INTEGRABILITY OF THE HORIZONTAL MEAN CURVATURE NEAR CHARACTERISTIC POINTS IN THE HEISENBERG GROUP

ANDREA PINAMONTI AND TOMMASO ROSSI

Abstract. We investigate the local integrability of the horizontal mean curvature near characteristic points of surfaces in the first Heisenberg group. We prove that, for every embedded surface of class $C^{2,\alpha}$, with $\alpha > 0$, the horizontal mean curvature is locally integrable with respect to (any) Riemannian surface measure, without any assumption on the structure of the characteristic set. We further establish that the horizontal unit normal belongs to BV_{loc} and characterize its $W^{1,1}_{\mathrm{loc}}$ regularity in terms of the vanishing one-dimensional Hausdorff measure of the characteristic set. Finally, we answer an open question by Danielli, Garofalo and Nhieu by constructing a C^2 surface with a unique characteristic point whose horizontal mean curvature fails to be locally integrable, thus showing that the additional Hölder regularity cannot, in general, be replaced by C^2 regularity.

1. Introduction

The horizontal mean curvature is one of the basic geometric quantities associated with hypersurfaces in sub-Riemannian manifolds. It enters the first-variation theory for horizontal perimeter and, in particular, the study of minimal and constant-mean-curvature surfaces [9, 10, 13, 14, 16, 17, 18] and references therein. It also appears in problems of a rather different nature, including short-time heat content asymptotics [1, 15, 19, 21] and tube-volume expansions [3, 5, 7, 8]. A recurring difficulty in these questions is that the natural curvature quantities, although well defined away from characteristic points, need not remain bounded as one approaches the characteristic set. Understanding their integrability is therefore a preliminary step in extending geometric identities beyond the non-characteristic setting.

In this paper, we focus on embedded surfaces in the first Heisenberg group $\mathbb{H}$. Recall that a point p of a surface S is *characteristic* if the horizontal distribution coincides with the tangent plane T_pS. We denote the set of such points by $C(S)$. If f is a local defining function near $p \in S$, namely there is an open neighborhood $\mathcal{U}$ of p such that $f|_{\mathcal{U}}$ is a submersion and $S \cap \mathcal{U} = f^{-1}(0) \cap \mathcal{U}$, then the horizontal unit normal on $\mathcal{U} \cap (S \setminus C(S))$ is

$$\nu := \frac{\nabla_H f}{\|\nabla_H f\|},$$

where $\nabla_H f$ denotes the horizontal gradient of f, and its divergence defines the horizontal mean curvature $\mathcal{H}$, up to a sign. The normalization is precisely where the difficulty arises: the denominator vanishes on $C(S)$, even if f and S are smooth and we do not have an immediate integrable bound on the derivatives of ν.

It is important here to distinguish the two measures naturally associated with S. For a compatible Riemannian extension of the horizontal metric, the horizontal perimeter measure σ_H has, away from characteristic points, the form $\sigma_H = W\,\sigma_R$, where W is the length of the horizontal projection of the Riemannian unit normal and σ_R is the induced Riemannian surface measure. The factor W compensates for the singular normalization and thus $\mathcal{H}$ is locally integrable with respect to σ_H, see [11, 20]. No analogous cancellation is available when σ_H is replaced by σ_R. The latter problem is more delicate, and its answer depends both on the regularity of the surface and on how its horizontal normal vanishes.

This distinction was emphasized by Danielli, Garofalo and Nhieu [11]. They constructed a C^2 surface in $\mathbb{H}$, with $C(S)$ being a line and whose horizontal mean curvature is not locally

integrable with respect to Riemannian surface measure, while establishing integrability in several examples with isolated characteristic points. They conjectured that, if S is C^2 and $C(S)$ consists of isolated points, $\mathcal{H}$ is locally integrable. Subsequent results identified geometric situations in which the answer is positive: nondegenerate characteristic points, certain mildly degenerate points, and, in particular, isolated characteristic points of real-analytic surfaces [20]. More recent work extends these finite-order conditions to the framework of 3D contact manifolds [4]; see also [5, 6] for related developments. These results show how the geometry of characteristic points may play a role in the integrability of $\mathcal{H}$, but they leave open a different issue: *can one obtain a general integrability theorem without prescribing the geometry, dimension, or local structure of $C(S)$?*

The purpose of this paper is to address that issue by replacing assumptions on the characteristic set with a quantitative regularity assumption on the surface. Our first main result asserts that *any positive Hölder regularity beyond C^2 is sufficient.*

Theorem 1.1. *Let $0 < \alpha \leq 1$, and let $S \subset \mathbb{H}^1$ be an embedded surface without boundary of class $C^{2,\alpha}$. Then*

$$\mathcal{H} \in L^1_{\mathrm{loc}}(S, \sigma_R),$$

for every Riemannian surface measure σ_R induced by a smooth ambient Riemannian metric.

Note that the latter statement includes noncompact and nonorientable surfaces. In fact, the proof yields information that is strictly finer than the integrability of the horizontal mean curvature. Since $\mathcal{H}$ is a trace of derivatives of the horizontal unit normal, it is natural to ask whether that normal admits a weak derivative across the characteristic set. The answer involves a distinction between bounded variation and Sobolev regularity. The former holds for every $C^{2,\alpha}$ surface, whereas the latter is governed exactly by the one-dimensional size of the characteristic set.

Theorem 1.2. *Let $0 < \alpha \leq 1$, and let $S \subset \mathbb{H}^1$ be an embedded surface without boundary of class $C^{2,\alpha}$. For every $p \in S$, let $\mathcal{U} \subset \mathbb{H}$ be an open neighborhood of p and $f \in C^{2,\alpha}(\mathcal{U})$ a defining function near p. Then,*

$$\nu \in \mathrm{BV}_{\mathrm{loc}}(S \cap \mathcal{U}; \mathbb{R}^3).$$

Moreover,

$$\nu \in W^{1,1}_{\mathrm{loc}}(S \cap \mathcal{U}; \mathbb{R}^3) \quad \Longleftrightarrow \quad \mathcal{H}^1(C(S) \cap \mathcal{U}) = 0.$$

In the above, the functional spaces and the one-dimensional Hausdorff measure $\mathcal{H}^1$ are those defined using any smooth Riemannian metric on S. In particular, the statement is independent of the chosen Riemannian metric.

Theorem 1.2 therefore identifies the precise obstruction to $W^{1,1}$ regularity: a characteristic set of positive length produces a nontrivial jump in the normalized horizontal normal. In particular, *the horizontal mean curvature may be integrable even though the horizontal unit normal is not a Sobolev map.* This separation between integrability of a curvature quantity and weak differentiability of the normal is one of the main features of the sub-Riemannian situation considered here. Additionally, as a byproduct of the proof, we obtain that the distributional derivative of ν has vanishing Cantor part.

A second important aspect is the regularity threshold. The Hölder assumption in Theorem 1.1 is sharp and cannot simply be dropped. Indeed, in Section 7 we construct an explicit C^2 surface with *exactly one characteristic point* for which

$$\mathcal{H} \notin L^1_{\mathrm{loc}}(S, \sigma_R).$$

Consequently, the conjecture formulated in [11] for arbitrary C^2 surfaces with isolated characteristic points fails in general. Combined with Theorem 1.1, this example gives a sharp distinction between the C^2 class and the classes $C^{2,\alpha}$, $\alpha > 0$.

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2. Preliminaries

Consider the Lie group $(\mathbb{R}^3, \star)$, where the group law is

$$(x,y,t)\star(x',y',t') = \left(x+x', y+y', t+t'+\frac{1}{2}(xy'-yx')\right), \qquad (x,y,t),(x',y',t')\in\mathbb{R}^3. \tag{1}$$

We endow this Lie group with the sub-Riemannian structure $(\mathcal{D}, g)$, where $\mathcal{D} = \operatorname{span}\{X_1, X_2\}$, with X_1, X_2 left-invariant vector fields defined as

$$X_1 = \partial_x - \frac{y}{2}\partial_t, \qquad X_2 = \partial_y + \frac{x}{2}\partial_t, \tag{2}$$

and g is obtained by declaring X_1, X_2 orthonormal. The resulting sub-Riemannian manifold is the first *Heisenberg group*, denoted with $\mathbb{H}$. We say that a curve $\gamma : [0,T] \to \mathbb{H}$ is *horizontal* if it is absolutely continuous and for a.e. $t \in [0,T]$,

$$\dot\gamma(t) \in \mathcal{D}_{\gamma(t)}.$$

For such curves we define the length as $\ell(\gamma) = \int_0^T \|\dot\gamma(t)\|_g dt$. The *sub-Riemannian distance* d_{SR} between two points $p, q \in \mathbb{H}$ is defined as

$$d_{SR}(p,q) = \inf\{\ell(\gamma) : \gamma \in \mathcal{C}_{p,q}\} \tag{3}$$

where $\mathcal{C}_{p,q}$ denotes the set of horizontal curves connecting p, q. By Chow-Rashevsky theorem, d_{SR} is finite, continuous and induces the topology of $\mathbb{R}^3$.

Characteristic points. Let $S \subset \mathbb{H}$ be a C^1 embedded hypersurface. We say that $p \in S$ is a *characteristic point* if

$$\mathcal{D}_p = T_pS. \tag{4}$$

We denote by $C(S) := \{p \in S : \mathcal{D}_p = T_pS\}$ the set of characteristic points. Note that $C(S) \subset S$ is a closed set. Given a surface $S \subset \mathbb{H}$ and a point $p \in S$, we say that $f \in C^1(\mathbb{H})$ is a locally defining function for S near p if there exists an open neighborhood $\mathcal{U} \subset \mathbb{H}$ of p such that $f|_{\mathcal{U}}$ is a submersion and $S \cap \mathcal{U} = f^{-1}(0) \cap \mathcal{U}$. In such case, defining the horizontal gradient of $f \in C^1(\mathbb{H})$ as the horizontal (i.e. tangent to $\mathcal{D}$) vector field

$$\nabla_H f := X_1 f X_1 + X_2 f X_2, \tag{5}$$

we have that $p \in C(S)$ if and only if

$$\nabla_H f(p) = 0 \qquad \Longleftrightarrow \qquad X_1 f(p) = X_2 f(p) = 0. \tag{6}$$

In particular, outside $C(S)$, we may define the (local) *horizontal unit normal* to S as

$$\nu(q) := \frac{\nabla_H f(q)}{\|\nabla_H f(q)\|}, \qquad \forall\, q \in (S \setminus C(S)) \cap \mathcal{U}. \tag{7}$$

Note that, by (6), ν is well-defined outside $C(S)$. In addition, from its definition, it follows that ν is transversal to S and $g(\nu, X)|_q = 0$ with $X \in \mathcal{D}_q \cap T_qS$, for every $q \in (S \setminus C(S)) \cap \mathcal{U}$.

Perimeter measures. Let $p \in S$ and $f \in C^1(\mathbb{H})$ be a local defining function for S near p. Let ν be the horizontal unit normal to S. Then the *horizontal perimeter measure* σ_H on S is the smooth measure with density on $\mathcal{U}$ given by $|\iota_\nu \mathcal{L}^3|$, where ι is the contraction of forms. Analogously, to define the *Riemannian perimeter measure* σ_{R}, consider any Riemannian metric on $\mathbb{R}^3$, then replace ν with the Riemannian unit normal, which is given by (7) defined with the Riemannian gradient. Note that σ_{R} coincides with the 2-dimensional Hausdorff measure on S induced by the Riemannian structure.

By the results of [2], $\sigma_H(C(S)) = 0$, and if S is of class C^2, $\sigma_R(C(S)) = 0$ for any Riemannian perimeter measure. We refer to [2, 12] for further results about the size of $C(S)$.

Horizontal mean curvature. Let $S \subset \mathbb{H}$ be a C^2 embedded surface. Recall that the divergence of a C^1 vector field $X \in \mathrm{Vec}(T\mathbb{H})$ is defined by the identity

$$\mathrm{div}(X)\, \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}t = \mathcal{L}_X(\, \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}t), \tag{8}$$

where $\mathcal{L}_X$ denotes the Lie derivative in the direction of X. Then, the *horizontal mean curvature* at $q \in S \setminus C(S)$ is defined as

$$\mathcal{H}(q) := \mathrm{div}(\nu)|_q = \mathrm{div}\left(\frac{\nabla_H f}{\|\nabla_H f\|}\right)\Big|_q = X_1\left(\frac{X_1 f}{\|\nabla_H f\|}\right)\Big|_q + X_2\left(\frac{X_2 f}{\|\nabla_H f\|}\right)\Big|_q, \tag{9}$$

where $f \in C^2(\mathbb{H})$ is a local defining function for S near q. Note that, up to a sign, the value of $\mathcal{H}$ does not depend on the choice of f. In addition, $\mathcal{H} \in C(S \setminus C(S))$ and locally integrable on every compact set $K \subset S$ satisfying $K \cap C(S) = \emptyset$. However, $\mathcal{H}$ is not well-defined at characteristic points and it may blow up as p approaches $C(S)$. In the sequel, we study the integrability of $\mathcal{H}$ with respect to a Riemannian perimeter measure on S. We stress that the choice of the Riemannian metric does not play any role concerning the integrability problem.

3. A change of variable

Let $S \subset \mathbb{H}$ be a surface of class $C^{2,\alpha}$, where $\alpha \in (0, 1]$ and let $p \in C(S)$. Up to a left-translation, which is an isometry of $\mathbb{H}$, we may assume that $p = (0, 0, 0)$. Then, a local defining function f for S near p must be the t-graph of a function $g \in C^{2,\alpha}(\mathbb{R}^2)$, i.e. $f(x, y, t) := t - g(x, y)$. Hence, there is a neighborhood $\mathcal{U}$ of p such that

$$S \cap \mathcal{U} = \{(x, y, t) \in \mathbb{H} : t = g(x, y)\} \cap \mathcal{U}, \qquad g(0) = g_x(0) = g_y(0) = 0.$$

Note that, since $f(x, y, t) = t - g(x, y)$, its horizontal gradient is

$$\nabla_H f(q) = \left(-g_x - \frac{y}{2}\right) X_1(q) + \left(-g_y + \frac{x}{2}\right) X_2(q), \qquad \forall q = (x, y, t) \in \mathbb{H}, \tag{10}$$

namely $X_1 f$ and $X_2 f$ do not depend on t. Thus, we can reduce the problem of integrability of $\mathcal{H}$ to a problem in $\mathbb{R}^2$. More precisely, define the vector field $V \in C^{1,\alpha}(\mathbb{R}^2; \mathbb{R}^2)$ by

$$V(x, y) := \left(-g_x - \frac{y}{2}, -g_y + \frac{x}{2}\right), \qquad \forall\, (x, y) \in \mathbb{R}^2, \tag{11}$$

and, setting $\mathcal{N} := \{(x, y) \in \mathbb{R}^2 : V(x, y) \neq 0\}$, define the vector field $N \in C^{1,\alpha}_{loc}(\mathcal{N}; \mathbb{R}^2)$ as

$$N(x, y) := \frac{V(x, y)}{|V(x, y)|}, \qquad \forall\, (x, y) \in \mathcal{N}, \tag{12}$$

where $|\cdot|$ denotes the standard Euclidean norm of $\mathbb{R}^2$. From (10), we see that $|V| = \|\nabla_H f\|$ and $C(S) \cap \mathcal{U} = \{(x, y, t) \in S \cap \mathcal{U} : V(x, y) = 0\}$, hence, in particular, $V(0) = 0$. Moreover, the (Euclidean) differential of V is

$$DV(0) = \begin{pmatrix} -g_{xx}(0) & -g_{xy}(0) - \frac{1}{2} \\ -g_{xy}(0) + \frac{1}{2} & -g_{yy}(0) \end{pmatrix},$$

which is always non-zero, since by the commutator relation $\partial_x V_2 - \partial_y V_1 = [X_1, X_2] f = 1$. This means that $\mathrm{rk}(DV(0)) \geq 1$.

Remark 3.1. Given $p = (0,0,0) \in C(S)$, if the rank of $DV(0)$ is maximal, p is a *non-degenerate* (and thus isolated) characteristic point. If instead the rank of $DV(0)$ is 1, then p is a *degenerate* characteristic point. We refer to [6, 4, 5] for further details around degenerate and non-degenerate characteristic points. The integrability problem around non-degenerate characteristic points is well-understood, see e.g. [20]. Around degenerate characteristic points, only partial results were known, see e.g. [20, 4].

Using the identity $|V| = \|\nabla_H f\|$, we deduce that the horizontal unit normal to S satisfies

$$\nu(x,y,t) = N_1(x,y)X_1(x,y,t) + N_2(x,y)X_2(x,y,t), \qquad \forall\,(x,y,t) \in (S \setminus C(S)) \cap \mathcal{U}, \quad (13)$$

where $N = (N_1, N_2)$. Thus, by its definition (9), $\mathcal{H}$ is independent of t and it holds

$$\mathcal{H}(x,y,t) = \mathrm{Tr}(DN(x,y)) = \partial_x N_1(x,y) + \partial_y N_2(x,y), \qquad \forall\,(x,y,t) \in (S \setminus C(S)) \cap \mathcal{U}. \quad (14)$$

Here DN is the classical derivative of N and it is well-defined whenever $V \neq 0$. This proves the following lemma.

Lemma 3.2. *Let $S \subset \mathbb{H}$ be a surface of class $C^{2,\alpha}$, where $\alpha \in (0,1]$ and let $p = (0,0,0) \in C(S)$. Let $f(x,y,t) = t - g(x,y)$ be a local defining $C^{2,\alpha}$-function for S on a neighborhood $\mathcal{U}$ of p. Then, there exists a constant $C > 0$ such that*

$$|\mathcal{H}(x,y,t)| \leq C|DN(x,y)|, \qquad \forall\,(x,y,t) \in (S \setminus C(S)) \cap \mathcal{U}$$

where N is defined in (12) and $|DN|$ denotes any (sub-multiplicative) matrix norm of DN.

The integrability of $\mathcal{H}$ (with respect to a Riemannian perimeter measure) is therefore implied by the integrability of $|DN|$ (with respect to the standard Lebesgue measure on $\mathbb{R}^2$). The next key lemma says that the integrability of the latter matrix norm is controlled by the behavior of the vector field $\widetilde{V}(u,v) = (u, b(u,v))$ for some function $b \in C^{1,\alpha}$.

Lemma 3.3. *Under the assumptions of Lemma 3.2, there exist an orthogonal transformation $R \in O(2)$ and a map $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ and an open, relatively compact neighborhood of the origin $U \subset \mathbb{R}^2$ such that*

(i) $\Phi(0) = 0$ and $\Phi|_U : U \to \Phi(U)$ is a $C^{1,\alpha}$-diffeomorphism. Moreover, defining $\widetilde{V}(u,v) := RV(\Phi^{-1}(u,v))$, for every $(u,v) \in \Phi(U)$, we have

$$\widetilde{V}(u,v) = (u, b(u,v)),$$

for some $b \in C^{1,\alpha}(\mathbb{R}^2)$, with $b(0) = 0$;

(ii) there is a constant $C > 0$ such that

$$\int_U |DN|\,\mathrm{d}x\,\mathrm{d}y \leq C \int_{\Phi(U)} |D\widetilde{N}|\,\mathrm{d}u\,\mathrm{d}v, \qquad \widetilde{N} := \frac{\widetilde{V}}{|\widetilde{V}|}.$$

Proof. (i) Since $\mathrm{rk}(DV(0)) \geq 1$, the image of $DV(0)$ is at least one-dimensional. Let $w \in \mathrm{Im}(DV(0))$, $|w| = 1$, and choose $R \in O(2)$ such that $Rw = e_1$. Thus, defining $Z := RV \in C^{1,\alpha}(\mathbb{R}^2;\mathbb{R}^2)$, the image of $DZ(0)$ contains e_1 and, after possibly applying the smooth diffeomorphism $(x,y) \mapsto (y,x)$, we may assume that $\partial_x Z_1(0) \neq 0$. Hence, the map

$$\Phi : \mathbb{R}^2 \to \mathbb{R}^2, \qquad \Phi(x,y) = (Z_1(x,y), y)$$

is of $C^{1,\alpha}$ class, fixes the origin and $D\Phi(0)$ is invertible, since

$$D\Phi(0) = \begin{pmatrix} \partial_x Z_1(0) & \partial_y Z_1(0) \\ 0 & 1 \end{pmatrix}.$$

As a consequence, there is a relatively compact neighborhood $U \subset \mathbb{R}^2$ of the origin such that $\Phi|_U$ defines a $C^{1,\alpha}$-diffeomorphism onto its image. Hence, letting $(u,v) = \Phi(x,y) \in \Phi(U)$, we explicitly compute

$$\widetilde{V}(u,v) = \widetilde{V}(\Phi(x,y)) = Z(x,y) = (Z_1(x,y), Z_2(x,y)) = (u, b(u,v)),$$

where $b \in C^{1,\alpha}(\mathbb{R}^2)$. Moreover, since $p = (0,0,0) \in C(S)$, $RV(0) = Z(0) = 0$, meaning that $b(0) = 0$, as claimed.

(*ii*) Note that $R \in O(2)$, $|Z| = |RV| = |V|$. Hence, it holds

$$\widetilde{N} \circ \Phi(x,y) = \frac{Z(x,y)}{|Z(x,y)|} = RN(x,y), \qquad \forall\, (x,y) \in U$$

or equivalently, $\widetilde{N} = R(N \circ \Phi^{-1})$ on $\Phi(U)$. As a consequence, by the chain rule, we have

$$|D\widetilde{N}| = |D(R(N \circ \Phi^{-1}))| = |(DN) \circ \Phi^{-1} \cdot D\Phi^{-1}|, \qquad \text{on } \Phi(U).$$

Thus, after the change of variable $(u,v) = \Phi(x,y)$, the integral of $|DN|$ over U becomes

$$\begin{aligned}\int_U |DN| \,\mathrm{d}x\,\mathrm{d}y &= \int_{\Phi(U)} |(DN) \circ \Phi^{-1}| \cdot |\det(D\Phi^{-1})| \,\mathrm{d}u\,\mathrm{d}v \\ &\le \int_{\Phi(U)} |(DN) \circ \Phi^{-1} \cdot D\Phi^{-1}| \cdot |(D\Phi) \circ \Phi^{-1}| \cdot |\det(D\Phi^{-1})| \,\mathrm{d}u\,\mathrm{d}v \\ &= \int_{\Phi(U)} |D\widetilde{N}| \cdot |(D\Phi) \circ \Phi^{-1}| \cdot |\det(D\Phi^{-1})| \,\mathrm{d}u\,\mathrm{d}v,\end{aligned}$$

where the inequality follows from the sub-multiplicativity of the fixed matrix norm. Choosing $C > 0$ as an upper bound for the continuous function $|(D\Phi) \circ \Phi^{-1}| \cdot |\det(D\Phi^{-1})|$ on $\Phi(U)$, we conclude the proof. □

Remark 3.4. Recall that, locally around $p = (0,0,0)$, $C(S) \cap \mathcal{U} = \{(x,y,t) \in S \cap \mathcal{U} : V(x,y) = 0\}$. Since $\Phi(0) = 0$, it holds that $V(x,y) = 0$ if and only if $\widetilde{V}(u,v) = 0$ with $(u,v) := \Phi(x,y)$ if and only if $b(0,v) = 0$ with $(0,v) = \Phi(x,y)$. Therefore, we have

$$C(S) \cap \mathcal{U} = \{(\Phi^{-1}(0,v), t) \in S \cap \mathcal{U} : (0,v) \in \Phi(U),\ b(0,v) = 0\}.$$

Hence, in the coordinates (u,v), the characteristic set is described by the zeros of the $C^{1,\alpha}$-function $h(v) := b(0,v)$.

4. Local integrability of $|D\widetilde{N}|$

In light of Lemma 3.3, we may assume that the vector field V, locally around the origin, has the form:

$$V(x,y) = (x, b(x,y)), \qquad \forall\, (x,y) \in U, \tag{15}$$

where $b \in C^{1,\alpha}(\mathbb{R}^2)$, with $b(0) = 0$.

Theorem 4.1. *Let $U \subset \mathbb{R}^2$ be open, $0 \in U$, and let $V \in C^{1,\alpha}(U;\mathbb{R}^2)$, $0 < \alpha \le 1$, defined as in* (15)*. Then, denoting by $N := \frac{V}{|V|}$, there are constants $C, r_* > 0$ such that*

$$\int_{Q_r} |DN| \,\mathrm{d}z < Cr, \qquad \forall\, 0 < r < r_*, \tag{16}$$

where $Q_r = (-r,r)^2 \subset \mathbb{R}^2$ is the standard square of edge length $2r$.

Proof. Throughout the proof, $C > 0$ denotes a constant, independent of r, whose value may change line by line.

For a vector field V as in (15), $|V(x,y)|^2 = x^2 + b(x,y)^2$. Hence, we write

$$N(x,y) = \frac{V(x,y)}{|V(x,y)|} = \frac{(x, b(x,y))}{(x^2 + b(x,y)^2)^{1/2}}, \qquad \forall\, (x,y) \in U \setminus \{V = 0\}.$$

Differentiating N, at points where $V \neq 0$, we obtain:

$$\partial_x N = \frac{(b(b - xb_x), -x(b - xb_x))}{(x^2 + b^2)^{3/2}}, \qquad \partial_y N = \frac{(-xbb_y, x^2 b_y)}{(x^2 + b^2)^{3/2}}.$$

Consequently,

$$|\partial_x N| = \frac{|b - xb_x|}{x^2 + b^2}, \qquad |\partial_y N| = \frac{|xb_y|}{x^2 + b^2}. \tag{17}$$

We set $h(y) := b(0, y)$ for every $y \in \mathbb{R}$. Since $b \in C^{1,\alpha}(\mathbb{R}^2)$, $h \in C^{1,\alpha}(\mathbb{R})$. We claim that $|V(x, y)|^2$ is controlled by $x^2 + h(y)^2$. Indeed, since b_x is bounded, we have

$$|b(x, y) - h(y)| \leq \int_0^x |b_u(s, y)| \, \mathrm{d}s \leq C|x|.$$

Combining the latter and the triangle inequality, we then obtain

$$\frac{1}{C}(x^2 + h(y)^2) \leq x^2 + b(x, y)^2 \leq C(x^2 + h(y)^2),$$

as claimed. For the numerators involved in (17), we exploit in a crucial way the Hölder regularity of b. More precisely, we estimate

$$|b_y(x, y)| \leq |h'(y)| + |b_y(x, y) - b_y(0, y)| \leq |h'(y)| + C|x|^\alpha.$$

Additionally, using the identity

$$b(x, y) - xb_x(x, y) = h(y) + \int_0^x [b_x(s, y) - b_x(x, y)] \, \mathrm{d}s,$$

we deduce that

$$|b - xb_x| \leq |h(y)| + C|x|^{1+\alpha}.$$

All in all, combining the previous bounds, we deduce that, for every $(x, y) \in U \setminus \{V = 0\}$,

$$|DN(x, y)| \leq C \frac{|h(y)| + |x|^{1+\alpha} + |x||h'(y)|}{x^2 + h(y)^2}. \tag{18}$$

We study the integrability on Q_r of the right-hand side of (18). Firstly, observe that

$$\int_{Q_r} \frac{|h(y)|}{x^2 + h(y)^2} \, \mathrm{d}x \, \mathrm{d}y = \int_{-r}^r \int_{-r}^r \frac{|h(y)|}{x^2 + h(y)^2} \, \mathrm{d}x \, \mathrm{d}y = \int_{(-r,r)\cap\{h\neq 0\}} 2 \arctan \frac{r}{|h(y)|} \, \mathrm{d}y \leq 2\pi r.$$

Secondly, recalling that $\alpha > 0$, we have:

$$\int_{Q_r} \frac{|x|^{1+\alpha}}{x^2 + h(y)^2} \, \mathrm{d}x \, \mathrm{d}y \leq 2r \int_{-r}^r |x|^{-1+\alpha} \, \mathrm{d}x = \frac{4}{\alpha} r^{1+\alpha}.$$

Finally, for the last term of (18), we observe that

$$\int_{Q_r} \frac{|x||h'(y)|}{x^2 + h(y)^2} \, \mathrm{d}x \, \mathrm{d}y = \int_{-r}^r |h'(y)| \log \left(1 + \frac{r^2}{h(y)^2}\right) \mathrm{d}y, \tag{19}$$

with the convention that the integrand of the right-hand side is 0 on the set $\{h = h' = 0\}$, so that it is defined almost everywhere. The term in (19) is bounded by Cr, thanks to Lemma 4.2, proved below, which holds since $h \in C^{1,\alpha}([-r, r])$. This concludes the proof. □

Lemma 4.2. *Let $h \in C^{1,\alpha}([-r, r])$ for some $\alpha \in (0, 1]$. Then, there exists $C > 0$ such that*

$$\int_{-r}^r |h'(y)| \log \left(1 + \frac{r^2}{h(y)^2}\right) \mathrm{d}y \leq Cr.$$

Proof. We define the function $F_r : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ by

$$F_r(t) := \log \left(1 + \frac{r^2}{t^2}\right). \tag{20}$$

Observe that $F_r \in C^1(\mathbb{R} \setminus \{0\})$ and, for every $p \in [1, +\infty)$, its L^p norm on $(-R, R)$ satisfies

$$\|F_r\|_{L^p(-R,R)} \leq C_{p,R}, \tag{21}$$

for a constant $C_{p,R} > 0$, depending only on p and $R > 0$. Using the function F_r, we write

$$\int_{-r}^{r} |h'(y)| \log\left(1 + \frac{r^2}{h(y)^2}\right) \mathrm{d}y = \int_{-r}^{r} |h'(y)|(F_r \circ h)(y)\, \mathrm{d}y.$$

Now, if the set $\mathcal{Z} := \{y \in (-r, r) : h'(y) = 0\}$ is empty, then the change of variable $t = h(y)$ is well-defined on $(-r, r)$ and gives

$$\int_{-r}^{r} |h'(y)| F_r(h(y))\, \mathrm{d}y = \int_{\min\{h(r),h(-r)\}}^{\max\{h(r),h(-r)\}} F_r(t)\, \mathrm{d}t \leq Cr,$$

by a direct computation, also using the fact that $h(0) = 0$. If instead $\mathcal{Z} \neq \emptyset$, since h' is continuous, the complement of $\mathcal{Z}$ is open and thus it can be written as a countable disjoint union of open intervals, namely

$$(-r, r) \setminus \mathcal{Z} = \bigsqcup_{j \in \mathbb{N}} I_j, \qquad I_j := (a_j, b_j),$$

with at least one between a_j, b_j belonging to $\mathcal{Z}$. If $h'(a_j) = 0$, by Hölder continuity of h', we deduce that

$$|h(b_j) - h(a_j)| = \int_{a_j}^{b_j} |h'(x)|\, \mathrm{d}x = \int_{a_j}^{b_j} |h'(x) - h'(a_j)|\, \mathrm{d}x \leq H|b_j - a_j|^{1+\alpha}, \tag{22}$$

where $H > 0$ is the Hölder seminorm of h' on $[-r, r]$. The same estimate holds if $h'(b_j) = 0$. Therefore, on the interval I_j, since $h|_{I_j}$ is strictly monotone, we can perform the change of variable $t = h(x)$ and obtain:

$$\int_{I_j} |h'(x)| F_r(h(x))\, \mathrm{d}x = \int_{\min\{h(a_j),h(b_j)\}}^{\max\{h(a_j),h(b_j)\}} F_r(t)\, \mathrm{d}t, \tag{23}$$

The right-hand side of (23) can be estimated using Hölder inequality:

$$\int_{\min\{h(a_j),h(b_j)\}}^{\max\{h(a_j),h(b_j)\}} F_r(t)\, \mathrm{d}t \leq \|F_r\|_{L^p([-R,R])} |h(b_j) - h(a_j)|^{1-\frac{1}{p}} \leq C_{p,R} H^{1-\frac{1}{p}} |b_j - a_j|^{(1+\alpha)\left(1-\frac{1}{p}\right)},$$

where, in the last inequality, we have used (22), together with (21), and $R > 0$ is a bound for the L^∞ norm of h. Since the latter inequality holds for every $p \in [1, \infty)$, we may choose p such that

$$1 - \frac{1}{p} = \frac{1}{1+\alpha},$$

so that we obtain

$$\int_{I_j} |h'(x)| F_r(h(x))\, \mathrm{d}x \leq C|b_j - a_j|,$$

where $C > 0$ is independent of ε and j. Since the latter estimate holds for every $j \in \mathbb{N}$, we conclude that

$$\int_{-r}^{r} |h'(x)| F_r(h(x))\, \mathrm{d}x = \sum_{j \in \mathbb{N}} \int_{I_j} |h'(x)| F_r(h(x))\, \mathrm{d}x \leq C \sum_{j \in \mathbb{N}} |b_j - a_j| \leq 2Cr.$$

This concludes the proof. □

5. Proof of Theorem 1.1

Recalling that $\mathcal{H} \in C(S \setminus C(S))$, we are left to prove the local integrability of $\mathcal{H}$ around characteristic points. So, fix $p = (0, 0, 0) \in C(S)$, $\mathcal{U} \subset S$ open neighborhood of p and assume that $S \cap \mathcal{U}$ is the graph of a function $g \in C^{2,\alpha}(\mathbb{R}^2)$. Observe that any Riemannian measure on S is mutually absolutely continuous with the Lebesgue measure in coordinates (x, y), namely

$$\sigma_R|_{\mathcal{U}} = \rho(x, y)\, \mathrm{d}x\, \mathrm{d}y,$$

where $\rho, \rho^{-1} \in L^\infty(\mathbb{R}^2)$. By Lemma 3.2, it is enough to prove the integrability of $|DN|$ with respect to the Lebesgue measure on a sufficiently small ball centered at the origin $B_r \subset \mathbb{R}^2$, where N is defined in (12). By the change of coordinates of Lemma 3.3, it holds

$$\int_{B_r} |DN| \,\mathrm{d}x \,\mathrm{d}y \leq C \int_{\Phi(B_r)} |D\widetilde{N}| \,\mathrm{d}u \,\mathrm{d}v, \qquad \widetilde{N} := \frac{\widetilde{V}}{|\widetilde{V}|},$$

where $\widetilde{V}(u,v) = (u, b(u,v))$ for some $b \in C^{1,\alpha}(\mathbb{R}^2)$. Since Φ is a $C^{1,\alpha}$-diffeomorphism, $\Phi(B_r)$ is an open set and $\Phi(B_r) \subset B_{Lr} \subset Q_{Lr}$, where $L > 0$ is the Lipschitz constant of Φ. Hence, applying Theorem 4.1, we deduce that, up to possibly shrinking r and updating the constant,

$$\int_{B_r} |DN| \,\mathrm{d}x \,\mathrm{d}y \leq Cr.$$

This proves that H is locally integrable close to $p \in C(S)$.

6. Proof of Theorem 1.2

We first show that $\nu \in \mathrm{BV}_{loc}(S;\mathbb{R}^3)$. As before, it is enough to check this property near characteristic points. Fix $p = (0,0,0) \in C(S)$, $\mathcal{U} \subset S$ open neighborhood of p and assume that $S \cap \mathcal{U}$ is the graph of a function $g \in C^{2,\alpha}(\mathbb{R}^2)$. According to (13), up to restricting $\mathcal{U}$, $\nu \in \mathrm{BV}_{loc}(\mathcal{U};\mathbb{R}^3)$ if and only if $N \in \mathrm{BV}_{loc}(B_r;\mathbb{R}^2)$, where N is defined in (12) and $B_r = \{|z| < r\} \subset \mathbb{R}^2$.

As a first step, we assume that $V(x,y) = (x, b(x,y))$ with $b \in C^{1,\alpha}(\mathbb{R}^2)$. Let $h(y) := b(0,y)$ for every $y \in \mathbb{R}$, $E = \{y \mid h(y) = 0\}$ and $a(y) := b_x(0,y)$. Recall that $|V(x,y)| = 0$ if and only if $x = 0$ and $y \in E$. Hence, outside of the set $\{0\} \times E$, $N \in C^{1,\alpha}$, its classical derivative DN is well-defined and, by Theorem 4.1, $DN \in L^1_{loc}(B_r;\mathbb{R}^{2,2})$. On the set $\{0\} \times E$, the differential of N has a jump part. Indeed, by the fundamental theorem of calculus, for every $(x,y) \in \mathbb{R}^2$, we have

$$|b(x,y) - h(y) - xa(y)| = \left| \int_0^x (b_x(s,y) - b_x(0,y)) \,\mathrm{d}s \right| \leq C|x|^{\alpha+1},$$

where, in the last step, we have used that $b \in C^{1,\alpha}(\mathbb{R}^2)$. As a consequence, we find that

$$b(x,y) = h(y) + xa(y) + O(|x|^{1+\alpha}), \qquad \text{as } x \to 0,\ y \in \mathbb{R}.$$

Therefore, for every $y \in E$, we compute

$$N^+(y) := \lim_{x \downarrow 0} N(x,y) = \lim_{x \downarrow 0} \frac{(x, b(x,y))}{(x^2 + b(x,y)^2)^{1/2}} = \lim_{x \downarrow 0} \frac{(1, b(x,y)/x)}{(1 + (b(x,y)/x)^2)^{1/2}} = \frac{(1, a(y))}{(1 + a(y)^2)^{1/2}},$$

and similarly

$$N^-(y) := \lim_{x \uparrow 0} N(x,y) = -\frac{(1, a(y))}{(1 + a(y)^2)^{1/2}}$$

While, for $y \in \mathbb{R} \setminus E$, since $h(y) \neq 0$, $N(x,y)$ admits a limit as $x \to 0$, so that

$$N^+(y) = N^-(y) := \lim_{x \to 0} N(x,y) = (0, \operatorname{sgn} h(y))$$

Define $B_r^\pm := B_r \cap \{\pm x > 0\}$. Then, given $\varphi \in C_c^\infty(B_r;\mathbb{R}^2)$, integration by parts shows that

$$\int_{B_r^\pm} \langle N, \partial_x \varphi \rangle \,\mathrm{d}x \,\mathrm{d}y = \mp \int_{-r}^r \langle N^\pm, \varphi \rangle \,\mathrm{d}y - \int_{B_r^\pm} \langle \partial_x N, \varphi \rangle \,\mathrm{d}x \,\mathrm{d}y.$$

Similarly, by integration by parts, we see that $\partial_y N$ does not carry a singular part, indeed

$$\int_{B_r} \langle N, \partial_y \varphi \rangle \,\mathrm{d}x \,\mathrm{d}y = -\int_{B_r} \langle \partial_y N, \varphi \rangle \,\mathrm{d}x \,\mathrm{d}y.$$

In conclusion, using that N^+ and N^- coincide on $\mathbb{R} \setminus E$, the distributional derivative $[DN]$ of N is given by

$$[DN] = DN\mathcal{L}^2 + ((N^+ - N^-) \otimes e_1)\mathcal{H}^1|_{\{0\} \times E}, \qquad \text{on } B_r. \tag{24}$$

Since DN is locally integrable and $|N^+ - N^-| = 2$, we finally conclude that $N \in \mathrm{BV}_{loc}(B_r;\mathbb{R}^2)$.

The second step consists in proving that N is of locally bounded variation for a general vector field V. We use the change of coordinates of Lemma 3.3 and, up to restricting r, write

$$V(x,y) = R^T \widetilde{V}(\Phi(x,y)), \qquad \forall\,(x,y) \in B_r,$$

where $R \in O(2)$, $\Phi|_{B_r}$ is the $C^{1,\alpha}$-diffeomorphism given by Lemma 3.3 and $\widetilde{V}(u,v) = (u, b(u,v))$, for some $b \in C^{1,\alpha}$. Then, on B_r, it holds

$$N = R^T \widetilde{N} \circ \Phi, \qquad \widetilde{N} := \frac{\widetilde{V}}{|\widetilde{V}|}.$$

Since $\Phi(B_r) \subset B_{Lr}$, where $L > 0$ is the Lipschitz constant of Φ, by the previous step, $\widetilde{N} \in \mathrm{BV}_{loc}(\Phi(B_r);\mathbb{R}^2)$. Therefore, as N is the composition of a function of locally bounded variation with a $C^{1,\alpha}$-diffeomorphism (hence bi-Lipschitz), we conclude that $N \in \mathrm{BV}_{loc}(B_r;\mathbb{R}^2)$. Furthermore, (24) implies that

$$[DN] = DN\mathcal{L}^2 + J\mathcal{H}^1|_\Sigma, \tag{25}$$

where $\Sigma := \Phi^{-1}(\{0\} \times E)$ and $J \in L^1_{loc}(\Sigma;\mathbb{R}^{2,2})$ is a suitable matrix-valued jump part.

We are now in position to prove also the second statement of the theorem. Assume that $\mathcal{H}^1(C(S)) = 0$. By Remark 3.4, $C(S) \cap \mathcal{U} = \{(x,y,t) \in S \cap \mathcal{U} : (x,y) \in \Phi^{-1}(\{0\} \times E)\}$. Thus, $\mathcal{H}^1(C(S)) = 0$ implies that $\mathcal{H}^1(\Phi^{-1}(\{0\} \times E)\}) = 0$. Therefore, exploiting (25), we deduce that $[DN] = DN\mathcal{L}^2$ and, thus, $N \in W^{1,1}_{loc}(B_r;\mathbb{R}^2)$. As in the first part of the proof, thanks to (13), the latter statement is equivalent to $\nu \in W^{1,1}_{loc}(\mathcal{U};\mathbb{R}^3)$, as claimed. Conversely, if $\nu \in W^{1,1}_{loc}(\mathcal{U};\mathbb{R}^3)$, then, after the change of coordinate of Lemma 3.3, $\widetilde{N} \in W^{1,1}_{loc}(\Phi(B_r);\mathbb{R}^2)$. However, from (24), we see that the jump part of $[D\widetilde{N}]$ always satisfy the identity $|\widetilde{N}^+ - \widetilde{N}^-| = 2$. Hence, to avoid contradictions, $\mathcal{H}^1(\Sigma) = 0$, or equivalently $\mathcal{H}^1(C(S)) = 0$.

Remark 6.1. Theorem 1.2 is sharp in the sense that there are examples where $\mathcal{H}^1(C(S)) > 0$ and ν is not $W^{1,1}_{loc}$. Let $f(x,y,t) := t - g(x,y)$, where $g \in C^\infty(\mathbb{R}^2)$ is defined by

$$g(x,y) := -\frac{x^2}{2} - \frac{xy}{2}, \qquad \forall\,(x,y) \in \mathbb{R}^2,$$

and consider the smooth surface

$$S = \{(x,y,t) \in \mathbb{H} : f(x,y,t) = 0\} = \{(x,y,t) \in \mathbb{H} : t = g(x,y)\}.$$

Then $\nabla_H f = x(X_1 + X_2)$, $\|\nabla_H f\| = \sqrt{2}|x|$ and $C(S) = \{(0,y,0) : y \in \mathbb{R}\}$. In particular, $C(S)$ is a smooth curve of positive $\mathcal{H}^1$-measure. A horizontal unit normal, as a function on S, is

$$\nu(x,y) = \frac{\mathrm{sgn}(x)}{\sqrt{2}}(X_1 + X_2) = \frac{\mathrm{sgn}(x)}{\sqrt{2}}\left(1, 1, \frac{x-y}{2}\right), \qquad x \neq 0.$$

A direct computation shows that the matrix-valued distributional derivative of ν is

$$D\nu = \frac{\mathrm{sgn}(x)}{\sqrt{2}}\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} + \sqrt{2}\left(1, 1, -\frac{y}{2}\right) \otimes e_1\, \mathcal{H}^1|_{\{x=0\}}.$$

Since this is a locally finite Radon measure, with a nonzero singular part,

$$\nu \notin W^{1,1}_{\mathrm{loc}}(S;\mathbb{R}^3).$$

7. A C^2 counterexample to the local integrability

In this section, we show that the additional Hölder regularity in Theorem 1.1 cannot, in general, be replaced by mere C^2 regularity. More precisely, we construct a C^2 surface whose characteristic set consists of a single isolated point and whose horizontal mean curvature fails to be locally integrable.

Proposition 7.1. *Let $S = \{(x,y,t) \in \mathbb{H} : t = g(x,y)\} \subset \mathbb{H}$ be the graph of the function*

$$g(x,y) := -\frac{x^2}{2} - \frac{xy}{2} - \int_0^y h(s)\,\mathrm{d}s, \tag{26}$$

where

$$h(y) := \begin{cases} e^{-1/y^2}\left(2 + \sin\left(y^4 e^{1/y^2}\right)\right), & y \neq 0, \\ 0, & y = 0. \end{cases} \tag{27}$$

Then, the surface S is of class C^2, the origin is its only characteristic point and

$$\mathcal{H} \notin L^1_{\mathrm{loc}}(S, \sigma_R), \tag{28}$$

where σ_R is any Riemannian perimeter measure.

Proof. Firstly, we claim that S is a surface of class C^2. Observe that $h(y) \to 0$ as $y \to 0$, so that h is continuous. Furthermore, $h(y)/y \to 0$ as $y \to 0$, and $h'(y) \to 0$ as $y \to 0$ since, for every $y \neq 0$, it holds

$$h'(y) = \frac{2e^{-1/y^2}}{y^3}\left(2 + \sin\left(y^4 e^{1/y^2}\right)\right) + (4y^3 - 2y)\cos\left(y^4 e^{1/y^2}\right). \tag{29}$$

Thus, $h \in C^1(\mathbb{R})$ and $h'(0) = 0$. This implies that $g \in C^2(\mathbb{R}^2)$ and proves the claim. Secondly, we show that $C(S) = \{(0,0,0)\}$. Let $f(x,y,t) := t - g(x,y)$, then

$$X_1 f(x,y,t) = x, \qquad X_2 f(x,y,t) = x + h(y).$$

Recalling (6), we have that

$$p \in C(S) \qquad \Longleftrightarrow \qquad (x, x + h(y)) = (0,0) \qquad \Longleftrightarrow \qquad x = 0,\ h(y) = 0.$$

Now, $h(0) = 0$, so that $(0,0,0) \in C(S)$. On the other hand, $h(y) > 0$ for every $y \neq 0$, since $2 + \sin\left(y^4 e^{1/y^2}\right) \geq 1$ for every $y \neq 0$.

To conclude the proof, we are left to prove (28) (with respect to the Lebesgue measure $\mathrm{d}x\,\mathrm{d}y$). Since S is the t-graph of (26), identity (14) holds and thus

$$\mathcal{H}(x,y) = \partial_x\left(\frac{x}{W}\right) + \partial_y\left(\frac{x + h(y)}{W}\right), \qquad W := \left(x^2 + (x + h(y))^2\right)^{1/2}.$$

Computing explicitly the derivatives yields

$$\mathcal{H}(x,y) = \frac{h'(y)x^2 + h(y)x + h(y)^2}{\left(x^2 + (x + h(y))^2\right)^{3/2}}. \tag{30}$$

Fix $0 < r < 1/4$ so small that $0 < h(y) < r/4$ whenever $0 < |y| < r$. For such y, set

$$I_r(y) := \int_{-r}^{r} \frac{x^2}{W(x,y)^3}\,\mathrm{d}x, \qquad J_r(y) := \int_{-r}^{r} \frac{|h(y)x + h(y)^2|}{W(x,y)^3}\,\mathrm{d}x.$$

The identity (30) and the reverse triangle inequality imply the lower bound

$$\int_{-r}^{r} |\mathcal{H}(x,y)|\,\mathrm{d}x \geq |h'(y)| I_r(y) - J_r(y). \tag{31}$$

On the one hand, the term $J_r(y)$ is uniformly bounded for every $0 < |y| < r$, indeed, performing the change of variable $x = h(y)s$ gives

$$J_r(y) = \int_{-r}^{r} \frac{|h(y)x + h(y)^2|}{\left(x^2 + (x + h(y))^2\right)^{3/2}}\,\mathrm{d}x \leq \int_{-\infty}^{+\infty} \frac{|s+1|}{\left(s^2 + (s+1)^2\right)^{3/2}}\,\mathrm{d}s < +\infty. \tag{32}$$

On the other hand, for the term $I_r(y)$, we observe that, for $2h(y) \leq x \leq r/2$, $W(x,y)$ is comparable to x, with absolute constants. Therefore

$$I_r(y) \geq c \int_{2h(y)}^{r/2} \frac{\mathrm{d}x}{x} = c \log \frac{r}{4h(y)}. \tag{33}$$

Therefore, combining the lower bound (31) with (32) and (33), we see that $\mathcal{H}$ is not locally integrable around the origin if

$$\int_0^r |h'(y)| \log \frac{r}{h(y)} \, \mathrm{d}y = \infty. \tag{34}$$

We now prove (34). Since $|\sin\theta| \leq 1$, it holds $e^{-1/y^2} \leq h(y) \leq 3e^{-1/y^2}$, which in turn implies that, for $0 < y < r$ sufficiently small,

$$\frac{c}{y^2} \leq \log \frac{r}{h(y)} \leq \frac{C}{y^2}, \tag{35}$$

for constants $0 < c < C$. Denoting by $\phi(y) := y^4 e^{1/y^2}$, for every $y \neq 0$, by (29), we have

$$|h'(y)| \geq 2y |\cos\phi(y)| - 4y^3 - \frac{6e^{-1/y^2}}{y^3}.$$

Combining this with (35), we obtain

$$|h'(y)| \log \frac{r}{h(y)} \geq c \frac{|\cos\phi(y)|}{y} - C\left(y + \frac{e^{-1/y^2}}{y^5}\right). \tag{36}$$

Since $y \mapsto y + y^{-5}e^{-1/y^2}$ is locally integrable on $\mathbb{R}$, (34) follows if we prove that

$$\int_0^r \frac{|\cos\phi(y)|}{y} \, \mathrm{d}y = \infty. \tag{37}$$

Using $|\cos\theta| \geq \cos^2\theta = (1 + \cos 2\theta)/2$, for every $\delta \in (0, r)$, we get

$$\int_\delta^r \frac{|\cos\phi(y)|}{y} \, \mathrm{d}y \geq \frac{1}{2}\int_\delta^r \frac{1 + \cos(2\phi(y))}{y} \, \mathrm{d}y \geq \frac{1}{2}\log\frac{r}{\delta} + \frac{1}{2}\int_\delta^r \frac{\cos(2\phi(y))}{y} \, \mathrm{d}y. \tag{38}$$

For $0 < y < r < 1/4$, $\phi'(y) = e^{1/y^2}(4y^3 - 2y) \neq 0$, so that we may write

$$\frac{\cos(2\phi(y))}{y} = q(y)\frac{\mathrm{d}}{\mathrm{d}y}\sin(2\phi(y)), \qquad q(y) := \frac{1}{2y\phi'(y)} = \frac{e^{-1/y^2}}{4y^2(2y^2-1)}.$$

Therefore, an integration by parts gives

$$\int_\delta^r \frac{\cos(2\phi(y))}{y} \, \mathrm{d}y = \left[q(y)\sin(2\phi(y))\right]_\delta^r - \int_\delta^r q'(y)\sin(2\phi(y)) \, \mathrm{d}y.$$

Since, both $q(y)$ and $q'(y)$ are absolutely integrable near zero, and $q(y) \to 0$ as $y \to 0$, we find that the latter integral is bounded uniformly as $\delta \to 0^+$. Consequently, the logarithmic term in the right-hand side of (38) positively diverges as $\delta \to 0^+$, while the remaining integral stays bounded. This finally proves (37), and thus (34). □

Remark 7.2. In the previous example, the vector field V as defined in (11) already has the form $V(x, y) = (x, b(x, y))$, with $b(x, y) = x + h(y)$. Then, as displayed by Theorem 4.1, what controls the local integrability of $\mathcal{H}$ is the function $h(y) = b(0, y)$, and in particular the integral

$$\int_0^r |h'(y)||\log(h(y))| \, \mathrm{d}y.$$

Then, the choice of h is motivated by Lemma 4.2: in order to obtain (34) (and find a function falling outside the hypothesis of that lemma), one needs to choose a function whose first derivative has fast oscillations, and is not Hölder, close to zero. We verify that indeed the function g is C^2, but it is not $C^{2,\alpha}$ near the origin for any $\alpha > 0$. For $y > 0$ small, $\phi(y)$ is strictly decreasing and unbounded as $y \to 0^+$. Hence, for every $j \in \mathbb{N}$, we choose $y_j, z_j \to 0$ such that $\phi(y_j) = 2\pi j$ and $\phi(z_j) = (2j + 1)\pi$. The mean value theorem applied to ϕ, together with $|\phi'(y)| \asymp y e^{1/y^2}$, gives

$$|y_j - z_j| \asymp \frac{e^{-1/y_j^2}}{y_j}, \qquad |h'(y_j) - h'(z_j)| \geq c y_j.$$

As a consequence, since $g_{yy}(x,y) = -h'(y)$, for every $\alpha > 0$, it holds

$$\frac{|g_{yy}(0,y_j) - g_{yy}(0,z_j)|}{|y_j - z_j|^\alpha} = \frac{|h'(y_j) - h'(z_j)|}{|y_j - z_j|^\alpha} \asymp y_j^\alpha e^{\alpha/y_j^2} |h'(y_j) - h'(z_j)| \geq c y_j^{1+\alpha} e^{\alpha/y_j^2} \xrightarrow{j\to\infty} \infty.$$

AI statement: The authors used ChatGPT 6 during the preparation of this manuscript for language editing, revision and to develop the counterexample of Section 7. The authors verified all mathematical content and references and assume responsibility for the final manuscript.

References

[1] A. Agrachev, L. Rizzi, and T. Rossi. Relative heat content asymptotics for sub-Riemannian manifolds. *Anal. PDE*, 17(9):2997–3037, 2024.

[2] Z. M. Balogh. Size of characteristic sets and functions with prescribed gradient. *J. Reine Angew. Math.*, 564:63–83, 2003.

[3] Z. M. Balogh, F. Ferrari, B. Franchi, E. Vecchi, and K. Wildrick. Steiner's formula in the Heisenberg group. *Nonlinear Anal.*, 126:201–217, 2015.

[4] D. Barilari, E. Bellini, and A. Pinamonti. Curvature measures and the sub-Riemannian Gauss-Bonnet theorem, 2025. ArXiv:2509.26460. `https://arxiv.org/abs/2509.26460`.

[5] D. Barilari, E. Bellini, and T. Rossi. Curvature distributions and the Steiner formula for characteristic surfaces in the Heisenberg group, 2026. ArXiv:2610.26706. `https://arxiv.org/abs/2610.26706`.

[6] D. Barilari, U. Boscain, and D. Cannarsa. On the induced geometry on surfaces in 3D contact sub-Riemannian manifolds. *ESAIM Control Optim. Calc. Var.*, 28:Paper No. 9, 28, 2022.

[7] D. Barilari and T. Bossio. Steiner and tube formulae in 3D contact sub-Riemannian geometry. *Commun. Contemp. Math.*, 26(7):Paper No. 2350034, 40, 2024.

[8] T. Bossio, L. Rizzi, and T. Rossi. Tubes in sub-Riemannian geometry and a Weyl's invariance result for curves in the Heisenberg groups. *Ann. Sc. Norm. Super. Pisa Cl. Sci.*, 2026. To appear.

[9] J.-H. Cheng, J.-F. Hwang, A. Malchiodi, and P. Yang. Minimal surfaces in pseudohermitian geometry. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 4(1):129–177, 2005.

[10] D. Danielli, N. Garofalo, and D. M. Nhieu. Sub-Riemannian calculus on hypersurfaces in Carnot groups. *Adv. Math.*, 215(1):292–378, 2007.

[11] D. Danielli, N. Garofalo, and D. M. Nhieu. Integrability of the sub-Riemannian mean curvature of surfaces in the Heisenberg group. *Proc. Amer. Math. Soc.*, 140(3):811–821, 2012.

[12] M. Derridj. Sur un théorème de traces. *Ann. Inst. Fourier (Grenoble)*, 22(2):73–83, 1972.

[13] G. Giovannardi, A. Pinamonti, J. Pozuelo, and S. Verzellesi. The prescribed mean curvature equation for t-graphs in the sub-Finsler Heisenberg group $\mathbb{H}^n$. *Adv. Math.*, 451:109788, 2024.

[14] G. Giovannardi, A. Pinamonti, and S. Verzellesi. Curvature estimates for minimal hypersurfaces in the Heisenberg group, 2025. ArXiv:2409.20359. `https://arxiv.org/abs/2409.20359`.

[15] M. Gordina, L. Luo, A. Pinamonti, and R. Sarkar. Heat content and spectrum for subordinated sub-Laplacians, 2026. ArXiv:2609.04045. `https://arxiv.org/abs/2609.04045`.

[16] R. K. Hladky and S. D. Pauls. Constant mean curvature surfaces in sub-Riemannian geometry. *J. Differential Geom.*, 79(1):111–139, 2008.

[17] F. Montefalcone. Stable H-minimal hypersurfaces. *J. Geom. Anal.*, 25(2):820–870, 2015.

[18] S. D. Pauls. Minimal surfaces in the Heisenberg group. *Geom. Dedicata*, 104:201–231, 2004.

[19] L. Rizzi and T. Rossi. Heat content asymptotics for sub-Riemannian manifolds. *J. Math. Pures Appl.*, 148:267–307, 2021.

[20] T. Rossi. Integrability of the sub-Riemannian mean curvature at degenerate characteristic points in the Heisenberg group. *Adv. Calc. Var.*, 16(1):99–110, 2023.

[21] J. Tyson and J. Wang. Heat content and horizontal mean curvature on the Heisenberg group. *Comm. Partial Differential Equations*, 43(3):467–505, 2018.

Dipartimento di Matematica, Università di Trento, Via Sommarive, 14, 38123 Povo TN, Italy
Email address: `andrea.pinamonti@unitn.it`

Department of Information Engineering, Computer Science and Mathematics, Università degli Studi dell'Aquila, via Vetoio 1, 67100 L'Aquila, Italy
Email address: `tommaso.rossi1@univaq.it`