in the smooth coefficient topology in (t, x), is constant on fixed endpoint collars in t, and has invertible endpoint operators. Suppose also that the coefficient path is independent of t for $b \in B_0$. Constant extension in t gives the data of [45, Assumption 3.11]. A Clifford-linear metric connection preserving the grading and constant in the cylinder direction on the endpoint collars and over B_0 satisfies Assumption 3.13. Such a connection is obtained from the fixed bundle by positive spinor comparison. The family and suspension classes agree under Bott periodicity by [45, Lemma 3.16 and Theorem 3.18].

The coefficient groups have the normalization of [12, Table 2 and Theorem 11.5]. For the local crossings in (8.1), the real chiral line and the real skew two-plane both have value 1 modulo two [28, §§4.2.1–4.2.2, Theorems 5.4–5.5]. In residues three and seven, orient $\mathbb{R} \times M$ with the $\mathbb{R}$-factor first. Under the chiral identification $c_M(\xi) = -c(dt)c_{\mathbb{R}\times M}(\xi)$, the suspension has chiral representative $\partial_t + T_t$. Thus a quaternionic line or a real line crossing from negative to positive contributes $+1$, respectively. Complex spectral flow is twice this integer in residue three and equals it in residue seven. These factors agree with the complexification homomorphisms in [12, §6, formula (6.11)].

If $g_* \in \mathcal{R}^{\mathrm{inv}}_\mu(M)$ and $(g, V) \in \mathcal{P}^{\mathrm{inv}}(M)$, define

$$\delta_{g_*}(g, V) \in KO_{n+1}$$

to be the relative real family index of a smooth coefficient path from $(g_*, 0)$ to (g, V). Any two such paths are homotopic relative to their endpoints by affine interpolation in the global metric

and potential coordinates. Norm-continuity of the bounded transform and homotopy invariance of the relative index therefore make this definition independent of the path. For $g \in \mathcal{R}^{\mathrm{inv}}(M)$, write $\delta_{g_*}(g) = \delta_{g_*}(g, 0)$. The index is additive under concatenation [28, Remark 5.2 and §6.2].

Lemma 8.1. *The subspace $\mathcal{R}^{\mathrm{inv}}_{\mu}(M)$ is a strong deformation retract of $\mathcal{R}^{\mathrm{inv}}(M)$. The same assertion holds for compatible pairs and for the chiral minimal loci defined above, also with a prescribed exterior of density μ. For an invertible base metric of density μ, these deformations preserve ordinary real index difference.*

Proof. For $(g, V) \in \mathcal{P}(M)$ and $0 \le t \le 1$, set

$$f = \frac{1}{n} \log \frac{\mu}{dv_g}, \qquad g_t = e^{2tf} g.$$

Let $\beta_t : \Sigma_g \to \Sigma_{g_t}$ be the pointwise conformal spinor isometry, and put

$$V_t = e^{-tf} \beta_t V \beta_t^{-1}.$$

The map β_t intertwines charge conjugation and chirality; since e^{-tf} is real, V_t is compatible. Conformal covariance [65, Proposition 1.3], in the normalization of [37, Section 2], gives

$$(D_{g_t} + V_t)\big(e^{-(n-1)tf/2} \beta_t \psi\big) = e^{-(n+1)tf/2} \beta_t (D_g + V) \psi.$$

The two multiplication maps are bounded isomorphisms on the corresponding Sobolev spaces. Hence they identify the kernels, including the two chiral kernels in even dimension. The map $(t, g, V) \mapsto (g_t, V_t)$ is continuous in the smooth topology, $dv_{g_1} = \mu$, and $(g_t, V_t) = (g, V)$ whenever $dv_g = \mu$. It preserves $V = 0$. On a prescribed exterior of density μ, one has $f = 0$ and $\beta_t = I$. These formulas give the asserted strong deformation retractions. The invertible base metric is fixed, so homotopy invariance of the relative real family index proves the last assertion. □

8.2. **Removal of multiplication potentials.**

Proposition 8.2. *The inclusions*

$$\mathcal{R}_{\mathrm{inv},\mu}(U) \longrightarrow \mathcal{P}_{\mathrm{inv},\mu}(U), \qquad \mathcal{R}_{\min,\mu}(U) \longrightarrow \mathcal{P}_{\min,\mu}(U), \qquad g \longmapsto (g, 0),$$

are weak homotopy equivalences; the second inclusion concerns the chiral block of quaternionic index $-a$ in dimension $n \equiv 4 \pmod 8$. Let B be a compact smooth parameter manifold, possibly with boundary, and let $B_0 \subset B$ be closed. Every smooth family in either pair space whose potential vanishes near B_0 is homotopic to a family with zero potential, relative to a neighborhood of B_0. In the invertible locus these homotopies preserve ordinary index difference. The assertions include $U = M$.

Proof. Write $L_b = D_{g_b} + V_b$. Direct positive spinor comparison with g_b puts the operators constructed below on $L^2(\Sigma_{g_b})$, with domain $H^1(\Sigma_{g_b})$. Comparison with one fixed reference metric identifies the parameter Hilbert spaces. Unless otherwise specified, operator norms are taken on L^2. If $V_b = 0$ for every b, the constant homotopy suffices.

First suppose that every V_b is supported in one compact subset of U. Choose $s \in C^\infty([0,1],[0,1])$ equal to zero and one on neighborhoods of the respective endpoints, and put

$$A_{b,t} = D_{g_b} + s(t) V_b, \qquad C_{b,t} = (1 - s(t)) V_b.$$

Thus $A_{b,t} + C_{b,t} = L_b$. Choose $\tau > 2 \sup_b \|V_b\|_\infty$, with $\tau > 0$. For $\eta > 0$, Theorem R applied to $A_{b,t}$ and $\boldsymbol{\mathcal{K}} = B \times [0,1] \times \{i\tau\}$ gives metrics $\widehat{g}_{b,t}$ whose compared Dirac operators $T_{b,t}$ satisfy

$$\sup_{b,t} \big\| (T_{b,t} - i\tau)^{-1} - (A_{b,t} - i\tau)^{-1} \big\| < \eta.$$

Indeed, $A_{b,t}$ is self-adjoint, and $s(t)V_b$ has common compact spatial support and vanishes near $(B \times \{0\}) \cup (B_0 \times [0,1])$. The metrics have density μ, agree with g_b on $M \setminus U$, and equal g_b near this parameter subset. If B has boundary, extend the smooth coefficients across it in the logarithmic metric

and fixed potential coordinates, and extend s constantly across its endpoint neighborhoods. The extensions retain the linear exterior and support conditions; a parameter cutoff retains the vanishing on a smaller neighborhood of B_0. Apply Theorem R on a compact smooth neighborhood of $B \times [0,1]$ with rounded corners in the extended parameter manifold, and restrict the resulting family to $B \times [0,1]$.

For bounded operators C, R put $F_C(R) = R(I + CR)^{-1}$ whenever the inverse exists. The resolvent identity gives

$$F_C((A - i\tau)^{-1}) = (A + C - i\tau)^{-1}, \qquad F_C(S) - F_C(R) = (I + SC)^{-1}(S - R)(I + CR)^{-1}.$$

Since $\|C_{b,t}\| < \tau/2$ and the resolvents of the self-adjoint operators $A_{b,t}$, $T_{b,t}$ at $i\tau$ have norm at most τ^{-1}, both inverse factors in the difference formula have norm at most two. Writing $N_{b,t} = T_{b,t} + C_{b,t}$, we obtain

$$\sup_{b,t} \left\| (N_{b,t} - i\tau)^{-1} - (L_b - i\tau)^{-1} \right\| < 4\eta. \tag{8.4}$$

Suppose first that every L_b is invertible. Its inverse is continuous in b as an operator from L^2 to H^1 [67, Chapter IV, Theorem 1.16]; hence

$$C_0 = \sup_b \|I - i\tau L_b^{-1}\| < \infty.$$

The inverse of $I + i\tau(L_b - i\tau)^{-1}$ is $I - i\tau L_b^{-1}$. Choose η so that $4\tau\eta C_0 < 1$. By (8.4), $I + i\tau(N_{b,t} - i\tau)^{-1}$ is invertible. The factorization on H^1

$$N_{b,t} = \left[I + i\tau(N_{b,t} - i\tau)^{-1} \right] (N_{b,t} - i\tau)$$

therefore proves that $N_{b,t}$ is invertible.

In the chiral minimal case, let Z_b be the chosen chiral block of L_b. Then $\ker Z_b = 0$, $\operatorname{ind}_{\mathbb{H}} Z_b = -a$, and $\dim_{\mathbb{H}} \ker L_b = a$. There is $\rho > 0$ such that

$$\operatorname{Spec}(L_b) \cap [-2\rho, 2\rho] \subset \{0\} \qquad (b \in B).$$

For each parameter, the spectral projection on a sufficiently small circle about zero has quaternionic rank a. By spectral continuity [67, Chapter IV, Theorem 3.16], this rank remains a nearby. Since every nearby kernel also has dimension a, the projection is onto the kernel. Compactness of B gives the stated uniform gap. Let $\gamma = \{z \in \mathbb{C} : |z| = \rho\}$, oriented counterclockwise, and set

$$R_b = (L_b - i\tau)^{-1}, \quad S_{b,t} = (N_{b,t} - i\tau)^{-1},$$
$$m = \max_{z \in \gamma} |i\tau - z|, \qquad \kappa = \sup_{b \in B,\ z \in \gamma} \left\| [I + (i\tau - z)R_b]^{-1} \right\|.$$

The number κ is finite. If $8m\kappa\eta < 1$, then $I + (i\tau - z)S_{b,t}$ is invertible with inverse norm at most 2κ. Applying the same difference identity with $C = (i\tau - z)I$ gives

$$\sup_{b,t,\ z \in \gamma} \left\| (N_{b,t} - z)^{-1} - (L_b - z)^{-1} \right\| < 8\kappa^2\eta.$$

Let P_b and $\widehat{P}_{b,t}$ be the corresponding spectral projections defined by (6.1). Contour integration yields

$$\sup_{b,t} \|\widehat{P}_{b,t} - P_b\| < 8\rho\kappa^2\eta.$$

Choose also $8\rho\kappa^2\eta < 1$. Both projections commute with J, being the spectral projections of J-commuting self-adjoint operators on $(-\rho, \rho)$. Lemma 6.1 therefore gives $\operatorname{rank}_{\mathbb{H}} \widehat{P}_{b,t} = a$. The chosen block $\widehat{Z}_{b,t}$ of $N_{b,t}$ has index $-a$, so

$$a + 2 \dim_{\mathbb{H}} \ker \widehat{Z}_{b,t} = \dim_{\mathbb{H}} \ker N_{b,t} \le \operatorname{rank}_{\mathbb{H}} \widehat{P}_{b,t} = a.$$

Thus $\widehat{Z}_{b,t}$ is injective.

Transport $C_{b,t}$ to $\Sigma_{\widehat{g}_{b,t}}$ by the direct unitary comparison. This preserves compatibility and the pointwise norm. The resulting pair homotopy starts at (g_b, V_b), ends at $(\widehat{g}_{b,1}, 0)$, and is stationary near B_0. Its density and prescribed exterior remain fixed.

Now assume only the support condition in (8.2). For each $\varepsilon > 0$ there is $\chi \in C_c^\infty(U)$, $0 \le \chi \le 1$, such that

$$\sup_b \|(1-\chi)V_b\|_\infty < \varepsilon.$$

Indeed, $x \mapsto \max_{b\in B} |V_b(x)|$ is continuous and vanishes on $M \setminus U$. The set $\{x : \max_{b\in B} |V_b(x)| \ge \varepsilon/2\}$ is compact in U, and χ may be chosen equal to one there. The interpolation $V_b \mapsto (1 - t + t\chi)V_b$ is smooth and stationary near B_0. For invertible L_b, choose $\varepsilon \sup_b \|L_b^{-1}\| < 1$ and apply [67, Chapter IV, Theorem 1.16]. For a chiral minimal family, take its uniform gap 2ρ as above. The source of Z_b is orthogonal to $\ker L_b$, so the spectral theorem gives $\|Z_b u\|_2 \ge 2\rho\|u\|_2$ for every u in its H^1 domain. If $E_{b,t}$ is the chiral block of $-t(1-\chi)V_b$, then $\|E_{b,t}\| < \varepsilon$. Hence

$$\|(Z_b + E_{b,t})u\|_2 \ge (2\rho - \varepsilon)\|u\|_2.$$

Taking $\varepsilon < 2\rho$ keeps the chosen block injective. The preceding compact-support construction applies to the endpoint $(g_b, \chi V_b)$. Reparametrization constant near the ends makes the concatenated homotopy smooth.

In the logarithmic metric and fixed potential coordinates, each pair locus and its metric locus is open in the corresponding real Fréchet space. Indeed, the operator map is continuous in $\mathcal{L}(H^1, L^2)$, and invertibility and injectivity of a Fredholm block are open in this topology [67, Chapter IV, Theorems 1.16 and 5.17]. Wockel's relative approximation theorem [87, Theorem 11, p. 99], applied componentwise to the trivial bundle over a compact smooth parameter manifold with one of these open sets as fibre, deforms any continuous family to a smooth one. If the family is smooth near a closed subset K, choose a closed neighborhood K' of K inside that smooth region. In the theorem, take the closed set to be B and the open set to be $B \setminus K'$; the resulting homotopy is fixed on K'. The prescribed exterior is retained because it is part of the ambient Fréchet space.

For a based sphere, a based self-map homotopic to the identity and constant near the basepoint first makes the family constant there. Relative smooth approximation and the preceding construction then give a based metric representative. Thus the inclusions are surjective on all based homotopy groups. If a based metric sphere bounds a pair disk, make the boundary map constant near its basepoint as above and smooth it within the metric locus relative to that neighborhood. Attach these based metric homotopies as an annulus, and reparametrize the disk radially so that its new boundary values are constant in the normal collar coordinate. The disk is now smooth and metric on a boundary collar. Smooth approximation relative to a smaller collar, followed by potential removal relative to that collar, gives a metric filling of the smooth boundary sphere. Attaching the reverse metric annulus gives a metric filling of the original sphere. Thus the inclusions are injective on the based homotopy groups. The same argument for points and intervals gives the bijection on path components. All deformations lie in the pair locus, so homotopy invariance preserves ordinary index difference on the invertible locus. □

9. Supported compatible potentials

Let (X, g) be a connected Riemannian spin manifold and let ∇ be its spin connection. For every smooth endomorphism V, there are differential operators P_1 and P_0, of orders at most one and zero and with smooth coefficients, such that

$$(D_g + V)^2 = \nabla^*\nabla + P_1 + P_0.$$

Aronszajn's unique continuation theorem [9], in the bundle formulation [61, Theorem 6.2.8], therefore implies that a solution of $(D_g + V)\phi = 0$ vanishing on a nonempty open subset is zero on X.

Proposition 9.1. *Let (W^n, g) be a compact connected Riemannian spin manifold, $n \ge 3$. If $\partial W \ne \emptyset$, suppose that $g = dt^2 + h$ near ∂W, that D_h is invertible, and put $X = W_\infty$. If $\partial W = \emptyset$, put $X = W$.*

Suppose $\alpha(W,h)=0$, with $\alpha(W,h)$ interpreted as $\alpha(W)$ in the closed case. For every nonempty open set $U \Subset \operatorname{int} W$ there is $V \in C_c^\infty(U;\mathscr{A}_n)$ such that D_g+V is invertible on X.

Proof. Choose a closed coordinate ball $K \Subset U$ with nonempty interior, and put

$$\mathscr{V}_K = \{V \in C^\infty(X;\mathscr{A}_n) \,:\, \operatorname{supp} V \subset K\}.$$

Equip $\mathscr{V}_K$ with the C^∞ topology. The operators D_g+V, $V\in\mathscr{V}_K$, are self-adjoint Fredholm operators with domain $H^1(\Sigma X)$, by Section 2. Their complex kernel dimensions have a minimum m. Choose $V_0\in\mathscr{V}_K$ attaining it, and put $L=D_g+V_0$ and $F=\ker L$. Elliptic regularity gives $F\subset C^\infty(\Sigma X)$ [66, Theorem 18.1.29 and pp. 91–93].

The elements of $\mathscr{V}_K$ are bounded self-adjoint operators on the separable Hilbert space $L^2(\Sigma X)$, and their inclusion in $\mathcal{L}(L^2(\Sigma X))$ is continuous. By Section 2, they are compact from $H^1(\Sigma X)$ to $L^2(\Sigma X)$, and the graph norms of D_g and L are equivalent to the H^1 norm. Anghel's theorem [7, Theorem 1.2] gives $\dim_{\mathbb{C}}\ker(D_g+V)=m$ for V in a neighborhood of V_0 in $\mathscr{V}_K$. Let P be the orthogonal projection onto F. Since $V_0+\mathscr{V}_K=\mathscr{V}_K$, the operator L has minimal nullity in $L+\mathscr{V}_K$. By [7, Theorem 1.3],

$$PBP=0 \qquad (B\in C_c^\infty(\operatorname{int}K;\mathscr{A}_n)). \tag{9.1}$$

For $x\in X$, let $e_x : F\to\Sigma_x X$ be evaluation. Lemma 6.5 gives

$$e_x^* Q e_x = 0 \qquad (x\in\operatorname{int}K,\ Q\in\mathscr{A}_{n,x}).$$

For every V in this neighborhood, the orthogonal projection onto $\ker(D_g+V)$ and the evaluation maps on that kernel satisfy the same identities. The index remains $\alpha(W,h)=0$ along D_g+tV_0, by Section 2.

Suppose $n\equiv 0$ or $4 \pmod 8$, and put $\mathbb{F}=\mathbb{R}$ or $\mathbb{H}$, respectively. In residue zero take the real fixed subspaces of J in the two chiral kernels; in residue four regard those kernels as quaternionic spaces. Their $\mathbb{F}$-dimensions are equal. If they are nonzero, choose nonzero kernel spinors ϕ^+ and ϕ^- of opposite chirality, fixed by J in residue zero. Their nonzero loci are open and dense by unique continuation. Choose a common nonzero point $x\in\operatorname{int}K$ and put $u=\phi^+(x)$, $v=\phi^-(x)$. Write $\langle\ ,\ \rangle_{\mathbb{F}}$ for the real or quaternionic Hermitian product, linear in the second argument. Between the corresponding real or quaternionic chiral fibres define the $\mathbb{F}$-linear map

$$Z(w)=\frac{v\langle u,w\rangle_{\mathbb{F}}}{|u|^2}.$$

Then $Zu=v$. In residue zero, extend Z complex-linearly from the real fixed space. By (2.3), $\mathcal{D}(Z)$ is compatible. Its pairing on ϕ^+ and ϕ^- is $\langle v,Zu\rangle=|v|^2>0$, contrary to the pointwise identity.

If $n\equiv 3$ or $7 \pmod 8$, the identity endomorphism is compatible. Thus $e_x^*e_x=0$ on $\operatorname{int}K$, and unique continuation gives $F=0$.

If $n\equiv 5 \pmod 8$, then $J^2=-I$ and $JL=-LJ$. If $F\neq 0$, choose $\phi\in F$ and $x\in\operatorname{int}K$ with $u=\phi(x)\neq 0$. The endomorphism

$$Q=uu^*-(Ju)(Ju)^*$$

is self-adjoint and satisfies $JQ=-QJ$. Antiunitarity and $J^2=-I$ give $\langle u,Ju\rangle=0$, hence $\langle Qu,u\rangle=|u|^4>0$, contradicting the pointwise identity.

If $n\equiv 6 \pmod 8$, charge conjugation exchanges the chiral kernels. If $\ker L^+\neq 0$, choose $\phi\in\ker L^+$ and $x\in\operatorname{int}K$ with $u=\phi(x)\neq 0$. In the charge coordinates of (2.3), pairing ϕ with $J\phi$ gives

$$u^T Z u = 0 \qquad (Z^T=Z).$$

Choose j with $u_j\neq 0$, and let E_{jj} be the matrix whose sole nonzero entry is 1 in position (j,j). The permitted choice $Z=E_{jj}$ gives $u_j^2=0$, a contradiction. Thus both chiral kernels vanish.

If $n\equiv 1 \pmod 8$, put $F_{\mathbb{R}}=F\cap\operatorname{Fix}J$. In real orthonormal fibre coordinates, iA is compatible for every real skew-symmetric matrix A. For $u,v\in e_x(F_{\mathbb{R}})$ the pointwise identity gives $u^TAv=0$ for every such A, hence $u_iv_j-u_jv_i=0$ for all i,j. Thus $\dim_{\mathbb{R}} e_x(F_{\mathbb{R}})\le 1$ on $\operatorname{int}K$. If $\phi,\psi\in F_{\mathbb{R}}$ are

independent, choose a connected ball $O \Subset \operatorname{int} K$ on which $\phi \neq 0$. On O one has $\psi = f\phi$, where $f = \phi^*\psi/|\phi|^2 \in C^\infty(O;\mathbb{R})$. Consequently

$$0 = L(f\phi) - fL\phi = c(df)\phi.$$

Since $c(df)^2 = -|df|^2 I$, one has $df = 0$. Unique continuation extends the resulting constant proportionality to X, a contradiction. Hence $\dim_{\mathbb{R}} F_{\mathbb{R}} \leq 1$. This dimension is even because the mod-two index vanishes. Thus $F_{\mathbb{R}} = 0$ and, since $F = F_{\mathbb{R}} \oplus_{\mathbb{R}} iF_{\mathbb{R}}$, also $F = 0$.

If $n \equiv 2 \pmod 8$, then $n \geq 10$. The local constancy established above and Lemma 9.2 give $\dim_{\mathbb{C}} \ker L^+ \leq 1$. This dimension is even because the mod-two index vanishes. Charge conjugation identifies the two chiral kernels, so $F = 0$.

Thus L is self-adjoint Fredholm with zero kernel, and its inverse $L^{-1} : L^2(\Sigma X) \to H^1(\Sigma X)$ is bounded. Since $\operatorname{supp} V_0 \subset K \Subset U$, the potential V_0 has the required support. □

Lemma 9.2. *Let (X^n, g) be a connected Riemannian spin manifold, either closed or the cylindrical completion of a compact spin manifold with invertible boundary Dirac operator. Let $K \Subset X$ be a closed coordinate ball with nonempty interior, and let $\mathscr{V}_K$ be the space of smooth compatible potentials supported in K, with its C^∞ topology. Suppose $n \equiv 2 \pmod 8$, $n \geq 10$, and $V_0 \in \mathscr{V}_K$. Put $L = D_g + V_0$. If $\dim_{\mathbb{C}} \ker(D_g + V)$ is constant in a neighborhood of V_0 in $\mathscr{V}_K$, then $\dim_{\mathbb{C}} \ker L^+ \leq 1$.*

Proof. Set $F = \ker L$, $F^+ = \ker L^+$, and let P be the orthogonal projection onto F in $L^2(\Sigma X)$. Local constancy and Lemma 6.4 give (9.1) at V_0 and at every sufficiently close potential in $\mathscr{V}_K$. In smooth orthonormal chiral frames with $J(u,v) = (-\bar{v}, \bar{u})$, the block Z in (2.3) ranges over all complex skew-symmetric matrices. Lemma 6.5, applied to positive kernel spinors and charge conjugates of positive kernel spinors, gives, for $\phi, \psi \in F^+$ and $x \in \operatorname{int} K$,

$$\psi(x)^T Z \phi(x) = 0 \qquad (Z^T = -Z).$$

Let $\wedge$ denote the complex exterior product in the spinor fibres. Then

$$\phi(x) \wedge \psi(x) = 0 \qquad (\phi, \psi \in F^+,\ x \in \operatorname{int} K). \tag{9.2}$$

The same identity holds for positive-chiral kernel spinors of $D_g + V$ whenever V is sufficiently close to V_0 in $\mathscr{V}_K$.

Suppose that $\phi, \psi \in F^+$ are linearly independent. Choose a connected coordinate ball $O \Subset \operatorname{int} K$ on which ϕ is nowhere zero. By (9.2),

$$\psi = f\phi, \qquad f = \frac{\phi^*\psi}{|\phi|^2} \in C^\infty(O;\mathbb{C}).$$

If $df = 0$ throughout O, unique continuation applied to $\psi - f\phi$, with f constant, contradicts linear independence. Choose $x_0 \in O$ with $df(x_0) \neq 0$, and a coordinate ball $O' \Subset O$ containing x_0. Extend f to a function in $C_c^\infty(X;\mathbb{C})$ which agrees with it near $\overline{O'}$. On O' the product rule gives

$$c(df)\phi = L(f\phi) - fL\phi = 0.$$

For $B \in C_c^\infty(O'; \mathscr{A}_n)$ and sufficiently small real t, let P_t be the orthogonal projection onto $\ker(L + tB)$. Local constancy and Section 6 give a C^1 family of projections in $\mathcal{L}(L^2, H^1)$. Let $U_t : F \to \operatorname{im} P_t$ be the polar frame (6.2), based at the inclusion of F. It intertwines chirality. A dot denotes differentiation at $t = 0$. The identity $P\dot{P}_0 P = 0$ and (6.2) give $P\dot{U}_0 = 0$.

The restriction $L : H^1(\Sigma X) \cap F^\perp \to F^\perp$ is an isomorphism. Define the reduced inverse

$$R = \left(L|_{H^1 \cap F^\perp}\right)^{-1}(I - P) : L^2(\Sigma X) \longrightarrow H^1(\Sigma X).$$

Then $LR = I - P$ on L^2, $RL = I - P$ on H^1, and $\Gamma R = -R\Gamma$. Put $\phi_t = U_t\phi$ and $\psi_t = U_t\psi$. Differentiation of $(L + tB)\phi_t = (L + tB)\psi_t = 0$ in the common domain yields

$$\dot{\phi} = -RB\phi, \qquad \dot{\psi} = -RB\psi. \tag{9.3}$$

The equations $L\dot\phi = -B\phi$ and $L\dot\psi = -B\psi$ imply that these derivatives are smooth by elliptic regularity [66, Theorem 18.1.29 and pp. 91–93]. The exterior product is a continuous bilinear map $L^2_{\mathrm{loc}}\times L^2_{\mathrm{loc}} \to L^1_{\mathrm{loc}}$, since $|u\wedge v| \le |u||v|$. Thus (9.2) can be differentiated in L^1_{loc}, giving $\phi\wedge(\dot\psi - f\dot\phi) = 0$. All terms are smooth, so $\dot\psi - f\dot\phi \in \mathbb{C}\phi$ pointwise on O'.

Let $\Pi_x : \Sigma^+_x X \to \Sigma^+_x X/\mathbb{C}\phi(x)$ be the quotient map. Since $B\psi = fB\phi$ on X, (9.3) gives

$$\Pi[R,f](B\phi) = 0 \quad \text{on } O', \qquad [R,f] = Rf - fR. \tag{9.4}$$

Fix smooth charge-adapted chiral frames on O', write $u(x) = \phi(x)$ in these frames, and define the hyperplane bundle

$$\mathcal{H}_{\phi,x} = \{y \in \Sigma^-_x X \,:\, u(x)^T y = 0\}.$$

For $y \in C^\infty_c(O';\mathcal{H}_\phi)$, set

$$w = \frac{\bar u}{|u|^2}, \qquad Z_y = yw^T - wy^T.$$

Then $Z_y^T = -Z_y$, $w^T u = 1$, and $y^T u = 0$, hence $Z_y u = y$. The endomorphism $B = \mathcal{D}(Z_y)$ from (2.3), extended by zero, belongs to $C^\infty_c(O';\mathscr{A}_n)$ and satisfies $B\phi = y$. Consequently (9.4) holds for every compactly supported smooth section of $\mathcal{H}_\phi$.

The operator R is locally pseudodifferential of order -1. Indeed, on a coordinate ball containing $\overline{O'}$, choose a properly supported two-sided elliptic parametrix Q for L [66, Theorem 18.1.24 and pp. 91–93]. Take Schwartz kernels relative to dv_g, and let S be the difference of the kernels of R and Q on the product of this ball with itself. The identities $LR = RL = I - P$ give $L_x S \in C^\infty$ and $L^t_y S \in C^\infty$; here L^t_y is the formal transpose on the dual spinor bundle with respect to dv_g. The kernel of P is smooth by elliptic regularity. The operator

$$L_x^* L_x + (L_y^t)^* L_y^t$$

has principal symbol $(|\xi|_x^2 + |\eta|_y^2)I$. Elliptic regularity therefore gives $S \in C^\infty$.

For a nonzero real covector ξ, the principal symbol of the negative-to-positive chiral component of R is $ic(\xi)/|\xi|^2$. Write σ_{-2} for the principal symbol of order -2, and let $\langle\ ,\ \rangle$ denote the complex-bilinear extension of g^{-1}. The composition formula [66, Theorem 18.1.8, p. 71] gives

$$\sigma_{-2}([R,f]) = \frac{c(df)}{|\xi|^2} - 2\frac{\langle\xi, df\rangle c(\xi)}{|\xi|^4}. \tag{9.5}$$

The inclusion of $\mathcal{H}_\phi$ and the quotient Π are smooth bundle maps. Since their composition with $[R,f]$ vanishes on every test section by (9.4), its principal symbol vanishes:

$$\Pi\left(\frac{c(df)}{|\xi|^2} - 2\frac{\langle\xi, df\rangle c(\xi)}{|\xi|^4}\right)\Big|_{\mathcal{H}_\phi} = 0. \tag{9.6}$$

At x_0, put $\eta = df(x_0) \neq 0$ and $r = 2^{n/2-1} = \dim_{\mathbb{C}} \Sigma^\pm_{x_0} X$. Since $c(\eta)\phi(x_0) = 0$ and $\phi(x_0) \neq 0$, the Clifford relation gives $g^{-1}(\eta,\eta) = 0$. Choose a maximal isotropic subspace containing η. In the exterior model [51, Lemma 20.9 and equation (20.11), pp. 304–305], $c(\eta)$ is a nonzero multiple of exterior multiplication by its first generator. Its rank on either parity is $2^{n/2-2} = r/2$. In particular, $A = c(\eta)|_{\Sigma^-_{x_0}X}$ has rank $r/2$. Choose a nonzero real covector ξ perpendicular to both $\operatorname{Re}\eta$ and $\operatorname{Im}\eta$. Then (9.6), evaluated at (x_0,ξ), gives

$$0 = \operatorname{rank}\big(\Pi_{x_0} A|_{\mathcal{H}_{\phi,x_0}}\big) \ge \operatorname{rank} A - 2 = r/2 - 2 \ge 8 - 2 > 0,$$

because restriction to a hyperplane and quotient by a line each lower rank by at most one. This contradiction proves the lemma. □

Remark 9.3. Let (M^n, g) be a closed connected Riemannian Spin^c-manifold, $1 \le n \le 4$, and let $\mathscr{C}$ be the space of unitary connections on its determinant line bundle. Denote the corresponding Dirac

operator by $D_{g,A}$ for $A \in \mathscr{C}$. For every $A \in \mathscr{C}$ and nonempty open set $U \subset M$, there is $A' \in \mathscr{C}$, equal to A outside U, such that

$$\dim_{\mathbb{C}} \ker D_{g,A'} = \min_{B \in \mathscr{C}} \dim_{\mathbb{C}} \ker D_{g,B};$$

see [71, Proposition 8.14, p. 430].

Let (X^n, g) now be a Riemannian spin manifold of arbitrary dimension. Regard its spin structure as a Spin^c-structure with trivial determinant line and product connection A_0. For a smooth real one-form β on X, put $a = i\beta$. The change $A_0 \mapsto A_0 + a$ gives

$$D_{g,A_0+a} = D_g + V_a, \qquad V_a = \tfrac{1}{2}c(a) = \tfrac{i}{2}c(\beta)$$

by [71, Section 2.3, p. 413]. Antiunitarity of J and (2.1) give $JV_a = -\epsilon_n V_a J$. Since J is invertible and $c(\beta)^2 = -|\beta|^2 I$, (2.2) implies

$$\left\{ \tfrac{i}{2}c(\xi) \,:\, \xi \in T^*_x X \right\} \cap \mathscr{A}_{n,x} = \{0\} \qquad (x \in X).$$

Suppose that X is closed and connected, $n = 4$, and $\alpha(X) = 0$, and let $U \subset X$ be nonempty and open. Then $\operatorname{ind}_{\mathbb{C}} D_g^+ = 0$. Let c^+ denote the restriction of Clifford multiplication from $\Sigma^+ X$ to $\Sigma^- X$. By [71, Theorem 1.4 and Proposition 8.14], with prescribed nonempty open set $U' \Subset U$, there is $\beta \in C_c^\infty(U; T^*X)$ for which

$$T = D_g^+ + \tfrac{i}{2}c^+(\beta) \,:\, H^1(\Sigma^+ X) \longrightarrow L^2(\Sigma^- X)$$

is invertible. The operator $C_\beta = c^+(\beta)T^{-1}$ is compact on $L^2(\Sigma^- X)$, and, for $t \in \mathbb{R}$,

$$D_g^+ + tc^+(\beta) = \left(I + (t - \tfrac{i}{2})C_\beta\right)T.$$

The spectrum of C_β has no nonzero accumulation point [67, Chapter III, Theorem 6.26, p. 185]. Hence $D_g^+ + tc^+(\beta)$ is invertible for all sufficiently small nonzero $t \in \mathbb{R}$. For such t, put $V_t = -t\Gamma c(\beta)$. In dimension four,

$$V_t^* = V_t, \qquad JV_t = V_t J, \qquad \Gamma V_t = -V_t \Gamma, \qquad V_t|_{\Sigma^+ X} = tc^+(\beta).$$

Thus $V_t \in C_c^\infty(U; \mathscr{A}_4)$ and $D_g + V_t$ is invertible.

10. Parameter families and kernel incidence

Throughout this section, M is closed and connected, and $n = \dim M$. The spin structure and the positive comparisons of Section 2 are fixed. All Sobolev spaces are taken on the resulting fixed bundles. Fix a reference metric g_0. The logarithmic coordinate of a metric g is the smooth g_0-self-adjoint endomorphism H determined by $g = g_0(e^H \cdot, \cdot)$. For a compatible potential W on $\Sigma_g M$, put $V = U_g^{-1} W U_g$, where U_g is the comparison (2.6) with reference metric g_0. A coefficient pair is $p = (H, V)$.

Separable Banach spaces of smooth sections also arise from weighted C^∞ norms [41, Definition 2.2.3 and Proposition 2.2.4].

Lemma 10.1. *Let $\mathcal{Q}$ be the real space of smooth compatible potential fields on $I \times M$, where $I = [0, 1]$, vanishing on fixed endpoint collars, or its subspace defined by $Q_t \psi_t = 0$ for a fixed smooth spinor family ψ_t. Put $\ell^2 = \ell^2(\mathbb{N}; \mathbb{R})$. There is a continuous linear map*

$$q \,:\, \ell^2 \longrightarrow \mathcal{Q}$$

with dense range. For every nonnegative integer k, it is bounded with values in $C^k(I \times M)$. For every real s, multiplication by $q(p)_t$ depends smoothly on (p, t) in $\mathcal{L}(H^s)$. The same conclusions hold for smooth potential fields on M, with or without the condition $Q\psi = 0$.

Let $\mathcal{Z}$ be a second-countable Banach manifold, and let F be a smooth Banach bundle section over $\ell^2 \times \mathcal{Z}$. At every zero, suppose that $D_z F$ is Fredholm of index d and that the composition of $D_p F$ with the quotient onto coker $D_z F$ *is surjective. Then $F^{-1}(0)$ is a smooth Banach submanifold, and the projection $\pi \,:\, F^{-1}(0) \to \ell^2$, $\pi(p, z) = p$, is Fredholm of index d.*

Proof. The stated constraints define a closed subspace of the separable Fréchet space of smooth fields. Choose a dense sequence (Q_j) and increasing C^k norms, including time derivatives when present. Choose $0 < a_j \le 2^{-j}(1 + \|Q_j\|_{C^j})^{-1}$ and set

$$q(p) = \sum_{j\ge 1} a_j p_j Q_j. \tag{10.1}$$

For each k, Cauchy–Schwarz gives

$$\|q(p)\|_{C^k} \le \left(\sum_{j\ge 1} a_j^2 \|Q_j\|_{C^k}^2 \right)^{1/2} \|p\|_{\ell^2} \le C_k \|p\|_{\ell^2}.$$

The series therefore converges in every smooth seminorm and preserves the constraints. Its range contains each Q_j. The multiplier estimate $\|Au\|_{H^s} \le C\|A\|_{C^k}\|u\|_{H^s}$, for sufficiently large k, gives the asserted smooth dependence; see [44, Propositions 12.9 and 12.13], applied in bundle charts.

At a zero, put $L = D_z F$, $C = D_p F$, and let $[C] : \ell^2 \to \operatorname{coker} L$ be the induced map. A complement of $\ker L$, together with finitely many parameter directions mapped onto $\operatorname{coker} L$, gives a bounded right inverse for (C, L). The Banach implicit function theorem [50, Theorem 13.20, p. 71] gives the zero submanifold. Its tangent projection satisfies

$$\ker D\pi \cong \ker L, \qquad \operatorname{im} D\pi = \ker[C], \qquad \operatorname{coker} D\pi \cong \operatorname{coker} L.$$

Thus $\operatorname{ind} D\pi = d$. The zero manifold is second countable and has at most countably many connected components. The regular-value conclusions of [82, Theorem 1.3 and Corollary 1.5, p. 862] apply to these components and hence to their union. □

Lemma 10.2. *Let p_t, $t \in I$, be a smooth coefficient path in logarithmic metric and linear potential coordinates, constant on fixed endpoint collars. Fix a finite-dimensional space of smooth potential directions. The path admits arbitrarily close approximations in any finite collection of spatial and parameter seminorms by smooth curves in finite-dimensional coefficient families analytic in their coefficient vectors. These families contain the fixed directions, and the approximations equal p_t on the prescribed collars.*

Suppose, moreover, that $L_p : H^1(E) \to L^2(E)$ is the associated real skew-adjoint first-order elliptic operator on a fixed Euclidean bundle E. Write $H^s = H^s(M; E)$, and let $\langle\ ,\ \rangle_{\mathbb{R}}$ denote the real L^2 inner product. Choose a basis $W_1, \dots, W_q$ of the fixed potential space, and let Q_j be the zeroth-order operator direction induced by W_j. If $L = i(D + V)$, then $Q_j = iW_j$. Put $Q(h) = \sum_{j=1}^q h_j Q_j$ for $h \in \mathbb{R}^q$. Let ψ_t be a smooth L^2-unit family with $L_{p_t}\psi_t = 0$, and let P_t be the real L^2 projection onto $\psi_t^{\perp_{\mathbb{R}}}$. Suppose that, for every $t \in I$, the map

$$\begin{aligned} \mathbb{R}^q \oplus (\psi_t^{\perp_{\mathbb{R}}} \cap H^1) &\longrightarrow \psi_t^{\perp_{\mathbb{R}}} \cap L^2, \\ (h, u) &\longmapsto P_t\big(Q(h)\psi_t + L_{p_t} u\big) \end{aligned}$$

is surjective. Then the approximations may be chosen as pairs $(\widetilde{p}_t, \varphi_t)$ with $L_{\widetilde{p}_t}\varphi_t = 0$ and $\|\varphi_t\|_2 = 1$, arbitrarily close to (p_t, ψ_t) in the prescribed seminorms and equal to it on the collars.

Proof. Choose a smooth scalar function σ, equal to zero and one on the two collars, and put $b_t = (1 - \sigma(t))p_0 + \sigma(t)p_1$. The remainder $p_t - b_t$ is flat at the inner boundaries of the collars. Cutting it off inside their complement changes any prescribed finite collection of seminorms by an arbitrarily small amount. A finite partition of unity in t and local Taylor polynomials of sufficiently high order approximate this remainder by $\sum_j f_j(t)e_j$, with every f_j supported off the collars. Adjoin p_0, p_1 and the prescribed potential directions to the span of the e_j.

Identify this coefficient space with $\mathbb{R}^N$. For every integer $s \ge 1$, the compared Dirac family is real analytic in z with values in $\mathcal{L}(H^s(\Sigma_{g_0}M), H^{s-1}(\Sigma_{g_0}M))$. The same holds for its real skew-adjoint realization $L(z)$ in the second assertion. Indeed, the comparison formula [61, Theorem 2.3.6, p. 25] involves matrix exponentials, inverses, positive square roots, their first spatial derivatives, and affine potentials. These operations are analytic at each sufficiently high finite C^k order.

For the second assertion, let $\iota : \mathbb{R}^q \to \mathbb{R}^N$ be the inclusion of the fixed potential directions, so that $L(z + \iota h) = L(z) + Q(h)$. Fix integers $s \geq 1$ and $k \geq 0$, and put

$$\mathcal{E}_t = \mathbb{R}^q \oplus (\psi_t^{\perp_{\mathbb{R}}} \cap H^s), \qquad \mathcal{F}_t = \psi_t^{\perp_{\mathbb{R}}} \cap H^{s-1}.$$

These are the fibres of smooth real Hilbert bundles $\mathcal{E}, \mathcal{F} \to I$. The maps $T_t(h, u) = P_t(Q(h)\psi_t + L_{p_t}u)$ from $\mathcal{E}_t$ to $\mathcal{F}_t$ are surjective. Indeed, for a target in H^{s-1}, an H^1 solution satisfies $L_{p_t}u = v - Q(h)\psi_t + c\psi_t$; elliptic regularity gives $u \in H^s$ [44, Theorem 16.1 and Remark 16.2]. Choose a bounded right inverse J_{t_0} of T_{t_0}. In smooth trivializations, $J_{t_0}(T_tJ_{t_0})^{-1}$ is a local right inverse. A partition of unity gives a smooth family of bounded right inverses $J_t : \mathcal{F}_t \to \mathcal{E}_t$ with $T_tJ_t = I$.

Let $\mathcal{B}_k$ be the Banach space of C^k maps $I \to \mathcal{L}(H^s, H^{s-1})$ vanishing on the prescribed collars, and let $\mathcal{W}_k$ be the Banach space of C^k sections of $\mathcal{F}$ vanishing there. Fix smooth trivializations on a finite cover of I and a subordinate partition of unity; the C^k section norms are computed in these trivializations. For $\Delta \in \mathcal{B}_k$ and $w \in \mathcal{W}_k$, write $(h_t, u_t) = J_t w_t$ and define

$$\mathcal{G}(\Delta, w)_t = P_t\big(L_{p_t} + \Delta_t + Q(h_t)\big)(\psi_t + u_t).$$

This is a smooth map $\mathcal{B}_k \times \mathcal{W}_k \to \mathcal{W}_k$. It satisfies $\mathcal{G}(0, 0) = 0$ and $D_w\mathcal{G}(0, 0) = I$. The Banach implicit function theorem [50, Theorem 13.20, p. 71] gives a solution $w = w(\Delta)$ near zero, with

$$\|w(\Delta)\|_{C^k(I,\mathcal{F})} \longrightarrow 0 \quad \text{as} \quad \|\Delta\|_{C^k(I,\mathcal{L}(H^s,H^{s-1}))} \longrightarrow 0.$$

Choose the coefficient approximation $z_0(t)$ so that $\Delta_t = L(z_0(t)) - L_{p_t}$ lies in this neighborhood. For smooth Δ, the pointwise derivative in w_t remains uniformly close to the identity. The same implicit function theorem, with parameter t, gives w smooth in t with values in H^{s-1}. Thus h and u are smooth in t, and they vanish on the collars.

Put $\widetilde{L}_t = L(z_0(t) + \iota h_t)$. The projected equation is $\widetilde{L}_t(\psi_t + u_t) = c_t\psi_t$ with $c_t \in \mathbb{R}$. Skew-adjointness and $u_t \perp_{\mathbb{R}} \psi_t$ give

$$0 = \langle \widetilde{L}_t(\psi_t + u_t), \psi_t + u_t\rangle_{\mathbb{R}} = c_t.$$

Moreover, $\|\psi_t + u_t\|_2^2 = 1 + \|u_t\|_2^2$. Normalize $\psi_t + u_t$ in L^2 to obtain φ_t. The $C^k(I; H^s)$ corrections tend to zero with Δ; choosing s large enough and applying Sobolev embedding [44, Theorem 12.16] gives every prescribed spatial seminorm. The potential correction lies in the fixed finite span, so its prescribed coefficient seminorms tend to zero as well.

Fix t_0, and let Π_{t_0} be the real L^2 projection onto $\ker \widetilde{L}_{t_0}$. Its range consists of smooth spinors. The orthogonal range and kernel decomposition of the skew-adjoint Fredholm operator, followed by elliptic regularity, shows that

$$\widetilde{L}_{t_0} + \Pi_{t_0} : H^{r+1} \longrightarrow H^r$$

is bijective for every integer $r \geq 0$. Its inverse remains smooth near t_0 [50, Theorems 11.7 and 11.17, pp. 59–60]. Since

$$\varphi_t = (\widetilde{L}_t + \Pi_{t_0})^{-1}\Pi_{t_0}\varphi_t$$

and $\Pi_{t_0}\varphi_t$ is smooth in the finite-dimensional space $\ker \widetilde{L}_{t_0}$, the family φ_t is smooth with values in every Sobolev space. It is therefore jointly smooth on $I \times M$. □

Quaternionic Hermitian products are linear in the second argument; their real parts define $\perp_{\mathbb{R}}$.

Lemma 10.3. *For $0 \neq u \in \mathbb{H}^r$, $r \geq 1$,*

$$\{Qu : Q \in \mathfrak{sp}(r)\} = u^{\perp_{\mathbb{R}}}.$$

For $y \in u^{\perp_{\mathbb{R}}}$, a preimage depends smoothly on (u, y).

Proof. The equality is (7.4), whose fibre calculation holds in every quaternionic Hermitian space. For $y \in u^{\perp_{\mathbb{R}}}$, put $s = |u|^2$ and $b = u^*y \in \operatorname{Im}\mathbb{H}$. Then

$$Q(u, y) = \frac{yu^* - uy^*}{s} - \frac{ubu^*}{s^2} \tag{10.2}$$

satisfies $Q(u,y)^* = -Q(u,y)$ and, since $y^*u = -b$, $Q(u,y)u = y + ub/s - ub/s = y$. The formula is smooth for $u \neq 0$. □

Fix a nonempty open set $U \subset M$. In residue four let S, T be the fixed chiral spinor bundles, ordered so that

$$A_g \,:\, H^1(S) \longrightarrow L^2(T), \qquad \operatorname{ind}_{\mathbb{H}} A_g = -a, \qquad a = |\operatorname{ind}_{\mathbb{H}} D^+|.$$

Thus $A_{g,V} = A_g + V|_S$, with the convention at $a = 0$ of Section 8. For a quaternionic Hermitian bundle E, write $\mathfrak{sp}(E) = \{Q \in \operatorname{End}_{\mathbb{H}}(E) \,:\, Q^* = -Q\}$. Put

$$L_{g,V} = \begin{cases} A_{g,V}, & n \equiv 4 \pmod 8, \\ i(D_g + V), & n \equiv 5 \pmod 8. \end{cases}$$

In residue five this is quaternionic-linear and skew-adjoint. The incidence equation is $L_{g,V}\psi = 0$ for $\|\psi\|_2 = 1$, modulo the free right action of the unit quaternions $\mathrm{Sp}(1)$. Its target is $L^2(T)$ in residue four and $\psi^{\perp_{\mathbb{R}}} \cap L^2(\Sigma M)$ in residue five.

Lemma 10.4. *Let P be a compact smooth d-manifold with corners and finitely many embedded boundary faces. The faces include P and all nonempty intersections of boundary faces. Let (g_p, V_p) be a smooth family, and put $L_p = L_{g_p,V_p}$. Suppose $\ker L_p = 0$ near a closed subset $P_0 \subset P$. For $n \equiv 4, 5 \pmod 8$, there is an arbitrarily small smooth compatible potential family W_p, in the $C^\infty(P \times M)$ topology, supported in a common compact subset of U and zero near P_0, such that, with $\widetilde{L}_p = L_{g_p,V_p+W_p}$,*

$$\mathscr{I} = \{(p,\ell) \,:\, p \in P,\ \ell \subset \ker \widetilde{L}_p,\ \dim_{\mathbb{H}} \ell = 1\}$$

is a compact smooth manifold with corners of dimension

$$d - r, \qquad r = 4(a+1) \text{ if } n \equiv 4\ (8), \qquad r = 3 \text{ if } n \equiv 5\ (8). \tag{10.3}$$

For $\pi \,:\, \mathscr{I} \to P$, $\pi(p,\ell) = p$, one has $\partial\mathscr{I} = \pi^{-1}(\partial P)$. For every face F, $\pi^{-1}(F)$ is a manifold with corners of dimension $\dim F - r$. An incidence with negative indicated dimension is empty.

The family W_p may be zero on any prescribed closed collar contained in an open set on which the original incidence equation is transverse to zero on every face, including all face intersections. In particular, transverse endpoint families and their face restrictions admit unchanged collars.

Proof. A projective chart at the line of an L^2-unit spinor ψ is given near zero by

$$u \longmapsto [\psi + u], \qquad u \in \psi^{\perp_{\mathbb{H}}} \cap H^1, \qquad \psi^{\perp_{\mathbb{H}}} = \{u \,:\, \langle \psi, u\rangle_{L^2,\mathbb{H}} = 0\}.$$

Normalization in L^2 is smooth on nonzero H^1 vectors. Changing a unit representative from ψ to ψq, with $q \in \mathrm{Sp}(1)$, changes the target vector from v to vq. The incidence equation therefore defines a smooth section of the corresponding Hilbert bundle.

At a zero in residue four, the vertical derivative is

$$L_p \,:\, \psi^{\perp_{\mathbb{H}}} \cap H^1(S) \longrightarrow L^2(T).$$

The unrestricted quaternionic index is $-a$. Removing the kernel line $\psi\mathbb{H}$ from the domain leaves the range unchanged and subtracts four from the real index. The resulting index is $-4(a+1)$. In residue five the target is the real orthogonal complement of ψ. In a local trivialization the equation is $\Pi_\psi L_p \psi$, where Π_ψ is the real orthogonal projection. At a zero, its derivative is

$$L_p \,:\, \psi^{\perp_{\mathbb{H}}} \cap H^1(\Sigma M) \longrightarrow \psi^{\perp_{\mathbb{R}}} \cap L^2(\Sigma M).$$

The unrestricted real index is zero. Restriction of the domain subtracts four, and restriction of the target adds one; the vertical index is -3.

In either residue, let η in the target annihilate the vertical image, with respect to the real L^2 product. The annihilator of $\psi^{\perp_{\mathbb{H}}} \cap H^1$ is represented by $\psi\mathbb{H}$. Hence $L_p^*\eta = \psi q_0$ distributionally for some $q_0 \in \mathbb{H}$. For every $q \in \mathbb{H}$,

$$\operatorname{Re}\langle \psi q, L_p^*\eta\rangle_{L^2} = \operatorname{Re}\langle L_p(\psi q), \eta\rangle_{L^2} = 0.$$

Taking $q = q_0$ gives $q_0 = 0$. Elliptic regularity makes η smooth [44, Theorem 16.1 and Remark 16.2]. In residue five, $L_p^* = -L_p$, so also $L_p\eta = 0$. In residue four, adjoining a zero opposite chiral component identifies sections of $\ker L_p$ and $\ker L_p^*$ with kernel spinors of $D_{g_p} + V_p$. In residue five, $\ker L_p = \ker(D_{g_p} + V_p)$. The unique continuation statement of Section 9 therefore applies to ψ and η.

Choose a closed coordinate ball $K_0 \Subset U$ with nonempty interior. In residue four, suppose in addition that η annihilates $B\psi$ for every $B \in C_c^\infty(\operatorname{int} K_0; \operatorname{Hom}_{\mathbb{H}}(S, T))$. These are precisely the chiral blocks of the permitted potential directions, by (2.3). Unique continuation gives a connected ball $O \Subset \operatorname{int} K_0$ on which $\psi \neq 0$. For $\chi \in C_c^\infty(O; \mathbb{R})$, the field

$$B(v) = \chi\eta\,\frac{\langle \psi, v\rangle_{\mathbb{H}}}{|\psi|^2}$$

extends by zero to a permitted direction and satisfies $B\psi = \chi\eta$. Consequently $\operatorname{Re}\langle \eta, \chi\eta\rangle_{L^2} = 0$ for every such χ, whence $\eta|_O = 0$. Unique continuation for L_p^* gives $\eta = 0$ on M.

In residue five, multiplication by i identifies the compatible potential directions with $\mathfrak{sp}(\Sigma M)$. Suppose η also annihilates $Q\psi$ for every $Q \in C_c^\infty(\operatorname{int} K_0; \mathfrak{sp}(\Sigma M))$. Multiplying local skew-adjoint fields by real test functions gives

$$\eta(x) \perp_{\mathbb{R}} \{Q\psi(x) \,:\, Q \in \mathfrak{sp}(\Sigma_x M)\} \qquad (x \in \operatorname{int} K_0).$$

On a connected ball $O \Subset \operatorname{int} K_0$ where $\psi \neq 0$, Lemma 10.3 gives

$$\eta = f\psi, \qquad f = \frac{\operatorname{Re}\langle \psi, \eta\rangle}{|\psi|^2} \in C^\infty(O; \mathbb{R}).$$

The operator L_p is first-order elliptic on the underlying real bundle, $L_p^* = -L_p$, and $\eta, \psi \in \ker L_p$. By [41, proof of Remark 1.1.14(3), p. 8] and the weak unique continuation statement of Section 9, one has $\eta = c\psi$ on M for some $c \in \mathbb{R}$. Since $\eta \in \psi^{\perp_{\mathbb{R}}}$ in L^2 and $\|\psi\|_2 = 1$, one has $c = 0$.

For a compact smooth coefficient family, the elliptic estimate [44, Theorem 16.7 and Remark 16.2] gives

$$\|\psi\|_{H^{s+1}} \leq C_s\big(\|L_p\psi\|_{H^s} + \|\psi\|_2\big), \qquad s \geq 0, \tag{10.4}$$

uniformly in p. Indeed, if $C_{s,p_0}\|L_p - L_{p_0}\|_{H^{s+1}\to H^s} < 1/2$, the estimate at p_0 holds with twice its constant; a finite cover gives C_s. Normalized kernel spinors therefore have uniform bounds at every Sobolev order. Given (p_j, ψ_j) in the normalized incidence, pass to $p_j \to p$. The uniform H^2 bound and the compact embedding $H^2 \hookrightarrow H^1$ [44, Theorem 16.10] give a subsequence converging in H^1 to ψ. Then

$$\|L_p\psi\|_2 \leq \|L_p(\psi - \psi_j)\|_2 + \|(L_p - L_{p_j})\psi_j\|_2 \longrightarrow 0, \qquad \|\psi\|_2 = 1.$$

Thus the normalized incidence and its quotient by $\mathrm{Sp}(1)$ are compact.

The vertical operators are Fredholm. The preceding annihilator arguments therefore imply that finitely many smooth potential directions supported in K_0 make the vertical differential surjective at any given zero. Surjectivity is open. A finite cover of the compact incidence gives finitely many such collections. Multiply each collection by a smooth parameter cutoff equal to one on the corresponding smaller parameter neighborhood and zero near P_0. Such cutoffs exist because the original incidence is disjoint from a neighborhood of P_0. At every original zero, the spinor and potential variables alone give a surjective differential, also after restriction to any face of P.

Let $y \in \mathbb{R}^N$ be the coefficients of these potential directions. For all sufficiently small y, every zero lies in the neighborhood where that differential is surjective. Otherwise a sequence of zeros with $y_j \to 0$ outside it would have a subsequence converging to an original zero, by (10.4) for the compact coefficient family $|y| \leq 1$, a contradiction. In a corner chart, extend the section smoothly to an open neighborhood. The Banach implicit function theorem [50, Theorem 13.20] gives a universal zero manifold with corners of dimension $d + N - r$. At a point of a face, a complement on which the differential is invertible can be chosen among variables tangent to the smallest face, together with spinor and coefficient variables. All boundary coordinates remain free. Thus the corner strata are the

inverse images of those of P. Sard's theorem [82, Theorem 1.2, p. 862], applied to the projection to $\mathbb{R}^N$ on the interiors of its restrictions over all faces of P, gives a common regular value arbitrarily close to zero. There are finitely many such restrictions and each has a countable atlas, so the union of the critical-value sets has measure zero. Its fibre and all face restrictions have the stated dimensions and are compact by (10.4). A regular fibre of negative dimension is empty.

For the relative assertion, let C be the prescribed closed collar and O an open set of transversality on every face. Choose

$$C \subset O_1 \subset \overline{O_1} \subset O_2 \subset \overline{O_2} \subset O$$

and a smooth function θ equal to zero on O_1 and one on $P \setminus O_2$. Multiply the parameter cutoffs above by θ. Over $\overline{O_2}$, the differential in the original parameter and spinor variables is surjective on each closed face. Compactness of the zero set over that closed face and openness of surjectivity preserve this property for sufficiently small y. There are finitely many faces, so one coefficient neighborhood suffices for all of them, including their intersections. On $P \setminus O_2$, $\theta = 1$ and the potential directions give surjectivity. The same argument with the smallest face and the same regular-value argument give the relative assertion, with $W_p = 0$ on C. Transversality of an endpoint family and all its face restrictions persists on a sufficiently small collar by the same compactness and openness argument. □

For a closed connected smooth hypersurface in a connected smooth manifold, both without boundary, the following conclusion follows from [64, Chapter 4, Lemma 4.4, p. 107].

Lemma 10.5. *Let X be path connected and locally path connected, and let $S \subset X$ be nonempty, closed, and path connected. Suppose that every point of S has an open product neighborhood*

$$(N, N \cap S) \cong (V \times (-1, 1), V \times \{0\}),$$

where V is path connected. Then $X \setminus S$ has at most two path components.

Proof. Write $\partial_X A$ for the boundary of a subset A in X. Let A be a path component of $X \setminus S$. Since $X \setminus S$ is locally path connected, A is both open and closed in $X \setminus S$. Thus A is open in X and $\partial_X A \subset S$. Connectedness of X and $S \neq \emptyset$ give $\partial_X A \neq \emptyset$.

For $s \in \partial_X A$, take a product neighborhood N of s, and denote the two path components of $N \setminus S$ by N^+ and N^-. The set A meets one of them and therefore contains it. Every point of $N \cap S$ belongs to the closure of that component, hence $N \cap S \subset \partial_X A$. Thus $\partial_X A$ is open and closed in S. Connectedness of S gives $\partial_X A = S$.

Fix one product neighborhood N_0 at a point of S. Every path component of $X \setminus S$ meets N_0 and contains one of the two path components of $N_0 \setminus S$. Distinct path components are disjoint, so there are at most two. □

11. Integer and vanishing index differences

Let M^n be a closed connected spin manifold, $n \geq 3$, with fixed spin structure and positive smooth density μ. The operators are untwisted. The fixed-domain comparisons and the spaces $\mathcal{R}_\mu(M)$ and $\mathcal{P}_\mu$ are those of Section 8, with their C^∞ topologies. The potentials are arbitrary smooth compatible endomorphisms. By Proposition 8.2, with $U = M$, the inclusion $g \mapsto (g, 0)$ induces a bijection on invertible path components and preserves ordinary index difference. If $\alpha(M) = 0$, then $\mathcal{R}_\mu^{\mathrm{inv}}(M)$, and hence $\mathcal{P}_\mu^{\mathrm{inv}}$, are nonempty by [3, Theorem 1.1] and Lemma 8.1.

11.1. **The two self-adjoint cases.** Suppose $n \equiv 3, 7 \pmod 8$, and put

$$\mathbb{F} = \begin{cases} \mathbb{H}, & n \equiv 3 \pmod 8, \\ \mathbb{R}, & n \equiv 7 \pmod 8. \end{cases}$$

Fix a reference metric in $\mathcal{R}_\mu(M)$. Let S be its spinor bundle, regarded as a right quaternionic bundle in residue three and as the real fixed bundle of charge conjugation in residue seven. Put

$$\mathscr{H} = L^2(M; S, \mu), \qquad \mathscr{D} = H^1(M; S).$$

All operators and potentials are represented on this bundle by the fixed-domain comparisons. The Dirac operator is self-adjoint and $\mathbb{F}$-linear. The compatible potentials are precisely the fibrewise self-adjoint $\mathbb{F}$-linear endomorphisms; in particular, real scalar multiplication is compatible. Real analyticity of an operator family is understood in $\mathcal{L}(\mathscr{D}, \mathscr{H})$.

Lemma 11.1. *Let $L = D_g + V$, let $\lambda \in \mathbb{R}$, and let $E \subset \ker(L - \lambda)$ have $\mathbb{F}$-dimension two. If P_E is the orthogonal projection onto E, then*

$$\dim_{\mathbb{R}}\{P_E Q|_E \,:\, Q \text{ is a smooth compatible potential}\} \geq 3.$$

Proof. Elliptic regularity gives $E \subset C^\infty(M; S)$. Denote the displayed real vector space by W. It contains I_E. If $\dim_{\mathbb{R}} W \leq 2$, the real or quaternionic spectral theorem gives an orthonormal basis ψ, ϕ of E in which every member of W is diagonal: W is spanned by I_E and at most one self-adjoint matrix. Hence

$$\int_M \langle \psi, Q\phi \rangle_{\mathbb{F}}\, \mu = 0 \qquad \text{for every smooth compatible } Q. \tag{11.1}$$

Apply Lemma 6.5 with both section spaces equal to E and with the bundle of self-adjoint $\mathbb{F}$-linear endomorphisms of S. It follows that

$$\langle \psi(x), A\phi(x) \rangle_{\mathbb{F}} = 0 \quad \text{for every } x \in M \text{ and every self-adjoint } A \in \mathrm{End}_{\mathbb{F}}(S_x).$$

The weak unique continuation statement of Section 9, applied to $D_g + (V - \lambda I)$, makes the nonzero sets of ψ and ϕ open and dense. Choose x in their intersection. The union of $\psi(x)^\perp$ and $\phi(x)^\perp$ is a proper subset of S_x. There is therefore $v \in S_x$ such that

$$\langle \psi(x), vv^* \phi(x) \rangle_{\mathbb{F}} = \langle \psi(x), v \rangle_{\mathbb{F}} \langle v, \phi(x) \rangle_{\mathbb{F}} \neq 0.$$

The endomorphism vv^* is self-adjoint, a contradiction. □

Lemma 11.2. *Let (g_t, V_t^0), $0 \leq t \leq 1$, be a real-analytic path in $\mathcal{P}_\mu$, and put $L_t^0 = D_{g_t} + V_t^0$. Suppose that L_0^0 and L_1^0 are invertible. There is a smooth compatible potential W_t, arbitrarily small in $C^1([0,1]; L^\infty)$ and zero on endpoint collars, such that $L_t = D_{g_t} + V_t^0 + W_t$ has the following properties. The two consecutive ordered eigenvalues which bound zero on the initial collar remain in a fixed bounded spectral interval. Every $\mathbb{F}$-eigenvalue in this interval is simple on a compact middle interval, and L_t is invertible on the remaining parameter intervals.*

Proof. Choose invertible endpoint collars for L_t^0 and a compact interval $J \Subset (0,1)$ whose complement lies in those collars. Let $\chi \in C^\infty([0,1]; \mathbb{R})$ be zero on smaller endpoint collars and one near J, with transition regions inside the invertible collars.

For every smooth compatible perturbation satisfying

$$\|W\|_{C^1([0,1];L^\infty)} \leq 1, \qquad W = 0 \text{ on the smaller endpoint collars,} \tag{11.2}$$

the uniform estimate (2.7) gives, for every L^2-unit eigenvector ψ of $L_t^0 + W_t$ with eigenvalue λ,

$$\|\psi\|_{H^1} \leq C_0\big(\|L_t^0 \psi\|_2 + \|\psi\|_2\big) \leq C_1(1 + |\lambda|).$$

The norms of $\dot{L}_t^0 : H^1 \to L^2$ are uniformly bounded. The finite spectral reductions (6.1) and (6.2), Lemma 6.4, and [67, Chapter II, Theorems 5.4 and 6.8] give locally Lipschitz ordered eigenvalues. At almost every parameter their derivatives are eigenvalues of the compression of $\dot{L}_t^0 + \dot{W}_t$ to the corresponding eigenspace. Thus

$$|\dot{\lambda}(t)| \leq \|\dot{L}_t^0 \psi\|_2 + \|\dot{W}_t\|_\infty \leq C_2(1 + |\lambda(t)|) \tag{11.3}$$

for a suitable L^2-unit eigenvector ψ. In the quaternionic case apply the same results to the complex representation, in which each eigenvalue occurs twice its quaternionic multiplicity. The constants depend only on the reference path and the bound (11.2).

The spectra are discrete and unbounded in both directions, as in Section 8. Number the eigenvalues at $t = 0$, with multiplicity over $\mathbb{F}$, so that $\lambda_0(0) < 0 < \lambda_1(0)$ are consecutive. On every interval of ordered continuation, Gronwall's inequality gives

$$1 + |\lambda_j(t)| \le (1 + |\lambda_j(s)|)e^{C_2|t-s|}.$$

Each continued eigenvalue is bounded and Lipschitz up to any finite endpoint of its interval of definition, and finite spectral reduction extends it across that endpoint. The same argument backwards from any parameter shows that every eigenvalue continues to $t = 0$. The local orderings therefore give a $\mathbb{Z}$-indexed ordering on $[0, 1]$. Choose

$$\Lambda > \max_{j=0,1}(1 + |\lambda_j(0)|)e^{C_2}.$$

Then $|\lambda_j(t)| < \Lambda$ for $j = 0, 1$, independently of the potential directions chosen below.

Let $\mathscr{K}$ be the set of tuples (t, λ, ψ, ϕ) with $t \in J$, $|\lambda| \le \Lambda+1$, and (ψ, ϕ) an orthonormal $\mathbb{F}$-two-frame in $\ker(L_t^0 - \lambda)$. Estimate (10.4), applied to the compact smooth family $L_t^0 - \lambda$, bounds these spinors in H^2. The eigen-equations and orthonormality are closed conditions in $J \times [-\Lambda - 1, \Lambda + 1] \times H^1 \times H^1$. Rellich compactness therefore makes $\mathscr{K}$ compact in this space and hence in the corresponding L^2 topology.

If $\mathscr{K}$ is empty, take $W = 0$. Otherwise Lemma 11.1 supplies three compatible potentials with linearly independent compressions at each frame. Their matrix entries are continuous in the L^2 topology. A finite subcover gives smooth compatible potentials $Q_1, \dots, Q_N$ on the fixed bundle whose compressions span a real vector space of dimension at least three on every frame in $\mathscr{K}$. Put

$$L_{t,p} = L_t^0 + \chi(t) \sum_{j=1}^{N} p_j Q_j, \qquad p = (p_1, \dots, p_N) \in \mathbb{R}^N. \tag{11.4}$$

There are an open interval $I \supset J$, on which $\chi = 1$, and $\rho > 0$ such that, for $t \in I$, $|p| < \rho$, and every two-plane $E \subset \ker(L_{t,p} - \lambda)$ with $|\lambda| \le \Lambda + \frac{1}{2}$,

$$\dim_{\mathbb{R}} \operatorname{span}\{P_E Q_j|_E \,:\, 1 \le j \le N\} \ge 3.$$

Indeed, a contrary sequence with $p \to 0$ and $\operatorname{dist}(t, J) \to 0$ has a subsequence of orthonormal frames converging in L^2 to a frame in $\mathscr{K}$. The compression matrices converge, and the locus of rank at most two is closed, contrary to the selection of the Q_j. The compactness follows from the same elliptic estimates, since the Q_j are fixed smooth endomorphisms.

Shrink ρ so that all perturbations in (11.4) with $|p| < \rho$ satisfy (11.2), are as small as prescribed, and preserve invertibility on the endpoint collars. The family is real analytic on $I \times \{p \,:\, |p| < \rho\}$. The set

$$\mathscr{C} = \{(t, p, \lambda) \,:\, t \in J,\ |p| \le \rho/2,\ |\lambda| \le \Lambda,\ \lambda \in \operatorname{Spec} L_{t,p}\}$$

is compact by norm-resolvent continuity. On open parameter sets $O_\alpha \subset I \times \{p \,:\, |p| < \rho\}$ there are analytic self-adjoint $\mathbb{F}$-matrices $B_\alpha(t, p)$ whose spectra are isolated clusters of the spectrum of $L_{t,p}$, contained in $(-\Lambda - \frac{1}{2}, \Lambda + \frac{1}{2})$. Finitely many such reductions suffice: for every $(t, p, \lambda) \in \mathscr{C}$, some α satisfies $(t, p) \in O_\alpha$ and $\lambda \in \operatorname{Spec} B_\alpha(t, p)$. The contour formulas (6.1) and the polar frames (6.2) give these reductions; their analytic dependence follows from [67, Chapter VII, Theorem 1.7]. They preserve the real or quaternionic structure. On each eigenspace, Lemma 6.4 identifies the compression of dB_α with that of $dL_{t,p}$; a change of spectral frame contributes a commutator whose compression to an eigenspace is zero.

Suppose first that $\mathbb{F} = \mathbb{R}$. On every two-plane in a repeated eigenspace of B_α, the potential derivatives span all of $\operatorname{Sym}_2(\mathbb{R})$. In an orthonormal basis, the difference of the two diagonal entries and the off-diagonal entry are therefore independent real linear functionals of the parameter variation. Thus B_α satisfies condition SAH2 of [85, §1] at every repeated eigenvalue. By [85, Theorem A], its repeated-eigenvalue locus has meager codimension two in O_α. The definition in [85, §2], applied to

$(t, p) \mapsto p$, makes the image of this locus meager in $\mathbb{R}^N$. The union over the finitely many reductions is meager.

Suppose now that $\mathbb{F} = \mathbb{H}$. The complex matrix representing $B_\alpha(t, p) - \lambda$ has twice its quaternionic nullity. Hence the locus of quaternionic nullity at least two is a real-analytic determinantal set. On compact semianalytic boxes contained in the reduction domains, take finite real-analytic stratifications compatible with all nullity loci [24, Proposition 2.10 and Corollary 2.11]. Finitely many such boxes cover $\mathscr{C}$. On each stratum $\mathcal{T}$ of nullity at least two, the kernels have constant rank and form a smooth vector bundle. The spectral frames lift this bundle to $\mathscr{D}$. At a point of $\mathcal{T}$, put $F = \ker(L_{t,p} - \lambda)$ and let P_F be the orthogonal projection onto F. Differentiating a local kernel section along $\mathcal{T}$ gives

$$P_F \, d(L_{t,p} - \lambda)[\xi]\big|_F = 0, \qquad \xi \in T\mathcal{T}. \tag{11.5}$$

The ambient real-linear map on the left has rank at least three: its potential directions have this rank after compression to any two-plane in F. Thus

$$\dim_{\mathbb{R}} \mathcal{T} \le (N + 2) - 3 = N - 1.$$

Sard's theorem [82, Theorem 1.2, p. 862], in local coordinates on each stratum, shows that the image of its projection to $\mathbb{R}^N$ has measure zero.

In either case choose p, arbitrarily close to zero, outside the projected repeated-eigenvalue loci, and put $W_t = \chi(t) \sum_j p_j Q_j$. Every eigenvalue in $[-\Lambda, \Lambda]$ is simple over $\mathbb{F}$ for $t \in J$. The two distinguished eigenvalues remain in this interval by (11.3), and the endpoint collars remain invertible. □

Remark 11.3. For a fixed metric and any bounded spectral interval, an arbitrarily small smooth compatible potential makes every eigenvalue in that interval simple over $\mathbb{F}$. The finite-dimensional construction above applies with t omitted and the prescribed interval in place of $[-\Lambda, \Lambda]$. In the real case the conclusion follows from [85, Theorem A]. In the quaternionic case, (11.5) gives $\dim_{\mathbb{R}} \mathcal{T} \le (N + 1) - 3 = N - 2$, and Sard's theorem gives the conclusion.

Let (g_t, V_t), $0 \le t \le 1$, be a smooth path in $\mathcal{P}_\mu$, and put $L_t = D_{g_t} + V_t$. If L_0 and L_1 are invertible, let $\mathrm{sf}_{\mathbb{F}}(L_\bullet)$ denote its spectral flow, counted positively from negative to positive and with multiplicity over $\mathbb{F}$. For $q_0, q_1 \in \mathcal{P}_\mu^{\mathrm{inv}}$, write $\mathrm{sf}_{\mathbb{F}}(q_0, q_1)$ for the spectral flow of a smooth pair path from q_0 to q_1. The common-domain continuity of Section 8.1, the contractibility of $\mathcal{P}_\mu$, and homotopy invariance [28, Remark 5.2(2) and Section 6.2] make it independent of the path. For pure metric endpoints, write $\mathrm{sf}_{\mathbb{F}}(g_0, g_1)$.

Lemma 11.4. *If $q_0, q_1 \in \mathcal{P}_\mu^{\mathrm{inv}}$ and $\mathrm{sf}_{\mathbb{F}}(q_0, q_1) = 0$, there is a smooth invertible pair path from q_0 to q_1. For pure metric endpoints its potential may vanish on endpoint collars.*

Proof. Write $q_j = (g_j, V_j)$ and $g_1 = g_0(G\cdot, \cdot)$, where G is positive and g_0-self-adjoint. Since $dv_{g_0} = dv_{g_1}$, one has $\det G = 1$. Transport both potentials to the fixed spinor bundle for g_0, and put

$$H = \log G, \qquad g_t = g_0(e^{tH}\cdot, \cdot), \qquad V_t^0 = (1 - t)V_0 + tV_1.$$

Then $\operatorname{tr} H = 0$, and (g_t, V_t^0) is an analytic path in $\mathcal{P}_\mu$. Put $L_t^0 = D_{g_t} + V_t^0$. Apply Lemma 11.2, and put $L_t = L_t^0 + W_t$. Let $(\lambda_j(t))_{j \in \mathbb{Z}}$ be its continuous ordered eigenvalue labels, counted over $\mathbb{F}$ and numbered so that $\lambda_0(0) < 0 < \lambda_1(0)$. These two labels are distinct on the compact middle interval.

Put $\nu(t) = \max\{j \in \mathbb{Z} : \lambda_j(t) < 0\}$. Choose a spectral-flow subdivision $0 = t_0 < \cdots < t_N = 1$ and numbers $a_i > 0$ with $\pm a_i \notin \operatorname{Spec} L_t$ for $t \in [t_{i-1}, t_i]$. The integer $k_i = \max\{j \in \mathbb{Z} : \lambda_j(t) < a_i\}$ is constant on that interval. The spectral projection onto $[0, a_i]$ satisfies

$$\dim_{\mathbb{F}} \operatorname{ran} \mathbf{1}_{[0,a_i]}(L_t) = k_i - \nu(t).$$

The spectral-projection formula [86, Section 3, equation (1)], with the multiplicities of Section 8.1, gives

$$\mathrm{sf}_{\mathbb{F}}(L_\bullet) = \sum_{i=1}^{N} \big(\nu(t_{i-1}) - \nu(t_i)\big) = \nu(0) - \nu(1).$$

The homotopy $L_t^0 + sW_t$, $0 \le s \le 1$, fixes the endpoints. Thus $\nu(1) = \nu(0) = 0$. On the final invertible collar, as on the initial collar, $\lambda_0(t) < 0 < \lambda_1(t)$. Together with simplicity on the middle interval, this gives $\lambda_0(t) < \lambda_1(t)$ for all $t \in [0,1]$.

Choose $\rho \in C^\infty([0,1];[0,1])$, supported in the invertible endpoint collars and equal to one near the endpoints. The continuous function

$$c_0(t) = (1-\rho(t))\frac{\lambda_0(t)+\lambda_1(t)}{2}$$

is zero near the endpoints and lies strictly between the two labels. Hence

$$\varepsilon = \min_{t\in[0,1]} \min\{c_0(t)-\lambda_0(t), \lambda_1(t)-c_0(t)\} > 0.$$

Relative smooth approximation [87, Corollary 12, p. 102], applied to $[0,1] \to \mathbb{R}$, gives a smooth c, equal to c_0 on smaller endpoint collars, with $\|c - c_0\|_\infty < \varepsilon/2$. Then

$$\lambda_0(t) < c(t) < \lambda_1(t), \qquad c(t) = 0 \text{ on the smaller endpoint collars.}$$

The operator $L_t - c(t)I$ is invertible. Its compatible potential $V_t^0 + W_t - c(t)I$ has the prescribed endpoint values. If $V_0 = V_1 = 0$, it vanishes on the smaller endpoint collars. □

Lemma 11.5. *Fix $g_* \in \mathcal{R}_\mu^{\mathrm{inv}}(M)$. For every $m \in \mathbb{Z}$ there is $g_m \in \mathcal{R}_\mu^{\mathrm{inv}}(M)$ such that $\mathrm{sf}_{\mathbb{F}}(g_*, g_m) = m$.*

Proof. For $m = 0$, take $g_m = g_*$. Suppose $m \neq 0$. By the two-sided unboundedness in Section 8.1, there is a bounded interval $\mathcal{I}$ whose boundary is disjoint from $\operatorname{Spec} D_{g_*}$ and which contains at least $|m|+1$ eigenvalues of each sign, counted over $\mathbb{F}$. By Remark 11.3, choose a compatible potential Q such that all eigenvalues of $A = D_{g_*} + Q$ in $\mathcal{I}$ are simple and

$$\|Q\|_\infty < \tfrac{1}{2}\min\Big\{\operatorname{dist}(0, \operatorname{Spec} D_{g_*}), \operatorname{dist}(\partial\mathcal{I}, \operatorname{Spec} D_{g_*})\Big\}.$$

The path $D_{g_*} + sQ$, $0 \le s \le 1$, has no eigenvalue at zero or at $\partial\mathcal{I}$. Its spectral projections on the negative and positive parts of $\mathcal{I}$ therefore have constant rank. In particular, A has at least $|m|+1$ eigenvalues of each sign in $\mathcal{I}$.

Let $\mu_1 < \cdots < \mu_{|m|+1}$ be the first $|m|+1$ positive eigenvalues of A, and let $\eta_1 < \cdots < \eta_{|m|+1}$ be the first $|m|+1$ positive eigenvalues of $-A$. They belong to $\mathcal{I}$ and $-\mathcal{I}$, respectively. Choose

$$c \in \begin{cases} (\eta_m, \eta_{m+1}), & m > 0, \\ (-\mu_{|m|+1}, -\mu_{|m|}), & m < 0. \end{cases}$$

The eigenvalues of $A + scI$ are $\lambda + sc$, with $\lambda \in \operatorname{Spec} A$. If $m > 0$, precisely the eigenvalues $-\eta_1, \dots, -\eta_m$ cross zero, all from negative to positive. If $m < 0$, precisely $\mu_1, \dots, \mu_{|m|}$ cross zero, all from positive to negative. Each has $\mathbb{F}$-multiplicity one, and $A + cI$ is invertible. Concatenation with the invertible path $D_{g_*} + sQ$, after stationary reparametrization, gives

$$q_m = (g_*, Q + cI) \in \mathcal{P}_\mu^{\mathrm{inv}}, \qquad \mathrm{sf}_{\mathbb{F}}((g_*, 0), q_m) = m.$$

Proposition 8.2, with $U = M$, a one-point parameter space, and empty relative subset, joins q_m through invertible pairs to $(g_m, 0)$ for some $g_m \in \mathcal{R}_\mu^{\mathrm{inv}}(M)$. Additivity [28, Remark 5.2(3) and Section 6.2] and path independence give $\mathrm{sf}_{\mathbb{F}}(g_*, g_m) = m$. □

Theorem 11.6. *If $n \equiv 3, 7 \pmod 8$ and $g_* \in \mathcal{R}_\mu^{\mathrm{inv}}(M)$, then*

$$\pi_0\mathcal{P}_\mu^{\mathrm{inv}} \longrightarrow \mathbb{Z}, \qquad [q] \longmapsto \mathrm{sf}_{\mathbb{F}}((g_*, 0), q),$$

and

$$\pi_0\mathcal{R}_\mu^{\mathrm{inv}}(M) \longrightarrow \mathbb{Z}, \qquad [g] \longmapsto \mathrm{sf}_{\mathbb{F}}(g_*, g),$$

are bijections of pointed sets. With the ordinary real index normalization, both maps are δ_{g_*}.

Proof. Spectral flow vanishes on an invertible path. By additivity, two pairs have equal invariants if and only if their relative spectral flow is zero. Lemma 11.4 then gives an invertible path between them, so the pair map is injective. Lemma 11.5 gives surjectivity. Proposition 8.2 identifies the metric and pair component sets. The normalization of Section 8.1, (8.1) and (8.3), gives

$$\delta_{g_*} = \mathrm{sf}_{\mathbb{F}}, \qquad \mathrm{sf}_{\mathbb{C}} = \begin{cases} 2\,\mathrm{sf}_{\mathbb{H}}, & n \equiv 3 \pmod 8, \\ \mathrm{sf}_{\mathbb{R}}, & n \equiv 7 \pmod 8. \end{cases}$$

□

Remark 11.7. The scalar correction in Lemma 11.4 is spatially constant. Theorem 11.6 concerns the full density slice $\mathcal{R}_\mu(M)$.

11.2. **The two complex chiral cases.** Suppose $n \equiv 2, 6 \pmod 8$. Let $\mathcal{B}_g$ be the bundle of independent chiral blocks in (2.3). By (2.4), it is a complex vector subbundle of $\mathrm{Hom}_{\mathbb{C}}(\Sigma_g^+ M, \Sigma_g^- M)$: in charge coordinates its matrices are alternating in residue two and symmetric in residue six. The compatible potential with block Z is $\mathcal{D}(Z)$.

Lemma 11.8. *In residues two and six, every nonempty fibre of the map*

$$\mathcal{P}_\mu^{\mathrm{inv}} \longrightarrow \mathcal{R}_\mu(M), \qquad (g, V) \longmapsto g,$$

is smoothly path connected. The paths may be stationary near their endpoints. If both endpoint potentials are supported in a compact set $K \subset M$, the connecting path may be supported in K.

Proof. Fix g and invertible endpoint potentials with blocks $Z_0, Z_1 \in C^\infty(M; \mathcal{B}_g)$. For $z \in \mathbb{C}$, put

$$Z(z) = (1 - z)Z_0 + zZ_1, \qquad A(z) = D_g^+ + Z(z) \,:\, H^1(\Sigma_g^+ M) \longrightarrow L^2(\Sigma_g^- M).$$

The endpoint operators $A(0)$ and $A(1)$ are isomorphisms. Moreover,

$$A(z)A(0)^{-1} = I + zT, \qquad T = (Z_1 - Z_0)A(0)^{-1} \in \mathcal{L}(L^2(\Sigma_g^- M)).$$

The operator T is compact by the compact embedding $H^1 \hookrightarrow L^2$ [44, Theorem 16.10]. Thus $I + zT$ is an entire Fredholm family, invertible at $z = 0$. The analytic Fredholm theorem [84, Theorem 1.1, p. 2] implies that

$$\mathcal{S} = \{z \in \mathbb{C} \,:\, A(z) \text{ is not invertible}\}$$

is closed and discrete. Since $0, 1 \notin \mathcal{S}$, choose a smooth curve $\gamma : [0, 1] \to \mathbb{C} \setminus \mathcal{S}$ from zero to one, constant near its endpoints. Then

$$D_g + \mathcal{D}(Z(\gamma(t))) = \begin{pmatrix} 0 & A(\gamma(t))^* \\ A(\gamma(t)) & 0 \end{pmatrix}$$

is self-adjoint and invertible. Complex linearity of $\mathcal{B}_g$ gives compatibility. If $\operatorname{supp} Z_0 \cup \operatorname{supp} Z_1 \subset K$, then $\operatorname{supp} \mathcal{D}(Z(\gamma(t))) \subset K$ for every t. □

Proposition 11.9. *Suppose $n \equiv 2, 6 \pmod 8$ and $\alpha(M) = 0$. Then $\mathcal{P}_\mu^{\mathrm{inv}}$ and $\mathcal{R}_\mu^{\mathrm{inv}}(M)$ are nonempty and smoothly path connected.*

Proof. Proposition 9.1, applied to the closed manifold M and a nonempty coordinate ball, gives an invertible potential over every metric. Let $(g_0, V_0), (g_1, V_1)$ be invertible pairs. Join g_0 to g_1 by a smooth path g_s in the density slice, constant near its endpoints. Positive spinor comparison identifies the potential bundles along this path.

For each s, choose a potential $V^{(s)}$ for which $D_{g_s} + V^{(s)}$ is invertible. Keeping its representative fixed gives invertible operators for parameters in an interval about s, by continuity in $\mathcal{L}(H^1, L^2)$ and [67, Chapter IV, Section 1.4, Theorem 1.16, p. 196]. Near zero and one choose V_0 and V_1, respectively. A Lebesgue subdivision $0 = s_0 < \cdots < s_k = 1$ subordinate to a finite such cover gives potentials

$W_1, \dots, W_k$ in the common bundle, with $W_1 = V_0$, $W_k = V_1$, such that $D_{g_s} + W_i$ is invertible for $s \in [s_{i-1}, s_i]$. At s_i, the potentials W_i and W_{i+1} lie in the same invertible fibre. Join them by Lemma 11.8.

Concatenate the paths $s \mapsto (g_s, W_i)$ with these fibre paths, after reparametrizing each piece to be constant near its endpoints. The resulting pair path is smooth. For pure metric endpoints its potential vanishes near zero and one. Proposition 8.2, with $U = M$, applied to this path with $B = [0, 1]$ and $B_0 = \{0, 1\}$, gives smooth path connectedness of $\mathcal{R}_\mu^{\mathrm{inv}}(M)$. □

11.3. **The two quaternionic incidence cases.** For $n \equiv 4, 5 \pmod 8$, let $L_{g,V}$ be the chiral operator or quaternionic skew-adjoint operator defined in Section 10. When $n = 4$ and $\alpha(M) = 0$, connectedness of $\mathcal{R}_\mu^{\mathrm{inv}}(M)$ follows from [2, Theorem 1.1] and Lemma 8.1.

Proposition 11.10. *Suppose $n \equiv 4, 5 \pmod 8$ and $\alpha(M) = 0$. Then $\mathcal{P}_\mu^{\mathrm{inv}}$ and $\mathcal{R}_\mu^{\mathrm{inv}}(M)$ are nonempty and smoothly path connected.*

Proof. Join two invertible pairs by a smooth path in the global metric and potential coordinates of Section 8, constant near its endpoints. Apply Lemma 10.4 with $P = [0, 1]$, $P_0 = \{0, 1\}$ and $U = M$. The original incidence is empty near P_0. In residue four, $\alpha(M) = 0$ gives $a = |\operatorname{ind}_{\mathbb{H}} D^+| = 0$ by (8.3). The perturbed incidence has dimension $1 - 4 = -3$ in residue four and $1 - 3 = -2$ in residue five; hence it is empty. The chiral operator in residue four is therefore injective of index zero, hence bijective. In residue five the full kernel is zero. Thus the perturbed path is invertible and unchanged near its endpoints. For pure metric endpoints its potential vanishes there. Proposition 8.2, with $U = M$, applied to this path with $B = [0, 1]$ and $B_0 = \{0, 1\}$, gives smooth path connectedness of $\mathcal{R}_\mu^{\mathrm{inv}}(M)$. □

Theorem 11.11. *Suppose $n \equiv 2, 4, 5, 6 \pmod 8$ and $\alpha(M) = 0$. Then $\mathcal{P}_\mu^{\mathrm{inv}}$ and $\mathcal{R}_\mu^{\mathrm{inv}}(M)$ are nonempty and path connected. For every $g_* \in \mathcal{R}_\mu^{\mathrm{inv}}(M)$, ordinary index difference induces bijections of pointed sets*

$$\delta_{g_*} : \pi_0 \mathcal{P}_\mu^{\mathrm{inv}} \longrightarrow KO_{n+1} = \{0\}, \qquad \delta_{g_*} : \pi_0 \mathcal{R}_\mu^{\mathrm{inv}}(M) \longrightarrow KO_{n+1} = \{0\},$$

with basepoints $[(g_*, 0)]$ and $[g_*]$, respectively.

Proof. Apply Propositions 11.9 and 11.10 and (8.3). □

12. Real chiral operators

Let M be a closed connected spin manifold of dimension $n \equiv 0 \pmod 8$, with $n \geq 8$ and $\alpha(M) = 0$, and fix a smooth positive density μ on M. Let g range over the metrics with volume density μ. In this section spinors, bundles, homomorphisms, and dimensions are real. The real half-spin bundles $\Sigma_g^{\pm}$ have rank

$$r = 2^{n/2-1}.$$

A compatible potential has the form

$$V_B = \begin{pmatrix} 0 & B^* \\ B & 0 \end{pmatrix}, \qquad B \in C^\infty\big(M; \operatorname{Hom}_{\mathbb{R}}(\Sigma_g^+, \Sigma_g^-)\big).$$

Here B^* is the fibrewise adjoint. Put

$$A_{g,B} = D_g^+ + B : H^1(\Sigma_g^+) \longrightarrow L^2(\Sigma_g^-).$$

Multiplication by B is compact between these spaces [44, Proposition 12.9 and Theorem 16.10]. Hence

$$\operatorname{ind}_{\mathbb{R}} A_{g,B} = \operatorname{ind}_{\mathbb{R}} D_g^+ = \alpha(M) = 0,$$

with the normalization of (8.3). The operator $D_g + V_B$ is invertible if and only if $A_{g,B}$ is an isomorphism. For a fixed metric g write

$$\mathcal{P}_g = C^\infty\big(M; \operatorname{Hom}_{\mathbb{R}}(\Sigma_g^+, \Sigma_g^-)\big), \qquad \mathcal{P}_g^{\mathrm{inv}} = \{B \in \mathcal{P}_g : A_{g,B} \text{ is an isomorphism}\}.$$

The map $(g, B) \mapsto (g, V_B)$ identifies the total space of these potentials over metrics with density μ with $\mathcal{P}_\mu$. Positive spin comparison from Section 2 preserves the real structures, the grading, and the L^2 inner products defined by μ.

All coefficient spaces carry the C^∞ topology. Potential variations are arbitrary smooth sections on M; relative families agree on prescribed collars in the parameter manifold.

Theorem 12.1. *For every metric g with density μ, the space $\mathcal{P}_g^{\mathrm{inv}}$ has exactly two path components. The space $\mathcal{P}_\mu^{\mathrm{inv}}$ also has exactly two path components. After the choice of an invertible basepoint, the ordinary real index difference induces a bijection of pointed sets from either component set to*

$$KO_{n+1} \cong KO_1 \cong \mathbb{Z}/2.$$

12.1. **Incidence and smooth Banach parameters.** Fix an integer $s > n/2 + 1$ and a reference metric with density μ. Write $\Sigma^\pm$ for its real half-spin bundles. Let P be a compact smooth manifold. After positive spin comparison, a smooth family $(g_t, B_t)_{t\in P}$ gives a smooth family

$$A_t \,:\, H^s(\Sigma^+) \longrightarrow H^{s-1}(\Sigma^-) \tag{12.1}$$

of Fredholm operators of index zero. Here A_t is the transported operator A_{g_t,B_t}, and A_t^* denotes its formal $L^2(\mu)$ adjoint. We identify operators, potentials, and spinors with their transports to these reference bundles. Elliptic regularity and the Fredholm alternative [44, Theorems 16.1 and 16.13; Remark 16.2] give $\ker A_t \subset C^\infty(\Sigma^+)$, $\ker A_t^* \subset C^\infty(\Sigma^-)$, and

$$\operatorname{ran} A_t = \{f \in H^{s-1}(\Sigma^-) \,:\, \langle f, \eta\rangle_{L^2} = 0 \text{ for every } \eta \in \ker A_t^*\}.$$

The L^2 orthogonal projection onto $\ker A_t^*$ therefore induces an isomorphism $\operatorname{coker} A_t \simeq \ker A_t^*$.

The unique continuation theorem in Section 9, applied to $(\psi, 0)$ and $(0, \eta)$ for $D_g + V_B$, applies respectively to $A_{g,B}$ and $A_{g,B}^*$. Thus the nonzero locus of every nonzero solution is open and dense. For every integer $j \geq 0$,

$$\|u\|_{H^{j+1}} \leq C_j\big(\|A_t u\|_{H^j} + \|u\|_{L^2}\big), \qquad u \in H^{j+1}(\Sigma^+), \tag{12.2}$$

with C_j uniform over the compact parameter manifold, by [44, Theorem 16.7 and Remark 16.2] and the perturbation estimate following (10.4).

Lemma 12.2. *Let $A = A_{g,B}$ and let $\psi \in \ker A$ satisfy $\|\psi\|_{L^2} = 1$. Put*

$$T_\psi = \{u \in H^s(\Sigma^+) \,:\, \langle u, \psi\rangle_{L^2} = 0\}.$$

The differential of $(B, \psi) \mapsto A_{g,B}\psi$ on the L^2 unit sphere is

$$(Q, u) \longmapsto Au + Q\psi, \qquad Q \in \mathcal{P}_g, \quad u \in T_\psi. \tag{12.3}$$

There is a finite-dimensional subspace $\mathcal{Q} \subset \mathcal{P}_g$ such that the restriction to $\mathcal{Q} \times T_\psi$ is surjective and has a bounded right inverse. The restriction to T_ψ is Fredholm of index -1.

Proof. Since $A\psi = 0$, one has $A(T_\psi) = \operatorname{ran} A$ and $\dim\ker(A|_{T_\psi}) = \dim\ker A - 1$. Thus $\operatorname{ind}(A|_{T_\psi}) = -1$.

Apply Lemma 6.5 with $F = \mathbb{R}\psi$, $E = \ker A^*$, $\mathcal{V} = \operatorname{Hom}_{\mathbb{R}}(\Sigma^+, \Sigma^-)$, and $U = M$. If $\eta \in E$ satisfies $\langle \eta, Q\psi\rangle_{L^2} = 0$ for every $Q \in \mathcal{P}_g$, then

$$\langle \eta(x), Q_x\psi(x)\rangle = 0 \qquad \big(x \in M, \quad Q_x \in \operatorname{Hom}_{\mathbb{R}}(\Sigma_x^+, \Sigma_x^-)\big).$$

For $\psi(x) \neq 0$, choose Q_x with $Q_x\psi(x) = \eta(x)$. Then $\eta(x) = 0$. The nonzero locus of ψ is dense, so $\eta = 0$. Consequently the potential directions map onto $\operatorname{coker} A$. Choose $Q_1, \dots, Q_N \in \mathcal{P}_g$ whose images span this cokernel, and put $\mathcal{Q} = \operatorname{span}_{\mathbb{R}}\{Q_1, \dots, Q_N\}$. The restriction of (12.3) to $\mathcal{Q} \times T_\psi$ is a surjective bounded operator between Hilbert spaces and therefore has a bounded right inverse. □

Lemma 10.1 supplies a separable Banach space of smooth potential variations, including variations which vanish on prescribed endpoint collars of a parameter interval.

12.2. **Evaluation at a spinor zero.** For nonzero Dirac eigenvalues of quaternionic multiplicity one in dimensions two and three, surjectivity of the differential of evaluation at a zero under conformal metric variation is proved in [61, Lemma 6.2.12].

Lemma 12.3. *Let $A = A_{g,B}$, let $\psi \in C^\infty(\Sigma^+)$ satisfy $A\psi = 0$ and $\|\psi\|_{L^2} = 1$, and let $x \in M$ satisfy $\psi(x) = 0$. The map*

$$(Q, u) \longmapsto u(x)$$

from the pairs $(Q, u) \in \mathcal{P}_g \times C^\infty(\Sigma^+)$ satisfying

$$Au + Q\psi = 0, \qquad \langle u, \psi\rangle_{L^2} = 0$$

to Σ^+_x is surjective.

Proof. Put $K = \ker A$ and $C = \ker A^*$, and let P_K, P_C be the L^2 orthogonal projections onto these spaces. Let

$$G : L^2(\Sigma^-) \longrightarrow H^1(\Sigma^+) \cap K^\perp$$

be the inverse of $A : H^1(\Sigma^+) \cap K^\perp \to C^\perp$, extended by zero on C. Then

$$AG = 1 - P_C, \qquad GA = 1 - P_K.$$

The first identity holds on $L^2(\Sigma^-)$ and the second on $H^1(\Sigma^+)$. Elliptic regularity gives bounded maps $G : H^j(\Sigma^-) \to H^{j+1}(\Sigma^+)$ for every integer $j \geq 0$ [44, Theorems 16.1, 16.7, and 16.13; Remark 16.2]. Thus G is continuous on smooth sections. The fibre metrics and μ identify distributional sections with continuous linear functionals on smooth sections. The transposes of G and P_K therefore act on distributions; denote them by G^* and P_K. They satisfy

$$A^*G^* = 1 - P_K.$$

Choose an open ball U with $\overline{U} \subset M \setminus \{x\}$ on which ψ is nowhere zero. Write ψ^* for its fibrewise metric dual. For $q \in C^\infty_c(U; \Sigma^-)$ with $q \perp C$, put

$$u = Gq, \qquad Q = -q \otimes \frac{\psi^*}{|\psi|^2}$$

on U, extending Q by zero. Then u is smooth, $Au = q = -Q\psi$, and $u \perp K$. The pairs $(0, u)$ with $u \in K \cap \psi^\perp$ also satisfy the two equations of the lemma.

Let $\zeta \in (\Sigma^+_x)^*$ annihilate every such value $u(x)$. Define $\delta_x\zeta$ by $\langle \delta_x\zeta, u\rangle = \zeta(u(x))$. Since $K = \mathbb{R}\psi \oplus (K \cap \psi^\perp)$ and $\psi(x) = 0$,

$$P_K(\delta_x\zeta) = 0.$$

Consequently $\eta = G^*(\delta_x\zeta)$ satisfies

$$A^*\eta = \delta_x\zeta. \tag{12.4}$$

Choose an orthonormal basis $\phi_1, \dots, \phi_m$ of C and set

$$\Lambda(q) = (\langle \phi_1, q\rangle_{L^2}, \dots, \langle \phi_m, q\rangle_{L^2}), \qquad q \in C^\infty_c(U; \Sigma^-).$$

The functional $q \mapsto \langle \eta, q\rangle$ vanishes on $\ker \Lambda$, since $\langle \eta, q\rangle = \zeta((Gq)(x))$. It is therefore $a \cdot \Lambda$ for some $a \in \mathbb{R}^m$. For $\phi = \sum_{j=1}^m a_j\phi_j \in C$ this gives $\eta|_U = \phi|_U$.

By (12.4), η is smooth on $M \setminus \{x\}$. This manifold is connected, since M is connected and $n \geq 2$. Unique continuation for A^* gives $\eta = \phi$ there. Hence $T = \eta - \phi$ is supported at x and satisfies $A^*T = \delta_x\zeta$. The principal symbol of A^* is invertible at every nonzero real covector. Lemma 2.3 gives $T = 0$ and $\zeta = 0$. Since Σ^+_x is finite dimensional, the evaluation map is surjective. □

Lemma 12.4. *Let $B_0 \in \mathcal{P}_g$ and $\psi_0 \in C^\infty(\Sigma^+)$ satisfy*

$$A_{g,B_0}\psi_0 = 0, \qquad \|\psi_0\|_{L^2} = 1, \qquad \dim\ker A_{g,B_0} = 1.$$

For every C^∞ neighborhood $\mathcal{U}$ of (B_0, ψ_0) in $\mathcal{P}_g \times C^\infty(\Sigma^+)$, there is a smooth path (B_t, ψ_t), $0 \le t \le 1$, in $\mathcal{U}$, beginning at (B_0, ψ_0), such that

$$A_{g,B_t}\psi_t = 0, \qquad \|\psi_t\|_{L^2} = 1, \qquad \dim\ker A_{g,B_t} = 1,$$

and ψ_1 is transverse to the zero section. The path is constant near both endpoints. For $n \ge 16$ the section ψ_1 is nowhere zero; for $n = 8$ its zero set is finite.

Proof. Put $A_0 = A_{g,B_0}$ and $Z = \psi_0^{-1}(0)$. If $Z = \emptyset$, the constant path suffices. For each $x \in Z$, Lemma 12.3 gives finitely many smooth pairs (Q, u) whose values $u(x)$ span Σ^+_x. These values still span the fibres near x. Since Z is compact, finitely many pairs suffice for all $x \in Z$. Let $\mathcal{E} \subset \mathcal{P}_g$ be a finite-dimensional space containing their potential components and the directions of Lemma 12.2. Choose a basis $Q_1, \dots, Q_N$ of $\mathcal{E}$, and put

$$A_b = A_0 + \sum_{j=1}^{N} b_j Q_j, \qquad b \in \mathbb{R}^N, \qquad \mathbb{S}^s = \{v \in H^s(\Sigma^+) : \|v\|_{L^2} = 1\}.$$

The differential of $(b, v) \mapsto A_b v$ on $\mathbb{R}^N \times \mathbb{S}^s$ at $(0, \psi_0)$ is split surjective, with Fredholm index $N - 1$. The implicit function theorem [50, Theorem 13.20] therefore gives a smooth manifold

$$\mathcal{I} = \{(b, v) : A_b v = 0,\ v \in \mathbb{S}^s\}$$

near $(0, \psi_0)$, of dimension $N - 1$. Choose a chart

$$\Theta : W \longrightarrow \mathcal{I}, \qquad \Theta(a) = (b(a), \psi_a), \qquad \Theta(0) = (0, \psi_0),$$

where W is an open ball about 0 in $\mathbb{R}^{N-1}$.

Choose an L^2-unit generator ϕ_0 of $\ker A_0^*$; this kernel has dimension one because $\operatorname{ind} A_0 = 0$. In the decompositions

$$H^1(\Sigma^+) = \mathbb{R}\psi_0 \oplus (H^1(\Sigma^+) \cap \psi_0^\perp), \qquad L^2(\Sigma^-) = \mathbb{R}\phi_0 \oplus \phi_0^\perp,$$

Lemma 6.6 gives $\dim\ker A_b \le 1$ for b sufficiently small. The unit spinor at each point of $\mathcal{I}$ makes this dimension equal to one.

Define

$$T_b u = A_b u + \langle u, \psi_0\rangle_{L^2}\phi_0.$$

The operator $T_0 : H^1(\Sigma^+) \to L^2(\Sigma^-)$ is bijective by the kernel and range decompositions of A_0. By [50, Theorem 11.7], it remains bijective for b in an open neighborhood $\mathcal{V}$ of 0. Take $\mathcal{V}$ inside the preceding neighborhood on which $\dim\ker A_b \le 1$, and shrink W so that $b(W) \subset \mathcal{V}$. Let $j \ge 0$ be an integer. If $b \in \mathcal{V}$, $f \in H^j(\Sigma^-)$, and $T_b u = f$, then $A_b u = f - \langle u, \psi_0\rangle_{L^2}\phi_0$. Elliptic regularity therefore makes

$$T_b : H^{j+1}(\Sigma^+) \longrightarrow H^j(\Sigma^-)$$

bijective for every integer $j \ge 0$ on the same neighborhood $\mathcal{V}$. Its inverse is smooth in b in each of these operator norms [50, Theorem 11.17]. The identity

$$\psi_a = \langle \psi_a, \psi_0\rangle_{L^2} T^{-1}_{b(a)}\phi_0$$

therefore makes $a \mapsto \psi_a$ smooth with values in every Sobolev space. Sobolev embedding gives a smooth map

$$F : W \times M \longrightarrow \Sigma^+, \qquad F(a, x) = \psi_a(x).$$

At $(0,x)$, $x \in Z$, the parameter derivative of F maps onto the fibre Σ^+_x. Compactness of Z and openness of surjectivity give an open neighborhood O of Z and, after shrinking W, transversality of F to the zero section on $W \times O$. If $M \setminus O$ is nonempty, shrink W further so that

$$\|\psi_a - \psi_0\|_{C^0} < \frac{1}{2} \min_{M \setminus O} |\psi_0| \qquad (a \in W).$$

Thus F is transverse to the zero section on $W \times M$. By continuity in the C^∞ topology, take W small enough that its image in $\mathcal{P}_g \times C^\infty(\Sigma^+)$ lies in $\mathcal{U}$. Parametric transversality [64, Chapter 3, §2, Theorem 2.7, pp. 79–80] gives parameters a_* arbitrarily close to 0 for which ψ_{a_*} is transverse to zero. Here W and M have no boundary, and F is smooth.

Choose a smooth $\rho : [0,1] \to [0,1]$ equal to 0 near 0 and to 1 near 1. Set

$$(b_t, \psi_t) = \Theta(\rho(t) a_*), \qquad B_t = B_0 + \sum_{j=1}^{N} b_{t,j} Q_j.$$

Convexity of W gives the required path. The zero set of a transverse section has codimension r. Since $r > n$ for $n \geq 16$, this zero set is empty in those dimensions. For $n = 8$, $r = n$, so it is a compact zero-dimensional submanifold and is finite. □

12.3. **Compatible first derivatives in dimension eight.** Let V be an oriented Euclidean space of dimension eight, let $E^\pm$ be its real half-spin modules, and let $c : V \to \mathrm{End}_{\mathbb{R}}(E^+ \oplus E^-)$ be Clifford multiplication. Thus

$$c(v)c(w) + c(w)c(v) = -2\langle v, w\rangle I, \qquad c(v)^* = -c(v) \quad (v, w \in V).$$

For $T \in \mathrm{Hom}_{\mathbb{R}}(V, E^+)$ its Clifford contraction is

$$C(T) = \sum_{i=1}^{8} c(e_i) T(e_i) \in E^-, \tag{12.5}$$

where (e_i) is any orthonormal basis of V. If a spinor ψ vanishes at x, its derivative $d\psi_x = (\nabla\psi)_x$ is independent of the connection. Identifying T_xM and the spinor fibres with V and $E^\pm$ gives $(D^+_g \psi)(x) = C(d\psi_x)$. For every compatible multiplication potential B, the equation $((D^+_g + B)\psi)(x) = 0$ is therefore equivalent to $C(d\psi_x) = 0$.

For a Euclidean space W, write

$$\mathrm{Sym}_0(W) = \{S \in \mathrm{End}_{\mathbb{R}}(W) : S^* = S,\ \mathrm{tr}\, S = 0\}, \qquad \mathfrak{so}(W) = \{K \in \mathrm{End}_{\mathbb{R}}(W) : K^* = -K\}.$$

Fix a unit $\phi \in E^-$. The map

$$f : V \longrightarrow E^+, \qquad f(v) = c(v)\phi$$

is an isomorphism, since $\dim E^+ = 8$ and

$$\langle f(v), f(w)\rangle = -\frac{1}{2}\langle \phi, (c(v)c(w) + c(w)c(v))\phi\rangle = \langle v, w\rangle.$$

For $A \in \mathrm{End}_{\mathbb{R}}(V)$, write $A_{ij} = \langle e_i, Ae_j\rangle$. Write $T = f \circ B$, with $B \in \mathrm{End}_{\mathbb{R}}(V)$, and put $S = (B + B^*)/2$ and $K = (B - B^*)/2$. The infinitesimal spin action on E^- is

$$\rho(K) = \frac{1}{2} \sum_{i<j} K_{ji} c(e_i) c(e_j).$$

The Clifford relations give

$$\begin{aligned} C(f \circ B) &= -(\mathrm{tr}\, B)\phi + \sum_{i<j} (B_{ji} - B_{ij}) c(e_i) c(e_j) \phi \\ &= -(\mathrm{tr}\, B)\phi + 4\rho(K)\phi. \end{aligned} \tag{12.6}$$

Since $\rho(K)$ is skew-adjoint, $\langle \rho(K)\phi, \phi\rangle = 0$. Therefore

$$\begin{aligned} L &= \{B \in \mathrm{End}_{\mathbb{R}}(V) \,:\, C(f\circ B) = 0\} = \mathrm{Sym}_0(V) \oplus \mathfrak{s}_\phi, \\ \mathfrak{s}_\phi &= \{K \in \mathfrak{so}(V) \,:\, \rho(K)\phi = 0\}. \end{aligned} \tag{12.7}$$

The Lie algebra $\mathfrak{s}_\phi$ is the infinitesimal stabilizer of ϕ.

Lemma 12.5. *Let $m \geq 3$, and let E be a real linear subspace of $\mathfrak{sl}_m(\mathbb{R}) = \{A \in \mathrm{End}_{\mathbb{R}}(\mathbb{R}^m) \,:\, \mathrm{tr}\, A = 0\}$ containing $\mathrm{Sym}_0(\mathbb{R}^m)$. Every invertible $A \in E$ can be joined by a straight segment in $E \cap GL_m(\mathbb{R})$ to an invertible symmetric trace-free matrix.*

Proof. By the real Schur theorem [54, Theorem 7.4.1], there is an orthogonal matrix Q such that Q^*AQ is block upper triangular. Its diagonal blocks are nonzero real numbers $\lambda_1, \dots, \lambda_p$ and real two-by-two matrices with eigenvalues $a_\ell \pm i\beta_\ell$, where $\beta_\ell \neq 0$ and $1 \leq \ell \leq q$. Thus $p + 2q = m$. Choose real numbers b_j, r_ℓ such that

$$\mathrm{sign}\, b_j = \mathrm{sign}\, \lambda_j, \qquad r_\ell \neq 0, \qquad \sum_{j=1}^{p} b_j + 2\sum_{\ell=1}^{q} r_\ell = 0. \tag{12.8}$$

If $q = 0$, take $b_j = \lambda_j$. If $q = 1$, then $p \geq 1$; choose the b_j with the indicated signs and nonzero sum, and put $r_1 = -\frac{1}{2}\sum_j b_j$. If $q \geq 2$, choose the b_j with the indicated signs and choose nonzero $r_1, \dots, r_{q-2}$. With

$$\sigma = -\frac{1}{2}\sum_j b_j - \sum_{\ell=1}^{q-2} r_\ell,$$

take $r_{q-1} \notin \{0, \sigma\}$ and $r_q = \sigma - r_{q-1}$.

Let B' be the block diagonal matrix whose blocks, in the Schur ordering, are b_j and $r_\ell I_2$, and put $B = QB'Q^*$. Then B is symmetric and trace-free, so $B \in E$. For $0 \leq t \leq 1$, the matrix

$$Q^*((1-t)A + tB)Q = (1-t)Q^*AQ + tB'$$

is block upper triangular. Its real diagonal blocks have the nonzero values $(1-t)\lambda_j + tb_j$. The eigenvalues of its two-by-two diagonal blocks are

$$(1-t)(a_\ell \pm i\beta_\ell) + tr_\ell.$$

They are nonreal for $t < 1$ and nonzero at $t = 1$. Thus $(1-t)A + tB$ is invertible. It belongs to E by linearity. □

Proposition 12.6. *Write $GL_8^{\pm}(\mathbb{R}) = \{B \in GL_8(\mathbb{R}) \,:\, \pm\det B > 0\}$. Both sets*

$$L^+ = \{B \in L \,:\, \det B > 0\}, \qquad L^- = \{B \in L \,:\, \det B < 0\}$$

are nonempty and path connected. For either sign and every basepoint $B_0 \in L^{\pm}$, inclusion induces a surjection

$$\pi_1(L^{\pm}, B_0) \longrightarrow \pi_1\big(GL_8^{\pm}(\mathbb{R}), B_0\big) \cong \mathbb{Z}/2.$$

The paths and loops can be chosen smooth and constant near each of finitely many prescribed concatenation points.

Proof. Choose orthonormal e_1, e_2, e_3, put $c_j = c(e_j)$ for $1 \leq j \leq 3$, and define e_4 by

$$f(e_4) = c_3c_1c_2\phi.$$

Since the c_j are orthogonal and f is an isometry, $|e_4| = 1$. Moreover,

$$\begin{aligned} \langle e_1, e_4\rangle &= \langle \phi, c_2c_3\phi\rangle = 0, \\ \langle e_2, e_4\rangle &= -\langle \phi, c_1c_3\phi\rangle = 0, \\ \langle e_3, e_4\rangle &= \langle \phi, c_1c_2\phi\rangle = 0, \end{aligned}$$

because each $c_i c_j$, $i \neq j$, is skew-adjoint. Thus e_1, e_2, e_3, e_4 are orthonormal, and

$$c(e_3)c(e_4)\phi = -c(e_1)c(e_2)\phi.$$

Define $K \in \mathfrak{so}(V)$ by

$$Ke_1 = e_2, \quad Ke_2 = -e_1, \quad Ke_3 = e_4, \quad Ke_4 = -e_3, \qquad K|_{\langle e_1,e_2,e_3,e_4\rangle^\perp} = 0.$$

Then

$$\rho(K)\phi = \frac{1}{2}\big(c(e_1)c(e_2) + c(e_3)c(e_4)\big)\phi = 0.$$

Extend to an oriented orthonormal basis of V. In this basis, $K = J \oplus J \oplus 0_4$, where

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

For $1 \leq k \leq 7$, let $\boldsymbol{D}_k$ be the convex set of diagonal trace-free matrices whose first k entries are positive and whose remaining entries are negative. By the spectral theorem [54, Theorem 8.1.1], the symmetric trace-free matrices with k positive eigenvalues are exactly

$$\{QDQ^{-1} \,:\, Q \in SO(8),\ D \in \boldsymbol{D}_k\}.$$

An orthogonal diagonalizing matrix can be chosen with determinant one by changing the sign of one eigenvector. Since $SO(8)$ is smoothly path connected [58, Section 1.3.2], this set is path connected.

For $1 \leq k \leq 5$, choose $d_5, \dots, d_8 \in \{1, -1\}$ with $k-1$ positive entries and $5-k$ negative entries. Put

$$c_0 = \sum_{j=5}^{8} d_j = 2k - 6, \quad a(t) = -\cos(\pi t), \quad b(t) = \sin(\pi t), \quad h(t) = -a(t) - c_0/2 \qquad (0 \leq t \leq 1).$$

The path

$$B(t) = \operatorname{diag}\big(a(t), a(t), 5 + h(t), -5 + h(t), d_5, d_6, d_7, d_8\big) + b(t)K \tag{12.9}$$

belongs to L, since its symmetric part has trace $2a(t) + 2h(t) + c_0 = 0$ and $K \in \mathfrak{s}_\phi$. Its two-by-two blocks have determinants

$$a(t)^2 + b(t)^2 = 1, \qquad h(t)^2 - 25 + b(t)^2 \leq -15,$$

because $|h(t)| \leq 3$ and $|b(t)| \leq 1$. The other diagonal entries are ± 1, so $B(t)$ is invertible. The endpoints are symmetric. At $t = 0$ the first pair is negative, the second pair has one positive entry, and the last four entries contain $k-1$ positive entries. At $t = 1$ the first pair is positive and the other signs are unchanged. Hence the endpoints have k and $k+2$ positive eigenvalues.

It follows that the symmetric signatures are joined in the chains

$$1 \longleftrightarrow 3 \longleftrightarrow 5 \longleftrightarrow 7, \qquad 2 \longleftrightarrow 4 \longleftrightarrow 6.$$

The determinant sign for signature $(k, 8-k)$ is $(-1)^k$. Every invertible symmetric trace-free matrix has $1 \leq k \leq 7$. Lemma 12.5, applied to L, therefore proves that L^+ and L^- are path connected. Their respective elements

$$S_+ = \operatorname{diag}(1,1,1,1,-1,-1,-1,-1), \qquad S_- = \operatorname{diag}(7,-1,-1,-1,-1,-1,-1,-1)$$

prove nonemptiness.

For $S \in \{S_+, S_-\}$, choose, in this order, one positive and one negative coordinate. Let $R(t)$ rotate their plane through πt and fix its orthogonal complement. Then

$$S(t) = R(t)SR(t)^{-1} \qquad (0 \leq t \leq 1)$$

is a loop of symmetric trace-free invertible matrices, since $R(1)$ commutes with S. Put $P = S(S^2)^{-1/2}$. The real polar deformation retraction [58, Theorem 2.17 and Proposition 2.19(1)] sends $S(t)$ to

$R(t)PR(t)^{-1}$. Right multiplication by P is a homeomorphism from the component of P in $O(8)$ onto $SO(8)$; it sends this loop to rotation through $2\pi t$ on the chosen plane and the identity on its complement. By [58, Proposition 13.10 and the following paragraph], this is the nonzero element of $\pi_1 SO(8) \cong \mathbb{Z}/2$. Thus the inclusion of $L^{\pm}$ induces a surjection at $S_{\pm}$. Changing basepoint along a path in $L^{\pm}$ proves the assertion at every $B_0 \in L^{\pm}$.

Each path above is a finite concatenation of smooth paths. Composing each piece with a smooth nondecreasing map $[0,1] \to [0,1]$ which equals 0 near 0 and 1 near 1 makes the concatenation smooth and constant near its junctions. □

12.4. **Paths with a selected harmonic spinor.**

Proposition 12.7. *Let $g_j \in \mathcal{R}_\mu(M)$ and $B_j \in \mathcal{P}_{g_j}$, $j = 0, 1$, and let $\psi_j \in C^\infty(\Sigma^+_{g_j})$ satisfy*

$$\ker(D^+_{g_j} + B_j) = \mathbb{R}\psi_j, \qquad \|\psi_j\|_{L^2} = 1.$$

There is a smooth path (g_t, B_t, ψ_t), $0 \le t \le 1$, constant near both endpoints and with the prescribed endpoint values, such that

$$g_t \in \mathcal{R}_\mu(M), \qquad (D^+_{g_t} + B_t)\psi_t = 0, \qquad \|\psi_t\|_{L^2} = 1.$$

If $g_0 = g_1$, the path can be chosen with $g_t = g_0$ for every t.

Proof. Put $I = [0,1]$. Lemma 12.4 joins each endpoint pair to a pair whose spinor is transverse to zero, through normalized solutions with one-dimensional kernels and with the metric fixed. These paths are constant near their endpoints. It suffices to join the resulting pairs and concatenate; thus assume that ψ_0 and ψ_1 are transverse to zero.

Choose a smooth path $g_t \in \mathcal{R}_\mu(M)$, constant near its endpoints, by the coordinates of Section 8. If $g_0 = g_1$, choose the constant path. Let $W = M \times I$, and let $E^{\pm} \to W$ have fibre $\Sigma^{\pm}_{g_t,x}$ at (x,t). Positive spin comparison identifies these bundles with pullbacks of the reference bundles. In these identifications, choose a smooth section Ψ^0 of E^+ equal to ψ_j on the endpoint collars.

The section Ψ^0 is transverse to zero on those collars. Choose finitely many sections of E^+, supported away from smaller collars, whose values span every fibre over the compact complement of the original collars. The family obtained by adding their real linear combinations to Ψ^0 is transverse to zero for sufficiently small parameters: the parameter derivative is surjective on that complement, and transversality persists on the remaining compact collar regions. Parametric transversality [64, Chapter 3, §2, Theorem 2.7, pp. 79–80], applied on $M \times (0,1)$, gives a smooth section Ψ transverse to zero and equal to ψ_j on smaller endpoint collars.

Nowhere vanishing sections. For $n \ge 16$, the rank $r = 2^{n/2-1}$ exceeds $n + 1 = \dim W$, so Ψ has no zeros. More generally, whenever Ψ is nowhere zero, write $\Psi_t = \Psi|_{M\times\{t\}}$ and let Ψ_t^* denote its fibrewise metric dual. Set

$$B_t^{\mathrm{can}} = -D^+_{g_t}\Psi_t \otimes \frac{\Psi_t^*}{|\Psi_t|^2}. \tag{12.10}$$

Then $(D^+_{g_t} + B_t^{\mathrm{can}})\Psi_t = 0$. Let $\theta_j : I \to [0,1]$ be smooth, supported in the respective endpoint collars and equal to one on smaller collars. There $\Psi_t = \psi_j$ and $g_t = g_j$, so $(B_j - B_t^{\mathrm{can}})\Psi_t = 0$. Hence

$$B_t = B_t^{\mathrm{can}} + \sum_{j=0}^{1} \theta_j(t)(B_j - B_t^{\mathrm{can}})$$

has the required endpoint values and satisfies $(D^+_{g_t} + B_t)\Psi_t = 0$. The spinors $\Psi_t / \|\Psi_t\|_{L^2}$ give the required path.

The zero curve in dimension eight. Suppose now that $n = 8$ and $Z = \Psi^{-1}(0)$ is nonempty. It is a compact one-dimensional submanifold of W, with boundary in ∂W, and is the product of the finite endpoint zero sets with the endpoint collars. Write $t : W \to I$ for the second projection. The function $t|_Z$ has nonzero differential on those collars. By the density of Morse functions [64, Chapter 6, §1, Theorem 1.2, p. 147], applied on $Z \setminus \partial Z$, there is an arbitrarily small smooth function $h : Z \to \mathbb{R}$, zero

near ∂Z, such that $t|_Z + h$ is Morse. Indeed, multiply the difference between a Morse approximation and $t|_Z$ by a cutoff equal to one off the collars; sufficiently small C^1 error preserves $d(t|_Z + h) \neq 0$ on the transition regions.

Choose h small enough that

$$i_u : Z \longrightarrow W, \qquad i_u(x,t) = (x, t + uh(x,t)), \qquad 0 \leq u \leq 1,$$

is an isotopy of embeddings fixed on the collars. This follows from openness of embeddings in the C^1 topology [64, Chapter 2, §1, Theorem 1.4, p. 37]; the support of h is a compact subset of the interior of W. Double W and Z across their product collars and extend the isotopy by the identity on the second copy. The isotopy extension theorem [64, Chapter 8, §1, Theorem 1.3, p. 180] applies to the doubled closed submanifold. Its extending vector field can be multiplied by a cutoff equal to one on the support of the prescribed velocity and supported in the interior of the first copy. Thus the ambient isotopy is the identity near ∂W. Parallel transport for any smooth connection on E^+ lifts it to a bundle isotopy. Transporting Ψ by this lift gives a transverse section with zero set $i_1(Z)$, unchanged on the endpoint collars. Retain the notation Ψ, Z for this section and its zero set. The function $t|_Z$ is now Morse.

Let

$$NZ = TW|_Z / TZ, \qquad q : TW|_Z \longrightarrow NZ$$

be the normal quotient bundle and its quotient map. The derivative of Ψ at its zeros induces an isomorphism

$$F_0 : NZ \longrightarrow E^+|_Z. \tag{12.11}$$

Fix an orientation of the real positive half-spin module. Since Spin(8) is connected, it induces an orientation on E^+. Orient NZ so that F_0 preserves orientation. For any bundle homomorphism $F : NZ \to E^+|_Z$, and any local parametrization $s \mapsto z(s) = (x(s), t(s))$ of Z, put

$$A_F(s) = F_{z(s)} q|_{T_{x(s)}M}, \qquad b_F(s) = F_{z(s)} q(\partial_t).$$

Since q annihilates TZ,

$$A_F(s)x'(s) + b_F(s)t'(s) = 0. \tag{12.12}$$

Let C_g denote the Clifford contraction of (12.5) for the metric g. Call F compatible if

$$C_{g_{t(s)}}(A_F(s)) = 0 \tag{12.13}$$

at every point of Z.

Compatible normal isomorphisms. Where $t'(s) \neq 0$, vertical inclusion induces an isomorphism

$$\iota_s : T_{x(s)}M \longrightarrow N_{z(s)}Z.$$

Thus an invertible vertical map $A(s)$ determines the normal isomorphism $F_s = A(s)\iota_s^{-1}$, with $b_F(s) = -A(s)x'(s)/t'(s)$. In a spin frame on a regular arc, let ϕ be the fixed unit negative half-spinor in (12.7), and identify $\mathbb{R}^8$ with the positive half-spin module by $v \mapsto c(v)\phi$. In these source and target bases, the compatible vertical matrices form the fixed space L in (12.7). The determinant sign of ι_s is constant on that arc; hence normal isomorphisms of the same orientation have vertical matrices in the same component L^+ or L^-.

At a critical point of $t|_Z$, the Morse lemma [64, Chapter 6, §1, Theorem 1.1] gives a parameter s with $t(s) = t_0 + s^2$ or $t(s) = t_0 - s^2$. The vector $x'(s)$ is nonzero on a sufficiently small interval. Choose an oriented orthonormal frame with $e_1(s) = x'(s)/|x'(s)|_{g_{t(s)}}$, a smooth unit section $\phi^-(s)$ of $E^-|_Z$, and the isometry $f_s(v) = c_{g_{t(s)}}(v)\phi^-(s)$. For nonzero constants $\lambda_2, \dots, \lambda_8$ with $\sum_{j=2}^8 \lambda_j = 0$, set

$$\begin{aligned} a(s) &= \frac{t'(s)}{|x'(s)|_{g_{t(s)}}}, & b(s) &= -f_s(e_1(s)), \\ A(s)e_1(s) &= a(s)f_s(e_1(s)), & A(s)e_2(s) &= (\lambda_2 - a(s))f_s(e_2(s)), \\ A(s)e_j(s) &= \lambda_j f_s(e_j(s)) & & (3 \leq j \leq 8). \end{aligned} \tag{12.14}$$

The matrix $f_s^{-1}A(s)$ is symmetric and trace-free, so $C_{g_{t(s)}}(A(s)) = 0$ by (12.6). Moreover, $A(s)x'(s) + b(s)t'(s) = 0$. Shrink the interval so that $\lambda_2 - a(s)$ and $\lambda_3, \dots, \lambda_8$ are nonzero. At its critical point $A(s)$ has rank seven and $b(s)$ spans the missing image direction. At its other points $A(s)$ is invertible. Consequently $(A(s), b(s))$ is surjective and has kernel $T_{z(s)}Z$ throughout the interval. It induces a compatible normal isomorphism.

At the critical point, its matrix in the quotient basis $q(\partial_t), q(e_2), \dots, q(e_8)$ and the target basis $f_s(e_1), \dots, f_s(e_8)$ is $\operatorname{diag}(-1, \lambda_2, \dots, \lambda_8)$. The choices

$$(6, -1, -1, -1, -1, -1, -1), \qquad (5/2, 5/2, -1, -1, -1, -1, -1)$$

have zero sum and opposite product signs. One therefore gives the orientation of F_0. The same orientation persists on the whole interval. This construction applies to both signs in $t(s) = t_0 \pm s^2$.

On endpoint collars retain F_0: there it is compatible because $D^+_{g_j}\psi_j = -B_j\psi_j = 0$ at every zero of ψ_j. Choose the preceding normal isomorphisms on disjoint neighborhoods of the finitely many critical points. The complementary pieces of Z are regular arcs. Their prescribed endpoint matrices lie in the same $L^{\pm}$ by the determinant comparison above. In a spin trivialization, each prescribed end germ can be joined to a constant germ inside a convex ball contained in $L^{\pm}$. For an end germ $A(s)$, choose s_0 in its domain and a smooth $[0,1]$-valued function χ, zero on the prescribed smaller collar and one near the other end. After restricting the germ, $(1 - \chi(s))A(s) + \chi(s)A(s_0)$ stays in that ball. Proposition 12.6 then joins the two constant matrices by a smooth path constant near its endpoints. This gives a smooth compatible normal isomorphism F_1 with the orientation of F_0, equal to F_0 on the endpoint collars.

The homotopy class of the normal isomorphism. Every component of Z is an interval or a circle. The oriented bundle NZ is trivial on each component, by [64, Chapter 4, §1, Corollary 1.2, p. 89] for intervals and [60, §1.2, p. 25] for circles. In a trivialization, $F_0^{-1}F_1$ is a map into $GL_8^+(\mathbb{R})$, equal to the identity on the endpoint collars. On an interval its relative homotopy class belongs to $\pi_1 GL_8^+(\mathbb{R}) = \mathbb{Z}/2$. On a circle its free homotopy class belongs to the same group, because this group is abelian; see [59, Proposition 4A.2 and Example 4A.3, p. 422].

Each component of Z contains a regular arc. On a small subarc, make the vertical matrix of F_1 constant, say A_*, by a homotopy in a convex ball of $L^{\pm}$, fixed outside that subarc. Choose a still smaller interval J on which the matrix is A_*. In its spin trivialization, let $J_s : \mathbb{R}^8 \to N_{z(s)}Z$ be vertical inclusion followed by q. If $\gamma : J \to L^{\pm}$ is a loop based at A_* and constant near the ends of J, replacing the constant vertical matrix by γ changes the normal isomorphism from F_1 to $\widetilde{F}_1$, with

$$F_1^{-1}\widetilde{F}_1 = J_s A_*^{-1}\gamma(s)J_s^{-1} \qquad (s \in J).$$

This is the identity near the ends of J. After trivializing NZ, the path of conjugating matrices J_s contracts to a constant path. Conjugation acts trivially on $\mathbb{Z}/2$. Proposition 12.6 therefore permits either change of the relative normal-isomorphism class. Choose γ to cancel the class of $F_0^{-1}F_1$ on each component. The resulting compatible F_1 is homotopic to F_0 through orientation-preserving isomorphisms relative to the endpoint collars.

Make this homotopy constant in its parameter near 0 and 1. Relative smoothing [87, Theorem 11, p. 99], applied to the bundle of oriented isomorphisms on each connected component of $Z \times [0,1]$, gives a smooth homotopy F_u, $0 \le u \le 1$, with the same endpoints and fixed on smaller endpoint collars. Each base component is a connected compact manifold with corners, and the fibre is the finite-dimensional Lie group $GL_8^+(\mathbb{R})$.

Tubular coordinates. Choose a tubular neighborhood of Z which is a product on the endpoint collars [64, Chapter 4, §6, Theorems 6.3–6.4, pp. 114–116], and identify E^+ on it with the pullback of $E^+|_Z$. Choose bundle metrics on NZ and $E^+|_Z$. In normal coordinates (z, y), Taylor's formula gives

$$\Psi(z, y) = F_0(z)y + R(z, y), \qquad |R(z, y)| \le C|y|^2,$$

with C independent of $z \in Z$. Compactness gives

$$\delta = \min_{\substack{z \in Z,\ 0 \le u \le 1 \\ v \in N_z Z,\ |v| = 1}} |F_u(z)v| > 0.$$

Shrink the tube so that $|R(z,y)| \le \delta|y|/2$. Let $\chi : [0,\infty) \to [0,1]$ be smooth, equal to one near zero and zero near the squared radius of the tube. Set

$$\Psi'(z,y) = F_{\chi(|y|^2)}(z)y + R(z,y) \tag{12.15}$$

inside the tube, and $\Psi' = \Psi$ outside it. These definitions agree near the tube boundary. On the endpoint collars $F_u = F_0$, so $\Psi' = \Psi$. For $y \neq 0$,

$$|\Psi'(z,y)| \ge \delta|y| - |R(z,y)| \ge \tfrac{1}{2}\delta|y| > 0. \tag{12.16}$$

Thus $(\Psi')^{-1}(0) = Z$ and its normal derivative is F_1. In particular it is transverse to zero. Write $\Psi'_t = \Psi'|_{M\times\{t\}}$. Since F_1 is compatible,

$$D^+_{g_t}\Psi'_t(x) = 0 \qquad \big((x,t) \in Z\big). \tag{12.17}$$

Division by the spinor. Put $G(x,t) = -D^+_{g_t}\Psi'_t(x)$; this is a smooth section of E^- vanishing on Z. Near an interior point of Z, choose bundle trivializations. Transversality implies that the eight components of Ψ', denoted $z_1,\dots,z_8$, together with one coordinate σ, are local coordinates on W. Restrict to a product of an interval in σ and a ball about 0 in z. Write $G_1,\dots,G_8$ for the components of G in the target trivialization. Since $G_i(\sigma,0) = 0$, Hadamard's formula is

$$G_i(\sigma,z) = \sum_{j=1}^{8} z_j \int_0^1 \frac{\partial G_i}{\partial z_j}(\sigma,\tau z)\,d\tau. \tag{12.18}$$

The integrals are jointly smooth in (σ,z) and define a homomorphism $B : E^+ \to E^-$ with $B\Psi' = G$ on this neighborhood. On $W \setminus Z$, let Ψ'^* denote the fibrewise metric dual of Ψ'. The homomorphism $G \otimes \Psi'^*/|\Psi'|^2$ has the same property. On the endpoint collars the prescribed B_j also satisfies $B_j\Psi' = G$. A smooth partition of unity subordinate to these neighborhoods gives a global solution of $B\Psi' = G$. Choose it so that only the prescribed endpoint solution occurs on each smaller endpoint collar. Then $B_t = B|_{M\times\{t\}}$ is jointly smooth, has the prescribed endpoint values, and

$$(D^+_{g_t} + B_t)\Psi'_t = 0.$$

The coordinates in (12.18) are coordinates on W, including at a critical point of $t|_Z$.

No slice Ψ'_t is identically zero. Otherwise its spatial derivative would vanish on that slice and its total derivative would have rank at most one, contrary to transversality to a bundle of rank eight. Therefore $N(t) = \|\Psi'_t\|_{L^2}$ is smooth and positive. Set $\psi_t = N(t)^{-1}\Psi'_t$. Since N depends only on t, the spinor equation and the normalized endpoint values are preserved. Concatenating with the endpoint paths from Lemma 12.4 gives the stated path. All pieces are constant on their endpoint collars. □

12.5. **Removing the additional kernel.**

Lemma 12.8. *Let $A = A_{g,B}$, and let $\psi, \xi \in \ker A$ be linearly independent over $\mathbb{R}$. Their values are linearly independent over $\mathbb{R}$ on an open dense subset of M.*

Proof. The operators A and its formal adjoint are real first-order elliptic operators with smooth kernel sections and weak unique continuation, by Subsection 12.1. The pointwise-independence argument in the proof of [41, Remark 1.1.14(3)] therefore applies. □

Proposition 12.9. *Let $I = [0,1]$, and let $(g_t, B_t, \psi_t)_{t\in I}$ be a smooth family, constant on fixed closed endpoint collars $I_\partial \subset I$, with*

$$A_t = D^+_{g_t} + B_t, \qquad A_t\psi_t = 0, \qquad \|\psi_t\|_{L^2} = 1.$$

Suppose $\ker A_t = \mathbb{R}\psi_t$ *for $t \in I_\partial$. For every C^∞ neighborhood $\mathcal{U}$ of zero in*

$$C^\infty\big(I \times M; \mathrm{Hom}_{\mathbb{R}}(\Sigma^+,\Sigma^-)\big),$$

there is $Q \in \mathcal{U}$ such that

$$Q|_{I_\partial \times M} = 0, \qquad Q_t \psi_t = 0, \qquad \ker(A_t + Q_t) = \mathbb{R}\psi_t \quad (t \in I).$$

Here the bundles are identified by positive spin comparison. For one fixed operator $A_{g,B}$ and any nonzero $\psi \in \ker A_{g,B}$, every C^∞ neighborhood of zero in $\mathcal{P}_g$ contains a potential Q satisfying $Q\psi = 0$ and $\ker(A_{g,B} + Q) = \mathbb{R}\psi$.

Proof. Let $\mathcal{Q}$ be the closed linear subspace of smooth potential fields satisfying $Q|_{I_\partial \times M} = 0$ and $Q_t \psi_t = 0$. Under $Q \mapsto V_Q$, these are compatible potential fields annihilating $(\psi_t, 0)$. Lemma 10.1 gives a continuous linear map $q : \ell^2 \to \mathcal{Q}$ with dense range, bounded in every C^k norm. Put

$$J = I \setminus I_\partial, \qquad H = H^s(\Sigma^+), \qquad Y = H^{s-1}(\Sigma^-), \qquad L_{p,t} = A_t + q(p)_t : H \longrightarrow Y.$$

If J is empty, take $Q = 0$. Otherwise J is an open interval. The space

$$\mathcal{S} = \{(t, \xi) \in J \times H : \langle \xi, \psi_t \rangle_{L^2} = 0,\ \|\xi\|_{L^2} = 1\}$$

is a smooth second-countable Hilbert submanifold. Indeed, the differentials in ξ of its two defining functionals are independent, as their values on ψ_t and ξ show. Define the smooth map

$$F : \ell^2 \times \mathcal{S} \longrightarrow Y, \qquad F(p, t, \xi) = L_{p,t}\xi. \tag{12.19}$$

At a zero of F, the spinors ψ_t and ξ are smooth orthonormal kernel vectors. The differential in the sphere fibre is the restriction of $L_{p,t}$ to

$$E_{t,\xi} = \{u \in H : \langle u, \psi_t \rangle_{L^2} = \langle u, \xi \rangle_{L^2} = 0\}.$$

Since $H = E_{t,\xi} \oplus \mathbb{R}\psi_t \oplus \mathbb{R}\xi$ and $\operatorname{ind}_{\mathbb{R}} L_{p,t} = 0$, this restriction has the same range as $L_{p,t}$ and has index -2. The tangent space of $\mathcal{S}$ is parametrized by

$$(\tau, u) \longmapsto \big(\tau, u - \tau\langle \xi, \partial_t \psi_t \rangle_{L^2} \psi_t\big), \qquad \tau \in \mathbb{R}, \quad u \in E_{t,\xi}.$$

In these coordinates,

$$D_{(t,\xi)}F(\tau, u) = L_{p,t}u + \tau(\partial_t L_{p,t})\xi.$$

Thus $D_{(t,\xi)}F$ is Fredholm of index -1.

Let $L^*_{p,t}$ denote the formal $L^2(\mu)$ adjoint and put $C_{p,t} = \ker L^*_{p,t}$. This finite-dimensional space represents the cokernel of $L_{p,t}|_{E_{t,\xi}}$. For $0 \neq \eta \in C_{p,t}$, Lemma 12.8 and unique continuation give a ball $U \Subset M$ on whose closure ψ_t, ξ are pointwise independent and η is nonzero. On U put

$$v = \xi - \frac{\langle \xi, \psi_t \rangle}{|\psi_t|^2}\psi_t.$$

Then $v \neq 0$ and $v \perp \psi_t$ pointwise. Let v^* denote the fibrewise metric dual of v. Choose $0 \leq \chi \in C^\infty_c(U)$, $\chi \neq 0$, and define

$$R^0 = \chi\, \eta \otimes \frac{v^*}{|v|^2},$$

extended by zero outside U. Thus $R^0\psi_t = 0$ and $R^0\xi = \chi\eta$.

Choose an open interval $J_t \Subset J$ containing t such that ψ_τ is nonzero on $\overline{U}$ for $\tau \in J_t$, and choose $\beta \in C^\infty_c(J_t)$ equal to one near t. Let ψ^*_τ denote the fibrewise metric dual of ψ_τ. Using the fixed bundle identifications, set on $J_t \times U$

$$P^\perp_\tau = \mathrm{id}_{\Sigma^+} - \frac{\psi_\tau \otimes \psi^*_\tau}{|\psi_\tau|^2}, \qquad R_\tau = \beta(\tau) R^0 P^\perp_\tau,$$

and extend R by zero. Then $R \in \mathcal{Q}$ and

$$\langle \eta, R_t \xi \rangle_{L^2} = \int_M \chi |\eta|^2 \, \mu > 0.$$

Consequently the map $R \mapsto [R_t\xi]$ from $\mathcal{Q}$ to the cokernel of $L_{p,t}|_{E_{t,\xi}}$ is surjective. Its composition with q has dense image, hence is surjective in that finite-dimensional space. The potential differential D_pF therefore maps onto the cokernel of $D_{(t,\xi)}F$.

By Lemma 10.1, $F^{-1}(0)$ is a second-countable Banach manifold and the projection

$$\pi \,:\, F^{-1}(0) \longrightarrow \ell^2, \qquad \pi(p,t,\xi) = p,$$

is Fredholm of index -1. Theorem 1.3 and Corollary 1.5 of [82, p. 862], applied to its countably many connected components, give a dense residual set of regular values of π. Every regular fibre is empty, since the index is negative.

Continuity of q gives $\varepsilon > 0$ such that $q(p) \in \mathcal{U}$ whenever $\|p\|_{\ell^2} < \varepsilon$. Choose a regular value with this norm bound and put $Q = q(p)$. No nonzero kernel vector orthogonal to ψ_t exists for $t \in J$. On I_∂, $Q_t = 0$ and the kernel is already $\mathbb{R}\psi_t$. Since $Q_t\psi_t = 0$, the stated equality of kernels follows for every $t \in I$.

For a fixed operator, first replace ψ by $\psi/\|\psi\|_{L^2}$. The fixed-operator part of Lemma 10.1 applies to $\mathcal{Q} = \{Q \,:\, Q\psi = 0\}$ and the sphere in $H^s(\Sigma^+) \cap \psi^\perp$. The spatial fields R^0 above give the same cokernel surjectivity. The incidence projection now has index -2, so the same regular-value theorem gives empty fibres for parameters arbitrarily close to zero. Rescaling ψ does not change its span or the equation $Q\psi = 0$. □

12.6. **The simple-kernel hypersurface and transverse paths.** Let $\mathcal{P}$ denote either $\mathcal{P}_g$, for a fixed metric g, or $\mathcal{P}_\mu$. Write A_p for the corresponding real chiral operator. Put

$$X = \{p \in \mathcal{P} \,:\, \dim\ker A_p \le 1\}, \qquad S = \{p \in \mathcal{P} \,:\, \dim\ker A_p = 1\}.$$

Since every A_p has index zero, $X \setminus S = \mathcal{P}^{\mathrm{inv}}$.

Proposition 12.10. *The set S is nonempty and path connected, both with the metric fixed and in the total density slice.*

Proof. Fix a metric g. Choose finitely many smooth sections of Σ^+_g whose values span every fibre. Parametric transversality [64, Chapter 3, §2, Theorem 2.7, pp. 79–80], applied to their real linear combinations, gives a smooth section ψ transverse to zero. Put $Z = \psi^{-1}(0)$. If $n \ge 16$, then $r > n$, so Z is empty. Whenever Z is empty, (12.10) gives a smooth potential B with $(D^+_g + B)\psi = 0$.

Suppose $Z \neq \emptyset$. Then $n = 8$ and Z is finite. For $x \in Z$, the derivative $T_{x,0} \,:\, T_xM \to \Sigma^+_{g,x}$ is an isomorphism. Let C_g denote the Clifford contraction of (12.5). In a spin frame, Proposition 12.6 gives an isomorphism $T_{x,1}$ of the same determinant sign with $C_g(T_{x,1}) = 0$. Two matrices in $GL_8(\mathbb{R})$ lie in the same path component if and only if their determinants have the same sign [60, §1.2, p. 24]. Choose a smooth path $T_{x,u}$, $0 \le u \le 1$, of isomorphisms between these derivatives. Apply (12.15) on disjoint coordinate balls about the points of Z, with these paths and the zero-dimensional submanifold Z. Estimate (12.16) gives a smooth section ψ' with zero set Z and derivative $T_{x,1}$ at each $x \in Z$. Thus $D^+_g\psi' = 0$ on Z.

At each point of Z, the eight components of ψ' in a local trivialization are coordinates. Formula (12.18), with no tangential coordinate, gives a smooth local homomorphism B satisfying $B\psi' = -D^+_g\psi'$. On $M \setminus Z$, (12.10) gives the same equation. A smooth partition of unity gives a global solution. Replace ψ by ψ' in this case.

The section ψ is not identically zero. Normalize it in L^2 and apply the fixed-operator assertion of Proposition 12.9. The resulting operator has kernel $\mathbb{R}\psi$. This gives a point of S over every metric g.

Given two points of S, choose L^2-unit generators of their kernels. Proposition 12.7 gives a smooth family of normalized solutions, constant on endpoint collars, with the prescribed endpoint operators. Proposition 12.9 gives a potential perturbation vanishing on those collars for which every kernel is the selected line. The resulting path lies in S. Both propositions allow the metric to remain fixed when the endpoint metrics coincide. □

Lemma 12.11. *The set X is open in $\mathcal{P}$, and S is closed in X. Near every point of S there is a local product chart in which S is the zero set of one real coordinate and its two local sides consist of invertible operators. The coordinate neighborhood and both sides can be chosen path connected. A smooth potential direction is transverse to S at the given point.*

Proof. Let $p_0 \in S$ and put $A_0 = A_{p_0}$, $K = \ker A_0$, and $C = \ker A_0^*$, where the adjoint is taken with respect to $L^2(\mu)$. These are real lines. Let P_C be the L^2 orthogonal projection onto C, and put

$$E_0 = H^s(\Sigma^+) \cap K^{\perp_{L^2}}, \qquad F_0 = H^{s-1}(\Sigma^-) \cap C^{\perp_{L^2}}.$$

Thus $H^s(\Sigma^+) = K \oplus E_0$ and $H^{s-1}(\Sigma^-) = C \oplus F_0$. The Fredholm alternative gives an isomorphism $d_0 = A_0|_{E_0} : E_0 \to F_0$. For p near p_0, write the blocks of A_p in these decompositions as

$$A_p = \begin{pmatrix} a(p) & b(p) \\ c(p) & d(p) \end{pmatrix}.$$

Then $d(p) : E_0 \to F_0$ is invertible. Choose L^2-unit generators $\psi_0 \in K$ and $\eta_0 \in C$. Define

$$\mathfrak{s}(p) = a(p) - b(p)d(p)^{-1}c(p) : K \longrightarrow C, \qquad \sigma(p) = \langle \eta_0, \mathfrak{s}(p)\psi_0 \rangle_{L^2}. \tag{12.20}$$

Lemma 6.6, applied with the invertible block d_0, gives $\ker A_p \simeq \ker \mathfrak{s}(p)$ and $\operatorname{coker} A_p \simeq \operatorname{coker} \mathfrak{s}(p)$. Consequently $\dim \ker A_p \le 1$, and A_p is invertible exactly when $\sigma(p) \neq 0$. Together with openness of invertibility at the remaining points, this proves that X is open and that S is closed in X.

At p_0 the blocks a, b, c vanish. For a smooth potential direction Q,

$$D\mathfrak{s}_{p_0}(Q) = P_C Q|_K, \qquad D\sigma_{p_0}(Q) = \langle \eta_0, Q\psi_0 \rangle_{L^2}.$$

Lemma 12.2 gives such a Q with $D\sigma_{p_0}(Q) \neq 0$. Rescale Q so that $D\sigma_{p_0}(Q) = 1$.

Take the affine potential coordinates and, in the total space, the logarithmic metric coordinates of Section 8, centered at p_0. Write $\mathcal{E}^\infty$ for their space of smooth coefficients and $\mathcal{E}^k$ for its C^k completion, with $k \ge s + 1$. In these coordinates write $p_0 + z$ for the point represented by z. The comparison formula [61, Theorem 2.3.6, p. 25] and the integer Sobolev product estimate imply that

$$z \longmapsto A_{p_0+z} \in \mathcal{L}\big(H^s(\Sigma^+), H^{s-1}(\Sigma^-)\big)$$

is smooth on a neighborhood of zero in $\mathcal{E}^k$. Indeed, the coefficients involve smooth matrix functions and at most one spatial derivative of the metric coefficients; multiplication by a C^j coefficient is bounded on H^j for every integer $0 \le j \le s$, by the integer Sobolev norm and the Leibniz rule [44, §12.1.2(3) and §12.1.4]. Thus σ extends smoothly to this neighborhood. Put

$$\ell(z) = D\sigma_{p_0}(z), \qquad \mathcal{E}_0^k = \ker \ell.$$

Since $\ell(Q) = 1$, one has $\mathcal{E}^k = \mathcal{E}_0^k \oplus \mathbb{R}Q$. The Banach inverse and implicit function theorems [50, Theorem 11.22 and Corollary 13.29, pp. 61–63, 72] give a convex neighborhood $V \subset \mathcal{E}_0^k$ of zero, $\varepsilon > 0$, and a smooth real function $\alpha : V \times (-\varepsilon, \varepsilon) \to \mathbb{R}$ such that

$$\Theta(h, u) = p_0 + h + \alpha(h, u)Q, \qquad \sigma(\Theta(h, u)) = u$$

is a product chart, with $\Theta(0, 0) = p_0$. Its inverse is

$$p_0 + z \longmapsto \big(z - \ell(z)Q, \sigma(p_0 + z)\big).$$

Since Q is smooth, both maps restrict to the smooth coefficient spaces. These restrictions are continuous in the C^∞ topology: the scalar functions ℓ, σ and α are continuous in the C^k topology, and the terms involving Q are scalar multiples of a fixed smooth field. Put $V^\infty = V \cap \mathcal{E}^\infty$. In this neighborhood the restriction identifies S with $V^\infty \times \{0\}$ and the two invertible sides with $V^\infty \times (0, \varepsilon)$ and $V^\infty \times (-\varepsilon, 0)$. These sets are path connected because V^∞ is convex. Finally, $\frac{d}{da}\big|_{a=0} \sigma(p_0 + aQ) = 1$; for sufficiently small $\delta > 0$, the segment $|a| < \delta$ meets S only at $a = 0$ and is transverse there. □

Proposition 12.12. *Any two points of $\mathcal{P}^{\mathrm{inv}}$ can be joined by a smooth coefficient path which is stationary on endpoint collars, has a one-dimensional kernel at finitely many interior times, and is invertible at every other time. Each singular time is a transverse simple crossing. In the total space, a prescribed smooth metric path, stationary on endpoint collars, may be retained. Consequently X is path connected.*

Proof. Put $I = [0, 1]$. Choose a smooth coefficient path p_t joining the given endpoints. In the fixed-metric space, compose affine interpolation of the potentials with a smooth map $I \to I$ equal to zero and one near the respective endpoints. In the total space, choose the prescribed metric path, or any smooth path given by the coordinates of Section 8, and interpolate the transported potentials in the same way. Choose $0 < \delta < 1/2$ so that p_t is constant on

$$I_\partial = [0, \delta] \cup [1 - \delta, 1], \qquad J = (\delta, 1 - \delta).$$

Write $A_t = A_{p_t}$ and put

$$H = H^s(\Sigma^+), \qquad Y = H^{s-1}(\Sigma^-),$$
$$\mathcal{Q} = \{Q \in C^\infty(I \times M; \mathrm{Hom}_{\mathbb{R}}(\Sigma^+, \Sigma^-)) \,:\, Q|_{I_\partial \times M} = 0\}.$$

Lemma 10.1 gives a continuous linear map $q : \ell^2 \to \mathcal{Q}$ with dense range and bounds in every C^k norm. For $a \in \ell^2$, set $L_{a,t} = A_t + q(a)_t : H \to Y$. The L^2 unit sphere $\mathbb{S} = \{\psi \in H : \|\psi\|_{L^2} = 1\}$ is a smooth second-countable Hilbert submanifold, with $T_\psi \mathbb{S} = \{u \in H : \langle u, \psi\rangle_{L^2} = 0\}$. Define the smooth map

$$F : \ell^2 \times J \times \mathbb{S} \longrightarrow Y, \qquad F(a, t, \psi) = L_{a,t}\psi. \tag{12.21}$$

At a zero, Lemma 12.2 gives $\mathrm{ind}(L_{a,t}|_{T_\psi \mathbb{S}}) = -1$. The differential

$$D_{(t,\psi)}F(\tau, u) = L_{a,t}u + \tau(\partial_t L_{a,t})\psi$$

therefore has index zero. The same lemma gives smooth spatial potential directions onto $\mathrm{coker}\, L_{a,t}$. Multiplication by a smooth time cutoff supported in J and equal to one near t extends these directions to $\mathcal{Q}$. The image of q in this finite-dimensional cokernel is dense, hence surjective. Thus $D_a F$ maps onto the cokernel of $D_{(t,\psi)}F$. By Lemma 10.1, $F^{-1}(0)$ is a second-countable Banach manifold and its projection to ℓ^2 is Fredholm of index zero. Theorem 1.3 and Corollary 1.5 of [82, p. 862], applied to its countably many connected components, give a regular value a.

The fixed-parameter incidence

$$\mathcal{N}_a = \{(t, \psi) \in J \times \mathbb{S} : L_{a,t}\psi = 0\}$$

is a zero-dimensional manifold. At every one of its points, $D_{(t,\psi)}F$ is an isomorphism. If $\dim \ker L_{a,t} \geq 2$, there is a nonzero $u \in \ker L_{a,t}$ orthogonal to ψ; then $(0, u)$ belongs to the kernel of this differential. Consequently every singular $L_{a,t}$ has a one-dimensional kernel.

For this fixed a, the family $L_{a,t}$ is smooth on the compact interval I. Elliptic regularity and (12.2), with $j = s$, give a uniform bound on $\|\psi\|_{H^{s+1}}$ for $(t, \psi) \in \mathcal{N}_a$. Given a sequence in $\mathcal{N}_a$, take a subsequence of times converging in I and, by the compact inclusion $H^{s+1}(\Sigma^+) \hookrightarrow H^s(\Sigma^+)$ [44, Theorem 16.10], a subsequence of spinors converging in H^s. The limit has L^2 norm one and satisfies the limiting equation, by operator-norm continuity in (12.1). Its time lies in J, since all operators on I_∂ are the prescribed invertible operators. Thus $\mathcal{N}_a$ is compact, hence finite. Each singular time contributes precisely the two normalized vectors of its real kernel line.

Let t_0 be a singular time, and choose L^2-unit generators $\psi \in \ker L_{a,t_0}$ and $\eta \in \ker L^*_{a,t_0}$. The spinor part of $D_{(t,\psi)}F$ has range $\mathrm{ran}\, L_{a,t_0}$, whose cokernel is the line $\mathbb{R}\eta$. Surjectivity of $D_{(t,\psi)}F$ therefore implies

$$\big\langle \eta, (\partial_t L_{a,t})|_{t=t_0}\psi \big\rangle_{L^2} \neq 0. \tag{12.22}$$

By (12.20), this is the nonvanishing derivative of a local defining function for S. The path with operators $L_{a,t}$ has all the asserted properties and retains the chosen metric path.

Any point of S can be joined to an invertible point by the segment in Lemma 12.11, contained in X. Concatenation with the paths just constructed joins any two points of X in X. □

12.7. **The two components and the real index.** The coefficient space $\mathcal{P}$ is contractible and metrizable: $\mathcal{P}_g$ is affine, and the coordinates of Section 8 identify $\mathcal{P}_\mu$ with a product of real Fréchet spaces. For a finite-dimensional real vector space E, put $\det E = \Lambda^{\dim E} E$, with $\det\{0\} = \mathbb{R}$. The family $A_p : H^s(\Sigma^+) \to H^{s-1}(\Sigma^-)$ is continuous in operator norm and consists of bounded real Fredholm operators of index zero, by (12.1). Its determinant line bundle has fibres

$$\lambda_p = \det\ker A_p \otimes \big(\det\operatorname{coker} A_p\big)^*. \tag{12.23}$$

This is a continuous real line bundle on all of $\mathcal{P}$ [88, Theorem 1]. The base is metrizable, hence paracompact; the bundle is therefore trivial by [60, Corollary 1.8]. Fix an orientation of it. At an invertible point p, the zero kernel and cokernel identify λ_p canonically with $\mathbb{R}$. Let $\varepsilon(p) \in \{1, -1\}$ be the sign of 1 in the chosen orientation. The function ε is locally constant on $\mathcal{P}^{\mathrm{inv}}$.

Lemma 12.13. *The two local sides of S have opposite signs of ε. These signs define a coorientation of S, and ε changes sign at every transverse simple crossing.*

Proof. Take a product chart as in Lemma 12.11. Each of its two sides is path connected, so ε is constant there. A short segment normal to S has invertible endpoints, exactly one singular point, real kernel and cokernel of dimension one, and a scalar Schur complement changing sign. By [88, Proposition 5.9, pp. 62–63], the canonical determinant orientations at its endpoints are opposite relative to the chosen orientation. Thus the signs on the two sides differ. They agree on overlaps because ε is defined on all of $\mathcal{P}^{\mathrm{inv}}$. The same argument applies at any transverse simple crossing. □

Proposition 12.14. *The invertible locus $\mathcal{P}^{\mathrm{inv}}$ has exactly two path components, distinguished by $\varepsilon \in \{1, -1\}$.*

Proof. Proposition 12.12 makes X path connected. It is open in the Fréchet coefficient space by Lemma 12.11, hence locally path connected. The set S is closed in X, nonempty, and path connected by Lemma 12.11 and Proposition 12.10; its product charts have path-connected factors. Lemma 10.5 therefore gives at most two path components of $X \setminus S = \mathcal{P}^{\mathrm{inv}}$. Both signs occur in every product chart by Lemma 12.13. They cannot be joined in the invertible locus, since ε is locally constant. There are consequently exactly two path components. □

For $p_0, p_1 \in \mathcal{P}^{\mathrm{inv}}$, let $\delta(p_0, p_1) \in KO_{n+1}$ be the relative real family index of a smooth coefficient path from p_0 to p_1, as in Section 8.1. Contractibility in the coefficient coordinates and homotopy invariance make this independent of the path. On metric endpoints it is the ordinary index difference.

Proposition 12.15. *For $p_0, p_1 \in \mathcal{P}^{\mathrm{inv}}$, the element $\delta(p_0, p_1) \in KO_{n+1} \cong \mathbb{Z}/2$ is zero if $\varepsilon(p_0) = \varepsilon(p_1)$ and is nonzero otherwise.*

Proof. Choose a path p_t, $t \in I = [0, 1]$, as in Proposition 12.12, and let N be its number of singular parameters. Put $A_t = A_{p_t}$, regarded as a closed operator from $L^2(\Sigma^+)$ to $L^2(\Sigma^-)$ with domain $H^1(\Sigma^+)$, and let A_t^* be its L^2 adjoint. Define

$$R_t = (1 + A_t^* A_t)^{-1/2}, \qquad G_t = A_t R_t.$$

By Section 8.1 and (8.3), G_t is the norm-continuous real chiral bounded transform representing $\delta(p_0, p_1)$. Functional calculus gives $R_t = (1 - G_t^* G_t)^{1/2}$, so R_t is norm-continuous on L^2. Each $R_t : L^2(\Sigma^+) \to H^1(\Sigma^+)$ is an isomorphism, and (12.2) gives

$$\sup_{t \in I} \|R_t\|_{L^2 \to H^1} < \infty.$$

For $t, \tau \in I$,

$$A_\tau (R_t - R_\tau) = G_t - G_\tau + (A_\tau - A_t) R_t.$$

The same elliptic estimate therefore proves continuity of R_t in $\mathcal{L}(L^2, H^1)$. The maps $A_t R_{ut}$, $0 \le u, t \le 1$, give a norm-continuous homotopy of bounded real Fredholm paths from $A_t R_0$ to G_t, with invertible

endpoints. Thus $\delta(p_0, p_1)$ is the relative class of $A_t : H^1(\Sigma^+) \to L^2(\Sigma^-)$ after the fixed source identification R_0.

Choose disjoint closed intervals about the singular parameters, each contained in one Schur chart and with invertible endpoints. On each interval, take the L^2 orthogonal splittings at its singular parameter, with kernel K and adjoint kernel C, and the same L^2-unit generators as in (12.20). The complementary block $d_t : H^1(\Sigma^+) \cap K^\perp \to C^\perp$ of A_t is invertible after shrinking the interval. If $d_t u = f$ and f is smooth, then $A_t u - f \in C$ is smooth, so elliptic regularity makes u smooth. Uniqueness therefore identifies this inverse on smooth right-hand sides with the inverse in (12.20). The resulting scalar Schur complements are equal, since $A_t(K)$ consists of smooth sections.

The triangular factors in (6.7) are homotopic to the identity by scaling their off-diagonal blocks. This homotopy preserves Fredholmness and invertibility at the endpoints. The relative class thus reduces to the scalar Schur family and an invertible summand. The latter has zero relative class. The scalar changes sign once, so its class is $1 \in KO_1 \cong \mathbb{Z}/2$, with the normalization of (8.1) [28, Proposition 4.9 and Theorem 5.5(3)]. The intervening invertible paths also have zero class. Additivity under concatenation gives $\delta(p_0, p_1) = N \pmod 2$ [28, Remark 5.2(2)–(7)]. Lemma 12.13 gives

$$\varepsilon(p_1) = (-1)^N \varepsilon(p_0),$$

and the assertion follows. □

Proof of Theorem 12.1. Proposition 12.14 applies to both $\mathcal{P}_g$ and $\mathcal{P}_\mu$. Choose an invertible basepoint p_* and orient the determinant bundle so that $\varepsilon(p_*) = 1$. The function $(1 - \varepsilon)/2$ takes the values zero and one on the two path components. By Proposition 12.15, it equals $\delta(p_*, \cdot)$ in $KO_{n+1} \cong \mathbb{Z}/2$. This proves the asserted bijections of pointed sets. □

Remark 12.16. By Proposition 8.2, with $U = M$, the inclusion $g \mapsto (g, 0)$ induces a bijection

$$\pi_0 \mathcal{R}_\mu^{\mathrm{inv}}(M) \longrightarrow \pi_0 \mathcal{P}_\mu^{\mathrm{inv}}$$

preserving ordinary index difference. The target has two elements by Proposition 12.14; in particular, $\mathcal{R}_\mu^{\mathrm{inv}}(M)$ is nonempty. For any $g_* \in \mathcal{R}_\mu^{\mathrm{inv}}(M)$, it follows that

$$\mathrm{inddiff}_{g_*} : \pi_0 \mathcal{R}_\mu^{\mathrm{inv}}(M) \longrightarrow KO_{n+1} \cong \mathbb{Z}/2$$

is a bijection of pointed sets.

13. Real skew-adjoint operators

Throughout this section, M is a closed connected spin manifold,

$$n \equiv 1 \pmod 8, \qquad n \geq 9, \qquad \alpha(M) = 0.$$

Fix the spin structure and a positive smooth density μ. Metrics belong to $\mathcal{R}_\mu(M)$. Write C for the charge conjugation J of Section 2. By (2.1),

$$C^2 = 1, \qquad C D_g = -D_g C.$$

Its fixed bundle $\Sigma_g^{\mathbb{R}} = \ker(C - 1)$ has real rank

$$r = 2^{(n-1)/2} = 2\ell, \qquad \ell = 2^{(n-3)/2}.$$

The Hermitian spinor metric restricts to a Euclidean metric on this bundle. Write $\langle\, , \,\rangle$ for the resulting fibre inner product and $\langle\, , \,\rangle_{L^2}$ for its integral against μ. Orthogonality of sections refers to this real L^2 inner product.

For a compatible potential V, in the sense of Definition 2.1, and $\xi \in T^*M$, put

$$K_g = iD_g, \qquad Q = iV, \qquad L_{g,Q} = K_g + Q, \qquad a_g(\xi) = ic_g(\xi). \tag{13.1}$$

These operators preserve $\Sigma_g^{\mathbb{R}}$. Transpose in a real spinor fibre is taken with respect to its Euclidean metric. For $x \in M$ and $\xi, \eta \in T_x^*M$,

$$a_g(\xi)^T = a_g(\xi), \qquad a_g(\xi)a_g(\eta) + a_g(\eta)a_g(\xi) = 2g^{-1}(\xi,\eta)I. \tag{13.2}$$

Tangent and cotangent vectors are identified by g in Clifford expressions. The identities $CV = -VC$ and $V^* = V$ give

$$C(iV) = iVC, \qquad (iV)^* = -iV.$$

Conversely, the complex-linear extension of a real skew-adjoint endomorphism Q gives a compatible potential $V = -iQ$. Let $\mathfrak{so}(\Sigma_g^{\mathbb{R}})$ denote the bundle of these real skew-adjoint endomorphisms, and put

$$\mathcal{Q}_g = C^\infty(M; \mathfrak{so}(\Sigma_g^{\mathbb{R}})), \qquad \mathcal{Q}_g^{\mathrm{inv}} = \{Q \in \mathcal{Q}_g \,:\, K_g + Q \text{ is invertible}\}.$$

The map $(g, V) \mapsto (g, iV)$ identifies $\mathcal{P}_\mu$ with the space of pairs (g, Q), where $g \in \mathcal{R}_\mu(M)$ and $Q \in \mathcal{Q}_g$. Write $L_p = K_g + Q$, whether $p = Q$ at a fixed metric or $p = (g, Q)$. Let $\mathcal{P}$ denote either $\mathcal{Q}_g$ or $\mathcal{P}_\mu$, with its C^∞ topology, and let $\mathcal{P}^{\mathrm{inv}}$ denote the invertible locus. Write sf_2 for the real mod-two spectral flow [28, Theorem 5.5(2) and §6.2].

Theorem 13.1. *The space $\mathcal{P}^{\mathrm{inv}}$ is nonempty and has exactly two path components. For $p_* \in \mathcal{P}^{\mathrm{inv}}$, the map*

$$\pi_0(\mathcal{P}^{\mathrm{inv}}) \longrightarrow KO_2 \cong \mathbb{Z}/2, \qquad [p] \longmapsto \mathrm{sf}_2(L_{p_*}, L_p) \tag{13.3}$$

is a bijection. The spectral flow is computed along any smooth path in $\mathcal{P}$ from p_* to p. For $\mathcal{P} = \mathcal{P}_\mu$, $p_* = (g_*, 0)$, and $p = (g, iV)$, its value is $\delta_{g_*}(g, V) \in KO_{n+1} \cong KO_2$, with the convention of Section 8.1.

13.1. **Fixed domains and the universal incidence.** For fixed g, put $E = \Sigma_g^{\mathbb{R}}$. For $\mathcal{P} = \mathcal{P}_\mu$, positive spinor comparison with a fixed $g_0 \in \mathcal{R}_\mu(M)$ identifies the real spinor bundles with $E = \Sigma_{g_0}^{\mathbb{R}}$, preserving their fibre metrics and L^2 inner products. Write $H^q = H^q(M; E)$. In these identifications each L_p is a real skew-adjoint elliptic operator on L^2, with domain H^1 and compact resolvent. Its coefficients depend smoothly on the metric and its first derivatives [61, Theorem 2.3.6, p. 25]; the domain and graph-norm assertions are those of Section 2. Its kernel consists of smooth sections, and

$$\dim_{\mathbb{R}} \ker L_p \equiv \alpha(M) = 0 \pmod 2.$$

For $Q = 0$, this is [16, Theorem 3.1, p. 147]. The real skew index is kernel parity and is invariant under Fredholm homotopies [15, Proposition 5.1, pp. 22–23]. The bounded transforms of $K_g + tQ$, $0 \le t \le 1$, are norm-continuous by Section 8.1, so the same parity formula holds for every Q.

The logarithmic metric and linear potential coordinates of Section 8 identify $\mathcal{P}_\mu$ with a real Fréchet vector space. Thus both choices of $\mathcal{P}$ are contractible. For an integer $s > n/2 + 1$, complete the coefficients in C^m, where $m \ge s+1$. In these coordinates the spinor equation is a smooth Banach-space equation with values in H^{s-1} and spinor variable in H^s. The integers m and s may be increased to prescribe any finite collection of smooth seminorms. In every finite parameter family below, the coefficients are smooth in the spatial and parameter variables. Kernel spinors which depend smoothly on the parameters in H^s are then jointly smooth, by the elliptic argument at the end of the proof of Lemma 10.2. Unique continuation for $L = i(D_g + V)$ follows from Section 9 and [61, Theorem 6.2.8].

For a nonzero spinor ψ, write $[\psi] = \mathbb{R}\psi$. Define

$$\mathcal{I}(\mathcal{P}) = \{(p, [\psi]) \,:\, 0 \ne \psi \in \ker L_p\}. \tag{13.4}$$

At an incidence point choose an L^2-unit representative ψ, and put $L = L_p$,

$$X_\psi = H^s \cap \psi^\perp, \qquad Y_\psi = H^{s-1} \cap \psi^\perp.$$

In local unit representatives, X_ψ is the spinor tangent space and Y_ψ is the equation target: skew-adjointness gives $\langle L_p\psi, \psi\rangle_{L^2} = 0$. The differential in potential and spinor directions is

$$T_\psi \,:\, \mathcal{Q}_g \oplus X_\psi \longrightarrow Y_\psi, \qquad T_\psi(R, u) = Lu + R\psi. \tag{13.5}$$

It extends to a bounded map with potential space $C^m(M; \mathfrak{so}(E))$.

Lemma 13.2. *The map* (13.5) *is surjective, even with* $R \in C_c^\infty(U; \mathfrak{so}(E))$ *for any fixed nonempty open set* $U \subset M$*. A finite-dimensional subspace of these potential directions suffices, and remains sufficient at nearby smooth incidence points.*

For a subspace $F \subset \ker L$*, let* P_F *be the real* L^2 *orthogonal projection onto* F*, and put*

$$W_F = \{P_F R|_F \,:\, R \in \mathcal{Q}_g\} \subset \mathfrak{so}(F).$$

Then

$$W_F v = v^\perp \cap F \qquad (0 \neq v \in F). \tag{13.6}$$

Proof. Put $K = \ker L$. The Fredholm alternative and elliptic regularity [44, Theorems 16.1 and 16.13; Remark 16.2] give

$$\ker(L|_{X_\psi}) = K \cap \psi^\perp, \qquad L(X_\psi) = Y_\psi \cap K^\perp.$$

In particular, $L : X_\psi \to Y_\psi$ has closed range and kernel and cokernel of dimension $\dim K - 1$.

Let $\eta \in K \cap \psi^\perp$ satisfy

$$\langle \eta, R\psi \rangle_{L^2} = 0 \quad \text{for every } R \in C_c^\infty(U; \mathfrak{so}(E)).$$

Unique continuation gives a ball $U_0 \Subset U$ on which ψ is nowhere zero. At every point of U_0, $\{R\psi \,:\, R^T = -R\} = \psi^\perp$ in the spinor fibre. Localization of R therefore gives $\eta = f\psi$ on U_0, with f a smooth real function. Hence

$$0 = L\eta = L(f\psi) = a_g(df)\psi.$$

By (13.2), $a_g(df)^2 = |df|_g^2 I$, so $df = 0$; cf. [41, Remark 1.1.14(3)]. Unique continuation on the connected manifold M gives $\eta = c\psi$. Since $\eta \perp \psi$, one has $\eta = 0$.

The images of the supported potential directions thus span the finite-dimensional cokernel of $L|_{X_\psi}$. Choose a finite span $\mathcal{E}$ with this property. The surjection

$$T_\psi \,:\, \mathcal{E} \oplus X_\psi \longrightarrow Y_\psi$$

has a bounded right inverse $\mathcal{G}$, since its domain and target are Hilbert spaces. After identifying nearby orthogonal complements, $\mathcal{G}(T\mathcal{G})^{-1}$ is a right inverse for every sufficiently close differential T. This proves the local assertion.

Skew-adjointness gives $W_F v \subset F \cap v^\perp$. If $\eta \in F \cap v^\perp$ is orthogonal to $W_F v$, then $\langle \eta, Rv \rangle_{L^2} = 0$ for every $R \in \mathcal{Q}_g$. The preceding proportionality argument, applied to $v, \eta \in \ker L$, gives $\eta = cv$, hence $\eta = 0$. Since F is finite dimensional, (13.6) follows. □

The vertical differential has Fredholm index zero. Hence, for a smooth coefficient family of dimension N whose incidence differential is surjective, the incidence is locally a smooth manifold of dimension N, by the Banach implicit function theorem [50, Theorem 13.20]. The fibre over a point with k-dimensional kernel is the space of real lines in $\ker L_p$, hence isomorphic to $\mathbb{R}P^{k-1}$.

Lemma 13.3. *At every incidence point and every* $x \in M$*,*

$$\{u(x) \,:\, (R, u) \in \ker T_\psi,\ R \in \mathcal{Q}_g,\ u \in C^\infty(M; E)\} + \mathbb{R}\psi(x) = E_x. \tag{13.7}$$

Consequently, near any incidence point, finitely many smooth potential directions give an incidence manifold on which evaluation in local unit representatives is transverse to the zero section and to every smooth submanifold invariant under positive scalar multiplication in the spinor fibres.

Let B *be a compact smooth* d*-manifold, possibly with boundary, and let* $b \mapsto (p_b, [\psi_b])$ *be a smooth incidence family. In the fixed spinor comparisons, let* $Z \subset B \times E$ *be a smooth submanifold of codimension* c*, with* $\partial Z = Z \cap (\partial B \times E)$*, whose projection to* $B \times M$ *is a submersion and whose fibres are invariant under multiplication by every nonzero real number. Suppose the given family avoids* Z *on a neighborhood of a closed subset* $A \subset B$*. There are arbitrarily close smooth incidence families, with the same metrics and unchanged on a neighborhood of* A*, whose evaluations are transverse to* Z *on the interior and the boundary of* B*. The potential perturbations lie in one finite span of smooth skew-adjoint multiplication endomorphisms. If* $c > n + d$*, these families avoid* Z*.*

Proof. First allow $R \in C^m(M; \mathfrak{so}(E))$ and $u \in X_\psi$. Let $w \in E_x$ satisfy

$$\langle w, \psi(x) \rangle = 0 \tag{13.8}$$

and suppose that $\lambda(R, u) = \langle w, u(x) \rangle$ vanishes on $\ker T_\psi$. Sobolev embedding makes λ continuous. Choose the bounded right inverse $\mathcal{G}$ from Lemma 13.2; then $\lambda = \lambda \mathcal{G} T_\psi$. Let $\Pi_\psi : H^{s-1} \to Y_\psi$ be the L^2 orthogonal projection, and define $\eta \in H^{1-s}$ by

$$\langle \eta, f \rangle = -\lambda(\mathcal{G} \Pi_\psi f), \qquad f \in H^{s-1}.$$

Since $L^* = -L$, there is $b \in \mathbb{R}$ such that

$$L\eta = \delta_x w + b\,\psi, \qquad \langle \eta, R\psi \rangle = 0 \quad \text{for every smooth skew } R. \tag{13.9}$$

Here $\delta_x w$ is the distribution $\langle \delta_x w, u \rangle = \langle w, u(x) \rangle$. Pairing the first equation with ψ, and applying (13.8), gives $b = 0$.

On $M \setminus \{x\}$, elliptic regularity makes η smooth. There $L\eta = L\psi = 0$, and the second identity in (13.9) holds for all potential directions supported in that punctured manifold. Since $M \setminus \{x\}$ is connected, the proportionality argument in Lemma 13.2 gives $\eta = c\psi$ there. Thus $T = \eta - c\psi$ is supported at x and satisfies $LT = \delta_x w$. The symbol $a_g(x, \xi)$ is invertible for every nonzero real ξ, by (13.2). Lemma 2.3 therefore gives $T = 0$ and $w = 0$. This proves (13.7) at the Sobolev level.

The coefficient component of $\mathcal{G}$ lies in a finite span of smooth potentials. If its argument is smooth, its spinor component is smooth by elliptic regularity. For $v \in \ker T_\psi$, choose smooth ambient pairs $v_i \to v$. Then

$$v_i - \mathcal{G} T_\psi v_i \in \ker T_\psi$$

are smooth and converge to v. Their evaluations converge in the finite-dimensional space E_x. The image of the smooth tangent pairs is a linear subspace, hence closed; this proves (13.7) as stated. If a target is invariant under positive fibre dilation, its tangent space contains the radial vector $\psi(x)$. Equation (13.7) supplies all remaining vertical directions, while evaluation covers the identity on M. This proves the first transversality assertion.

For the family over B, choose at every (b, x) both potential directions spanning the equation cokernel and the potential components of smooth tangent pairs satisfying (13.7). Surjectivity is open. Compactness of $B \times M$ therefore gives finitely many smooth reference-bundle potentials $R_1, \dots, R_N$ which suffice at all these points. Near the original section, put

$$\mathcal{J} = \left\{ (b, z, [\varphi]) : \left(L_{p_b} + \sum_{j=1}^{N} z_j R_j \right) \varphi = 0 \right\}, \qquad \sigma(b) = (b, 0, [\psi_b]).$$

The differential in $(z, [\varphi])$ is surjective and Fredholm of index N. The Banach implicit function theorem makes $\mathcal{J}$ a smooth manifold of dimension $N + d$ near $\sigma(B)$. The projection $\pi_B : \mathcal{J} \to B$ is a submersion: for each $\dot{b}$, its equation in the vertical variables is

$$T_\psi(R, u) = -(\partial_b L)(\dot{b})\psi,$$

and the right-hand side is orthogonal to ψ by skew-adjointness. The selected directions solve this equation. Its fibres have dimension N. Equation (13.7), restricted to $\ker d\pi_B$, holds near $\sigma(B) \times M$. Taking B to be a point gives the local finite-dimensional transversality assertion.

Choose finitely many smooth sections $v_1, \dots, v_a$ spanning $\sigma^* \ker d\pi_B$, by local frames and a partition of unity. A vertical Riemannian metric gives fibrewise exponential maps Θ_b on a neighborhood of the zero section. Choose an open neighborhood O of A on which the given family avoids Z, and choose $\chi \in C^\infty(B; [0, 1])$ which vanishes on a neighborhood of A and is positive on $B \setminus O$. For ε in a sufficiently small ball in $\mathbb{R}^a$, put

$$\sigma_\varepsilon(b) = \Theta_b \left(\chi(b) \sum_{j=1}^{a} \varepsilon_j v_j(b) \right).$$

These are sections of π_B. Where $\chi > 0$, their parameter derivatives span the vertical tangent space. At an intersection with Z, radiality and (13.7) give every vertical normal direction. The full evaluation covers the identity on $B \times M$, so it is transverse to Z. Where $\chi = 0$, the original family is unchanged and disjoint from Z.

The unit representatives form a double cover of B. The homotopy $t \mapsto \sigma_{t\varepsilon}$, $0 \le t \le 1$, identifies the nearby unit covers with this one [59, Proposition 1.30]. These identifications are smooth in local covering charts. Evaluation is therefore a smooth map on a fixed double cover, and invariance of Z under multiplication by -1 makes transversality and avoidance descend to the projective family. The submersion $Z \to B \times M$ implies that its restriction over $\partial B \times M$ has the same codimension c. Apply parametric transversality [64, Chapter 3, §2, Theorem 2.7, pp. 79–80] separately over the interior and the boundary. A residual dense set of sufficiently small ε gives transversality on both. If $c > n + d$, both inverse images are empty.

The perturbed potentials are $Q_b + \sum_j z_j(b, \varepsilon) R_j$, and the metrics are g_b. Parameter-dependent elliptic regularity, as above, makes the local unit spinors jointly smooth. Their dependence on ε is continuous in every smooth seminorm. Taking ε sufficiently small gives the asserted approximation. □

13.2. **Currents and connectedness of the full incidence.** Put $I = [0, 1]$. For $g \in \mathcal{R}_\mu(M)$ and a smooth real spinor ψ, define its current $j_g(\psi)$ and squared length $\rho(\psi)$ by

$$g(j_g(\psi), X) = \langle a_g(X^\flat)\psi, \psi\rangle, \qquad \rho(\psi) = |\psi|^2, \qquad X \in TM. \tag{13.10}$$

At fixed g, put $j = j_g$. Let $(e_i)_{i=0}^{n-1}$ be a local orthonormal frame, let $(e^i)_{i=0}^{n-1}$ be its dual coframe, and put $a_i = a_g(e^i)$. By (13.2), the vectors $a_i\psi$ are orthogonal and have length $|\psi|$. Bessel's inequality and compatibility of the spin connection give

$$|j(\psi)| \le \rho(\psi), \qquad \operatorname{div}_g j(\psi) = 2\langle K_g\psi, \psi\rangle. \tag{13.11}$$

For the second identity, a normal orthonormal frame at a point gives

$$\operatorname{div}_g j(\psi) = 2\sum_{i=0}^{n-1} \langle a_i \nabla_{e_i}\psi, \psi\rangle = 2\langle K_g\psi, \psi\rangle.$$

If $(K_g + Q)\psi = 0$ with $Q^T = -Q$, then $\operatorname{div}_g j(\psi) = 0$. Conversely, suppose ψ is nowhere zero and $\operatorname{div}_g j(\psi) = 0$. Put $y = -K_g\psi$, so that $\langle y, \psi\rangle = 0$ pointwise. With $\psi^T(v) = \langle \psi, v\rangle$, the endomorphism

$$Q_g(\psi) = \frac{y\psi^T - \psi y^T}{|\psi|^2} \tag{13.12}$$

is smooth and skew-adjoint, and satisfies $Q_g(\psi)\psi = y$. Hence $(K_g + Q_g(\psi))\psi = 0$.

In an orthonormal spin frame, the real spinor fibre is $\Delta_n = \mathbb{R}^{2^\ell}$, and $(a_0, \dots, a_{n-1})$ is a symmetric Clifford system. Write $j(v) = (\langle a_i v, v\rangle)_{i=0}^{n-1}$ for its fibre current. The corresponding polynomial is

$$F(v) = |v|^4 - 2\sum_{i=0}^{n-1} \langle a_i v, v\rangle^2 = |v|^4 - 2|j(v)|^2, \qquad v \in \Delta_n.$$

The parameters in [49, §4] are $m = n - 1$ and $m_2 = \ell - n$.

Lemma 13.4. *Suppose $n \ge 17$. Every incidence point can be joined, in any prescribed neighborhood and with its metric fixed, to an incidence point whose spinor is nowhere zero and satisfies*

$$|j(\psi)| < |\psi|^2$$

everywhere. The incidence path can be chosen smooth. Every nowhere-zero spinor with this strict inequality and divergence-free current admits a smooth deformation, at the same metric, through nowhere-zero spinors with divergence-free current to a spinor with zero current.

Proof. Since $m_2 = \ell - n > 0$, the unit locus $|j(v)| = 1$ is the focal manifold $M_- = F^{-1}(-1) \cap S^{2\ell-1}$ of [49, §4.2, Theorem (i)]. It is a smooth $S^{\ell-1}$-bundle over S^{n-1}. For $x \in M$, the cone

$$C_x = \{v \in \Sigma^{\mathbb{R}}_{g,x} \setminus \{0\} \,:\, |j_g(v)| = |v|^2\}$$

has codimension

$$2\ell - (\ell + n - 1) = \ell - n + 1 > n + 1. \tag{13.13}$$

The current is Spin(n)-equivariant. Consequently $C = \bigcup_{x\in M} C_x$ is a smooth subbundle of $\Sigma^{\mathbb{R}}_g \setminus \{0\}$, invariant under every nonzero real dilation. Its projection to M is a submersion. The zero section has codimension $2\ell > n + 1$. In the finite-dimensional incidence manifold of Lemma 13.3, apply transversality first to the zero section and then to C. Nonvanishing is preserved by the second perturbation if it is sufficiently small. The perturbed spinor section is transverse to both targets and avoids them, since their codimensions exceed $\dim M$. A sufficiently small convex coordinate ball in this incidence manifold lies in the prescribed neighborhood and joins the original point to the perturbed one. The metric remains fixed.

Let ψ_0 be nowhere zero, with divergence-free current and $|j(\psi_0)| < |\psi_0|^2$. Put $j_0 = j(\psi_0)$, write $(j_0)_i = g(j_0, e_i)$, and solve

$$\frac{\partial \psi_s}{\partial s} = -\frac{a_g(j_0^\flat)\psi_s}{2|\psi_s|^2}, \qquad \psi_{s=0} = \psi_0. \tag{13.14}$$

The Clifford relation gives

$$\frac{\partial}{\partial s}\langle a_i \psi_s, \psi_s\rangle = -(j_0)_i, \qquad j(\psi_s) = (1 - s) j_0.$$

For $\rho_s = |\psi_s|^2$, it follows that

$$\frac{d\rho_s}{ds} = -\frac{(1 - s)|j_0|^2}{\rho_s}, \qquad \rho_s^2 = \rho_0^2 - (2s - s^2)|j_0|^2.$$

As long as the solution exists with $s \in I$,

$$0 < \min_M\big(\rho_0^2 - |j_0|^2\big) \le \rho_s^2 \le \max_M \rho_0^2.$$

The solution therefore extends smoothly over $M \times I$. Moreover,

$$\operatorname{div}_g j(\psi_s) = (1 - s)\operatorname{div}_g j_0 = 0, \qquad j(\psi_1) = 0.$$

Formula (13.12) supplies smooth incidence potentials $Q_g(\psi_s)$ along this deformation. □

Lemma 13.5. *For $n \ge 17$, the unit zero-current fibre*

$$Z = \{v \in \Delta_n \,:\, |v| = 1,\ j(v) = 0\}$$

is $(\ell - n - 1)$-connected, and hence is n-connected. The associated bundle of pointwise unit zero-current spinors admits a smooth section over M. Any two prescribed sections at the endpoint metrics extend to a smooth section along a prescribed smooth metric path on $M \times I$ that is stationary on endpoint collars. The extension can equal the prescribed sections on smaller endpoint collars.

Proof. The equations defining Z identify it with $M_+ = F^{-1}(1) \cap S^{2\ell-1}$, the smooth compact focal manifold of [49, §4.2, Theorem (ii)], with $m = n - 1$ and $m_2 = \ell - n > 0$. Its sphere-bundle description [80, §1, p. 2] is

$$S^{\ell-n} \longrightarrow Z \longrightarrow S^{\ell-1}. \tag{13.15}$$

The homotopy exact sequence [59, Theorem 4.41] implies that Z is $(\ell - n - 1)$-connected. Since $n \equiv 1 \pmod 8$, $n \ge 17$, and $\ell = 2^{(n-3)/2}$, one has $\ell - n - 1 \ge n$. The current equations are Spin(n)-invariant, so Z defines an associated smooth bundle, also along a varying metric.

The fibre is nonempty and path connected, and $\pi_{q-1}(Z) = 0$ for $2 \le q \le n+1$. The bases M and $(M \times I, M \times \partial I)$ have dimension at most $n+1$. Relative obstruction theory [59, §4.3, pp. 416–419] gives a continuous section and the prescribed relative extension, respectively. In the second case prescribe the endpoint sections on stationary collars before extending over the remaining cylinder. Relative smooth approximation [87, Theorem 11, p. 99], fixing smaller closed collars, gives smooth sections with the same collar values. For positive functions f, $j(f\psi) = f^2 j(\psi)$; hence positive amplitudes can be joined independently within the zero-current sections. □

For $n = 9$, one has $\ell = 8$ and $m_2 = -1$. By [49, §4.2, Theorem (i)], $F|_{S^{15}} = -1$, so $|j(v)| = |v|^2$ for every $v \in \Delta_9$. On unit spinors the current is the smooth Spin(9)-equivariant octonionic Hopf fibration [78, §4]:

$$S^7 \longrightarrow S^{15} \xrightarrow{j} S^8. \tag{13.16}$$

The fibre over $v \in S^8$ is the unit sphere in the eight-dimensional positive eigenspace of $a_g(v^\flat)$.

Lemma 13.6. *Let $n = 9$. Given two nowhere-zero incidence spinors and a smooth path between their metrics that is stationary on endpoint collars, there is a smooth nowhere-zero spinor path along these metrics with divergence-free current and the prescribed endpoint spinors. It can be stationary on endpoint collars. For each fixed metric there also exists a nowhere-zero spinor with divergence-free current.*

Proof. Let g_t, $t \in I$, be the prescribed metric path. The unit-spinor fibre S^{15} is 14-connected, whereas $\dim(M \times I) = 10$. Relative obstruction theory [59, §4.3, pp. 416–419] and smooth approximation [87, Theorem 11, p. 99] give a smooth unit-spinor path with the prescribed normalized endpoint values on smaller stationary collars. Interpolation of the logarithms of the endpoint lengths gives a smooth nowhere-zero spinor path ψ_t^0 with the prescribed endpoint values on these collars.

Let dv_{g_t} be the volume form for the spin orientation, and put

$$\alpha_t^0 = \iota_{j_{g_t}(\psi_t^0)} dv_{g_t}.$$

These eight-forms are nowhere zero, and their endpoint values are closed by (13.11). Write d_x for exterior differentiation on M. Choose $\chi \in C^\infty(I;[0,1])$, equal to zero near 0 and one near 1, with its transition outside the stationary collars, and set

$$\gamma_t = (1 - \chi(t))\alpha_0^0 + \chi(t)\alpha_1^0.$$

Then $d_x \gamma_t = 0$ and $\gamma_t = \alpha_t^0$ on the collars. Extend the data constantly to $t \le 0$ and $t \ge 1$.

Let $\mathrm{pr}_M : M \times \mathbb{R} \to M$ be the projection, put $\mathcal{B} = \mathrm{pr}_M^* \Lambda^7 T^*M$, and let $J^1\mathcal{B}$ be its first-jet bundle. For local sections β of $\mathcal{B}$, consider the open relation

$$\gamma_t + d_x \beta \neq 0 \tag{13.17}$$

in $J^1\mathcal{B}$. For a nonzero principal covector $\xi = \xi_x + \xi_t\, dt$, a derivative variation $\xi \otimes \eta$, with $\eta \in \Lambda^7 T_x^*M$, changes $d_x\beta$ by $\xi_x \wedge \eta$. If $\xi_x \neq 0$, the latter map has rank $\binom{8}{7} = 8$. The forbidden set in the principal affine slice is therefore empty or an affine subspace of codimension eight. Its complement is connected and has convex hull equal to the whole slice. If $\xi_x = 0$, every principal slice through an admissible jet is entirely admissible. Thus (13.17) is ample in every principal direction of $M \times \mathbb{R}$.

Spatial alternation

$$\mathcal{A} : T^*M \otimes \Lambda^7 T^*M \longrightarrow \Lambda^8 T^*M, \qquad \mathcal{A}(L) = \sum_{i=0}^{8} e^i \wedge L(e_i),$$

has the smooth right inverse

$$S(\omega)(X) = \tfrac{1}{8} \iota_X \omega, \qquad \omega \in \Lambda^8 T^*M, \quad X \in TM.$$

The jet with value zero, spatial derivative $S(\alpha_t^0 - \gamma_t)$, and time derivative zero is a smooth formal solution of (13.17). These derivative components are taken relative to the zero-section jet $j^1 0$. On the stationary collars this formal solution equals $j^1 0$.

Choose $0 < \varepsilon < 1/2$ so that this equality holds on a neighborhood of the closed set

$$A = M \times \big((-\infty, \varepsilon] \cup [1-\varepsilon, \infty)\big).$$

The relative open-ample theorem [47, Theorem 18.4.1 and Remark 1, p. 171], also in the formulation [72, Theorem 3.5], gives a homotopy through the relation to $j^1\beta$, with $\beta \in C^\infty(M \times \mathbb{R}; \mathcal{B})$, fixed near A. In particular, $\beta = 0$ there. Put $\beta_t = \beta(\cdot, t)$. The resulting homotopy of nowhere-zero eight-forms on M, restricted to $t \in I$, joins α_t^0 to

$$\alpha_t = \gamma_t + d_x\beta_t, \qquad d_x\alpha_t = 0.$$

After a reparametrization constant near the endpoints of the homotopy parameter $s \in I$, relative smooth approximation [87, Theorem 11, p. 99] in the bundle of nonzero eight-forms over $M \times I \times I$ gives a smooth homotopy $\alpha_{s,t}$ with

$$\alpha_{0,t} = \alpha_t^0, \qquad \alpha_{1,t} = \alpha_t,$$

unchanged on the time collars. Both endpoint families and the collar values are smooth and are fixed by this approximation.

Define $Y_{s,t}$ and $\nu_{s,t}$ by

$$\iota_{Y_{s,t}} dv_{g_t} = \alpha_{s,t}, \qquad \nu_{s,t} = Y_{s,t}/|Y_{s,t}|_{g_t}.$$

Along g_t, the equivariant fibration (13.16) defines a smooth bundle from unit spinors to unit tangent vectors. Choose a smooth connection on it. Horizontal lifting in s, starting at $\psi_t^0/|\psi_t^0|$, gives unit spinors $\widehat{\psi}_{s,t}$ with $j_{g_t}(\widehat{\psi}_{s,t}) = \nu_{s,t}$. The lifts exist for $s \in I$ because the fibre is compact. They are smooth and constant in s on the time collars. Set

$$\psi_t = \sqrt{|Y_{1,t}|_{g_t}}\, \widehat{\psi}_{1,t}.$$

Then $j_{g_t}(\psi_t) = Y_{1,t}$, so $\operatorname{div}_{g_t} j_{g_t}(\psi_t) = 0$. On the time collars, $Y_{1,t} = j_{g_t}(\psi_t^0)$ and $|Y_{1,t}|_{g_t} = |\psi_t^0|^2$; hence $\psi_t = \psi_t^0$ there.

For a fixed metric, the same obstruction argument on M gives a smooth nowhere-zero spinor. Its current eight-form is homotopic through nowhere-zero eight-forms to a nowhere-zero exact eight-form by [47, Corollary 20.3.2], with $n = 9$ and $p = 8$. The same relative smoothing, Hopf lift, and length rescaling give a nowhere-zero spinor with divergence-free current. □

Proposition 13.7. *For $\mathcal{P} = \mathcal{Q}_g$ and for $\mathcal{P} = \mathcal{P}_\mu$, the full projective incidence $\mathcal{I}(\mathcal{P})$ is nonempty and path connected. The connecting paths may be chosen smooth and stationary on endpoint collars.*

Proof. For $n \geq 17$, Lemma 13.5 gives a smooth unit zero-current spinor at any fixed metric. For $n = 9$, apply the existence assertion of Lemma 13.6. Formula (13.12) gives an incidence point in either case. Kernel parity gives a kernel of dimension at least two.

Let $(p_i, [\eta_i])$, $i = 0, 1$, be two incidence points, with metrics g_i. At these fixed metrics, join them by smooth incidence paths to points $(p_i', [\psi_i])$ with nowhere-zero spinors: for $n \geq 17$, apply Lemma 13.4, so that the strict current inequality also holds; for $n = 9$, apply Lemma 13.3 to the zero section, of codimension $16 > 9$. Let Q_i' be the potentials at these new points.

The logarithmic metric coordinates of Section 8 give a smooth path in $\mathcal{R}_\mu(M)$ from g_0 to g_1. Compose it with a smooth map $I \to I$ equal to the respective endpoints on endpoint collars. For $\mathcal{P} = \mathcal{Q}_g$, take the constant path $g_t = g$. If $n \geq 17$, deform each ψ_i to a nowhere-zero zero-current spinor by Lemma 13.4, and join the resulting spinors along this metric path by Lemma 13.5. Reverse the second deformation. If $n = 9$, apply Lemma 13.6. Stationary reparametrizations and concatenation give a smooth path (g_t, ψ_t) of nowhere-zero spinors with divergence-free current, equal to (g_i, ψ_i) on endpoint collars.

For $i = 0, 1$, put

$$B_i = Q_i' - Q_{g_i}(\psi_i).$$

Then $B_i^T = -B_i$ and $B_i\psi_i = 0$. Choose $\theta_i \in C^\infty(I;[0,1])$, equal to one near i and supported in the corresponding stationary collar. Extend $\theta_i B_i$ by zero outside that collar, and set

$$Q_t = Q_{g_t}(\psi_t) + \theta_0(t)B_0 + \theta_1(t)B_1.$$

This is a smooth skew-adjoint potential path satisfying

$$(K_{g_t} + Q_t)\psi_t = 0, \qquad Q_{t=i} = Q_i' \quad (i = 0, 1).$$

It joins $(p_0', [\psi_0])$ to $(p_1', [\psi_1])$. Concatenate it with the first preliminary incidence path and the reverse of the second, after stationary reparametrization. The resulting path joins the original incidence points. All metrics are constant in the fixed-metric case. □

13.3. **Compression on arbitrary kernel subspaces.** Fix $g \in \mathcal{R}_\mu(M)$, $Q \in \mathcal{Q}_g$, and put $L = K_g + Q$. Let $F \subset \ker L$ be a real subspace, with its L^2 inner product. This inner product identifies F with F^*, and hence $\Lambda^2 F$ with $\Lambda^2 F^*$. We identify these spaces with $\mathfrak{so}(F)$ by

$$u \wedge v \longleftrightarrow uv^T - vu^T, \qquad u^T(w) = \langle u, w\rangle_{L^2}, \qquad u, v, w \in F.$$

Thus a skew endomorphism A represents the alternating form $(u, v) \mapsto \langle u, Av\rangle_{L^2}$. On $\mathfrak{so}(F)$ the inner product is $\langle A, B\rangle = -\frac{1}{2}\operatorname{tr}(AB)$. Let

$$e_x : F \longrightarrow \Sigma^{\mathbb{R}}_{g,x}, \qquad e_x(\psi) = \psi(x), \qquad U_x = \operatorname{im}(e_x^*) \subset F.$$

The adjoint is taken with respect to the L^2 inner product on F and the Euclidean inner product in the spinor fibre. Retain the notation W_F of Lemma 13.2.

Lemma 13.8. *For every real subspace $F \subset \ker L$,*

$$W_F = \operatorname{span}_{x\in M} \Lambda^2 U_x. \tag{13.18}$$

In particular, W_F is spanned by decomposable skew endomorphisms. If $k = \dim_{\mathbb{R}} F \geq 2$, then $\operatorname{rank} e_x \geq 2$ *on a dense open subset of M. For even $k \geq 2$,*

$$\dim W_F \geq \begin{cases} 1, & k = 2, \\ 4, & k = 4, \\ k+1, & k \geq 6. \end{cases} \tag{13.19}$$

Proof. For $a, b \in \Sigma^{\mathbb{R}}_{g,x}$, the exterior-square identity is

$$e_x^*(ab^T - b\,a^T)e_x = (e_x^*a)(e_x^*b)^T - (e_x^*b)(e_x^*a)^T.$$

Since e_x^* maps onto U_x, Lemma 6.5, with the real skew endomorphism bundle, gives (13.18).

The real proportionality argument in the proof of Lemma 13.2 shows that independent elements of $\ker L$ cannot be pointwise proportional on a nonempty open set. Hence, if $k \geq 2$, the open set $\{x : \operatorname{rank} e_x \geq 2\}$ is dense. In particular, W_F contains a nonzero decomposable endomorphism.

For $0 \neq v \in F$, equation (13.6) gives a surjection

$$\varepsilon_v : W_F \longrightarrow v^\perp \cap F, \qquad \varepsilon_v(A) = Av.$$

Thus $\dim W_F \geq k - 1$, which proves the assertion for $k = 2$. If $k = 4$ and $\dim W_F = 3$, every ε_v is an isomorphism. A nonzero decomposable $A \in W_F$ has rank two and therefore has a nonzero kernel vector v, a contradiction. Hence $\dim W_F \geq 4$.

For even $k \geq 6$, (13.18) and $\dim W_F \geq k - 1$ give two independent decomposable endomorphisms $A, B \in W_F$. Since

$$\dim(\ker A \cap \ker B) \geq k - 4 > 0,$$

there is $0 \neq v \in \ker A \cap \ker B$. Then $\dim\ker \varepsilon_v \geq 2$, and $\dim W_F \geq (k-1) + 2 = k + 1$. □

13.4. **Compression on a four-dimensional real kernel.**

Lemma 13.9. *Suppose $F = \ker L$, $\dim_{\mathbb{R}} F = 4$, and $\dim W_F = 4$. Put $\mathcal{A} = W_F^{\perp} \subset \Lambda^2 F^*$. There are symplectic forms $\alpha, \beta \in \mathcal{A}$ and a complex structure $J = \alpha^{-1}\beta$ on F such that*

$$\alpha \wedge \beta = 0, \qquad \alpha \wedge \alpha = \beta \wedge \beta \neq 0, \qquad J^2 = -I.$$

The set

$$M_2 = \{x \in M \,:\, \operatorname{rank} e_x = 2\}$$

is dense and open. Over M_2, the spaces U_x form a smooth J-complex line subbundle of $M_2 \times F$. For every nonempty open $O \subset M_2$ and every $\delta > 0$, there are distinct $x, y \in O$ with $d_g(x, y) < \delta$ such that $\alpha|_{U_x \times U_y}$ is nondegenerate.

Proof. The two-plane $\mathcal{A}$ has no nonzero decomposable member. Indeed, let $v \wedge w \in \mathcal{A}$, with $v \neq 0$ and $w \perp v$. For every $B \in W_F$,

$$0 = \langle v \wedge w, B\rangle = -\langle w, Bv\rangle_{L^2}.$$

Equation (13.6) gives $w = 0$.

Choose $0 \neq \mathrm{vol}_F \in \Lambda^4 F^*$, and define the symmetric bilinear form q on $\Lambda^2 F^*$ by

$$\eta \wedge \zeta = q(\eta, \zeta)\mathrm{vol}_F.$$

It has signature $(3, 3)$, and its null cone consists of the decomposable forms. Thus $q|_{\mathcal{A}}$ is definite. Replace vol_F by its negative if necessary so that this restriction is positive, and choose a q-orthonormal basis α, β of $\mathcal{A}$. The linear-algebra correspondence of [42, §1] gives a complex structure J on F for which $\Omega = \alpha - i\beta$ is a nondegenerate complex bilinear alternating form. Its complex bilinearity gives

$$\alpha(Ju, v) = \alpha(u, Jv) = \beta(u, v), \qquad u, v \in F.$$

Consequently $J = \alpha^{-1}\beta$.

By (13.18), U_x is isotropic for both α and β. Since α is symplectic, $\dim U_x \leq 2$. If $\dim U_x = 2$, then $\{v \in F \,:\, \alpha(v, u) = 0 \text{ for all } u \in U_x\} = U_x$. Hence $\alpha(JU_x, U_x) = \beta(U_x, U_x) = 0$ gives $JU_x = U_x$. Lemma 13.8 makes M_2 dense and open. The constant-rank theorem gives the asserted smooth subbundle.

The map $x \mapsto U_x$ is not constant on any nonempty open subset of M_2. Otherwise, if its value were U, a nonzero $v \in F \cap U^{\perp}$ would satisfy $e_x v = 0$ throughout that subset. Unique continuation would give $v = 0$. Given O and δ, choose a coordinate ball in O of diameter less than δ, and then x, y in that ball with $U_x \neq U_y$. The form Ω pairs these distinct complex lines nontrivially. If its value on complex bases is $a + ib \neq 0$, the matrix of $\operatorname{Re}\Omega = \alpha$ in the corresponding real bases is

$$\begin{pmatrix} a & -b \\ -b & -a \end{pmatrix}, \qquad \det = -(a^2 + b^2) \neq 0.$$

□

For $F = \ker L$, let P be its L^2 orthogonal projection, put $\Pi = I - P$, and define the reduced inverse

$$G = \left(L|_{H^1 \cap F^{\perp}}\right)^{-1}\Pi \,:\, L^2 \longrightarrow H^1.$$

As a bounded operator on L^2, $G^T = -G$. Moreover, $LG = \Pi$ on L^2, and $GL = \Pi$ on H^1. For $T \in \mathcal{Q}_g$ sufficiently small in C^0, the operator

$$C_T = \Pi(L + T)|_{H^1 \cap F^{\perp}} \,:\, H^1 \cap F^{\perp} \longrightarrow L^2 \cap F^{\perp}$$

is invertible. Lemma 6.6 defines the Schur map

$$\Phi(T) = PT|_F - PTC_T^{-1}\Pi T|_F \in \mathfrak{so}(F),$$

with $\dim\ker\Phi(T) = \dim\ker(L + T)$. Compressions below are regarded as endomorphisms of F.

Lemma 13.10. *Let F, α be as in Lemma 13.9. There are smooth skew potentials $R, S \in \mathcal{Q}_g$, supported in disjoint coordinate balls in M_2, such that*

$$d\Phi_0(R) = d\Phi_0(S) = 0, \qquad \langle \alpha, d^2\Phi_0(R,S)\rangle \neq 0. \tag{13.20}$$

Proof. Differentiation at $T = 0$ gives

$$d\Phi_0(R) = PRP, \qquad d^2\Phi_0(R,S) = -P(RGS + SGR)P. \tag{13.21}$$

The two factors PT and $\Pi T|_F$ vanish at zero, and $C_0^{-1}\Pi = G$, which gives the second identity.

Let $G(x,y) : \Sigma^{\mathbb{R}}_{g,y} \to \Sigma^{\mathbb{R}}_{g,x}$ be the Schwartz kernel relative to $d\mu(y) = dv_g(y)$. The parametrix argument in the proof of Lemma 9.2, applied to $LG = GL = I - P$, shows that G is a classical pseudodifferential operator of order -1. The kernel of P is smooth. With Fourier phase $e^{i\langle x-y,\xi\rangle}$, the principal symbols are

$$\sigma_1(L)(x,\xi) = i a_g(\xi), \qquad \sigma_{-1}(G)(x,\xi) = -i\frac{a_g(\xi)}{|\xi|_g^2}.$$

Put $\omega_{n-1} = \mathrm{vol}(S^{n-1})$ for the Euclidean unit sphere, and write $d = d_g(x,y)$. For sufficiently close distinct x, y, let $\nu_{x,y}$ be the terminal unit covector of the unique short minimizing geodesic from y to x, and let $\tau_{y,x}$ be spin parallel transport along that geodesic. The Euclidean kernel formula [61, Corollary 5.3.4, p. 55], with spectral parameter zero and $K = iD$, gives

$$G(x,y) = \frac{d^{1-n}}{\omega_{n-1}} a_g(\nu_{x,y})\tau_{y,x} + O(d^{2-n}). \tag{13.22}$$

Indeed, in normal coordinates centred at y, the symbol after subtraction of its principal homogeneous part, cut off near $\xi = 0$, has order at most -2. The low-frequency terms have smooth kernels. Freezing the principal part at y adds terms of the form $(x_j - y_j)b_j(x,y,\xi)$, with b_j of order -1. Lemma 2.4, with $m = 2$ and $m = 1$, respectively, bounds the resulting kernels by Cd^{2-n}. The density and frame comparison factors are normalized to the identity on the diagonal. Their differences from the identity have the same coordinate-difference factorization and contribute $O(d^{2-n})$. On a smaller relatively compact coordinate ball these estimates are uniform in x, y.

Choose such a ball with closure in M_2. Since $a_g(\nu_{x,y})\tau_{y,x}$ is an isometry, (13.22) gives, for $u \in \Sigma^{\mathbb{R}}_{g,y}$,

$$|G(x,y)u| \geq d^{1-n}\big(\omega_{n-1}^{-1} - Cd\big)|u|.$$

Thus $G(x,y)$ is invertible for all sufficiently close distinct x, y in this ball. Lemma 13.9 supplies such a pair with nondegenerate $\alpha|_{U_x\times U_y}$.

For $z \in M_2$, put $E_z = \operatorname{im} e_z \subset \Sigma^{\mathbb{R}}_{g,z}$ and let π_z be the orthogonal projection onto E_z. These spaces form a smooth rank-two subbundle. The map

$$E_y^{\perp} \longrightarrow E_x^{\perp}, \qquad q \longmapsto (I - \pi_x)G(x,y)q$$

has rank at least $r - 4 > 0$: restriction of the invertible map $G(x,y)$ to a codimension-two subspace and projection with a two-dimensional kernel each lower its rank by at most two. Choose $p \in E_x^{\perp}$, $q \in E_y^{\perp}$ such that

$$c = \langle p, G(x,y)q\rangle \neq 0.$$

The maps $e_z^*|_{E_z} : E_z \to U_z$ are isomorphisms. Hence there are $a \in E_x$, $b \in E_y$ with

$$r_0 = e_x^* a, \qquad s_0 = e_y^* b, \qquad r_0^T \alpha\, s_0 \neq 0.$$

Choose coordinate balls $B_x, B_y \Subset M_2$, with disjoint closures, about x, y. Extend a, p smoothly on B_x with $a_z \in E_z$, $p_z \in E_z^{\perp}$, and extend b, q smoothly on B_y with $b_w \in E_w$, $q_w \in E_w^{\perp}$. Put

$$R_z^0 = p_z a_z^T - a_z p_z^T \quad (z \in B_x), \qquad S_w^0 = q_w b_w^T - b_w q_w^T \quad (w \in B_y).$$

They map the respective evaluation images into their orthogonal complements. For sufficiently small $\varepsilon > 0$, choose nonnegative smooth functions ρ_ε, σ_ε, with integrals one against μ, supported in $B_g(x,\varepsilon) \Subset B_x$ and $B_g(y,\varepsilon) \Subset B_y$, respectively. Extend $R_\varepsilon = \rho_\varepsilon R^0$ and $S_\varepsilon = \sigma_\varepsilon S^0$ by zero to M. For every $z \in M$,

$$e_z^* R_\varepsilon(z) e_z = e_z^* S_\varepsilon(z) e_z = 0.$$

Integration gives $PR_\varepsilon P = PS_\varepsilon P = 0$.

On $B_x \times B_y$, put $\boldsymbol{\mathcal{K}}(z,w) = e_z^* R_z^0 G(z,w) S_w^0 e_w$. The supports are disjoint, and this kernel is smooth. In the operator norm on F,

$$\left\| \iint_{B_x \times B_y} \rho_\varepsilon(z)\sigma_\varepsilon(w) \boldsymbol{\mathcal{K}}(z,w)\, d\mu(z)\, d\mu(w) - \boldsymbol{\mathcal{K}}(x,y) \right\| \leq \sup_{\substack{z \in \operatorname{supp} \rho_\varepsilon \\ w \in \operatorname{supp} \sigma_\varepsilon}} \| \boldsymbol{\mathcal{K}}(z,w) - \boldsymbol{\mathcal{K}}(x,y) \| \longrightarrow 0.$$

Since $e_x^* R_x^0 = -r_0 p^T$ and $S_y^0 e_y = q s_0^T$,

$$PR_\varepsilon G S_\varepsilon P \longrightarrow e_x^* R_x^0 G(x,y) S_y^0 e_y = -c\, r_0 s_0^T. \tag{13.23}$$

All three operators $R_\varepsilon, G, S_\varepsilon$ are skew-adjoint, so

$$(PR_\varepsilon G S_\varepsilon P)^T = -PS_\varepsilon G R_\varepsilon P.$$

Equations (13.21) and (13.23) now give

$$\left\langle \alpha, d^2\Phi_0(R_\varepsilon, S_\varepsilon) \right\rangle \longrightarrow c\, r_0^T \alpha s_0 \neq 0.$$

For sufficiently small positive ε, the choices $R = R_\varepsilon$, $S = S_\varepsilon$ satisfy (13.20). □

13.5. **Finite analytic families and avoidance.** In the logarithmic metric and linear potential coordinates of Section 8, a finite coefficient family is an open subset of a finite-dimensional affine subspace of $\mathcal{P}$. Write its injective affine parametrization as $p : \mathcal{U} \to \mathcal{P}$, where $\mathcal{U} \subset \mathbb{R}^N$ is open, and put $L(z) = L_{p(z)}$. The family contains a potential direction if that constant vector belongs to its translation space. Thus the direction is tangent to the family at every parameter. By Lemma 10.2, $z \mapsto L(z)$ is real analytic in $\mathcal{L}(H^1, L^2)$.

Lemma 13.11. *Let p_t, $t \in I = [0,1]$, be a smooth path in $\mathcal{P}$, stationary on prescribed endpoint collars. There are a finite-dimensional space $\mathcal{V}$ of smooth potential directions and a C^m-open neighborhood $\mathcal{N}$ of $p(I)$ with the following properties. For every $p \in \mathcal{N}$ and every L^2-unit vector $\psi \in \ker L_p$, the map*

$$\mathcal{V} \oplus (H^1 \cap \psi^\perp) \longrightarrow L^2 \cap \psi^\perp, \qquad (R,u) \longmapsto R\psi + L_p u$$

is surjective. For every subspace $F \subset \ker L_p$ of even dimension at least four, let $W_F(\mathcal{V}) = \{P_F R|_F : R \in \mathcal{V}\}$, where P_F is the L^2-orthogonal projection onto F. Then

$$\dim W_F(\mathcal{V}) \geq \begin{cases} 4, & \dim F = 4, \\ \dim F + 1, & \dim F \geq 6. \end{cases}$$

The path admits arbitrarily close C^∞ approximations in finite coefficient families contained in $\mathcal{N}$ and containing $\mathcal{V}$, with the prescribed collars fixed. For a smooth incidence path $(p_t, [\psi_t])$ stationary on these collars, the approximation can be chosen in the incidence of the finite family, with the same collars.

Proof. On a sufficiently small C^m-open neighborhood $\mathcal{N}_0$ of the compact path, the elliptic estimate gives

$$\|\psi\|_{H^1} \leq C \|\psi\|_{L^2} \qquad (p \in \mathcal{N}_0,\ \psi \in \ker L_p).$$

A finite cover by spectral neighborhoods also gives an upper bound $K \geq 2$ for $\dim\ker L_p$ on $\mathcal{N}_0$. For $1 \leq k \leq K$, let

$$\mathcal{F}_k = \{(t, v_1, \dots, v_k) : v_i \in \ker L_{p_t}, \quad \langle v_i, v_j\rangle_{L^2} = \delta_{ij}\}.$$

This set is compact in $I \times (L^2)^k$. Indeed, Rellich compactness gives, after passage to a subsequence, $t_j \to t$, strong L^2 convergence and weak H^1 convergence of each frame vector. Continuity in $\mathcal{L}(H^1, L^2)$ preserves the kernel equation, and strong L^2 convergence preserves orthonormality.

For every $(t, v, w) \in \mathcal{F}_2$, equation (13.6) gives a potential R with $\langle w, Rv\rangle_{L^2} \neq 0$. These pairings are continuous on $\mathcal{F}_2$, so a finite set of potentials has a nonzero pairing at every such pair. Its span maps onto $\ker L_{p_t} \cap v^\perp$ by $R \mapsto P_{\ker L_{p_t}} Rv$: a nonzero vector orthogonal to this image would contradict the preceding pairings. Since

$$\operatorname{coker}\big(L_{p_t} : H^1 \cap v^\perp \longrightarrow L^2 \cap v^\perp\big) \cong \ker L_{p_t} \cap v^\perp,$$

this is the required surjectivity for the incidence differential.

For each even $4 \leq k \leq K$, apply Lemma 13.8 to the span of a frame in $\mathcal{F}_k$. The required compression rank is expressed by a nonzero minor for finitely many potential directions. These minors are continuous on $\mathcal{F}_k$. Finite covers of the finitely many sets $\mathcal{F}_k$ give finitely many such potentials. Let $\mathcal{V}$ be their span together with the potentials chosen on $\mathcal{F}_2$. It satisfies the surjectivity and compression inequalities along the path.

They persist on a C^m-open neighborhood $\mathcal{N} \subset \mathcal{N}_0$ of the path. Otherwise, after passage to a subsequence, there are coefficients $q_j \to p_t$ in C^m and either orthonormal kernel pairs (v_j, w_j) with $\langle w_j, Rv_j\rangle_{L^2} = 0$ for every $R \in \mathcal{V}$, or orthonormal kernel k-frames for a fixed even $k \geq 4$ whose compression rank is below the stated bound. The uniform H^1 estimate gives a strong L^2 limit frame in $\ker L_{p_t}$. Every selected pairing or compressed entry converges, since the multiplication operators are bounded on L^2. The vanishing pairings or minors contradict the corresponding nonzero pairing or minor at the limit. The limiting kernel may have larger dimension; the selection included all its subspaces of the given dimension.

A smooth incidence path has a smooth L^2-unit lift over I, stationary on its prescribed collars, since a real line bundle over I is trivial. Lemma 10.2 applies with $\mathcal{V}$; its surjectivity hypothesis is the one established above. It gives the coefficient approximation and its incidence correction, with the collars fixed. Sufficiently close approximations have image in $\mathcal{N}$. □

Let $p : \mathcal{U} \to \mathcal{N}$ be a finite coefficient family containing $\mathcal{V}$ as in Lemma 13.11. Its incidence is

$$\mathcal{I}_{\mathcal{U}} = \{(z, [\psi]) : z \in \mathcal{U},\ 0 \neq \psi \in \ker L(z)\}.$$

It is a smooth N-dimensional manifold by Lemma 13.2. Fix $z_0 \in \mathcal{U}$ and put $F_0 = \ker L(z_0)$. In the orthogonal splitting $L^2 = F_0 \oplus F_0^\perp$, the complementary block is an isomorphism $H^1 \cap F_0^\perp \to L^2 \cap F_0^\perp$ near z_0. Lemma 6.6 gives a real analytic skew Schur map $\Phi : \mathcal{U}_0 \to \mathfrak{so}(F_0)$ on a neighborhood $\mathcal{U}_0$ of z_0, together with a graph isomorphism $\ker\Phi(z) \cong \ker L(z)$. The kernel-dimension loci in $\mathcal{U}_0$ are therefore semianalytic, being defined by finitely many minors. They admit a compatible locally finite stratification by real analytic submanifolds, by [24, Proposition 2.10 and Corollary 2.11]. Over a stratum T of positive kernel dimension k, the kernels form a smooth vector bundle. Its projectivization is a smooth submanifold of $\mathcal{I}_{\mathcal{U}}$, of dimension $\dim T + k - 1$. A countable Schur-chart cover gives a countable collection of coefficient strata and their projective lifts.

For any countable subcollection of these submanifolds, simultaneous transversality is residual in the space of smooth maps from a point or an interval, by [64, Chapter 3, §2, Theorem 2.1]. If prescribed endpoint collars avoid the submanifolds, the same assertion holds for paths fixed on those collars: the local perturbations in the proof of that theorem are supported off the collars. A transverse map whose source dimension is smaller than their codimensions misses every submanifold in the subcollection.

Lemma 13.12. *Let $p : \mathcal{U} \to \mathcal{N}$, $\mathcal{U} \subset \mathbb{R}^N$, be a finite coefficient family containing $\mathcal{V}$ as in Lemma 13.11. A smooth stratum of constant kernel dimension k has codimension at least four for $k = 4$, and at least $k + 1$ for even $k \geq 6$. Its projective lift in $\mathcal{I}_{\mathcal{U}}$ has codimension at least two for $k \geq 6$.*

Proof. At $z \in \mathcal{U}$, put $F = \ker L(z)$. Skew-adjointness identifies $\operatorname{coker} L(z)$ with F by the L^2 pairing. The reduced derivative is

$$\Lambda_z : T_z\mathcal{U} \longrightarrow \mathfrak{so}(F), \qquad v \longmapsto P_F\, dL_z(v)|_F;$$

see [41, Lemma 1.1.4]. By [41, Remark 1.1.14(1)], the constant-kernel locus near z is contained in a submanifold of codimension $\operatorname{rank}\Lambda_z$. The image of Λ_z contains $W_F(\mathcal{V})$, whose dimension has the lower bounds of Lemma 13.11. For $k \geq 6$, the projective lift has dimension at most

$$N - (k+1) + (k-1) = N - 2,$$

whereas $\dim \mathcal{I}_{\mathcal{U}} = N$. □

Lemma 13.13. *At a four-dimensional kernel there are a C^m-open coefficient neighborhood and finitely many smooth potential directions such that, in every finite coefficient family containing these directions, each smooth stratum of constant kernel dimension four in that neighborhood has codimension at least five.*

Proof. Let p_0 be the given coefficient and put $F_0 = \ker L_{p_0}$. On a C^m-open neighborhood of p_0, take the Schur map $\Phi : \mathcal{N}_0 \to \mathfrak{so}(F_0)$ for the fixed splitting $L^2 = F_0 \oplus F_0^{\perp}$. Here $\Phi(p_0) = 0$, the complementary block is invertible, and the four-kernel locus is $\Phi^{-1}(0)$. If $\dim W_{F_0} \geq 5$, choose five potentials whose differentials under Φ are independent at p_0. They remain independent on a smaller neighborhood. The tangent space of any smooth stratum in $\Phi^{-1}(0)$ lies in the kernel of the restricted derivative, so its codimension is at least five.

It remains to consider $\dim W_{F_0} = 4$, by Lemma 13.8. Choose potentials $B_1, \dots, B_4$ with independent compressions, a form $\alpha \in W_{F_0}^{\perp}$ from Lemma 13.9, and the corresponding potentials R, S from Lemma 13.10. For p near p_0, put

$$C_i(p) = d\Phi_p(B_i), \qquad \mathcal{W}_p = \operatorname{span}\{C_1(p), \dots, C_4(p)\}.$$

The $C_i(p)$ remain independent. On $\mathfrak{so}(F_0)$, let $\langle A, B\rangle = -\frac{1}{2}\operatorname{tr}(AB)$, and let Π_p be the orthogonal projection onto $\mathcal{W}_p$. Define the Gram matrix and the projection coefficients by

$$\mathsf{G}_{ij}(p) = \langle C_i(p), C_j(p)\rangle, \qquad d_j^V(p) = \langle C_j(p), d\Phi_p(V)\rangle, \qquad b^V(p) = \mathsf{G}(p)^{-1} d^V(p) \quad (V = R, S).$$

Put

$$\widehat{R}(p) = R - \sum_{i=1}^{4} b_i^R(p) B_i, \qquad \widehat{S}(p) = S - \sum_{i=1}^{4} b_i^S(p) B_i, \qquad \widehat{\alpha}(p) = \alpha - \Pi_p \alpha.$$

Thus $d\Phi_p(\widehat{V}(p)) = (I - \Pi_p) d\Phi_p(V)$ for $V = R, S$. These quantities depend smoothly on p. At p_0, one has $b^R = b^S = 0$ and $\widehat{\alpha} = \alpha$. Lemma 13.10 and continuity give, on a smaller neighborhood,

$$\Big\langle \widehat{\alpha}(p),\, d^2\Phi_p\big(\widehat{R}(p), \widehat{S}(p)\big) \Big\rangle \neq 0. \tag{13.24}$$

The second differential is taken in the fixed affine coefficient coordinates, with its two arguments held fixed at p.

Restrict now to any finite coefficient family containing $B_1, \dots, B_4, R, S$, and denote the restriction of Φ by Ψ. Let T be a smooth four-kernel stratum and $p \in T$. Since $\Psi|_T = 0$, $T_pT \subset \ker d\Psi_p$. If $\operatorname{rank} d\Psi_p \geq 5$, this gives the required bound. Otherwise $\operatorname{im} d\Psi_p = \mathcal{W}_p$, since it contains this four-dimensional space. Consequently

$$d\Psi_p(\widehat{R}(p)) = d\Psi_p(\widehat{S}(p)) = 0, \qquad \widehat{\alpha}(p) \perp \operatorname{im} d\Psi_p.$$

The image here includes all metric directions of the family. If T had codimension four at p, equality of dimensions would give $T_pT = \ker d\Psi_p$. For a local parametrization f of T, choose vectors u, v with $df(u) = \widehat{R}(p)$ and $df(v) = \widehat{S}(p)$. The identity $\Psi \circ f = 0$ gives

$$d^2\Psi_p(\widehat{R}(p), \widehat{S}(p)) + d\Psi_p\big(d^2 f(u, v)\big) = 0.$$

The affine inclusion of the family identifies the first term with $d^2\Phi_p(\widehat{R}(p), \widehat{S}(p))$. Pairing with $\widehat{\alpha}(p)$ contradicts (13.24). Thus the codimension is at least five. □

Proposition 13.14. *The simple incidence*

$$\mathcal{I}_2(\mathcal{P}) = \{(p, [\psi]) \in \mathcal{I}(\mathcal{P}) \,:\, \dim_{\mathbb{R}} \ker L_p = 2\}$$

is nonempty and path connected. A connecting path between prescribed simple incidence points can be chosen smooth and stationary on endpoint collars.

Proof. Let B be either a point or I. For B a point, choose an element of $\mathcal{I}(\mathcal{P})$, which is nonempty by Proposition 13.7. For $B = I$, that proposition gives a smooth incidence path between any two prescribed simple incidence points, stationary on endpoint collars. Apply Lemma 13.11 to this path, or to the constant path through the chosen point. It gives a finite family $p : \mathcal{U} \to \mathcal{N}$ containing $\mathcal{V}$, and a smooth map $\gamma : B \to \mathcal{I}_{\mathcal{U}}$, with the prescribed collars fixed in the interval case.

By Lemma 13.12, the projective lifts of the strata with kernel dimension at least six have codimension at least two. The preceding transversality statement gives an arbitrarily close perturbation of γ avoiding all of them, since $\dim B \leq 1$. In the interval case the perturbation is relative to the simple endpoint collars. For the perturbed map, again denoted by γ, the kernels have dimension two or four.

Let C_4 be the four-kernel subset of its coefficient image. It is compact by upper semicontinuity and the absence of larger kernels. A finite cover of C_4 by the neighborhoods of Lemma 13.13 gives finitely many additional potential directions. Adjoin them to the affine coefficient space. The map γ still lies in this enlarged family. If C_4 is empty, no additional directions are needed. Choose an open neighborhood $O \subset \mathcal{N}$ of the compact coefficient image on which every kernel has dimension at most four. If O_4 is the union of the selected four-kernel neighborhoods, replace O by

$$O \cap \big(\{p \,:\, \dim_{\mathbb{R}} \ker L_p < 4\} \cup O_4\big).$$

It still contains the image. Every four-kernel point in it belongs to a neighborhood of Lemma 13.13, and the enlarged family contains the corresponding directions. Since $O \subset \mathcal{N}$, the original space $\mathcal{V}$ gives incidence surjectivity throughout O.

Restrict the enlarged family to O, and let N' be its dimension. Its incidence is a smooth N'-manifold. Every four-kernel stratum has codimension at least five; its projective lift has dimension at most $N' - 5 + 3 = N' - 2$. A second perturbation, relative to the endpoint collars when $B = I$, avoids all these lifts. No larger kernel occurs in O. Incidence and even kernel parity therefore give kernel dimension exactly two. For B a point this proves nonemptiness; for $B = I$ it gives the required connecting path. □

13.6. **The simple-kernel hypersurface and the ordinary real index.** Write

$$Z_4 = \{p \in \mathcal{P} \,:\, \dim_{\mathbb{R}} \ker L_p \geq 4\}, \qquad X = \mathcal{P} \setminus Z_4, \qquad S = \{p \in X \,:\, \dim_{\mathbb{R}} \ker L_p = 2\}.$$

Lemma 13.15. *The space X is path connected. The subset S is a nonempty path-connected embedded hypersurface, closed in X. The complement $X \setminus S = \mathcal{P}^{\mathrm{inv}}$ has at most two path components.*

Proof. Upper semicontinuity of kernel dimension makes X open in the locally convex Fréchet coefficient space. Hence X is locally path connected. The global linear coordinates of $\mathcal{P}$ join any two points of X by a smooth coefficient path stationary on endpoint collars. Lemma 13.11 gives a nearby path in a finite family with the same collars and the stated compression bounds. Every coefficient stratum of kernel dimension at least four has codimension at least four, by Lemma 13.12. Relative transversality of the coefficient path therefore gives a path in X between the prescribed endpoints.

At $p \in S$, choose an ordered L^2-orthonormal basis of $F = \ker L_p$. The Schur map then has the form

$$\Phi(q) = f(q) J_0, \qquad J_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Equation (13.6) gives a smooth potential R with $df_p(R) \neq 0$. The scalar product-chart construction in the proof of Lemma 12.11 applies to f on the C^m completion and restricts to smooth coefficients. The hypersurface factor is convex. The sets $f = 0$, $f > 0$ and $f < 0$ are respectively the local two-kernel locus and its two path-connected invertible sides, since the complementary block remains invertible.

Noninvertibility is closed, so S is closed in X. The continuous surjection $\mathcal{I}_2(\mathcal{P}) \to S$ and Proposition 13.14 make S nonempty and path connected. Lemma 10.5 now applies to (X, S) and gives the asserted bound. □

In the fixed spinor comparisons, coefficient paths and homotopies give continuous families in $\mathcal{L}(H^1, L^2)$. Their graph norms are uniformly equivalent to the H^1 norm on compact parameter sets, by Section 2. They are therefore Riesz continuous, and hence gap continuous, by [28, §6.1, equation (6.1)]. For such a path with invertible endpoints, let

$$\mathrm{sf}_2(L_t) \in KO_2 \cong \mathbb{Z}/2$$

denote the real spectral flow of [28, §6.2]. It is homotopy invariant, additive under concatenation and direct sums, and zero on invertible paths; see [28, Remark 5.2 and §6.2]. The bounded transform $L_t(I - L_t^2)^{-1/2}$ has the same kernel and phase as L_t, with identical choices of complex structures on their kernels. The phase formulas of [28, Theorems 5.4 and 6.4] give equal spectral flows. By [28, Theorem 5.5(2)], this value is the $\mathbb{Z}/2$-valued spectral flow of [32]. The two-plane crossing formula [28, Remark 5.2(5), equation (5.1), and §6.2] gives the class of $(\mathbb{R}^2, J_0)$. Under [28, §2.4.4, equation (2.20)],

$$[(\mathbb{R}^2, J_0)] \longmapsto \dim_{\mathbb{C}}(\mathbb{R}^2, J_0) \pmod 2 = 1 \in KO_2. \tag{13.25}$$

Thus every transverse real skew two-plane crossing has spectral flow one.

Proof of Theorem 13.1. Choose $p_0 \in S$, and let R be a transverse potential direction in the scalar Schur chart of Lemma 13.15. For sufficiently small $\varepsilon > 0$, the curve $p(t) = p_0 + tR$, $-\varepsilon \le t \le \varepsilon$, has invertible endpoints on opposite local sides. Put $F = \ker L_{p_0}$. A smooth orthogonal trivialization of the rank-two spectral cluster, equal at $t = 0$ to the chosen basis of F, gives a matrix $b(t)J_0$. Its derivative satisfies

$$b(0) = 0, \qquad b'(0)J_0 = P_F R|_F = d\Phi_{p_0}(R) \neq 0.$$

The complementary operator remains invertible. The spectral reduction of [28, Lemma 6.3], direct-sum additivity and (13.25) give spectral flow one.

An invertible path from $p(\varepsilon)$ to $p(-\varepsilon)$ would give, by concatenation, a loop of spectral flow one. The global linear coefficient coordinates contract this loop with its invertible base point fixed. Homotopy invariance would give spectral flow zero, a contradiction. Thus the two local sides lie in different path components. Lemma 13.15 gives the upper bound two. There are exactly two nonempty components.

For $p_*, p \in \mathcal{P}^{\mathrm{inv}}$, two coefficient paths from p_* to p differ by a loop in the contractible coefficient space and therefore have equal spectral flows. This defines (13.3). The invariant is constant on invertible paths, and the local crossing gives both values in $\mathbb{Z}/2$. The map on components is consequently a bijection.

For $n \equiv 1 \pmod 8$, the real skew family in (8.3) is $L = i(D_g + V)$. Section 8.1 identifies its spectral flow with the ordinary relative family index in $KO_{n+1} \cong KO_2$, with normalization (8.1). For a path from $(g_*, 0)$ to (g, iV) in the real skew coefficient coordinates, its value is $\delta_{g_*}(g, V)$. □

The inequality $\#\pi_0 \mathcal{R}_\mu^{\mathrm{inv}}(M) \ge 2$, when this space is nonempty, follows from [39, Theorem 3.3 and Corollary 2.4] and Lemma 8.1. Proposition 8.2, with $U = M$, identifies its components with those of $\mathcal{P}_\mu^{\mathrm{inv}}$ and preserves the ordinary index difference. Theorem 13.1 therefore proves the $n \equiv 1 \pmod 8$ case of Theorem C, with two components.

14. Components and positive scalar curvature

Proof of Theorem C. Since D_{g_*} is invertible, $\alpha(M) = 0$. For $n \equiv 3, 7 \pmod 8$, the bijection is Theorem 11.6; for $n \equiv 2, 4, 5, 6 \pmod 8$, it is Theorem 11.11. For $n \equiv 0, 1 \pmod 8$, Theorems 12.1 and 13.1, respectively, give the bijections for invertible pairs. Proposition 8.2, with $U = M$, identifies these pair components with the metric components of density μ and preserves index difference. The component counts are the cardinalities of KO_{n+1}. □

Write $\mathcal{R}^+(M)$ for the space of smooth metrics of positive scalar curvature and put $\mathcal{R}^+_\mu(M) = \mathcal{R}^+(M) \cap \mathcal{R}_\mu(M)$.

Lemma 14.1. *Let M^n be a closed connected oriented manifold of positive dimension, and let μ be a positive smooth density. Every $h \in \mathcal{R}^+(M)$ can be joined in $\mathcal{R}^+(M)$ to a metric h_μ with $dv_{h_\mu} = \mu$. If M is spin with fixed spin structure, then h and h_μ lie in the same path component of $\mathcal{R}^{\mathrm{inv}}(M)$.*

Proof. Set

$$a = \left(\frac{\int_M \mu}{\int_M dv_h} \right)^{2/n}, \qquad h' = ah.$$

Then $\int_M dv_{h'} = \int_M \mu$. Identify positive densities with positive n-forms by the orientation of M. The forms $\omega_t = (1-t)\mu + t\, dv_{h'}$, $0 \le t \le 1$, have constant cohomology class. Moser's theorem [74, Theorem 2, Section 4, pp. 292–293] gives a smooth isotopy ϕ_t with $\phi_0 = \mathrm{id}_M$ and $\phi_t^* \omega_t = \mu$. Write $\mathrm{scal}(k)$ for the scalar curvature of a metric k. For

$$h_t = \phi_t^*(a^t h), \qquad 0 \le t \le 1,$$

one has

$$\mathrm{scal}(h_t) = a^{-t}\big(\mathrm{scal}(h) \circ \phi_t\big) > 0.$$

Moreover, $h_0 = h$ and $dv_{h_1} = \phi_1^* dv_{h'} = \mu$. Take $h_\mu = h_1$. In the spin case, D_{h_t} is invertible for every t by the Schrödinger–Lichnerowicz formula [61, Theorem 2.1.3]. □

Proof of Corollary 1.1. By Lemma 14.1, choose $g_+ \in \mathcal{R}^+_\mu(M)$. The restriction of δ_{g_+} to $\mathcal{R}^+(M)$ is the index difference of [26, Section 3.3.1], with g_+ as first endpoint. For even $n \ge 6$, [26, Theorem D], with homotopy degree zero, gives the surjectivity of

$$\delta_{g_+} : \pi_0 \mathcal{R}^+(M) \longrightarrow KO_{n+1}.$$

For $n = 2m+1 \ge 7$, put $d = 2m$. Let $\gamma_d \to B\mathrm{Spin}(d)$ be the universal real spin d-plane bundle, and let $MT\mathrm{Spin}(d)$ be the Thom spectrum of $-\gamma_d$. Write ko for the connective real K-theory spectrum and $ko_j = \pi_j(ko)$ for $j \in \mathbb{Z}$. Taking components in the map of [26, Theorem C] gives

$$\pi_2 MT\mathrm{Spin}(2m) \longrightarrow \pi_0 \mathcal{R}^+(M) \xrightarrow{\delta_{g_+}} KO_{2m+2},$$

whose composite is induced by the spin Thom homomorphism. By [26, Theorem 5.3.1], with $d = 2m$ and $k = 2 \le 2m+1$, this homomorphism is onto $ko_{2m+2} \cong KO_{2m+2}$. Thus δ_{g_+} is surjective also in odd dimensions $n \ge 7$. For $n = 5$, its target is $KO_6 = 0$ and its domain is nonempty.

Let $g \in \mathcal{R}^{\mathrm{inv}}_\mu(M)$. Choose $h \in \mathcal{R}^+(M)$ with $\delta_{g_+}(h) = \delta_{g_+}(g)$. Lemma 14.1 gives $h_\mu \in \mathcal{R}^+_\mu(M)$ joined to h through invertible metrics. Hence

$$\delta_{g_+}(h_\mu) = \delta_{g_+}(h) = \delta_{g_+}(g).$$

Theorem C gives $[h_\mu] = [g]$ in $\pi_0 \mathcal{R}^{\mathrm{inv}}_\mu(M)$. □

If M is simply connected and $n \ge 5$, then $\alpha(M) = 0$ implies $\mathcal{R}^+(M) \ne \emptyset$ by [83, Theorem A]. If $\alpha(M) \ne 0$, both $\mathcal{R}^+_\mu(M)$ and $\mathcal{R}^{\mathrm{inv}}_\mu(M)$ are empty.

15. Invertible boundary extensions

15.1. **The relative index and the bordism group.**

Proof of Theorem I. The relative index is independent of the product extension [40, Section 5.4] and vanishes for an invertible extension. Conversely, suppose that $\alpha(W, h) = 0$. Let g be a product extension of h, and let $U \Subset \mathrm{int}\, W$ be nonempty and open. Put $X = W_\infty$ if $\partial W \ne \emptyset$, and $X = W$ otherwise. Extend all product metrics to X by the fixed metric $dt^2 + h$ on the cylindrical ends.

Choose a nonempty open set $U_0 \Subset U$ and put $K = \overline{U_0}$. Proposition 9.1 gives a compatible potential $V \in C_c^\infty(U_0; \mathscr{A}_n)$ such that $D_g + V$ is invertible on X. Corollary 5.2, with the same open set, gives an invertible metric $\widehat{g}$ with $dv_{\widehat{g}} = dv_g$ and $\widehat{g} = g$ on $X \setminus U_0$.

The positive g-self-adjoint endomorphism $S = g^{-1}\widehat{g}$ has a smooth g-self-adjoint logarithm $H = \log S$. Since $dv_{\widehat{g}}/dv_g = (\det S)^{1/2} = 1$,

$$\operatorname{supp} H \subset K \Subset U, \qquad \operatorname{tr} H = \log\det S = 0.$$

For $g_t = g(e^{tH}\cdot, \cdot)$, $0 \le t \le 1$, one has

$$g_0 = g, \qquad g_1 = \widehat{g}, \qquad dv_{g_t} = e^{t\operatorname{tr} H/2} dv_g = dv_g, \qquad g_t = g \quad \text{on } X \setminus U_0.$$

Put $\mathscr{D} = H^1(\Sigma_g X)$ and $\mathscr{H} = L^2(\Sigma_g X)$. With reference metric g in (2.6), set

$$L_t = U_{g_t}^{-1} D_{g_t} U_{g_t} \,:\, \mathscr{D} \longrightarrow \mathscr{H}.$$

The frame map is $e^{-tH/2}$ and the density factor is one. In fixed local frames, the coefficients in [61, Theorem 2.3.6] are finite sums of products of entries of $e^{\pm tH/2}$, g_t, g_t^{-1}, their first spatial derivatives, and fixed smooth coefficients. For every integer $r \ge 0$, the map $\mathbb{C} \ni z \mapsto e^{zH}$ is entire in the spatial C^{r+1} norm on K. Substituting $z \in \mathbb{C}$ for t in the coefficient formulas therefore gives coefficients holomorphic in the spatial C^r norm on K. The coefficients of $L_z - D_g$ are supported in K, so the resulting family $L_z : \mathscr{D} \to \mathscr{H}$ is holomorphic in operator norm near $[0,1]$. For real t, it is Fredholm by Section 2. The openness of the Fredholm set gives a connected complex neighborhood Ω of $[0,1]$ on which every L_z is Fredholm. Since L_1 is invertible, $L_1^{-1} : \mathscr{H} \to \mathscr{D}$ is bounded, and

$$A(z) = L_z L_1^{-1} \in \mathcal{L}(\mathscr{H}), \qquad A(1) = I,$$

is a holomorphic Fredholm family on Ω. By the analytic Fredholm theorem [84, Theorem 1.1, p. 2], the noninvertible parameters form a discrete subset of Ω. As $0 \in \Omega$, there is an $\varepsilon \in (0,1)$ such that D_{g_t} is invertible for $0 < t < \varepsilon$. Since H is smooth and compactly supported, $g_t \to g$ in C^∞ as $t \to 0$. □

Let R_n^{inv} be the relative spin bordism group of [40, Section 5.2], and let $\theta_n : R_n^{\mathrm{inv}} \to KO_n$ be its Dirac index homomorphism [40, Section 5.4].

Corollary 15.1. *For $n \ge 3$, $\theta_n : R_n^{\mathrm{inv}} \to KO_n$ is an isomorphism.*

Proof. Surjectivity is proved in [40, Section 5.4]. Let $[M,h] \in \ker\theta_n$. The empty representative is zero. If $M \ne \emptyset$, write $M = \bigsqcup_{j=1}^r M_j$ for its connected components. Attach $r-1$ one-handles to $M \times [0,1]$, joining $M_j \times \{1\}$ to $M_{j+1} \times \{1\}$ for $1 \le j < r$, along pairwise disjoint interior n-discs. The spin structures on the attaching discs extend over each handle after a choice of attaching spin identifications. Denote the trace by T, its connected upper face by N, and its side face by $C = \partial M \times [0,1]$. The inherited collars identify ∂N with ∂M, including their spin structures. With the induced orientations, rounding the corners gives

$$\partial T = (-M) \cup_{\partial M} C \cup_{\partial N} N.$$

If $C \ne \emptyset$, its cylindrical completion is $C_\infty = \partial M \times \mathbb{R}$, with metric $h + dt^2$. Put $\lambda_h = \operatorname{dist}(0, \operatorname{spec} D_h) > 0$. The product formula [40, proof of Proposition 2.1(3)] gives

$$\|D_{h+dt^2}\psi\|_{L^2}^2 \ge \lambda_h^2 \|\psi\|_{L^2}^2, \qquad \psi \in C_c^\infty(\Sigma C_\infty).$$

Thus $h + dt^2 \in \mathcal{R}^{\mathrm{inv}}(C)$. For $C = \emptyset$ the side metric is empty. The defining relation [40, Section 5.2] therefore gives $[M,h] = [N,h]$, and

$$\alpha(N,h) = \theta_n([N,h]) = \theta_n([M,h]) = 0.$$

Theorem I gives an invertible product extension of h over N. By [40, Corollary 5.6], $[N,h] = 0$. Hence $\ker\theta_n = 0$. □

16. Spectral matrices on Berger products

16.1. **The orbit of a middle-weight projector.** For $a \in \frac{1}{2}\mathbb{Z}_{\geq 0}$, let $V_a = \operatorname{Sym}^{2a}\mathbb{C}^2$, with an $SU(2)$-invariant Hermitian inner product. Fix an integer $k \geq 1$ and put $j = k - \frac{1}{2}$. Write ρ for the action on V_j. Choose self-adjoint generators J_1, J_2, J_3 with $[J_1, J_2] = iJ_3$ and cyclic permutations. Choose an orthonormal basis v_m, $m = -j, -j+1, \dots, j$, such that

$$J_3 v_m = m v_m, \qquad (J_1 + iJ_2)v_m = \sqrt{(j-m)(j+m+1)}\, v_{m+1} \quad (m < j),$$

and $(J_1 + iJ_2)v_j = 0$. Let P_m be orthogonal projection onto $\mathbb{C}v_m$.

Lemma 16.1. *For $k \geq 1$ and every nonempty open $O \subset SU(2)$,*

$$\operatorname{span}_{\mathbb{R}}\{\rho(x)P_{1/2}\rho(x)^{-1} \,:\, x \in O\} = \operatorname{Herm}_{\mathbb{C}}(V_j). \tag{16.1}$$

The same equality holds with $P_{-1/2}$.

Proof. For the conjugation representation, the Clebsch–Gordan decomposition [51, Exercise 11.11, p. 151] is

$$\operatorname{End}_{\mathbb{C}}(V_j) = \bigoplus_{\ell=0}^{2k-1} V_\ell.$$

The summands are the eigenspaces of $C(A) = \sum_{a=1}^{3}[J_a, [J_a, A]]$, with respective eigenvalues $\ell(\ell+1)$ [10, Section 2.1, equations (2.4)–(2.5)]. Since $C(A^*) = C(A)^*$, adjoint preserves each summand. Its one-dimensional weight-zero subspace therefore contains a nonzero Hermitian vector T. Write $Tv_m = q_m v_m$, with $q_m \in \mathbb{R}$. In $V_\ell = \operatorname{Sym}^{2\ell}\mathbb{C}^2$, the weight-zero monomial is $z^\ell w^\ell$. The Weyl transformation $(z, w) \mapsto (w, -z)$ gives

$$q_{-m} = (-1)^\ell q_m. \tag{16.2}$$

The diagonal entries of $C(T) = \ell(\ell+1)T$ are

$$\begin{aligned} \ell(\ell+1)q_m = {} & (j-m)(j+m+1)(q_m - q_{m+1}) \\ & + (j+m)(j-m+1)(q_m - q_{m-1}). \end{aligned} \tag{16.3}$$

Terms with indices outside $[-j, j]$ are omitted. If $\operatorname{tr}(P_{1/2}T) = 0$, then $q_{1/2} = q_{-1/2} = 0$ by (16.2). For $j > 1/2$, the coefficient $(j-m)(j+m+1)$ is positive for $m = 1/2, 3/2, \dots, j-1$; successive substitution in (16.3) gives $q_{3/2} = \cdots = q_j = 0$. Equation (16.2) gives $q_m = 0$ also for $m < 0$. For $j = 1/2$, the two initial entries are all the entries. In either case $T = 0$, a contradiction.

Thus $P_{1/2}$ has nonzero projection onto every irreducible summand. Since the decomposition is multiplicity free, [36, Section 3, equations (12)–(13)] implies that its orbit has complex span $\operatorname{End}_{\mathbb{C}}(V_j)$. Every orbit element is Hermitian, so its real span is $\operatorname{Herm}_{\mathbb{C}}(V_j)$. For $H \in \operatorname{Herm}_{\mathbb{C}}(V_j)$, the function

$$x \longmapsto \operatorname{tr}\big(H\rho(x)P_{1/2}\rho(x)^*\big)$$

is real analytic on connected $SU(2)$. If it vanishes on O, it vanishes everywhere, whence $H = 0$. The trace pairing is nondegenerate on $\operatorname{Herm}_{\mathbb{C}}(V_j)$, so (16.1) follows. A Weyl representative w_0 exchanges $P_{1/2}$ and $P_{-1/2}$; the equality for $P_{1/2}$ on Ow_0 gives the equality for $P_{-1/2}$ on O. □

16.2. **The Berger kernel and its evaluation.** Identify S^3 with $SU(2)$. Choose left-invariant fields X_1, X_2, X_3 with $[X_1, X_2] = 2X_3$ and cyclic permutations, and let ω_i be the dual coframe. For $t > 0$, put

$$g_t = t^2\omega_1^2 + \omega_2^2 + \omega_3^2, \qquad S^3_t = (SU(2), g_t). \tag{16.4}$$

The oriented orthonormal frame is $E_1 = t^{-1}X_1$, $E_2 = X_2$, $E_3 = X_3$. Let

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In the corresponding spin frame take $c_a = c(E_a^\flat)$, with $c_1 = i\sigma_3$, $c_2 = i\sigma_1$, and $c_3 = i\sigma_2$; then $c_1c_2c_3 = I$. Hitchin's formulas [65, Propositions 3.1–3.2], in these conventions, give

$$D_t = t^{-1}c_1X_1 + c_2X_2 + c_3X_3 + (t/2 + t^{-1})I. \tag{16.5}$$

Besides the positive eigenvalues $t/2 + p/t$, $p \in \mathbb{Z}_{\geq 1}$, the eigenvalues are

$$\frac{t}{2} \pm \sqrt{4pq + \frac{(p-q)^2}{t^2}}, \qquad p, q \in \mathbb{Z}_{\geq 1}. \tag{16.6}$$

The matrix-coefficient map

$$\ell \otimes v \longmapsto \bigl[x \longmapsto \ell(\rho(x)v)\bigr]$$

identifies $V_j^* \otimes V_j$ with its Peter–Weyl summand. Left-invariant fields act on the second factor; thus V_j^* is the multiplicity space in $V_j^* \otimes V_j \otimes \mathbb{C}^2$. Take $d\rho(X_1) = -2iJ_3$, $d\rho(X_2) = -2iJ_1$, and $d\rho(X_3) = -2iJ_2$. On the pair

$$v_m \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad v_{m+1} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad -j \leq m < j,$$

the operator matrix is

$$(t/2 + t^{-1})I + \begin{pmatrix} 2m/t & 2\sqrt{(j-m)(j+m+1)} \\ 2\sqrt{(j-m)(j+m+1)} & -2(m+1)/t \end{pmatrix}. \tag{16.7}$$

This block corresponds to $p = j + m + 1$ and $q = j - m$ in (16.6).

Let $v_m^\vee$ be the basis of V_j^* dual to v_m, and let $P_m^\vee$ be orthogonal projection onto $\mathbb{C}v_m^\vee$. Write $\rho^\vee(x)\ell = \ell \circ \rho(x)^{-1}$ for the contragredient representation; its matrix in this basis is $\bar\rho(x)$. Denote the total volume of g_t by $\mathrm{Vol}(g_t)$.

Lemma 16.2. *If k is prime, $k \equiv 3 \pmod 4$, and $t = 4k$, then*

$$\ker D_t \cong V_{k-1/2}^*, \qquad \dim_{\mathbb{C}} \ker D_t = 2k.$$

Let $e_a(x) : \ker D_t \to \mathbb{C}$ be evaluation at x followed by the a-th spinor coordinate, $a = 1, 2$. Under a unitary identification $\ker D_t \cong V_j^*$, each family $\{e_a(x)^* e_a(x) : x \in SU(2)\}$ is the conjugation orbit of $k\,\mathrm{Vol}(g_t)^{-1} P_{1/2}^\vee$ under $\rho^\vee$.

Proof. For $t = 4k$ with k prime and $k \equiv 3 \pmod 4$, the only zero in (16.6) has $p = q = k$; this is the observation of Chowla recorded in [65, p. 32, Remarks (1)(a)].

Thus $j = k - 1/2$ and $m = -1/2$. The block (16.7) is $\left(\begin{smallmatrix} t/2 & 2k \\ 2k & t/2 \end{smallmatrix}\right)$, whose zero eigenvector at $t = 4k$ is $(1, -1)/\sqrt{2}$. For $\ell \in V_j^*$ the corresponding spinor is

$$\Psi_\ell(x) = \sqrt{\frac{k}{\mathrm{Vol}(g_t)}} \begin{pmatrix} \ell(\rho(x)v_{-1/2}) \\ -\ell(\rho(x)v_{1/2}) \end{pmatrix}. \tag{16.8}$$

For Haar measure dx of total mass one, Schur orthogonality [36, Section 3, equation (9)] gives

$$\int_{SU(2)} \rho(x) P_m \rho(x)^*\, dx = \frac{I}{2k}.$$

Since $dv_{g_t} = \mathrm{Vol}(g_t)\, dx$,

$$\int_{SU(2)} |\ell(\rho(x)v_m)|^2\, dv_{g_t} = \frac{\mathrm{Vol}(g_t)}{2k}\, \|\ell\|^2.$$

Thus (16.8) is an L^2 isometry on V_j^*. Under this identification,

$$e_1(x)^* e_1(x) = \frac{k}{\mathrm{Vol}(g_t)}\, \rho^\vee(x) P_{-1/2}^\vee \rho^\vee(x)^*. \tag{16.9}$$

The formula for $e_2(x)^* e_2(x)$ has $P_{1/2}^\vee$ in place of $P_{-1/2}^\vee$. A unitary intertwiner $V_j^* \cong V_j$ sends $\mathbb{C}v_m^\vee$ to $\mathbb{C}v_{-m}$. The two middle-weight projectors are conjugate by a Weyl representative, and Lemma 16.1 applies to each. $\square$

16.3. **Flat factors and compatible compression.** In the rest of this section, k is prime and $k \equiv 3 \pmod 4$. Let $n \geq 3$ and put $d = n-3$. Give $T^d = \mathbb{R}^d/(2\pi\mathbb{Z})^d$ its flat metric and its spin structure with constant spinors. For $d = 0$, take T^0 to be a point, with volume one, spinor space $\mathbb{C}$, and $D_{T^0} = 0$. Set

$$X_{n,k} = S^3_{4k} \times T^{n-3}. \tag{16.10}$$

Hitchin gives the product formula for harmonic spinors [65, p. 12, Remark (4)]. In the product spinor identification, with an extra factor $\mathbb{C}^2$ when d is odd,

$$D^2_{X_{n,k}} = D^2_{S^3_{4k}} \otimes I + I \otimes D^2_{T^d}, \tag{16.11}$$

where identity on the extra factor is understood. The torus kernel consists of parallel spinors. Set

$$W_n = \begin{cases} \ker D_{T^d}, & d \text{ even}, \\ \ker D_{T^d} \otimes \mathbb{C}^2, & d \text{ odd}. \end{cases}$$

In particular, $W_3 = \mathbb{C}$. The product formula and Lemma 16.2 give

$$E_{n,k} = \ker D_{X_{n,k}} \cong V^*_{k-1/2} \otimes W_n, \qquad \dim_{\mathbb{C}} W_n = 2^{\lfloor n/2 \rfloor - 1}. \tag{16.12}$$

Evaluation identifies parallel torus spinors with the full torus spinor fibre. For the L^2 inner product on parallel sections, multiplication of evaluation by $\mathrm{Vol}(T^d)^{1/2}$ makes this identification unitary. Thus the product spinor fibre is identified with $\mathbb{C}^2 \otimes W_n$. Let Q be projection onto the first coordinate line in $\mathbb{C}^2$. Then $Q \otimes T$ is a Hermitian fibre endomorphism for every $T \in \mathrm{Herm}_{\mathbb{C}}(W_n)$. Put $E = E_{n,k}$, and let P_E be the L^2 orthogonal projection onto E.

Theorem 16.3. *For every nonempty open $U \subset X_{n,k}$, the map*

$$C^\infty_c(U; \mathscr{A}_n) \longrightarrow \mathcal{A}_n(E), \qquad V \longmapsto P_E V|_E \tag{16.13}$$

is onto. It has a linear right inverse whose image is supported in one compact subset of U. There is a full potential section at $D_{X_{n,k}}$ with the same support property.

Proof. Put $j = k - 1/2$, and let $e(x,y) : E \to \Sigma_{(x,y)} X_{n,k}$ be evaluation. For unrestricted Hermitian multiplication endomorphisms, Lemma 6.5 identifies the compression image with the real span of $e(x,y)^* Q' e(x,y)$, where $(x,y) \in U$ and Q' ranges over Hermitian fibre endomorphisms. Choose a nonempty product neighbourhood $U_3 \times U_T \Subset U$. Equation (16.9) and the torus normalization give, for $x \in U_3$, $y \in U_T$, and $T \in \mathrm{Herm}_{\mathbb{C}}(W_n)$,

$$e(x,y)^*(Q \otimes T) e(x,y) = \frac{k}{\mathrm{Vol}(g_{4k})\,\mathrm{Vol}(T^d)} \rho^\vee(x) P^\vee_{-1/2} \rho^\vee(x)^* \otimes T.$$

By Lemma 16.1, these matrices span

$$\mathrm{Herm}_{\mathbb{C}}(V_j^*) \otimes_{\mathbb{R}} \mathrm{Herm}_{\mathbb{C}}(W_n) \cong \mathrm{Herm}_{\mathbb{C}}(E).$$

Thus unrestricted compression is onto.

On Hermitian endomorphisms put $\mathcal{J}(A) = JAJ^{-1}$ and, in even dimension, $\mathcal{G}(A) = \Gamma A \Gamma$. These are commuting real-linear involutions, since $J^2 = \pm I$, $\Gamma^2 = I$, and $J\Gamma = \pm \Gamma J$. The projection onto (2.2) is

$$\Pi = \frac{I + \epsilon_n \mathcal{J}}{2} \quad (n \text{ odd}), \qquad \Pi = \frac{I + \epsilon_n \mathcal{J}}{2} \frac{I - \mathcal{G}}{2} \quad (n \text{ even}).$$

Since P_E commutes with J and, in even dimension, with Γ, compression intertwines these projections on the bundle and on E. For $A \in \mathcal{A}_n(E)$, an unrestricted preimage V therefore gives a compatible preimage ΠV, with $\mathrm{supp}(\Pi V) \subset \mathrm{supp}\, V$.

Choose compatible preimages $V_1, \dots, V_N$ of a real basis of $\mathcal{A}_n(E)$. Their supports lie in one compact subset of U, and they define a linear right inverse s of compression. Choose a circle γ centered at zero in the resolvent set of $D_{X_{n,k}}$, enclosing no other eigenvalue. For A near zero, γ remains in the resolvent

set of $D_{X_{n,k}} + s(A)$. The enclosed spectral projection commutes with J and, in even dimension, with Γ. Let U_A be the polar frame of Lemma 6.1, based at the inclusion of E. It intertwines these structures, and

$$F(A) = U_A^*(D_{X_{n,k}} + s(A))U_A \in \mathcal{A}_n(E)$$

is smooth. By Lemma 6.4, $F(0) = 0$ and $dF_0 = I$. The inverse function theorem [50, Theorem 11.22 and Corollary 11.35] gives the full potential section $A \mapsto s(F^{-1}(A))$ near zero. The complementary spectrum lies outside γ. □

Corollary 16.4. *For every $n \geq 3$ and every nonempty open $U \subset S^3 \times T^{n-3}$, there are full metric sections $A \mapsto g(A)$ with balanced zero modules of arbitrarily large complex dimension, such that*

$$dv_{g(A)} = dv_{g(0)}, \qquad \operatorname{supp}\big(g(A) - g(0)\big) \Subset U.$$

Every smaller balanced zero module occurs after fixing an invertible complementary block.

Proof. Choose a nonempty open $U_0 \Subset U$ and apply Proposition 6.3 to the potential section of Theorem 16.3 supported in U_0. The metric family has fixed volume density and equals the product metric outside U_0; hence its differences have support in $\overline{U_0} \subset U$. There are arbitrarily large primes $3 \pmod 4$ [8, Theorem 7.1, p. 147], and (16.12) gives $\dim_{\mathbb{C}} E_{n,k} = k\, 2^{\lfloor n/2 \rfloor}$.

For even n, the grading on the extra $\mathbb{C}^2$ factor gives equal chiral ranks. In residues one and two, the skew matrix sizes are respectively

$$k\, 2^{\lfloor n/2 \rfloor}, \qquad k\, 2^{\lfloor n/2 \rfloor - 1};$$

both are even, since the least dimensions in these residues are nine and ten. The other cases of (2.4) impose no further restriction. Hence $E_{n,k}$ is balanced.

By (2.4), every balanced zero module E' of smaller complex dimension admits a J-invariant and, in even dimension, Γ-invariant orthogonal embedding in $E_{n,k}$. Write $E_{n,k} = E' \oplus E''$ for the resulting decomposition. The complement is balanced: in the chiral cases subtract equal ranks, and in the skew cases subtract even sizes. Fix a sufficiently small invertible $B \in \mathcal{A}_n(E'')$ and restrict the full metric section to matrices $A \oplus B$, $A \in \mathcal{A}_n(E')$. Its value at $0 \oplus B$ has zero module E'. Choose a radius smaller than both $\operatorname{dist}(0, \operatorname{spec} B)$ and the original spectral radius. For $A \oplus B$ in the original parameter domain and $\|A\|$ below this radius, the enclosed spectral subspace is the image of E' under the original frame, and its restriction matrix is A. The density and support conditions remain valid relative to the metric at $0 \oplus B$. □

17. Kernel dimensions and the real index

The bases B of the parameter disk bundles below are compact smooth manifolds without boundary. Vector bundles carry fixed smooth Euclidean, Hermitian, or quaternionic Hermitian metrics. Matrix bundles have the associated Hilbert–Schmidt inner products, and direct sums have the product inner product. Write $|\cdot|_{\mathrm{fib}}$ for these fibre norms. For a real vector bundle $N \to B$ and $\eta > 0$, put

$$D_\eta(N) = \{(b, A) \,:\, b \in B,\ A \in N_b,\ |A|_{\mathrm{fib}} \leq \eta\}.$$

This is a compact smooth manifold with possibly empty boundary. The norm of a section is its supremum fibre norm. Operator norms between orthogonal summands are denoted by $\|\cdot\|_{\mathrm{op}}$. For each of the smooth matrix decompositions below, compactness and finite rank give a constant $C_{\mathrm{fib}} \geq 1$ such that

$$|(T_1, \dots, T_s)|_{\mathrm{fib}} \leq C_{\mathrm{fib}} \|T\|_{\mathrm{op}}$$

for every matrix T with components $(T_1, \dots, T_s)$. For an individual matrix, its Hilbert–Schmidt norm dominates its operator norm.

Let P be a compact parameter space and E a fixed zero module as in Definition 2.1. A *structural perturbation* of $T_0 : P \to \mathcal{A}_n(E)$ is a continuous family $T : P \to \mathcal{A}_n(E)$; its size is $\sup_{p \in P} \|T(p) - T_0(p)\|_{\mathrm{op}}$. Such perturbations may mix the index-minimal and auxiliary summands of E.

Lemma 17.1. *Let B be a compact smooth manifold without boundary, let $\eta > 0$, and let $N, E \to B$ be real vector bundles with* $\operatorname{rank} E = \dim B$. *Suppose that E admits a smooth section with exactly one transverse zero. On $D_\eta(N)$ let $s_0(b, A) = (A, 0)$ be the section of the pullback of $N \oplus E$. Every continuous section s satisfying $\|s - s_0\| < \eta/2$ has a zero in the interior of $D_\eta(N)$.*

Proof. For $\pi : N \to B$, let $\mathcal{E} = \pi^*(N \oplus E)$ over the full total space N. Its rank equals $\dim D_\eta(N)$. Let σ be a section of E with unique transverse zero b_0. For $\varepsilon > 0$, the section

$$s_\varepsilon(b, A) = (A, \varepsilon\sigma(b))$$

has its unique zero at $(b_0, 0)$. At this point its vertical differential is

$$T_{b_0}B \oplus N_{b_0} \longrightarrow N_{b_0} \oplus E_{b_0}, \qquad (\xi, v) \longmapsto (v, \varepsilon d\sigma_{b_0}\xi),$$

where $d\sigma_{b_0} : T_{b_0}B \to E_{b_0}$ is the vertical differential of σ at its zero. This map is an isomorphism.

Suppose that s has no zero. Compactness and smooth approximation give a smooth nowhere-zero section s' with $\|s' - s_0\| < \eta/2$. In the homotopy $H_t = (1 - t)s_\varepsilon + ts'$, the N-component on $\partial D_\eta(N)$ has norm at least $\eta - \|s' - s_0\| > \eta/2$. Let $\mathcal{W}$ be obtained by rounding the corners of $D_\eta(N) \times [0, 1]$ in a collar on which H has no zero. The map $H|_{\partial\mathcal{W}}$ into the full total space of $\mathcal{E}$ is transverse to its closed zero section and meets it exactly once. It extends over the compact manifold $\mathcal{W}$, so its mod-two intersection number is zero by the Boundary Theorem [57, Chapter II, Section 4, p. 80], a contradiction. Finally, $|s|_{\text{fib}} \geq \eta - \|s - s_0\| > \eta/2$ on $\partial D_\eta(N)$. □

17.1. **Integer indices.** Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{H}$, and put $d_\mathbb{F} = \dim_\mathbb{R} \mathbb{F}$. Suppose that the fixed index-minimal module has chiral parts $\mathbb{F}^a$ and 0, with $a \geq 0$, in domain and range order. If the nonzero minimal chiral part lies in the range, take adjoints throughout. Fix $r \geq 1$, $c > 0$, and $0 < \eta < c/4$. Put

$$B = \operatorname{Gr}_r(\mathbb{F}^{a+r}), \qquad L \to B, \tag{17.1}$$

where L is the tautological bundle. The auxiliary domain and range are both the fixed space $\mathbb{F}^{a+r}$. For $(L, A) \in D_\eta(\operatorname{End}_\mathbb{F}(L))$, define

$$Z_{L,A} = cP_{L^\perp} + AP_L. \tag{17.2}$$

After adjoining the minimal domain summand, the full chiral map is

$$\widehat{Z}_{L,A} : \mathbb{F}^{a+r} \oplus \mathbb{F}^a \longrightarrow \mathbb{F}^{a+r}, \qquad \widehat{Z}_{L,A}(z, u) = Z_{L,A}z.$$

Its complementary block is $cI : L^\perp \to L^\perp$. Its Schur complement is $(A, 0)$ in

$$\operatorname{Hom}_\mathbb{F}(L \oplus \mathbb{F}^a, L) = \operatorname{End}_\mathbb{F}(L) \oplus \operatorname{Hom}_\mathbb{F}(\mathbb{F}^a, L). \tag{17.3}$$

Proposition 17.2. *Every sufficiently small continuous $\mathbb{F}$-linear perturbation of the family $\widehat{Z}_{L,A}$ has a parameter at which the full chiral kernel and cokernel have dimensions $r + a$ and r over $\mathbb{F}$.*

Proof. Choose an injective map $B_0 : \mathbb{F}^a \to \mathbb{F}^{a+r}$, and put $L_0 = (\operatorname{im} B_0)^\perp$. The bundle $E = \operatorname{Hom}_\mathbb{F}(\mathbb{F}^a, L)$ has real rank $d_\mathbb{F}ar = \dim_\mathbb{R} B$. Its section $s(L) = P_L B_0$ vanishes exactly at L_0: the equality $P_L B_0 = 0$ is equivalent to $L \subset (\operatorname{im} B_0)^\perp$, and both spaces have dimension r.

For $X \in \operatorname{Hom}_\mathbb{F}(L_0, \operatorname{im} B_0)$, the map $j_X = \binom{I}{X}(I + X^*X)^{-1/2}$ is an isometry from L_0 onto $L = \operatorname{graph}(X)$. In this local trivialization,

$$j_X^* B_0 = (I + X^*X)^{-1/2} X^* B_0.$$

Its derivative at zero is $X \mapsto X^*B_0$. Since $B_0 : \mathbb{F}^a \to \operatorname{im} B_0$ is an isomorphism, this is a real-linear isomorphism onto $\operatorname{Hom}_\mathbb{F}(\mathbb{F}^a, L_0)$. Thus s has one transverse zero.

Let δ bound the perturbation in operator norm, and choose

$$\delta < c/2, \qquad C_{\text{fib}}\left(\delta + \frac{2\delta^2}{c}\right) < \eta/2.$$

Lemma 6.6 gives a section of (17.3) whose distance from $(A, 0)$ is less than $\eta/2$ in the fibre norm. Lemma 17.1, with $N = \mathrm{End}_{\mathbb{F}}(L)$ and the bundle E above, gives a zero of this section. Its domain has dimension $r + a$ and its range dimension r. The complementary map is invertible, so these are exactly the full chiral kernel and cokernel dimensions. For $a = 0$, the base B is a point and $E = 0$. □

In residue zero $a = |\widehat{A}(M)|$, and the complex nullity is $|\widehat{A}(M)| + 2r$. In residue four $a = |\widehat{A}(M)|/2$, and the complex nullity is $|\widehat{A}(M)| + 4r$.

17.2. **The nonzero index in residue one.** Fix $r \geq 1$, $c > 0$, and $0 < \eta < c/4$. On the real forms of the modules, the index-minimal summand is $\mathbb{R}$ and the auxiliary space is $(\mathbb{C}^{r+1})_{\mathbb{R}}$. Let $B = \mathbb{C}P^r$ parameterize the complex hyperplanes $L \subset \mathbb{C}^{r+1}$, and let J_0 be multiplication by i. For $(L, A) \in D_\eta(\mathfrak{so}(L_{\mathbb{R}}))$, take the real skew map

$$A \oplus cJ_0 \oplus 0 \quad \text{on} \quad L_{\mathbb{R}} \oplus (L^\perp)_{\mathbb{R}} \oplus \mathbb{R}.$$

Multiplication by i on the complexification gives a compatible family. After eliminating $(L^\perp)_{\mathbb{R}}$, the residual matrix bundle is

$$\mathfrak{so}(L_{\mathbb{R}} \oplus \mathbb{R}) \cong \mathfrak{so}(L_{\mathbb{R}}) \oplus L_{\mathbb{R}}. \tag{17.4}$$

Here the second component is the map $\mathbb{R} \to L_{\mathbb{R}}$ in the off-diagonal block; its negative adjoint is the opposite block.

The real bundle $L_{\mathbb{R}}$ has rank $2r = \dim_{\mathbb{R}} \mathbb{C}P^r$. Fix $v \in \mathbb{C}^{r+1} \setminus \{0\}$. The section $L \mapsto P_L v$ vanishes exactly at $L_0 = v^\perp$. For $X \in \mathrm{Hom}_{\mathbb{C}}(L_0, \mathbb{C}v)$, the graph formula in the preceding proof gives the derivative $X \mapsto X^* v$. Its inverse sends $w \in L_0$ to

$$X_w(u) = \frac{v\langle w, u\rangle}{|v|^2}, \qquad u \in L_0,$$

with the Hermitian product linear in its second argument. Thus the zero is transverse over $\mathbb{R}$.

The complementary block has inverse norm $1/c$. For a structural perturbation with the bounds in the proof of Proposition 17.2, Lemmas 6.6 and 17.1, with $N = \mathfrak{so}(L_{\mathbb{R}})$ and $E = L_{\mathbb{R}}$, give a zero residual matrix on a real space of dimension $2r + 1$. The complex nullity is exactly $2r + 1$.

17.3. **The nonzero index in residue two.** Fix $r \geq 1$, $c > 0$, and $0 < \eta < c/4$. Let $W = \mathbb{H}^{r+1}$, viewed as a complex Hermitian space, and let $K : W \to W$ be its quaternionic antiunitary, with $K^2 = -I$. Put $U = W \oplus \mathbb{C}$, and let

$$R : U \longrightarrow U^*, \qquad Rx = \langle x, \cdot\rangle,$$

be the Riesz antiunitary. On $U \oplus U^*$ define

$$\Gamma(x, \lambda) = (x, -\lambda), \qquad J(x, \lambda) = (-R^{-1}\lambda, Rx).$$

Then $J^2 = -I$ and $J\Gamma = -\Gamma J$. For $\beta \in \Lambda^2_{\mathbb{C}} U^*$, put $Z_\beta x = \beta(x, \cdot)$. The operator $\mathcal{D}(Z_\beta)$ of (2.3) satisfies (2.2). The fixed index-minimal summand is $\mathbb{C} \oplus \mathbb{C}^*$, and the auxiliary summand is $W \oplus W^*$.

Let $B = \mathbb{H}P^r$ parameterize the quaternionic hyperplanes $L \subset W$, and put $Q = L^\perp$. Thus $\dim_{\mathbb{C}} L = 2r$ and $\dim_{\mathbb{C}} Q = 2$. The form

$$\omega(u, v) = \langle Ku, v\rangle_{\mathbb{C}}, \qquad u, v \in W,$$

is complex bilinear and skew. Its associated map $W \to W^*$ is the composition of two antiunitaries and is therefore unitary. Since Q is K-invariant, the form $c\omega|_Q$ has inverse norm $1/c$. In particular, $W \oplus W^*$ is balanced. For $(L, A) \in D_\eta(\Lambda^2_{\mathbb{C}} L^*)$, define the reference form on $U = (L \oplus \mathbb{C}) \oplus Q$ by

$$\beta_{L,A} = A \oplus 0 \oplus c\omega|_Q.$$

Take the dual decomposition $U^* = (L \oplus \mathbb{C})^* \oplus Q^*$, extending each functional by zero on the other summand. In $U \oplus U^*$ the retained and complementary summands are

$$(L \oplus \mathbb{C}) \oplus (L \oplus \mathbb{C})^*, \qquad Q \oplus Q^*.$$

Both are invariant under J and Γ. The complementary block of $\mathcal{D}(Z_{\beta_{L,A}})$ is self-adjoint and has inverse norm $1/c$. Lemma 6.6 preserves the identities (2.2); hence, for a sufficiently small structural perturbation, its Schur complement has the form $\mathcal{D}(Z_S)$, where S is a complex skew form in

$$\Lambda^2_{\mathbb{C}}(L \oplus \mathbb{C})^* \cong \Lambda^2_{\mathbb{C}} L^* \oplus L^*. \tag{17.5}$$

Here the second component of a form γ is $u \mapsto \gamma(u, 1)$, for $u \in L$. The constants C_{fib} include the norm factors of the off-diagonal blocks in (17.4) and (17.5).

The real rank of L^* is $4r = \dim_{\mathbb{R}} \mathbb{H}P^r$. Fix $v \in W \setminus \{0\}$, and put $\ell(w) = \langle v, w\rangle_{\mathbb{C}}$ and $v\mathbb{H} = \mathrm{span}_{\mathbb{C}}\{v, Kv\}$. The section $L \mapsto \ell|_L$ vanishes exactly at $L_0 = (v\mathbb{H})^{\perp}$: for a K-invariant subspace, orthogonality to v implies orthogonality to Kv. In quaternionic graph coordinates its derivative is

$$X \longmapsto \ell \circ X, \qquad X \in \mathrm{Hom}_{\mathbb{H}}(L_0, v\mathbb{H}).$$

If $\ell \circ X = 0$, then for every $w \in L_0$ both $\ell(Xw)$ and $\ell(KXw) = \ell(XKw)$ vanish. In the complex basis (v, Kv) of $v\mathbb{H}$, these two equalities imply $Xw = 0$. Thus $X = 0$. The domain and range both have real dimension $4r$, so the derivative is an isomorphism.

For a structural perturbation with the bounds in the proof of Proposition 17.2, the identity $\|\mathcal{D}(Z_S)\|_{\mathrm{op}} = \|Z_S\|_{\mathrm{op}}$ and Lemma 6.6 give a residual section within $\eta/2$ of $(A, 0)$. Lemma 17.1, with $N = \Lambda^2_{\mathbb{C}} L^*$ and $E = L^*$ regarded as real bundles, gives a zero skew form on $L \oplus \mathbb{C}$. Both chiral kernels then have dimension $2r + 1$, and the total complex nullity is $4r + 2$.

17.4. **Zero index and the index minimum.** If the fixed index-minimal module is zero, let E_0 be a balanced zero module of the prescribed dimension in (2.4). Fix $\eta > 0$. On the ball $D_\eta(\mathcal{A}_n(E_0))$, regarded as a disk over a point, take the reference matrix A. Lemma 17.1, with B a point, $N = \mathcal{A}_n(E_0)$, and $E = 0$, gives a zero after every sufficiently small structural perturbation. A fixed invertible complement may be eliminated by Lemma 6.6.

If the prescribed nullity is the index minimum, no auxiliary zero block is needed. After elimination of the invertible complement, the residual matrix space is respectively $\mathrm{Hom}_{\mathbb{F}}(\mathbb{F}^a, 0)$ or its adjoint, $\mathfrak{so}(1)$, $\mathrm{Skew}_1(\mathbb{C})$, or the zero space. Each is $\{0\}$. The kernel therefore has exactly the prescribed dimension, including nullity zero.

Proposition 17.3. *For each admissible nullity in* (1.9)*, there exist a fixed balanced zero module E_{aux}, a real vector bundle $N \to B$ over a compact manifold, a number $\eta > 0$, and a family $T_0 : D_\eta(N) \to \mathcal{A}_n(E_{\mathrm{aux}})$ such that, after adjoining the fixed index-minimal module, every sufficiently small structural perturbation has a parameter with exactly that nullity. Let V be a finite-dimensional real inner product space, and put $\underline{V} = B \times V$. On $D_\eta(N \oplus \underline{V})$ the same conclusion holds simultaneously with continuous V-valued equations whose reference map is $(b, A, X) \mapsto X$, provided the combined perturbation is sufficiently small.*

Proof. The preceding constructions give every nullity in (1.9). For the additional equations, let $S = (S_N, S_E)$ be the residual section in $N \oplus E$, and let F be the continuous V-valued map. On the round disk bundle

$$D_\eta(N \oplus \underline{V}) = \{(b, A, X) : |A|^2_{\mathrm{fib}} + |X|^2 \le \eta^2\},$$

the section $s = (S_N, F, S_E)$ takes values in the pullback of $N \oplus \underline{V} \oplus E$. Its reference section is $s_0(b, A, X) = (A, X, 0)$. If $\|s - s_0\| < \eta/2$, Lemma 17.1 gives a common zero. The complementary inverse bounds and the fibre norm comparison constants are fixed before the perturbation size is chosen. □

18. Spectral subspaces under surgery

Spectral surgery for compact families is given in [77, Theorem 4.24], extending [21, Theorem 1.2] and [38, Theorem 6, third arXiv version]. An eigenbundle isomorphism is obtained in [77, Theorem 4.26 and Remark 4.27]. The proposition below gives a uniform norm estimate for the compressed operators and preserves the spinor structures. For smooth families over compact parameter domains, the connected-sum construction in odd dimensions also appears in [18, Section 7, Remark 4].

Proposition 18.1. *Let B be a compact topological space, let $b \mapsto g_b$ be a C^2-continuous family of smooth metrics on a closed spin manifold N^n, and fix a spin surgery of codimension $q \geq 3$, with resulting manifold $\widetilde{N}$. Write $D_b = D_{g_b}$. Let $\alpha, \beta : B \to \mathbb{R}$ be continuous, with $\alpha_b < \beta_b$ and $\alpha_b, \beta_b \notin \operatorname{Spec}(D_b)$. Suppose that*

$$E_b = \mathbf{1}_{(\alpha_b, \beta_b)}(D_b) L^2(\Sigma_{g_b})$$

has fixed rank. For every $\eta > 0$ and every neighborhood V of the surgery sphere, there is a C^2-continuous family of smooth metrics $\widetilde{g}_b$ on $\widetilde{N}$, equal to g_b outside V, such that, with $\widetilde{D}_b = D_{\widetilde{g}_b}$, one has $\alpha_b, \beta_b \notin \operatorname{Spec}(\widetilde{D}_b)$ and the space

$$\widetilde{E}_b = \mathbf{1}_{(\alpha_b, \beta_b)}(\widetilde{D}_b) L^2(\Sigma_{\widetilde{g}_b})$$

has the same rank as E_b. There is a continuous bundle isometry $W : E \to \widetilde{E}$ such that

$$\sup_{b \in B} \left\| W_b^* \widetilde{D}_b W_b - D_b|_{E_b} \right\| < \eta.$$

If E is invariant under the spinor structures, W intertwines their restrictions. If $\alpha_b > 0$ and there is a sign-reversing spinor structure, the same conclusion holds on the direct sum of the bands (α_b, β_b) and $(-\beta_b, -\alpha_b)$: W preserves each summand and intertwines all the spinor structures.

Proof. Put $k = \operatorname{rank}_{\mathbb{C}} E$ and $\Lambda = \sup_{b \in B} \max(|\alpha_b|, |\beta_b|)$. For $k = 0$, the spectral surgery theorem gives the assertion. Assume $k > 0$. The spectral bundles are continuous by [77, Theorems 3.3–3.4]. Norm-resolvent continuity and compactness give

$$d := \inf_{b \in B} \operatorname{dist}\big(\{\alpha_b, \beta_b\}, \operatorname{Spec}(D_b)\big) > 0.$$

The constants C below are independent of b and of the surgery radius.

For $D_b \psi = \lambda \psi$ with $\lambda \in (\alpha_b, \beta_b)$, the square formula and Kato's inequality give, for $u_\tau = (|\psi|^2 + \tau^2)^{1/2}$ and $\tau > 0$,

$$\Delta_b u_\tau \leq C_0 u_\tau, \qquad C_0 = \Lambda^2 + \frac{1}{4} \sup_{b \in B} \left\| \operatorname{scal}_{g_b} \right\|_\infty .$$

With the nonnegative Laplacian convention, this is

$$\partial_i (a_b^{ij} \partial_j u_\tau) + d_b u_\tau \geq 0, \qquad a_b^{ij} = \sqrt{\det g_b}\, g_b^{ij}, \quad d_b = C_0 \sqrt{\det g_b}.$$

On a fixed finite atlas, compactness in C^2 gives common ellipticity and boundedness constants. The local boundedness estimate [52, Theorem 8.17, p. 194], with $p = 2$ and zero right-hand side, applies on nested coordinate balls. Uniform equivalence of the coordinate and metric measures and $\tau \to 0$ give

$$\|\psi\|_\infty \leq C_1 \|\psi\|_2 . \tag{18.1}$$

For an orthonormal eigenbasis $\psi_1, \dots, \psi_k$ of E_b and $a_1, \dots, a_k \in \mathbb{C}$, Cauchy–Schwarz gives, pointwise,

$$\left| \sum_{j=1}^k a_j \psi_j \right| \leq \left(\sum_{j=1}^k |a_j|^2 \right)^{1/2} \left(\sum_{j=1}^k |\psi_j|^2 \right)^{1/2} \leq C_1 \sqrt{k} \left\| \sum_{j=1}^k a_j \psi_j \right\|_2 .$$

Fix tubular neighborhoods T_r of the surgery sphere with respect to a background metric. Choose a smooth function $0 \leq \chi_\rho \leq 1$ equal to zero on T_ρ and to one outside $T_{2\rho}$, with $|d\chi_\rho|_{g_b} \leq C/\rho$. Take $0 < \rho \leq 1$ small enough that $T_{2\rho} \subset V$. Theorem 4.24 of [77], with a sufficiently small spectral error $0 < \varepsilon < d/3$, gives a C^2-continuous family $\widetilde{g}_b$ equal to g_b outside $T_{\rho/2}$, with the same band ranks and with α_b, β_b outside the new spectra. On the common exterior, identify the spinor bundles by the spin structure of the surgery. Set

$$A_b = D_b|_{E_b}, \qquad C_{\rho,b} : E_b \longrightarrow L^2(\Sigma_{\widetilde{g}_b}), \quad C_{\rho,b}\psi = \chi_\rho \psi,$$

extending by zero across the modified region. The extension is smooth, since χ_ρ vanishes on a neighborhood of that region. The map $C_{\rho,b}$ is continuous in b, has norm at most one, and takes values in $\boldsymbol{\mathcal{D}}(\widetilde{D}_b)$. The volume of $T_{2\rho}$ is $O(\rho^q)$ uniformly. Hence

$$\begin{aligned} 0 \le I - C_{\rho,b}^* C_{\rho,b} &\le C\rho^q I, \\ R_b := \widetilde{D}_b C_{\rho,b} - C_{\rho,b} A_b &= d\chi_\rho \cdot |_{E_b}, \qquad \|R_b\| \le C\rho^{q/2-1}. \end{aligned} \tag{18.2}$$

Let P_b be the orthogonal projection onto $\widetilde{E}_b$. We have $\mathrm{Spec}(A_b) \subset [\alpha_b + d, \beta_b - d]$, whereas the spectrum of $\widetilde{D}_b$ on $\mathrm{im}(I - P_b)$ lies outside (α_b, β_b). For an orthonormal eigenbasis e_j of A_b, with eigenvalues λ_j, the spectral theorem gives

$$\left\|(I - P_b)C_{\rho,b}\, e_j\right\| \le d^{-1} \left\|(\widetilde{D}_b - \lambda_j)(I - P_b)C_{\rho,b}\, e_j\right\| \le d^{-1} \left\|R_b\, e_j\right\|.$$

Consequently $\left\|(I - P_b)C_{\rho,b}\right\| \le \sqrt{k}\, d^{-1} \|R_b\|$. Put $G_b = C_{\rho,b}^* P_b C_{\rho,b}$. Since

$$G_b = C_{\rho,b}^* C_{\rho,b} - \big((I - P_b)C_{\rho,b}\big)^* \big((I - P_b)C_{\rho,b}\big),$$

we have $\|G_b - I\| \le C(\rho^q + \rho^{q-2})$. For small ρ, $\frac{1}{2} I \le G_b \le I$. Set

$$W_b = P_b C_{\rho,b} G_b^{-1/2}.$$

Then $W_b^* W_b = I$, and equality of ranks makes W_b onto $\widetilde{E}_b$. Continuity of the factors gives continuity of W. The identity

$$W_b^* \widetilde{D}_b W_b = G_b^{1/2} A_b G_b^{-1/2} + G_b^{-1/2} C_{\rho,b}^* P_b R_b G_b^{-1/2} \tag{18.3}$$

and the bounds

$$\left\|G_b^{1/2} - I\right\| \le \|G_b - I\|, \qquad \left\|G_b^{-1/2} - I\right\| \le \sqrt{2}\, \|G_b - I\|$$

give

$$\left\|W_b^* \widetilde{D}_b W_b - A_b\right\| \le 3\Lambda \|G_b - I\| + 2\|R_b\| \le C\rho^{q/2-1}.$$

Since $q \ge 3$, decreasing ρ gives the prescribed η.

Suppose a spinor structure preserves E_b. If it reverses the sign of D_b, each old band eigenvalue λ has $-\lambda$ in the same band. Every new band eigenvalue μ satisfies $|\mu - \lambda| < \varepsilon$ for an old band eigenvalue λ. As $-\lambda$ has distance at least d from the endpoints and $\varepsilon < d$, also $-\mu \in (\alpha_b, \beta_b)$. The new band is therefore invariant. The common exterior spin identification, the real cutoff, and the spectral projections intertwine the structures. The real functional calculus in the proof of Lemma 6.1 gives the same property for $G_b^{-1/2}$ and W_b.

For the two nonzero bands, put

$$I_{b,+} = (\alpha_b, \beta_b), \qquad I_{b,-} = (-\beta_b, -\alpha_b), \qquad E_{b,\pm} = \mathbf{1}_{I_{b,\pm}}(D_b) L^2(\Sigma_{g_b}).$$

Define $\widetilde{E}_{b,\pm}$ in the same way with $\widetilde{D}_b$, and let $P_{b,\pm}$ be the corresponding orthogonal projections. Let S_b be a sign-reversing spinor structure and $\widetilde{S}_b$ its counterpart. Spectral symmetry gives

$$S_b E_{b,+} = E_{b,-}, \qquad \widetilde{S}_b \widetilde{E}_{b,+} = \widetilde{E}_{b,-}.$$

Thus the negative endpoints avoid both spectra and both negative bands have rank k. For $\sigma \in \{+,-\}$, set

$$C_{b,\sigma} = \chi_\rho|_{E_{b,\sigma}}, \qquad G_{b,\sigma} = C_{b,\sigma}^* P_{b,\sigma} C_{b,\sigma}.$$

Let T_b and $\widetilde{T}_b$ be corresponding spinor structures on the old and new L^2 spaces, with $T_b(E_{b,\sigma}) = E_{b,\tau}$ for $\sigma, \tau \in \{+,-\}$. Naturality on the common exterior gives

$$C_{b,\tau} T_b = \widetilde{T}_b C_{b,\sigma}, \qquad P_{b,\tau} \widetilde{T}_b = \widetilde{T}_b P_{b,\sigma}.$$

Hence $G_{b,\tau}T_b = T_b G_{b,\sigma}$. In particular, $G_{b,-} = S_b G_{b,+} S_b^{-1} \geq \frac{1}{2} I$. We may therefore set

$$W_{b,\sigma} = P_{b,\sigma} C_{b,\sigma} G_{b,\sigma}^{-1/2}. \tag{18.4}$$

The real functional calculus in the proof of Lemma 6.1 gives $W_{b,\tau}T_b = \widetilde{T}_b W_{b,\sigma}$, including antiunitary T_b. Taking $T_b = S_b$ gives

$$W_{b,-} = \widetilde{S}_b W_{b,+} S_b^{-1}. \tag{18.5}$$

Put $A_{b,\pm} = D_b|_{E_{b,\pm}}$ and $\widetilde{A}_{b,\pm} = \widetilde{D}_b|_{\widetilde{E}_{b,\pm}}$. Then

$$W_{b,-}^* \widetilde{A}_{b,-} W_{b,-} - A_{b,-} = -S_b \big(W_{b,+}^* \widetilde{A}_{b,+} W_{b,+} - A_{b,+} \big) S_b^{-1}.$$

The two errors have the same norm. Thus $W_{b,+} \oplus W_{b,-}$ preserves both summands, intertwines all the spinor structures, and satisfies the required estimate. □

18.1. **Spin bordisms.** Let X_n be the underlying spin manifold of $X_{n,k}$ in Section 16; thus $X_n = S^3 \times T^{n-3}$ with the product spin structure specified there. The unique spin structure on S^3 extends over D^4. Hence X_n bounds $D^4 \times T^{n-3}$ with the same torus spin structure and is spin bordant to S^n. For $n \geq 5$, the spin-bordism surgery theorem [55, proof of Theorem B, pp. 431–432], applied to the simply connected target S^n, gives a finite sequence of spin surgeries of codimension at least three from X_n to S^n; see also [38, proof of Proposition 7, p. 7, third arXiv version]. Disconnected intermediate manifolds are permitted in [77, Theorem 4.24 and Section 5].

For $n = 4$, fix $p \in S^3$. Surgery on $S^1 \times \{p\} \subset S^1 \times S^3$ with the prescribed spin structure gives S^4 [38, proof of Proposition 7, p. 6, third arXiv version]. Choose the normal framing so that the spin structure induced on the surgery circle relative to this framing is bounding; it then extends over the attached handle [21, Section 1].

For $n = 3$, $X_3 = S^3$. Connected-sum surgery on $M \sqcup S^n$ has codimension $n \geq 3$ and gives $M \# S^n \cong M$ with its original spin structure. The attachments are independent of the parameters. For a sequence of $m \geq 1$ surgeries, let E_j be the successive spectral bundles and A_j the corresponding compressed Dirac operators, $0 \leq j \leq m$. Let $W_j : E_{j-1} \to E_j$, $1 \leq j \leq m$, be the isometries of Proposition 18.1. Put $U_0 = I$, $U_j = W_j \cdots W_1$ for $1 \leq j \leq m$, and $W = U_m$. Then

$$W^* A_m W - A_0 = \sum_{j=1}^{m} U_{j-1}^* (W_j^* A_j W_j - A_{j-1}) U_{j-1},$$

so

$$\| W^* A_m W - A_0 \| \leq \sum_{j=1}^{m} \| W_j^* A_j W_j - A_{j-1} \|.$$

At each stage the family remains compact and C^2-continuous, with the band endpoints outside every spectrum. The proposition therefore applies successively. Choosing each error less than η/m gives total error less than η, uniformly on B. For $m = 0$, take $W = I$; the error is zero.

19. Finite spectral prescription

19.1. **Nonzero matrix blocks.** By (2.1), the multiplicity field for a nonzero eigenvalue is

$$\mathbb{F}_n = \begin{cases} \mathbb{H}, & n \equiv 2, 3, 4 \pmod 8, \\ \mathbb{C}, & n \equiv 1, 5 \pmod 8, \\ \mathbb{R}, & n \equiv 0, 6, 7 \pmod 8. \end{cases} \tag{19.1}$$

For $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ and $m \geq 1$, write

$$\mathrm{Herm}_m(\mathbb{F}) = \{ H \in \mathrm{Mat}_m(\mathbb{F}) : H^* = H \},$$

regarded as a real vector space.

Lemma 19.1. *Let $H \in \mathrm{Herm}_m(\mathbb{F}_n)$. If $n \not\equiv 3,7 \pmod 8$, assume H positive definite. Then H is the restriction of a compatible matrix on a balanced zero module. The module and the isometric inclusion of the selected subspace can be chosen independently of H, compatibly with its multiplicity field. If charge conjugation or chirality changes the spectral sign, its image subspace carries the block determined by that identity.*

Proof. For $n \equiv 3,7 \pmod 8$, take H itself. For $n \equiv 0,4 \pmod 8$, put $Z = H$. The $\mathbb{F}_n$-linear isometries $f_\pm(u) = (u, \pm u)/\sqrt{2}$ satisfy

$$\mathcal{D}(H) f_\pm = \pm f_\pm H.$$

The same formula applies in residue six, since a real symmetric H is complex symmetric. Here $J(x,y) = (\bar{y}, \bar{x})$ induces conjugation on the positive subspace: $J f_+(u) = f_+(\bar{u})$.

In residue two, represent the quaternionic Hermitian matrix H on $\mathbb{C}^{2m}$ and put

$$J_0 = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \qquad Z = HJ_0.$$

The identities $H^* = H$ and $HJ_0 = J_0\bar{H}$ give $Z^T = -J_0\bar{H} = -HJ_0 = -Z$. For $f_+(u) = (-J_0 u, u)/\sqrt{2}$ and $f_-(u) = (J_0 u, u)/\sqrt{2}$,

$$\mathcal{D}(Z) f_+(u) = (-J_0 H u, Hu)/\sqrt{2} = f_+(Hu),$$

$$\mathcal{D}(Z) f_-(u) = (-J_0 H u, -Hu)/\sqrt{2} = -f_-(Hu).$$

The charge conjugation $J(x,y) = (-\bar{y}, \bar{x})$ satisfies $J f_+(u) = f_+(-J_0\bar{u})$. Thus the positive subspace has its quaternionic structure, commuting with H.

In residue five, take $T = \mathrm{diag}(H, -\bar{H})$ and $J(u,v) = (-\bar{v}, \bar{u})$. Then $T^* = T$, $J^2 = -I$, and

$$JT(u,v) = (H\bar{v}, \bar{H}\bar{u}) = -TJ(u,v).$$

The two coordinate inclusions give H and $-\bar{H}$.

In residue one write $H = A + iB$, where $A, B \in \mathrm{Mat}_m(\mathbb{R})$, $A^T = A$, and $B^T = -B$. Set

$$R(H) = \begin{pmatrix} A & -B \\ B & A \end{pmatrix}, \qquad J_0 = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}.$$

Then $R(H)^T = R(H)$ and $R(H)J_0 = J_0 R(H)$, so $J_0 R(H)$ is real skew. On the complexification, the isometries $f_+(u) = (u, -iu)/\sqrt{2}$ and $f_-(u) = (u, iu)/\sqrt{2}$ satisfy

$$-iJ_0 R(H) f_+ = f_+ H, \qquad -iJ_0 R(H) f_- = -f_-\bar{H}.$$

Every displayed inclusion is independent of H. The underlying modules satisfy the balance conditions of (2.4). □

19.2. **Separation of spectral intervals.**

Proof of Theorem P. Let $\lambda_1, \dots, \lambda_s$ be the distinct prescribed nonzero values, with multiplicities $m_1, \dots, m_s$ over $\mathbb{F}_n$. In the symmetric classes list only the positive values. The case $s = 0$ is allowed. Put

$$V = \bigoplus_{j=1}^{s} \mathrm{Herm}_{m_j}(\mathbb{F}_n), \qquad H_j = \lambda_j I + X_j, \quad X = (X_j)_{j=1}^s \in V. \tag{19.2}$$

Choose pairwise disjoint closed intervals $I_0, I_1, \dots, I_s$ contained in $(-\Lambda, \Lambda)$, with $I_0 = -I_0$, $0 \in \mathrm{int}\, I_0$, and $\lambda_j \in \mathrm{int}\, I_j$ for $1 \le j \le s$. In the symmetric classes include $I_{-j} = -I_j$ and require all these intervals to be disjoint. Let $\mathcal{I} = \{0, 1, \dots, s\}$, with $-1, \dots, -s$ adjoined in the symmetric classes. For $\ell \in \mathcal{I}$, let γ_ℓ be the positively oriented circle with diameter I_ℓ.

Fix $M_0 > \Lambda + 3$. By [4, Theorem 1.1], the prescribed spin manifold admits a metric $g_{\min}$ whose complex nullity is

$$a(M) = \begin{cases} |\widehat{A}(M)|, & n \equiv 0, 4 \pmod 8, \\ \alpha(M), & n \equiv 1 \pmod 8, \\ 2\alpha(M), & n \equiv 2 \pmod 8, \\ 0, & n \equiv 3, 5, 6, 7 \pmod 8. \end{cases} \tag{19.3}$$

Under the constant spinor comparison, $D_{t^2 g} = t^{-1} D_g$ for $t > 0$. Rescale $g_{\min}$ so that its nonzero eigenvalues have absolute value greater than $3M_0$, and put $K_{\min} = \ker D_{g_{\min}}$.

For $h > a(M)$, take the zero family constructed in Section 17 for Proposition 17.3, with $c = M_0$ in the nonzero-index cases. Write B for its base, N for its normal bundle, and E for the remaining residual matrix bundle. If $a(M) = 0$, then B is a point and $E = 0$. For $h = a(M)$, take B to be a point and $N = E = 0$. In residues zero and four, when the nonzero minimal chiral summand lies in the range, take the adjoint chiral matrices throughout, as in Section 17. This interchanges kernel and cokernel and preserves their sum.

Put $\underline{V} = B \times V$, with the product Hilbert–Schmidt norm on V. Choose $\eta > 0$ so small that $\eta < M_0/4$ and the spectrum of every variable block is contained in the interior of its chosen interval. Let

$$Y = D_\eta(N \oplus \underline{V}), \qquad y = (b, A, X), \qquad \pi : Y \longrightarrow B$$

be the round disk bundle and its projection. Combine the auxiliary zero family with the nonzero blocks of Lemma 19.1. Each summand is balanced. Choose a strictly larger balanced module F_{aux} from Corollary 16.4 containing their structural orthogonal direct sum. Its complement is balanced by (2.4); on that complement fix an allowed matrix whose singular values are M_0. The resulting matrices $B_y \in \mathcal{A}_n(F_{\mathrm{aux}})$ depend smoothly on $y \in Y$ and satisfy $\|B_y\| \le M_0$.

Let $Q \mapsto g(Q)$ be the full metric section on F_{aux} given by Corollary 16.4. On a sufficiently small closed matrix ball its complementary spectrum has distance at least $d_{\mathrm{aux}} > 0$ from zero. Choose a fixed $\epsilon > 0$ such that ϵB_y lies in that ball for every y and $d_{\mathrm{aux}}/\epsilon > 3M_0$. Set

$$g_y^{\mathrm{aux}} = \epsilon^2 g(\epsilon B_y). \tag{19.4}$$

Its selected matrix is B_y, and every eigenvalue outside this cluster has absolute value greater than $3M_0$. On the disjoint union with $(M, g_{\min})$, the full spectral subspace for $(-2M_0, 2M_0)$ is identified with

$$F = F_{\mathrm{aux}} \oplus K_{\min}, \qquad T_y = B_y \oplus 0_{K_{\min}}, \qquad \|T_y\| \le M_0.$$

The spectral subspace inside γ_0 is a smooth subbundle $F_0 \subset B \times F$, independent of A and X, of complex rank h. It is the residual module of Section 17, including $K_{\min}$. For $h = a(M)$, it is $B \times K_{\min}$. Put $F_1 = F_0^{\perp}$. Both summands are invariant under charge conjugation and, in even dimensions, under chirality.

For $\ell \in \mathcal{I}$, write $|\gamma_\ell|$ for the contour length and set

$$d_\ell = \inf_{y \in Y,\ z \in \gamma_\ell} \operatorname{dist}(z, \operatorname{Spec}(T_y)) > 0, \qquad \kappa_\ell = \frac{|\gamma_\ell|}{\pi d_\ell^2}.$$

The spectra lie in the interiors of the selected intervals or at $\pm M_0$; compactness of Y gives the strict lower bounds.

Apply the finite spin-surgery sequence of Section 18 to the auxiliary manifold, and then form its connected sum with $(M, g_{\min})$. The parameter space Y is compact, the initial metric family is smooth, and the endpoints $\pm 2M_0$ avoid its spectrum. Proposition 18.1 and the finite composition estimate identify the entire resulting band with F. For any sufficiently small $\delta > 0$, the resulting matrix $\widetilde{T}_y$ satisfies

$$\sup_{y \in Y} \|\widetilde{T}_y - T_y\| < \delta.$$

These matrices are continuous and retain all the spinor identities. The metrics are smooth and depend continuously on y in C^2.

Set

$$\mu = \min\bigl(\{M_0\} \cup \{\operatorname{dist}(0, I_j) \,:\, 1 \le j \le s\}\bigr) > 0.$$

The block $T_y|_{F_{1,b}}$ is invertible, with inverse norm at most μ^{-1}. For $\delta < \mu/2$, let $\widehat{S}_y$ be the Schur complement of $\widetilde{T}_y$ on $F_{0,b}$ relative to $F = F_{0,b} \oplus F_{1,b}$. Lemma 6.6 gives

$$\widehat{S}_y \in \mathcal{A}_n(F_{0,b}), \qquad \bigl\|\widehat{S}_y - T_y|_{F_{0,b}}\bigr\| \le \delta + \frac{2\delta^2}{\mu} \le 2\delta.$$

The identifications of Section 17 give smooth real-linear isomorphisms

$$\Phi_b \,:\, \mathcal{A}_n(F_{0,b}) \longrightarrow N_b \oplus E_b, \qquad \Phi_b(T_y|_{F_{0,b}}) = (A, 0).$$

For $h > a(M)$ and nonzero index, these are (17.3), (17.4), and (17.5): take the independent chiral block in even dimensions and multiply by $-i$ in residue one. Take the adjoint in the opposite integer-index case. At zero index, $N_b = \mathcal{A}_n(F_{0,b})$, $E_b = 0$, and Φ_b is the identity. For $h = a(M)$, the residual matrix space is $\{0\}$. For $S \in N_b \oplus E_b$, put $\|S\|_{\mathrm{op}} = \|\Phi_b^{-1} S\|$. Thus $S_0(y) = \Phi_b(\widehat{S}_y)$ is a section of $\pi^*(N \oplus E)$ satisfying

$$\|S_0(y) - (A, 0)\|_{\mathrm{op}} \le 2\delta.$$

Let $P_\ell(y)$ and $\widetilde{P}_\ell(y)$ be the orthogonal Riesz projections of T_y and $\widetilde{T}_y$ for γ_ℓ. For $\delta < d_\ell/2$, the resolvent identity gives

$$\|(z - \widetilde{T}_y)^{-1} - (z - T_y)^{-1}\| \le \frac{2\delta}{d_\ell^2}, \qquad z \in \gamma_\ell.$$

Integration yields

$$q_\ell := \sup_{y \in Y} \|\widetilde{P}_\ell(y) - P_\ell(y)\| \le \kappa_\ell \delta.$$

Put $\delta_{\mathrm{polar}} = \min_{\ell \in \mathcal{I}} (2\kappa_\ell)^{-1}$. For $\delta < \delta_{\mathrm{polar}}$, the projections have the same ranks by [67, Chapter I, Theorem 6.32].

For $1 \le j \le s$, let U_j be the fixed isometric inclusion from Lemma 19.1, extended into F. Then $P_j = U_j U_j^*$. Put $G_j = U_j^* \widetilde{P}_j U_j$. Since $\|G_j - I\| \le q_j < 1/2$,

$$(1 - q_j) I \le G_j \le I.$$

The polar formula (6.2) therefore defines the continuous isometry

$$\widetilde{U}_j = \widetilde{P}_j U_j G_j^{-1/2}$$

onto $\operatorname{im} \widetilde{P}_j$. The spectral theorem gives

$$\|G_j^{-1/2}\| \le \sqrt{2}, \qquad \|G_j^{-1/2} - I\| \le 2q_j, \qquad \|\widetilde{U}_j - U_j\| \le 4q_j.$$

Thus $\widetilde{H}_j = \widetilde{U}_j^* \widetilde{T}_y \widetilde{U}_j$ satisfies

$$\|\widetilde{H}_j - H_j\| \le \delta + 2M_0 \|\widetilde{U}_j - U_j\| \le C_j \delta, \qquad C_j = 1 + 8M_0 \kappa_j. \tag{19.5}$$

The polar isometry intertwines the sign-preserving spinor structures, as in the proof of Lemma 6.1. Consequently $\widetilde{H}_j \in \mathrm{Herm}_{m_j}(\mathbb{F}_n)$; the sign-reversing structures determine the negative partners.

Choose $C_{\mathrm{fib}} \ge 1$ so that, for every $b \in B$, $S = (S_N, S_E) \in N_b \oplus E_b$, and $W = (W_j)_{j=1}^s \in V$,

$$|(S_N, W, S_E)|_{\mathrm{fib}} \le C_{\mathrm{fib}} \left(\|S\|_{\mathrm{op}}^2 + \sum_{j=1}^{s} \|W_j\|_{\mathrm{op}}^2 \right)^{1/2}.$$

Such a uniform constant exists because the bundles have finite rank over the compact base. Put $K_0 = (4 + \sum_{j=1}^{s} C_j^2)^{1/2}$. Choose the total surgery error so that

$$0 < \delta < \min\left\{ \frac{\mu}{2},\ \frac{1}{2}\min_{\ell \in \mathcal{I}} d_\ell,\ \frac{\eta}{2C_{\mathrm{fib}}K_0},\ \frac{M_0 - \Lambda}{2},\ \delta_{\mathrm{polar}} \right\}.$$

The continuous section with components S_0 and $(\widetilde{H}_j - \lambda_j I)_{j=1}^{s}$, ordered in $\pi^*(N \oplus \underline{V} \oplus E)$, differs from $(A, X, 0)$ by less than $C_{\mathrm{fib}}K_0\delta < \eta/2$. The constructions of Section 17 give a compact smooth base B without boundary, $\operatorname{rank} E = \dim B$, and a section of E with one transverse zero. Lemma 17.1, applied with $N \oplus \underline{V}$, therefore gives a parameter with $S_0 = 0$ and

$$\widetilde{H}_j = \lambda_j I \quad (1 \le j \le s). \tag{19.6}$$

For $h = a(M)$ this is the same lemma with B a point and $N = E = 0$. If also $V = 0$, the parameter space is a point.

At this parameter, $S_0 = 0$ implies $\widehat{S}_y = 0$. Lemma 6.6 gives

$$\ker \widetilde{T}_y \cong \ker \widehat{S}_y = F_{0,b}, \qquad \dim_{\mathbb{C}} \ker \widetilde{T}_y = h.$$

Every kernel vector of the Dirac operator belongs to the full band and to the spectral subspace inside γ_0. The latter has rank h, so it is exactly the kernel.

Each nonzero projection retains its reference rank. Equation (19.6) makes its compressed operator the prescribed scalar matrix. A block of size m_j contributes complex multiplicity $2m_j$ over $\mathbb{H}$ and m_j over $\mathbb{R}$ or $\mathbb{C}$. In the symmetric classes the negative partners have the same multiplicities. The spectrum on the direct sum of all prescribed nonzero spectral subspaces is Ξ, with complex multiplicities.

Put $r = \dim_{\mathbb{C}} F$. At this parameter, let $\nu_1 \le \cdots \le \nu_r$ and $\widetilde{\nu}_1 \le \cdots \le \widetilde{\nu}_r$ be the eigenvalues of T_y and $\widetilde{T}_y$, counted with complex multiplicity. The ordered-eigenvalue inequality [67, Chapter II, Theorem 6.10, p. 126] gives

$$|\widetilde{\nu}_k - \nu_k| < \delta \qquad (1 \le k \le r).$$

The remaining reference eigenvalues equal M_0 or $-M_0$. The corresponding ordered entries therefore have absolute value greater than $M_0 - \delta > \Lambda$. Their number, together with the ranks of the zero projection and all the prescribed nonzero projections, equals r. Proposition 18.1 preserves the full spectral subspace of dimension r, so these eigenvalues exhaust the spectrum in $(-2M_0, 2M_0)$. There is no further eigenvalue in $(-\Lambda, \Lambda)$. The final spin manifold is $M \# S^n \cong M$ with its prescribed spin structure. This proves (1.10). □

Corollary 19.2. *For every closed connected spin manifold M of dimension at least three,*

$$\sup_{g} \dim_{\mathbb{C}} \ker D_g = \infty.$$

Here g ranges over the smooth Riemannian metrics on M.

Proof. Each row of (1.9) contains arbitrarily large integers. Apply Theorem P with an empty nonzero multiset. □

20. Local harmonic spinors in dimension four

Let (M^4, g) be a closed, connected Riemannian spin manifold, and let $U \subset M$ be a nonempty open set. Order the chiral spinor bundles S, T as in Section 8, so that

$$A_g : H^1(S) \longrightarrow L^2(T), \qquad \operatorname{ind}_{\mathbb{H}} A_g = -a, \qquad a \in \mathbb{Z}_{\ge 0}.$$

Both bundles are right quaternionic line bundles. Their Hermitian products are linear in the second argument; adjoints and orthogonal projections refer to these products and the corresponding L^2 products. When the metric varies, the bundles are identified by (2.6). By Lemma 2.2 and (2.3), the real-linear bundle map

$$\Lambda^3 T^*M \longrightarrow \operatorname{Hom}_{\mathbb{H}}(S, T), \qquad \tau \longmapsto c_g(\tau)|_S \tag{20.1}$$

is an isomorphism. A section B of $\operatorname{Hom}_{\mathbb{H}}(S, T)$ corresponds to the compatible self-adjoint potential $\left(\begin{smallmatrix} 0 & B^* \\ B & 0 \end{smallmatrix}\right)$.

20.1. **A flat ball with prescribed density.**

Lemma 20.1. *For every $p \in U$, the metric g can be approximated in C^1 by smooth metrics g_1 which are flat on a neighborhood of p, equal g outside U, and satisfy $dv_{g_1} = dv_g$. If A_g is injective, g_1 can be chosen so that A_{g_1} is injective.*

Proof. Take oriented normal coordinates x at p on a coordinate box with closure in U, and write $dv_g = J(x)\,dx$. Then $J = 1 + O(|x|^2)$ and $dJ = O(|x|)$. On a smaller box put

$$y_1 = \int_0^{x_1} J(t, x_2, x_3, x_4)\,dt, \qquad y_i = x_i \quad (i > 1).$$

The Jacobian determinant is $J > 0$, and

$$y - x = O(|x|^3), \qquad Dy - I = O(|x|^2), \qquad D^2 y = O(|x|).$$

Thus y is a coordinate system with $dv_g = dy$. The matrix $G(y)$ of g satisfies

$$\det G = 1, \qquad G - I = O(|y|^2), \qquad dG = O(|y|).$$

For sufficiently small $r > 0$, choose a smooth function χ_r with values in $[0, 1]$, equal to zero on $|y| \le r/3$ and to one on $|y| \ge 2r/3$, with $|d\chi_r| \le C/r$. Put

$$G_r = \exp(\chi_r \log G).$$

This matrix is positive symmetric, equals G near the boundary of the box, and equals I near p. Since $\operatorname{tr} \log G = \log \det G = 0$, one has $\det G_r = 1$. Lemma 3.4 gives

$$\|G_r - G\|_{C^0} \le Cr^2, \qquad \|G_r - G\|_{C^1} \le Cr,$$

with C independent of r. Extension by g defines the required metrics g_r.

The operator $A_g : H^1(S) \to L^2(T)$ has closed range and finite-dimensional kernel [66, Theorem 19.5.1]; the density dv_g identifies sections with half-density-valued sections. Under (2.6), the local formula for the Dirac operator [61, Theorem 2.3.6] gives

$$\|A_{g_r} - A_g\|_{H^1 \to L^2} \le C' \|g_r - g\|_{C^1} \longrightarrow 0.$$

If A_g is injective, [66, Theorem 19.1.5, pp. 182–183] implies that A_{g_r} is injective for all sufficiently small r. □

20.2. **Cokernel-orthogonal point sources.**

Lemma 20.2. *Suppose that A_{g_1} is injective and that g_1 is flat on a ball contained in U. There is $B \in C_c^\infty(U; \operatorname{Hom}_{\mathbb{H}}(S, T))$ such that* $\ker(A_{g_1} + B) \ne 0$.

Proof. Put $A = A_{g_1}$, and let A^* be its formal L^2 adjoint. The space $E = \ker A^*$ consists of smooth sections and has quaternionic dimension a. Let P_E denote the L^2 orthogonal projection onto E. For $p \in M$, write $\mathrm{ev}_p : E \to T_p$ for evaluation at p. Select $m > a$ distinct points $p_1, \dots, p_m$ in the flat ball. The quaternionic-linear map

$$\bigoplus_{i=1}^m T_{p_i} \longrightarrow E, \qquad (v_i) \longmapsto \sum_{i=1}^m \mathrm{ev}_{p_i}^*\, v_i$$

has a nonzero kernel. Choose a nonzero vector (v_i) in this kernel and put $\mathcal{I} = \{i : v_i \ne 0\}$. Then

$$\eta = \sum_{i \in \mathcal{I}} \delta_{p_i} v_i \ne 0, \qquad P_E \eta = 0.$$

The positive self-adjoint operator A^*A on $L^2(S)$ has domain $H^2(S)$ and is invertible. Set

$$R = (A^*A)^{-1} A^*.$$

The elliptic calculus [66, Theorems 18.1.8 and 18.1.24, pp. 71 and 87; bundle calculus, pp. 91–93] gives R of order -1. On smooth sections, $RA = I$ and $AR = I - P_E$. These identities extend to distributions: A and R act continuously on distributions, and P_E has a smooth kernel. Hence

$$\psi = R\eta, \qquad A\psi = \eta. \tag{20.2}$$

In particular, ψ is smooth away from the p_i, $i \in \mathcal{I}$.

Choose oriented Euclidean coordinates (x_0, x_1, x_2, x_3) and parallel quaternionic unit frames on the flat ball. Write $x = x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k} \in \mathbb{H}$, where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are the standard quaternionic units. The frames may be chosen so that A is one of the operators

$$\mathscr{D} = \partial_0 + \mathbf{i}\partial_1 + \mathbf{j}\partial_2 + \mathbf{k}\partial_3, \qquad \overline{\mathscr{D}} = \partial_0 - \mathbf{i}\partial_1 - \mathbf{j}\partial_2 - \mathbf{k}\partial_3,$$

acting by left multiplication on quaternion-valued functions. Suppose first that $A = \mathscr{D}$. For $\Delta_{\mathbb{R}^4} = \sum_{j=0}^{3} \partial_j^2$, the fundamental solution is $\Phi_0(x) = -1/(4\pi^2|x|^2)$ [52, Section 2.4, equation (2.12), p. 17]. Since $\mathscr{D}\overline{\mathscr{D}} = \Delta_{\mathbb{R}^4}$,

$$E_0(x) = \overline{\mathscr{D}}\Phi_0(x) = \frac{\bar{x}}{2\pi^2|x|^4}, \qquad \mathscr{D}E_0 = \delta_0.$$

For each $i \in \mathcal{I}$, elliptic regularity gives $\psi = E_0(x - p_i)v_i + h_i$ near p_i, where h_i is smooth and $\mathscr{D}h_i = 0$. Choose disjoint balls about these points, with closures in the flat ball, such that

$$|h_i(x)| < \frac{|v_i|}{4\pi^2|x - p_i|^3} \qquad (x \neq p_i).$$

Put

$$u(x) = x_0 + x_1\mathbf{i} + x_2\mathbf{j} - x_3\mathbf{k}, \qquad \mathscr{D}u = 1 - 1 - 1 + 1 = 0. \tag{20.3}$$

For $\theta \in S^3 \subset \mathbb{H}$ and $s \in [0, 1]$, define

$$\begin{aligned} Q_s(\theta) = \theta_0 &+ (\cos(\pi s)\theta_1 - \sin(\pi s)\theta_2)\mathbf{i} \\ &+ (\sin(\pi s)\theta_1 + \cos(\pi s)\theta_2)\mathbf{j} - \theta_3\mathbf{k}. \end{aligned}$$

Then $|Q_s(\theta)| = 1$, $Q_0(\theta) = u(\theta)$, and $Q_1(\theta) = \bar{\theta}$. In one of the chosen balls write $x - p_i = \rho\theta$ and take $0 < r_1 < r_2 < r_3$ smaller than its radius. Choose smooth $\chi, \zeta : [0, \infty) \to [0, 1]$ with

$$\chi = 0\ (\rho \leq r_1), \quad \chi = 1\ (\rho \geq r_2), \qquad \zeta = 0\ (\rho \leq r_2), \quad \zeta = 1\ (\rho \geq r_3).$$

For $\rho > 0$, set

$$b(\rho) = (1 - \chi(\rho))\rho + \frac{\chi(\rho)}{2\pi^2\rho^3}, \qquad \phi_i(p_i + \rho\theta) = b(\rho)Q_{\chi(\rho)}(\theta)v_i + \zeta(\rho)h_i(p_i + \rho\theta).$$

For $\rho \leq r_1$ this is $u(x - p_i)v_i$, so it extends smoothly across p_i. For $\rho \geq r_3$ it equals ψ. If $\zeta = 0$, then $|\phi_i| = b(\rho)|v_i| > 0$. If $\zeta \neq 0$, then $\chi = 1$ and

$$|\phi_i| \geq |E_0(x - p_i)v_i| - |h_i(x)| > \tfrac{1}{2}|E_0(x - p_i)v_i| > 0.$$

For $A = \overline{\mathscr{D}}$, the fundamental solution is $x/(2\pi^2|x|^4)$. Replace $Q_s(\theta)$ by $Q_s(\bar{\theta})$; the inner map is $u(\bar{x})$ and $\overline{\mathscr{D}}(u(\bar{x})) = 1 - 1 - 1 + 1 = 0$.

Set $\phi = \phi_i$ in these balls and $\phi = \psi$ outside them. This is a smooth section, and $A\phi = 0$ near the centers, near the ball boundaries, and outside the balls. For a fibre vector $w \in S_x$, put $\phi^*(w) = \langle \phi(x), w \rangle_{\mathbb{H}}$. On each punctured ball define

$$B = -\frac{(A\phi)\phi^*}{|\phi|^2}. \tag{20.4}$$

This is a quaternionic-linear homomorphism and $B\phi = -A\phi$, since $\phi^*(\phi) = |\phi|^2$. It vanishes near the centers and ball boundaries. Its extension by zero therefore belongs to $C_c^\infty(U; \mathrm{Hom}_{\mathbb{H}}(S, T))$ and satisfies $(A + B)\phi = 0$. Since ϕ_i is nonzero on a punctured ball, $\phi \neq 0$. □

20.3. **A chiral kernel of quaternionic dimension one.**

Proposition 20.3. *Let $B_0 \in C_c^\infty(U;\mathrm{Hom}_{\mathbb{H}}(S,T))$, and suppose that $L = A_g + B_0$ has nonzero kernel. There is a $B_1 \in C_c^\infty(U;\mathrm{Hom}_{\mathbb{H}}(S,T))$ for which the spaces $F_1 = \ker(A_g + B_1)$ and $E_1 = \ker(A_g + B_1)^*$ satisfy*

$$\dim_{\mathbb{H}} F_1 = 1, \qquad \dim_{\mathbb{H}} E_1 = a + 1.$$

There is a finite-dimensional real subspace $\mathscr{V} \subset C_c^\infty(U;\mathrm{Hom}_{\mathbb{H}}(S,T))$ such that

$$\mathscr{V} \longrightarrow \mathrm{Hom}_{\mathbb{H}}(F_1,E_1), \qquad V \longmapsto P_{E_1} V|_{F_1},$$

is an isomorphism, where P_{E_1} is the L^2 orthogonal projection. The associated compatible self-adjoint operator admits a full potential section whose potentials have a common compact support in U.

Proof. Multiplication by B_0 is compact from $H^1(S)$ to $L^2(T)$, so $\mathrm{ind}_{\mathbb{H}} L = -a$ [66, Corollary 19.1.8, p. 183]. Put $F = \ker L$, $E = \ker L^*$, and $k = \dim_{\mathbb{H}} F$; then $\dim_{\mathbb{H}} E = k + a$. Both spaces consist of smooth sections. The operators L^*L and LL^* have smooth coefficients and scalar principal symbol $|\xi|_g^2 I$. By the unique continuation theorem for systems [9, Remark 3, p. 248], an element of F or E vanishing on a nonempty open set vanishes identically. Compression defines the real-linear map

$$\mathcal{C} \,:\, C_c^\infty(U;\mathrm{Hom}_{\mathbb{H}}(S,T)) \longrightarrow \mathrm{Hom}_{\mathbb{H}}(F,E), \qquad \mathcal{C}(B) = P_E B|_F, \tag{20.5}$$

where P_E is the L^2 orthogonal projection onto E. Let $e_F(x) : F \to S_x$ and $e_E(x) : E \to T_x$ be evaluation. In local quaternionic unit frames put $v_x = e_F(x)^*(1)$ and $u_x = e_E(x)^*(1)$. Lemma 6.5 gives $u_x q v_x^* \in \mathrm{im}\,\mathcal{C}$ for every $x \in U$ and $q \in \mathbb{H}$.

If $k = 1$, put $M_0 = 0$. If $k \geq 2$, choose distinct $x_1, \dots, x_{k-1} \in U$ such that the vectors $v_j = v_{x_j}$ and $u_j = u_{x_j}$ are independent in their respective spaces. Indeed, suppose that the previous spans are $V \subsetneq F$ and $W \subsetneq E$. The condition $v_x\mathbb{H} \not\subset V$ defines an open subset of U. If its complement contained a nonempty open set, every element of $V^\perp$ would vanish there, contrary to unique continuation. Thus this subset is dense. The condition $u_x\mathbb{H} \not\subset W$ is open and dense for the same reason, applied to L^*. The intersection of these sets contains a point distinct from the previous choices. Set

$$M_0 = \sum_{j=1}^{k-1} u_j v_j^* \qquad (k \geq 2). \tag{20.6}$$

The map M_0 lies in $\mathrm{im}\,\mathcal{C}$ and has rank $k-1$: both column maps are injective, and the adjoint of the source column map is surjective. In either case put $\ell = \ker M_0$ and $C = (\mathrm{im}\,M_0)^\perp$. Then $\dim_{\mathbb{H}} \ell = 1$ and $\dim_{\mathbb{H}} C = a + 1$.

The real-linear map

$$B \longmapsto P_C \mathcal{C}(B)|_\ell \in \mathrm{Hom}_{\mathbb{H}}(\ell, C) \tag{20.7}$$

is surjective. Indeed, for an L^2-unit generator ψ of ℓ, the residue-four argument in the proof of Lemma 10.4, applied to L and the compatible potential $\left(\begin{smallmatrix} 0 & B_0^* \\ B_0 & 0 \end{smallmatrix}\right)$, gives

$$\{P_E(B\psi) \,:\, B \in C_c^\infty(U;\mathrm{Hom}_{\mathbb{H}}(S,T))\} = E.$$

Projecting this equality onto C and identifying $\mathrm{Hom}_{\mathbb{H}}(\ell, C)$ with C by $\Xi \mapsto \Xi\psi$ proves the surjectivity of (20.7). If $k = 1$, take $B_1 = B_0$.

Suppose $k \geq 2$. Choose B_* with $\mathcal{C}(B_*) = M_0$ and $V_1, \dots, V_d \in C_c^\infty(U;\mathrm{Hom}_{\mathbb{H}}(S,T))$ whose images under (20.7) form a real basis, where $d = 4(a+1)$. For $q = (q_1, \dots, q_d) \in \mathbb{R}^d$, put $B(q) = B_* + \sum_{\nu=1}^d q_\nu V_\nu$. Give $H^1(S)$ the graph inner product of L. Since $LF = 0$, its orthogonal decomposition is $F \oplus (F^\perp \cap H^1(S))$, with $F^\perp$ taken in $L^2(S)$. The restriction

$$L_c = L|_{F^\perp \cap H^1(S)} \,:\, F^\perp \cap H^1(S) \longrightarrow E^\perp \subset L^2(T)$$

is bijective with bounded inverse. Write $B(q) = (B_{ij}(q))$ in these domain and range decompositions. For small (t, q), Lemma 6.6 gives the finite-dimensional Schur complement

$$\mathscr{S}(t,q) = t\mathcal{C}(B(q)) - t^2 B_{12}(q)(L_c + tB_{22}(q))^{-1} B_{21}(q). \tag{20.8}$$

Define

$$N(t,q) = C(B(q)) - tB_{12}(q)(L_c + tB_{22}(q))^{-1}B_{21}(q).$$

This is smooth and equals $\mathscr{S}(t,q)/t$ for $t \neq 0$.

Write $N = (N_{ij})$ relative to $F = \ell \oplus \ell^{\perp}$ and $E = C \oplus \operatorname{im} M_0$. At $(t,q) = (0,0)$, one has $N = M_0$; hence N_{22} is invertible and the other three blocks vanish. On a neighborhood of $(0,0)$ the second Schur complement is

$$s(t,q) = N_{11} - N_{12}N_{22}^{-1}N_{21} \in \operatorname{Hom}_{\mathbb{H}}(\ell, C).$$

Consequently

$$s(0,0) = 0, \qquad d_q s(0,0)[\dot{q}] = P_C C\Big(\sum_{\nu=1}^{d} \dot{q}_\nu V_\nu\Big)|_\ell.$$

This derivative is an isomorphism. The implicit function theorem [50, Theorem 13.20, p. 71] gives a smooth $q(t)$ with $q(0) = 0$ and $s(t, q(t)) = 0$. For sufficiently small $t \neq 0$, both complementary blocks are invertible. Lemma 6.6, applied twice, gives

$$\dim_{\mathbb{H}} \ker(L + tB(q(t))) = 1, \qquad \dim_{\mathbb{H}} \operatorname{coker}(L + tB(q(t))) = a + 1.$$

Fix such a t and put $B_1 = B_0 + tB(q(t))$.

In either case, for the resulting B_1, the same application of Lemma 10.4 shows that $\dot{B} \mapsto P_{E_1}\dot{B}|_{F_1}$ is onto $\operatorname{Hom}_{\mathbb{H}}(F_1, E_1)$. Apply Lemma 6.4 to the self-adjoint family

$$\mathcal{D}(A_g + B) = \begin{pmatrix} 0 & (A_g + B)^* \\ A_g + B & 0 \end{pmatrix}.$$

Its kernel at $B = B_1$ is $F_1 \oplus E_1$. A circle centered at zero separates this kernel from the remaining spectrum. The polar frame of Lemma 6.1 preserves the chiral quaternionic structures, and the derivative of the chiral restriction matrix is $\dot{B} \mapsto P_{E_1}\dot{B}|_{F_1}$. Choose $d = 4(a+1)$ potential directions on which this derivative is an isomorphism. The inverse function theorem [50, Theorem 11.22 and Corollary 11.35, pp. 61–63] gives a full potential section with chiral column $q \mapsto q$ near zero in $\mathbb{H}^{a+1}$, after choosing quaternionic unitary coordinates on F_1 and E_1. The chiral block of every potential in this section lies in B_1 plus the real span of the chosen directions, so all have support in one compact subset of U. □

Proof of the four-dimensional assertion of Theorem L. Fix $\varepsilon > 0$. If $\ker A_g \neq 0$, put $g' = g$ and apply Proposition 20.3 with $B_0 = 0$. Otherwise, Lemma 20.1 gives a metric g' with $A_{g'}$ injective, flat on a ball in U, and

$$\|g' - g\|_{C^0_g} < \varepsilon/2, \qquad dv_{g'} = dv_g, \qquad g' = g \text{ on } M \setminus U.$$

Lemma 20.2 and Proposition 20.3 then apply to g'.

In either case there is a full potential section with chiral column $q \mapsto q$, of quaternionic ranks 1 and $a+1$, whose potentials have a common compact support in U. Restrict the parameter domain to a small closed ball on which one circle centered at zero separates this spectral subspace from the remaining spectrum. Apply Proposition 6.3. In its application of Theorem R, choose the error small enough that

$$\sup_q \|h_q - g'\|_{C^0_g} < \varepsilon/2.$$

The resulting metric section has chiral column $q \mapsto q$, $dv_{h_q} = dv_g$, and $h_q = g$ on $M \setminus U$. Thus $\|h_q - g\|_{C^0_g} < \varepsilon$ throughout its parameter domain. At $q = 0$, the spectral subspace is the whole kernel, whence

$$\dim_{\mathbb{H}} \ker A_{h_0} = 1, \qquad \dim_{\mathbb{H}} \ker A^*_{h_0} = a + 1, \qquad \dim_{\mathbb{C}} \ker D_{h_0} = 2a + 4.$$

The derivative of the normal column is the identity. This proves (1.11) and (1.12). □

21. Local harmonic spinors in dimension five

Fix a closed connected spin five-manifold (M, g) and write $\Sigma M = \Sigma_g M$. This is a right quaternionic bundle of rank two. Hermitian products are linear in the second variable. The symbol $*$ denotes the complex adjoint, fibrewise or in L^2; for a quaternionic-linear map it also equals the quaternionic adjoint. For a closed complex subspace $V \subset L^2(\Sigma M)$, let P_V be the orthogonal projection onto V. Write dv_g for the volume form of the spin orientation. The symbol i denotes complex scalar multiplication, whereas $\mathbf{i} \in \operatorname{Im}\mathbb{H}$ is a fixed unit quaternion. By (2.1), charge conjugation satisfies $J^2 = -I$ and $JD_g = -D_g J$. Put

$$K = iD_g, \qquad \gamma(X) = ic_g(X).$$

Then K is quaternionic-linear and skew-adjoint; $\gamma(X)$ is quaternionic Hermitian, with $\gamma(X)^2 = |X|^2 I$.

21.1. The spinor current.

Lemma 21.1. *For a smooth spinor field ϕ, define the real vector field j_ϕ by*

$$g(j_\phi, X) = \operatorname{Re}\langle \phi, \gamma(X)\phi\rangle. \tag{21.1}$$

It satisfies

$$|j_\phi| = |\phi|^2, \qquad \operatorname{div} j_\phi = 2\operatorname{Re}\langle \phi, iD_g\phi\rangle. \tag{21.2}$$

At a point x with $\phi(x) \neq 0$, the equation $(D_g\phi)(x) + c_g(\tau_x)\phi(x) = 0$ has a real three-form solution $\tau_x \in \Lambda^3 T_x^* M$ if and only if $\operatorname{div} j_\phi(x) = 0$. On an open set where ϕ is nowhere zero and the divergence vanishes, τ can be chosen smoothly.

Proof. In a quaternionic orthonormal spin frame and an orthogonal identification $T_xM \simeq \mathbb{H} \oplus \mathbb{R}$, the endomorphisms $\gamma(z, t)$ have the form

$$\gamma(z,t) = \begin{pmatrix} t & z \\ \bar z & -t \end{pmatrix}, \qquad (z,t) \in \mathbb{H} \oplus \mathbb{R}.$$

Consequently the map in (21.1) is

$$(u, v) \longmapsto (2u\bar v,\ |u|^2 - |v|^2) \in \mathbb{H} \oplus \mathbb{R}.$$

Its squared norm is $4|u|^2|v|^2 + (|u|^2 - |v|^2)^2 = (|u|^2 + |v|^2)^2$. At a point choose a normal orthonormal frame $e_1, \dots, e_5$. Since $\nabla\gamma = 0$ and the $\gamma(e_j)$ are Hermitian, differentiation gives

$$\operatorname{div} j_\phi = 2\sum_j \operatorname{Re}\langle \phi, \gamma(e_j)\nabla_{e_j}\phi\rangle = 2\operatorname{Re}\langle \phi, iD_g\phi\rangle.$$

For a unit spinor η, the one-form α_η of [23, Definition 4.8 and Lemma 4.11] satisfies $j_\eta^\flat = -\alpha_\eta$ when its complex structure j_1 is identified with $-iI$.

By Lemma 2.2 and (2.4), $\tau \mapsto ic_g(\tau)$ identifies $\Lambda^3 T^*M$ with $\mathfrak{sp}(\Sigma M)$. By Lemma 10.3, a nonzero $u \in \mathbb{H}^2$ admits a skew-adjoint solution of $Qu = y$ precisely when $\operatorname{Re}(u^* y) = 0$; the displayed solution in that lemma is smooth in (u, y) and vanishes when $y = 0$. Apply it with $u = \phi$ and $y = -iD_g\phi$. The compatibility condition is the second equality in (21.2). □

21.2. A nonzero spinor with divergence-free current.

Proposition 21.2. *For every nonempty open $U \subset M$, there is $\tau \in C_c^\infty(U; \Lambda^3 T^*M)$ such that* $\ker(D_g + c_g(\tau)) \neq 0$.

Proof. If D_g has a kernel, take $\tau = 0$. Otherwise choose $p \in U$ and $0 \neq v \in \Sigma_p M$. Let $\delta_p v$ denote the spinor-valued point mass at p with coefficient v, relative to dv_g. The inverse of D_g extends to distributions, and $\psi = D_g^{-1}(\delta_p v)$ is smooth and harmonic away from p. In normal coordinates centered

at p and the Bourguignon–Gauduchon trivialization, Hermann's expansion [61, Corollary 5.3.4 and Theorem 5.3.10, pp. 55 and 58–60], with $n = 5$ and $\lambda = 0$, gives

$$\psi(x) = -\frac{1}{\omega_4}\frac{c_p(x)v}{|x|^5} + O(|x|^{-3}), \qquad \omega_4 = \mathrm{vol}(S^4). \tag{21.3}$$

Since $|c_p(x)v| = |x|\,|v|$, there is $r_0 > 0$ such that

$$|\psi(x)| \geq \frac{|v|}{2\omega_4}|x|^{-4} > 0 \qquad (0 < |x| < r_0).$$

Choose a closed coordinate ball $B \Subset U$ about p, of radius less than r_0, and a boundary collar on which $\psi \neq 0$. On the compact exterior $M \setminus \operatorname{int} B$ the current is divergence-free. Stokes' theorem gives

$$\int_{\partial B} \iota_{j_\psi}\, dv_g = 0. \tag{21.4}$$

Choose a quaternionic orthonormal trivialization on B. Since $\pi_4(S^7) = 0$ [59, Corollary 4.9, p. 349], the normalized collar spinor extends continuously to B, with its prescribed values on a smaller collar. Relative smooth approximation [87, Theorem 11, p. 99] gives a smooth such extension. Extend $\log|\psi|$ smoothly from that collar to B and exponentiate. The resulting smooth nowhere-zero spinor ϕ_0 equals ψ near ∂B. Put

$$\alpha_0 = \iota_{j_{\phi_0}}\, dv_g \in \Omega^4(B).$$

This form is nowhere zero and is closed near ∂B. On a collar $C \simeq S^4 \times [0,1)$ its de Rham class is determined by the integral over ∂B. This integral vanishes by (21.4). Hence $\alpha_0 = d\beta_\partial$ on C for a smooth three-form β_∂. Extend a cutoff multiple of β_∂ by zero to obtain $\beta_0 \in \Omega^3(B)$ with $d\beta_0 = \alpha_0$ on a smaller collar.

The Levi–Civita connection identifies a first jet of a three-form with its value and covariant derivative. For a local orthonormal frame $e_1, \ldots, e_5$ with dual $e^1, \ldots, e^5$, set

$$\mathcal{A} \,:\, T^*B \otimes \Lambda^3 T^*B \longrightarrow \Lambda^4 T^*B, \qquad \mathcal{A}(L) = \sum_{j=1}^{5} e^j \wedge L(e_j),$$
$$\mathcal{S} \,:\, \Lambda^4 T^*B \longrightarrow T^*B \otimes \Lambda^3 T^*B, \quad \mathcal{S}(\alpha)(X) = \tfrac{1}{4}\iota_X \alpha.$$

Then $\mathcal{A}(\nabla\beta) = d\beta$ and $\mathcal{A}\mathcal{S} = I$. Thus the value β_0 and formal derivative

$$L_0 = \nabla\beta_0 + \mathcal{S}(\alpha_0 - d\beta_0) \tag{21.5}$$

define a section of $J^1(\Lambda^3 T^*B)$ with $\mathcal{A}(L_0) = \alpha_0 \neq 0$. It equals $j^1\beta_0$ on the fixed collar.

The first-order relation $\mathcal{A}(L) \neq 0$ is open and ample. Indeed, in a principal affine subspace with covector $\xi \in T^*_x B \setminus \{0\}$, the derivative varies by $\xi \otimes \zeta$, where $\zeta \in \Lambda^3 T^*_x B$, and its alternating part by $\xi \wedge \zeta$. The latter map has rank $\binom{4}{3} = 4$. The excluded set is therefore empty or an affine subspace of codimension four. Its complement is connected and its convex hull is the whole principal subspace [47, Theorem 20.3.1 and Corollary 20.3.2].

Apply the relative form of the open-ample theorem [47, Theorem 18.4.1 and Remark 1] on $\operatorname{int} B$, fixing an outer annulus that is relatively closed in $\operatorname{int} B$ and has a neighborhood on which (21.5) is holonomic. It gives a continuous homotopy of formal solutions from (β_0, L_0) to $j^1\beta_1$, with β_1 smooth on $\operatorname{int} B$, fixed on that annulus. There $\beta_1 = \beta_0$, so β_1 extends smoothly to B. The alternating parts form a continuous homotopy through nonzero four-forms from α_0 to $d\beta_1$, fixed on a boundary collar. Reparameterize this homotopy to be constant near its endpoints. Apply relative smooth approximation [87, Theorem 11, p. 99] to the pullback of $\Lambda^4 T^*B \setminus 0$ over $B \times [0,1]$, relative to smaller closed endpoint collars and a smaller closed spatial collar. The section is smooth near these fixed sets. The resulting smooth family α_t, $0 \leq t \leq 1$, is fixed on the spatial collar and satisfies

$$\alpha_t \neq 0, \qquad \alpha_{t=0} = \alpha_0, \qquad \alpha_{t=1} = d\beta_1. \tag{21.6}$$

Write $\alpha_t = \iota_{Y_t} dv_g$. The normalized currents $Y_t/|Y_t|$ define a homotopy in the unit tangent sphere bundle. On B the map $\phi \mapsto j_\phi$ on unit spheres is the quaternionic Hopf bundle $S^7 \to S^4$ [59, Example 4.46, p. 378]. Choose a smooth connection on this sphere bundle. Horizontal lifting from $\phi_0/|\phi_0|$ gives a smooth unit-spinor family $\widehat{\phi}_t$ with $j_{\widehat{\phi}_t} = Y_t/|Y_t|$. The lifts exist for $0 \le t \le 1$ because the fibre S^3 is compact. Their time derivative is zero on the fixed collar, so $\widehat{\phi}_t = \psi/|\psi|$ there. Put $\phi = \sqrt{|Y_1|}\,\widehat{\phi}_1$. Since $j_{t\phi} = t^2 j_\phi$ for $t > 0$, $j_\phi = Y_1$. On the fixed collar, $|Y_1| = |j_\psi| = |\psi|^2$ and $\widehat{\phi}_1 = \psi/|\psi|$. Thus ϕ is nowhere zero on B, equals ψ on the collar, and satisfies $d(\iota_{j_\phi} dv_g) = d^2\beta_1 = 0$.

Extend ϕ by ψ outside B. Lemma 21.1, with $y = -iD_g\phi$, gives $Q = ic_g(\tau)$ on B and $(iD_g + Q)\phi = 0$. On the boundary collar $D_g\phi = 0$, so the solution in Lemma 10.3 gives $Q = 0$. Extension by zero yields $\tau \in C_c^\infty(U; \Lambda^3 T^*M)$ and $(D_g + c_g(\tau))\phi = 0$ on M. Since ϕ is nowhere zero on B, it is a nonzero element of this kernel. □

21.3. **Exact quaternionic nullity.**

Proposition 21.3. *Let $U \subset M$ be nonempty and open. The potential in Proposition 21.2 can be chosen so that $\dim_{\mathbb{C}} \ker(D_g + c_g(\tau)) = 2$. At this potential three real variations supported in U, after a smooth change of parameters, give the complete zero-cluster matrix $q \mapsto q \in \mathfrak{sp}(1)$ for $i(D_g + c_g(\tau))$.*

Proof. Choose τ_0 as in Proposition 21.2. Put $L_0 = D_g + c_g(\tau_0)$, $K = iL_0$, and $F = \ker K$, with $\dim_{\mathbb{H}} F = k \ge 1$. These operators have domain H^1, and $K^* = -K$. Their kernels consist of smooth sections. Unique continuation holds for every $D_g + c_g(\tau)$ with τ smooth: its square has scalar principal symbol $|\xi|_g^2 I$, and its smooth lower-order coefficients satisfy the local inequalities of [9, Remark 3, p. 248] for a system with a common scalar principal operator.

For

$$\mathcal{C} \,:\, C_c^\infty(U; \mathfrak{sp}(\Sigma M)) \longrightarrow \mathfrak{sp}(F), \qquad Q \longmapsto P_F Q|_F,$$

Lemma 6.5 gives

$$vqv^* \in \operatorname{im} \mathcal{C}, \qquad v = G_x^* u, \quad x \in U, \quad u \in \Sigma_x M, \quad q \in \operatorname{Im}\mathbb{H},$$

where $G_x \,:\, F \to \Sigma_x M$ is evaluation. Indeed, uqu^* is skew-adjoint. The vectors $G_x^* u$ span F: an element orthogonal to all of them vanishes on U, hence on M.

If $k = 1$, put $\ell = F$. If $k \ge 2$, choose $k-1$ quaternionically independent vectors of this form and put

$$M_0 = \sum_{j=1}^{k-1} v_j \mathbf{i} v_j^*, \qquad \ell = \ker M_0. \tag{21.7}$$

If $W \,:\, \mathbb{H}^{k-1} \to F$ has columns v_j, then $M_0 = W \operatorname{diag}(\mathbf{i}, \dots, \mathbf{i}) W^*$. The map W is injective and W^* is surjective. Thus M_0 has rank $k-1$, $\dim_{\mathbb{H}} \ell = 1$, and $M_0|_{\ell^\perp} \,:\, \ell^\perp \to \ell^\perp$ is invertible.

In either case let ψ be an L^2-unit generator of ℓ. Unique continuation gives a ball $O \Subset U$ on which $\psi \neq 0$. Choose a real $\rho \in C_c^\infty(O)$ with $\int_M \rho\, dv_g = 1$. For $q \in \operatorname{Im}\mathbb{H}$, set

$$Q_q(x) = \rho(x) \frac{\psi(x) q \psi(x)^*}{|\psi(x)|^4} \quad (x \in O), \qquad Q_q = 0 \quad \text{on } M \setminus O. \tag{21.8}$$

These are smooth skew-adjoint multipliers, and

$$\langle \psi, Q_q \psi \rangle_{L^2,\mathbb{H}} = \int_M \rho q \, dv_g = q.$$

Hence $P_\ell \mathcal{C}(Q_q)|_\ell = q$ in the frame ψ.

Suppose $k \ge 2$. Choose $Q^\circ \in C_c^\infty(U; \mathfrak{sp}(\Sigma M))$ with $\mathcal{C}(Q^\circ) = M_0$, and put $Q(q) = Q^\circ + Q_q$. Equip H^1 with the graph norm

$$\|u\|_K^2 = \|u\|_2^2 + \|Ku\|_2^2.$$

By (2.7) this norm is equivalent to the H^1 norm, and $H^1 = F \oplus (F^\perp \cap H^1)$ is an orthogonal decomposition for it. The complementary operator

$$K_c \,:\, F^\perp \cap H^1 \longrightarrow F^\perp$$

is bijective and has a bounded inverse. Write $Q_{ij}(q)$ for the blocks of $Q(q)$ relative to $F \oplus F^\perp$. For small (t, q), Lemma 6.6, with these fixed graph domains, gives the skew-adjoint Schur matrix

$$S(t, q) = tC(Q(q)) - t^2 Q_{12}(q)(K_c + tQ_{22}(q))^{-1} Q_{21}(q).$$

The smooth skew-adjoint matrix

$$N(t, q) = C(Q(q)) - tQ_{12}(q)(K_c + tQ_{22}(q))^{-1} Q_{21}(q)$$

satisfies $S(t, q) = tN(t, q)$ and $N(0, 0) = M_0$.

Write $N_{ij}(t, q)$ for its blocks in $F = \ell \oplus \ell^\perp$. At $(0, 0)$ the blocks N_{11}, N_{12}, N_{21} vanish, and $N_{22} = M_0|_{\ell^\perp}$ is invertible. Lemma 6.6 gives the residual matrix

$$s(t, q) = N_{11}(t, q) - N_{12}(t, q) N_{22}(t, q)^{-1} N_{21}(t, q) \in \mathfrak{sp}(\ell).$$

It satisfies

$$s(0, 0) = 0, \qquad D_q s(0, 0)\dot{q} = P_\ell C(Q_{\dot{q}})|_\ell = \dot{q}.$$

The derivative of the Schur product is zero at $(0, 0)$, since both off-diagonal blocks of M_0 vanish. The implicit function theorem on the real spaces $\mathbb{R} \times \operatorname{Im}\mathbb{H}$ and $\mathfrak{sp}(\ell)$ [50, Theorem 13.20, p. 71] gives a smooth $q(t)$ with $q(0) = 0$ and $s(t, q(t)) = 0$. For a sufficiently small fixed $t \neq 0$, both complementary operators remain invertible. The two kernel isomorphisms of Lemma 6.6 give

$$\ker(K + tQ(q(t))) \simeq \ell.$$

By Lemma 2.2, this operator is $K_1 = i(D_g + c_g(\tau_1))$ for a smooth three-form τ_1 compactly supported in U. If $k = 1$, take $\tau_1 = \tau_0$ and $K_1 = K$. In both cases $E = \ker K_1$ is a quaternionic line.

Define R_η, $\eta \in \operatorname{Im}\mathbb{H}$, by (21.8) with an L^2-unit generator of E, and identify E with $\mathbb{H}$ through that generator. Consider the self-adjoint family

$$T_\eta = -i(K_1 + R_\eta) = D_g + c_g(\tau_1) - iR_\eta$$

on the common domain H^1. A small symmetric spectral circle separates the entire zero eigenspace E of T_0 from the remaining spectrum. It remains in the resolvent for small η. Since $JT_\eta = -T_\eta J$, the cluster projection commutes with J. Lemma 6.1 gives a smooth quaternionic isometric frame U_η based on E. Put

$$f(\eta) = iU_\eta^* T_\eta U_\eta \in \mathfrak{sp}(E).$$

Lemma 6.4, applied to T_η, gives

$$f(0) = 0, \qquad Df(0)\dot{\eta} = iP_E(-iR_{\dot{\eta}})|_E = P_E R_{\dot{\eta}}|_E = \dot{\eta}.$$

The inverse function theorem [50, Theorem 11.22 and Corollary 11.35, pp. 61–63] gives a smooth change of parameters for which the complete matrix of $i(D_g + c_g(\tau(q)))$ is q. The corresponding self-adjoint matrix is $-iq \in \mathcal{A}_5(E)$. Choose a real basis η_1, η_2, η_3 of $\operatorname{Im}\mathbb{H}$ and define $\sigma_j \in C_c^\infty(U; \Lambda^3 T^*M)$ by $c_g(\sigma_j) = -iR_{\eta_j}$. Since $\eta \mapsto R_\eta$ is real-linear,

$$\tau(q) \in \tau_1 + \operatorname{span}_{\mathbb{R}}\{\sigma_1, \sigma_2, \sigma_3\}.$$

The union of the supports of $\tau_1, \sigma_1, \sigma_2, \sigma_3$ is a compact subset of U containing every $\operatorname{supp} \tau(q)$. □

Proof of the five-dimensional assertion of Theorem L. Let $U \subset M$ be nonempty and open, and let $\varepsilon > 0$. Propositions 21.2 and 21.3 give a full potential section on the fixed metric g, with self-adjoint matrix $-iq$, where $q \in \mathfrak{sp}(1) = \operatorname{Im}\mathbb{H}$. Restrict its parameter domain to a small closed ball on which one

symmetric circle separates the full zero cluster from the remaining spectrum. The entire potential family has one common compact support in U.

Apply Proposition 6.3, with the self-adjoint coordinate $A = -iq$. In its application of Theorem R require also

$$\sup_q \|h_q - g\|_{C^0_g} < \varepsilon.$$

The resulting smooth metric family satisfies

$$h_q = g \text{ on } M \setminus U, \qquad dv_{h_q} = dv_g,$$

and its complete matrix for iD_{h_q} is q. At $q = 0$ the whole kernel is the selected quaternionic line, so $\dim_{\mathbb{C}} \ker D_{h_0} = 2$. The derivative of its normal matrix is the identity. □

22. Spaces of Dirac-minimal metrics

Let M, g_*, $\mu = dv_{g_*}$, and the nonempty open set U be as in (8.2). All spaces carry the C^∞ topology. In the first three subsections, $n = \dim M \in \{4, 5\}$. For $n = 4$, put $a = |\operatorname{ind}_{\mathbb{H}} D^+|$ and let $A_{g,V}$ be the chosen chiral block of quaternionic index $-a$. Put

$$(\mathcal{Z}, r) = \begin{cases} (\mathcal{R}_{\min,\mu}(U), 4a + 4), & n = 4, \\ (\mathcal{R}_{\mathrm{inv},\mu}(U), 3), & n = 5, \end{cases} \qquad L_{g,V} = \begin{cases} A_{g,V}, & n = 4, \\ i(D_g + V), & n = 5. \end{cases}$$

22.1. **Relative incidence families.**

Lemma 22.1. *The space $\mathcal{Z}$ is nonempty. Every smooth map $S^j \to \mathcal{Z}$, $0 \le j \le r - 2$, extends to a smooth map $D^{j+1} \to \mathcal{Z}$.*

Proof. Logarithmic metric coordinates extend the sphere to a smooth family in $\mathcal{R}_\mu(U)$ over D^{j+1}, constant in the normal coordinate on a boundary collar. Since $j + 1 < r$, Lemma 10.4 gives supported potentials, zero on that collar, for which every $L_{g,V}$ is injective. Proposition 8.2, relative to the collar, gives the required metric family. The same argument with a one-point parameter space proves nonemptiness. For continuous sphere maps, relative smooth approximation in the open set $\mathcal{Z}$, as in Proposition 8.2, gives a homotopy to a smooth map. Attaching this homotopy to a filling of the smooth map gives a continuous filling with the original boundary. □

22.2. **Oriented incidence.**

Lemma 22.2. *Let $p \mapsto (g_p, V_p)$ be a family as in Lemma 10.4, transverse on every face of its oriented parameter manifold P, with $\dim P \in \{r, r + 1\}$. Put $L_p = L_{g_p,V_p}$. At every incident parameter, $\ker L_p$ is a quaternionic line. The incidence manifold has a canonical orientation compatible with boundary restriction.*

Proof. Let $\mathcal{I}$ be the incidence manifold of pairs (p, ℓ), where $\ell \subset \ker L_p$ is a quaternionic line, and let $\pi : \mathcal{I} \to P$ be the projection. If $\dim_{\mathbb{H}} \ker L_p \ge 2$, the fibre over p contains a smoothly embedded $\mathbb{H}P^1$ of real dimension four. This contradicts $\dim \mathcal{I} = \dim P - r \le 1$.

At $(p, \ell) \in \mathcal{I}$ one has $\ell = \ker L_p$. In dimension four put $C_p = \ker L_p^*$, so that $\dim_{\mathbb{H}} C_p = a + 1$. Define

$$N_p = \begin{cases} \operatorname{Hom}_{\mathbb{H}}(\ell, C_p), & n = 4, \\ \mathfrak{sp}(\ell), & n = 5. \end{cases}$$

Here $\mathfrak{sp}(\ell)$ is the real space of skew-adjoint quaternionic endomorphisms of ℓ. In the fixed positive spinor identifications let $\dot{L}_p(v)$ denote the parameter derivative in the direction $v \in T_pP$, and let P_{C_p} and P_ℓ denote the orthogonal projections. The vertical derivative on a projective slice is injective, with cokernel N_p, by the calculation in Lemma 10.4. Its parameter derivative on this cokernel is

$$\nu_p(v) = \begin{cases} P_{C_p} \dot{L}_p(v)|_\ell, & n = 4, \\ P_\ell \dot{L}_p(v)|_\ell, & n = 5. \end{cases}$$

Transversality gives the exact sequence

$$0 \longrightarrow T_{(p,\ell)}\boldsymbol{\mathcal{I}} \xrightarrow{d\pi} T_pP \xrightarrow{\nu_p} N_p \longrightarrow 0.$$

Thus π is an injective immersion. Since $\boldsymbol{\mathcal{I}}$ is compact, π identifies it with an embedded submanifold of P, with normal bundle N having fibres N_p.

For $n = 4$, orthonormal quaternionic frames identify N_p with the oriented real space $\mathbb{H}^{a+1}$. The frame changes belong to the connected group $\mathrm{Sp}(1)\times\mathrm{Sp}(a+1)$ and preserve this orientation. A global right quaternionic structure on N is unnecessary. For $n = 5$, unit frames of ℓ identify N_p with $\mathrm{Im}\,\mathbb{H}$, oriented by (i, j, k). Frame changes act by the adjoint representation of $\mathrm{Sp}(1)$ in $SO(3)$ and preserve this orientation. Orient $\boldsymbol{\mathcal{I}}$ by

$$\mathrm{or}(TP) = \mathrm{or}(T\boldsymbol{\mathcal{I}}) \otimes \mathrm{or}(N).$$

The incidence is empty on parameter faces of codimension at least two, since their expected dimensions are negative. At a boundary point, face transversality gives the same exact sequence with $T\partial\boldsymbol{\mathcal{I}}$ and $T\partial P$. If $w \in T_pP$ points outward, choose $z \in T_p\partial P$ with $\nu_p(z) = \nu_p(w)$. Then $w - z$ lies in $T\boldsymbol{\mathcal{I}}$ and still points outward. With this common outward vector, the outward-normal-first boundary conventions give

$$\mathrm{or}(T\partial P) = \mathrm{or}(T\partial\boldsymbol{\mathcal{I}}) \otimes \mathrm{or}(N).$$

No factor is moved past N; this formula holds for both values of r. □

22.3. **Local integer classes.**

Proposition 22.3. *For every basepoint g_b in the indicated space there are homomorphisms*

$$\begin{aligned} \lambda_4 &: \pi_{4a+3}(\mathcal{R}_{\min,\mu}(U), g_b) \longrightarrow \mathbb{Z}, \\ \lambda_5 &: \pi_2(\mathcal{R}_{\mathrm{inv},\mu}(U), g_b) \longrightarrow \mathbb{Z}. \end{aligned} \tag{22.1}$$

They are defined by the signed number of kernel lines in a transverse disk filling with supported compatible potentials.

Proof. Give D^r its standard orientation and S^{r-1} its boundary orientation. For a smooth based map $f : S^{r-1} \to \mathcal{Z}$, choose a smooth filling g_p, $p \in D^r$, in $\mathcal{R}_\mu(U)$, equal to the boundary family on a collar. Lemma 10.4 gives supported compatible potentials, zero on that collar, with transverse incidence. The incidence is a compact zero-manifold, oriented by Lemma 22.2. Let $\lambda(f)$ be its signed cardinality; here $\lambda = \lambda_n$.

For two such fillings write $g_p^{(i)} = g_*(e^{H_p^{(i)}}\cdot,\cdot)$, $i = 0, 1$. Their logarithms agree on a common boundary collar. The interpolation

$$H_{p,t} = (1-t)H_p^{(0)} + tH_p^{(1)}, \qquad 0 \le t \le 1,$$

preserves self-adjointness, zero trace, and vanishing on $M \setminus U$. Interpolate the potentials in the fixed positive spinor identifications. Compatibility is real-linear. Reparametrize in t to make this family constant near its ends, and orient $D^r \times [0, 1]$ by $dt \wedge \mathrm{or}(D^r)$. The relative part of Lemma 10.4 leaves the transverse endpoint families and the side collar fixed. Its incidence is a compact oriented one-manifold whose boundary is the final zero-manifold minus the initial one. Hence the signed counts agree.

Let $f_t : S^{r-1} \to \mathcal{Z}$ be a smooth homotopy, stationary near $t = 0, 1$. Extend it in logarithmic coordinates over $D^r \times [0, 1]$, keeping the side collar fixed, and take zero potential on that collar. There $\ker L_{g,0} = 0$. Relative transversality on all faces gives a compact oriented one-dimensional incidence with boundary only at $t = 0, 1$. Its endpoint counts equal the previously defined values, by independence of filling. Thus $\lambda(f_0) = \lambda(f_1)$. This also holds when the basepoint moves.

For addition, choose representatives constant near the basepoint by the based approximation in Proposition 8.2. Form the oriented boundary connected sum of two collared fillings, attaching a

one-handle in the regions where both families are $(g_b, 0)$. Extend by $(g_b, 0)$ on the handle. Smoothing its corners adds no incidence. The boundary represents the sum in $\pi_{r-1}(\mathcal{Z}, g_b)$, and its signed count is the sum of the two counts. Thus λ is a homomorphism. Relative smooth approximation as in Proposition 8.2 extends the definition and homotopy invariance to continuous maps. □

Proposition 22.4. *The homomorphisms in* (22.1) *are split-surjective. Every C^0 neighborhood of every metric in $\mathcal{R}_\mu(U)$ contains a sphere map with signed count one, based at one of its own points.*

Proof. Theorem L gives a transverse metric disk arbitrarily C^0-close to the prescribed metric, with the same density and exterior. After restriction and the inverse function theorem, its entire small chiral matrix in dimension four is the quaternionic column $q \in \mathbb{H}^{a+1}$, of source rank one and target rank $a + 1$. In dimension five its entire small iD matrix is $q \in \mathfrak{sp}(1)$. The complementary spectral family is invertible throughout the disk. Hence $\ker L_{g_q,0} = 0$ for $q \neq 0$. Orient the disk by these normal coordinates. Its single incidence has sign +1.

Write the parameter disk as D^r_δ, of radius $\delta > 0$. Choose a smooth radial map $R : D^r_\delta \to D^r_\delta$ equal to the identity near zero, with $R^{-1}(0) = \{0\}$ and $R(q) = \delta q/|q|$ near the boundary. The family $g_{R(q)}$ has a constant boundary collar and the same single transverse incidence, with the same sign. With zero potential it is therefore an admissible filling in Proposition 22.3. Its boundary sphere has $\lambda = 1$ and lies in the prescribed C^0 neighborhood.

Choose a boundary metric as basepoint. By Lemma 22.1, it can be joined to any prescribed g_b by a path in $\mathcal{Z}$. Homotopy invariance with moving basepoint preserves the signed count, even if this path leaves the chosen neighborhood. For the transported class x one has $\lambda(x) = 1$. Since $r - 1 \geq 2$, the map $\mathbb{Z} \to \pi_{r-1}(\mathcal{Z}, g_b)$, $m \mapsto mx$, is a homomorphism and splits λ. □

Proposition 22.5. *At index zero, the normal disks of Theorem L in dimensions four and five represent primitive ordinary real index-difference classes.*

Proof. At index zero, the normal disks have parameter dimensions $r = 4$ for $n = 4$ and $r = 3$ for $n = 5$. Restrict each disk so that one spectral circle separates its complete normal block from the remaining spectrum. The complementary family is invertible on the whole disk and represents the zero relative class [45, Definition 2.18]. Thus the full real Dirac family, with the identifications of Section 8.1, and its normal block represent the same class

$$x_n \in KO^{-n}(D^r, S^{r-1}) \cong KO^{-(n+r)}(\mathrm{pt}) = KO^{-8}(\mathrm{pt}).$$

Complexification preserves the Morita reductions in (8.3). In particular,

$$\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \cong M_2(\mathbb{C}), \qquad M_2(\mathbb{C})e \cong \mathbb{C}^2, \qquad e = \mathrm{diag}(1, 0).$$

For a right quaternionic multiplicity space V with underlying real space $V_{\mathbb{R}}$, its complexification is a right $M_2(\mathbb{C})$-module, and the primitive corner $(V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C})e$ has complex dimension $2 \dim_{\mathbb{H}} V$. Quaternionic-linear operators preserve this corner. Left multiplication $L_q : x \mapsto qx$ on $\mathbb{H}$ therefore gives one defining complex block.

In dimension four the normal chiral map is $A(q) = L_q : \mathbb{H} \to \mathbb{H}$, $q \in D^4$. On S^3 its complex clutching map is the defining representation $\mathrm{Sp}(1) = SU(2) \to U(2)$. The associated complex rank-two Hopf bundle has second Chern number ± 1 [60, Proposition 3.13(c), p. 91, and §3.2, p. 93]. Its reduced complex class is therefore primitive.

In dimension five put $K(v) = L_v$ for $v \in \mathrm{Im}\,\mathbb{H}$. Let S be an irreducible real Cl_4-Hilbert module with a commuting left $\mathbb{H}$-action, and let $V = \mathbb{H}$ be the right quaternionic multiplicity line. On $E = V \otimes_{\mathbb{H}} S$, the four Clifford generators $c_1, \dots, c_4$ act on S, and K acts on V; these actions commute. Put $\gamma = c_1 c_2 c_3 c_4$. Then

$$\gamma^2 = I, \qquad \gamma^* = \gamma, \qquad \gamma c_j = -c_j \gamma \quad (1 \leq j \leq 4).$$

On $E \oplus E$ put

$$C_j = \begin{pmatrix} c_j & 0 \\ 0 & c_j \end{pmatrix}, \qquad \iota = \begin{pmatrix} \gamma & 0 \\ 0 & -\gamma \end{pmatrix}, \qquad 1 \leq j \leq 4,$$

$$F_K = \begin{pmatrix} 0 & -K \\ K & 0 \end{pmatrix}, \qquad e_5 = \begin{pmatrix} 0 & -\gamma \\ \gamma & 0 \end{pmatrix}.$$

The generators $C_1, \dots, C_4, e_5$ square to $-I$ and anticommute with each other and with ι. The operator F_K is odd, self-adjoint, and commutes with these generators. They define a $\mathrm{Cl}^{5,0}$-cycle in the convention of [45, Definition 2.1]. On the positive γ-corner, the grading is $\mathrm{diag}(I, -I)$, the fifth generator is U_-, and the operator is $F_-(K)$, as in (8.3).

Take e_5 as the first generator in Ebert's Bott map [45, §2.4, (2.23)] and then forget it. The resulting family is

$$B(s, v) = F_K + s(e_5\iota) = \begin{pmatrix} 0 & sI - K \\ sI + K & 0 \end{pmatrix}, \qquad B(s, v)^2 = (s^2 + |v|^2)I.$$

On $[-1, 1] \times D^3$ this family is invertible away from $(0, 0)$, so excision reduces its relative class to a small four-ball. For $\varepsilon \in \{1, -1\}$ put $p_\varepsilon = (I + \varepsilon\gamma)/2$. This is a primitive idempotent in the corresponding simple summand of Cl^0_4, and $p_\varepsilon S$ is a quaternionic line. On its doubled corner the grading is $\mathrm{diag}(\varepsilon, -\varepsilon)$. With the positive grading summand first, the chiral map is

$$sI + \varepsilon K(v) = L_{s+\varepsilon v}.$$

For the dimension-four corner of Section 8.1, $\varepsilon = -1$. The two choices differ by the coordinate map $(s, v) \mapsto (s, -v)$, of degree -1. Each therefore gives the four-dimensional clutching generator up to sign. Both local classes have primitive complex images in total degree eight.

Let $\beta \in K^{-2}(\mathrm{pt})$ be the complex Bott generator and $b_{\mathbb{R}} \in KO^{-8}(\mathrm{pt})$ the real Bott generator. Complexification satisfies $c(b_{\mathbb{R}}) = \beta^4$ [12, (6.11) and Theorem 11.5]. Thus $c : KO^{-8}(\mathrm{pt}) \to K^{-8}(\mathrm{pt})$ is an isomorphism, and both x_4 and x_5 are primitive, up to sign. The factor two for complexification of the degree-four coefficient generator does not occur in this total degree.

Choose a basepoint $z_0 \in S^{r-1}$ for the boundary family $f : S^{r-1} \to \mathcal{Z}$, and put $g_b = f(z_0)$. Put $H_z = \log(g_b^{-1} f(z))$. Each H_z is g_b-self-adjoint and trace-free, vanishes on $M \setminus U$, and $H_{z_0} = 0$. Choose a smooth $\chi : \mathbb{R} \to [0, 1]$ with $\chi(t) = 0$ for $t \le -1/2$ and $\chi(t) = 1$ for $t \ge 1/2$. The paths

$$g_{z,t} = g_b(e^{\chi(t)H_z}\cdot, \cdot)$$

have fixed density and exterior. They are constant for $|t| \ge 1/2$ and for $z = z_0$, and their endpoint operators are invertible. Contractibility of $\mathcal{R}_\mu(U)$ identifies their relative path class with x_n. The pair $(S^{r-1}, \{z_0\})$ is finite CW, M is closed, and the full real Dirac family is odd, Clifford-linear, and of Dirac type. The positive spinor identifications and compatible connection of Section 8.1 are constant in the cylinder direction on the ends and over z_0. Thus Ebert's Assumptions 3.11 and 3.13 apply; his Lemma 3.16 and Theorem 3.18 identify the path and cylinder classes under Bott periodicity [45]. These are the ordinary based real index-difference classes. □

22.4. **The comparison maps.** Let $n \equiv 4$ or $5 \pmod 8$ and let $y = (g, V)$ belong to the supported pair space of (8.2); the pair agrees with the prescribed data on $M \setminus U$. In residue four put $j = \mathrm{ind}_{\mathbb{H}} D^+$ and $a = |j|$. Take $\varepsilon = +$ if $j \le 0$ and $\varepsilon = -$ otherwise. Thus $A_y = (D_g + V)^\varepsilon$ has quaternionic index $-a$. Set $Y_a(U) = \mathcal{P}_{\min,\mu}(U)$ in this case and $Y_5(U) = \mathcal{P}_{\mathrm{inv},\mu}(U)$ in residue five. Fix $b = (g_b, V_b)$ in the indicated locus; its existence follows from Lemma 10.4 with parameter space a point.

Identify the spinor bundles by the positive comparison with g_b, including the density factor. Write $g = g_b(e^H\cdot, \cdot)$, where H is g_b-self-adjoint and $\mathrm{tr}\, H = 0$. In these coordinates the path from b to y is

$$c_t(y) = (tH, (1 - t)V_b + tV), \qquad 0 \le t \le 1. \tag{22.2}$$

It is constant at b when $y = b$ and preserves the prescribed data on $M \setminus U$. Without the density constraint the same formula applies to all g_b-self-adjoint H.

Write $\mathrm{Sp} = \mathrm{colim}_m \mathrm{Sp}(m)$ and $\mathrm{U}_{\mathrm{st}} = \mathrm{colim}_m \mathrm{U}(m)$ for the standard block inclusions, with the direct limit topologies and $\mathrm{U}(m) \subset \mathrm{Sp}(m)$ by $\mathbb{C} \subset \mathbb{H}$. In $\mathbb{H}^{r+a} = \mathbb{H}^a \oplus \mathbb{H}^r$ take the last r coordinate vectors as the standard frame. Its stabilizer is $\mathrm{Sp}(a)$ on the first summand. Thus $\mathrm{Sp}(r + a)/\mathrm{Sp}(a)$ is the

space of ordered orthonormal r-frames. Appending a unit coordinate vector gives the inclusions in $\mathrm{Sp}\,/\,\mathrm{Sp}(a) = \mathrm{colim}_r\,\mathrm{Sp}(r+a)/\,\mathrm{Sp}(a)$. Here $\simeq$ denotes an equivalence of weak homotopy types.

In residue four let $S = \Sigma^{\varepsilon}_{g_b}$ and $T = \Sigma^{-\varepsilon}_{g_b}$ be the source and target spinor bundles. Let $\mathscr{F}_{-a}$ be the space of bounded quaternionic Fredholm operators $H^1(S) \to L^2(T)$ of index $-a$, with the operator norm, and let $\mathscr{I}_a$ be its injective locus. The coefficients of A_y depend continuously on y in these fixed spaces; thus $y \mapsto A_y$ is norm-continuous. Let $\mathscr{H}_b$ be the homotopy fibre of $\mathscr{I}_a \hookrightarrow \mathscr{F}_{-a}$ over A_b. Its points are pairs (L, γ) with $L \in \mathscr{I}_a$ and a path γ in $\mathscr{F}_{-a}$ from L to A_b; its basepoint is the constant path at A_b paired with A_b. Then

$$\mathscr{I}_a \simeq B\,\mathrm{Sp}(a), \qquad \mathscr{F}_{-a} \simeq B\,\mathrm{Sp}, \qquad \mathscr{H}_b \simeq \mathrm{Sp}\,/\,\mathrm{Sp}(a). \tag{22.3}$$

Indeed polar decomposition identifies the first space, up to homotopy, with the isometric embeddings of codimension a. The orthogonal complement map to the Grassmannian of quaternionic a-planes is a locally trivial bundle with contractible Hilbert isometry fibres [75, Theorem II.6(i)–(ii)]. For the middle identification take the negative of the reduced quaternionic Fredholm index [15, Theorem A(k) and §5, p. 23]. On the injective locus its value is $[C] - a$, where C is the cokernel bundle and a denotes the trivial quaternionic bundle of rank a. The inclusion therefore corresponds to stabilization $B\,\mathrm{Sp}(a) \to B\,\mathrm{Sp}$, whose homotopy fibre is $\mathrm{Sp}\,/\,\mathrm{Sp}(a)$.

The based map $Y_a(U) \to \mathscr{H}_b$ sends y to $(A_y, t \mapsto A_{c_{1-t}(y)})$. In the weak homotopy types of (22.3) it is denoted by

$$\kappa_a \;:\; Y_a(U) \longrightarrow \mathrm{Sp}\,/\,\mathrm{Sp}(a). \tag{22.4}$$

The path gives a nullhomotopy of the stabilized cokernel class; no frame of C is required. For $a = 0$ the sign is that of the ordinary quaternionic index difference: reversal of the ambient path and negation of the reduced index each negate the suspended class. On the unit disk of $\mathbb{H}$ the scalar family $L_q v = qv$ has boundary frame $q \mapsto L_q$, the clutching map of Proposition 22.5. Orient this disk by the positive generator for the ordinary relative index.

In residue five set $\mathcal{H} = L^2(\Sigma_{g_b})$ and $A_y = i(D_g + V)$ as in Section 2. This is a quaternionic skew-adjoint operator on $\mathcal{H}$ with domain $H^1(\Sigma_{g_b})$. Its Cayley transform is

$$u(A_y) = (A_y + I)(A_y - I)^{-1} = (D_g + V - i)(D_g + V + i)^{-1}. \tag{22.5}$$

Let Sp_{c} be the norm-topological group of quaternionic unitaries on $\mathcal{H}$ which differ from I by a compact operator. Then $u(A_y) \in \mathrm{Sp}_{\mathrm{c}}$ since $u(A_y) - I = 2(A_y - I)^{-1}$ and the resolvent is compact. The resolvent identity on the fixed domain gives norm continuity in y. Choose an orthonormal quaternionic basis of $\mathcal{H}$. The finite block inclusion $\mathrm{Sp} \to \mathrm{Sp}_{\mathrm{c}}$ is a weak equivalence by [75, Theorem II.14(i), pp. 109–110]: take the ground field $\mathbb{H}$ and the compact ideal ($p = \infty$); polar retraction commutes with the inclusions.

Let $\mathcal{V} = \{u \in \mathrm{Sp}_{\mathrm{c}} \;:\; u + I \text{ is invertible}\}$. The principal logarithm maps this open set homeomorphically onto the compact quaternionic skew-adjoint operators Z on $\mathcal{H}$ with $\|Z\| < \pi$. Write $u_y = u(A_y)$. For $y \in Y_5(U)$ both u_y and u_b belong to $\mathcal{V}$, since A_y and A_b are invertible. Let η_y be the concatenation of the paths

$$t \longmapsto u(A_{c_t(y)}), \qquad s \longmapsto \exp\bigl((1-s)\log u_y + s \log u_b\bigr) \quad (0 \le t, s \le 1).$$

It starts and ends at u_b. Left translation by u_b^{-1} gives the based map

$$\kappa_5 \;:\; Y_5(U) \longrightarrow \Omega\,\mathrm{Sp}_{\mathrm{c}} \simeq \Omega\,\mathrm{Sp} \simeq \mathrm{Sp}\,/\mathrm{U}_{\mathrm{st}}, \qquad \kappa_5(y) = u_b^{-1}\eta_y. \tag{22.6}$$

Here the loops are based at I and the last equivalence is Bott periodicity [25, Theorems I–II, (1.3) and (1.6)]. The relative Fredholm class in (22.34) identifies the induced map on π_k with real index difference in KO_{n+k+1}; (22.5) is the Cayley convention of Proposition 22.23.

Lemma 22.6. *Let a continuous family of chiral pairs over a disk D^d, $d \ge 1$, be minimal on ∂D^d. Suppose continuous families of bounded quaternionic-linear isomorphisms of source and target over D^d put its operator at p in the form $F_p \oplus G_p$, where $F_p \;:\; \mathbb{H}^r \to \mathbb{H}^{r+a}$ and G_p is invertible between fixed Hilbert spaces. The free homotopy class of the boundary map under* (22.4) *is represented by the frame $p \mapsto F_p(F_p^{*}F_p)^{-1/2}$ in $\mathrm{Sp}(r+a)/\,\mathrm{Sp}(a)$.*

Proof. The given disk and the cone (22.2) on its boundary are homotopic relative to the boundary in the contractible pair space. After fixed Hilbert isometries, the coordinate changes are homotopic over the disk to the identity by [75, Theorem II.6]. Left and right multiplication by these isomorphisms preserve Fredholmness and injectivity. The same argument reduces G_p to the identity. On the boundary F_p is injective, and $C_p = \ker F_p^*$ represents its quotient cokernel. In the Stiefel model of (22.3) the corresponding trivialization is

$$\mathbb{H}^r \oplus C_p \longrightarrow \mathbb{H}^{r+a}, \qquad (v, w) \longmapsto F_p(F_p^* F_p)^{-1/2} v + w.$$

Its first summand is the stated frame. For a based boundary class, choose a boundary point and send its frame to the standard frame by a constant element of $\mathrm{Sp}(r + a)$. □

22.5. **Boundary domains and supported perturbations.** Let W be a compact connected spin manifold of dimension n with a product metric on a boundary collar. Denote the inward unit normal by ν. In residue four take $E = \Sigma^\varepsilon$, $F = \Sigma^{-\varepsilon}$ for either chirality ε, and let $A = D^\varepsilon + R$ with $R : E \to F$ quaternionic linear. In residue five take $E = F = \Sigma$ and $A = i(D + V)$ with V a compatible potential. This differential expression is quaternionic linear and formally skew-adjoint. Write $A^\dagger$ for the formal adjoint in either case.

Fix a quaternionic self-adjoint adapted tangential operator B for A and ν [20, §3.2]. Write $K = \ker B$ and $H^{1/2}_{<0}(B)$ for the closed span in $H^{1/2}(\partial W; E)$ of the eigensections of B with negative eigenvalue. Let $\gamma : H^1(W; E) \to H^{1/2}(\partial W; E)$ be the trace. The low and full APS realizations $A_{\mathrm{low}}, A_{\mathrm{full}}$ in L^2 have domains

$$\mathcal{D}_{\mathrm{low}} = \{v \in H^1(W; E) : \gamma v \in H^{1/2}_{<0}(B)\},$$
$$\mathcal{D}_{\mathrm{full}} = \{v \in H^1(W; E) : \gamma v \in H^{1/2}_{<0}(B) \oplus K\}.$$

These are elliptic spectral boundary conditions: the cuts are zero and a sufficiently small positive number. Since W is compact, the realizations are Fredholm; the adjoint boundary conditions are the Green annihilators of the trace spaces. The kernels of these realizations and of their adjoints are smooth up to the boundary [20, Theorems 4.6, 4.9, 5.3 and Example 4.21]. Trace induces $\mathcal{D}_{\mathrm{full}}/\mathcal{D}_{\mathrm{low}} \cong K$. Hence $\mathrm{ind}_{\mathbb{H}} A_{\mathrm{full}} - \mathrm{ind}_{\mathbb{H}} A_{\mathrm{low}} = \dim_{\mathbb{H}} K$ by [20, Corollary 6.2]; the complex indices and dimensions there are twice their quaternionic values here.

In residue five assume in addition the following product form on the collar, with t the inward normal coordinate:

$$A = \sigma(\partial_t + B), \qquad \sigma = ic(\nu), \qquad \sigma^* = \sigma, \quad \sigma^2 = I, \quad \sigma B = -B\sigma. \tag{22.7}$$

Suppose $\sigma|_K$ has quaternionic signature (N, N). Since σ interchanges the positive and negative spectral subspaces, the Green annihilator of $H^{1/2}_{<0}(B)$ is $H^{1/2}_{<0}(B) \oplus K$. Thus $A^\dagger = -A$ and $\dim_{\mathbb{H}} K = 2N$ give

$$A^*_{\mathrm{low}} = -A_{\mathrm{full}}, \qquad \mathrm{ind}_{\mathbb{H}} A_{\mathrm{low}} = -N, \qquad \mathrm{ind}_{\mathbb{H}} A_{\mathrm{full}} = N. \tag{22.8}$$

Indeed the two indices are opposite and their difference is $2N$.

Let P be a compact smooth manifold of dimension d, possibly with boundary, and let $\emptyset \neq O \Subset \mathrm{int}\, W$ be open. Choose the fixed collar disjoint from $\overline{O}$, shrinking it if necessary. Suppose the metric and coefficients of a smooth family parametrized by P are fixed on this collar, and keep B fixed. Positive spinor comparison then identifies the domains above for all parameters.

Lemma 22.7. *Let the potential vary by quaternionic-linear multiplication operators in the chiral case and by quaternionic skew-adjoint multiplication operators for A in residue five. The following perturbations may be taken arbitrarily small in C^∞ with support in a common compact subset of O. They may vanish near any closed subset of P on a neighborhood of which the required conclusion holds.*

For a chiral family with a fixed elliptic APS domain and index $-r$, $r \geq 0$, injectivity can be attained if $d < 4(r + 1)$. For index r the same bound gives surjectivity. If $m = \dim_{\mathbb{H}} K$ and the low index is $-r$, low injectivity and full surjectivity can be attained by the same perturbation when

$$0 \leq r \leq m, \qquad d < 4(r + 1), \qquad d < 4(m - r + 1). \tag{22.9}$$

For (22.7)*, low injectivity and full surjectivity can be attained if* $d < 4N + 3$*. On a closed manifold the bounds are* $d < 4(r + 1)$ *in the chiral case and* $d < 3$ *in the skew case.*

Proof. Let $\mathcal{D}$ be the domain of the realization. At a unit kernel vector ψ, a slice for right $\mathrm{Sp}(1)$ on the L^2-unit sphere in $\mathcal{D}$ has tangent space $\mathcal{D} \cap \psi^{\perp_{\mathbb{H}}}$. Orthogonality is in L^2; the subscripts $\mathbb{H}$ and $\mathbb{R}$ refer to the quaternionic inner product and its real part. Since $\psi\mathbb{H} \subset \ker A$,

$$A(\mathcal{D} \cap \psi^{\perp_{\mathbb{H}}}) = A(\mathcal{D}).$$

Consequently a real annihilator ϕ of the vertical image belongs to the kernel of the Hilbert adjoint. The spinors ψ and ϕ are smooth. In the interior the equations are $A\psi = 0$, $A^{\dagger}\phi = 0$. Since $A^{\dagger}A$ and $AA^{\dagger}$ are connection Laplacians plus terms of order at most one, Aronszajn's unique continuation theorem applies on the connected interior [61, Theorem 6.2.8]. In particular ψ is nowhere zero on some ball in O.

For chiral index $-r$ the vertical real index is $-4r - 4$, since the slice removes one quaternionic line from the domain. Suppose that ϕ also annihilates $Q\psi$ for every smooth quaternionic homomorphism $Q : E \to F$ supported in O. On the ball just chosen, $Q(x)\psi(x)$ ranges over F_x. Multiplication by scalar test functions gives $\phi = 0$ there, hence $\phi = 0$ on W.

For the low skew realization the trace space is Green-isotropic. Hence $\mathrm{Re}\langle v, Av\rangle_{L^2} = 0$ for $v \in \mathcal{D}_{\mathrm{low}}$, and the incidence equation at v takes values in $v^{\perp_{\mathbb{R}}}$. At a zero its vertical real index is $4\,\mathrm{ind}_{\mathbb{H}} A_{\mathrm{low}} - 4 + 1 = -4N - 3$. An annihilator $\phi \in \psi^{\perp_{\mathbb{R}}}$ belongs to $\ker A^{*}_{\mathrm{low}} = \ker A_{\mathrm{full}}$ by (22.8). If it also annihilates every supported skew variation $Q\psi$, Lemma 10.3 and scalar test functions give $\phi = f\psi$ on the ball, with f smooth and real. The two interior equations now give $ic(df)\psi = 0$. Clifford multiplication by a nonzero covector is invertible; hence f is constant, say c. Unique continuation gives $\phi = c\psi$ on W, and real orthogonality gives $c = 0$.

For normalized kernel vectors the elliptic boundary estimate gives a uniform H^2 bound on compact parameter sets. Thus the kernel incidence is compact in $P \times H^1$ before passage to $\mathrm{Sp}(1)$-orbits. By the argument of Lemma 10.4 there is a finite-dimensional real space of smooth potentials compactly supported in O and a ball about zero in this space on which the universal differential is onto at every zero, also over ∂P. The incidence projection to the ball has index $d - 4(r + 1)$ or $d - 4N - 3$, respectively; on ∂P replace d by $d - 1$. A common regular value arbitrarily close to zero has empty fibre when the indicated indices are negative. The relative assertion follows by the parameter cutoff of that lemma, since injectivity and surjectivity are open on these fixed Fredholm domains.

For chiral surjectivity apply the same argument to the adjoint with its elliptic Green-annihilator domain. The full index is $m - r$. Add the perturbation space for the low operator to the adjoints of the perturbations for A^{*}_{full}. In this sum a common regular value gives both conclusions under (22.9). For the skew pair full surjectivity is equivalent to low injectivity by (22.8). The closed assertions are Lemma 10.4 and its chiral adjoint form. □

22.6. **Auxiliary spin domains with large boundary kernel.**

Lemma 22.8. *Let* $n \equiv 4$ *or* $5 \pmod 8$*. There is a connected separating hypersurface* Σ *in the spin sphere* S^n *with the following properties. The metric is a product on a collar of* Σ*. On the closures* W_0*,* W_1 *of the complementary components there are compatible potentials supported in the interiors. On either side let* B *denote the tangential Dirac operator for the product collar and the inward normal.*

In residue four fix one chirality. Then $\dim_{\mathbb{H}} \ker B = 2r$*, the two low APS operators are injective of quaternionic index* $-r$*, and the two full APS operators are surjective of index* r*. In residue five the involution* σ *of* (22.7) *has signature* (N, N) *on* $\ker B$*; the corresponding low operators are injective of index* $-N$ *and the full operators are surjective of index* N*. The construction gives arbitrarily large* r*, respectively* N*. In each example the nonzero spectrum of* B *is separated from zero. An interior ball of* W_1 *has Euclidean metric and zero potential.*

Proof. Put $m = n - 3$ and $T^m = (\mathbb{R}/2\pi\mathbb{Z})^m$. Embed T^m as an oriented hypersurface in $\mathbb{R}^{m+1}$: starting with $S^1 \subset \mathbb{R}^2$, pass from $T^j \subset \mathbb{R}^{j+1}$ to the boundary $T^j \times S^1$ of its tube in $\mathbb{R}^{j+2}$. The oriented normal

line and two constant normal vectors frame the normal bundle of T^m in $\mathbb{R}^{m+3}$. Place its framed tube $T^m \times D^3$ in a ball of S^n. Changing the framing by a map $T^m \to SO(3)$ changes the induced torus spin structure by the pullback of the spin double cover. For $\ell_j \in \{0, 1\}$, rotation in a fixed plane through the angle $\sum_{j=1}^m \ell_j \theta_j$, in angular coordinates θ_j, has monodromy $(-1)^{\ell_j}$ on the jth circle. We may thus choose the periodic spin structure on every circle. Restriction to the core identifies spin structures on the tube with those on the framed core, since $T^m \times D^3$ retracts onto T^m.

For an integer $b \geq 1$ embed a handlebody H_g of genus $g = 4b - 1$ in $\operatorname{int} D^3$. Take

$$W_0 = T^m \times H_g, \qquad \Sigma = T^m \times \partial H_g, \qquad W_1 = S^n \setminus \operatorname{int} W_0.$$

The common boundary is connected; its exterior collar meets every component of W_1, so W_1 is connected. The spin structure on ∂H_g bounds. On a hyperelliptic curve C of genus g, Hitchin's Proposition 2.4 gives a square root κ of the canonical bundle with $h^0(C, \kappa) = (g + 1)/2 = 2b$ [65]. Its spin structure bounds by [11, Proposition 4.1]. The two spin structures are therefore equivalent under an orientation-preserving diffeomorphism $\partial H_g \to C$ [30, Lemma 2.15, $r = 2$]. Pull back a metric in the conformal class of C. The two chiral surface kernels have complex dimensions $h^0(C, \kappa)$ and $h^1(C, \kappa) = h^0(C, \kappa)$ by [65, Theorem 2.2] and Serre duality. Extend this metric over H_g as a product near its boundary. Give W_0 the product of this metric and the flat torus metric. Extend it across Σ over W_1, with the same product collar and a Euclidean metric on an interior ball.

Let D_T, D_C be the Dirac operators of the torus and the surface, and let Γ_C denote surface chirality. The periodic flat torus has precisely its parallel spinors in the kernel. In residue four, m is odd and the boundary operator, up to sign and unitary identification, is $D_T \otimes \Gamma_C + I \otimes D_C$. For a torus eigenvalue μ and a unit surface eigenspinor v of eigenvalue $\lambda > 0$, the surface plane spanned by $v, \Gamma_C v$ gives the matrix $\left(\begin{smallmatrix} \lambda & \mu \\ \mu & -\lambda \end{smallmatrix}\right)$. On $\ker D_C$ the eigenvalues $\mu, -\mu$ occur with equal multiplicity. Thus $\eta(B) = 0$. The complex torus kernel has dimension $2^{(m-1)/2}$; the product formula [65, p. 12, Remark (4)] gives

$$\dim_{\mathbb{H}} \ker B = \tfrac{1}{2}\, 2^{(m-1)/2}(4b) = 2r, \qquad r = b\, 2^{(n-4)/2}. \tag{22.10}$$

The positive-degree Pontryagin forms on W_0 vanish: its curvature is pulled back from H_g, of dimension three. The integrals of the top component of $\widehat{A}$ over W_0 and W_1 add to $\widehat{A}(S^n) = 0$, since TS^n is stably trivial. Thus both integrals vanish. For the positive chiral operator the product APS formula [13, Theorem 4.2] gives on each side

$$\operatorname{ind}_{\mathbb{C}} D^+_{\text{low}} = \int_{W_i} \widehat{A} - \tfrac{1}{2}\big(\eta(B) + \dim_{\mathbb{C}} \ker B\big) = -2r.$$

Reversing the normal changes B to $-B$ under the boundary identification. Changing chirality likewise reverses the tangential sign and negates the interior index density. Neither change affects the value $-2r$. The quaternionic low index is $-r$ and adjoining $\ker B$ to the trace space gives full index r.

In residue five m is even. Each torus half-spinor kernel has complex dimension $2^{m/2-1}$. Let $B^{\pm}$ be the chiral parts of the intrinsic boundary Dirac operator. In each total chirality the product formula has two summands of complex dimension $2^{m/2-1}(2b)$. Both chiral kernels are quaternionic in boundary dimension $4 \pmod 8$; hence

$$\dim_{\mathbb{H}} \ker B^+ = \dim_{\mathbb{H}} \ker B^- = 2^{m/2-1}(2b) =: N. \tag{22.11}$$

The involution $ic(\nu)$ for the inward normal ν is boundary chirality up to sign. The indices now follow from (22.8).

Start with zero potentials. Apply Lemma 22.7 on each side with parameter space a point and perturbations supported in an interior ball, disjoint in W_1 from the prescribed Euclidean ball. This gives the asserted injectivity and surjectivity without changing B. The gap off $\ker B$ follows from compact resolvent on the closed hypersurface Σ. □

22.7. **Connected sums with a fixed APS boundary condition.** For an elliptic realization A_D on a compact manifold, with the boundary conditions of Section 22.5, its self-adjoint double is

$$\mathscr{D}_D = \begin{pmatrix} 0 & A_D^* \\ A_D & 0 \end{pmatrix}, \qquad \operatorname{dom} \mathscr{D}_D = \operatorname{dom} A_D \oplus \operatorname{dom} A_D^*. \tag{22.12}$$

Both summands of the domain, with their graph norms, embed continuously in H^1 by elliptic regularity. Thus the double has compact resolvent. Its grading is $+I$ on the source of A_D and $-I$ on the target.

Let P be compact. On compact n-manifolds M_1, M_2, $n \geq 3$, let $D_{i,p}$, $p \in P$, be continuous families of smooth, self-adjoint first-order elliptic differential realizations with fixed elliptic boundary trace conditions and fixed collar data; an empty boundary is allowed. Continuity is in the operator norm from the fixed H^1 domain to L^2 after bundle and density identification. Fix an interior Euclidean ball of radius $R > 0$ in each piece and spinor identifications in which the local operators are the same fixed sum of signed Euclidean spin Dirac operators. Fix oriented gluing charts and compatible spin lifts on the annuli. Match corresponding summands and take the same signed spin Dirac operators on the neck. The boundary trace condition on the sum is the direct sum of the original conditions.

For $0 < \rho < \min(R, 2^{-4})$ put $t_2 = \rho$, $t_1 = \rho^4/2$, $t_{-1} = t_1/2$ and $s_\rho = t_{-2} = 2^{-9}\rho^{16}$. Fix a smooth even function $h > 0$ with $h(s) = |s|$ for $|s| \geq 1$, $h \leq 1$ on $[-1, 1]$ and $|h'| \leq 1$. Replace the balls of radius s_ρ by the cylinder $[-s_\rho, s_\rho] \times S^{n-1}$ with metric $dt^2 + s_\rho^2 h(t/s_\rho)^2 g_{S^{n-1}}$, where $g_{S^{n-1}}$ has sectional curvature one. Denote the glued realization by $D_{\rho,p}$ and the disjoint sum by $D_p^0 = D_{1,p} \oplus D_{2,p}$. The Green form and the elliptic boundary conditions are unchanged, so $D_{\rho,p}$ is self-adjoint with compact resolvent. For a self-adjoint operator D and a Borel set $I \subset \mathbb{R}$ write $E_I(D) = \operatorname{ran} \mathbf{1}_I(D)$.

Proposition 22.9. *Let $\Lambda, \epsilon > 0$, let k be a nonnegative integer and suppose $\dim_{\mathbb{C}} E_{(-\Lambda,\Lambda)}(D_p^0) \leq k$ for every $p \in P$. There is a $\rho_0 > 0$, depending only on R, Λ, ϵ, k, such that, for $0 < \rho < \rho_0$ and $|\lambda| < \Lambda - 2\epsilon$,*

$$\dim_{\mathbb{C}} E_{\{\lambda\}}(D_p^0) \leq \dim_{\mathbb{C}} E_{[\lambda-\epsilon,\lambda+\epsilon]}(D_{\rho,p}), \tag{22.13}$$

$$\dim_{\mathbb{C}} E_{[\lambda-\epsilon,\lambda+\epsilon]}(D_{\rho,p}) \leq \dim_{\mathbb{C}} E_{[\lambda-2\epsilon,\lambda+2\epsilon]}(D_p^0). \tag{22.14}$$

For positive radii the family is norm-resolvent continuous in (p, ρ), and its resolvent is smooth for smooth coefficient families. If the realizations and the gluing identifications are quaternionic-linear, the inequalities hold in quaternionic dimension.

Proof. These are the two inequalities in the proof of Bär's closed gluing theorem [19, Theorem 2.1, pp. 7–13]. On the ith piece take its scalar cutoff χ_i, zero for $r \leq t_{-1}$, one for $r \geq t_1$, with $0 \leq \chi_i \leq 1$ and $|d\chi_i| \leq 4/t_1$; here r is the Euclidean radius. Extend the cut sections by zero and put

$$Tu = \chi_1 u_1 + \chi_2 u_2, \quad u = (u_1, u_2), \qquad Jv = (\chi_1 v, \chi_2 v).$$

The supports on the connected sum are disjoint. Each transplant leaves the trace on its own original boundary unchanged. Thus T and J map the respective operator domains into one another, also for nonlocal APS conditions.

Take ρ_0 at most the minimum of $R, 2^{-4}, (100\Lambda^2)^{-1}, (2\Lambda)^{-1}, (2(k+1))^{-1}$ and $2^{-17}\epsilon^2/(k+1)^2$. Bär's annular and neck estimates [19, Propositions 3.2 (Corollaries 1–2) and 3.3; (2.2)–(2.7)] apply on each signed summand with spectral parameter μ or $-\mu$ for an eigenvalue $\mu \in (-\Lambda, \Lambda)$. The annuli are Euclidean, the spherical Dirac eigenvalues have absolute value at least $(n-1)/2 \geq 1$, and the warping function on $|t| \leq \rho$ is at most ρ with derivative of absolute value at most one. No boundary trace enters these estimates. Suppress p in the operator notation. For $u \in E_{\{\lambda\}}(D^0)$ they give

$$\|Tu\|^2 \geq \tfrac{1}{2}\|u\|^2, \qquad \|(D_\rho - \lambda)Tu\|^2 \leq 2^{11}\rho\|u\|^2. \tag{22.15}$$

For v in the span of at most $k+1$ orthogonal eigenspinors of D_ρ in $[\lambda - \epsilon, \lambda + \epsilon]$, they give

$$\|Jv\|^2 \geq \tfrac{1}{2}\|v\|^2, \qquad \|(D^0 - \lambda)Jv\| \leq \tfrac{5}{4}\epsilon\|v\|. \tag{22.16}$$

Indeed, Proposition 3.3 of [19] gives $\|v\|_{\{|t|<t_1\}} \leq \rho(k+1)\|v\| \leq \|v\|/2$; outside this region $\chi_1^2 + \chi_2^2 = 1$. Moreover J is a contraction and each of the two cutoff commutators has norm on v at most $2^{11/2}\rho^{1/2}(k+1)\|v\| \leq \epsilon\|v\|/8$.

By (22.15) the spectral projection of Tu onto the interval of radius ϵ about λ is injective as a map of u: the squared residual quotient is at most $2^{12}\rho < \epsilon^2$. This proves (22.13). Put $l = \dim_{\mathbb{C}} E_{[\lambda-2\epsilon,\lambda+2\epsilon]}(D^0)$; then $l \le k$. If the glued spectral space in (22.13) contained $l+1$ orthogonal eigenspinors, the projection of their cutoffs by J onto that l-dimensional space would be injective on their span by (22.16), since $25\epsilon^2/8 < (2\epsilon)^2$. This proves (22.14).

Locally about each positive radius, radial diffeomorphisms identify the necks smoothly and are the identity outside the gluing balls. After the corresponding bundle and density identifications the H^1 trace domain is fixed. The coefficients on the neck depend smoothly on ρ; elsewhere the family has the prescribed dependence on p. Inversion of $D_{\rho,p} - i : \operatorname{dom} D_{\rho,p} \to L^2$ gives the asserted resolvent regularity. In the quaternionic case each spectral space has twice its quaternionic dimension over $\mathbb{C}$. □

Let X have a fixed boundary collar, an interior Euclidean ball with zero potential, and an injective low APS operator of quaternionic index $-r$. Let $(A_p)_{p\in P}$ be a continuous family of closed chiral operators on a spin manifold M, of the same chirality as on X, with quaternionic index $-a$, zero kernel and a common Euclidean ball with zero potential. In the skew case take an invertible closed family over P with the same local gluing data and put $a = 0$. In each case glue the underlying spin operators and take the double (22.12) on the resulting source and target bundles.

Corollary 22.10. *For all sufficiently small common radii $\rho > 0$ the low APS operator on $X \#_\rho M$ is injective of quaternionic index $-(r + a)$. The nonzero spectrum of its double is bounded away from zero uniformly in p and ρ.*

Proof. The kernel of the disjoint double lies in the target summand and has quaternionic dimension $r + a$. Its rank is constant; by norm-resolvent continuity, the local spectral projections about zero and compactness of P give $\delta > 0$ such that $E_{[-4\delta,4\delta]}(D_p^0) = \ker D_p^0$ for every p. The chiral double has the spin Dirac model on the gluing balls. For a skew operator $A = iD$ there, the double is $\left(\begin{smallmatrix} 0 & -i \\ i & 0 \end{smallmatrix}\right) \otimes D$, unitarily equivalent over $\mathbb{C}$ to $D \oplus (-D)$. Thus Proposition 22.9 applies with $\lambda = 0$, $\epsilon = \delta$, $\Lambda = 4\delta$ and $k = 2(r + a)$.

Let $\Pi_\rho = \mathbf{1}_{[-\delta,\delta]}(D_{\rho,p})$. By the lower inequality and its proof, $\Pi_\rho T : \ker D_p^0 \to \operatorname{ran} \Pi_\rho$ is injective; by the upper inequality it is onto. The transplant T preserves the source and target summands. The double anticommutes with its grading, so Π_ρ commutes with it. Hence $\operatorname{ran} \Pi_\rho$ lies in the target summand. The double preserves this spectral space and sends the target to the source; it therefore vanishes on the spectral space. Thus the glued source kernel is zero, the adjoint kernel has quaternionic dimension $r + a$, and there is no other spectrum in $[-\delta, \delta]$. □

22.8. **Low APS coordinates and their uniform bounds.** Let $A : H^1(W; E) \to L^2(W; F)$ and B be as in Section 22.5. Put $\Sigma = \partial W$ and $X = H^{1/2}(\Sigma; E|_\Sigma)$, with norm $\|x\|_X = \|(I + B^2)^{1/4} x\|_{L^2}$. Denote the negative, zero and positive spectral projections of B by $Q_{\text{allow}}, P, Q_{\text{forb}}$, respectively, and their ranges in X by $X_{\text{allow}}, K, X_{\text{forb}}$. Let γ denote the trace. Suppose that A_{low} is injective. Its range is $C^\perp$, where $C = \ker A_{\text{low}}^* \subset L^2(W; F)$. Let P_C be orthogonal projection onto C and put $Gf = A_{\text{low}}^{-1}(f - P_C f)$, where $A_{\text{low}}^{-1} : C^\perp \to \boldsymbol{D}_{\text{low}}$. The domain $\boldsymbol{D}_{\text{low}}$ carries its H^1 norm.

Lemma 22.11. *There are bounded quaternionic-linear maps*

$$w : K \oplus X_{\text{forb}} \to H^1(W; E), \qquad \beta : K \to C, \qquad \eta : X_{\text{forb}} \to C,$$
$$H : K \oplus X_{\text{forb}} \to X_{\text{allow}}, \qquad J : L^2(W; F) \to X_{\text{allow}}$$

with the following properties. For $f \in L^2(W; F)$, $k \in K$ and $b \in X_{\text{forb}}$, the equations $Au = f$, $P\gamma u = k$ and $Q_{\text{forb}}\gamma u = b$ are equivalent to

$$u = Gf + w(k, b), \qquad \beta k + \eta b = P_C f. \tag{22.17}$$

In this case $Q_{\text{allow}}\gamma u = Jf + H(k, b)$. The map

$$H^1(W; E) \longrightarrow C^\perp \oplus K \oplus X_{\text{forb}}, \qquad u \longmapsto \big((I - P_C)Au, P\gamma u, Q_{\text{forb}}\gamma u\big) \tag{22.18}$$

is a bounded isomorphism with inverse $(h,k,b)\mapsto Gh+w(k,b)$. *In these coordinates* $A(h,k,b)=(h,\beta k+\eta b)$ *in* $C^{\perp}\oplus C$.

On fixed manifolds and bundles with fixed collar and domain, these maps depend continuously on A *in the* $H^1\to L^2$ *operator norm, and smoothly for smooth families of coefficients. Over fillings with the same collar data, the norms of* β,η,H,J *are uniformly bounded if the nonzero spectrum of the low double* (22.12) *stays outside* $(-\delta,\delta)$ *for some* $\delta>0$.

Proof. Take a bounded quaternionic-linear right inverse of the H^1 trace on the collar, multiply by a fixed collar cutoff equal to one near Σ, and restrict to $K\oplus X_{\mathrm{forb}}$. Denote the resulting lift by $\mathcal{L}$, so that $\gamma\mathcal{L}(k,b)=k+b$. The elliptic estimate for the low domain [20, Theorem 4.9 and Example 4.21] and the bounded inverse theorem give $G:L^2(W;F)\to\mathcal{D}_{\mathrm{low}}$ bounded. Define

$$w=(I-GA)\mathcal{L},\quad \beta k+\eta b=P_C A\mathcal{L}(k,b),$$
$$H=Q_{\mathrm{allow}}\gamma w,\qquad\qquad J=Q_{\mathrm{allow}}\gamma G.$$

For $k=P\gamma u$, $b=Q_{\mathrm{forb}}\gamma u$ one has $u-\mathcal{L}(k,b)\in\mathcal{D}_{\mathrm{low}}$. Thus $u=GAu+w(k,b)$. Conversely $Gh+w(k,b)$ has the prescribed zero and positive traces and its image under A is $h+\beta k+\eta b$ for $h\in C^{\perp}$. This proves both formulas.

For fixed domains regard $S=A_{\mathrm{low}}$ as a bounded operator from $\mathcal{D}_{\mathrm{low}}$ with a fixed H^1 inner product to $L^2(W;F)$, and denote its bounded adjoint by $S^{\#}$. Since S is injective with closed range, $G=(S^{\#}S)^{-1}S^{\#}$ and $P_C=I-SG$. Bounded inversion gives the stated parameter dependence, with C and $C^{\perp}$ the ranges of the corresponding projections.

For the last assertion the spectral theorem for the low double gives $\|Gf\|_{L^2}\le\delta^{-1}\|f\|_{L^2}$. Fix a cutoff ζ on the common collar $\mathcal{C}$, equal to one near Σ and zero near its inner edge. For $v\in\mathcal{D}_{\mathrm{low}}$, extend ζv by zero to one fixed filling of this collar. Applying its elliptic estimate to $A(\zeta v)=\zeta Av+[A,\zeta]v$ and taking traces gives $\|\gamma v\|_X\le c(\|Av\|_{L^2(\mathcal{C})}+\|v\|_{L^2(\mathcal{C})})$, with c independent of the filling on which v was given. For the adjoint domain the same estimate holds with trace in $H^{1/2}(\Sigma;F|_\Sigma)$ and the Green-annihilator boundary condition. In particular,

$$\|\gamma Gf\|_X\le c(1+\delta^{-1})\|f\|_{L^2(W;F)}. \tag{22.19}$$

Since $A\mathcal{L}$ is fixed on $\mathcal{C}$ and vanishes elsewhere, the formulas for β,η,H,J give the asserted bounds. □

The full domain is $b=0$ in (22.18). Its kernel and cokernel therefore fit into the exact sequence

$$0\longrightarrow\ker A_{\mathrm{full}}\xrightarrow{P\gamma}K\xrightarrow{\beta}C\longrightarrow\operatorname{coker}A_{\mathrm{full}}\longrightarrow 0, \tag{22.20}$$

where the last map sends c to its class modulo the range. Suppose A_{full} is surjective. Put $E_0=\ker\beta\subset K$, and let $e=-(\beta|_{E_0^{\perp}})^{-1}\eta:X_{\mathrm{forb}}\to E_0^{\perp}$. For $x=x_{\mathrm{allow}}+k+b$ in $X_{\mathrm{allow}}\oplus K\oplus X_{\mathrm{forb}}$, there is a solution of $Au=0$ with trace x if and only if

$$k-eb\in E_0,\qquad x_{\mathrm{allow}}=H(k,b). \tag{22.21}$$

Indeed (22.17) then gives $u=w(k,b)$. In the skew case (22.7), Green's formula on two elements of $\ker A_{\mathrm{full}}$ reduces to their σ-pairing on K, since σ exchanges the positive and negative spectral subspaces. Thus $E_0=P\gamma\ker A_{\mathrm{full}}$ is null. By (22.8) and (22.20) it has quaternionic dimension N, so it is maximal null.

Lemma 22.12. *In Corollary 22.10 denote the original cap by* Y, *the glued manifold by* $Y_{p,\rho}$, *and their low adjoint kernels by* C_Y *and* $C_{p,\rho}$. *For all sufficiently small* $\rho>0$ *there are quaternionic isometries* $j_{p,\rho}:C_Y\hookrightarrow C_{p,\rho}$, *continuous in* (p,ρ), *such that*

$$j_{p,\rho}^{*}\beta_{p,\rho}\longrightarrow\beta_Y,\qquad j_{p,\rho}^{*}\eta_{p,\rho}\longrightarrow\eta_Y \tag{22.22}$$

in operator norm, uniformly in p *as* $\rho\downarrow 0$. *On the common boundary,* $\gamma j_{p,\rho}\to\gamma|_{C_Y}$ *in operator norm from* C_Y *to* $H^{1/2}(\Sigma;F|_\Sigma)$.

Proof. Use the cutoff on Y in Proposition 22.9, obtained by scaling one fixed radial cutoff by $t_1 = \rho^4/2$. Write $T_\rho\phi$ for the extension by zero of $\chi_\rho\phi$ to $Y_{p,\rho}$. It has the original adjoint boundary condition. Let $S = A_{Y_{p,\rho},\mathrm{low}}$ and $P_{p,\rho}$ be the L^2 projection onto $\ker S^*$. In the target summand of the double, Corollary 22.10 gives $\|(I - P_{p,\rho})v\| \le \delta^{-1}\|S^*v\|$ on $\operatorname{dom} S^*$, with $\delta > 0$ independent of p, ρ. Here the norms are L^2. On the cap modes $\phi \in C_Y$, smoothness and finite dimension give a common pointwise bound $c\|\phi\|$ on the gluing ball. The cutoff derivative is $O(t_1^{-1})$ on an annulus of volume $O(t_1^n)$, and $A_Y^\dagger\phi = 0$. Hence

$$\|S^*T_\rho\phi\| \le ct_1^{(n-2)/2}\|\phi\|, \qquad 0 \le \|\phi\|^2 - \|T_\rho\phi\|^2 \le ct_1^n\|\phi\|^2.$$

Put $J_0 = P_{p,\rho}T_\rho$. Orthogonality gives $\|I - J_0^*J_0\| \le c(t_1^n + \delta^{-2}t_1^{n-2}) = o(1)$. Thus $j_{p,\rho} = J_0(J_0^*J_0)^{-1/2}$ is an isometry. The cutoff and the spectral projection are continuous for positive small radii by Proposition 22.9 and the common gap; hence so is $j_{p,\rho}$.

On C, $T_\rho\phi = \phi$ and $S^*(T_\rho\phi - J_0\phi) = 0$. The adjoint collar estimate in the proof of (22.19) therefore bounds the $H^{1/2}$ trace of this difference by its L^2 norm times a fixed constant. This norm is $o(1)\|\phi\|$, uniformly in p. Since $(J_0^*J_0)^{-1/2} = I + o(1)$ on C_Y, the asserted trace convergence follows, as does $\|j_{p,\rho}\phi - \phi\|_{L^2(C)} = o(1)\|\phi\|$. For $d = (k, b) \in K \oplus X_{\mathrm{forb}}$ the difference of the pairings of $\beta_{p,\rho}k + \eta_{p,\rho}b$ with $j_{p,\rho}\phi$ and of $\beta_Y k + \eta_Y b$ with ϕ is bounded by

$$\|j_{p,\rho}\phi - \phi\|_{L^2(C)}\,\|A\mathcal{L}d\|_{L^2(C)} = o(1)\|\phi\|\,\|d\|.$$

This proves (22.22). □

22.9. **Boundary evolution operators.** Let Σ be closed and connected, $S \to \Sigma$ a quaternionic Hermitian bundle and B a self-adjoint Dirac-type operator on S, linear over $\mathbb{H}$. Fix a nonempty open set $O \subset \Sigma$ and put $\mathcal{V} = C_c^\infty(O; \operatorname{End}_{\mathbb{H}} S)$. For a quaternionic subspace D of a finite sum of eigenspaces of B, let P_D be L^2 orthogonal projection onto D, and put $\pi_D(V) = P_D V|_D$. Denote evaluation by $e_x : D \to S_x$. For a set of matrices $\mathcal{A}$, let $\mathrm{Lie}_{\mathbb{R}}(\mathcal{A})$ denote the real Lie algebra it generates.

Lemma 22.13. (i) $\mathrm{Lie}_{\mathbb{R}}\big(\pi_D(\mathcal{V})\big) = \operatorname{End}_{\mathbb{H}} D$.

(ii) *Let σ on D and σ_x on S_x be quaternionic self-adjoint involutions, the latter smooth in x, with $e_x\sigma = \sigma_x e_x$. Put $\mathcal{V}_\sigma = \{V \in \mathcal{V} : V(x)^*\sigma_x + \sigma_x V(x) = 0\}$. Then*

$$\mathrm{Lie}_{\mathbb{R}}\big(\pi_D(\mathcal{V}_\sigma)\big) = \mathfrak{g}_\sigma := \{Z \in \operatorname{End}_{\mathbb{H}} D : Z^*\sigma + \sigma Z = 0\}.$$

In particular, for $D = \ker B$ and signature (N, N), this is $\mathfrak{sp}(N, N)$ in an orthonormal eigenbasis of σ.

Proof. The assertions are immediate for $D = 0$. Every nonzero element of D has dense nonzero locus in O: the containing spectral sum is B-invariant with bounded spectrum, so Bär's unique continuation theorem for semibounded spectral subspaces applies.[1] Consequently the union $\mathcal{R} = \bigcup_{x\in O} \operatorname{im} e_x^*$ spans D over $\mathbb{H}$: a section orthogonal to it vanishes on O.

We first prove (ii). The intertwining identity gives $\pi_D(V)^*\sigma + \sigma\pi_D(V) = 0$ for $V \in \mathcal{V}_\sigma$. Write $\mathfrak{l} = \mathrm{Lie}_{\mathbb{R}}(\pi_D(\mathcal{V}_\sigma))$. For each $v = e_x^*z \in \mathcal{R}$ and $p \in \operatorname{Im}\mathbb{H}$,

$$X_v(p) := vpv^*\sigma \in \mathfrak{l}.$$

Indeed extend $z \in S_x$ to a smooth local section s and multiply $sps^*\sigma_{(\cdot)}$ by smooth nonnegative bumps of integral one supported in O and shrinking to x. These multipliers are in $\mathcal{V}_\sigma$. Their compressions converge to $e_x^*(zpz^*\sigma_x)e_x = X_v(p)$, and $\mathfrak{l}$ is closed in the finite-dimensional matrix space.

For $v, w \in \mathcal{R}$ put $a = v^*\sigma w$. If p, q are imaginary quaternions, then

$$[X_v(p), X_w(q)] = (vcw^* - w\bar{c}v^*)\sigma, \qquad c = paq.$$

For $a \neq 0$ these c span $\mathbb{H}$ over $\mathbb{R}$: conjugation by a preserves $\operatorname{Im}\mathbb{H}$, and products of imaginary quaternions span $\mathbb{H}$. The same matrices therefore belong to $\mathfrak{l}$ for all $c \in \mathbb{H}$ even if $a = 0$. In fact we may suppose $v \neq 0$ and write $w = e_y^*z$ with $y \in O$. Since σv is a nonzero element of D, take $y_j \to y$

[1] C. Bär, *Localization and semibounded energy—a weak unique continuation theorem*, J. Geom. Phys. **34** (2000), 155–161.

in O with $e_{y_j}\sigma v \neq 0$. In a local frame take $z_j \to z$ with $z_j \notin (e_{y_j}\sigma v)^{\perp}$. Then $w_j = e_{y_j}^{*} z_j \to w$ and $v^{*}\sigma w_j \neq 0$, giving the assertion by passage to the limit. Choose a quaternionic basis from $\mathcal{R}$ and let U be its column matrix in an orthonormal basis. The real span of the displayed matrices contains $UCU^{*}\sigma$ for every $C^{*} = -C$ (take diagonal and off-diagonal entries in turn). Since U is invertible and $\mathfrak{g}_{\sigma} = \{H\sigma \,:\, H^{*} = -H\}$, this proves (ii).

In (i) take $\sigma = I$ and $\sigma_x = I$: the generated algebra contains all skew-adjoint matrices. Localization of ss^{*} as above also gives vv^{*} for every $v \in \mathcal{R}$. Conjugation by quaternionic unitaries preserves the algebra, since the connected compact symplectic group is generated by exponentials of its Lie algebra. Its action is transitive on the unit sphere. Thus the algebra contains every orthogonal projection onto a quaternionic line. The spectral theorem gives all self-adjoint matrices, and (i) follows. □

Write K for the chosen space D and $m = \dim_{\mathbb{H}} K$. In case (i) put $G = \mathrm{GL}_{\mathbb{H}}(K)$ and in case (ii) put $G = \{T \in \mathrm{GL}_{\mathbb{H}}(K) \,:\, T^{*}\sigma T = \sigma\}$. Write $\mathfrak{g}$ for its Lie algebra. For the chiral spin operator, a multiplier V in normal Clifford coordinates corresponds to the off-diagonal block $c(\nu)V$ and its adjoint in the opposite block. For (22.7) the added block is σV, which is skew-adjoint precisely when $V^{*}\sigma + \sigma V = 0$ on the fibres. For a smooth time-dependent multiplier $V(t)$ in the corresponding class let $\mathcal{E}(V) = U(1)$, where $U' = -\pi_K(V(t))U$, $U(0) = I$. This equation preserves G.

Lemma 22.14. *Fix a nonempty open interval $I_0 \Subset (0,1)$. On a neighborhood of the identity in G, the map $\mathcal{E}$ has a continuous right inverse $T \mapsto V_T$ with $V_I = 0$, taking values in smooth multipliers supported in a fixed compact subset of $I_0 \times O$. Continuity is for the C^{∞} topology.*

The neighborhood may be taken uniform for data over a compact Hausdorff parameter space, provided the controlled spaces form a continuous finite-rank bundle with local frames continuous in $C^{\infty}(\Sigma; S)$, and the volume density, fibre metrics and any prescribed involutions vary continuously in C^{∞}. The right inverse then depends continuously on both the parameter and the endpoint in the corresponding group bundle.

Proof. The assertion for $K = 0$ follows with $V_I = 0$. For fixed data choose multipliers $V_1', \dots, V_b'$ such that some iterated brackets Y_j of the $-\pi_K(V_i')$ span $\mathfrak{g}$ over $\mathbb{R}$ (Lemma 22.13). Write ℓ_j for the length of Y_j. The commutator words of Boarotto–Lerario[2] for a system without drift give products $W_j(s)$ of exponentials of the generators with arguments $\pm s$, analytic in s and equal to $I + s^{\ell_j} Y_j + O(s^{\ell_j+1})$ at zero. Define

$$D_j(z) = \begin{cases} W_j(z^{1/\ell_j}), & z \geq 0, \\ W_j((-z)^{1/\ell_j})^{-1}, & z < 0. \end{cases}$$

Analyticity gives $D_j'(z) = Y_j + O(|z|^{1/\ell_j})$ for $z \neq 0$, and $D_j'(0) = Y_j$. On an ordered list of disjoint subintervals of I_0, a fixed smooth bump of integral one times a multiplier of amplitude a has endpoint $\exp(-a\pi_K(V_i'))$. Place the factors in reverse product order. Separate lists for the positive and negative words, with amplitudes of absolute values $(\max\{z,0\})^{1/\ell_j}$ and $(\max\{-z,0\})^{1/\ell_j}$, realize $D_j(z)$ by controls continuous in every C^{∞} seminorm and zero at $z = 0$.

For the family assertion choose such a list locally at each parameter. The same smooth multipliers extend nearby: in the second case apply the fibrewise projection $V \mapsto (V - \sigma_x V^{*} \sigma_x)/2$ with the varying involution. The spanning property is open. A finite cover and parameter cutoffs equal to one on smaller sets covering the base give a single finite list of global bracket words $Y_{p,1}, \dots, Y_{p,d}$. This includes fixed data by taking the base to be a point. The real linear map $L_p : \mathbb{R}^d \to \mathfrak{g}_p$ with these columns is onto; put $R_p = L_p^{*}(L_p L_p^{*})^{-1}$ using continuous fibre inner products. For $z = R_p x$, the map

$$x \longmapsto \Phi_p(x) = \log\bigl(D_{p,d}(z_d) \cdots D_{p,1}(z_1)\bigr)$$

has derivative I at $x = 0$. Its derivative in x is continuous in (p, x) by the word expansions, whose coefficients are continuous in p. On a uniform ball $\|x\| \leq r$ one has $\|d\Phi_p - I\| \leq 1/2$. For $\|y\| < r/2$, the

[2] F. Boarotto and A. Lerario, *Homotopy properties of horizontal path spaces and a theorem of Serre in subriemannian geometry*, Comm. Anal. Geom. **25** (2017), 269–301, proof of Proposition 2, formula (2).

contraction $x \mapsto y + x - \Phi_p(x)$ of this ball has a unique fixed point $x_p(y)$ continuous in (p, y). Taking $y = \log T$ and placing the controls of the words in successive intervals gives V_T. Their supports lie in a fixed finite union of compact sets in $I_0 \times O$. Since $\|x_p(y)\| \le 2\|y\|$, each C^∞ seminorm of V_T is $O(\|T - I\|^{1/\ell})$ for $\ell = \max_j \ell_j$, uniformly in p. □

Lemma 22.15. *Let P be compact Hausdorff, $A \subset P$ closed, and V_p^0 a continuous C^∞ family of permitted multipliers with endpoints H_p. Suppose V_p^0 vanishes on a common final interval of positive length. Let $T_{p,s}$, $0 \le s \le 1$, be a continuous homotopy in G with $T_{p,0} = H_p$ and $T_{p,s} = H_p$ on A. There are continuous C^∞ controls $V_{p,s}$ with $\mathcal{E}(V_{p,s}) = T_{p,s}$, $V_{p,0} = V_p^0$ and $V_{p,s} = V_p^0$ on A.*

Proof. Put $Q_{p,s} = T_{p,s} H_p^{-1}$ and choose a subdivision $0 = s_0 < \cdots < s_d = 1$ so fine that $Q_{p,t} Q_{p,s_{j-1}}^{-1}$ belongs to the neighborhood of Lemma 22.14 for every p and $s_{j-1} \le t \le s_j$. Such a subdivision exists by compactness. Set $\tau_j(s) = \min(s, s_j)$. In successive subintervals of the reserved interval add section controls for $Q_{p,\tau_j(s)} Q_{p,\tau_{j-1}(s)}^{-1}$, $j = 1, \dots, d$. Affine rescaling of a pulse multiplies its amplitude by the inverse time scale and preserves its endpoint. The ordered endpoint product telescopes to $Q_{p,s}$, so the new endpoint is $Q_{p,s} H_p$. All added controls vanish at $s = 0$ and on A, since the section at the identity is zero. □

The groups $\mathrm{GL}_m(\mathbb{H})$ and $\mathrm{Sp}(N, N)$ are connected; their polar decompositions retract them onto $\mathrm{Sp}(m)$ and $\mathrm{Sp}(N) \times \mathrm{Sp}(N)$, respectively. Thus every individual endpoint in either group is attained from the identity. Now take $K = \ker B$. For two fillings satisfying the hypotheses of Proposition 22.17, let $\beta_{i,p} : K \to C_{i,p}$ be as in Lemma 22.11. For a compact family of controls with endpoints T_p put $F_p k = (\beta_{0,p} k, \beta_{1,p} T_p k)$. If $\|F_p k\| \ge \delta \|k\|$ for every p, k and some $\delta > 0$, the reduction $M_{L,p}$ of Proposition 22.17 satisfies $\|M_{L,p} - F_p\| < \delta/2$ for all sufficiently large cylinder lengths L, uniformly in p. In particular the glued operators are injective.

To reserve a final interval for a family vanishing near both endpoints, choose a smooth nondecreasing $\phi : [0,1] \to [0,1]$ with $\phi(0) = 0$ and $\phi = 1$ on that interval. Put $\phi_s(t) = (1 - s)t + s\phi(t)$ and replace $V_p(t)$ by $\phi_s'(t) V_p(\phi_s(t))$. Its fundamental matrix at t is $U_p(\phi_s(t))$, where U_p is that of V_p. This gives a compact continuous homotopy of smooth controls with the same endpoints. At $s = 1$ the final interval is free. Under the preceding lower bound a single sufficiently large L gives injective operators throughout this homotopy. Its final control may thus serve as the initial family V_p^0 in Lemma 22.15.

22.10. **The inhomogeneous problem on a long cylinder.** Let B be the fixed self-adjoint quaternionic tangential Dirac-type operator on the closed hypersurface Σ. Write $K = \ker B$, $P = 1_{\{0\}}(B)$, $Q = I - P$, $Q_+ = 1_{(0,\infty)}(B)$ and $Q_- = 1_{(-\infty,0)}(B)$. Give $X = H^{1/2}(\Sigma)$ the norm $\|x\|_X = \|(I + B^2)^{1/4} x\|_{L^2}$ and put $X_\pm = Q_\pm X$. Fix $\gamma_0 > 0$ with $|B| \ge \gamma_0$ on $K^\perp$. Let $z \mapsto V_z$ be a continuous map from a compact space Z to the smooth quaternionic multipliers on $[0,1] \times \Sigma$, with the C^∞ topology. On K let U_z be the solution of

$$U_z' + P V_z P \, U_z = 0, \qquad U_z(0) = I, \qquad T_z = U_z(1).$$

In the skew case let σ be the normal involution of (22.7) and assume $V_z^* \sigma + \sigma V_z = 0$. For smooth gluing take V_z zero near both ends. The mixed problem itself requires no endpoint vanishing.

In normal Clifford coordinates the physical operator is $\partial_s + B + L^{-1} V_z(s/L, x)$ on $[0, L] \times \Sigma$. Set $t = s/L$ and $\epsilon = L^{-1}$. A physical right-hand side $g(s)$ in these coordinates becomes $f(t) = L g(Lt)$; hence $\|f\|_{L^2(dt)} = \sqrt{L}\, \|g\|_{L^2(ds)}$, with the same spatial L^2 norm. All time norms below are taken with respect to dt. Control derivatives are taken at fixed L in smooth finite-dimensional parameters, with uniform coefficient bounds through the prescribed order.

Lemma 22.16. *For all sufficiently small $\epsilon > 0$ the map*

$$\begin{aligned} \mathcal{C}_{\epsilon,z} : H^1([0,1] \times \Sigma) &\longrightarrow L^2([0,1] \times \Sigma) \oplus K \oplus X_+ \oplus X_-, \\ w &\longmapsto ((\partial_t + \epsilon^{-1} B + V_z) w, P w(0), Q_+ w(0), Q_- w(1)) \end{aligned} \tag{22.23}$$

is a bounded isomorphism. For data (f, k, a, d) its remaining traces $k' = Pw(1)$, $b = Q_-w(0)$, $c = Q_+w(1)$ are

$$k' = T_z k + E_{\epsilon,z}(k, a, d) + F_{\epsilon,z} f, \tag{22.24}$$

$$(b, c) = B_{\epsilon,z}(k, a, d) + J_{\epsilon,z} f, \tag{22.25}$$

where the low outputs lie in K and the ordered high outputs in $X_- \oplus X_+$. The operator norms satisfy

$$\|E_{\epsilon,z}\| + \|B_{\epsilon,z}\| \le C\epsilon, \qquad \|F_{\epsilon,z}\| \le C, \qquad \|J_{\epsilon,z}\| \le C\sqrt{\epsilon}.$$

For $f = 0$, write $u = Pw$, $v = Qw$. Then

$$\|u - U_z k\|_{C_t^0 K} \le C\epsilon(\|k\| + \|a\|_X + \|d\|_X), \tag{22.26}$$

$$\|v\|_{C_t^0 X} \le C(\|a\|_X + \|d\|_X + \epsilon\|k\|), \qquad \|v\|_{L_t^1 X} \le C\epsilon(\|k\| + \|a\|_X + \|d\|_X). \tag{22.27}$$

These bounds are uniform over the family and hold for the indicated control derivatives. The norm of the inverse into H^1 may depend on ϵ. The same estimates hold on each fixed subinterval of positive length, with its own propagator.

Proof. Write $B_\pm$ for the restrictions of B to the positive and negative spectral spaces. Extend the high source by zero beyond $t = 1$. With tangential operator B/ϵ, the half-cylinder formula of Atiyah, Patodi and Singer [13, Proposition 2.5, (2.6)–(2.11)] restricts to $[0, 1]$. The free incoming solution and the zero-incoming Green operator are, respectively,

$$v_{\mathrm{in}}(t) = e^{-tB_+/\epsilon} a + e^{-(1-t)|B_-|/\epsilon} d,$$

$$(G_\epsilon h)(t) = \int_0^t e^{-(t-s)B_+/\epsilon} Q_+ h(s)\, ds - \int_t^1 e^{-(s-t)|B_-|/\epsilon} Q_- h(s)\, ds.$$

Put $A = \|a\|_X + \|d\|_X$. The gap bounds $\|v_{\mathrm{in}}\|_{C_t^0 X}$ by A, $\|v_{\mathrm{in}}\|_{L_t^1 X}$ by $\epsilon A/\gamma_0$, and its two outgoing traces by $e^{-\gamma_0/\epsilon} A$ in sum. Convolution with the exponential kernels gives

$$\|G_\epsilon\|_{C_t^0 X \to C_t^0 X} + \|G_\epsilon\|_{L_t^1 X \to L_t^1 X} \le C\epsilon, \qquad \|G_\epsilon\|_{L_t^2 L^2 \to L_t^2 L^2} \le \epsilon/\gamma_0, \tag{22.28}$$

using orthogonality of the two signs in L^2.

For $\lambda \neq 0$ let h_λ be the component of h in the λ-eigenspace and $y_\lambda = (G_\epsilon h)_\lambda$. The one-sided exponential has L^1 norm $\epsilon/|\lambda|$ and squared L^2 norm $\epsilon/(2|\lambda|)$ on the half-line. Thus $\|(\lambda/\epsilon) y_\lambda\|_2 \le \|h_\lambda\|_2$ and $|\lambda| |y_\lambda(t)|^2 \le (\epsilon/2)\|h_\lambda\|_2^2$. Since $y_\lambda' = h_\lambda - (\lambda/\epsilon) y_\lambda$ and $(1 + \lambda^2)^{1/2} \le C|\lambda|$, summation gives

$$\|\partial_t G_\epsilon h\|_{L^2} + \epsilon^{-1}\|B G_\epsilon h\|_{L^2} \le C\|h\|_{L^2}, \qquad \|G_\epsilon h\|_{C_t^0 X} \le C\sqrt{\epsilon}\, \|h\|_{L^2}. \tag{22.29}$$

The same bound on spectral tails gives convergence in $C_t^0 X$. The graph norm of B is equivalent to the spatial H^1 norm. For an incoming mode of absolute eigenvalue μ, its squared L_t^2 norm is its squared amplitude times $\epsilon(1 - e^{-2\mu/\epsilon})/(2\mu)$; the squared norms after applying ∂_t or B have the additional factors μ^2/ϵ^2 or μ^2, respectively. Thus $v_{\mathrm{in}} \in H^1$ exactly for incoming data in $X_+ \oplus X_-$ at fixed ϵ. Already a single nonzero incoming mode can have H^1 norm of order $\epsilon^{-1/2}$.

Let $R_z g(t) = U_z(t) \int_0^t U_z(s)^{-1} g(s)\, ds$. The uniform bounds on U_z, U_z^{-1} give $R_z : L_t^1 K \to C_t^0 K$ bounded uniformly. The low and high equations are

$$u = U_z k + R_z(Pf - PV_z v), \qquad v = v_{\mathrm{in}} + G_\epsilon Q\{f - V_z(u + v)\}.$$

Substitution gives

$$(I + \mathcal{K}_{\epsilon,z}) v = v_{\mathrm{in}} + G_\epsilon Q\{f - V_z(U_z k + R_z P f)\},$$

$$\mathcal{K}_{\epsilon,z} = G_\epsilon Q V_z (I - R_z P V_z).$$

Multiplication by V_z is uniformly bounded on X and on L^2. Consequently (22.28) gives $\|\mathcal{K}_{\epsilon,z}\| \le C\epsilon$ on each of $C_t^0 X, L_t^1 X, L_t^2 L^2$. For small ϵ the inverses of $I + \mathcal{K}_{\epsilon,z}$ on these spaces are given by the same Neumann series.

When $f = 0$, the $L^1_t X$ inverse and the incoming estimate give $\|v\|_{L^1_t X} \le C\epsilon(A + \|k\|)$; the low equation then gives (22.26). The $C^0_t X$ inverse gives the remaining bound in (22.27). The high integrals at the outgoing ends are bounded by $C\epsilon\|u + v\|_{C^0_t X}$, in addition to the exponentially small free traces. This proves the estimate for $B_{\epsilon,z}$; evaluating $-R_z P V_z v$ at 1 proves the one for $E_{\epsilon,z}$.

For data $(f, 0, 0, 0)$ the L^2 inverse gives $\|v\|_{L^2} \le C\epsilon\|f\|_{L^2}$, and the low equation gives $\|u\|_{C^0_t K} \le C\|f\|_{L^2}$. Apply (22.29) to $Q\{f - V_z(u + v)\}$. It gives $v \in H^1$ and the $C\sqrt{\epsilon}$ estimate for $J_{\epsilon,z}$; the low equation gives $u \in H^1$ and the estimate for $F_{\epsilon,z}$. For homogeneous data the high solution is the sum of $v_{\text{in}} \in H^1$ and a Green term with L^2 source, and the low equation again gives $u \in H^1$. Superposition constructs a bounded inverse for fixed ϵ; the L^2 Neumann equation gives uniqueness. The forward map is bounded by the H^1 trace theorem.

Derivatives of U_z, R_z are uniformly bounded to the given order. Every derivative of $\boldsymbol{\mathcal{K}}_{\epsilon,z}$ retains the left factor G_ϵ. Differentiating the inverse by $\partial S^{-1} = -S^{-1}(\partial S)S^{-1}$ and using the same three norms proves the differentiated bounds, including the $C^0_t X$ bound for the differentiated source Green terms by (22.29). On $[t_0, t_0 + h]$, $h > 0$ fixed, affine rescaling replaces ϵ, V, f by ϵ/h, $hV(t_0 + h\,\cdot)$ and $hf(t_0 + h\,\cdot)$; the low transfer is $U_z(t_0 + h)U_z(t_0)^{-1}$. □

Join compact fillings W_0, W_1 to the ends of the cylinder. In the common boundary identification the low APS domains allow X_+ on W_0 and X_- on W_1. Assume the two low realizations are injective. For a parameter p let $A_{i,p,\text{low}}$ denote these realizations, $i = 0, 1$. Write $C_{i,p} = \ker A^*_{i,p,\text{low}}$, and denote by $\beta_{i,p}, \eta_{i,p}, H_{i,p}, J_{i,p}$ the boundary maps of Lemma 22.11. Assume the exterior coefficients vary continuously in fixed collar and bundle charts, and these maps have uniformly bounded norms.

For the matching of low modes without a prescribed transfer on the cylinder, see Cappell–Lee–Miller, *Self-adjoint elliptic operators and manifold decompositions I*, Theorem A, and Nicolaescu, *On the Cappell–Lee–Miller gluing theorem*, §2 and Corollary 5.3.

Proposition 22.17. *For all sufficiently large finite L, bounded quaternionic source and target isomorphisms put the glued operator $H^1 \to L^2$ in the direct sum of an identity and the finite block*

$$M_{L,p,z} : K \longrightarrow C_{0,p} \oplus C_{1,p}, \qquad M_{L,p,z} = (\beta_{0,p}, \beta_{1,p} T_z) + O(L^{-1}). \tag{22.30}$$

The error and its control derivatives to the specified order are uniform, including for the thin families of Corollary 22.10. The index is $\dim_{\mathbb{H}} K - \dim_{\mathbb{H}} C_{0,p} - \dim_{\mathbb{H}} C_{1,p}$.

The reduction preserves the Fredholm pair with its injective locus and identifies the actual cokernel bundles there. On a compact subset where the limiting map is injective, the straight homotopy to it remains injective for large L. In the skew case these are statements about the underlying quaternionic Fredholm operators; Proposition 22.23 computes the skew class.

Proof. Write A_i for the exterior operators, $C_i = C_{i,p}$, and P_{C_i} for orthogonal projection onto C_i. On the cylinder take $f = (\partial_t + \epsilon^{-1}B + V_z)w$; in the skew case the normal identification first multiplies the physical expression by σ. The closed H^1 domain is the space of triples (u_0, w, u_1) with matching traces at both seams: equal traces cancel the jump terms in the distributional first derivatives. The L^2 target is the direct sum on the three pieces. Use the coordinates

$$e = (h_0, f, h_1) \in \boldsymbol{\mathcal{H}}_p := C_0^\perp \oplus L^2([0,1] \times \Sigma) \oplus C_1^\perp, \qquad h_i = (I - P_{C_i})A_i u_i, \qquad k = Pw(0).$$

The map $(u_0, w, u_1) \mapsto (e, k)$ is a bounded isomorphism, as follows.

Write $h = (a, d) \in X_+ \oplus X_-$ for the incoming high traces. Combine (22.24)–(22.25) into

$$(k', b, c) = (T_z k, 0, 0) + R_K k + R_H h + Sf \quad \text{in } K \oplus X_- \oplus X_+.$$

Here (R_K, R_H) are the columns of $(E_{\epsilon,z}, B_{\epsilon,z})$ under the splitting into k and h, and $S = (F_{\epsilon,z}, J_{\epsilon,z})$ with its high output split into two components. In particular $\|R_K\| + \|R_H\| \le C\epsilon$. The allowed exterior traces are $a = J_0 h_0 + H_0(k, b)$ and $d = J_1 h_1 + H_1(k', c)$, by (22.18). Define

$$D(\ell, b, c) = (H_0(0, b), H_1(\ell, c)), \qquad H^0 k = (H_0(k, 0), H_1(T_z k, 0)).$$

Then both matching conditions reduce to the single equation

$$(I - DR_H)h = (H^0 + DR_K)k + (J_0h_0, J_1h_1) + DSf.$$

The boundary bounds make D, H^0 uniformly bounded. Thus $I - DR_H$ has a uniformly bounded inverse for small ϵ. For arbitrary (e, k) this determines h; the mixed cylinder inverse gives w and (k', b, c). The exterior inverses at (h_0, k, b) and (h_1, k', c) then give u_0, u_1 with matching traces. These bounded operations give its inverse for each finite L.

Split the target into $\mathcal{H}_p$ and $C_0 \oplus C_1$. In the new coordinates the glued operator takes the form

$$(e, k) \longmapsto (e, Ne + Mk),$$

where the last component is $(\beta_0 k + \eta_0 b, \beta_1 k' + \eta_1 c)$ and N, M are its two columns. The target shear $(e, y) \mapsto (e, y - Ne)$ gives $I_{\mathcal{H}_p} \oplus M$. When $e = 0$ the matching equation gives $h = (I - DR_H)^{-1}(H^0 + DR_K)k$, so $\|h\| \le C\|k\|$ and $\|(k' - T_z k, b, c)\| \le C\epsilon\|k\|$. This gives (22.30). Differentiation of this equation and the bounded inverse formula preserves the $O(\epsilon)$ error at fixed L. Exterior derivatives have the same property if the corresponding exterior maps and their derivatives are uniformly bounded in local bundle charts. For shrinking positive radii only uniform boundary bounds, supplied by Lemma 22.11 and Corollary 22.10, enter the estimate.

For varying $L \ge 1$, take a fixed cylinder coordinate $r \in [0, 1]$ and write $s = \psi_L(r) = r + (L-1)\chi(r)$, with χ smooth nondecreasing, zero near 0 and one near 1. These charts are translations on fixed end collars. After normal identification the pulled-back expression is $(\psi_L')^{-1}\partial_r + B + L^{-1}V_z(\psi_L(r)/L)$. As a map $H^1 \to L^2$ on fixed spaces it is norm-continuous at each finite L; the mixed map in these coordinates has the same endpoint traces and is invertible by the lemma and the pointwise change of variables. Inverting it gives continuous solution and trace maps in the fixed chart. Thus one can use its source coordinate instead of f throughout the matching equation. The homogeneous columns, hence M, are unchanged. The exterior projections and maps are norm-continuous in local bundle charts by Lemma 22.11. This proves continuity of the reduction without choosing global frames of C_0, C_1 or passing to an exterior at radius zero.

The kernel is $\ker M$ in these coordinates; the quotient cokernel is carried isomorphically onto $\operatorname{coker} M$ by $[(e, y)] \mapsto [y - Ne]$, with inverse $[y] \mapsto [(0, y)]$. This proves the index formula and identifies the cokernel bundles on the injective locus. The shear contracts by scaling N; after Hilbert trivializations the other infinite-dimensional coordinate isomorphisms do not change the class of the Fredholm pair by contractibility of the quaternionic Hilbert general linear group, as in Lemma 22.6. Finally, write $F = (\beta_0, \beta_1 T_z)$. On the indicated compact subset let $\|Fk\| \ge \delta\|k\|$, $\delta > 0$. For $\|M - F\| < \delta/2$ every point of the straight homotopy has the same lower bound $\delta/2$. □

For $K \ne 0$, let $k \in K$ be a unit vector and fix $0 \ne \phi \in C_c^\infty(0, 1)$. On the physical cylinder put $w_L(s) = \phi(s/L)k$ and extend by zero to both fillings. For the glued operator A_L, in the physical L^2 norms,

$$\frac{\|A_L w_L\|_2}{\|w_L\|_2} \le \frac{1}{L}\left(\frac{\|\phi'\|_2}{\|\phi\|_2} + \|V_z\|_\infty\right).$$

Thus a lower bound for the finite matching maps need not give a uniform lower bound for A_L.

22.11. **Insertion inside a prescribed ball.** Let U be a nonempty open subset of a closed connected spin n-manifold M and fix a coordinate ball $D \Subset U$. Let $a_p = (g_p, Q_p)$, $p \in P$, be a family of compatible pairs continuous in the smooth coefficient topology, with P compact Hausdorff. In dimension 4 (mod 8) assume the chosen chiral operator is injective of quaternionic index $-a$; in dimension 5 (mod 8) assume the skew operator is invertible.

Take the caps W_0, W_1 of Lemma 22.8, with $\dim_{\mathbb{H}} K = 2r$ in residue four and $2N$ in residue five. Their full APS operators are surjective, so (22.20) gives surjections $\beta_i : K \to C_i$. Choose a control V_*, supported away from both ends of $[0, 1]$, whose endpoint T_* makes

$$F_* = (\beta_0, \beta_1 T_*) : K \longrightarrow C_0 \oplus C_1 \quad \text{an isomorphism.} \tag{22.31}$$

Indeed $E_i = \ker \beta_i$ have half the dimension of K. In the chiral case choose T_* carrying E_0 to a complement of E_1. In the skew case write the maximal null planes as graphs of unitaries $U_i : K^+ \to K^-$, where $K^\pm = \ker(\sigma|_K \mp I)$: projection of a null plane to K^+ is injective and both have dimension N. The isometry $\operatorname{diag}(I, -U_1 U_0^{-1})$ takes E_0 to a complement of E_1. Lemmas 22.14–22.15 and connectedness of the two control groups realize the chosen endpoint, leaving a final interval free.

For connected sums of closed Dirac operators see Bär [19, Theorem 2.1 and proof of Theorem 4.1]; for compact families of metrics see [77, Theorem 4.24]. Local deformations through metrics with invertible Dirac operator are given in [56, Lemma 18 and Corollary 19].

Proposition 22.18. *There are a finite L_0 and a radius $\rho > 0$ such that a_p is homotopic in its injective or invertible locus, by changes supported in D, to the family*

$$W_0 \cup ([0, L_0] \times \Sigma, V_*) \cup (W_1 \#_\rho M_p).$$

Here M_p carries the locally prepared pair, the product collars and B are fixed, and at length L the normal-coordinate multiplier is $L^{-1} V_*(s/L)$. The last piece has injective low APS operator of index $-(r + a)$ in the chiral case and $-N$ in the skew case. With this same ρ, every $L \geq L_0$ stays in the stated locus. This remains true under sufficiently small compatible perturbations of the exterior coefficients and the controls on $[0, 1]$, with the product collars fixed and the controls zero near both ends. The neighborhood is independent of L. If the perturbations agree with the original data off D, the perturbed endpoint can also be reached through pairs agreeing with a_p off D.

Proof. A common Euclidean ball. In oriented coordinates centered at $0 \in D$ write $G_p = g_p(0)$. For a cutoff χ supported in D and equal to one near zero, the flow of $-\frac{1}{2}\chi(x)(\log G_p)x$ equals $G_p^{-t/2}x$ on a smaller common ball for $0 \leq t \leq 1$, by compactness. Pull back the pairs with the spin lift starting at the identity, and keep the notation g_p, Q_p for their endpoint coefficients. The resulting homotopy is continuous in p, preserves kernels, and gives $g_p(0) = I$ at its endpoint. In the fixed comparison (2.6) the operators A_p of the pulled-back pairs satisfy $\|A_p u\|_2 \geq c\|u\|_{H^1}$, $c > 0$, by closed range and operator-norm continuity.

Let B_s denote the coordinate ball of radius s about zero and let g_e be its Euclidean metric. For $B_{2s} \Subset D$ take a rescaled cutoff $0 \leq \chi_s \leq 1$ equal to one on B_s and zero off B_{2s}. In the same spinor comparison interpolate by the pairs $((1 - t\chi_s)g_p + t\chi_s g_e, (1 - t\chi_s)Q_p)$, $0 \leq t \leq 1$. The metric change is $O(s)$ in supremum norm and its first derivatives are $O(1)$, since $|g_p(x) - I| \leq C|x|$ and the cutoff derivative is $O(s^{-1})$. The local Dirac formula [61, Theorem 2.3.6] therefore gives a leading coefficient change of size $O(s)$ and a bounded zeroth-order change supported in B_{2s}, including the potential. Since $n > 2$, Hölder and the Sobolev inequality give $\|1_{B_{2s}} u\|_2 \leq Cs\|u\|_{H^1}$. Thus the whole interpolation changes A_p by $O(s) : H^1 \to L^2$, uniformly in p. Choose s so the change is less than $c/2$. The operators remain injective, hence invertible in the skew case. Write $a'_p = (g'_p, Q'_p)$ for the prepared pairs.

The length of the cylinder. On $W_1 \#_\rho M_p$, Corollary 22.10 gives the asserted low index, injectivity and uniform gap for all sufficiently small positive radii. Lemma 22.11 then bounds the boundary maps uniformly. For the isometries $j_{p,\rho} : C_1 \hookrightarrow C_{1,p,\rho}$ of Lemma 22.12, put $F_{p,\rho} = (\beta_0, \beta_{1,p,\rho} T_*)$. Set $\delta = \|F_*^{-1}\|^{-1} > 0$. Since $(I \oplus j^*_{p,\rho})F_{p,\rho} \to F_*$ uniformly and $I \oplus j^*_{p,\rho}$ is a contraction, there is $\rho_1 > 0$ such that

$$\|F_{p,\rho} k\| \geq \tfrac{1}{2}\delta\|k\|, \qquad 0 < \rho < \rho_1.$$

The chiral target has quaternionic dimension $\dim_{\mathbb{H}} K + a$. Choose L_0 so the uniform reduction error of Proposition 22.17 is less than $\delta/4$ for all $L \geq L_0$, both for these fillings and for the original pair of caps. The first reduction is injective; the second, being square, is invertible. The first is also invertible in the skew case by index zero. The two original caps at L_0 thus give an invertible pair b on the standard spin sphere: lengthening its separating collar does not change the underlying spin manifold or the gluing chart in W_1.

A path on the sphere. On a smaller closed disk identify that chart with an equally oriented stereographic chart by a fixed sphere diffeomorphism [64, Chapter 8, §3, Theorem 3.1]. In stereographic

coordinates multiply the round metric by a smooth positive radial factor that makes it Euclidean on the disk and is one near the opposite pole. The resulting zero-potential pair b_{rd} is invertible by round positive scalar curvature and conformal covariance [65, Proposition 1.3]. Interpolate the pairs b_{rd}, b in metric and compatible potential coordinates relative to the disk, stationary near the time endpoints. The closed case of Lemma 22.7, with perturbations in a ball disjoint from the disk, makes this sphere path b_t invertible relative to those endpoints: its parameter dimension is $1 < 4$ in the index-zero chiral case and $1 < 3$ in the skew case.

Use the standard orthogonal reflection on the angular spheres and a fixed spin lift of the oriented neck identification. The disjoint doubles for (a'_p, b_t) have a common gap off zero by compactness and constant kernel rank. Apply Proposition 22.9 at zero, uniformly over $P \times [0, 1]$. In the chiral case the a old zero modes lie in the target. As in the proof of Corollary 22.10, cutoff followed by projection onto the small symmetric spectral interval injects them into the glued double; the upper bound makes this map onto. The cutoff preserves the grading and the symmetric projection commutes with it, so the whole small spectral space lies in the target. In particular the glued chiral operator is injective. In the skew case the upper bound excludes zero. Fix one $\rho > 0$ below this pathwise threshold and ρ_1, small enough for the common gluing disks. At fixed radius the neck and charts are constant, so the glued coefficients vary continuously in (p, t). The b endpoint has precisely the displayed decomposition; the choice of L_0 still applies for every $L \geq L_0$.

The round attachment. Include a Euclidean annulus from M_p in the replacement disk. For b_{rd} this disk, including the neck, has metric $ds^2 + f(s)^2 g_{S^{n-1}}$, with arclength s measured from its smooth pole, $f > 0$ off the pole, and $f' = 1$ on the outer Euclidean collar. Solve $dr/r = ds/f$, with $r = f$ on that collar. Near zero $f(s) = sF(s^2)$, F smooth with $F(0) = 1$; hence $1/f - 1/s$ is smooth odd and r/s extends to a positive smooth even function. Thus r is a radial diffeomorphism onto the original disk and $u = \log(f/r)$ is smooth, zero on its outer collar. This gives one spin identification with M, equal to the identity off the disk, for the entire glued path. At its round endpoint the new metric is $e^{2u} g'_p$, with u extended by zero. The potential is zero wherever u can be nonzero. Conformal covariance for pairs (Lemma 8.1) therefore joins a'_p to this endpoint through $e^{2hu} g'_p$, $0 \leq h \leq 1$, fixing the exterior. The preparation, this conformal path, and the glued sphere path now concatenate in that order, starting from a_p.

For perturbation stability keep this positive ρ fixed. Low injectivity is open on the fixed exterior domains, their boundary maps vary continuously by Lemma 22.11, and the finite endpoint T varies continuously by its ODE. In the C^∞ coefficient topology a neighborhood of the compact data therefore retains a positive lower bound for the limiting matching maps and common bounds on the boundary maps and control multiplier norms. The estimates proving Proposition 22.17 give one C/L error bound there, hence injectivity for every L beyond a common $L_1 \geq L_0$. For $L_0 \leq L \leq L_1$ ordinary openness on the compact parameter set $P \times [L_0, L_1]$ gives the same conclusion after shrinking the neighborhood. Interpolate to a sufficiently close perturbation at L_0 in compatible coefficient coordinates; this fixes the exterior of D whenever the perturbation does. □

For a filling with a parameter collar, the homotopy gives an annulus with the original family on its outer boundary. If that family has prescribed density μ, complete the cylinder construction first and then normalize the assembled family by Lemma 8.1, including the potential rescaling. This fixes both the normalized outer boundary and its unchanged spatial exterior; the normalized metric need not retain the displayed product cylinder.

22.12. **Rectangular completion and cancellation.** For $m \geq 1$ the standard inclusion induces a map

$$\pi_i \operatorname{Sp}(m) \longrightarrow \pi_i \operatorname{Sp} \quad \text{bijective for } 0 \leq i \leq 4m+1, \qquad \text{surjective for } i = 4m+2 \tag{22.32}$$

by [75, Theorem II.12 and its proof, pp. 108–109]. Here Sp has the direct limit topology. For integers $a \geq 0$ and $r \geq \max(1, a)$ put $V = \operatorname{Sp}(r+a)/\operatorname{Sp}(a)$ and $V_\infty = \operatorname{Sp}/\operatorname{Sp}(a)$. In each stage $\operatorname{Sp}(a)$ acts on the first a coordinates; the last r coordinate vectors form the standard ordered frame in V. The inclusion $V \to V_\infty$ is $(4(r+a)+2)$-connected: compare the homotopy sequences of the finite-stage

bundles with common fibre $\mathrm{Sp}(a)$ [59, Theorem 4.41] and apply (22.32). Passage to the limit is valid since compact subsets of this increasing union of compact manifolds lie in finite stages.

Let $K = \mathbb{H}^{2r}$, let $0 \le k \le 4r+1$, and put $d = k+1$. Suppose that E_0, E_1 are subbundles of $D^d \times K$ of ranks $r, r-a$, respectively. Each subbundle and its orthogonal complement are trivial over the disk. Choose independent orthonormal frames in the source and target so that E_0, E_1 become fixed subspaces of K. These changes of coordinates carry extensions and homotopies relative to $S^k = \partial D^d$ to extensions and homotopies relative to the same boundary. Let q be the orthogonal projection of K onto $E_1^\perp$, identified with the quotient by E_1. Put

$$G = \mathrm{GL}_{2r}(\mathbb{H}), \qquad \mathcal{U}_a = \{T \in G \,:\, qT|_{E_0} \text{ is injective}\}.$$

For an injection $F : E_0 \to E_1^\perp$ write $\nu(F) = F(F^*F)^{-1/2} \in V$, in the chosen frames.

Lemma 22.19. (a) *Suppose $T_0 : S^k \to \mathcal{U}_a$ and $H : D^d \to G$ satisfy $H|_{S^k} = T_0$. If $\nu(qT_0|_{E_0}) : S^k \to V$ extends over the disk after inclusion in V_∞, then T_0 has an extension $T : D^d \to \mathcal{U}_a$ homotopic to H in G relative to S^k.*

(b) *Fix $T_* \in \mathcal{U}_a$ and use $\nu(qT_*|_{E_0})$ as basepoint in V and V_∞. If $1 \le k \le 4r+1$, every class in $\pi_k(V_\infty)$ is represented by $\nu(qT|_{E_0})$ for a map $T : (S^k, *) \to (\mathcal{U}_a, T_*)$ based homotopic in G to the constant T_*.*

Proof. Let $\mathcal{J}$ be the space of quaternionic-linear injections $E_0 \to K$. Polar normalization retracts $\mathcal{J}$ onto the Stiefel manifold of r-frames in K, which is $(4r+2)$-connected [59, Examples 4.53–4.54]. Restriction defines a locally trivial bundle $\pi : G \to \mathcal{J}$, $\pi(T) = T|_{E_0}$. Indeed, fix $L_0 \in \mathcal{J}$ and choose $W : E_0^\perp \to K$ so that $[L_0, W]$ is invertible. For L near L_0 the matrix $[L, W]$ is invertible, and every element of $\pi^{-1}(L)$ has the unique form

$$[L, LC + WQ] = [L, W] \begin{pmatrix} I & C \\ 0 & Q \end{pmatrix}, \qquad C : E_0^\perp \to E_0, \quad Q \in \mathrm{GL}(E_0^\perp). \tag{22.33}$$

The two coordinates are continuous by matrix inversion. We apply the homotopy lifting property for CW pairs [59, Proposition 4.48] to this bundle.

(a) Put $L_0 = T_0|_{E_0}$ on S^k. By the connectivity of $V \to V_\infty$, the frame $\nu(qL_0)$ extends over D^d in V; for $k = 0$ this follows from path connectedness. Write $qL_0 = \nu(qL_0)P_0$ with $P_0 > 0$. In the radial coordinate $s \in [0,1]$ on the disk, extend P_0 by $(1-s)I + sP_0$ and $(I-q)L_0$ by $s(I-q)L_0$. Multiplying the extended frame by the positive factor and adding the E_1 component gives a map $L : D^d \to \mathcal{J}$ with $L|_{S^k} = L_0$ and qL injective. Put πH and L on the two ends of $D^d \times [0,1]$ and L_0 on $S^k \times [0,1]$. This map on the boundary sphere S^d extends to the whole cylinder since $d \le 4r+2$ and $\pi_d \mathcal{J} = 0$. Lift the resulting homotopy from H, prescribing T_0 on $S^k \times [0,1]$. Its endpoint T satisfies $\pi T = L$, hence $T(D^d) \subset \mathcal{U}_a$.

(b) Put $L_* = T_*|_{E_0}$ and $P_* = ((qL_*)^*(qL_*))^{1/2}$. Choose a based representative $F : S^k \to V$ of the given stable class by the connectivity of $V \to V_\infty$. Then

$$L = FP_* + (I-q)L_* : S^k \longrightarrow \mathcal{J}$$

satisfies $L(*) = L_*$ and $\nu(qL) = F$. Since $\pi_k \mathcal{J} = 0$, there is a based homotopy from the constant L_* to L. Lift it from the constant T_* relative to the basepoint. Its endpoint T belongs to $\mathcal{U}_a$ at every point of S^k and has the required quotient frame. □

22.13. **Quaternionic isotropic subspaces.** Let $N \ge 0$ and give $K = \mathbb{H}^N \oplus \mathbb{H}^N$ the Hermitian form with matrix $\sigma = \mathrm{diag}(I, -I)$. Let E_0, E_1 be maximal totally isotropic subspaces. Write

$$G_N = \mathrm{Sp}(N,N) = \{T \in \mathrm{GL}_{2N}(\mathbb{H}) \,:\, T^*\sigma T = \sigma\}, \qquad \mathcal{U}_N = \{T \in G_N \,:\, TE_0 \cap E_1 = 0\}.$$

Stabilization adjoins a copy of $\mathbb{H} \oplus \mathbb{H}$ with the same split form, extends T by the identity, and adjoins the lines $\{(x,x) : x \in \mathbb{H}\}$ and $\{(x,-x) : x \in \mathbb{H}\}$ to E_0 and E_1, respectively.

Lemma 22.20. *Up to homotopy, the inclusion $\mathcal{U}_N \hookrightarrow G_N$ is the diagonal $\mathrm{Sp}(N) \hookrightarrow \mathrm{Sp}(N)^2$. Its homotopy fibre has the homotopy type of $\Omega\,\mathrm{Sp}(N)$, the loop space at I. For $d \geq 2$ there is an isomorphism*

$$\pi_d(G_N, \mathcal{U}_N) \;\cong\; \pi_d\,\mathrm{Sp}(N)$$

under which the image of an absolute class (α, β) is $\beta - \alpha$. Stabilization is an isomorphism for $2 \leq d \leq 4N+1$; the relative pointed set in degree one is trivial.

Let $d \geq 1$ and let $H : D^d \to G_N$ have boundary restriction $T_0 : S^{d-1} \to \mathcal{U}_N$. If $d \leq 4N+1$ and the relative class of H vanishes after stabilization, there is a homotopy in G_N, fixed on S^{d-1}, from H to a map $D^d \to \mathcal{U}_N$.

Proof. For $N = 0$ both spaces are points. Suppose $N \geq 1$ and write $S = \mathrm{Sp}(N)$. A maximal totally isotropic subspace is the graph of an element of S, as in Section 22.11: on an isotropic subspace the projections to the two summands induce an isometry of their images, which extends to a unitary. Write $E_i = \operatorname{graph} R_i$. With $D_0 = \operatorname{diag}(I, R_0)$ and $D_1 = \operatorname{diag}(I, -R_1)$, the change $T \mapsto D_1^{-1} T D_0$ reduces E_0, E_1 to $\operatorname{graph} I, \operatorname{graph}(-I)$, respectively. The basepoint $D_1 D_0^{-1} \in \mathcal{U}_N$ becomes I.

In these coordinates let $\ell(T) \in S$ have graph TE_0. For block entries T_{ij} it is $(T_{21} + T_{22})(T_{11} + T_{12})^{-1}$; the denominator is invertible since the positive projection on TE_0 is an isomorphism. Thus $T \in \mathcal{U}_N$ precisely when $I + \ell(T)$ is invertible. Write $T = uP$ for its polar decomposition and put $T_t = uP^{1-t}$, $0 \leq t \leq 1$. Since $\sigma P \sigma = P^{-1}$ and $u\sigma = \sigma u$ by the polar decomposition following Lemma 22.15, $T_t \in G_N$ and $u \in S \times S$. The path

$$T \longmapsto \operatorname{diag}\big(I, \ell(T)\ell(T_t)^{-1}\big)\; T_t$$

has constant image plane TE_0, fixes $S \times S$ pointwise, and ends in $S \times S$. It therefore retracts the pair onto its intersection with $S \times S$.

On $S \times S$, $\ell(a, b) = b\,a^{-1}$. The coordinates $(a, b) \mapsto (a, b\,a^{-1})$ identify the resulting pair of spaces with

$$(S \times S,\, S \times \mathcal{C}), \qquad \mathcal{C} = \{L \in S \,:\, I + L \text{ is invertible}\}.$$

The Cayley chart $L \mapsto (I - L)(I + L)^{-1}$ identifies $\mathcal{C}$ with the vector space of quaternionic skew-adjoint matrices (Macías-Virgós, Pereira-Sáez and Tanré, Introduction, (1)). It sends I to zero. Hence $S \times \{I\}$ is a deformation retract of $S \times \mathcal{C}$; in the original compact coordinates it is the diagonal. The second projection gives the asserted homotopy fibre and relative groups. On absolute groups it sends (α, β) to $\beta - \alpha$. The initial change of coordinates, after right translation by the inverse basepoint, is conjugation by D_1^{-1}; this has no effect on homotopy groups since $S \times S$ is connected. Other basepoints of $\mathcal{U}_N$ give the same identifications by transport, since $\mathcal{U}_N$ is homotopy equivalent to the connected, simply connected group S.

The second projection commutes with $L \mapsto L \oplus I$, so (22.32) gives the stated range. If the stable class of H vanishes in that range, its finite relative class is zero; this also holds for $d = 1$, since $\pi_1 S = 0$. The compression criterion for (D^d, S^{d-1}) [59, §4.1, p. 343] gives the required homotopy relative to S^{d-1}. □

22.14. **Quaternionic skew operators and finite boundary relations.** On a separable infinite-dimensional quaternionic Hilbert space $\mathscr{H}$, let Sp_c be the group of unitaries U for which $U - I$ is compact, with the norm topology. Put $\mathrm{Sp}_c^{\circ} = \mathcal{V} = \{U \in \mathrm{Sp}_c \,:\, U + I \text{ is invertible}\}$. As in (22.5), write $\mathrm{Cay}(A) = (A + I)(A - I)^{-1}$ for a skew-adjoint operator A. Let (P, P_0) be a compact CW pair and $(A_p)_{p \in P}$ a norm-resolvent continuous family of skew-adjoint operators with compact resolvent, invertible on P_0. Its Cayley transform takes (P, P_0) to $(\mathrm{Sp}_c, \mathcal{V})$, since $\mathrm{Cay}(A_p) - I = 2(A_p - I)^{-1}$ and $\ker(\mathrm{Cay}(A_p) + I) = \ker A_p$. Quaternionic clutching and real Bott periodicity give

$$[\mathrm{Cay}(A_\bullet)] \in K\,\mathrm{Sp}^{-1}(P, P_0) \simeq KO^{-5}(P, P_0). \tag{22.34}$$

Here $\mathcal{V}$ is contractible and $\mathrm{Sp} \to \mathrm{Sp}_c$ is a weak equivalence, as in (22.6). In the degree-five Fredholm model [15, Theorem A(k), $k = 5$] the first four Clifford generators act through $\mathrm{Cl}_{0,4} = \mathrm{Mat}_2(\mathbb{H})$

[12, §4, Table 1]. Their product ω is a self-adjoint involution anticommuting with each generator. Multiplication by ω identifies the skew-adjoint operators anticommuting with these generators with the skew-adjoint operators in their commutant: an operator anticommuting with all four commutes with their product. Restriction to the range of a diagonal matrix unit e_{11} identifies the commutant with the quaternionic-linear operators on one multiplicity space.

In dimension $n \equiv 5 \pmod 8$, take $A = i(D + V)$ as in (22.5). For the exponential convention of [15, §2, (2.7)], the signs agree as follows. For $x \in \mathbb{R}$ and $0 \le t \le 1$, put

$$\phi_t(x) = (1-t)2\arctan x + t\pi x(1+x^2)^{-1/2}, \qquad v_t(ix) = -e^{i\phi_t(x)}.$$

Then $v_0(A) = \operatorname{Cay}(A)$ and $v_1(A) = -\exp(\pi A(1-A^2)^{-1/2})$. The functions $v_t - 1$ vanish at infinity uniformly in t, vary continuously in the uniform norm on $C_0(i\mathbb{R})$, and satisfy $v_t(-ix) = \overline{v_t(ix)}$. The continuous functional calculus therefore gives a norm-continuous homotopy in Sp_c for norm-resolvent continuous families. Since $|\phi_t| < \pi$ and ϕ_t vanishes only at 0, the -1 eigenspace throughout is $\ker A$.

Let S be a densely defined closed quaternionic skew-symmetric operator with closed range. Suppose that $C = \ker S^*$ is finite dimensional and that S has an invertible skew-adjoint extension A_0 with compact inverse R. Write $K : C \hookrightarrow \mathscr{H}$ for the inclusion and $S_{\max} = -S^*$. For $u \in \operatorname{dom} S_{\max}$ there are unique $f \in \mathscr{H}$ and $x \in C$ such that

$$f = S_{\max} u, \qquad u = Rf + Kx.$$

Indeed $S_{\max} R = I$, so $u - Rf \in \ker S_{\max} = C$. Put $\Gamma_0 u = x$ and $\Gamma_1 u = K^* f$. On $C \oplus C$ take the Hermitian form $h((x,y),(x',y')) = \langle y, x'\rangle + \langle x, y'\rangle$. For a maximal null subspace $\mathcal{B}$ let $A_{\mathcal{B}}$ be the restriction of $S_{\max}$ to the vectors with $(\Gamma_0 u, \Gamma_1 u) \in \mathcal{B}$. Denote by $W_{\mathcal{B}} = \operatorname{Cay}(\mathcal{B})$ the unitary $y - x \mapsto y + x$ for $(x,y) \in \mathcal{B}$, and put $\ker \mathcal{B} = \{x : (x,0) \in \mathcal{B}\}$. Indeed $(x,y) \mapsto (y-x, y+x)/\sqrt{2}$ takes $\mathcal{B}$ to the graph of a unitary, as in the proof of Lemma 22.20. These definitions include the vertical subspace $\{0\} \oplus C$.

On the underlying complex space, the boundary maps $(\Gamma_0, -i\Gamma_1)$ for $-iS$ and its reference extension $-iA_0$ are those of Schmüdgen, *Unbounded Self-adjoint Operators on Hilbert Space*, Example 14.6, at $\mu = 0$. For the equality of the complex family indices of an extension and its boundary relation, see Ivanov, Theorem 5.4, and Prokhorova, (16.3)–(16.4) and Theorem 16.3.

Lemma 22.21. *The operator $A_{\mathcal{B}}$ is skew-adjoint with compact resolvent. Its Cayley transform is homotopic in Sp_c to $W = I + K(W_{\mathcal{B}} - I)K^*$, with -1 eigenspace $K \ker \mathcal{B}$ throughout. For a family this equality holds at each parameter. The homotopy is norm-continuous in K, R and the orthogonal projection onto $\mathcal{B}$ in local quaternionic Hilbert bundle charts.*

Proof. Since $R^* = -R$, for $u, v \in \operatorname{dom} S_{\max}$,

$$\langle S_{\max} u, v\rangle + \langle u, S_{\max} v\rangle = \langle \Gamma_1 u, \Gamma_0 v\rangle + \langle \Gamma_0 u, \Gamma_1 v\rangle. \tag{22.35}$$

The trace (Γ_0, Γ_1) sends $RKy + Kx$ to (x,y). Its kernel is $\operatorname{dom} S$: $x = 0$ and $K^* f = 0$ mean $u = Rf$ and $f \in C^\perp = \operatorname{ran} S$. Hence the skew-adjoint extension criterion of Schubert, Seifert, Voigt and Waurick (Proposition 1.9 and Theorem 2.5) applies to $-S$ on the underlying real Hilbert space: the trace $(y+x, y-x)/\sqrt{2}$ takes the real Green form in (22.35) to the positive-minus-negative form, and takes $\mathcal{B}$ to the graph of $W_{\mathcal{B}}^{-1}$. Thus $A_{\mathcal{B}}$ is real skew-adjoint and, since its domain and action are quaternionic-linear, quaternionic skew-adjoint.

For a linear relation write $\mathcal{B}^{-1} = \{(y,x) : (x,y) \in \mathcal{B}\}$. The domain formula gives, as an equality of relations,

$$A_{\mathcal{B}}^{-1} = R + Q_0, \qquad Q_0 = K\mathcal{B}^{-1}K^*; \tag{22.36}$$

thus $(f,q) \in Q_0$ means $q = Kx$ and $(x, K^* f) \in \mathcal{B}$. For $0 \le t \le 1$ put $Q_t = tR + Q_0$ and $E_0 = (I+W)/2$. For $(f, q_0) \in Q_0$, $W(f - q_0) = f + q_0$; conversely every $z \in \mathscr{H}$ occurs as $z = f - q_0$, since $W_{\mathcal{B}}$ has domain C and W is the identity on $C^\perp$. Thus $f = E_0 z$. For $q = q_0 + tRf$ one obtains $f - q = (I - tRE_0)z$. If the latter vanishes, $f = q$; $\operatorname{Re}\langle f, q\rangle = 0$ by skewness of R and nullity of $\mathcal{B}$, so $f = q = 0$ and $z = 0$. Compactness of RE_0 and the Fredholm alternative now give

$$E_t = (I - Q_t)^{-1} = E_0(I - tRE_0)^{-1}. \tag{22.37}$$

The operator $V_t = 2E_t - I$ sends $f - q$ to $f + q$ on Q_t. These vectors have equal norm. Since $E_0 - I$ has finite rank and RE_0 is compact, $V_t - I$ is compact. The isometry V_t is therefore onto by the Fredholm alternative. Its -1 eigenspace is $\ker E_t = \ker E_0 = K \ker \mathcal{B}$, since $I - tRE_0$ fixes $\ker E_0$ pointwise.

At $t = 0$ one has $V_0 = W$. At $t = 1$, (22.36) gives $V_1 = \mathrm{Cay}(A_{\mathcal{B}})$, whence $(A_{\mathcal{B}} - I)^{-1} = (V_1 - I)/2$ is compact. In the coordinates $(y - x, y + x)/\sqrt{2}$ the orthogonal projection onto $\mathcal{B}$ has lower left block $W_{\mathcal{B}}/2$. Thus W and (22.37) vary continuously in the stated topologies. Unitary changes of frame on C do not alter W. □

22.15. **A fixed midpoint restriction.** Let W_0, W_1 be compact caps with common boundary Σ. Insert $[0, L] \times \Sigma$ between them and denote the resulting closed manifold by M_L. Let p range over a compact CW complex P. Consider the quaternionic formally skew-adjoint Dirac-type operators $A_{L,p}$ of (22.7). On the caps the coefficients are independent of L and vary continuously with p in the C^∞ topology, in fixed bundle coordinates and with fixed product collars. On the cylinder let

$$A_{L,p} = \sigma\big(\partial_s + B + L^{-1}V_p(s/L)\big), \qquad \sigma^* = \sigma, \quad \sigma^2 = I, \quad B^* = B, \quad B\sigma = -\sigma B, \quad V_p^*\sigma + \sigma V_p = 0. \tag{22.38}$$

Here B and σ are fixed and the smooth quaternionic endomorphisms V_p vary continuously in the C^∞ topology, with support away from the two ends. Write $K_0 = \ker B$, P_K for its orthogonal projection, and $Q = I - P_K$. Suppose that $\sigma|_{K_0}$ has quaternionic signature (N, N) and both low APS realizations are injective. By (22.8) the full realizations $A_{i,p,\mathrm{full}}$ on W_i are surjective. By (22.20) and the Green formula following (22.21), the spaces $E_{i,p} = P_K \gamma_i(\ker A_{i,p,\mathrm{full}})$, where γ_i is the trace on W_i, are maximal null N-planes in K_0.

Let T_-, T_+ be the propagators of $u' + P_K V_p P_K u = 0$ on $[0, 1/2]$, $[1/2, 1]$, in the direction of increasing time, and put $T = T_+ T_-$. Since $P_K \sigma = \sigma P_K$, differentiation of the σ-pairing along solutions gives $T_\pm^* \sigma T_\pm = \sigma$. Put $m = L/2$ and denote by γ_m the trace at $s = m$. Set

$$\mathrm{dom}\, S_{L,p} = \{u \in H^1(M_L) \,:\, P_K \gamma_m u = 0\}, \qquad S_{L,p} u = A_{L,p} u. \tag{22.39}$$

On the cut manifold denote the left and right midpoint traces by γ_l, γ_r. Put

$$\mathcal{Q} = K_0 \oplus K_0, \qquad h_{\mathcal{Q}}(x, y) = \langle \sigma x_l, y_l \rangle - \langle \sigma x_r, y_r \rangle, \qquad \Delta = \{(z, z) \,:\, z \in K_0\},$$

where $x = (x_l, x_r)$ and $y = (y_l, y_r)$. For piecewise H^1 sections put $\Gamma u = (P_K \gamma_l u, P_K \gamma_r u)$, and denote the orthogonal projection onto $F \subset \mathcal{Q}$ by Π_F.

For limits of Cauchy data under unperturbed cylindrical extension, see Nicolaescu, *The Maslov index, the spectral flow, and decompositions of manifolds*, Theorem 4.9 and Corollary 4.12.

Proposition 22.22. *For all sufficiently large L and all $p \in P$, $S_{L,p}$ is injective and Fredholm of quaternionic index $-2N$. The domain of $S_{\max,L,p} = -S_{L,p}^*$ consists of the piecewise H^1 sections with $Q\gamma_l u = Q\gamma_r u$. On this domain Γ induces an isometry of the quotient by $\mathrm{dom}\, S_{L,p}$, with its Green form, onto $(\mathcal{Q}, h_{\mathcal{Q}})$. The domain of $A_{L,p}$ corresponds to Δ.*

For fixed such L, the kernels $C_{L,p} = \ker S_{\max,L,p}$ form a continuous quaternionic bundle of rank $2N$. The restriction of Γ maps $C_{L,p}$ isomorphically onto a maximal null plane $\Lambda_{L,p}$. There is a constant c, independent of L, p, such that

$$\big\| \Pi_{\Lambda_{L,p}} - \Pi_{(T_- E_{0,p}) \oplus (T_+^{-1} E_{1,p})} \big\| \le c/L. \tag{22.40}$$

Proof. Suppress L, p. On the cut manifold the domain of S has trace space $\{(v, v) \,:\, v \in QH^{1/2}(\Sigma)\}$. This is the transmission condition of [20, Example 4.23] on the nonzero spectral subspace of the doubled tangential operator: the two copies have opposite tangential operators. In the notation of [20, Definition 4.7], retain the interchange of the two copies on the nonzero modes and take $W_+ = 0$, $W_- = K_0 \oplus K_0$. The resulting condition for the underlying complex operator $-iA$ is elliptic. For full transmission the added trace space is Δ. The closed operator on M_L is skew-adjoint with compact resolvent by [20, Theorems 4.9, 4.11 and Remark 5.7]; its index is zero. Corollary 6.2 of the same reference gives $\mathrm{ind}_{\mathbb{H}}\, S = -\dim_{\mathbb{H}} \Delta = -2N$, since the complex indices and dimensions are twice

the quaternionic ones. The domain of S is dense in L^2, as it contains smooth sections supported off the cut.

By [20, Theorems 4.6, 4.9 and Remark 4.8(3)] the adjoint condition is the Green annihilator of this trace space and has piecewise H^1 domain. The high part of the Green pairing against (v, v) is $\langle \sigma(\gamma_l u - \gamma_r u), v\rangle$. Since σ preserves $K_0^\perp$, the annihilator is precisely $Q\gamma_l u = Q\gamma_r u$, with both low traces free. Thus $S_{\max}$ acts by A on each side. Independent smooth collar lifts of the two low values give surjectivity of Γ. Its kernel is $\operatorname{dom} S$, and Green's formula descends to h_Q; equality of the two full traces amounts to $\Gamma u \in \Delta$. The graph norm of $S_{\max}$ and the piecewise H^1 norm are equivalent by the bounded inverse theorem.

Let X_+ and X_- be the strictly positive and strictly negative spectral subspaces of B in $H^{1/2}(\Sigma)$. In the common cylinder coordinates the allowed high trace spaces are X_+ on W_0 and X_- on W_1, as in Section 22.10. Write H_i, e_i for the cap maps in (22.21). Their norms are bounded on P by Lemma 22.11 and surjectivity of the finite maps β_i in (22.20). The planes $E_i = \ker \beta_i$ vary continuously.

Put $x = (k, \ell) \in K_0^2$, where k is the low value at $s = 0$ and ℓ the right low value at m. On the left half prescribe the incoming high traces $a_0 \in X_+$ at 0 and $d_l \in X_-$ at m; on the right half prescribe $a_r \in X_+$ at m and $d_1 \in X_-$ at L. Put $h = (a_0, d_l, a_r, d_1)$. Denote the outgoing high traces by $(b_0, c_l) \in X_- \oplus X_+$ on the left and $(b_r, c_1) \in X_- \oplus X_+$ on the right, and the final low traces by k_l, k_1, respectively. Lemma 22.16 on the two half-intervals gives the unique homogeneous solutions and

$$\|(b_0, c_l, b_r, c_1)\| + \|(k_l, k_1) - (T_- k, T_+ \ell)\| \le \frac{c}{L}(\|x\| + \|h\|).$$

Here and below finite products carry their Hilbert norms, and the high trace norm is that of $H^{1/2}$. The two cap equations for the allowed traces and the midpoint high matching equations are

$$h = (H_0(k, b_0), b_r, c_l, H_1(k_1, c_1)).$$

Their right side is $G_L x + D_L h$, with $\|G_L\| \le c$ and $\|D_L\| \le c/L$. Thus $h = (I - D_L)^{-1} G_L x$ is uniquely determined and $\|h\| \le c'\|x\|$.

Let p_i be orthogonal projection of K_0 onto $E_i^\perp$. After substitution of this h, the remaining cap equations (22.21) are

$$F_L x = (p_0(k - e_0 b_0), p_1(k_1 - e_1 c_1)) = 0, \qquad F_\infty x = (p_0 k, p_1 T_+ \ell), \qquad \|F_L - F_\infty\| \le c/L. \tag{22.41}$$

Both maps have target $E_0^\perp \oplus E_1^\perp$. By Lemmas 22.11 and 22.16, restriction to the two specified low values is a linear isomorphism $C_{L,p} \to \ker F_L$: the half-cylinder solutions and the cap solutions are unique and all remaining traces match. Its inverse varies continuously in p, in piecewise H^1 for fixed L, by the cap formulas and bounded inversion in (22.23).

The right inverse $R_\infty(u, v) = (u, T_+^{-1} v)$ of F_∞ is uniformly bounded on P. Hence $R_L = R_\infty(F_L R_\infty)^{-1}$ is a uniformly bounded right inverse of F_L for large L. The map

$$D_L^0 = (I - R_L F_L)|_{\ker F_\infty} : \ker F_\infty \longrightarrow \ker F_L$$

differs from the inclusion by $O(L^{-1})$ in norm. It is injective for large L, hence bijective since both kernels have dimension $2N$. Consequently $C_{L,p}$ is a continuous bundle of rank $2N$.

In the coordinates x, Γ is the restriction of $J_L x = (k_l, \ell)$. The half-cylinder bounds give $\|J_L - J_\infty\| \le c/L$, where $J_\infty(k, \ell) = (T_- k, \ell)$ has uniformly bounded inverse. Thus J_L is injective, and $\ker S = \{u \in C_{L,p} : \Gamma u = 0\} = 0$. The Green pairing vanishes on $C_{L,p}$. Its image under Γ has half the dimension of $\mathcal{Q}$, hence is maximal null. Finally $J_L D_L^0$ and $J_\infty|_{\ker F_\infty}$ differ by $O(L^{-1})$ and are uniformly bounded below. For each of these maps Z the projection onto its image is $Z(Z^*Z)^{-1}Z^*$. Since $\ker F_\infty = E_0 \oplus T_+^{-1} E_1$, the projection formula gives (22.40). □

22.16. **The relative Cayley class.** Let $\mathcal{Q}_\pm$ be the eigenspaces of $\sigma \oplus (-\sigma)$ for ± 1 on the midpoint space $\mathcal{Q}$, and $\pi_\pm$ their orthogonal projections. For a maximal null plane $B \subset \mathcal{Q}$ let $U_B : \mathcal{Q}_+ \to \mathcal{Q}_-$ be the unitary with graph B. For a domain plane Δ and a Cauchy plane Λ put

$$C(\Delta, \Lambda) = -U_\Delta^{-1} U_\Lambda \in \operatorname{Sp}(\mathcal{Q}_+). \tag{22.42}$$

Then π_+ identifies $\Delta \cap \Lambda$ with $\ker(I + C(\Delta, \Lambda))$. For the real family index in terms of full Cauchy data, including relative disk families, see Nicolaescu, *Generalized symplectic geometries and the index of families of elliptic problems*, Theorems 6.2 and 6.10. For the complex analogue of (22.42) with ordered pair (Λ, Δ), see Prokhorova, (12.4)–(12.5).

Proposition 22.23. *Under the hypotheses of Proposition 22.22, let $P_0 \subset P$ be a closed subcomplex and suppose $T_p E_{0,p} \cap E_{1,p} = 0$ for $p \in P_0$. For all sufficiently large L, the closed operator $A_{L,p}$ is invertible on P_0. Its relative class* (22.34) *is represented by the stabilization of* (22.42) *for*

$$(\Delta, (T_p E_{0,p}) \oplus E_{1,p}) \quad \text{in } (K_0 \oplus K_0, \sigma \oplus (-\sigma)). \tag{22.43}$$

For fixed E_0, E_1, let T vary over $G_N = \mathrm{Sp}(N, N)$ in this formula and put $\mathcal{U}_N = \{T : TE_0 \cap E_1 = 0\}$. The resulting map of pairs to $(\mathrm{Sp}_c, \mathcal{V})$ induces

$$\pi_d(G_N, \mathcal{U}_N) \longrightarrow KO_{5+d},$$

an isomorphism for $1 \le d \le 4N + 1$. The image of an absolute class $(\alpha, \beta) \in \pi_d(\mathrm{Sp}(N)^2)$ is the stabilization of $\beta - \alpha$.

Proof. Fix a sufficiently large L. Write D for the maximal piecewise H^1 domain of Proposition 22.22, with low trace $\Gamma : D \to \mathcal{Q}$. Write $\mathscr{H}$ for the quaternionic L^2 space in fixed unitary bundle coordinates. On the closed manifold $A_{L,p} - I : H^1(M_L) \to \mathscr{H}$ is a continuous family of bounded isomorphisms. Its inverses are continuous in norm. Put $\Lambda^0_p = T_{-,p} E_{0,p} \oplus T^{-1}_{+,p} E_{1,p}$. For a graph unitary U_B, the lower left block of the orthogonal projection onto B is $U_B/2$. Thus (22.40) gives $\|U_{\Lambda_{L,p}} - U_{\Lambda^0_p}\| \le c/L$ uniformly in p. The graphs of the unitaries

$$U_{\Lambda^0_p} \exp\Big(t \log(U^{-1}_{\Lambda^0_p} U_{\Lambda_{L,p}}) \Big), \qquad 0 \le t \le 1,$$

stay $O(L^{-1})$ from Λ^0_p in projection norm; the logarithm is the principal one near I. A vector in $\Delta \cap \Lambda^0_p$ is (z, z) with $T_{+,p} z \in T_p E_{0,p} \cap E_{1,p}$. If $N > 0$ and $P_0 \neq \emptyset$, the least singular value of $U_{\Lambda^0_p} - U_\Delta$ consequently has positive minimum on P_0. For large L the path stays transverse to Δ there. The low trace identifies $\ker A_{L,p}$ with $\Lambda_{L,p} \cap \Delta$, by Proposition 22.22; compact resolvent and skew-adjointness give invertibility on P_0. When $N = 0$ this holds on P and both relative classes are zero.

The stabilized class of (22.42) is invariant under a continuous simultaneous change of Green coordinates. Indeed, in orthonormal bases let $g = kH$ be the polar decomposition of an isometry of the signed form. The path kH^t consists of such isometries by the polar decomposition in Lemma 22.20. It preserves intersections of the two planes up to isomorphism. For the compact factor $k = \mathrm{diag}(a, b)$ the graph formula becomes aCa^{-1}. Extend a by the identity to the ambient infinite-dimensional quaternionic Hilbert space. Its full unitary group is norm-contractible [75, Theorem II.6(ii)]; conjugating by a contraction to I preserves compact perturbations of I and the spectrum. This gives a homotopy of pairs.

Write $S_{\max,p} = -S^*_{L,p}$ for the maximal operator and $U_p = U_{\Lambda_{L,p}}$. The reference plane $\mathrm{graph}(-U_p)$ is maximal null. The extension with this trace is skew-adjoint by the boundary form and the extension criterion in the proof of Lemma 22.21. Consider the continuous family of bounded maps

$$D \longrightarrow \mathscr{H} \oplus \mathcal{Q}_-, \qquad u \longmapsto (S_{\max,p} u,\ \pi_- \Gamma u + U_p \pi_+ \Gamma u).$$

Since $S_{L,p}$ is injective with closed range, $S_{\max,p}$ is onto. On its kernel the second component is $2U_p \pi_+ \Gamma$, an isomorphism by Proposition 22.22. Thus the map is bijective. Write B_p for its bounded inverse. Then $R_p f = B_p(f, 0)$ is the inverse of the reference extension; it depends norm-continuously on p as a map to D and is compact as a map to $\mathscr{H}$. Set $F = \mathcal{Q}_+$ and $Z_p \xi = B_p(0, 2U_p \xi)$. Then $\pi_+ \Gamma Z_p = I_F$, and $K_p = Z_p (Z_p^* Z_p)^{-1/2} : F \to \mathscr{H}$ is an isometry onto $\ker S_{\max,p}$, with the adjoint of Z_p taken in $\mathscr{H}$.

For $u \in D$ put $f = S_{\max,p} u$ and set $x = K_p^*(u - R_p f)$, $y = K_p^* f$. The map

$$u \longmapsto \big((x + y)/\sqrt{2}, (x - y)/\sqrt{2} \big)$$

and the low trace Γ are surjective, have the same kernel dom $S_{L,p}$, and induce the same signed Green form by (22.35). They therefore differ by a Green isometry on the boundary quotient; it is continuous in p, since it is given by the displayed map composed with a fixed right inverse of Γ. In the (x, y) coordinates write $\mathcal{B}_p$ for the closed extension plane. It varies continuously by this isometry. The Cauchy plane $y = 0$ has graph unitary I in the signed coordinates. For the domain plane its graph unitary U takes $x + y$ to $x - y$. Thus $-U^{-1}$ takes $y - x$ to $y + x$, and equals $W_p = \mathrm{Cay}(\mathcal{B}_p)$, including for multivalued relations. Lemma 22.21 identifies the actual Cayley class with $I + K_p(W_p - I)K_p^*$. After adjoining F, the isometries $\xi \mapsto (\cos\theta\, K_p\xi, \sin\theta\, \xi)$, $0 \le \theta \le \pi/2$, move this representative to $I \oplus W_p$ without changing its spectrum away from 1. By the invariance just proved, the resulting relative class is represented in the original coordinates by $C(\Delta, \Lambda_{L,p})$.

The path of graphs above replaces $\Lambda_{L,p}$ by Λ_p^0. The Green isometry $T_{+,p} \oplus T_{+,p}$ fixes Δ and takes Λ_p^0 to the plane in (22.43). Likewise, if $E_{i,p} = g_{i,p}E_i^*$ for continuous $g_{i,p} \in G_N$, then $g_{1,p}^{-1} \oplus g_{1,p}^{-1}$ fixes Δ and replaces $(T_p, E_{0,p}, E_{1,p})$ by $(g_{1,p}^{-1}T_p g_{0,p}, E_0^*, E_1^*)$.

For fixed end planes make the two constant unitary changes of Lemma 22.20, so $E_0 = \operatorname{graph} I$ and $E_1 = \operatorname{graph}(-I)$ in $K_0 = \mathbb{H}_+^N \oplus \mathbb{H}_-^N$. Write $TE_0 = \operatorname{graph} L_T$. In the doubled space take positive coordinates (x_l^+, x_r^-) and negative coordinates (x_l^-, x_r^+). For $\Lambda = (TE_0) \oplus E_1$ one obtains

$$U_\Delta = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \qquad U_\Lambda = \operatorname{diag}(L_T, -I), \qquad C = \begin{pmatrix} 0 & I \\ -L_T & 0 \end{pmatrix}. \tag{22.44}$$

For $0 \le \theta \le \pi/2$ put $s = \sin\theta$, $c = \cos\theta$ and

$$C_\theta = \operatorname{diag}(I, L_T) \begin{pmatrix} sI & cI \\ -cI & sI \end{pmatrix}.$$

This unitary path joins C to $\operatorname{diag}(I, L_T)$, and

$$\ker(I + C_\theta) = \{(-c(1+s)^{-1}y, y) \,:\, y \in \ker(I + L_T)\}.$$

Hence this homotopy stays in $\mathcal{V}$ on $\mathcal{U}_N$. At the normalized basepoint $T = I$ this path lies in the contractible space $\mathcal{V}$; basepoint transport there is unique. In the coordinates of Lemma 22.20, $T \mapsto L_T$ induces its relative isomorphism to $\pi_d \operatorname{Sp}(N)$ for $d \ge 2$: on compact blocks (a, b) it is $b\,a^{-1}$, and the transverse chart contracts to I. Thus our map is that isomorphism followed by stabilization. The range is (22.32); in degree one both pointed sets are trivial. Finally (22.34) on (D^d, S^{d-1}) gives KO_{5+d}, equal by periodicity to KO_{n+k+1} when $n \equiv 5 \pmod 8$ and $d = k + 1$. □

22.17. **Homotopy groups of compatible pairs.** For an ample Clifford Hilbert space and a fixed invertible pseudo-Dirac operator, Ebert treats paths of pseudo-Dirac operators with the same principal symbol, starting at the fixed operator and ending at an invertible one [45, Definition 4.2 and Propositions 4.3, 4.6]. In the following proposition the potentials are multiplication operators and the coefficients are prescribed outside the given open set.

Proposition 22.24. *Let M be a closed connected spin manifold of dimension n and let $U \subset M$ be nonempty and open. If $n \equiv 4 \pmod 8$, put $a = |\operatorname{ind}_{\mathbb{H}} D^+|$. Then $Y_a(U)$ is nonempty and* (22.4) *is a weak homotopy equivalence. If $n \equiv 5 \pmod 8$, then $Y_5(U)$ is nonempty and* (22.6) *is a weak homotopy equivalence.*

Proof. First omit the density condition, keeping the prescribed exterior. Denote the resulting minimal or invertible locus by Y and its comparison map by κ. By [4, Theorem 1.1], a D-minimal metric agreeing with the prescribed metric outside a ball in U exists; with zero potential it belongs to Y. Any two points can be joined by a smooth coefficient path stationary near its endpoints. The closed case of Lemma 22.7, with perturbations in the ball, deforms this path into Y relative to its endpoints, since $1 < 4(a + 1)$ in residue four and $1 < 3$ in residue five. Both targets are path connected and simply connected: for $\operatorname{Sp}/\operatorname{Sp}(a)$ this follows from the finite Stiefel fibrations, and for $\Omega \operatorname{Sp}$ from $\pi_1 \operatorname{Sp} = \pi_2 \operatorname{Sp} = 0$ by (22.32) and $\operatorname{Sp}(1) = S^3$. Thus changes of target basepoint along paths with given endpoints induce the same identifications on homotopy groups.

Fix $k \geq 1$, set $d = k + 1$, and let $f : S^k \to Y$ be based at a chosen pair. Choose disjoint coordinate balls $D_0, D_1 \Subset U$. By Lemma 22.8 choose caps with tangential operator B. Put $K = \ker B$; its quaternionic dimension is $2r$ in residue four and its signature is (N, N) in residue five. Choose the caps so that

$$\begin{cases} r \geq \max(1, a), \quad d < 4(r - a + 1), \quad k \leq 4r + 1, \quad n \equiv 4 \pmod 8, \\ d < 4N + 3, \quad d \leq 4N + 1, \qquad\qquad\qquad n \equiv 5 \pmod 8. \end{cases} \tag{22.45}$$

Proposition 22.18, applied in D_0, joins f in Y to a sphere of pairs with fixed first cap, tangential operator B, cylinder length L_0 and control V_*. Write $T_* = \mathcal{E}(V_*)$ for its endpoint. The exterior cap is connected and contains D_1; its low realization is injective. For $i = 0, 1$, let C_i denote the cokernel of the low operator on the ith cap and let $\beta_i : K \to C_i$ be the boundary map in (22.20). The limiting matching map $(\beta_0, \beta_1 T_*)$ is uniformly bounded below by the estimate in the proof of Proposition 22.18. The control vanishes on a common final interval.

On the exterior cap take logarithmic metric and linear multiplier coordinates relative to one member of the sphere. The fixed spatial collar and the prescribed exterior of U amount to closed linear conditions on the coordinate differences. Smooth approximation in these coordinates [87, Theorem 11] gives a smooth parameter family arbitrarily close to the given one. The straight interpolation preserves low injectivity and the matching lower bound by Proposition 22.18 and Lemma 22.11. In residue four the full exterior index is $r - a$. Lemma 22.7, with parameter space S^k and perturbations in D_1, makes the full problem surjective, since $k < 4(r - a + 1)$. Take the perturbation small enough for the same interpolation to preserve the two lower bounds. In residue five full surjectivity follows from (22.8). All these interpolations stay in Y at every sufficiently large length, by Proposition 22.17; the original inserted sphere may first be lengthened from L_0 by Proposition 22.18.

Extend the smooth exterior family over D^d in the same coefficient coordinates: multiply its coordinate difference from a fixed family member by a smooth radial function equal to zero near the centre and to one on a boundary collar. Apply Lemma 22.7 in D_1, relative to a smaller boundary collar. In residue four the low index is $-(r + a)$ and the full index is $r - a$; the two inequalities in (22.9) follow from $r \geq a$ and $d < 4(r - a + 1)$. In residue five the relevant inequality is $d < 4N + 3$. The cap families on the disk now have injective low and surjective full realizations. Thus the two maps β_i are surjective by (22.20); put $E_i = \ker \beta_i$.

Injectivity. The exterior disk with the fixed control V_* is a coefficient filling with boundary in Y at every sufficiently large length. Its boundary represents f up to the paths just constructed. Since the ambient coefficient space is contractible by (22.2), this filling computes the comparison of its boundary. Suppose $\kappa_*[f] = 0$.

In residue four $\dim_{\mathbb{H}} K = 2r$, and E_0, E_1 have ranks $r, r - a$. Choose unitary maps $g_i : D^d \to \mathrm{Sp}(2r)$ taking fixed planes $\bar{E}_i$ to E_i; frames of E_i and $E_i^\perp$ exist over the disk. Put $H = g_1^{-1} T_* g_0$ and let q denote the orthogonal projection onto $\bar{E}_1^\perp$. The boundary frame of $qH|_{\bar{E}_0}$ represents $\kappa_*[f]$. Indeed, the finite block of Proposition 22.17 is homotopic to $(\beta_0, \beta_1 T_*)$ by straight interpolation, through injections on S^k at sufficiently large length. Put $B_i = \beta_i g_i|_{\bar{E}_i^\perp} : \bar{E}_i^\perp \to C_i$; these are isomorphisms. In the source coordinates $g_0(\bar{E}_0^\perp \oplus \bar{E}_0)$ and the target coordinates $B_0^{-1} \oplus B_1^{-1}$ the limiting block is

$$\begin{pmatrix} I & 0 \\ qH|_{\bar{E}_0^\perp} & qH|_{\bar{E}_0} \end{pmatrix}.$$

Subtracting $qH|_{\bar{E}_0^\perp}$ times the first row from the second splits off the identity over the disk. Lemma 22.6 gives the stable class of the polar frame of $qH|_{\bar{E}_0}$ on S^k. By Lemma 22.19(a), H is homotopic in $\mathrm{GL}_{\mathbb{H}}(K)$ relative to S^k to a map H' with $qH'|_{\bar{E}_0}$ injective everywhere.

In residue five $\dim_{\mathbb{H}} K = 2N$ and E_0, E_1 are maximal null planes. In the fixed positive and negative splitting of K, write $E_i(p) = \mathrm{graph}\, U_i(p)$ and $\bar{E}_i = \mathrm{graph}\, \bar{U}_i$ for fixed reference planes. The isometries $g_i = \mathrm{diag}(I, U_i \bar{U}_i^{-1})$ satisfy $g_i \bar{E}_i = E_i(p)$. Put $H = g_1^{-1} T_* g_0$. In these coordinates $H : (D^d, S^k) \to (G_N, \mathcal{U}_N)$, with $G_N, \mathcal{U}_N$ as in Lemma 22.20 for $\bar{E}_0, \bar{E}_1$. Proposition 22.23 identifies its stabilized relative class with $\kappa_*[f]$: the simultaneous change $g_1^{-1} \oplus g_1^{-1}$ fixes the diagonal and takes

$(T_*E_0)\oplus E_1$ to $(H\bar{E}_0)\oplus\bar{E}_1$. Since $d\le 4N+1$, Lemma 22.20 supplies a homotopy relative to S^k from H to $H' : D^d\to\mathcal{U}_N$.

In either case multiply this homotopy by g_1 on the left and g_0^{-1} on the right. It starts at T_*, is stationary on S^k, and ends at $T=g_1H'g_0^{-1}$ with $TE_0\cap E_1=0$ on D^d. Lemma 22.15, applied to V_* relative to S^k, gives controls with endpoint T and boundary value V_*. On the compact disk there is a $\delta>0$ such that $\|(\beta_0u,\beta_1Tu)\|\ge\delta\|u\|$ for all $u\in K$. At a sufficiently large finite length the error in (22.30) is less than $\delta/2$. The disk therefore lies in Y. Its boundary is the prescribed inserted sphere at this length. The preceding length changes and homotopies give an annulus in Y from that boundary to f; attaching it gives a filling of f.

Surjectivity. Apply the insertion and the small exterior perturbation above to one base pair, with ranks as in (22.45). Keep both caps fixed. The reference control V_* then has transverse endpoint T_*. Fix $b\in S^k=\partial D^d$. In residue four choose orthonormal frames of E_0 and $E_1^\perp$ and let $q : K\to E_1^\perp$ be orthogonal projection. Then Lemma 22.19(b) represents any prescribed stable frame class by the boundary frame of $qH|_{E_0}$ for a disk map $H : D^d\to\mathrm{GL}_{\mathbb{H}}(K)$ with $H(b)=T_*$ and $H(p)E_0\cap E_1=0$ on S^k: the based nullhomotopy in the lemma extends its sphere of endpoints to the disk.

In residue five take the normalized end-plane coordinates of Lemma 22.20; transport endpoints back by the inverse constant changes of coordinates before lifting them. By (22.32), a prescribed class in $\pi_k\Omega\,\mathrm{Sp}=\pi_d\,\mathrm{Sp}$ has a representative $g : (D^d,S^k)\to(\mathrm{Sp}(N),I)$. Put $H(p)=\mathrm{diag}(I,g(p))T_*$. Then $H=T_*$ on S^k. Writing $T_*E_0=\mathrm{graph}\,L_*$ gives $H(p)E_0=\mathrm{graph}(g(p)L_*)$. Join L_* to I by a path A_t in the contractible chart $\{A\in\mathrm{Sp}(N) : I+A \text{ is invertible}\}$ of that lemma. The disks $g(p)A_t$ then have boundary A_t throughout. Proposition 22.23 gives the prescribed relative class with coefficient $+1$.

Regard D^d as the closed unit ball. In either residue apply Lemma 22.15 to the fixed V_* with parameter space D^d, closed subset $\{b\}$ and endpoint homotopy $(p,s)\mapsto H((1-s)b+sp)$. The resulting controls have endpoint $H(p)$ and equal V_* at b. In residue five $H=T_*$ on S^k; the controls on S^k need agree with V_* only at b. Over S^k the limiting matching map is uniformly bounded below, so a common finite length gives a sphere in Y based at the lengthened reference pair. The disk of coefficients has this sphere as its actual boundary. In residue four the reduction above and Lemma 22.6 give its prescribed frame class, and in residue five Proposition 22.23 gives its prescribed relative class. Transport along the insertion, perturbation and lengthening paths gives surjectivity at the original basepoint.

Finally, the strong deformation retraction of Lemma 8.1 preserves the prescribed exterior. For a normalized base pair, (22.2) restricts to the same path on the density slice since its metric logarithm is trace free there. Thus (22.4) and (22.6) on the slice are the restrictions of the unrestricted maps. The isomorphisms just proved descend to the density slice. □

22.18. **Naturality of localization.**

Corollary 22.25. *Fix a metric g_* on the closed connected spin manifold M^n and put $\mu=dv_{g_*}$. For nonempty open sets $U_0\subset U_1\subset M$, with the exterior data $(g_*,0)$ of (8.2), the following inclusions are weak homotopy equivalences:*

$$Y_a(U_0)\longrightarrow Y_a(U_1),\quad \mathcal{R}_{\min,\mu}(U_0)\longrightarrow\mathcal{R}_{\min,\mu}(U_1),\quad n\equiv 4 \pmod 8,\quad a=|\operatorname{ind}_{\mathbb{H}}D^+|,$$
$$Y_5(U_0)\longrightarrow Y_5(U_1),\quad \mathcal{R}_{\mathrm{inv},\mu}(U_0)\longrightarrow\mathcal{R}_{\mathrm{inv},\mu}(U_1),\quad n\equiv 5 \pmod 8.$$

Proof. In either residue write Y_i and X_i for the indicated pair and metric spaces on U_i, and $j : Y_0\hookrightarrow Y_1$, $\iota : X_0\hookrightarrow X_1$ for their inclusions. Fix $b\in Y_0$ by Proposition 22.24. Use the positive spinor comparison with the metric of b on both spaces. If $y\in Y_0$, the ambient path $c_t(y)$ in (22.2) agrees with $(g_*,0)$ on $M\setminus U_0$ for every t. Hence its operator path is the same for both open sets. In residue four this gives the same point of $\mathscr{H}_b$ in (22.4); in residue five the endpoints and their principal logarithms agree, giving the same based loop in (22.6). Thus the two maps κ_0,κ_1 into the common operator model satisfy $\kappa_1 j=\kappa_0$. Both are weak homotopy equivalences by Proposition 22.24, and so is j.

For $h=0,1$ put $e_h : X_h\to Y_h$, $e_h(g)=(g,0)$. Then $e_1\iota=je_0$. Both e_h are weak homotopy equivalences by Proposition 8.2; hence ι is one as well. □

22.19. **Homotopy groups of metric spaces.**

Proof of Theorem T. By Proposition 22.24 the supported pair spaces are nonempty. Proposition 8.2 gives a bijection on components, so the corresponding metric spaces are nonempty. Fix a base metric g_b in the indicated locus and take $(g_b, 0)$ as base pair. The inclusion $g \mapsto (g, 0)$ of Proposition 8.2 followed by the comparison of Proposition 22.24 is a weak homotopy equivalence. The two operator targets are the homotopy fibre $\mathscr{H}_b$ of (22.3) in residue four and $\Omega \operatorname{Sp}_{\mathrm{c}}$ in (22.6) in residue five.

In the logarithmic coordinates about g_b, the metric locus is open in the Fréchet space of smooth trace-free g_b-self-adjoint endomorphisms of TM vanishing on $M \setminus U$. It is covered by convex open subsets of this space; hence it has CW type [59, Corollary 4G.3]. The inclusion of g_b is a cofibration, by radial contraction in a convex chart with a continuous cutoff supported in the chart. Thus the weak identifications of the operator targets in (22.3) and (22.6) give based maps from the metric locus to the indicated stable spaces [59, Proposition 4.22]. This proves (1.14) and (1.15); for nested open sets the assertion is Corollary 22.25.

With $\operatorname{Sp}(0) = \{I\}$, compare the homogeneous fibrations with common fibre $\operatorname{Sp}(a)$ and total spaces $\operatorname{Sp}(a+1)$ and $\operatorname{Sp}(m)$, $m \geq a+1$. By their homotopy exact sequences and (22.32), the map from the first base $\operatorname{Sp}(a+1)/\operatorname{Sp}(a) = S^{4a+3}$ to $\operatorname{Sp}/\operatorname{Sp}(a)$ induces isomorphisms through degree $4a+5$; the direct limit has the homotopy groups of the finite stages in this range, since compact subsets of the quotient lie in finite stages. Thus $\operatorname{Sp}/\operatorname{Sp}(a)$ is $(4a+2)$-connected and its homotopy group in degree $4a+3$ is $\mathbb{Z}$. For $a = 0$ the same range gives $\pi_1 \operatorname{Sp} = \pi_2 \operatorname{Sp} = 0$ and $\pi_3 \operatorname{Sp} = \mathbb{Z}$ since $\operatorname{Sp}(1) = S^3$. Consequently $\Omega \operatorname{Sp}$ is simply connected and $\pi_2(\Omega \operatorname{Sp}) = \mathbb{Z}$.

For a k-sphere of invertible metrics, $k \geq 1$, the path (22.2) from g_b has zero potential. At chiral index zero, (22.4) gives its ordinary index difference in $\mathrm{KSp}^0(D^{k+1}, S^k) \cong KO_{k+5}$; in residue five (22.6) and (22.34) give it in $\mathrm{KSp}^{-1}(D^{k+1}, S^k) \cong KO_{k+6}$. Bott periodicity identifies the respective groups with KO_{n+k+1}, proving (1.16).

In dimensions four and five the signed counts (22.1) now have infinite cyclic source. They are onto by Proposition 22.4, hence are isomorphisms. The sphere of count one given there lies in the prescribed C^0 neighborhood. Based at one of its points, it represents a generator of the full locus. Its class at any other base metric is obtained by transport along a path in that locus. At index zero its ordinary index difference is primitive in KO_8 by Proposition 22.5. □

22.20. **Positive scalar curvature and the ordinary Dirac operator.** Let M be a closed spin manifold. The Lichnerowicz formula gives the inclusion $i : \mathcal{R}^+(M) \hookrightarrow \mathcal{R}^{\mathrm{inv}}(M)$. Both spaces carry the C^∞ topology, with no restriction on the density or the exterior metric. For $g_0 \in \mathcal{R}^+(M)$ put

$$\mathcal{F}_{g_0}(M) = \operatorname{hofib}_{g_0}(i),$$

based at $(g_0, \mathrm{const}_{g_0})$, and let $p : \mathcal{F}_{g_0}(M) \to \mathcal{R}^+(M)$ be the projection.

Proposition 22.26. *Let $M = S^1 \times S^{11}$ with either spin structure on S^1, and let $g_0 \in \mathcal{R}^+(M)$. For every integer $j \geq 1$, the group $\pi_{4j}(\mathcal{F}_{g_0}(M))$ has an infinite cyclic direct summand.*

Proof. Put $r = 4j$, $m = 12 + r + 1$ and $C = C^*_{\mathbb{R},\mathrm{r}}\mathbb{Z}$, the reduced real group C^*-algebra of $\mathbb{Z}$. Projection $M \to S^1 = B\mathbb{Z}$ is an isomorphism on fundamental groups. The discrete group $\mathbb{Z}$ is torsion free, has rational homological dimension one and acts properly by isometric translations on $\mathbb{R}$. Its real Baum–Connes assembly map is an isomorphism by [63, Theorem 1.1] and [81, Corollary 2.13]. Choosing a basepoint of S^1 and using suspension in reduced KO-homology gives

$$KO_m(C) \cong KO_m(S^1) \cong KO_m \oplus KO_{m-1}.$$

Here $KO_{m-1} \cong \mathbb{Z}$ and $2KO_m = 0$ by real Bott periodicity, since $m \equiv 1$ or $5 \pmod 8$. Let $G = \pi_r(\mathcal{R}^+(M), g_0)$ and let $\lambda : G \to \mathbb{Z}$ be the higher real index difference based at g_0 for $M \to B\mathbb{Z}$, followed by projection onto $KO_{m-1} \cong \mathbb{Z}$. By Ebert–Randal-Williams [46, Theorem C], the higher index differences span $KO_m(C) \otimes \mathbb{Q}$: the dimension is even and at least six, the map on fundamental groups is split surjective, and $r \geq 1 - 12 - 1$. Thus $\lambda(G) \neq 0$.

Since D_{g_0} is invertible, $\operatorname{ind}_{\mathbb{H}} D^+ = 0$. Apply (1.16) with $U = M$ and $\mu = dv_{g_0}$, and then Lemma 8.1. This gives $\pi_r(\mathcal{R}^{\mathrm{inv}}(M), g_0) \cong KO_m$. By exactness of the homotopy fibre sequence [59, Theorem 4.41], $p_*\pi_r(\mathcal{F}_{g_0}(M)) = \ker i_* \supset 2G$. In particular λp_* has nonzero image $d\mathbb{Z}$ for some $d > 0$. Choose $y \in \pi_r(\mathcal{F}_{g_0}(M))$ with $\lambda p_*(y) = d$. The homomorphism $z \mapsto (\lambda p_*(z)/d)y$ retracts this abelian group onto $\mathbb{Z}y$. □

Appendix A. The obstruction groups in degrees at most two

Let Ω_n^{Spin} be the bordism group of closed spin n-manifolds and R_n^{inv} the group of Dahl–Große [40, §5.2]. For $0 \le n \le 2$ let $j_n : \Omega_n^{\mathrm{Spin}} \to R_n^{\mathrm{inv}}$ send the class of a closed spin manifold X to $[X, -]$. Its composite with the relative index $\theta_n : R_n^{\mathrm{inv}} \to KO_n$ is the ordinary real spin index α_n, which is surjective [40, §§5.2, 5.4]. In these degrees α_n is an isomorphism: the spin bordism groups are $\mathbb{Z}, \mathbb{Z}/2, \mathbb{Z}/2$, respectively [73, p. 201]; real Bott periodicity gives the same groups for KO_0, KO_1, KO_2.

It remains to show that each j_n is surjective. A compact zero-manifold is closed. A nonempty zero-manifold has zero Dirac operator on a nonzero spinor space, so the boundary of a representative of R_1^{inv} must be empty.

Let $[W, h] \in R_2^{\mathrm{inv}}$. On each component S of ∂W the Dirac operator of $h|_S$ is invertible, so $\alpha_1([S]) = 0$. Since α_1 is injective, S has the spin structure induced from the disk [73, p. 201]. Write L for its length. A torpedo metric g_{tor} on $\mathbb{R}^2$ as in [56, Appendix A, p. 29] has invertible Dirac operator and equals $dt^2 + d\varphi^2$ on its cylindrical end, with $\varphi \in \mathbb{R}/2\pi\mathbb{Z}$. Truncating $(L/2\pi)^2 g_{\mathrm{tor}}$ in this end gives a spin disk whose cylindrical completion is the scaled plane. Constant rescaling multiplies the Dirac operator by $2\pi/L$ under the spinor identification. Since the boundary has length L and the bounding spin structure, it is spin isometric to $-S$.

Let C be the disjoint union of these disks and put $\widehat{W} = W \cup_{\partial W} C$. If ∂W is empty, then $\widehat{W} = W$. Otherwise $C = \widehat{W} \setminus \operatorname{int} W$ has invertible cylindrical completion and induces h on ∂W. Thus [40, Lemma 5.5] gives $[W, h] = [\widehat{W}, -] = j_2([\widehat{W}])$. It follows that j_n is surjective for $0 \le n \le 2$; since $\theta_n j_n = \alpha_n$ is an isomorphism, so is θ_n.

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