SHARP SCHRÖDINGER INEQUALITIES FOR EMBEDDED HYPERSURFACES

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Abstract. Let $n \geq 2$, let h denote the second fundamental form, and put $\Delta = -\operatorname{div}\nabla$. Every closed connected smooth embedded minimal hypersurface $M^n \subset \mathbb{S}^{n+1}$ which is not totally geodesic satisfies $\Delta + |h|^2 \geq n$ in quadratic-form sense on $H^1(M)$. In particular, $\int_M |h|^2 \, dv_g \geq n \operatorname{Vol}(M)$, as conjectured by Perdomo, with equality precisely for minimal Clifford hypersurfaces.

Let $Q > 0$ be self-adjoint on $\mathbb{R}^{n+1}$, and put $a = \lambda_{\min}(Q)$ and $F(x) = \frac{1}{2}\langle Qx, x\rangle$. On a hypersurface with unit normal ν, write $d\mu_F = e^{-F} dv_g$, $\Delta_F = \Delta + \langle \nabla F, \nabla\rangle$ and $U = |h|^2 + \langle Q\nu, \nu\rangle$. Every connected complete smooth properly embedded hypersurface $M^n \subset \mathbb{R}^{n+1}$ without boundary, stationary for the weighted area $e^{-F} dv_g$ and not a hyperplane, satisfies $\Delta_F + U \geq 2a$ in quadratic-form sense on $C_c^\infty(M)$ in $L^2(M, d\mu_F)$. Its weighted volume is finite, and $\int_M U \, d\mu_F \geq 2a\mu_F(M)$. Equality in this average inequality holds precisely for the cylinders $\{x + y : x \in E,\ |x| = \sqrt{p/a},\ y \in E^\perp\}$, where $1 \leq p \leq n$, E is a linear subspace of $\ker(Q - a\mathrm{Id})$ and $\dim E = p + 1$. For $Q = \frac{1}{2}\mathrm{Id}$, this gives the Gaussian average-curvature inequalities conjectured by Li and Zhao.

In the spherical case, Morse index $n + 3$ characterizes the minimal Clifford hypersurfaces. Every smooth entire graph stationary for the quadratic density in a Q-eigendirection with eigenvalue $\beta \leq 2a$ is a hyperplane through the origin.

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1. Introduction

Write $\mathbb{S}^m(r) \subset \mathbb{R}^{m+1}$ for the sphere of radius $r > 0$ centred at the origin and $\mathbb{S}^m = \mathbb{S}^m(1)$. Every hypersurface carries its induced metric g, volume measure dv_g, and Levi-Civita connection ∇. We put $\mathrm{Vol}(M) = \int_M dv_g$ and $\Delta = -\operatorname{div}\nabla$. For a function, ∇ denotes its gradient. All function spaces are real. The space $H^1(M)$ consists of square-integrable functions with square-integrable weak gradient, and $C_c^\infty(M)$ denotes the smooth functions with compact support.

Let $X : M^n \hookrightarrow \mathbb{S}^{n+1}$ be a closed connected smooth embedded minimal hypersurface, $n \geq 2$, with second fundamental form h in $\mathbb{S}^{n+1}$. Put $S = |h|^2$. Simons' identity [40, Theorem 5.3.1, (5.3.2), p. 93] gives

$$\int_M S(S-n)\, dv_g = \int_M |\nabla h|^2\, dv_g. \tag{1.1}$$

Perdomo's conjecture [36, Conjecture B, p. 560] asks for $\int_M S\, dv_g \geq n\,\mathrm{Vol}(M)$ when M is not totally geodesic, with equality precisely for minimal Clifford hypersurfaces; see also [25, arXiv v3, Conjecture 1.5, p. 3]. The quadratic-form inequality below implies this average inequality.

1.1. Minimal hypersurfaces of the sphere.

Theorem 1.1. *Let $n \geq 2$, and let $X : M^n \hookrightarrow \mathbb{S}^{n+1}$ be a closed connected smooth minimal embedding which is not totally geodesic. Then*

$$\int_M \left(|\nabla f|^2 + Sf^2\right) dv_g \geq n \int_M f^2\, dv_g \qquad (f \in H^1(M)). \tag{1.2}$$

In particular,

$$\int_M S\, dv_g \geq n\,\mathrm{Vol}(M). \tag{1.3}$$

Equality in (1.3) *holds if and only if there is an integer p, $1 \leq p \leq n-1$, such that, after an ambient isometry,*

$$X(M) = \mathbb{S}^p\left(\sqrt{p/n}\right) \times \mathbb{S}^{n-p}\left(\sqrt{(n-p)/n}\right). \tag{1.4}$$

Ge–Li [25, arXiv v3, Theorem 1.4, p. 3] obtained $\int_M S\, dv_g \geq \delta(n)\,\mathrm{Vol}(M)$ for a constant $\delta(n) > 0$ depending only on n. Wang–Wang [43, arXiv v1, Theorem 1.2, p. 2] obtained the coefficient n under invariance by reflection in $n+2$ mutually perpendicular great hyperspheres. The classification under the pointwise identity $S \equiv n$ is due to Chern–do Carmo–Kobayashi [16, Main theorem, p. 60] and Lawson [29, Theorem 1, p. 189].

For $n = 2$, the hypersurface M in Theorem 1.1 is orientable by Lemma 3.1 and has genus $\mathfrak{g} \geq 1$ [40, §5.4, p. 96]. Gauss–Bonnet gives

$$\int_M S\, dv_g = 2\,\mathrm{Vol}(M) + 8\pi(\mathfrak{g}-1).$$

Thus the average inequality holds in this dimension, and its equality case is Brendle's classification of embedded minimal tori [7, Theorem 1, p. 178]. The embedding hypothesis in Theorem 1.1 is essential. For $n \geq 3$, closed connected oriented minimal immersions with exactly two principal curvatures and non-Clifford image satisfy $\int_M S\, dv_g < n\,\mathrm{Vol}(M)$ [37, Theorem 2.1].

1.2. Quadratic Gaussian densities.

Let Q be a positive-definite self-adjoint endomorphism of $\mathbb{R}^{n+1}$, and set

$$a = \lambda_{\min}(Q), \qquad b = \lambda_{\max}(Q), \qquad F(z) = \tfrac{1}{2}\langle Qz, z\rangle. \tag{1.5}$$

For a smooth hypersurface $X : M^n \to \mathbb{R}^{n+1}$, let ν be a local unit normal. Identify tangent vectors with their images under dX, and denote the Euclidean connection by $\bar{D}$. For $p \in M$ and $\xi, \eta \in T_pM$, put

$$h(\xi,\eta) = \langle \bar{D}_\xi \nu, dX(\eta)\rangle, \qquad H = \mathrm{tr}_g\, h, \qquad S = |h|^2. \tag{1.6}$$

Stationarity for the density e^{-F} under compactly supported variations is equivalent to

$$H = \langle QX, \nu \rangle \tag{1.7}$$

for every local choice of ν [14, arXiv v1, Proposition 2, p. 6, and (7), p. 7]. Write Y^T for the tangential component of an ambient vector Y. Define

$$\begin{gathered} d\mu_F = e^{-F\circ X} dv_g, \qquad \Delta_F = -\operatorname{div}\nabla + \langle (QX)^T, \nabla\rangle, \qquad \mathcal{L}_F = -\Delta_F, \\ q_\nu = \langle Q\nu, \nu\rangle, \qquad U = S + q_\nu. \end{gathered} \tag{1.8}$$

Theorem 1.2. *Let $n \geq 2$, and let $X : M^n \hookrightarrow \mathbb{R}^{n+1}$ be a connected complete smooth proper embedding without boundary satisfying* (1.7)*. Assume that $X(M)$ is not a hyperplane. Then*

$$\int_M \left(|\nabla f|^2 + Uf^2\right) d\mu_F \geq 2a \int_M f^2 \, d\mu_F \qquad (f \in C_c^\infty(M)). \tag{1.9}$$

The weighted volume is finite, and

$$\int_M U \, d\mu_F \geq 2a \, \mu_F(M). \tag{1.10}$$

Equality in (1.10) *holds if and only if there are an integer $1 \leq p \leq n$ and a linear subspace $E \subset \ker(Q - a\mathrm{Id})$, $\dim E = p+1$, such that*

$$X(M) = \{x + y \,:\, x \in E,\ |x| = \sqrt{p/a},\ y \in E^\perp\}. \tag{1.11}$$

In the compact equality case, $p = n$ and $Q = a\mathrm{Id}$.

The integral in (1.10) may be infinite. The Friedrichs realization of $\Delta_F + U$ in $L^2(M, d\mu_F)$ has spectrum in $[2a, \infty)$ [28, Chapter VI, §2.3, pp. 325–326]. Since $q_\nu \leq b$, one also has $\Delta_F + S \geq 2a - b$ in quadratic-form sense on $C_c^\infty(M)$. Moreover, $F \circ X \in L^1(M, d\mu_F)$ by Lemma 6.1, a consequence of Cheng–Vieira–Zhou [17, arXiv v1, Theorem 1.1(ii), p. 3].

The second variation of weighted area has quadratic form $\int_M (|\nabla f|^2 - Uf^2)\, d\mu_F$ for $f \in C_c^\infty(M)$ [14, arXiv v1, Proposition 15, (31)–(32), p. 23]; see also [27, arXiv v2, §3, p. 6]. The ambient Bakry–Émery tensor, the sum of the ambient Ricci tensor and the ambient Hessian of F, is Q and is parallel. The Simons and normal-component equations are therefore specializations of [15, arXiv v1, (23), Theorem 3 and Proposition 2, pp. 10–12]; see Lemma 5.1.

For $Q = \frac{1}{2}\mathrm{Id}$, equation (1.7) is the self-shrinker equation. Put

$$d\gamma = e^{-|X|^2/4} dv_g, \qquad \gamma(M) = \int_M d\gamma, \qquad \Delta_\gamma = \Delta_F.$$

For the hypersurfaces of Theorem 1.2,

$$\Delta_\gamma + S \geq \tfrac{1}{2}, \qquad \int_M S \, d\gamma \geq \tfrac{1}{2}\gamma(M), \tag{1.12}$$

where the first inequality is in quadratic-form sense on $C_c^\infty(M)$. Equality in the average inequality holds precisely, up to an orthogonal transformation, for $\mathbb{S}^p(\sqrt{2p}) \times \mathbb{R}^{n-p}$, $1 \leq p \leq n$. Cheng–Zhou [18, arXiv v2, Theorem 1.3, p. 3] prove that properness, polynomial volume growth and finite Gaussian volume are equivalent for complete immersed self-shrinkers. The entropy identity of Lemma 7.5, from [19, proof of Lemma 7.10, (7.13)–(7.27), pp. 786–788], therefore identifies the second inequality in (1.12) and its equality case with the conjectures of Li–Zhao [32, arXiv v1, Conjectures 1.6–1.7, p. 4], for $n \geq 2$.

Cao–Li [10, arXiv v2, Theorem 1.1, p. 2] and Ding–Xin [21, Proposition 3.1] classify complete self-shrinkers without boundary with $S \leq \frac{1}{2}$, under polynomial volume growth or properness, respectively. For complete properly embedded self-shrinkers, Li–Zhao [33, arXiv v3, Theorems 1.2–1.3, p. 5] obtain a hyperplane or a generalized round cylinder under $S < \frac{3}{4}$, and under $S \leq \frac{3}{4}$ when $n = 2$. Their normalization and that of Cao–Li are related to the present one by

$$\widehat{X} = X/\sqrt{2}, \qquad \widehat{g} = \tfrac{1}{2}g, \qquad \widehat{S} = 2S, \qquad e^{-|\widehat{X}|^2/2} = e^{-|X|^2/4}. \tag{1.13}$$

Thus their bounds $\widehat{S} < 3/2$ and $\widehat{S} \le 3/2$ become the stated bounds on S. For the scalar drift Laplacian, Ding–Xin [20, Theorem 1.3, p. 445] in the compact case and Brendle–Tsiamis [9, arXiv v3, Theorem 1.1, p. 1] in the properly embedded case give

$$\int_M |\nabla f|^2 \, d\gamma \ge \tfrac{1}{4} \int_M f^2 \, d\gamma$$

for every locally H^1 function f such that $\int_M (f^2 + |\nabla f|^2) \, d\gamma < \infty$ and $\int_M f \, d\gamma = 0$.

1.3. **Tangent-ball curvatures and weak inequalities.** For a hypersurface in Theorem 1.1 or Theorem 1.2, choose a global unit normal ν as in Lemma 3.1. Let k, ℓ be the tangent-ball curvatures defined in (3.2) for the normals $\nu, -\nu$, respectively. The signed suprema are retained in the sphere; in Euclidean space each is replaced by its maximum with zero. By Lemmas 3.2 and 3.3, k, ℓ are locally semiconvex and $w := k + \ell > 0$. Their distributional Hessians have positive semidefinite singular parts [24, Theorems 6.8–6.9]; see also [34, Section 2 and Appendix].

In the spherical case put $F = 0$, $d\mu_F = dv_g$, $\Delta_F = \Delta$ and $V = S - n$. In the quadratic Gaussian case retain (1.8) and put $V = U - 2a$. For a locally Lipschitz function z, the weighted distribution $\mathcal{L}_F z = -\Delta_F z$ is defined, as in (2.2), by

$$\langle \mathcal{L}_F z, \eta \rangle_F = -\int_M \langle \nabla z, \nabla \eta \rangle \, d\mu_F, \qquad \eta \in C_c^\infty(M).$$

This distribution includes the weight. When it is a measure, the density of its absolutely continuous part is taken with respect to $d\mu_F$.

Let $\kappa_{\max}(p)$ denote the largest principal curvature at $p \in M$. If ϕ is a C^2 function near p with $\phi \ge k$ and $\phi(p) = k(p) = \kappa_{\max}(p)$, then $d\phi_p$ vanishes on the $\kappa_{\max}(p)$-principal eigenspace, by Lemma 3.5. The same assertion applies to ℓ after reversing ν. The contact inequalities of Propositions 4.1 and 5.2, together with Lemma 2.5, give

$$\mathcal{L}_F w + V w \, d\mu_F \ge 2 w^{-1} |\nabla w|^2 \, d\mu_F \tag{1.14}$$

as measures.

Put $u = w^{-1}$. Proposition 2.4 gives the nonnegative locally finite Radon measure $\mathfrak{m} = \Delta_F u + V u \, d\mu_F$ and the identity

$$\int_M \left(|\nabla f|^2 + V f^2 \right) d\mu_F = \int_M w^{-2} |\nabla (wf)|^2 \, d\mu_F + \int_M w f^2 \, d\mathfrak{m} \qquad (f \in C_c^\infty(M)).$$

When u is a positive smooth solution of $(\Delta_F + V)u = 0$, this is the ground-state transformation [38, Lemma 2.4]. For ground-state transformations with measure potentials, see [31, Theorem 2.3]. For a closed connected smooth embedded non-totally-umbilic hypersurface of constant trace mean curvature H in $\mathbb{S}^{n+1}$, Theorem 4.2 gives the same identity with $F = 0$ and $V = S - n - H(k - \ell)$. Reversing the normal interchanges k, ℓ and changes H to $-H$, so this potential is unchanged.

Tangent-ball curvatures and their flow inequalities appear in [2, 5, 3]. For compact connected embedded Euclidean hypersurface flows with outward normal, Theorem 2 of [5] gives unnormalized viscosity inequalities for a smooth symmetric speed, homogeneous of degree one and strictly increasing in each principal curvature on a symmetric convex cone: concavity gives the interior inequality, and convexity the exterior inequality. Positive speed is an additional hypothesis of its quotient Corollary 3. Two-point functions also occur in Brendle's theorem [7] and in the constant-mean-curvature theorem of Andrews–Li [6]. Proposition 3 of [6] is the dimension-two antecedent of the finite-contact calculation in Proposition 4.1; its mean curvature equals $H/2$ in the trace convention (1.6). The reflected tangent-space identity occurs in [4, Lemma 2.2]. For the cubic identity at a diagonal contact, see Brendle [7, proof of Proposition 8, p. 186] and [4, Lemma 2.4]. For compact embedded mean-convex hypersurfaces evolving by mean curvature flow in Euclidean space, Brendle [8, arXiv v1, Propositions 3, 5 and Corollary 6] proves semiconvexity and a distributional contact inequality on the set where the inscribed curvature exceeds the largest principal curvature.

1.4. Graphical and spectral consequences.

Theorem 1.3. *Let $n \geq 2$, let Q satisfy* (1.5)*, and let $X : M^n \hookrightarrow \mathbb{R}^{n+1}$ be connected, complete, smooth, properly embedded, without boundary, and satisfy* (1.7)*. Suppose that a global unit normal ν and a nonzero constant vector $v \in \mathbb{R}^{n+1}$ satisfy*

$$Qv = \beta v, \qquad \beta \leq 2a, \qquad \langle \nu, v \rangle > 0 \quad \text{on } M. \tag{1.15}$$

Then $X(M)$ is a hyperplane through the origin. In particular, every entire smooth graph in such an eigendirection is a hyperplane through the origin.

For $Q = a\mathrm{Id}$, dilation by $\sqrt{2a}$ reduces the entire-graph conclusion to Wang's theorem [42, Theorem 1.1], and the positive-angle conclusion to Ding–Xin–Yang [22, arXiv v1, Theorem 3.1]. Hurtado–Palmer–Rosales [26, Theorem 4.26] treat entire graphs of constant weighted mean curvature for product densities $e^{\eta(x)+\mu(t)}$, under a radial decay hypothesis and $\mathrm{Hess}\,\eta(\xi,\xi) \leq c \leq \mu''(t)$ for a constant $c \in \mathbb{R}$ and every unit horizontal vector ξ. In orthonormal coordinates $(x,t) \in \mathbb{R}^n \times \mathbb{R}$ with $Q = \mathrm{diag}(A, \beta)$, put $\eta(x) = -\langle Ax, x\rangle/2$ and $\mu(t) = -\beta t^2/2$. The radial bound, with Euclidean gradients, is $\langle \nabla(\eta+\mu), \nabla|(x,t)|\rangle \leq -a|(x,t)|$ away from the origin, and the Hessian condition is $-\lambda_{\min}(A) \leq c \leq -\beta$. Thus that theorem gives the entire-graph conclusion when $\beta = a$. Theorem 1.3 also includes the range $a < \beta \leq 2a$.

The Morse index of a closed two-sided minimal hypersurface of $\mathbb{S}^{n+1}$ is the number of negative eigenvalues, counted with multiplicity, of $\Delta - S - n$. Perdomo's estimate [36, Theorem 3.1], together with Theorem 1.1, gives the embedded index-$(n+3)$ classification in Corollary 7.3. The implication from the average inequality to this classification is recorded in [36, p. 561] and [25, p. 4]. For $n = 2$, the classification is Urbano's theorem [41, Theorem, p. 989]. For the hypersurfaces of Theorem 1.1, Chen–Wang [13, Corollary 1.3] obtain the lower bound $n+4$ for the index when the first positive eigenvalue of Δ is less than n.

On a compact hypersurface in Theorem 1.1 or Theorem 1.2, consider $L^2(M, dv_g)$ or $L^2(M, d\mu_F)$, respectively. Let λ_j be the eigenvalues of Δ or Δ_F and ζ_j those of $\Delta + S$ or $\Delta_F + U$ in the corresponding space, in nondecreasing order and with multiplicity, starting at $j = 0$. Proposition 7.2 gives

$$\zeta_j \geq c + \left(\frac{\min_M w}{\max_M w}\right)^2 \lambda_j \quad (j \geq 0), \qquad c = \begin{cases} n & \text{in the sphere,} \\ 2a & \text{in Euclidean space.} \end{cases}$$

The comparison follows from the quadratic-form min–max principle [28, Chapter VII, §6.4, Remark 6.2].

2. Positive supersolutions and quadratic forms

Let (M, g) be a smooth n-dimensional Riemannian manifold without boundary, let $F \in C^\infty(M)$, and put

$$d\mu_F = e^{-F} dv_g, \quad \mathcal{L}_F = \mathrm{div}\,\nabla - \langle \nabla F, \nabla \rangle, \quad \Delta_F = -\mathcal{L}_F. \tag{2.1}$$

All function spaces are real. The space $H^1(M, d\mu_F)$ consists of functions whose value and weak gradient are square integrable with respect to $d\mu_F$, with norm $\|f\|^2_{H^1(d\mu_F)} = \int_M (f^2 + |\nabla f|^2)\, d\mu_F$. On a closed manifold it coincides with $H^1(M)$, with an equivalent norm. The notation $K \Subset M$ means that K is a compact subset of M. A C^2 function ϕ is an upper test for a continuous function z at p if $\phi(p) = z(p)$ and $\phi \geq z$ on a neighbourhood of p.

The unweighted case is $F = 0$. For locally Lipschitz z, define

$$\langle \mathcal{L}_F z, \eta \rangle_F = -\int_M \langle \nabla z, \nabla \eta \rangle\, d\mu_F, \qquad \eta \in C_c^\infty(M). \tag{2.2}$$

When $\mathcal{L}_F z$ is a measure, it includes the weight: a density a represents the measure $a\, d\mu_F$.

Lemma 2.1. *Let $K \Subset M$, and let $z \geq 0$ be Lipschitz with support in K. There are nonnegative $z_j \in C_c^\infty(M)$, all supported in one compact set, such that $z_j \to z$ uniformly and in $H^1(M, d\mu_F)$.*

Proof. Choose finitely many relatively compact coordinate balls U_i and nonnegative $\rho_i \in C_c^\infty(U_i)$ with $\sum_i \rho_i = 1$ near K. Apply the mollification theorem [24, Theorem 4.1(ii), (vi)] to the zero extensions of $\rho_i z$ in the coordinate spaces. Nonnegative convolution kernels preserve nonnegativity; sufficiently small radii keep the supports in fixed compact subsets of the charts. The sum of the pullbacks converges uniformly and in the coordinate H^1 norms to $\sum_i \rho_i z = z$. On these compact sets the metric and the positive smooth weight give equivalent coordinate and weighted H^1 norms. □

If z is locally Lipschitz and $\mathcal{L}_F z$ is a locally finite signed Radon measure, then (2.2) also holds for compactly supported Lipschitz tests. Indeed, Lemma 2.1, applied to the positive and negative parts of a test η, gives smooth tests η_j supported in a fixed compact set K' and converging uniformly and in H^1 to η. Uniform convergence and $|\mathcal{L}_F z|(K') < \infty$ give convergence of the measure integrals; Cauchy–Schwarz gives convergence of the gradient integrals. In particular, measure inequalities may be tested against nonnegative compactly supported Lipschitz functions.

A function z is locally semiconvex if, on every sufficiently small coordinate ball, $z + C|x|^2/2$ is convex for some $C \geq 0$. Write Sym_n for the space of real symmetric $n \times n$ matrices. A point p is an Alexandrov point of z if, in coordinates centred at p, there are $\xi \in \mathbb{R}^n$ and $B \in \mathrm{Sym}_n$ such that

$$z(x) = z(0) + \xi \cdot x + \tfrac{1}{2} x^T B x + o(|x|^2). \tag{2.3}$$

Then $\xi = Dz(0)$, and B is the coordinate Alexandrov Hessian. A locally semiconvex function is locally Lipschitz, and almost every point is an Alexandrov point [24, Theorems 6.7(i) and 6.9].

Lemma 2.2. *If z is locally semiconvex, its distributional coordinate Hessian is a locally finite symmetric matrix measure. Its singular part is positive semidefinite, and its absolutely continuous density is the Alexandrov Hessian almost everywhere. The distribution $\mathcal{L}_F z$ is a locally finite signed Radon measure whose singular part relative to $d\mu_F$ is nonnegative.*

Proof. The Hessian assertions follow from [24, Theorems 6.8–6.9], applied to $z + C|x|^2/2$ on a smaller coordinate ball. Subtracting the smooth quadratic changes only the absolutely continuous part. In coordinates $x = (x^1, \dots, x^n)$, let dx denote Lebesgue measure, let $(g^{ij}) = (g_{ij})^{-1}$, and let Γ_{ij}^k be the Christoffel symbols of g. Repeated indices are summed from 1 to n. Write

$$\mathcal{H} = (\partial_{ij} z) = A\, dx + \mathcal{H}^s, \qquad \omega = e^{-F}\sqrt{\det(g_{ij})},$$

where A is the absolutely continuous density and $\mathcal{H}^s \perp dx$. Equation (2.2) gives $\mathcal{L}_F z = \partial_i(\omega g^{ij} \partial_j z)$, hence

$$\mathcal{L}_F z = \omega g^{ij} \mathcal{H}_{ij} + \omega\big(-g^{ij}\Gamma_{ij}^k \partial_k z - g^{ij}(\partial_i F)(\partial_j z)\big)\, dx. \tag{2.4}$$

The second term has a locally bounded density. Since $\omega > 0$, singularity relative to dx is equivalent to singularity relative to $d\mu_F = \omega\, dx$. Put $\sigma = \operatorname{tr} \mathcal{H}^s$. This is a nonnegative locally finite measure, and $\mathcal{H}^s = C\, d\sigma$ for a positive semidefinite matrix density C. Since (g^{ij}) is positive definite,

$$(\mathcal{L}_F z)^s = \omega g^{ij} C_{ij}\, d\sigma \geq 0.$$

□

In almost-everywhere inequalities, $\mathcal{L}_F z$ denotes the density of its absolutely continuous part with respect to $d\mu_F$.

Lemma 2.3. *Let z be locally semiconvex and let $G \in L^1_{\mathrm{loc}}(M, d\mu_F)$. If $\mathcal{L}_F z \geq G$ almost everywhere, then $\mathcal{L}_F z \geq G\, d\mu_F$ as measures.*

Fix an Alexandrov point p and coordinates centred at p, and let B be the matrix in (2.3)*. Let $\Psi : \mathrm{Sym}_n \to \mathbb{R}$ be continuous. If $\Psi(D^2\phi(0)) \geq 0$ for every upper test ϕ for z at p, then $\Psi(B) \geq 0$. The value $z(p)$ and differential $dz(p)$ may occur as fixed parameters in Ψ.*

Proof. The first assertion follows from the nonnegative singular part in Lemma 2.2. For the second, (2.3) shows that, for every $\varepsilon > 0$,

$$\phi_\varepsilon(x) = z(0) + \xi \cdot x + \tfrac{1}{2} x^T B x + \varepsilon |x|^2$$

is an upper test on a sufficiently small ball. Its value and differential at p equal those of z, and $D^2\phi_\varepsilon(0) = B + 2\varepsilon\mathrm{Id}$. Thus $\Psi(B + 2\varepsilon\mathrm{Id}) \geq 0$; continuity gives $\Psi(B) \geq 0$. □

For ground-state transformations with a positive solution and with measure potentials, see [38, Lemma 2.4] and [31, Theorem 2.3], respectively.

Proposition 2.4. *Let $w > 0$ be locally Lipschitz, suppose that $\mathcal{L}_F w$ is a locally finite signed Radon measure, and let $V \in L^\infty_{\mathrm{loc}}(M)$. Assume*

$$\mathcal{L}_F w + V w \, d\mu_F \geq 2w^{-1}|\nabla w|^2 \, d\mu_F. \tag{2.5}$$

Then $u = w^{-1}$ is locally Lipschitz and

$$\mathfrak{m} = (\Delta_F + V)u = \Delta_F u + V u \, d\mu_F \tag{2.6}$$

is a nonnegative locally finite Radon measure. For every $f \in C^\infty_c(M)$,

$$\mathcal{Q}_V(f) := \int_M (|\nabla f|^2 + V f^2) \, d\mu_F = \int_M w^{-2}|\nabla(wf)|^2 \, d\mu_F + \int_M wf^2 \, d\mathfrak{m}. \tag{2.7}$$

Proof. On every compact set, w takes values in a compact subinterval of $(0, \infty)$. Hence $u = w^{-1}$ is locally Lipschitz. Extend $t \mapsto t^{-1}$ smoothly from that interval to $\mathbb{R}$, with bounded derivative and value zero near the origin. The local Sobolev chain rule [24, Theorem 4.4(ii)] gives $\nabla u = -w^{-2}\nabla w$. By (2.5), the measure

$$\mathfrak{n} = \mathcal{L}_F w + \left(V w - 2w^{-1}|\nabla w|^2\right) d\mu_F$$

is nonnegative and locally finite. For $\eta \in C^\infty_c(M)$, the function η/w^2 is a compactly supported Lipschitz test. Hence

$$\begin{aligned}\int_M \frac{\eta}{w^2} \, d\mathfrak{n} &= -\int_M \langle \nabla w, \nabla(\eta/w^2)\rangle \, d\mu_F + \int_M \left(\frac{V\eta}{w} - \frac{2\eta|\nabla w|^2}{w^3}\right) d\mu_F \\ &= \int_M \left(\langle \nabla u, \nabla\eta\rangle + V u\eta\right) d\mu_F. \end{aligned} \tag{2.8}$$

Thus $\mathfrak{m} = w^{-2}\mathfrak{n}$, which proves (2.6). In particular, $\mathcal{L}_F u = V u \, d\mu_F - \mathfrak{m}$ is a locally finite signed Radon measure. The extension of (2.2) to Lipschitz tests therefore applies to u.

For $f \in C^\infty_c(M)$, test (2.6) with the compactly supported Lipschitz function f^2/u. The Sobolev product rule gives

$$\begin{aligned}\mathcal{Q}_V(f) - \int_M u^2|\nabla(f/u)|^2 \, d\mu_F &= \int_M \left(\langle \nabla u, \nabla(f^2/u)\rangle + V f^2\right) d\mu_F \\ &= \int_M \frac{f^2}{u} \, d\mathfrak{m}. \end{aligned}$$

Substitute $u = w^{-1}$ to obtain (2.7). □

Lemma 2.5. *Let $V \in L^\infty_{\mathrm{loc}}(M)$, let k, ℓ be locally semiconvex, let $w = k + \ell > 0$, and let $\kappa_1, \dots, \kappa_n$ be the eigenvalues, counted with multiplicity, of a smooth symmetric endomorphism of TM. At each point where k and ℓ are differentiable, choose an orthonormal eigenbasis $e_1, \dots, e_n$ and put $k_i = dk(e_i)$ and $\ell_i = d\ell(e_i)$. No smooth choice of eigenbasis is assumed. Suppose $k \geq \kappa_i$ and $\ell \geq -\kappa_i$, and assume almost everywhere that*

$$\mathcal{L}_F k + V k \geq 2\sum_i \frac{k_i^2}{k - \kappa_i}, \qquad \mathcal{L}_F \ell + V \ell \geq 2\sum_i \frac{\ell_i^2}{\ell + \kappa_i}. \tag{2.9}$$

A zero denominator is required to have zero numerator, and that summand is set equal to zero. Then (2.5) *holds as a measure inequality.*

Proof. Within each eigenspace the denominators in (2.9) are constant, and the sums of squared directional derivatives are the squared norms of the orthogonal projections of the gradients. Thus both sums and the zero-denominator condition are independent of the chosen orthonormal eigenbasis. For $A, B > 0$ and $r, s \in \mathbb{R}$,

$$\frac{r^2}{A} + \frac{s^2}{B} - \frac{(r+s)^2}{A+B} = \frac{(Br - As)^2}{AB(A+B)} \geq 0. \tag{2.10}$$

If $A = 0$ and $r = 0$, the inequality $r^2/A + s^2/B \geq (r+s)^2/(A+B)$ is an equality under the stated convention. The case $B = 0$, $s = 0$ is identical. Apply this inequality with $A = k - \kappa_i$, $B = \ell + \kappa_i$, $r = k_i$ and $s = \ell_i$. Since $A + B = w > 0$, the two denominators do not vanish simultaneously. Summing over i gives (2.5) almost everywhere. The function w is locally semiconvex, and $2w^{-1}|\nabla w|^2 - Vw \in L^\infty_{\mathrm{loc}}(M)$. Lemma 2.3, applied to w, gives the measure inequality. □

Lemma 2.6. *Under the hypotheses of Proposition 2.4, suppose that $V \in L^1(M, d\mu_F)$ and that there are $\chi_j \in C_c^\infty(M)$ with $0 \leq \chi_j \uparrow 1$, equal to one eventually on every compact set, such that $\int_M |\nabla \chi_j|^2 \, d\mu_F \to 0$. Then*

$$\int_M V \, d\mu_F = \int_M |\nabla \log w|^2 \, d\mu_F + \int_M w \, d\mathfrak{m}. \tag{2.11}$$

Both terms on the right are finite. On a closed manifold, the identity follows directly by taking $f = 1$ in (2.7).

Proof. Apply (2.7) to χ_j:

$$\int_M (|\nabla \chi_j|^2 + V\chi_j^2) \, d\mu_F = \int_M |\nabla \chi_j + \chi_j \nabla \log w|^2 \, d\mu_F + \int_M w\chi_j^2 \, d\mathfrak{m}. \tag{2.12}$$

By dominated convergence and the energy hypothesis, the left side tends to $q := \int_M V \, d\mu_F < \infty$. For each point, $\chi_j = 1$ on a neighbourhood of that point for all sufficiently large j. Fatou's lemma for the first term and monotone convergence for the second give

$$\int_M |\nabla \log w|^2 \, d\mu_F + \int_M w \, d\mathfrak{m} \leq q.$$

Thus both integrals are finite. In particular,

$$\left| 2 \int_M \chi_j \langle \nabla \chi_j, \nabla \log w \rangle \, d\mu_F \right| \leq 2 \|\nabla \chi_j\|_{L^2(d\mu_F)} \|\nabla \log w\|_{L^2(d\mu_F)} \longrightarrow 0. \tag{2.13}$$

Expand the square in (2.12). The gradient energy and the mixed term tend to zero, while monotone convergence applies to $\chi_j^2 |\nabla \log w|^2$ and $w\chi_j^2$. Taking limits gives (2.11). □

Lemma 2.7. *On a closed manifold, multiplication by a Lipschitz function w is bounded on $H^1(M, d\mu_F)$. If $\min_M w > 0$, this multiplication operator is invertible. On a connected manifold, a locally Lipschitz function with zero weak gradient is constant.*

Proof. For smooth f, the Sobolev product rule [24, Theorem 4.4(i)] gives $\nabla(wf) = w\nabla f + f\nabla w$, and hence

$$\|wf\|_{H^1(d\mu_F)} \leq \big(\|w\|_\infty + \|\nabla w\|_\infty\big) \|f\|_{H^1(d\mu_F)}. \tag{2.14}$$

Smooth density [24, Theorem 4.2], applied in finitely many charts, extends this estimate to $H^1(M, d\mu_F)$. If $\min_M w > 0$, then w^{-1} is Lipschitz, and multiplication by w^{-1} is the bounded inverse. For the last assertion, Morrey's inequality [24, Theorem 4.10(i)], with an exponent greater than n, gives constancy almost everywhere on each relatively compact coordinate ball. Continuity and connectedness give constancy on M. □

3. Tangent-ball curvatures

Let $X : M^n \hookrightarrow N$ be a smooth connected embedded hypersurface without boundary, with $n \geq 2$. Suppose either that $N = \mathbb{S}^{n+1} \subset \mathbb{R}^{n+2}$ and M is closed, or that $N = \mathbb{R}^{n+1}$ and X is proper. Let g be the induced metric, ∇ its Levi–Civita connection, and d_M its distance. Write $\exp_x$ and $\exp^N_Y$ for the exponential maps of M and N, respectively.

Lemma 3.1. *The complement $N \setminus X(M)$ has exactly two connected components. There is a smooth global unit normal pointing from either specified component to the other.*

Proof. The image $X(M)$ is a closed subset of N, by compactness or properness. Since N is simply connected, the separation theorem [23, Section 3.2, p. 509] gives two components with common boundary $X(M)$. The local unit normals directed toward either chosen component agree on overlaps and define a smooth global unit normal. □

Fix such a normal ν. Let $\bar{D}$ be the flat connection of the containing Euclidean space, and put

$$h(v, z) = \langle \bar{D}_v \nu, dX(z) \rangle = g(Bv, z), \qquad H = \operatorname{tr} B, \qquad S = |h|^2.$$

The eigenvalues of B, counted with multiplicity, are denoted by $\kappa_1, \dots, \kappa_n$; put $\kappa_{\max} = \max_i \kappa_i$. Identify tangent vectors with their images under dX. For $x \neq y$, define

$$d(x, y) = X(y) - X(x), \quad A(x, y) = \tfrac{1}{2}|d(x, y)|^2, \quad c(x, y) = -\frac{\langle \nu(x), d(x, y) \rangle}{A(x, y)}. \tag{3.1}$$

In the sphere, $A = 1 - \langle X(x), X(y) \rangle$. Define

$$\begin{aligned} \mathbb{S}^{n+1} &: \quad k(x) = \sup_{y \neq x} c(x, y), && \ell(x) = \sup_{y \neq x} [-c(x, y)], \\ \mathbb{R}^{n+1} &: \quad k(x) = \max\{0, \sup_{y \neq x} c(x, y)\}, && \ell(x) = \max\{0, \sup_{y \neq x} [-c(x, y)]\}. \end{aligned} \tag{3.2}$$

Put $w = k + \ell$. Reversing ν interchanges k and ℓ and leaves w unchanged. The spherical envelopes are not truncated at zero.

For the extension of the quotient to tangent directions, see [5, Section 4, p. 28]. Brendle [8, Proposition 3] proves semiconvexity on $\{k > \kappa_{\max}\}$ for compact embedded mean-convex hypersurfaces moving by mean curvature flow in $\mathbb{R}^{n+1}$.

Lemma 3.2. *The functions k, ℓ are finite, locally Lipschitz, and locally semiconvex. Moreover,*

$$k \geq \max_i \kappa_i, \qquad \ell \geq -\min_i \kappa_i. \tag{3.3}$$

In the spherical case, $k(p) > \max_i \kappa_i(p)$ implies that the supremum is attained at a point distinct from p. In the Euclidean case, the same holds provided also that $k(p) > 0$.

Proof. For $|v| = 1$, the Gauss formula along $\gamma(t) = \exp_x(tv)$ gives

$$A(x, \gamma(t)) = \tfrac{1}{2}t^2 + O(t^3), \qquad -\langle \nu(x), X(\gamma(t)) - X(x) \rangle = \tfrac{1}{2} h_x(v, v) t^2 + O(t^3). \tag{3.4}$$

Thus $c(x, \gamma(t)) \to h_x(v, v)$, which gives (3.3).

Fix $p \in M$. Choose a relatively compact convex normal neighbourhood of p with a smooth orthonormal frame. All x derivatives below are taken in fixed coordinates on this neighbourhood. Choose $\varepsilon > 0$ so that, for $d_M(p, x) \leq \varepsilon/4$, the map $\exp_x$ is defined and injective on $\{u \in T_x M : |u| \leq \varepsilon\}$, with image in that neighbourhood. The radius may also be chosen so that every pair with $d_M(p, x) \leq \varepsilon/4$ and $d_M(p, y) < \varepsilon/2$ is joined there by its unique short geodesic. In the chosen frame, let v_x correspond to $v \in \mathbb{S}^{n-1}$. Consider

$$(x, v, t) \longmapsto c(x, \exp_x(t v_x)), \qquad 0 < t \leq \varepsilon. \tag{3.5}$$

Both numerator and denominator vanish to second order at $t = 0$. For any smooth P with $P(x, v, 0) = \partial_t P(x, v, 0) = 0$,

$$\frac{P(x, v, t)}{t^2} = \int_0^1 (1 - s) \partial_t^2 P(x, v, st)\, ds.$$

The divided denominator equals $1/2$ at $t = 0$. Consequently, (3.5) extends smoothly to $t = 0$, with value $h_x(v_x, v_x)$. On the compact parameter set, this family and its first two x derivatives are uniformly bounded.

Take the fixed far set $\{y : d_M(p, y) \geq \varepsilon/2\}$. Every remaining competitor occurs in (3.5), since its distance from x is less than $3\varepsilon/4$. Properness makes the intersection of the far set with the inverse image of any closed bounded ambient set compact. On this intersection, $d_M(x, y) \geq \varepsilon/4$; injectivity and compactness therefore give a positive lower bound for $A(x, y)$. The quotient and its first two x derivatives are uniformly bounded there.

In the Euclidean case, put $P(x, y) = \langle \nu(x), X(y) - X(x)\rangle$. There are constants $C, c_1, R > 0$, independent of x in the fixed compact set and of y with $|X(y)| \geq R$, such that

$$|P| + |\partial_x P| + |\partial_x^2 P| \leq C|X(y)|, \quad A \geq c_1|X(y)|^2, \quad |\partial_x A| + |\partial_x^2 A| \leq C|X(y)|.$$

After increasing C, differentiation of $c = -P/A$ gives

$$|c(x, y)| + |\partial_x c(x, y)| + |\partial_x^2 c(x, y)| \leq C|X(y)|^{-1}. \tag{3.6}$$

The near and far families thus have common bounds for their values and first two coordinate derivatives. Every member is a quotient or a directional limit of quotients, and every competitor occurs in one of the families. Their supremum is the untruncated envelope. In Euclidean space, adjoin the constant zero function. On a smaller convex coordinate ball, all members f_α have a common Lipschitz bound and satisfy $D^2 f_\alpha \geq -C\mathrm{Id}$. The supremum is finite and Lipschitz, and

$$\sup_\alpha f_\alpha(x) + \tfrac{C}{2}|x|^2 = \sup_\alpha \big(f_\alpha(x) + \tfrac{C}{2}|x|^2\big)$$

is convex. This proves the assertions for k. The same derivative bounds hold for $-c$, and hence for ℓ.

In either strict case of the statement, $k(p) = \sup_{y \neq p} c(p, y)$. Choose $y_j \neq p$ with $c(p, y_j) \to k(p)$. After passage to a subsequence, properness gives one of three alternatives: $y_j \to q \neq p$, $y_j \to p$, or, in Euclidean space, $|X(y_j)| \to \infty$. The second has upper limit at most $\max_i \kappa_i(p)$ by the smooth diagonal extension; the third has limit zero by (3.6). The strict hypotheses exclude both. In the first case, continuity gives $c(p, q) = k(p)$. □

Lemma 3.3. *In the spherical case, $w > 0$ everywhere unless the image is a round totally umbilic hypersphere. In the Euclidean case, $w > 0$ everywhere unless the image is an affine hyperplane.*

Proof. In the sphere, $w(p)$ is the supremum minus the infimum of $c(p, \cdot)$. If $w(p) = 0$, there is a $t \in \mathbb{R}$ such that $c(p, y) = t$ for every $y \neq p$. Thus

$$\langle \nu(p) - tX(p), X(y)\rangle = -t \qquad (y \in M). \tag{3.7}$$

Since $|\nu(p) - tX(p)|^2 = 1 + t^2$, this affine section of $\mathbb{S}^{n+1}$ is a round n-sphere of radius $(1 + t^2)^{-1/2}$. The immersion into that section is a local diffeomorphism. Its image is open and, by compactness, closed, so it is the entire section.

In Euclidean space, $k, \ell \geq 0$. If $w(p) = 0$, then $c(p, y) = 0$ for every $y \neq p$, and $X(M) \subset X(p) + \nu(p)^\perp$. The image is open in this hyperplane by the inverse function theorem and closed by properness; hence it is the entire hyperplane. On a compact nonexceptional hypersurface, continuity gives $\min_M w > 0$. □

Denote the two components of $N \setminus X(M)$ by Ω_- and Ω_+, with ν pointing from Ω_- to Ω_+.

Lemma 3.4. *Let $\mathcal{B} \subset N \setminus X(M)$ be connected and meet Ω_-. Let $Y = X(q) \in \partial\mathcal{B}$, and let $G : U \to \mathbb{R}$ be smooth on a neighbourhood U of Y, with gradient $\nabla^N G$ in N. Suppose that $\mathcal{B} \cap U = \{z \in U : G(z) < 0\}$, $G(Y) = 0$, $\nabla^N G(Y) \neq 0$, and $\nabla^N G(Y) \perp T_q M$. Then $\nabla^N G(Y)/|\nabla^N G(Y)| = \nu(q)$.*

Proof. Connectedness gives $\mathcal{B} \subset \Omega_-$. Put $V = \nabla^N G(Y)/|\nabla^N G(Y)|$. For small $t > 0$,

$$G(\exp_Y^N(-tV)) = -t|\nabla^N G(Y)| + O(t^2) < 0,$$

so $\exp_Y^N(-tV) \in \Omega_-$. Since $V \in \{\nu(q), -\nu(q)\}$ and $\exp_Y^N(t\nu(q)) \in \Omega_+$, one has $V = \nu(q)$. □

3.1. **Covariant identities.** Put $c_0 = 1$ in the sphere and $c_0 = 0$ in Euclidean space. The Gauss and Weingarten formulas are

$$\bar{D}_v dX(z) = dX(\nabla_v z) - h(v,z)\nu - c_0 g(v,z)X, \qquad \bar{D}_v \nu = dX(Bv). \tag{3.8}$$

With the convention $R(v,z)t = \nabla_v\nabla_z t - \nabla_z\nabla_v t - \nabla_{[v,z]}t$, the Gauss and Codazzi equations read

$$\begin{aligned} R(v,z)t &= c_0\big(g(z,t)v - g(v,t)z\big) + h(z,t)Bv - h(v,t)Bz, \\ (\nabla_v h)(z,t) &= (\nabla_z h)(v,t). \end{aligned} \tag{3.9}$$

In particular, ∇h is fully symmetric. Write $\mathscr{D} = \operatorname{div}\nabla$, also denoting by $\mathscr{D}$ the trace of the covariant Hessian of a tensor. Simons' identity in a space of constant curvature [39, (1.20), p. 278] is

$$\mathscr{D}h = \nabla^2 H + (nc_0 - S)h + Hh^2 - c_0 Hg, \tag{3.10}$$

where $(h^2)(v,z) = g(B^2v, z)$. In [39], the normal $-\nu$ gives the second fundamental form h and mean curvature H/n.

3.2. **Diagonal contacts.**

Lemma 3.5. *Let k be the envelope for ν, let ϕ be a C^2 upper test at p, and suppose $k(p) = \kappa_{\max}(p)$. Put $E = \ker(k(p)\mathrm{Id} - B_p)$. Then*

$$d\phi|_E = 0. \tag{3.11}$$

Let Z be a smooth tangent field and put $L_Z = \mathscr{D} - \nabla_Z$. For any orthonormal eigenbasis with $e_1 \in E$, write $\phi_i = d\phi(e_i)$ and $h_{ij;r} = (\nabla_{e_r}h)(e_i, e_j)$. At p,

$$L_Z\phi \geq (L_Z h)_{11} + 2\sum_{j=1}^{n}\sum_{\kappa_i<k}\frac{h_{1i;j}^2}{k-\kappa_i} \geq (L_Z h)_{11} + 2\sum_{\kappa_i<k}\frac{\phi_i^2}{k-\kappa_i}. \tag{3.12}$$

Proof. Fix a unit vector $v \in E$. The directional limits give

$$\kappa_{\max}(p) \leq \sup_{y\neq p} c(p,y) \leq k(p) = \kappa_{\max}(p).$$

For the reversed normal, the least principal curvature is therefore $-k(p) = \inf_{y\neq p}[-c(p,y)]$. Brendle's cubic identity [7, proof of Proposition 8], in the form [4, arXiv v2, proof of Lemma 2.4], applied to $-\nu$ and v, gives

$$(\nabla_v h)(v,v) = 0. \tag{3.13}$$

The C^2 tensor $T = \phi g - h$ is positive semidefinite near p, and $T_p(v,\cdot) = 0$. For a local unit extension V of v, the scalar $T(V,V)$ has a minimum equal to zero at p. Differentiation gives

$$d\phi(z) = (\nabla_z h)(v,v) \qquad (z \in T_pM). \tag{3.14}$$

Taking $z = v$ and applying (3.13) gives $d\phi(v) = 0$. Since v is arbitrary, (3.11) follows.

For $v = e_1$, equation (3.14) gives $(\nabla_Z T)_{11} = 0$. The local null-eigenvector inequality [1, arXiv v1, (3.3)–(3.4), proof of Theorem 3.2], applied to $T \geq 0$ with $T_p(e_1,\cdot) = 0$, gives, for every real array Γ^i_j,

$$0 \leq (L_Z T)_{11} - 4\sum_{j,i}\Gamma^i_j(\nabla_{e_j}T)_{1i} + 2\sum_{j,i,r}\Gamma^i_j\Gamma^r_j T_{ir}.$$

At p, $T_{ir} = (k-\kappa_i)\delta_{ir}$. Choose

$$\Gamma^i_j = \begin{cases} -h_{1i;j}/(k-\kappa_i), & \kappa_i < k, \\ 0, & \kappa_i = k. \end{cases}$$

For $\kappa_i < k$, one has $(\nabla_{e_j}T)_{1i} = -h_{1i;j}$. Substitution gives the first inequality in (3.12). Retaining the terms with $j = 1$ and applying Codazzi and (3.14), $h_{1i;1} = h_{11;i} = \phi_i$, gives the second. □

4. The spherical inequality

Let $X : M^n \hookrightarrow \mathbb{S}^{n+1}$ be a smooth embedding. Assume that M is closed and connected, that its trace mean curvature H is constant, and that it is not totally umbilic. Let k, ℓ be the spherical envelopes in (3.2), and put $w = k + \ell$. By Lemma 3.3, $\min_M w > 0$.

Since H is constant, (3.8) and (3.10) give

$$\mathscr{D}X = -nX - H\nu, \qquad \mathscr{D}\nu = -S\nu - HX, \qquad \mathscr{D}h = (n - S)h + H(h^2 - g). \tag{4.1}$$

Proposition 4.1. *Let ϕ be a C^2 upper test for k at p. In an orthonormal principal basis $e_1, \dots, e_n$ at p, put $\phi_i = d\phi(e_i)$. Then*

$$\mathscr{D}\phi + (S - n)k - H(k^2 - 1) \geq 2 \sum_{\kappa_i < k} \frac{\phi_i^2}{k - \kappa_i}. \tag{4.2}$$

Moreover, $\phi_i = 0$ whenever $\kappa_i = k$.

Proof. If $k(p) = \kappa_{\max}(p)$, Lemma 3.5 with $Z = 0$ and (4.1) give (4.2) and $\phi_i = 0$ for $\kappa_i = k$.

Suppose that $k(p) > \kappa_{\max}(p)$. By Lemma 3.2, there is $q \neq p$ with $c(p, q) = k(p)$. Write

$$Y = X(q), \quad A = 1 - \langle X(p), Y\rangle > 0, \quad d = Y - X(p), \quad b_i = \langle e_i, Y\rangle. \tag{4.3}$$

Here $k = k(p)$ and $\kappa_i = \kappa_i(p)$. The function

$$Z(x, y) = \phi(x)(1 - \langle X(x), X(y)\rangle) + \langle \nu(x), X(y)\rangle \tag{4.4}$$

is nonnegative for x near p and every $y \in M$, and vanishes at (p, q). Its value and its first derivatives in x give

$$\langle \nu(p), Y\rangle = -kA, \qquad A\phi_i = (k - \kappa_i)b_i. \tag{4.5}$$

On $\mathbb{S}^{n+1}$ put

$$G(z) = k(1 - \langle X(p), z\rangle) + \langle \nu(p), z\rangle, \qquad \mathcal{B} = \{G < 0\}.$$

Put $a = \nu(p) - kX(p)$. Then $G(z) = k + \langle a, z\rangle$, $|a| = \sqrt{1 + k^2}$, and $-k/|a| \in (-1, 1)$. Thus $\mathcal{B}$ is a nonempty connected spherical cap. The cap is disjoint from $X(M)$. Since $\nabla^{\mathbb{S}} G(X(p)) = \nu(p)$, it meets Ω_- near $X(p)$; connectedness gives $\mathcal{B} \subset \Omega_-$. At Y,

$$\nabla^{\mathbb{S}} G(Y) = \nu(p) + kd =: W, \qquad |W|^2 = 1 - 2k^2A + 2k^2A = 1. \tag{4.6}$$

Since $G \circ X$ has a minimum at q, one has $W \perp T_qM$. Moreover, $G(Y) = 0$ and $W \neq 0$ give $Y \in \partial\mathcal{B}$. Lemma 3.4 gives $\nu(q) = W$.

By [4, Lemma 2.2], the reflected vectors

$$\widetilde{e_i} = e_i - \frac{b_i}{A} d \tag{4.7}$$

form an orthonormal basis of T_qM. Put $v_i = (e_i, \widetilde{e_i}) \in T_{(p,q)}(M \times M)$. In the spatial Hessian formula of [3, Section 2, p. 335], take ambient curvature 1 and $F(h) = \operatorname{tr} h$. Put $\dot{F}^{ij} = \partial F/\partial h_{ij}$. Then $\dot{F}^{ij} = \delta_{ij}$, $F(p) = F(q) = H$, and $\nabla F = 0$. The inequality $\dot{F}^{ij}(p)h_{ij}(q) \geq F(q)$ in that formula is an equality. Since $|d|^2 = 2A$ and $\langle \nu(p), \nu(q)\rangle = 1 - k^2A$, the formula becomes

$$\begin{aligned} \sum_i \operatorname{Hess}_{M\times M} c(v_i, v_i) &= \frac{H\big(1 - \langle \nu(p), \nu(q)\rangle\big)}{A} + (n - S)k - H + \frac{2}{A^2} \sum_i (k - \kappa_i)b_i^2 \\ &= (n - S)k + H(k^2 - 1) + \frac{2}{A^2} \sum_i (k - \kappa_i)b_i^2. \end{aligned} \tag{4.8}$$

The function $(x, y) \mapsto \phi(x) - c(x, y)$ has a local minimum equal to zero at (p, q). Consequently, (4.5) and (4.8) give

$$\mathscr{D}\phi \geq \sum_i \operatorname{Hess}_{M\times M} c(v_i, v_i) = (n - S)k + H(k^2 - 1) + 2 \sum_i \frac{\phi_i^2}{k - \kappa_i}. \tag{4.9}$$

All the denominators are positive in this case. □

Theorem 4.2. *Let $M^n \hookrightarrow \mathbb{S}^{n+1}$ be a closed connected hypersurface of constant trace mean curvature H, and suppose that it is not totally umbilic. Put*

$$w = k + \ell, \qquad V = S - n - H(k - \ell), \qquad u = w^{-1}. \tag{4.10}$$

Then $(\Delta + V)u = \mathfrak{m}$ is a nonnegative finite Radon measure and

$$\int_M (|\nabla f|^2 + Vf^2)\, dv_g = \int_M w^{-2}|\nabla(wf)|^2\, dv_g + \int_M wf^2\, d\mathfrak{m} \tag{4.11}$$

for every $f \in C^\infty(M)$. In particular, $\Delta + V \geq 0$ on $H^1(M)$. The function V and the measure $\mathfrak{m}$ are unchanged by reversing the normal.

Proof. Let p be a common Alexandrov point of k and ℓ. Choose an orthonormal principal basis $e_1, \dots, e_n$ at p, and put $k_i = dk_p(e_i)$ and $\ell_i = d\ell_p(e_i)$. The quadratic upper tests from Lemma 2.3 have values $k(p)$, $\ell(p)$ and differentials dk_p, $d\ell_p$, respectively. Lemma 3.5 and normal reversal therefore give $k_i = 0$ when $k = \kappa_i$, and $\ell_i = 0$ when $\ell = -\kappa_i$. In each sum below, a term with zero denominator is defined to be zero. Proposition 4.1 and normal reversal, followed by passage to the Alexandrov Hessians, give

$$\mathscr{D}k + (S - n)k - H(k^2 - 1) \geq 2\sum_i \frac{k_i^2}{k - \kappa_i}, \tag{4.12}$$

$$\mathscr{D}\ell + (S - n)\ell + H(\ell^2 - 1) \geq 2\sum_i \frac{\ell_i^2}{\ell + \kappa_i}.$$

The common Alexandrov points have full measure. Since

$$(S - n)(k + \ell) - H(k^2 - \ell^2) = Vw, \tag{4.13}$$

addition and (2.10) yield

$$\mathscr{D}w + Vw \geq 2w^{-1}|\nabla w|^2 \quad \text{almost everywhere.}$$

The functions k, ℓ are locally semiconvex by Lemma 3.2. Moreover, $2w^{-1}|\nabla w|^2 - Vw \in L^\infty(M)$, since M is compact and $\min_M w > 0$. Lemma 2.3 therefore gives the measure inequality (2.5). Proposition 2.4, with $F = 0$, gives $\mathfrak{m}$ and (4.11). Its nonnegative right side implies $\Delta + V \geq 0$ on $C^\infty(M)$. Since $V \in L^\infty(M)$, the quadratic form is continuous on $H^1(M)$; smooth density [24, Theorem 4.2] extends this inequality to $H^1(M)$.

Normal reversal sends (H, k, ℓ) to $(-H, \ell, k)$, leaving w, V, u, and $\mathfrak{m}$ unchanged. □

Proof of Theorem 1.1. A minimal totally umbilic hypersurface has $h = (H/n)g = 0$. Hence Theorem 4.2 applies under the hypotheses of Theorem 1.1. With $H = 0$, it gives $V = S - n$ and (1.2). Taking $f = 1$ in (4.11) gives

$$\int_M (S - n)\, dv_g = \int_M |\nabla \log w|^2\, dv_g + \int_M w\, d\mathfrak{m} \geq 0. \tag{4.14}$$

If equality holds, then $\nabla w = 0$ almost everywhere and $\mathfrak{m} = 0$, since $\min_M w > 0$. By Lemma 2.7, w is constant. Thus

$$0 = \mathfrak{m} = (S - n)w^{-1}\, dv_g,$$

and continuity gives $S = n$. Proposition 4.3 identifies the image with (1.4). Conversely, Lemma 4.4 gives $S = n$ on every minimal Clifford hypersurface. □

Let B be a parallel self-adjoint endomorphism of TM, where (M, g) is any connected Riemannian manifold. For a smooth curve $\gamma : [0, 1] \to M$, let $P_t : T_{\gamma(0)}M \to T_{\gamma(t)}M$ denote parallel transport [35, Section 3.3.2]. Then

$$\frac{d}{dt}\left(P_t^{-1} B_{\gamma(t)} P_t\right) = P_t^{-1}(\nabla_{\dot\gamma} B)P_t = 0. \tag{4.15}$$

Thus the eigenvalues and their multiplicities are constant.

Proposition 4.3 (Chern–do Carmo–Kobayashi and Lawson). *Let $M^n \hookrightarrow \mathbb{S}^{n+1}$ be closed, connected, and minimal. If $S \equiv n$, its image is congruent to a product* (1.4).

Proof. This is the codimension-one case of [16, Main theorem, p. 60]. □

Lemma 4.4. *Let $p, q \geq 1$ be integers with $p + q = n$, and let $r, t > 0$ satisfy $r^2 + t^2 = 1$. On $\mathbb{S}^p(r) \times \mathbb{S}^q(t) \subset \mathbb{S}^{n+1}$, write $X = (rx, ty)$ with $x \in \mathbb{S}^p(1)$ and $y \in \mathbb{S}^q(1)$, and choose $\nu = (tx, -ry)$. Then*

$$k = t/r, \qquad \ell = r/t, \qquad S - n - H(k - \ell) = 0. \tag{4.16}$$

If $H = 0$, then $r^2 = p/n$, $t^2 = q/n$, and $S = n$; equality holds in (1.3).

Proof. The principal curvatures are $\alpha = t/r$ and $\beta = -r/t$, with multiplicities p and q. For a distinct second point (rx', ty'), put $A_1 = 1 - \langle x, x' \rangle$ and $A_2 = 1 - \langle y, y' \rangle$. Then

$$c = \frac{rt(A_1 - A_2)}{r^2 A_1 + t^2 A_2}. \tag{4.17}$$

Since $A_1, A_2 \geq 0$ and $r^2 A_1 + t^2 A_2 > 0$, one has $\beta \leq c \leq \alpha$. Taking $y' = y$ and $x' \neq x$ gives $c = \alpha$; taking $x' = x$ and $y' \neq y$ gives $c = \beta$. Hence $k = \alpha$ and $\ell = -\beta$. Since $\alpha\beta = -1$,

$$H(k - \ell) = (p\alpha + q\beta)(\alpha + \beta) = p\alpha^2 + q\beta^2 - n = S - n.$$

Finally, $H = 0$ is equivalent to $pt^2 = qr^2$, which gives $r^2 = p/n$ and $S = n$. □

5. The quadratic Gaussian inequality

Let $Q > 0$ be self-adjoint, and retain (1.5)–(1.8). Let $X : M^n \hookrightarrow \mathbb{R}^{n+1}$ be complete, connected, properly embedded, and without boundary. Choose the normal from Lemma 3.1, and assume $H = \langle QX, \nu \rangle$.

Let X_t be a smooth variation equal to X outside a fixed compact set, with $X_0 = X$ and $\dot{X}_0 = f\nu$, where $f \in C_c^\infty(M)$. Put $g_t = X_t^* g_{\mathrm{Eucl}}$ and $J_t = dv_{g_t}/dv_g$. The relative weighted area

$$\mathcal{A}(t) = \int_M \left(e^{-F \circ X_t} J_t - e^{-F \circ X} \right) dv_g$$

is finite because the integrand has fixed compact support. The first-variation formula [14, arXiv v1, Proposition 2, (4), p. 6], with mean curvature vector $-H\nu$, gives

$$\mathcal{A}'(0) = \int_M (H - \langle QX, \nu \rangle) f \, d\mu_F. \tag{5.1}$$

Thus stationarity is equivalent to (1.7).

Lemma 5.1 (Cheng–Mejia–Zhou). *Let $P_T v = v - \langle v, \nu \rangle \nu$ be the orthogonal projection onto TM. Put $Z = P_T QX = \nabla(F \circ X)$ and $Q_T = P_T Q|_{TM}$. On symmetric two-tensors, set $\mathcal{L}_F = \mathscr{D} - \nabla_Z$. Identify self-adjoint endomorphisms with their symmetric bilinear forms. Then*

$$\mathcal{L}_F h = hQ_T + Q_T h - Uh, \tag{5.2}$$

$$\tfrac{1}{2}\mathcal{L}_F S = |\nabla h|^2 + 2\,\mathrm{tr}(h^2 Q_T) - US, \tag{5.3}$$

$$\mathcal{L}_F \nu = Q\nu - U\nu. \tag{5.4}$$

In (5.4), *$\mathcal{L}_F$ acts on the ambient components of ν.*

Proof. The ambient curvature vanishes, $\bar{D}^2 F = Q$, and $\bar{D}Q = 0$. In [15], take $f = F$; its operator Δ_f is $\mathcal{L}_F$ here. The tensor equation [15, arXiv v1, (23), p. 11] gives (5.2), and the scalar equation [15, arXiv v1, Theorem 3, p. 10] gives (5.3). For every constant ambient vector v, [15, arXiv v1, Proposition 2, (27), p. 12] gives

$$\mathcal{L}_F \langle \nu, v \rangle = \langle Qv, \nu \rangle - U\langle \nu, v \rangle.$$

Since Q is self-adjoint, this is (5.4). □

Let k, ℓ be the nonnegative Euclidean envelopes in (3.2). Assume that $X(M)$ is not a hyperplane. Then $w = k + \ell > 0$ by Lemma 3.3.

Proposition 5.2. *Let ϕ be a C^2 upper test for k at p. In an orthonormal principal basis $e_1, \dots, e_n$ at p, put $\phi_i = d\phi(e_i)$. Then*

$$\mathcal{L}_F\phi + (U - 2a)k \geq 2 \sum_{\kappa_i < k} \frac{\phi_i^2}{k - \kappa_i}. \tag{5.5}$$

If $k(p) = \kappa_i(p)$, then $\phi_i = 0$. If $k(p) = 0$, then $\nabla\phi(p) = 0$.

Proof. If $k(p) = 0$, then $\phi \geq k \geq 0$ near p and $\phi(p) = 0$. Hence $\nabla\phi(p) = 0$ and $\mathcal{L}_F\phi(p) = \mathscr{D}\phi(p) \geq 0$, which proves (5.5).

Suppose that $k(p) = \kappa_{\max}(p) > 0$. For a unit vector e_1 in the maximal eigenspace, (5.2) gives

$$(\mathcal{L}_F h)_{11} = (2\langle Qe_1, e_1\rangle - U)k. \tag{5.6}$$

Lemma 3.5, with $Z = (QX)^T$, gives the zero components and

$$\mathcal{L}_F\phi + (U - 2a)k \geq 2k(\langle Qe_1, e_1\rangle - a) + 2 \sum_{\kappa_i < k} \frac{\phi_i^2}{k - \kappa_i}. \tag{5.7}$$

Since $k > 0$ and $Q \geq a\mathrm{Id}$, this implies (5.5).

Suppose now that $k(p) > \max\{0, \kappa_{\max}(p)\}$. By Lemma 3.2, there is $q \neq p$ with $c(p, q) = k(p)$. Write

$$Y = X(q), \quad d = Y - X(p), \quad A = |d|^2/2, \quad b_i = \langle e_i, d\rangle, \quad H_x = H(p), \quad H_y = H(q). \tag{5.8}$$

Here $k = k(p)$, $\kappa_i = \kappa_i(p)$, and $H_i = dH_p(e_i)$. The function

$$\mathcal{Z}(x, y) = \tfrac{\phi(x)}{2}|X(y) - X(x)|^2 + \langle \nu(x), X(y) - X(x)\rangle \tag{5.9}$$

is nonnegative for x near p and every $y \in M$, and vanishes at (p, q). Its value and first derivatives in x give

$$\langle \nu(p), d\rangle = -kA, \qquad A\phi_i = (k - \kappa_i)b_i. \tag{5.10}$$

Its derivatives in y give $W := \nu(p) + kd \perp T_qM$, and (5.10) gives $|W| = 1$.

The open ball $\mathcal{B}$ of centre $X(p) - \nu(p)/k$ and radius $1/k$ is disjoint from $X(M)$. For all sufficiently small $t > 0$, one has $X(p) - t\nu(p) \in \mathcal{B} \cap \Omega_-$. Since $\mathcal{B}$ is connected, Lemma 3.1 gives $\mathcal{B} \subset \Omega_-$. At $Y \in \partial\mathcal{B}$, the gradient of $z \mapsto k|z - X(p)|^2/2 + \langle \nu(p), z - X(p)\rangle$ is W. Lemma 3.4 gives

$$\nu(q) = \nu(p) + kd. \tag{5.11}$$

By [4, Lemma 2.2], the reflected vectors

$$\widetilde{e}_i = e_i - \frac{b_i}{A}d \tag{5.12}$$

form an orthonormal basis of T_qM. Put $v_i = (e_i, \widetilde{e}_i)$. Then

$$\langle e_i, \widetilde{e}_j\rangle = \delta_{ij} - \frac{b_i b_j}{A}, \qquad \langle d, \widetilde{e}_i\rangle = -b_i.$$

The function $\Psi(x, y) = \phi(x) - c(x, y)$ has a minimum zero at (p, q), and $\mathrm{Hess}\,\mathcal{Z} = A\,\mathrm{Hess}\,\Psi$ there. The spatial derivative formulas [5, (8)–(10), p. 29], with unit chord $-d/|d|$, give

$$T := \sum_i \mathrm{Hess}_{M\times M}\,\mathcal{Z}(v_i, v_i) \geq 0, \tag{5.13}$$

$$T = A(\mathscr{D}\phi + Sk - H_x k^2) - 2\sum_i \phi_i b_i + \sum_i H_i b_i + H_x - H_y.$$

Put $Z_i = \langle QX(p), e_i\rangle$. Differentiating (1.7) at p gives

$$H_i = \langle Qe_i, \nu(p)\rangle + \kappa_i Z_i. \tag{5.14}$$

Stationarity and (5.11) give

$$H_y - H_x = \langle Qd, \nu(p)\rangle + k\langle QX(p), d\rangle + k\langle Qd, d\rangle, \tag{5.15}$$

$$\sum_i H_i b_i = \langle Qd, \nu(p)\rangle + kAq_\nu + \sum_i \kappa_i Z_i b_i. \tag{5.16}$$

Indeed, (5.10) gives

$$d = \sum_i b_i e_i - kA\nu(p), \qquad \langle QX(p), d\rangle = \sum_i Z_i b_i - kAH_x. \tag{5.17}$$

Subtracting (5.15) from (5.16) and applying (5.10) yields

$$\sum_i H_i b_i + H_x - H_y = -A\sum_i Z_i\phi_i + k^2AH_x + kAq_\nu - k\langle Qd, d\rangle. \tag{5.18}$$

Substitution in (5.13), with $e = d/|d|$, gives

$$\frac{T}{A} = \mathcal{L}_F\phi + (U - 2\langle Qe, e\rangle)k - 2\sum_i \frac{\phi_i^2}{k - \kappa_i}. \tag{5.19}$$

Hence

$$\mathcal{L}_F\phi + (U - 2a)k = \frac{T}{A} + 2k(\langle Qe, e\rangle - a) + 2\sum_i \frac{\phi_i^2}{k - \kappa_i}. \tag{5.20}$$

Here $T \geq 0$, $A > 0$, $k > 0$, and $\langle Qe, e\rangle \geq a$. All denominators are positive. This proves (5.5). □

Proposition 5.3. *For every hypersurface in this section whose image is not a hyperplane, put $V = U - 2a$ and $u = w^{-1}$. Then*

$$\mathcal{L}_F w + (U - 2a)w \geq 2w^{-1}|\nabla w|^2 \tag{5.21}$$

distributionally. The distribution $\mathfrak{m} = (\Delta_F + U - 2a)u$ is a nonnegative locally finite Radon measure, and

$$\int_M \left(|\nabla f|^2 + (U - 2a)f^2\right) d\mu_F = \int_M w^{-2}|\nabla(wf)|^2\, d\mu_F + \int_M wf^2\, d\mathfrak{m} \tag{5.22}$$

for every $f \in C_c^\infty(M)$.

Proof. The functions k, ℓ are locally semiconvex by Lemma 3.2. Normal reversal preserves S, q_ν, U and $\mathcal{L}_F$, and interchanges k, ℓ. At common Alexandrov points, Lemma 2.3 and Proposition 5.2 give (2.9) with $V = U - 2a$. A zero denominator has zero numerator by that proposition. Lemma 2.5 gives (5.21) as a measure inequality. The function $V = U - 2a$ is smooth, and continuity gives $\min_K w > 0$ for every nonempty compact $K \Subset M$. Proposition 2.4 therefore gives $\mathfrak{m}$ and (5.22). □

Quadratic-form assertion of Theorem 1.2. Equation (5.22) gives (1.9). On the complexification of $L^2(M, d\mu_F)$, let $T_0 = \Delta_F + U$ with domain $C_c^\infty(M; \mathbb{C})$. This domain is dense, and weighted integration by parts makes T_0 symmetric, with

$$q(f) := \langle T_0 f, f\rangle_{L^2(d\mu_F)} = \int_M \left(|\nabla f|^2 + U|f|^2\right) d\mu_F \geq 2a\|f\|^2_{L^2(d\mu_F)}. \tag{5.23}$$

The inequality follows by applying (1.9) to the real and imaginary parts of f. By [28, Chapter VI, Corollary 1.28], q is closable. Its closure has lower bound $2a$ and defines the Friedrichs extension T of T_0 [28, Chapter VI, Theorem 2.6 and §2.3]. Thus $\sigma(T) \subset [2a, \infty)$. Finally, $q_\nu \leq b$ gives, for real $f \in C_c^\infty(M)$,

$$\int_M (|\nabla f|^2 + Sf^2)\, d\mu_F \geq (2a - b)\int_M f^2\, d\mu_F. \tag{5.24}$$

□

6. Weighted exhaustion and rigidity

Let $Q > 0$ be self-adjoint, let $a = \lambda_{\min}(Q)$, and let $F(z) = \frac{1}{2}\langle Qz, z\rangle$. Let $X : M^n \hookrightarrow \mathbb{R}^{n+1}$, $n \geq 2$, be a connected complete smooth proper embedding without boundary satisfying (1.7).

Lemma 6.1. *The weighted volume of M is finite, and $F \circ X \in L^1(M, d\mu_F)$.*

Proof. Write F for $F \circ X$. Since $F \geq a|X|^2/2$ and X is proper, F is nonnegative and proper. The identities $\nabla F = (QX)^T$ and, for tangent vectors v, z, $\operatorname{Hess} F(v, z) = \langle Qv, z\rangle - Hh(v, z)$ give

$$\mathscr{D}F = \operatorname{tr} Q - q_\nu - H^2, \qquad |\nabla F|^2 = |QX|^2 - H^2, \tag{6.1}$$

$$\mathscr{L}_F F = \operatorname{tr} Q - q_\nu - |QX|^2 \leq C_0 - 2aF, \qquad C_0 = \operatorname{tr} Q - a. \tag{6.2}$$

Here $q_\nu \geq a$ and $Q^2 \geq aQ$. In particular,

$$\mathscr{D}F - \tfrac{1}{2}|\nabla F|^2 + aF = \operatorname{tr} Q - q_\nu - \tfrac{1}{2}H^2 - \tfrac{1}{2}|QX|^2 + aF \leq C_0, \qquad \mathscr{D}F \leq C_0. \tag{6.3}$$

If M is noncompact, Cheng–Vieira–Zhou [17, arXiv v1, Theorem 1.1(ii), p. 3], with $\alpha = \frac{1}{2}$, $\beta = a$ and upper bound C_0, gives $\int_M e^{-F/2}\, dv_g < \infty$. The same integral is finite when M is compact. The inequalities

$$e^{-F} \leq e^{-F/2}, \qquad Fe^{-F} \leq \frac{2}{e}\, e^{-F/2}$$

give both conclusions. □

Choose a smooth nonincreasing $\chi : [0, \infty) \to [0, 1]$, equal to one on $[0, 1]$ and zero on $[2, \infty)$. For $R > 0$, set

$$\chi_R = \chi(|X|/R), \qquad |\nabla \chi_R| \leq \|\chi'\|_\infty / R. \tag{6.4}$$

The functions χ_R are smooth also at $X = 0$, where they are locally constant. They have compact support, increase to one, and are eventually one on every compact set. Lemma 6.1 gives

$$\int_M |\nabla \chi_R|^2\, d\mu_F \leq R^{-2}\|\chi'\|_\infty^2 \mu_F(M) \longrightarrow 0. \tag{6.5}$$

Average inequality in Theorem 1.2. Suppose $X(M)$ is not a hyperplane, and let $w, \mathfrak{m}$ be as in Proposition 5.3. If $\int_M S\, d\mu_F = \infty$, then (1.10) follows from $a \leq q_\nu \leq b$ and $\mu_F(M) < \infty$. Otherwise $V = U - 2a \in L^1(M, d\mu_F)$. Lemma 2.6, applied to the sequence $(\chi_j)_{j \geq 1}$ from (6.4), gives

$$\int_M (U - 2a)\, d\mu_F = \int_M |\nabla \log w|^2\, d\mu_F + \int_M w\, d\mathfrak{m} \geq 0. \tag{6.6}$$

□

Lemma 6.2. *Suppose $X(M)$ is not a hyperplane. Equality in* (1.10) *implies $U = 2a$ and $\nabla h = 0$.*

Proof. Equality implies $S \in L^1(M, d\mu_F)$. Equation (6.6) therefore gives $\nabla \log w = 0$ almost everywhere and $\int_M w\, d\mathfrak{m} = 0$. Since $\log w$ is locally Lipschitz, Lemma 2.7 and connectedness give $w = c > 0$ for a constant c. Since $\int_M w\, d\mathfrak{m} = 0$, one has $\mathfrak{m} = 0$. Thus

$$0 = \mathfrak{m} = c^{-1}(U - 2a)\, d\mu_F.$$

Since U is smooth, $U = 2a$ everywhere. Thus

$$0 \leq S = 2a - q_\nu \leq a. \tag{6.7}$$

Equation (5.3) becomes

$$\tfrac{1}{2}\mathscr{L}_F S = |\nabla h|^2 + 2\operatorname{tr}\big(h^2(Q_T - a\mathrm{Id})\big) \geq |\nabla h|^2. \tag{6.8}$$

Indeed, an orthonormal eigenbasis (e_i) of Q_T, with eigenvalues $q_i \geq a$, gives

$$\operatorname{tr}(h^2(Q_T - a\mathrm{Id})) = \sum_i (q_i - a)|Be_i|^2 \geq 0. \tag{6.9}$$

For the density $e^{-F\circ X}$, the weighted Laplacian in [26, arXiv v1, Theorem 2.2, p. 7] is $\mathcal{L}_F$. Equations (6.4)–(6.5) give condition (iv) of that theorem, with the sequence $(\chi_j)_{j\geq 1}$. Thus every smooth function v bounded above with $\mathcal{L}_F v \geq 0$ is constant. Applied to S, this gives $\mathcal{L}_F S = 0$; equation (6.8) then gives $\nabla h = 0$. □

Proposition 6.3. *Let $X : M^n \hookrightarrow \mathbb{R}^{n+1}$ be a smooth proper embedding of a connected manifold without boundary. Assume that the induced metric is complete and that the shape operator is parallel. Then $X(M)$ is an affine hyperplane, or there are an integer $1 \leq p \leq n$, a linear subspace $E_0 \subset \mathbb{R}^{n+1}$ of dimension $n-p$, a point $c \in E_0^\perp$ and $r > 0$ such that*

$$X(M) = \{c + r\omega + z : \omega \in E_0^\perp,\ |\omega| = 1,\ z \in E_0\}.$$

In the nonhyperplane case, (1.7) *and $U = 2a$ imply* (1.11).

Proof. Equation (4.15) gives constant principal curvatures. By the Euclidean classification [11, §2, p. 3], the image is an open subset of an affine hyperplane, a hypersphere or a spherical cylinder. Properness makes the image closed in the ambient space, hence in that model. Each model is connected, so the image is the whole model.

In the nonhyperplane case, put $E_1 = E_0^\perp$. Equation (1.7) is unchanged by normal reversal. With the outward spherical normal ω, it reads

$$\frac{p}{r} = \langle Q(c + r\omega + z), \omega\rangle \qquad (\omega \in E_1,\ |\omega| = 1,\ z \in E_0). \tag{6.10}$$

Subtracting the case $z = 0$ from (6.10) gives $\langle Qz, \omega\rangle = 0$ for all $z \in E_0$ and all unit $\omega \in E_1$. Thus $Q(E_0) \subset E_0$; self-adjointness gives $Q(E_1) \subset E_1$. The equations at ω and $-\omega$ are

$$p/r = \langle Qc, \omega\rangle + r\langle Q\omega, \omega\rangle, \qquad p/r = -\langle Qc, \omega\rangle + r\langle Q\omega, \omega\rangle.$$

Subtracting gives $\langle Qc, \omega\rangle = 0$ on the unit sphere of E_1. Since $Qc \in E_1$, one has $Qc = 0$, and positive definiteness gives $c = 0$. Adding gives $\langle Q\omega, \omega\rangle = p/r^2$. Polarization yields $Q|_{E_1} = (p/r^2)\mathrm{Id}$. On the cylinder $S = q_\nu = p/r^2$. Therefore $U = 2a$ gives $r^2 = p/a$ and $E_1 \subset \ker(Q - a\mathrm{Id})$, as in (1.11). □

Equality in Theorem 1.2. Equality gives $\nabla h = 0$ and $U = 2a$ by Lemma 6.2. Since $X(M)$ is not a hyperplane, Proposition 6.3 gives (1.11). Conversely, let E and p be as in (1.11), and put $r = \sqrt{p/a}$. The product is connected, complete and properly embedded. Its outward principal curvatures are $1/r$ with multiplicity p and zero with multiplicity $n-p$. The subspaces $E, E^\perp$ are Q-invariant, and $Q|_E = a\mathrm{Id}$. Hence $H = p/r = ar = \langle QX, \nu\rangle$ and $S = q_\nu = a$. Lemma 6.1 gives finite weighted volume, so the product attains equality. If it is compact, $p = n$ and $E = \mathbb{R}^{n+1}$, whence $Q = a\mathrm{Id}$. □

If $h = 0$, Proposition 6.3 gives an affine hyperplane $X(M) = x_0 + \nu^\perp$ with constant unit normal ν. Stationarity gives

$$0 = \langle Q(x_0 + y), \nu\rangle \qquad (y \in \nu^\perp). \tag{6.11}$$

Subtracting the case $y = 0$ gives $\langle Qy, \nu\rangle = 0$ for every $y \in \nu^\perp$. By self-adjointness, $Q\nu = \beta\nu$, with $\beta = \langle Q\nu, \nu\rangle > 0$. At $y = 0$, equation (6.11) gives $\beta\langle x_0, \nu\rangle = 0$. Thus $X(M) = \nu^\perp$. On this hyperplane $k = \ell = 0$.

7. Spectral and graphical consequences

7.1. **Graphs in eigendirections.** For $Q = a\mathrm{Id}$, the dilation $X \mapsto \sqrt{2a}\,X$ reduces Theorem 1.3 to the hemisphere theorem of Ding–Xin–Yang [22, arXiv v1, Theorem 3.1, p. 8].

Proof of Theorem 1.3. Rescale v so that $|v| = 1$ and put $\theta = \langle \nu, v\rangle > 0$. Equation (5.4) and $Qv = \beta v$ give

$$\mathcal{L}_F \theta = (\beta - U)\theta. \tag{7.1}$$

Suppose that $X(M)$ is not a hyperplane. Equation (7.1) gives

$$\mathcal{L}_F(\theta^{-1}) + (\beta - U)\theta^{-1} = 2\theta^{-3}|\nabla\theta|^2.$$

For $w = \theta^{-1}$ and $V = \beta - U$, the hypothesis (2.5) holds with equality and $\mathfrak{m} = (\Delta_F + \beta - U)\theta = 0$. Proposition 2.4 therefore gives

$$\int_M \theta^2 |\nabla(f/\theta)|^2 \, d\mu_F = \int_M \left(|\nabla f|^2 + (\beta - U) f^2 \right) d\mu_F \qquad (f \in C_c^\infty(M)). \tag{7.2}$$

By Theorem 1.2,

$$0 \le \int_M \theta^2 |\nabla(f/\theta)|^2 \, d\mu_F \le 2 \int_M |\nabla f|^2 \, d\mu_F + (\beta - 2a) \int_M f^2 \, d\mu_F. \tag{7.3}$$

Take $f = \chi_R$ from (6.4). Its weighted Dirichlet energy tends to zero by (6.5). If $\beta < 2a$, the right side tends to $(\beta - 2a)\mu_F(M) < 0$, a contradiction. If $\beta = 2a$, then, for every compact $K \subset M$ and all sufficiently large R,

$$\int_K |\nabla \log \theta|^2 \, d\mu_F \le 2 \int_M |\nabla \chi_R|^2 \, d\mu_F \longrightarrow 0.$$

Thus θ is constant by connectedness, and (7.1) gives $U = 2a$. Since $\mu_F(M) < \infty$, equality holds in (1.10). Theorem 1.2 then gives (1.11); in particular, $\nu(x) \in \ker(Q - a\mathrm{Id})$ for every $x \in M$. Since $v \in \ker(Q - 2a\mathrm{Id})$ and $a > 0$, self-adjointness of Q gives $\langle \nu, v \rangle = 0$, a contradiction. Hence $X(M)$ is a hyperplane, and (6.11) places it through the origin. □

Corollary 7.1. *Let $n \ge 2$, let Q satisfy* (1.5)*, and let v be a unit eigenvector of Q with eigenvalue $\beta \le 2a$. Let $f \in C^\infty(v^\perp)$. If the graph*

$$X(y) = y + f(y)v, \qquad y \in v^\perp,$$

satisfies (1.7)*, its image is a hyperplane through the origin.*

Proof. The graph is diffeomorphic to $v^\perp$ and is properly embedded by [30, Proposition 5.7, p. 101]. Since the ambient Euclidean metric is complete, its induced metric is complete by [30, Problem 13-18(b), p. 346]. With ∇f denoting the Euclidean gradient on $v^\perp$, choose the unit normal

$$\nu = \frac{v - \nabla f}{\sqrt{1 + |\nabla f|^2}}, \qquad \langle \nu, v \rangle = \frac{1}{\sqrt{1 + |\nabla f|^2}} > 0.$$

Theorem 1.3 gives the conclusion. □

7.2. **Compact spectral comparison.** Let (M, g) be a closed connected smooth Riemannian manifold of positive dimension, let $F \in C^\infty(M)$, and put

$$d\mu_F = e^{-F} dv_g, \qquad \Delta_F = -\operatorname{div} \nabla + \langle \nabla F, \nabla \rangle.$$

For $P \in L^\infty(M; \mathbb{R})$, the real quadratic form

$$\mathfrak{q}_P(f) = \int_M \left(|\nabla f|^2 + P f^2 \right) d\mu_F, \qquad f \in H^1(M),$$

is densely defined, closed and bounded below in $L^2(M, d\mu_F)$. Denote its associated self-adjoint operator by $\Delta_F + P$ [28, Chapter VI, Theorem 2.6, p. 323]. The weight has positive upper and lower bounds. Rellich's theorem therefore gives the compact embedding $H^1(M) \hookrightarrow L^2(M, d\mu_F)$, and these operators have compact resolvent. Let

$$0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \cdots$$

be the eigenvalues of Δ_F, counted with multiplicity.

Proposition 7.2. *Let $c \in \mathbb{R}$, let $P \in L^\infty(M; \mathbb{R})$, and suppose that a positive Lipschitz function w satisfies*

$$\int_M \left(|\nabla f|^2 + (P - c) f^2 \right) d\mu_F \ge \int_M w^{-2} |\nabla(wf)|^2 \, d\mu_F \qquad (f \in H^1(M)). \tag{7.4}$$

Let ζ_j be the eigenvalues of $\Delta_F + P$, counted with multiplicity in nondecreasing order. Then

$$\zeta_j \ge c + \vartheta \lambda_j \quad (j \ge 0), \qquad \vartheta = \left(\frac{\min_M w}{\max_M w} \right)^2. \tag{7.5}$$

Proof. Put $u = w^{-1}$. By Lemma 2.7, multiplication by u is an automorphism of $H^1(M)$. For $f = uv \neq 0$, (7.4) gives

$$\frac{\int_M (|\nabla f|^2 + (P-c)f^2)\, d\mu_F}{\int_M f^2\, d\mu_F} \geq \frac{\int_M u^2|\nabla v|^2\, d\mu_F}{\int_M u^2 v^2\, d\mu_F} \geq \vartheta\, \frac{\int_M |\nabla v|^2\, d\mu_F}{\int_M v^2\, d\mu_F}. \tag{7.6}$$

The map $E \mapsto u^{-1}E$ is a bijection on the set of $(j+1)$-dimensional subspaces of $H^1(M)$. Taking the supremum over $0 \neq f \in E$ and then the infimum over these subspaces gives (7.5) by the form min–max principle [28, Chapter VII, §6.4, Remark 6.2, p. 423]. □

On each closed hypersurface of Theorem 1.1 or 1.2, put $w = k + \ell$ and $\vartheta = (\min_M w / \max_M w)^2$. The function w is positive and Lipschitz by Lemmas 3.2 and 3.3. The nonnegative measure terms in (4.11), with $H = 0$, and (5.22), respectively, give (7.4) for smooth f. Both sides of this inequality are continuous on $H^1(M)$ by Lemma 2.7; density of smooth functions extends it to $H^1(M)$. Taking $(F, P, c) = (0, S, n)$ in the spherical case and $(P, c) = (U, 2a)$ in the quadratic case, with the density (1.8), gives

$$\zeta_j(\Delta + S) \geq n + \vartheta\lambda_j(\Delta), \qquad \zeta_j(\Delta_F + U) \geq 2a + \vartheta\lambda_j(\Delta_F). \tag{7.7}$$

On a minimal Clifford hypersurface, w is constant and $S = n$ by Lemma 4.4. On the compact equality model (1.11), the outward normal gives $k = \sqrt{a/n}$, $\ell = 0$, and $U = 2a$. Thus $\vartheta = 1$ and equality holds for every j in the respective comparisons.

7.3. **The embedded low-index classification.** For a closed two-sided minimal hypersurface of $\mathbb{S}^{n+1}$, its Morse index $\mathrm{ind}(M)$ is the number of negative eigenvalues, counted with multiplicity, of

$$J = \Delta - S - n. \tag{7.8}$$

Perdomo's theorem [36, Theorem 3.1, p. 564], with ambient sphere dimension $N = n + 1$, gives

$$\mathrm{ind}(M) = n + 3 \quad \Longrightarrow \quad \int_M S\, dv_g \leq n\,\mathrm{Vol}(M) \tag{7.9}$$

for every closed oriented minimally immersed $M^n \to \mathbb{S}^{n+1}$.

Corollary 7.3. *Let $n \geq 2$. A closed connected embedded minimal hypersurface of $\mathbb{S}^{n+1}$ has Morse index $n + 3$ if and only if it is a minimal Clifford hypersurface.*

Proof. By Lemma 3.1, M is two-sided and oriented. The equator has index 1 [36, p. 560]. Suppose that $\mathrm{ind}(M) = n + 3$. Then M is not totally geodesic, and (7.9) and Theorem 1.1 give

$$n\,\mathrm{Vol}(M) \leq \int_M S\, dv_g \leq n\,\mathrm{Vol}(M).$$

The equality case of Theorem 1.1 gives the conclusion. Conversely, every minimal Clifford hypersurface has index $n + 3$ [36, p. 560]. □

7.4. **Self-shrinkers and entropy.** Set $Q = \frac{1}{2}\mathrm{Id}$. For an immersion $X : M^n \to \mathbb{R}^{n+1}$, put

$$d\gamma = e^{-|X|^2/4}\, dv_g, \qquad \gamma(M) = \int_M d\gamma, \qquad \Delta_\gamma = -\mathscr{D} + \tfrac{1}{2}\langle X^\top, \nabla\rangle.$$

Stationarity becomes $H = \frac{1}{2}\langle X, \nu\rangle$.

Corollary 7.4. *Let $M^n \subset \mathbb{R}^{n+1}$, $n \geq 2$, be a connected complete smooth properly embedded self-shrinker without boundary, other than a hyperplane. Then $\gamma(M) < \infty$,*

$$\int_M \left(|\nabla f|^2 + Sf^2\right) d\gamma \geq \frac{1}{2}\int_M f^2\, d\gamma \qquad (f \in C_c^\infty(M)), \tag{7.10}$$

and

$$\int_M S\, d\gamma \geq \frac{1}{2}\gamma(M). \tag{7.11}$$

Equality in (7.11) *holds precisely for $\mathbb{S}^p(\sqrt{2p}) \times \mathbb{R}^{n-p}$, $1 \leq p \leq n$, up to an orthogonal transformation.*

Proof. Here $F = |X|^2/4$, $a = q_v = 1/2$ and $U = S + 1/2$. Lemma 6.1 gives $\gamma(M) < \infty$. Substitution in (1.9) and (1.10) gives the two inequalities. Since $\ker(Q - a\mathrm{Id}) = \mathbb{R}^{n+1}$, (1.11) gives the stated cylinders. Among them, precisely the model with $p = n$ is compact. □

For complete immersed self-shrinkers, properness, polynomial volume growth and finiteness of $\gamma(M)$ are equivalent by Cheng–Zhou [18, Theorem 1.3].

Lemma 7.5 (Colding–Minicozzi). *Let $X : M^n \to \mathbb{R}^{n+1}$, $n \geq 2$, be a smooth complete immersion without boundary. Suppose that, for every local unit normal ν, the mean curvature H satisfies $H = \langle X, \nu\rangle/2$. Assume that there are constants $C > 0$ and $d \geq 0$ such that*

$$\mathrm{Vol}_g\big(\{p \in M : |X(p)| < R\}\big) \leq C(1 + R)^d \qquad (R > 0).$$

For $y \in \mathbb{R}^{n+1}$ and $t > 0$, define

$$\mathcal{F}_{y,t}(M) = (4\pi t)^{-n/2} \int_M e^{-|X-y|^2/(4t)}\, dv_g, \qquad \Lambda(M) = \sup_{y\in\mathbb{R}^{n+1},\, t>0} \mathcal{F}_{y,t}(M). \tag{7.12}$$

Then

$$\Lambda(M) = \mathcal{F}_{0,1}(M) = (4\pi)^{-n/2}\gamma(M) < \infty. \tag{7.13}$$

Proof. Cheng–Zhou's theorem gives $\gamma(M) < \infty$. The comparison in [19, proof of Lemma 7.10, (7.13)–(7.27), pp. 786–788] gives $\mathcal{F}_{y,t}(M) \leq \mathcal{F}_{0,1}(M)$ for every $y \in \mathbb{R}^{n+1}$ and $t > 0$. Taking the supremum, with equality at $(y, t) = (0, 1)$, gives (7.13). □

For M as in Corollary 7.4, Cheng–Zhou's theorem supplies polynomial volume growth. Thus (7.11) is equivalent to

$$\int_M S e^{-|X|^2/4}\, dv_g \geq \frac{1}{2}(4\pi)^{n/2}\Lambda(M), \tag{7.14}$$

with equality precisely for the cylinders of Corollary 7.4, and for $\mathbb{S}^n(\sqrt{2n})$ in the closed case. These are the inequalities and equality cases of Li–Zhao [32, arXiv v1, Conjectures 1.6–1.7, p. 4], in dimensions $n \geq 2$.

7.5. **The operator on a punctured cone.** Let $L^n \subset \mathbb{S}^{n+1}$ be a smooth minimal hypersurface. Let g_L be its induced metric, let S_L be the squared norm of its second fundamental form in $\mathbb{S}^{n+1}$, and put $dv_L = dv_{g_L}$ and $\Delta_L = -\operatorname{div}_{g_L} \nabla_{g_L}$. On $C = (0, \infty) \times L$, with $X_C(r, p) = rp \in \mathbb{R}^{n+2}$, let g_C be the induced metric and let S_C be the squared norm of the Euclidean second fundamental form of X_C. Put $d\gamma_C = e^{-|X_C|^2/4} dv_{g_C}$. Then

$$g_C = dr^2 + r^2 g_L, \qquad d\gamma_C = e^{-r^2/4} r^n\, dr\, dv_L, \qquad S_C = r^{-2} S_L.$$

The weighted Laplacian is

$$\Delta_{\gamma,C} := -\operatorname{div}_{g_C} \nabla_{g_C} + \frac{r}{2}\partial_r = -\partial_r^2 - \frac{n}{r}\partial_r + r^{-2}\Delta_L + \frac{r}{2}\partial_r.$$

For $v \in C^\infty(L)$, regard rv as the function $(r, p) \mapsto rv(p)$ on C. Then

$$(\Delta_{\gamma,C} + S_C - \tfrac{1}{2})(rv) = r^{-1}(\Delta_L + S_L - n)v \qquad (r > 0). \tag{7.15}$$

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