# PRIME GEODESIC PERIODS ON MANIFOLDS OF COMPLEX PROJECTIVE TYPE

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Abstract. Let $M^{2n}$, $n \geq 2$, be a closed connected Riemannian manifold all of whose geodesics are closed, and assume $H^2(M;\mathbb{Q}) \neq 0$. We prove that all prime unit-speed geodesics have the same period for $2 \leq n \leq 8$. For every $n \geq 2$, the same conclusion holds if each nontrivial stabilizer of the geodesic circle action has prime-power order. For $n \geq 3$, unequal prime periods imply the existence of a proper connected critical manifold whose generic stabilizer order has at least two distinct odd prime divisors and whose inverse image in an auxiliary circle bundle has total rational Betti number at least eight. The period restrictions follow from reversal in equivariant Morse theory, the cohomology of fixed manifolds of a lifted geodesic flow, and index and nullity formulas under iteration.

## 1. Introduction

A Riemannian manifold $M$ is called *Besse* if all its geodesics are closed. The *prime period* of a unit-speed geodesic is its least positive period. Write $SM$ for the unit tangent bundle and $\varphi_\tau$, $\tau \in \mathbb{R}$, for the unit-speed geodesic flow. Wadsley's theorem gives a common period for this flow on a closed connected Besse manifold [36]. Let $L$ be the least such period. The flow defines an effective, locally free action

$$\mathbb{R}/L\mathbb{Z} \curvearrowright SM.$$

A Besse metric is called *Zoll* if all its prime periods are equal; equivalently, this action is free.

**Theorem A** (Equality of prime periods). *Let $(M^{2n}, g)$ be a closed connected smooth Besse manifold, where $2 \leq n \leq 8$. If $H^2(M;\mathbb{Q}) \neq 0$, then all prime unit-speed geodesics have the same period.*

A manifold is of *complex projective type* if it is homotopy equivalent to $\mathbb{CP}^n$. For a closed connected Besse manifold $M^{2n}$, $n \geq 2$, with $H^2(M;\mathbb{Q}) \neq 0$, Proposition 2.1 and Lemma 2.2 give

$$\pi_1(M) = 0, \quad H^*(M;\mathbb{Z}) = \mathbb{Z}[h]/(h^{n+1}), \quad |h| = 2, \quad M \simeq \mathbb{CP}^n. \tag{1.1}$$

The integral cohomology ring of the universal cover is that of a compact rank-one symmetric space, by Bott, Samelson, and McCleary [5, 34, 26]; see [3, Theorem 1.1]. Thus Theorem A applies, in particular, to Besse manifolds with the rational homology groups of $\mathbb{CP}^n$. For Kähler Besse manifolds, simple connectedness and the integral cohomology ring in (1.1) are also recorded in [18, Proposition 6].

Assume henceforth that $M^{2n}$ is closed, connected and Besse, that $n \geq 2$, and that $H^2(M;\mathbb{Q}) \neq 0$. For a space $Y$ and a field $\Bbbk$, put

$$b_j(Y;\Bbbk) = \dim_\Bbbk H^j(Y;\Bbbk), \qquad b_j(Y) = b_j(Y;\mathbb{Q}), \qquad \beta_\Bbbk(Y) = \sum_{j\geq 0} b_j(Y;\Bbbk).$$

For a Riemannian manifold $X$, let $\Lambda X = H^1(\mathbb{R}/\mathbb{Z}, X)$ be the Sobolev free loop space, with $X$ as its subspace of constant loops. The circle $S^1 = \mathbb{R}/\mathbb{Z}$ acts by translation of the parameter; reversal is $\rho\gamma(\tau) = \gamma(-\tau)$. For a compact Lie group $G$, let $EG \to BG$ be a universal principal bundle. For a $G$-space $Y$, put

$$Y_G = EG \times_G Y, \qquad H^*_G(Y;\Bbbk) = H^*(Y_G;\Bbbk).$$

The same convention applies to pairs and to local systems on the corresponding Borel constructions.

2020 *Mathematics Subject Classification.* Primary 53C22; Secondary 58E10, 55N91.

*Key words and phrases.* Closed geodesics, Besse manifolds, complex projective spaces, equivariant Morse theory, periodic geodesic flow.

An *underlying critical manifold* $F \subset SM$ is a connected component of $\mathrm{Fix}(\varphi_\ell)$ for some $\ell > 0$. The choice of $\ell$ specifies a *represented component* at length $\ell$; its parametrizing circle acts by $[\tau]\cdot v = \varphi_{\ell\tau}(v)$, for $\tau \in \mathbb{R}$ and $v \in F$. Write $L/q_F$ for the generic prime period of $F$; thus $q_F \in \mathbb{Z}_{\geq 1}$ is its generic stabilizer order for the geodesic circle action. Unless another period is specified, an underlying $F$ is represented at $L/q_F$. Put $d_F = (\dim F - 1)/2$. Let $I$ be the Morse index of the regular critical manifold $SM$ at period $L$. Let $Q \longrightarrow M$ be the principal circle bundle with first Chern class $h$ in (1.1), and put

$$P(F) = Q \times_M F.$$

The map $F \longrightarrow M$ in this fibre product is the footpoint projection.

**Theorem B** (Restrictions on unequal prime periods). *Let $(M^{2n}, g)$ be a closed connected smooth Besse manifold with $n \geq 3$ and $H^2(M;\mathbb{Q}) \neq 0$. There is an integer $r \geq 0$ such that*

$$I = 4nr + 1.$$

*Put $t = 2r + 1$. If the prime periods are unequal, then $r > 0$ and there is a proper connected critical manifold $F \subset SM$ such that*

$$q_F \nmid t, \qquad \beta_{\mathbb{Q}}(P(F)) \geq 8.$$

*The integer $q_F$ has at least two distinct odd prime divisors, and*

$$d_F \leq \begin{cases} n-3, & n \text{ even}, \\ n-2, & n \text{ odd}. \end{cases}$$

**Corollary 1.1.** *Let $(M^{2n}, g)$ be a closed connected smooth Besse manifold with $n \geq 2$ and $H^2(M;\mathbb{Q}) \neq 0$. If every nontrivial stabilizer of the geodesic circle action has prime-power order, then all prime unit-speed geodesics have period $L$.*

1.1. **The Berger conjecture.** Berger's conjecture asserts that all prime geodesics on a simply connected Besse manifold have the same length. Gromoll and Grove proved it for the two-sphere [10], and Radeschi and Wilking for manifolds homeomorphic to spheres of dimension at least four [33, Theorem A]. The three-dimensional case is treated in Dittmer's preprint [8, Theorem A]. A Besse metric on the real projective plane has constant curvature, by Pries [32]. Lin and Schmidt prove the same rigidity for a Besse manifold of dimension at least two, other than three, admitting an even-degree smooth covering by a smooth homotopy sphere [17, Theorem 2].

On a simply connected spin closed manifold admitting a simple Zoll metric, Mazzucchelli and Suhr characterize Zoll metrics by the equality of two min–max values on a finite-dimensional loop space [25, Theorem 1.2]. Here a Zoll metric is *simple* if its prime geodesics are embedded. Ginzburg, Gürel, and Mazzucchelli give a characterization by equality of two specified $S^1$-equivariant spectral values on a simply connected spin closed manifold admitting a Zoll metric [9, Theorem 1.6].

For a Besse manifold homotopy equivalent to $\mathbb{CP}^n$, Adelstein and Vargas Pallete prove that the metric is Blaschke under the following additional hypotheses: every prime geodesic has length twice the diameter, and the cut locus of every point contains a smooth closed submanifold of codimension two at distance equal to the diameter [1, Theorem 1.1]. Their theorem also has quaternionic and octonionic versions. For a Besse convex contact sphere, Mazzucchelli and Radeschi prove that each nonempty fixed-point set of a time map of the Reeb flow is an integral homology sphere [24, Theorem A].

Radeschi and Wilking give a holonomy criterion for orientability of the negative bundles on an orientable Besse manifold and prove rational $S^1$-equivariant perfection of the energy functional on a simply connected Besse manifold [33, Theorems B and D]. For every nonconstant closed geodesic $\gamma$ on an orientable Besse manifold, Wilking's index-parity theorem gives $\mathrm{ind}(\gamma) \equiv \dim M + 1 \pmod 2$ [38, Theorem 3]. Klingenberg studied the loop spaces and geodesic manifolds of projective spaces [15]. Ottosen and Bökstedt compute $H^*_{S^1}(\Lambda\mathbb{CP}^n;\mathbb{F}_p)$ as a graded module over $H^*(BS^1;\mathbb{F}_p) = \mathbb{F}_p[u]$, with $|u| = 2$, for every prime $p$ [29].

1.2. **Reversal and iteration.** Put $U = SM$ and $P = Q \times_M U$. Let $H$ be the structure circle of $P \to U$, let $j \in H$ have order two, and put $\widehat{\rho}(q, v) = (q, -v)$. By Lemmas 2.3 and 2.5, the cohomology ring is the exterior algebra

$$H^*(P; \mathbb{Z}) = \Lambda_{\mathbb{Z}}(a, b), \qquad |a| = 2n - 1, \quad |b| = 2n + 1.$$

For the reductions $\bar{a}, \bar{b}$ modulo two, the second Steenrod square satisfies

$$\mathrm{Sq}^2 \bar{a} = (n + 1 \bmod 2)\bar{b}.$$

By Theorems 4.1 and 4.8, the geodesic flow has an $H$-equivariant lift $\Phi$ satisfying

$$\widehat{\rho}\Phi_\tau\widehat{\rho} = \Phi_{-\tau} \quad (\tau \in \mathbb{R}), \qquad \Phi_L = j.$$

Let $\mathcal{L}$ be the rank-one rational local system on $\Lambda M$ with monodromy $-1$ on a generator of $\pi_1(\Lambda M) \cong \mathbb{Z}$. Its restriction to the connected based loop space is acyclic [2, Corollary 1]. Normalize the lift of reversal to $\mathcal{L}$ to be the identity over the constant loops. The connecting homomorphism is then a reversal-equivariant isomorphism of $\mathbb{Q}[u]$-modules

$$H^{k-1}(M \times BS^1; \mathbb{Q}) \xrightarrow{\simeq} H^k_{S^1}(\Lambda M, M; \mathcal{L}) \qquad (k \in \mathbb{Z}),$$

where $u$ is a fixed generator of $H^2(BS^1; \mathbb{Q})$ and reversal sends $u$ to $-u$ (Proposition 3.3).

Every represented critical component $F$ is reversal invariant, and $q_F$ is odd (Theorem 5.9). Put $B_F = F/S^1$. For an indeterminate $s$, set

$$P_F(s) = \sum_{j \geq 0} \dim_{\mathbb{Q}} H^{2j}(B_F; \mathbb{Q}) s^j, \qquad J_F(s) = \frac{P_F(s)}{1 + s}.$$

The coefficient of $s^j$ in $J_F$ is the dimension of the positive eigenspace of reversal on $H^{2j}(B_F; \mathbb{Q})$ (Theorem 5.9). Let $i(F)$ be the Morse index of the represented component, put $h_F = (i(F) - 1)/2$, and let $\varepsilon_F$ be the character of reversal on the orientation line of its negative bundle. If $\kappa_F$ is the reversal character on the one-dimensional space $H^0(F; \mathcal{L})$, then $\varepsilon_F = \kappa_F$ (Theorem 5.9). The orbit space $B_F$ and the polynomials $P_F, J_F$ depend only on the underlying manifold; $i(F), h_F, \varepsilon_F, \kappa_F$ refer to the represented period. Define $J_\pm$ by summing $s^{h_F} J_F(s)$ over all represented components with $\varepsilon_F = \pm 1$. For a positive integer $m$, write $[m]_s = 1 + s + \cdots + s^{m-1}$. Corollary 5.10 gives

$$J_-(s) = \frac{J_U(s)}{1 - s^{2n}}, \qquad J_+(s) = s^n J_-(s), \qquad J_U(s) = \frac{[n]_s[n + 1]_s}{1 + s}. \tag{1.2}$$

Each coefficient is a finite sum of nonnegative integers. For a formal Laurent series $A(s)$, let $[s^j]A$ denote the coefficient of $s^j$, and put $\operatorname{supp} A = \{j \in \mathbb{Z} : [s^j]A \neq 0\}$. The supports of $J_-$ and $J_+$ are the respective unions of the coefficient intervals

$$I_b^- = [2nb, 2nb + 2n - 2] \cap \mathbb{Z}, \qquad I_b^+ = n + I_b^- \qquad (b \in \mathbb{Z}_{\geq 0}).$$

An interval for a summand of $J_\pm$ means an interval $I_b^\pm$. Consecutive intervals in either series are separated by one zero coefficient.

A nonempty fixed set of a prime-power subgroup of the lifted flow circle is connected and has rational cohomology exterior on two odd generators (Lemma 6.1). Whenever $\beta_{\mathbb{Q}}(P(F)) = 4$, one has

$$[s^{2j}]J_F(s) - [s^{2j+1}]J_F(s) \geq 0 \qquad (j \geq 0)$$

by Lemma 8.1. For $n \geq 3$, every coefficient interval without a regular summand contains the support of a summand $s^{h_F} J_F(s)$ for which one of these differences is negative (Theorem 8.9). Theorem B follows from Theorem 5.4, Corollary 5.5, and Theorem 8.11.

1.3. **The dimension restriction.** For an underlying critical manifold $F$ and an integer $m \geq 1$, let $F^m \subset U$ be the connected component of the fixed-point set of the geodesic flow at time $mL/q_F$ containing $F$, represented at that period. Its index and dimension are given by Lemma 3.4, from Bott's iteration theorem [6] and the endpoint formula [20, Theorem 3, formula (2.6)]. An order $m$ is nonresonant when $F^m = F$ as underlying manifolds.

Let $n \geq 3$, and let $t = 2r + 1$ be as in Theorem B. Let $F \subsetneq U$ be an underlying critical manifold of dimension $2d_F + 1$, and suppose $b_2(B_F) \geq 3$. Assume that for every integer $\ell \geq 1$ the summand $s^{h_{F^\ell}} J_{F^\ell}(s)$ is supported in one coefficient interval of its character series. Let $m$ be nonresonant, and let $b$ be the index of the interval containing the support of $s^{h_{F^m}} J_{F^m}(s)$. Let $a$ be the difference between $h_{F^m}$ and the least integer in this interval. By Corollary 5.11, $a \in 2\mathbb{Z}$. If

$$2n - 2d_F - 6 < a < d_F + 5,$$

then Proposition 7.3 and Corollary 7.4 give

$$\frac{mL}{q_F} = \begin{cases} (2b+1)L/t, & m \text{ odd}, \\ 2(b+1)L/t, & m \text{ even}. \end{cases}$$

In particular, the same support containment holds at orders $\ell$ for which $F \subsetneq F^\ell$.

If the support of $s^{h_F} J_F(s)$ meets two consecutive coefficient intervals, then $d_F \geq 7$ (Lemma 8.6). For $2 \leq n \leq 8$, this support is contained in one interval for every represented $F$ (Corollary 8.7). Corollary 9.4 and Proposition 9.5 give equality of prime periods in this range.

For $\dim F = 7$, the strict inequality is $2n - 12 < a < 8$. Its even solutions are $4, 6$ when $n = 7$, and $6$ when $n = 8$; there are none when $n = 9$. At $n = 9$, the support of $s^{h_F} J_F(s)$ can meet two coefficient intervals only if $d_F = 7$ (Remark 9.6).

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## 2. Projective topology and circle bundles

### 2.1. The cohomology ring.

**Proposition 2.1.** *Let $M^{2n}$ be a closed connected Besse manifold, with $n \geq 2$ and $H^2(M;\mathbb{Q}) \neq 0$. Then $M$ is simply connected and*

$$H^*(M;\mathbb{Z}) = \mathbb{Z}[h]/(h^{n+1}), \qquad |h| = 2. \tag{2.1}$$

*Proof.* The universal cover $\widetilde{M}$ is compact and Besse [3, Corollary 4.7]; hence the covering is finite. By the Bott–Samelson–McCleary theorem, its integral cohomology ring is that of a compact rank-one symmetric space [3, Theorem 1.1]. The finite-cover transfer makes $H^2(M;\mathbb{Q}) \longrightarrow H^2(\widetilde{M};\mathbb{Q})$ injective [11, §3.G]. Since $2n \geq 4$, nonvanishing in degree two leaves only the complex projective case. Thus

$$H^*(\widetilde{M};\mathbb{Z}) = \mathbb{Z}[\widetilde{h}]/(\widetilde{h}^{n+1}), \qquad |\widetilde{h}| = 2.$$

Let $\gamma$ be a deck transformation. It fixes the nonzero image of $H^2(M;\mathbb{Q})$, hence fixes $\widetilde{h}$ and all its powers. Its Lefschetz number is

$$\sum_{k=0}^{2n} (-1)^k \operatorname{tr}\big(\gamma^* | H^k(\widetilde{M};\mathbb{Q})\big) = n + 1.$$

The Lefschetz fixed-point theorem [11, Theorem 2C.3] gives a fixed point of $\gamma$. A deck transformation with a fixed point is the identity. Therefore $\pi_1(M) = 0$. □

**Lemma 2.2.** *Let $M^{2n}$ be a closed simply connected smooth manifold satisfying* (2.1)*. There is a homotopy equivalence $f : M \longrightarrow \mathbb{CP}^n$ such that $f^* c_1(\mathcal{O}(1)) = h$, where $\mathcal{O}(1)$ is the hyperplane line bundle on $\mathbb{CP}^n$.*

*Proof.* Give $M$ a finite CW structure of dimension $2n$. The class $h$ is represented by a map $M \longrightarrow \mathbb{CP}^\infty = K(\mathbb{Z}, 2)$ [11, Theorem 4.57]. Cellular approximation deforms this map into the $2n$-skeleton $\mathbb{CP}^n$ [11, Theorem 4.8]. For the resulting map $f$, put $u = c_1(\mathcal{O}(1))$. Since $f^*(u^k) = h^k$ for $0 \le k \le n$, $f$ induces an integral cohomology isomorphism, hence an integral homology isomorphism by the universal coefficient theorem. Both spaces are simply connected CW complexes, so $f$ is a homotopy equivalence by the homological Whitehead theorem [11, Corollary 4.33]. □

For the remainder of this section, let $M^{2n}$ be a closed simply connected smooth manifold with $n \ge 2$ satisfying (2.1). Choose the fundamental class $[M] \in H_{2n}(M; \mathbb{Z})$ with $\langle h^n, [M] \rangle = 1$, and orient $M$ accordingly. Let

$$q : Q \longrightarrow M, \qquad c_1(Q) = h,$$

be the principal circle bundle with this Chern class. Choose a Riemannian metric on $M$. For a vector bundle $E$ with a fibre metric, write $S(E)$ for its unit sphere bundle. Put

$$U = SM, \qquad p : U \longrightarrow M, \qquad P = Q \times_M U = S(q^* TM), \qquad \pi : P \longrightarrow U. \tag{2.2}$$

Here $p$ is the footpoint projection and $\pi$ is the projection onto $U$. Let $r : P \longrightarrow Q$ be the other projection. Write $H = \{z \in \mathbb{C} : |z| = 1\}$ for the structure circle of $Q$. It acts on $P$ by $z \cdot (\xi, v) = (z\xi, v)$; put $j = -1 \in H$. The maps

$$\rho(x, v) = (x, -v), \qquad \widehat{\rho}(\xi, v) = (\xi, -v)$$

are involutions on $U$ and $P$. The second commutes with $H$.

### 2.2. **Cohomology and tangent classes.**

**Lemma 2.3.** *The manifold $Q$ is homotopy equivalent to $S^{2n+1}$. Moreover,*

$$H^*(P; \mathbb{Z}) = \Lambda_{\mathbb{Z}}(a, b), \qquad |a| = 2n - 1, \quad |b| = 2n + 1. \tag{2.3}$$

*The class $a$ restricts to a generator on a sphere fibre of $P \longrightarrow Q$, and $b$ is the pullback of a generator of $H^{2n+1}(Q; \mathbb{Z})$. Both classes are fixed by $\widehat{\rho}^*$. Let $e(TM)$ be the Euler class, let $w(TM)$ be the total Stiefel–Whitney class, and let $\bar{h} \in H^2(M; \mathbb{F}_2)$ be the reduction of $h$ modulo two. Then*

$$e(TM) = (n + 1)h^n, \qquad w(TM) = (1 + \bar{h})^{n+1}. \tag{2.4}$$

*Proof.* Let $f$ be as in Lemma 2.2, and let $q_0 : S(\mathcal{O}(1)) \longrightarrow \mathbb{CP}^n$ be the unit-circle bundle, with Chern class $c_1(\mathcal{O}(1))$. Classification of complex line bundles gives

$$Q \cong f^* S(\mathcal{O}(1))$$

as principal circle bundles [12, Proposition 3.10]. The induced $H$-equivariant bundle map $Q \longrightarrow S(\mathcal{O}(1)) \cong S^{2n+1}$ covers the homotopy equivalence $f$ and is an isomorphism on each fibre. It is therefore a homotopy equivalence.

The sphere bundle $r : P \longrightarrow Q$ has Euler class $q^* e(TM) = 0$, since $H^{2n}(Q; \mathbb{Z}) = 0$. The Gysin sequence [11, §4.D] gives

$$H^k(P; \mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & k = 0, 2n - 1, 2n + 1, 4n, \\ 0, & \text{otherwise.} \end{cases}$$

Restriction to a sphere fibre is an isomorphism in degree $2n - 1$, and $r^*$ is an isomorphism in degree $2n + 1$. Let $r_!$ denote integration along the oriented sphere fibres. Choose $a$ with $r_! a = 1$ and write

$b = r^*\omega$, where $\omega$ generates $H^{2n+1}(Q;\mathbb{Z})$. The projection formula gives $r_!(ab) = \pm\omega$, so $ab$ generates $H^{4n}(P;\mathbb{Z})$. The vanishing groups give $a^2 = b^2 = 0$. The map $\widehat{\rho}$ covers $\mathrm{id}_Q$ and restricts on each sphere fibre to the antipodal map, of degree $(-1)^{2n} = 1$. The two isomorphisms above imply $\widehat{\rho}^*a = a$ and $\widehat{\rho}^*b = b$.

The Euler characteristic is $\chi(M) = n+1$. The Euler–Poincaré formula [27, Corollary 11.12] and $\langle h^n, [M]\rangle = 1$ give $e(TM) = (n+1)h^n$. Reduction modulo two gives $H^*(M;\mathbb{F}_2) = \mathbb{F}_2[\bar{h}]/(\bar{h}^{n+1})$. Corollary 11.15 of [27] gives $w(TM) = (1+\bar{h})^{n+1}$. □

2.3. **Two free quotients.** For indeterminates $C, S$, define

$$K_n(C,S) = \frac{(C+S)^{n+1} + C^{n+1}}{S} = \sum_{i=0}^{n} \binom{n+1}{i} C^i S^{n-i} \quad \text{in } \mathbb{F}_2[C,S]. \tag{2.5}$$

This polynomial is homogeneous of degree $n$ and monic in $S$.

For a real vector bundle $E$, let $\mathbb{P}_{\mathbb{R}}(E)$ be the bundle of real lines in its fibres. The *characteristic homomorphism* of a principal $G$-bundle $X \longrightarrow Y$ is the map $H^*(BG;\mathbb{F}_2) \longrightarrow H^*(Y;\mathbb{F}_2)$ induced by a classifying map.

**Proposition 2.4.** *The action of $H \times \langle\widehat{\rho}\rangle$ on $P$ is free. Let $c \in H^2(BH;\mathbb{F}_2)$ and $s \in H^1(B\langle\widehat{\rho}\rangle;\mathbb{F}_2)$ be the generators. The same letters denote their pullbacks to $B(H \times \langle\widehat{\rho}\rangle)$. The kernel of its characteristic homomorphism is*

$$\ker\big(\mathbb{F}_2[c,s] \longrightarrow H^*(P/(H\times\langle\widehat{\rho}\rangle);\mathbb{F}_2)\big) = \big(K_n(c,s^2), c^{n+1}\big). \tag{2.6}$$

*For the subgroup $\langle j, \widehat{\rho}\rangle$, let $t_1, s \in H^1(B\langle j,\widehat{\rho}\rangle;\mathbb{F}_2)$ correspond to the characters taking the value 1 on $j$ and $\widehat{\rho}$, respectively, and zero on the other generator. Denote their images under the characteristic homomorphism by the same letters. Then*

$$H^*(P/\langle j,\widehat{\rho}\rangle;\mathbb{F}_2) = \mathbb{F}_2[s,t_1]/\big(K_n(t_1^2,s^2), t_1^{2n+2}\big). \tag{2.7}$$

*Proof.* For $z \in H$ and $\epsilon \in \{0,1\}$, an equality $(z\xi, (-1)^\epsilon v) = (\xi, v)$ implies $z = 1$ and $\epsilon = 0$. The first quotient is the real projectivized tangent bundle $\mathbb{P}_{\mathbb{R}}(TM)$. The characteristic classes $c, s$ map to the pullback of $\bar{h}$ and to $w_1(\lambda)$, respectively, where $\lambda$ is the tautological line bundle. Writing $w_i(TM)$ for the degree-$i$ Stiefel–Whitney class, the real projective bundle theorem [12, §3.1, pp. 78–79] gives

$$H^*(\mathbb{P}_{\mathbb{R}}(TM);\mathbb{F}_2) = H^*(M;\mathbb{F}_2)[s]\Big/\left(s^{2n} + w_1(TM)s^{2n-1} + \cdots + w_{2n}(TM)\right).$$

By (2.4), the last relation is $K_n(\bar{h}, s^2)$, and (2.6) follows.

Put $\bar{Q} = Q/\langle j\rangle$ and let $\bar{q} : \bar{Q} \longrightarrow M$ be the induced circle bundle. The pullback description in the proof of Lemma 2.3 identifies $\bar{Q}$ with the pullback of $\mathbb{RP}^{2n+1} \longrightarrow \mathbb{CP}^n$. The resulting map $\bar{Q} \longrightarrow \mathbb{RP}^{2n+1}$ is a homotopy equivalence over $f$ and identifies $Q \longrightarrow \bar{Q}$ with the pullback of the antipodal double cover. Hence

$$H^*(\bar{Q};\mathbb{F}_2) = \mathbb{F}_2[t_1]/(t_1^{2n+2}), \qquad \bar{q}^*\bar{h} = t_1^2$$

by [11, Example 3E.2]. The second quotient is $\mathbb{P}_{\mathbb{R}}(\bar{q}^*TM)$. Its character $t_1$ pulls back from $\bar{Q}$, whereas $s$ is the first Stiefel–Whitney class of the tautological line. The projective bundle theorem and $w(\bar{q}^*TM) = (1+t_1^2)^{n+1}$ give (2.7) with these character classes. □

2.4. **A Steenrod square.** In this subsection, $a, b$ denote the reductions modulo two of the classes in (2.3). Write $\mathrm{Sq}^i$ for the Steenrod squares.

**Lemma 2.5.** *In $H^*(P;\mathbb{F}_2)$ one has*

$$\mathrm{Sq}^2 a = (n+1 \bmod 2)b. \tag{2.8}$$

*Proof.* Put $G = H \times \langle\widehat{\rho}\rangle$, $R = H^*(BG;\mathbb{F}_2) = \mathbb{F}_2[c,s]$, and $K = K_n(c,s^2)$. Let $E_r^{p,q}$ be the cohomological Serre spectral sequence of $P \longrightarrow P_G \longrightarrow BG$, with

$$d_r : E_r^{p,q} \longrightarrow E_r^{p+r,q-r+1}.$$

Its local coefficients are constant, since $H$ is connected and $\widehat{\rho}^*$ is the identity by Lemma 2.3. Thus

$$E_2 = R \otimes \Lambda(a,b).$$

Give $Q$ the $G$-action induced by $G \longrightarrow H$. The projection $r$ is $G$-equivariant. Since $H$ acts freely on $Q$ and $\widehat{\rho}$ acts trivially,

$$Q_G \simeq M \times B\langle\widehat{\rho}\rangle.$$

The edge homomorphism for $Q \longrightarrow Q_G \longrightarrow BG$ sends $c$ to $\bar{h}$ and $s$ to the generator of $H^1(B\langle\widehat{\rho}\rangle;\mathbb{F}_2)$; its kernel is $(c^{n+1})$. Its spectral sequence has only the fibre degrees 0 and $2n+1$. Hence a generator $\omega \in H^{2n+1}(Q;\mathbb{F}_2)$ transgresses to $c^{n+1}$. Since $r^*\omega = b$, naturality implies that $b$ survives to $E_{2n+2}$.

By multiplicativity and bidegrees, all differentials before $d_{2n}$ vanish. Since $G$ acts freely on $P$, the edge kernel is $(K, c^{n+1})$ by Proposition 2.4. Its degree-$2n$ part is $\mathbb{F}_2K$, and the only differential into $E_r^{2n,0}$ is $d_{2n}$ on $a$. Consequently

$$d_{2n}a = K, \qquad d_{2n}(ab) = Kb.$$

Multiplication by $K$ is injective in $R$, so

$$E_{2n+1} = E_{2n+2} = R/(K) \otimes \Lambda(b).$$

For a graded algebra $A$, write $A^k$ for its degree-$k$ part. Naturality with respect to $r$ gives

$$d_{2n+2}b = [c^{n+1}] \quad \text{in } E_{2n+2}^{2n+2,0} = (R/(K))^{2n+2}. \tag{2.9}$$

The class $[c^{n+1}]$ is nonzero, since $K$ is monic in $s$ of degree $2n > 0$.

Write $\mathrm{Sq}^2 a = \epsilon b$, with $\epsilon \in \mathbb{F}_2$. By the Steenrod transgression theorem [22, Theorem 3.4], $\mathrm{Sq}^2 a$ transgresses to $\mathrm{Sq}^2 K$. Equation (2.9) therefore gives

$$[\mathrm{Sq}^2 K] = \epsilon[c^{n+1}] \quad \text{in } (R/(K))^{2n+2}. \tag{2.10}$$

Put $C = c$ and $S = s^2$. On $\mathbb{F}_2[C,S]$, the Cartan formula makes $D = \mathrm{Sq}^2$ a derivation: $\mathrm{Sq}^1 C = \mathrm{Sq}^1 S = 0$, $DC = C^2$, and $DS = S^2$. Applying $D$ to $SK_n = (C+S)^{n+1} + C^{n+1}$ and cancelling $S$ gives

$$DK_n = \begin{cases} CK_n + C^{n+1}, & n \text{ even}, \\ SK_n, & n \text{ odd}. \end{cases}$$

Since $[c^{n+1}] \neq 0$, equation (2.10) yields $\epsilon = n+1 \pmod 2$. □

## 3. Equivariant Morse theory and reversal

Let $g$ be a Besse metric on the manifold $M$ of Section 2, and let $\varphi_\tau$ be its unit-speed geodesic flow on $U = SM$. For the footpoint projection $p : U \to M$, the canonical contact form is

$$\alpha_v(\xi) = g(v, dp_v(\xi)), \qquad \xi \in T_vU.$$

Its Reeb vector field $X$ generates $\varphi$. Wadsley's theorem gives a least positive number $L$ such that $\varphi_L = \mathrm{id}$ [36]; see also [3, Theorem 4.6]. The energy $E : \Lambda M \to \mathbb{R}$ is

$$E(\gamma) = \frac{1}{2}\int_0^1 |\dot{\gamma}(t)|^2\, dt.$$

For each positive integer $m$, define the iteration map $I_m : \Lambda M \to \Lambda M$ by $I_m(\gamma)(t) = \gamma(mt)$. Reversal conjugates the circle action to its inverse. Thus it also acts on the circle Borel construction, with complex conjugation on $BS^1$.

A nonconstant critical component at energy $\ell^2/2$ is identified with a connected component $F$ of $\mathrm{Fix}(\varphi_\ell)$ by

$$j_\ell : F \longrightarrow \Lambda M, \qquad j_\ell(v)(t) = p\varphi_{\ell t}(v).$$

Every prime period is $L/q$ for some positive integer $q$. Thus $\ell/L \in \mathbb{Q}$, and $\varphi_\ell$ has finite order. The fixed set is smooth, and $j_\ell$ identifies its tangent space with the kernel of the Hessian [33, §1.1]. On $F$, the parametrizing circle $S^1 = \mathbb{R}/\mathbb{Z}$ acts by $[t]\cdot v = \varphi_{\ell t}(v)$. Its finite ineffective kernel is retained in the Borel construction. Write $B_F = F/S^1$, endowed with the effective quotient orbifold structure. The quotient map $F_{S^1} \to B_F$ induces an isomorphism

$$H^*(B_F;\mathbb{Q}) \xrightarrow{\sim} H^*_{S^1}(F;\mathbb{Q}).$$

Let $\nu_F^- \to F$ be the negative bundle of the energy Hessian; its rank is the Morse index $i(F)$. Under this isomorphism, write $u_F \in H^2(B_F;\mathbb{Q})$ for the pullback of the fixed generator $u \in H^2(BS^1;\mathbb{Q})$ by the Borel projection.

### 3.1. **The critical manifolds and their negative bundles.**

**Lemma 3.1.** *Every nonconstant critical manifold $F$ is an odd-dimensional contact submanifold of $U$. Its orbit space $B_F$ is a compact symplectic orbifold satisfying Poincaré duality over $\mathbb{Q}$. For its symplectic form $\omega_F$, there is a real number $\lambda_F \neq 0$ such that*

$$[\omega_F] = \lambda_F(u_F \otimes 1) \quad \text{in } H^2(B_F;\mathbb{R}).$$

*If $\dim B_F = 2d$, then $u_F^j \neq 0$ for $0 \le j \le d$. If $F$ is invariant under reversal, the induced involution on $B_F$ is free and antisymplectic.*

*Proof.* Let $\ell$ be the represented length of $F$. At $x \in F$, put $P = D\varphi_\ell|_{\ker \alpha_x}$. The fixed subspace of the finite-order symplectic map $P$ is symplectic, and

$$T_xF = \mathbb{R}X_x \oplus \ker(P - \mathrm{id}).$$

Hence $\alpha|_F$ is a contact form. Its Reeb flow is periodic. Let $\pi_F : F \to B_F$ be the quotient map. The orbifold Boothby–Wang theorem [14, Theorem 1.1], applied to the effective flow circle, gives a symplectic form $\omega_F$ with $\pi_F^*\omega_F = d(\alpha|_F)$; its class is a nonzero real multiple of the effective Euler class. If the parametrizing circle has ineffective kernel of order $k$, the quotient homomorphism to the effective circle has degree $k$. On the second cohomology of their classifying spaces it multiplies compatible generators by $k$. The effective Euler class therefore corresponds to a nonzero rational multiple of $u_F$. The compact oriented orbifold $B_F$ satisfies Poincaré duality over $\mathbb{R}$ by [35, §8, Theorem 3], and hence over $\mathbb{Q}$ by extension of scalars. Moreover, $\langle [\omega_F]^d, [B_F]\rangle > 0$, so $u_F^j \neq 0$ for $0 \le j \le d$.

Reversal satisfies $\rho^*\alpha = -\alpha$, whence $\rho^*\omega_F = -\omega_F$. A fixed point of the quotient involution would give a unit-speed geodesic with $\gamma(-t) = \gamma(t + c)$ for all $t$ and some $c$. Differentiation at $t = -c/2$ gives $-\dot\gamma(c/2) = \dot\gamma(c/2)$, which is impossible. □

**Lemma 3.2.** *For every nonconstant represented critical component $F$, the negative bundle is orientable, also after passage to the circle Borel construction. Its index is a positive odd integer, and*

$$H^{\mathrm{odd}}(B_F;\mathbb{Q}) = 0.$$

*The energy is rationally $S^1$-equivariantly perfect relative to the constant loops.*

*Proof.* The sphere bundle $S^{2n-1} \longrightarrow U \longrightarrow M$ has simply connected base and fibre, so $\pi_1(U) = 0$. The evaluation fibration $\mathrm{ev}_0 : \Lambda M \to M$, $\gamma \mapsto \gamma(0)$, with its constant-loop section, gives

$$\pi_1(\Lambda M) \cong \pi_2(M) \cong \mathbb{Z}.$$

Let $\ell$ be the represented length of $F$. On the based-loop fibre of evaluation, $I_m$ is the $m$-fold loop product. Thus $(I_m)_*$ on $\pi_1(\Lambda M)$ is multiplication by $m$. Choose such an $m$ with $m\ell \in L\mathbb{Z}$, and let

$j_{m\ell} : U \to \Lambda M$ parametrize the regular critical manifold at length $m\ell$. If $\iota : F \hookrightarrow U$ is the inclusion, then

$$I_m \circ j_\ell = j_{m\ell} \circ \iota, \qquad m(j_\ell)_* = (j_{m\ell})_* \iota_* = 0.$$

Since $\pi_1(\Lambda M)$ is torsion free, $(j_\ell)_* = 0$.

For a loop $c : S^1 \to F$, choose a filling $\widehat{c} : D^2 \to \Lambda M$ of $j_\ell \circ c$. In local coordinates, parallel transport is defined by a linear differential equation with $L^1$ coefficients. These coefficients depend continuously on the curve in the $H^1$ topology, so its holonomy depends continuously on the curve. An oriented orthonormal trivialization of $(\mathrm{ev}_0 \circ \widehat{c})^* TM$ identifies parallel transport around each loop with a continuous map $A : D^2 \to SO(2n)$. Its restriction to $\partial D^2$ is the holonomy loop associated with $c$, which is therefore nullhomotopic. The manifold $M$ is oriented and Besse; the holonomy criterion [33, Theorem B] therefore implies that the negative bundle is orientable along $c$. Thus it is orientable on $F$. The connected circle preserves an orientation, so the Borel negative bundle is orientable as well.

Wilking's index-parity theorem gives $i(F) \equiv \dim M + 1 \equiv 1 \pmod 2$ [38, Theorem 3]; hence $i(F) > 0$. All negative bundles are orientable, so [33, §5.1, Proposition 5.4 and Corollary 5.5] gives $H^{\mathrm{odd}}_{S^1}(F; \mathbb{Q}) = 0$ and rational $S^1$-equivariant perfection relative to $M$. □

Let $\mathcal{L}$ be the rank-one rational local system on $\Lambda M$ whose monodromy is $-1$ on a generator of $\pi_1(\Lambda M) = \mathbb{Z}$. Fix a point $x_0 \in M$, regarded as a constant loop. Reversal inverts the generator and therefore preserves this sign representation. Its lift to $\mathcal{L}$ is uniquely determined by requiring the identity on the fibre at $x_0$; this lift has square one. The constant loop $x_0$ supplies a section of the Borel fibration, so

$$\pi_1(\Lambda M) \longrightarrow \pi_1((\Lambda M)_{S^1})$$

is an isomorphism. Thus $\mathcal{L}$ extends to the Borel construction, with the same normalization of its reversal lift. Its restriction to $M \times BS^1$ is constant, and the normalized lift acts as the identity on these coefficients. The local system is trivial on $F_{S^1}$: the map $\pi_1(F) \longrightarrow \pi_1(F_{S^1})$ is onto, and its composite with the map to $\pi_1((\Lambda M)_{S^1})$ is zero by the preceding proof.

For every nonconstant component $F$, set

$$h_F = \frac{i(F) - 1}{2}, \qquad P_F(s) = \sum_{j \geq 0} \dim H^{2j}(B_F; \mathbb{Q}) s^j.$$

Let $o_F$ be the rational orientation local system of $\nu_F^-$. If $F$ is reversal invariant, define $\varepsilon_F, \kappa_F \in \{\pm 1\}$ by the action of reversal on the one-dimensional spaces

$$H^0(F; o_F), \qquad H^0(F; \mathcal{L}|_F),$$

respectively. These signs are independent of the choice of a basis. For such $F$, put

$$Q_F(s) = \sum_{j \geq 0} \mathrm{tr}(\rho^* | H^{2j}(B_F; \mathbb{Q})) s^j.$$

### 3.2. **The two global reversal traces.**

**Proposition 3.3.** *The groups $H^*_{S^1}(\Lambda M, M; \mathbb{Q})$ and $H^*_{S^1}(\Lambda M, M; \mathcal{L})$ are concentrated in odd degrees and have a common dimension series $D_n(s) \in \mathbb{Z}[[s]]$. Let $O_n(s)$ and $T_n(s)$ be their reversal trace series, respectively, with degree $2j+1$ contributing to the coefficient of $s^j$. Then*

$$D_n(s) = \frac{[n+1]_s}{1-s}, \tag{3.1}$$

$$O_n(s) = -\frac{(1-s^n)(1-s^{n+1})}{(1+s^n)(1-s^2)}, \tag{3.2}$$

$$T_n(s) = \frac{1-s^{n+1}}{1-s^2}. \tag{3.3}$$

*For a reversal-invariant represented component, the corresponding summands of these three series are*

$$s^{h_F} P_F(s), \qquad \varepsilon_F s^{h_F} Q_F(s), \qquad \varepsilon_F \kappa_F s^{h_F} Q_F(s).$$

*A pair of exchanged components contributes zero to both traces. Moreover $Q_F(1) = 0$ if $\rho(F) = F$.*

*Proof.* The geodesic-flow circle acts smoothly on the compact manifold $U$, so it has finitely many orbit types [31, Theorem 1.7.25, p. 38]. All stabilizers are finite. Hence there are finitely many critical lengths in $(0, L]$, and each fixed manifold has finitely many components. By Lemma 3.4, addition of $L$ raises every index by $I + 2n - 1 > 0$, where $I = i(U)$ at length $L$. Thus only finitely many critical components contribute in each degree. For each integer $k$, the relative Thom groups at all sufficiently high critical levels vanish in degrees $k$ and $k + 1$. The relative equivariant cohomology of the energy sublevels therefore stabilizes in degree $k$, with either coefficient system. The energy descends to the Borel construction, and every singular simplex has bounded energy. Thus the realizations of the singular sublevel pairs exhaust the realization of the full pair by CW subcomplexes. The Milnor exact sequence [11, Theorem 3F.8] identifies these stable groups with the cohomology of the full pair: its derived inverse-limit term vanishes by eventual constancy in degree $k - 1$. For $\mathcal{L}$, take the $(-1)$-eigenspaces of the deck involution in the ordinary sequence for the double cover defined by the sign character.

Put $\Omega M = \{\gamma \in \Lambda M : \gamma(0) = x_0\}$. This is a path-connected $H$-space. The evaluation fibration and its section identify $\pi_1(\Omega M)$ with $\pi_1(\Lambda M)$, so $\mathcal{L}|_{\Omega M}$ is nontrivial. Since $1 - (-1) = 2 \in \mathbb{Q}^\times$, [2, Corollary 1] gives $H_*(\Omega M; \mathcal{L}) = 0$. The homology spectral sequence with local coefficients [2, Proposition 5], applied first to evaluation and then to the circle Borel fibration, gives

$$H_*((\Lambda M)_{S^1}; \mathcal{L}) = 0.$$

The sign system is self-dual. The universal coefficient isomorphism over $\mathbb{Q}$ therefore gives $H^*_{S^1}(\Lambda M; \mathcal{L}) = 0$.

Since $\mathcal{L}$ restricts to the constant system on $M \times BS^1$, the connecting homomorphism is a reversal-equivariant isomorphism of $\mathbb{Q}[u]$-modules

$$H^{q-1}(M \times BS^1; \mathbb{Q}) \xrightarrow{\cong} H^q_{S^1}(\Lambda M, M; \mathcal{L}). \tag{3.4}$$

On the left, reversal fixes $h$ and sends the generator $u \in H^2(BS^1; \mathbb{Q})$ to $-u$. The monomials $h^a u^b$, with $0 \le a \le n$ and $b \ge 0$, give

$$D_n(s) = \sum_{a=0}^{n} \sum_{b \ge 0} s^{a+b}, \qquad T_n(s) = \sum_{a=0}^{n} \sum_{b \ge 0} (-1)^b s^{a+b},$$

which are (3.1) and (3.3). The local coefficient system is trivial on every critical Borel construction. By Lemma 3.2, $H^{k-i(F)}_{S^1}(F; \mathbb{Q}) = 0$ for every even $k$. Hence the Morse spectral sequence with coefficients in $\mathcal{L}$ collapses. Its $E_1$-terms and the ordinary $E_1$-terms have the same dimensions, so the ordinary dimension series is also $D_n$. The ordinary dimension formula also follows from [29, Proposition C.2] and the cohomology sequence of the pair.

By Lemma 2.2 and smooth approximation, there is a smooth homotopy equivalence $f : M \to \mathbb{CP}^n$ with a smooth homotopy inverse and smooth inverse homotopies. By [4, Theorem 1], composition with these maps and homotopies induces smooth maps and homotopies on the $H^1$ free-loop spaces. They are $O(2)$-equivariant and preserve constant loops, so they give an equivariant homotopy equivalence of pairs. Thus the ordinary trace agrees with that for the Fubini–Study metric on $\mathbb{CP}^n$ of holomorphic sectional curvature four.

By [29, Proposition 2.2 and Lemmas 2.3 and 2.6], the prime orbit quotient $B$ is the flag manifold of two ordered orthogonal complex lines. Under this identification, for orthonormal $z, w \in \mathbb{C}^{n+1}$, the geodesic

$$\gamma_{z,w}(t) = [\cos t\, z + \sin t\, w], \qquad 0 \le t \le \pi,$$

corresponds to $(\mathbb{C}(z+iw), \mathbb{C}(z-iw))$. Since $\rho\gamma_{z,w} = \gamma_{z,-w}$, reversal interchanges these lines. Let $L_1, L_2$ be the tautological line bundles and put $x_i = c_1(L_i^*)$. With this convention, [29, Theorem 3.2 and Corollary 3.3] gives

$$H^*(B;\mathbb{Q}) = \mathbb{Q}[x_1,x_2]/(H_n, H_{n+1}), \qquad H_r = \sum_{a=0}^{r} x_1^a x_2^{r-a} \quad (r \geq 0). \tag{3.5}$$

The tautological generators in that reference are $-x_1, -x_2$; the ideal is unchanged since $H_r(-x_1,-x_2) = (-1)^r H_r(x_1,x_2)$. Give $R = \mathbb{Q}[x_1,x_2]$ the grading $|x_1| = |x_2| = 1$, and grade $H^{2*}(B;\mathbb{Q})$ by half the cohomological degree. Since $R/(H_n, H_{n+1})$ is finite dimensional, $H_n, H_{n+1}$ form a regular sequence in the Cohen–Macaulay ring $R$. Both polynomials are fixed by reversal. Give the corresponding Koszul generators the trivial action. With $R(-d)^j = R^{j-d}$, the $\rho$-equivariant Koszul resolution is

$$0 \longrightarrow R(-2n-1) \longrightarrow R(-n) \oplus R(-n-1) \longrightarrow R \longrightarrow H^{2*}(B;\mathbb{Q}) \longrightarrow 0.$$

The graded trace on $R$ is $\sum_{a\geq 0} s^{2a} = (1-s^2)^{-1}$. Additivity of traces gives

$$Q_n(s) = \sum_{j\geq 0} \operatorname{tr}(\rho^* | H^{2j}(B;\mathbb{Q}))s^j = \frac{(1-s^n)(1-s^{n+1})}{1-s^2}. \tag{3.6}$$

Let $F_m$ be the critical manifold of the $m$th iterates. Its orbit quotient is $B$. Klingenberg's negative-bundle decomposition [15, §1.4, Lemma; §1.6, Theorem] gives

$$i_m := i(F_m) = 2n(m-1) + 1 = 2nm - (2n-1). \tag{3.7}$$

See also [30, Theorem 3.4]. The negative subspaces are unchanged by a constant rescaling of the metric. Let $J$ denote the complex structure on $\mathbb{CP}^n$, and let $\gamma : \mathbb{R}/(m\pi\mathbb{Z}) \to \mathbb{CP}^n$ be the unit-speed parametrization of an $m$th iterate. The summand along $\mathbb{R}J\dot\gamma$ has the basis

$$J\dot\gamma, \qquad \cos(2kt/m)J\dot\gamma, \qquad \sin(2kt/m)J\dot\gamma, \qquad 1 \leq k \leq m-1.$$

The complementary negative subbundle consists of fields in $(\mathbb{C}\dot\gamma)^\perp$. Parallel transport, the curvature operator, and the periodic boundary condition preserve $J$, so this is a complex bundle. The prime holonomy on $(\mathbb{C}\dot\gamma)^\perp$ is $-\mathrm{id}$ [15, §1.3, Proposition] and commutes with $J$. For $\bar\gamma(t) = \gamma(-t)$, $J\dot{\bar\gamma}(t) = -J\dot\gamma(-t)$. Thus reversal acts on the displayed scalar coefficients by $f(t) \mapsto -f(-t)$. The constant and cosine modes have sign $-1$, and the sine modes have sign $+1$; their determinant is $(-1)^m$. On the complementary negative subbundle, reversal commutes with $J$ and preserves the real orientation. Hence $\varepsilon_{F_m} = (-1)^m$. Combining (3.6) and (3.7) gives

$$O_n(s) = \sum_{m\geq 1} (-1)^m s^{n(m-1)} Q_n(s) = -\frac{Q_n(s)}{1+s^n},$$

which is (3.2).

The equivariant Thom isomorphism gives the three local summands. Reversal exchanges the two summands of an exchanged pair, so their trace is zero. Traces are additive in the short exact sequences of the relative Morse filtration, which is finite in each degree.

Finally, $Q_F(1)$ is the Lefschetz number of the quotient involution: odd rational cohomology vanishes by Lemma 3.2. The quotient is a compact triangulable orbifold, and the involution is free by Lemma 3.1. The Lefschetz fixed-point theorem [11, Theorem 2C.3] gives $Q_F(1) = 0$. □

### 3.3. **Finite-order iteration and endpoints.**

**Lemma 3.4.** *Put $N = 2n-1$. Let $F$ be represented at length $\ell$, with index $J$, and put $2k = \operatorname{codim}_U F$. For each integer $m \geq 1$, let $F^m$ be the component of* $\mathrm{Fix}(\varphi_{m\ell})$ *containing $F$, represented at length $m\ell$. There is a multiset of rational numbers $0 < \beta_1, \dots, \beta_k < 1$, called the* effective Bott weights *of $F$ at length $\ell$, such that*

$$i(F^m) = m(J+N) - N - 2\sum_{j=1}^{k} \lfloor m\beta_j \rfloor \qquad (m \in \mathbb{Z}_{\geq 1}). \tag{3.8}$$

*Moreover,*

$$\dim F^m - \dim F = 2\#\{j \,:\, m\beta_j \in \mathbb{Z}\}.$$

*If $q \in \mathbb{Z}_{>0}$ and $q\ell = L$, then $q\beta_j \in \mathbb{Z}$ for all $j$. If $I$ is the index of $U$ represented at length $L$, the indices of $F$ at lengths $\ell + L$ and $\ell$ differ by $I + N$.*

*Proof.* Choose $v \in F$ and put $c(t) = p\varphi_t(v)$ for $0 \le t \le \ell$. Write $D_t$ for Levi-Civita covariant differentiation along $c$ and $\mathcal{R}$ for the curvature tensor, with Jacobi equation $D_t^2 V + \mathcal{R}(V, \dot{c})\dot{c} = 0$. The oriented normal bundle of $c$ is trivial. Choose a smooth periodic orthonormal normal frame and repeat it on all iterates. In this frame,

$$D_t x = \dot{x} + A(t)x, \qquad A(t)^T = -A(t),$$

and let $K(t) = K(t)^T$ represent $V \mapsto \mathcal{R}(V, \dot{c})\dot{c}$. The matrices $A$ and $K$ are smooth and $\ell$-periodic. The normal index form on $H^1(\mathbb{R}/m\ell\mathbb{Z}, \mathbb{R}^N)$ is

$$\mathfrak{q}_m(x) = \int_0^{m\ell} \left( |\dot{x} + Ax|^2 - \langle Kx, x \rangle \right) dt.$$

Let $I_d$ denote the $d \times d$ identity matrix. The periodic Sturm coefficients are $I_N$, $A$, and $A^T A - K$; the leading coefficient is positive definite. In the phase coordinates $(\dot{x} + Ax, x)$, the fundamental solution $\Gamma : [0, \ell] \to \mathrm{Sp}(2N, \mathbb{R})$ satisfies

$$\dot{\Gamma} = \begin{pmatrix} 0 & -I_N \\ I_N & 0 \end{pmatrix} \begin{pmatrix} I_N & -A \\ A & K \end{pmatrix} \Gamma, \qquad \Gamma(0) = I_{2N}.$$

Put $P = \Gamma(\ell)$ and define

$$\Gamma^m(t + a\ell) = \Gamma(t)P^a \qquad (0 \le t \le \ell, \quad a = 0, \dots, m-1).$$

Let $i_1$ be Long's Maslov-type index at 1, with its lower-semicontinuous extension at degenerate endpoints [20, Definition 1, formula (1.4)]. The Sturm–Maslov comparison [19, Theorem 6.1] gives

$$i(F^m) = i_1(\Gamma^m).$$

Indeed, the tangential index form is $\int_0^{m\ell} |\dot{f}|^2 \, dt$ on periodic real-valued $H^1$ functions and has index zero.

The endpoint $P$ represents $D\varphi_\ell$ on $\ker \alpha_{\dot{c}(0)}$. Since $\ell/L \in \mathbb{Q}$, there is an integer $q_0 > 0$ such that $q_0\ell \in L\mathbb{Z}$, and $P^{q_0} = I_{2N}$. Thus $P$ is symplectically conjugate to

$$I_2^{\diamond d} \diamond R(2\pi\theta_1) \diamond \cdots \diamond R(2\pi\theta_k), \qquad d = N - k,$$

where $\diamond$ denotes symplectic direct sum,

$$R(a) = \begin{pmatrix} \cos a & -\sin a \\ \sin a & \cos a \end{pmatrix}, \qquad \theta_j \in \mathbb{Q} \cap (0, 1).$$

Here $2d = \dim F - 1$ by Lemma 3.1. Let $S_P^{\pm}(\omega)$ be the splitting numbers at $\omega \in \mathbb{C}$, $|\omega| = 1$ [20, equation (2.5)]. Their values on these summands are [19, Theorem 4.11 and Corollary 4.14]

$$S_P^+(1) = d, \qquad S_P^-(e^{2\pi i t}) = \#\{j \,:\, \theta_j = t\} \quad (0 < t < 1).$$

This includes $R(\pi) = -I_2$, for which $S_{-I_2}^-(-1) = 1$. Consequently,

$$C(P) := \sum_{0<t<1} S_P^-(e^{2\pi i t}) = k.$$

In local symplectic trivializations of $(\ker \alpha)|_F$, the return matrices vary smoothly with $v \in F$ and satisfy $P^{q_0} = I_{2N}$. Their spectral multiplicities are locally constant, and so are their splitting numbers

[19, Lemma 4.5]. Hence the multiset $\{\theta_1, \dots, \theta_k\}$ is constant on $F$. The iteration formula of Long and Zhu [21, Theorem 2.1, formula (2.1)] gives

$$i(F^m) = m(J + d - k) + 2\sum_{j=1}^{k} \lceil m\theta_j \rceil - N.$$

Set $\beta_j = 1 - \theta_j$. Since $\lceil m\theta_j \rceil = m - \lfloor m\beta_j \rfloor$ for every integer $m$, this is (3.8).

By Lemma 3.1,

$$\dim F^m = 1 + \dim \ker(P^m - I_{2N}).$$

A rotation summand belongs to this kernel precisely when $m\beta_j \in \mathbb{Z}$, which proves the dimension formula. If $q\ell = L$, then $P^q = I_{2N}$ and $q\beta_j \in \mathbb{Z}$.

Let a geodesic have prime period $L/r$, where $r \in \mathbb{Z}_{>0}$, index $J_0$, and effective Bott weights $\beta_1^0, \dots, \beta_{k_0}^0$. For $m \geq 1$, write $i_m$ for its index at length $mL/r$. Then $i_r = I$ and $r\beta_j^0 \in \mathbb{Z}$, so

$$\begin{aligned} i_{m+r} - i_m &= r(J_0 + N) - 2\sum_{j=1}^{k_0} r\beta_j^0 \\ &= i_r + N = I + N. \end{aligned}$$

Every represented period is an integer multiple of the prime period of each of its geodesics. Since $\varphi_{\ell+L} = \varphi_\ell$, the component $F$ is the same at both lengths, and its indices differ by $I + N$. □

## 4. The common return of the lifted flow

Let $\pi : P \longrightarrow U$, $H$, $j$, and $\widehat{\rho}$ be as in (2.2). For a critical manifold $F$, put $P(F) = \pi^{-1}(F)$.

**Theorem 4.1.** *The geodesic flow admits an $H$-equivariant lift $\Phi_t$ preserving a $\widehat{\rho}$-invariant connection, such that*

$$\widehat{\rho}\Phi_t\widehat{\rho} = \Phi_{-t}, \qquad \Phi_L \in \{\mathrm{id}, j\}.$$

*If $n$ is even, its common return is $\Phi_L = j$.*

For odd $n$, one also has $\Phi_L = j$ (Theorem 4.8).

### 4.1. The reversible lift.

**Lemma 4.2.** *The bundle $P \to U$ admits a $\widehat{\rho}$-invariant connection and an $H$-equivariant lift $\Phi_t$ preserving it, with*

$$\widehat{\rho}\Phi_t\widehat{\rho} = \Phi_{-t}, \qquad \Phi_L \in \{\mathrm{id}, j\}.$$

*The groups $H$ and $\{\Phi_t\}$ generate a two-dimensional torus $T$.*

*Proof.* Let $A_0$ be the pullback of a real connection form on $Q \to M$, normalized by $A_0(V) = 1$, where $V$ generates $z \mapsto e^{it}z$. Then $\widehat{\rho}^* A_0 = A_0$. Write $dA_0 = \pi^*\Omega_0$ and put

$$\Omega = \frac{1}{L}\int_0^L \varphi_t^*\Omega_0 \, dt.$$

This form is closed and flow invariant. Since $\rho\varphi_t = \varphi_{-t}\rho$ and $\rho^*\Omega_0 = \Omega_0$, it is also $\rho$-invariant. Each $\varphi_t$ is isotopic to the identity, so $\Omega$ is cohomologous to $\Omega_0$. Choose a real one-form $\eta$ with $d\eta = \Omega - \Omega_0$ and replace it by $(\eta + \rho^*\eta)/2$. Then

$$A = A_0 + \pi^*\eta, \qquad dA = \pi^*\Omega, \qquad \widehat{\rho}^* A = A.$$

Cartan's formula gives $d\iota_X\Omega = 0$. Since $H^1(U; \mathbb{R}) = 0$, there is a smooth function $f$ with $df = -\iota_X\Omega$. One has $df(X) = 0$ and $d(f \circ \rho) = -df$. Thus $f \circ \rho + f$ is constant. Subtracting half this constant gives the unique primitive satisfying $f \circ \rho = -f$. The connection-preserving infinitesimal lift [28, Theorem 3.1] is

$$\widetilde{X} = X^{\mathrm{hor}} + (f \circ \pi)V,$$

where $X^{\mathrm{hor}}$ is the horizontal lift for $A$; indeed,

$$\mathcal{L}_{\widetilde{X}} A = \pi^*(\iota_X \Omega + df) = 0.$$

The field $\widetilde{X}$ is $H$-invariant. Preservation of $A$ by $\widehat{\rho}$, together with $\rho_* X = -X$ and $f \circ \rho = -f$, gives $\widehat{\rho}_* \widetilde{X} = -\widetilde{X}$. Since $P$ is compact, its flow $\Phi_t$ is complete and reversible.

The map $\Phi_L$ is a connection-preserving gauge transformation. As $U$ is connected, it is a constant scalar $\lambda \in H$ [37, Lemma 3.2.4]. Reversibility and the commutation of $\widehat{\rho}$ with $H$ give $\lambda = \lambda^{-1}$. Hence $\Phi_L \in \{\mathrm{id}, j\}$ and $\Phi_{2L} = \mathrm{id}$. Since $L$ is the least positive period of the base flow, the homomorphism

$$H \times (\mathbb{R}/2L\mathbb{Z}) \longrightarrow \mathrm{Diff}(P), \qquad (z, [t]) \longmapsto z\Phi_t$$

has kernel $\{(1, [0]), (\lambda, [L])\}$. Its image is therefore a two-dimensional torus. □

Choose $A_0, \eta, f$ as in the proof of Lemma 4.2, and let $A$ and $\Phi$ be the resulting connection and lift.

**Lemma 4.3.** *Let $F$ be a connected reversal-invariant critical component represented at length $\ell$. Then*

$$\Phi_\ell|_{P(F)} = \kappa_F \,\mathrm{id}\,. \tag{4.1}$$

*Proof.* The restriction of $\Phi_\ell$ to $P(F)$ is a connection-preserving gauge transformation. Connectedness and reversibility give $\Phi_\ell|_{P(F)} = \lambda\,\mathrm{id}$ with $\lambda \in \{1, -1\}$, as in Lemma 4.2.

For a connection $B$ on a principal $H$-bundle and a loop $c$ in its base, define holonomy by $\mathrm{PT}_c^B(a) = \mathrm{Hol}_B(c)\, a$ for parallel transport, where $a$ lies over $c(0)$. For $u \in F$, put $c_u(t) = \varphi_{\ell t}(u)$, $0 \le t \le 1$. Since $df(X) = 0$, one has

$$\mathrm{PT}^A_{c_u}(a) = e^{-i\ell f(u)} \Phi_\ell(a) \qquad (\pi(a) = u).$$

Thus

$$\mathrm{Hol}_A(c_u) = \lambda e^{-i\ell f(u)}. \tag{4.2}$$

Parallel transport along $H^1$ loops defines a continuous map $\mathrm{Hol}_Q : \Lambda M \to H$ for the connection whose pullback is $A_0$. Continuity follows from continuous dependence for the parallel-transport equation with $L^1$ coefficients. Consider the double cover

$$\mathscr{C}_Q = \{(\gamma, z) \in \Lambda M \times H \,:\, z^2 = \mathrm{Hol}_Q(\gamma)\}.$$

Write $\mathrm{wind}(v) = \deg(v)$ for a map $v : S^1 \to H$. Let $\Gamma : S^1 \to \Lambda M$ have smooth adjoint $\Gamma^\vee : S^1 \times S^1 \to M$, $\Gamma^\vee(s, t) = \Gamma(s)(t)$. The holonomy–curvature formula [37, Proposition 3.2.13] gives[1]

$$\mathrm{wind}(\mathrm{Hol}_Q \circ \Gamma) \equiv \langle h, (\Gamma^\vee)_*[S^1 \times S^1] \rangle \pmod 2.$$

Under $\pi_1(\Lambda M) \cong \pi_2(M)$, a generator can be represented, by relative smooth approximation, by a jointly smooth loop of based loops at a point $x \in M$, with constant initial loop. Its adjoint is constant on the two coordinate circles and factors through $(S^1 \times S^1)/(S^1 \vee S^1) \cong S^2$. The resulting sphere represents a generator of $\pi_2(M)$, so its pairing with $h$ is $\pm 1$ by the Hurewicz theorem. Thus the rational sign system associated with $\mathscr{C}_Q$ is $\mathcal{L}$. Reversal lifts to this cover by $(\gamma, z) \mapsto (\rho\gamma, z^{-1})$. It fixes both roots over every constant loop and therefore induces the normalized lift on $\mathcal{L}$.

Since $A_0$ is pulled back from $Q$, the cover

$$E_{A_0} = \{(u, z) \in F \times H \,:\, z^2 = \mathrm{Hol}_{A_0}(c_u)\}$$

defines $\mathcal{L}|_F$, with reversal $(u, z) \mapsto (\rho u, z^{-1})$. Put $b(u) = \int_{c_u} \eta$. The change-of-connection formula (cf. [28, §4, equations (2) and (4)]) gives

$$\mathrm{Hol}_A(c_u) = \mathrm{Hol}_{A_0}(c_u) e^{-ib(u)}.$$

[1] Our holonomy is the inverse of that in [37, Definition 3.2.11]; its winding has the same parity.

Since $c_{\rho u} = \rho \circ c_u^{-1}$ and $\rho^* \eta = \eta$, one has $b \circ \rho = -b$. Consequently the real function $a_F = b - \ell f|_F$ satisfies $a_F \circ \rho = -a_F$, and

$$\mathrm{Hol}_{A_0}(c_u) = \lambda e^{i a_F(u)}.$$

Choose $\zeta \in H$ with $\zeta^2 = \lambda$ and put $s(u) = \zeta e^{i a_F(u)/2}$. Then $u \mapsto (u, s(u))$ is a section of $E_{A_0} \to F$. Since $\lambda^2 = 1$,

$$s(u)^{-1} = \zeta^{-2} s(\rho u) = \lambda s(\rho u).$$

Thus normalized reversal acts on $H^0(F; \mathcal{L}|_F)$ by $\lambda$, and $\kappa_F = \lambda$. □

4.2. **Fixed manifolds with total Betti number four.** For a group $G$ acting on a space $Y$ and a subgroup $K \leq G$, write $Y^K = \{y \in Y : ky = y \text{ for all } k \in K\}$. For $g \in G$, put $Y^g = Y^{\langle g \rangle}$.

**Lemma 4.4.** *Let $a \in T$ have order $p^v$, where $p$ is prime and $v \in \mathbb{Z}_{\geq 1}$. If $P^a$ is nonempty, then it is connected and*

$$\beta_{\mathbb{F}_p}(P^a) = 4.$$

*Every nonempty fixed component is $P(F)$ for a critical component $F \subset U$.*

*Proof.* The projection of $a$ to $U$ is a finite-order time map. On the full preimage of each connected component $F$ of its fixed set, $a$ is a connection-preserving gauge transformation. Its scalar is constant, so this preimage is either fixed pointwise or disjoint from $P^a$. If the projected map is the identity, its fixed set is the connected manifold $U$.

Each nonempty $P(F)$ is connected. The contact orientation of $F$ and the circle orientation make it a closed oriented manifold of positive even dimension. Its free $H$-action gives $\chi(P(F)) = 0$. Put $N = P(F)$ and $d = \dim N$. Poincaré duality gives $b_0(N; \mathbb{F}_p) = b_d(N; \mathbb{F}_p) = 1$. Since $d > 0$ is even and $\chi(N) = 0$,

$$\beta_{\mathbb{F}_p}(N) = 2 \sum_{r \text{ even}} \dim_{\mathbb{F}_p} H^r(N; \mathbb{F}_p) \geq 4.$$

By Lemma 2.3, $\beta_{\mathbb{F}_p}(P) = 4$. Apply Smith's inequality [23, Chapter IV, formula (1.4)] successively along the subgroup chain of $\langle a \rangle$. It gives $\beta_{\mathbb{F}_p}(P^a) \leq \beta_{\mathbb{F}_p}(P)$. If $c$ is the number of connected components of $P^a$, then $4c \leq \beta_{\mathbb{F}_p}(P^a) \leq 4$. Thus $c = 1$ and $\beta_{\mathbb{F}_p}(P^a) = 4$. □

**Lemma 4.5.** *Let $F$ be a reversal-invariant critical component and suppose $\beta_{\mathbb{Q}}(P(F)) = 4$. There are positive integers $a < b$, with $a + b$ odd, such that*

$$H^*(P(F); \mathbb{Q}) = \Lambda_{\mathbb{Q}}(\alpha, \beta), \qquad |\alpha| = 2a - 1, \quad |\beta| = 2b - 1. \tag{4.3}$$

*The lifted reversal fixes both generators. Choose degree-two generators $x, y$ of $H^*(BT; \mathbb{Q})$ with reversal signs $-1, +1$. The characteristic homomorphism $H^*(BT; \mathbb{Q}) \to H_T^*(P(F); \mathbb{Q}) \cong H^*(B_F; \mathbb{Q})$ is onto, with a kernel generated by a homogeneous regular sequence:*

$$H^*(B_F; \mathbb{Q}) = \mathbb{Q}[x, y]/(f, g), \qquad |f| = 2a, \quad |g| = 2b. \tag{4.4}$$

*In particular,*

$$P_F(s) = [a]_s [b]_s, \qquad Q_F(s) = \frac{(1 - s^a)(1 - s^b)}{1 - s^2}. \tag{4.5}$$

*The dimension of $P(F)$ is divisible by four.*

*Proof.* Put $N = P(F)$. It is connected and oriented of positive even dimension, and its free $H$-action gives $\chi(N) = 0$. Since $\beta_{\mathbb{Q}}(N) = 4$, the even cohomology is spanned by the unit and the orientation class, and the odd cohomology has total dimension two. Poincaré duality pairs its two generators nontrivially. If they have the same degree, the middle-dimensional pairing is alternating and nondegenerate. Odd squares vanish over $\mathbb{Q}$. Thus $H^*(N; \mathbb{Q})$ is exterior on generators of degrees $2a - 1, 2b - 1$, initially with $a \leq b$.

The torus $T$ acts locally freely on $N$: the vertical generator $V$ is nonzero, and $\pi_*\widetilde{X} = X$ is nowhere zero, so $V$ and $\widetilde{X}$ are pointwise independent. Its orbit space is $B_F$. Thus the rational cohomology of $N_T$ is that of $B_F$ and is finite dimensional. Reversal inverts the flow direction and fixes the $H$-direction; its two eigenlines in $H^2(BT;\mathbb{Q})$ have signs $-1,+1$. Choose $x, y$ on those lines. The Borel spectral sequence has

$$E_2 = \mathbb{Q}[x,y] \otimes \Lambda(\alpha,\beta).$$

Put $p = 2a-1$ and $q = 2b-1$. The fibre degrees are $0, p, q, p+q$, and all odd base columns vanish. Thus the only possible differentials into the bottom row are $d_{p+1}\alpha = f$ and $d_{q+1}\beta = g$, the latter taken modulo $f$ if $a < b$; the top-to-bottom differential has odd base degree. A differential between the two odd rows also has odd base degree and vanishes. Consequently $E_\infty^{*,0} = \mathbb{Q}[x,y]/(f,g)$. This quotient is finite dimensional, so $(f,g)$ has height two and $f, g$ form a regular sequence in $\mathbb{Q}[x,y]$. If $a = b$, the differential is the Koszul differential of $f, g$; if $a < b$, the two transgressions give its successive one-generator Koszul complexes. In both cases $E_\infty$ is concentrated in fibre degree zero. This gives (4.4).

Choose the two odd generators to be reversal eigenvectors, with signs $e_1, e_2$. When their degrees coincide, choose an eigenbasis whose product is nonzero; the nondegenerate middle pairing permits this. The transgressions have the corresponding signs. A representative of the later transgression with its prescribed sign is obtained by averaging over the involution. The Koszul resolution gives the trace polynomial

$$\frac{(1-e_1 s^a)(1-e_2 s^b)}{(1+s)(1-s)}.$$

It is a polynomial and vanishes at $s = 1$, by Proposition 3.3. If both signs were negative, it would have a pole at 1. If exactly one were positive, its value at 1 would be $a$ or $b$, and would be nonzero. Hence $e_1 = e_2 = 1$. The ordinary Hilbert series and this trace yield (4.5).

Write $\dim F = 2d+1$, and let $\theta$ be the restriction of the canonical contact form to $F$. Reversal acts on the contact orientation $\theta \wedge (d\theta)^d$ by $(-1)^{d+1}$. Its lift preserves the orientation of $H$, so it has the same orientation sign on $N$. It fixes the product of the two odd generators. Thus $d$ is odd and $\dim N = 2d+2$ is divisible by four. Since $\dim N = 2a+2b-2$, the integer $a+b$ is odd. In particular $a \neq b$. □

4.3. **The product identity when $\Phi_L = \mathrm{id}$.** Suppose in this subsection that $\Phi_L = \mathrm{id}$ and put

$$A = \Phi_{L/2}, \qquad N_+ = P^A, \qquad N_- = P^{jA}.$$

The commuting involutions $A, j, \widehat{\rho}$ generate $E \cong (\mathbb{Z}/2\mathbb{Z})^3$; set $K = \langle j, \widehat{\rho}\rangle$. The subgroup $K$ acts freely by Proposition 2.4. The involutions $A\widehat{\rho}$ and $jA\widehat{\rho}$ are also free: a fixed point would give a geodesic orbit preserved by reversal, contrary to Lemma 3.1. Thus the only possible nontrivial isotropy subgroups for $E$ are $\langle A\rangle$ and $\langle jA\rangle$. Their fixed sets are disjoint, since $j$ acts freely.

**Proposition 4.6.** *Both $N_+$ and $N_-$ are nonempty. For $i \in \{+,-\}$ there are integers $a_i < b_i$ such that the rational and mod-two cohomology of $N_i$ are exterior on generators in degrees $2a_i - 1, 2b_i - 1$. Their integral cohomology has no two-primary torsion, and*

$$1 \le a_i \le n, \qquad b_i \le n+1, \qquad a_i + b_i \text{ is odd}, \qquad a_i + b_i \le 2n-1, \tag{4.6}$$

$$a_+ b_+ + a_- b_- = n(n+1), \tag{4.7}$$

$$a_+ + a_- \le \begin{cases} n, & n \text{ even}, \\ n+1, & n \text{ odd}. \end{cases} \tag{4.8}$$

*Proof.* Let $z, t, s \in H^1(BE;\mathbb{F}_2)$ be dual to $A, j, \widehat{\rho}$, respectively. By Proposition 2.4,

$$H^*(P/K;\mathbb{F}_2) = \mathbb{F}_2[s,t]/\big(K_n(t^2,s^2), t^{2n+2}\big).$$

The indicated generators are the characteristic classes of the $K$-cover. Since $K_n$ is monic of degree $2n$ in $s$, the monomials $s^u t^v$, $0 \le u < 2n$, $0 \le v < 2n+2$, form a basis. All lift to $H_E^*(P;\mathbb{F}_2)$, because the two characters extend to $E$. The $E$-Borel construction is homotopy equivalent to the $(E/K)$-Borel construction of $P/K$. The Leray–Hirsch theorem [11, Theorem 4D.1] gives

$$H_E^*(P;\mathbb{F}_2) \text{ is free over } \mathbb{F}_2[z] \text{ of rank } 4n(n+1), \tag{4.9}$$

and this algebra is generated by $z, s, t$.

Put $X = P/K$ and $\bar{E} = E/K \cong \mathbb{Z}/2\mathbb{Z}$. A point $[p] \in X$ is fixed by $\bar{E}$ precisely when $Ap = kp$ for some $k \in K$. The elements $A\widehat{\rho}$ and $jA\widehat{\rho}$ act freely, so

$$X^{\bar{E}} = (N_+/K) \sqcup (N_-/K).$$

The localization theorem for the $\bar{E}$-action on the compact smooth manifold $X$ [23, Chapter IV, Theorem 2.1] gives

$$H_E^*(P;\mathbb{F}_2)[z^{-1}] \cong H_E^*(N_+ \sqcup N_-;\mathbb{F}_2)[z^{-1}].$$

Let $N \in \{N_+, N_-\}$ be nonempty. By Lemma 4.4, $N$ is connected and $\beta_{\mathbb{F}_2}(N) = 4$. The involutions commute with $\widehat{\rho}$, so $N$ and $F = N/H$ are reversal invariant. The rational total Betti number of $N$ is at least four by orientability and Euler characteristic zero, and at most four by universal coefficients. Equality excludes two-primary torsion in its integral cohomology. Its rational and mod-two Betti numbers agree in every degree. Lemma 4.5 gives distinct odd generator degrees $p = 2a-1 < q = 2b-1$, with $a + b$ odd. The same exterior multiplication holds modulo two: over $\mathbb{Z}_{(2)}$ the groups are free, odd squares vanish by graded commutativity, and Poincaré duality makes the product of the two odd generators a unit multiple of the top class.

Write $E = \langle g \rangle \times K$, where $g = A$ or $jA$ fixes $N$ pointwise. As $K$ acts freely,

$$H_E^*(N;\mathbb{F}_2) = \mathbb{F}_2[z] \otimes B, \qquad B = H^*(N/K;\mathbb{F}_2). \tag{4.10}$$

For $g = jA$, the character trivial on $g$ and restricting to $t$ on $K$ is $t + z$; the other $K$-character is still $s$. The localized restriction is onto. Its source is generated over $\mathbb{F}_2[z, z^{-1}]$ by $s, t$. Every element of its image is a Laurent polynomial in $z$ with coefficients in the algebra generated by the two $K$-classes. In (4.10), $z$ is independent of $B$. Comparison of the coefficient of $z^0$ shows that $B$ is generated by those two characteristic classes.

All four nonzero cohomology groups of $N$ are one dimensional over $\mathbb{F}_2$, so $K$ acts trivially. Let $\alpha, \beta$ be its cohomology generators, with $|\alpha| = p$ and $|\beta| = q$. The Borel spectral sequence starts with

$$R \otimes \Lambda_{\mathbb{F}_2}(\alpha, \beta), \qquad R = \mathbb{F}_2[s,t].$$

Surjectivity of the base edge homomorphism forces $d_{p+1}\alpha = f_N \neq 0$. Otherwise $E_\infty^{0,p} \neq 0$: every differential with target $E_r^{0,p}$ has zero source, and $d_{p+1} : E_{p+1}^{0,p} \to E_{p+1}^{p+1,0}$ is the only possible nonzero differential from this bidegree. A degree-$p$ class from the base has filtration $p > 0$, a contradiction. There is one possible differential between the two odd rows, $d_{q-p+1}\beta = h_N\alpha$, with $h_N \in R^{q-p+1}$. If it preceded $d_{p+1}$ and $h_N \neq 0$, the $\alpha$-row would become $R/(h_N)$. Its subsequent $R$-linear differential into the polynomial domain $R$ would then be zero, contradicting $f_N \neq 0$. If it came later, the nonzero polynomial $f_N$ would already have removed the $\alpha$-row. The pages cannot coincide, since that would give $q = 2p$, whereas $q$ is odd. The between-row differential is therefore zero. At page $p+1$, the Leibniz rule also gives $d_{p+1}(\alpha\beta) = f_N\beta$. Since $f_N$ is nonzero in the polynomial domain $R$, this differential removes both the $\alpha$-row and the top row. The remaining rows are copies of $R/(f_N)$ in fibre degrees zero and $q$, and their only possible differential is $d_{q+1}\beta = g_N$ modulo $f_N$. Finite dimensionality of $B$ forces $R/(f_N, g_N)$ to be finite dimensional. Hence $(f_N, g_N)$ has height two, and $g_N$ is a non-zero-divisor modulo $f_N$. The spectral sequence is therefore concentrated in the bottom row at its limit, giving

$$B = R/(f_N, g_N), \qquad \dim_{\mathbb{F}_2} B = (p+1)(q+1) = 4ab. \tag{4.11}$$

The ranks over $\mathbb{F}_2(z)$ in localized restriction, together with (4.9), give

$$\sum_{N_i \neq \emptyset} a_i b_i = n(n+1). \tag{4.12}$$

For the involution $g$ fixing $N$, one has $P^g = N$. The compact smooth manifold $P$ is finitistic. The mod-two cohomology rings of $P$ and $N$ are those of $S^{2n-1} \times S^{2n+1}$ and $S^{2a-1} \times S^{2b-1}$, respectively. Since $N$ is nonempty and connected, the product case of Su's theorem [7, Chapter VII, Theorem 8.1(1), p. 410] gives $2a-1 \leq 2n-1$ and $2b-1 \leq 2n+1$. Thus $a \leq n$ and $b \leq n+1$.

Equation (4.12) implies that some $N_i$ is nonempty. If exactly one were nonempty, the degree bounds and $ab = n(n+1)$ would force $(a,b) = (n, n+1)$. Its dimension would be $4n = \dim P$. This fixed component would then be open and closed in the connected manifold $P$, and the fixing involution would be the identity. This is impossible, since both $A$ and $jA$ cover the nonidentity map $\varphi_{L/2}$. Thus both fixed sets are nonempty, proving (4.7). Each $N_i$ is a proper fixed manifold and has dimension divisible by four. Thus $\dim N_i \leq 4n-4$, or equivalently $a_i + b_i \leq 2n-1$.

Put $F_i = N_i/H$. They are distinct components of $\mathrm{Fix}(\varphi_{L/2})$, distinguished by their lifted return scalars $+1, -1$. For the free $H$-action on $N_i$, the mod-two Borel base is $\mathbb{F}_2[c]$, $|c| = 2$. Odd base columns vanish, so there is no differential between the two odd fibre rows or from the top row to the bottom row. Finite dimensionality forces a first nonzero transgression $c^{k_i}$. It removes its source row and the top row, leaving two copies of $\mathbb{F}_2[c]/(c^{k_i})$; any later transgression is zero in this quotient. Consequently

$$\beta_{\mathbb{F}_2}(F_i) = 2k_i \geq 2a_i, \qquad k_i \in \{a_i, b_i\}. \tag{4.13}$$

By Lemma 2.3, the mod-two Euler class of $TM$ is $(n+1)\bar{h}^n$. The sphere-bundle Gysin sequence gives

$$\beta_{\mathbb{F}_2}(U) = \begin{cases} 2n, & n \text{ even}, \\ 2n+2, & n \text{ odd}. \end{cases} \tag{4.14}$$

The Smith inequality for $\varphi_{L/2}$ therefore yields

$$2(a_+ + a_-) \leq \beta_{\mathbb{F}_2}(F_+) + \beta_{\mathbb{F}_2}(F_-) \leq \beta_{\mathbb{F}_2}(U).$$

This proves (4.8). □

4.4. **Even** $n$.

*Proof of Theorem 4.1.* Let $\Phi$ be the lift of Lemma 4.2. Suppose $n$ is even and $\Phi_L = \mathrm{id}$. Proposition 4.6 gives

$$n(n+1) = \sum_i a_i b_i \leq (n+1) \sum_i a_i \leq n(n+1).$$

Equality forces $a_+ + a_- = n$ and $b_+ = b_- = n+1$. Since $a_+, a_- > 0$, one has $a_i < n$.

Choose $g \in \{\Phi_{L/2}, j\Phi_{L/2}\}$ and write $N_i = P^g$. Choose distinct points $p_0, x \in N_i$ and put $Y = P \setminus \{p_0\}$ and $Z = N_i \setminus \{p_0\}$. The finite-dimensional paracompact manifold $Y$ is finitistic, and $Y^g = Z$. Excision and the mod-two fundamental classes show that puncturing removes the top cohomology group and preserves every lower group. Since $Y$ and $Z$ are connected,

$$H^r(Y, \{x\}; \mathbb{F}_2) = \begin{cases} \mathbb{F}_2, & r = 2n-1,\ 2n+1, \\ 0, & \text{otherwise}, \end{cases}$$

$$H^r(Z, \{x\}; \mathbb{F}_2) = \begin{cases} \mathbb{F}_2, & r = 2a_i - 1,\ 2n+1, \\ 0, & \text{otherwise}. \end{cases}$$

Lemma 2.5 and naturality give

$$\mathrm{Sq}^2 : H^{2n-1}(Y, \{x\}; \mathbb{F}_2) \longrightarrow H^{2n+1}(Y, \{x\}; \mathbb{F}_2), \qquad \mathrm{Sq}^2 \neq 0.$$

The ambient and fixed-set degree gaps are 2 and $2(n - a_i + 1) > 2$. Theorem 11.1 of [7, Chapter VII, pp. 425–426] therefore gives $2n+1 \leq 2n-1$, a contradiction. Thus $\Phi_L = j$. □

4.5. **Odd $n$.**

**Lemma 4.7.** *Assume $\Phi_L = \mathrm{id}$, and let $F_i = N_i/H$ be the two half-period components of Proposition 4.6. Let $J_i$ be the index of $F_i$ represented at $L/2$, and let $I$ be the index of the regular component at $L$. Then*

$$I = 2J_i + 2(a_i + b_i) - 2n - 3, \qquad h_i := \frac{J_i - 1}{2} = \frac{I + 2n + 1 - 2(a_i + b_i)}{4}. \tag{4.15}$$

*Proof.* Let $\alpha$ be the canonical contact form on $U$. The dimension of $F_i$ is $2a_i + 2b_i - 3$. In the contact distribution $\ker \alpha|_{F_i}$, the symplectic complement of $TF_i \cap \ker \alpha$ has real rank $2(2n + 1 - a_i - b_i)$. The derivative of $\varphi_{L/2}$ is $-\mathrm{id}$ on this complement, since $\varphi_{L/2}$ is an involution with fixed manifold $F_i$. All effective weights in Lemma 3.4 are therefore $1/2$. At $m = 2$, (3.8) reads

$$I = 2(J_i + 2n - 1) - (2n - 1) - 2(2n + 1 - a_i - b_i),$$

which gives (4.15). □

**Theorem 4.8.** *If $n \geq 3$ is odd, the reversible lift of Lemma 4.2 also satisfies $\Phi_L = j$.*

Suppose $n \geq 3$ is odd and $\Phi_L = \mathrm{id}$. Denote the two half-period fixed manifolds of Proposition 4.6 by $N_1, N_2$. Put $F_i = N_i/H$, represent $F_i$ at length $L/2$, and write $J_i = i(F_i)$ and $Q_i(s) = Q_{F_i}(s)$ for $i = 1, 2$. Let $I$ be the index of the regular component represented at $L$.

**Lemma 4.9.** *Up to interchange, the pairs $(a_i, b_i)$ in Proposition 4.6 have one of the following forms:*

(i) *$a_1, a_2$ are even, $a_1 + a_2 = n + 1$, and $b_1 = b_2 = n$;*

(ii) *$n + 1 = ad$, with $a \geq 3$ odd and $d \geq 2$ even, and*

$$(a_1, b_1) = (a, n + 1 - d), \qquad (a_2, b_2) = (n + 1 - a, n + 1).$$

*Proof.* The product identity and $b_i \leq n + 1$ give $a_1 + a_2 \geq n$; the Smith bound gives $a_1 + a_2 \leq n + 1$. If the sum were $n$, equality in the product bound would force $b_1 = b_2 = n + 1$. Both $a_i$ would then be odd, whereas their sum $n$ is odd. Thus $a_1 + a_2 = n + 1$. Put $d_i = n + 1 - b_i \geq 0$. The product identity becomes

$$a_1 d_1 + a_2 d_2 = n + 1 = a_1 + a_2. \tag{4.16}$$

The $a_i$ have the same parity, and $d_i$ has parity opposite to $a_i$. If the $a_i$ are even, both $d_i$ are positive odd integers; (4.16) forces $d_1 = d_2 = 1$. If the $a_i$ are odd, both $d_i$ are even. They cannot both be positive, since the left side would be at least $2(n + 1)$, and they cannot both be zero. After interchange, $d_1 = d \geq 2$ and $d_2 = 0$. Writing $a = a_1$ gives $ad = n + 1$ and the second pair of formulas. Properness of the second fixed manifold says $2(n + 1) - a \leq 2n - 1$, so $a \geq 3$. □

Write

$$C_0 = \frac{I + 1}{4}, \qquad h_i = \frac{J_i - 1}{2}.$$

Equation (4.15), together with the oddness of $J_i, n, a_i + b_i$, gives $I \equiv 3 \pmod 4$. Thus $C_0$ is an integer and

$$h_i = C_0 + \frac{n - a_i - b_i}{2}. \tag{4.17}$$

One has

$$[s^r]\big([a_i]_s[b_i]_s\big) = a_i \qquad (a_i - 1 \leq r \leq b_i - 1).$$

For the polynomial $s^{h_i} P_{F_i}(s)$, these degrees form the interval

$$\mathcal{I}_i = \left[ C_0 + \frac{n + a_i - b_i - 2}{2}, C_0 + \frac{n - a_i + b_i - 2}{2} \right] \cap \mathbb{Z}. \tag{4.18}$$

Both intervals have center $C_0 + (n - 2)/2$. For $J \in \mathcal{I}_1 \cap \mathcal{I}_2$,

$$\sum_{i=1}^{2} [s^{J - h_i}] P_{F_i}(s) = a_1 + a_2 = n + 1.$$

By Lemma 3.2, $h_i \geq 0$, so every such $J$ is nonnegative. Proposition 3.3 and nonnegativity of the local dimensions give $[s^J]D_n \geq n+1$. Since $[s^J]D_n = J+1$ for $0 \leq J < n$, one has $J \geq n$. Hence

$$[s^J]D_n = \sum_{i=1}^{2} [s^{J-h_i}]P_{F_i}(s) = n+1 \qquad (J \in \mathcal{I}_1 \cap \mathcal{I}_2). \tag{4.19}$$

Every other shifted local rank polynomial has coefficient zero in these degrees.

The local-system signs of $F_1, F_2$ are opposite, by Lemma 4.3: one lifted return is $+1$ and the other is $-1$. For odd $n$ the two global trace series become

$$T_n(s) = 1 + s^2 + \cdots + s^{n-1}, \qquad O_n(s) = -T_n(s) + 2\sum_{\ell \geq 1} (-1)^{\ell-1} s^{\ell n} T_n(s). \tag{4.20}$$

For $J \geq n$, one has $[s^J]T_n(s) = 0$. Writing $J = \ell n + r$ with $0 \leq r < n$, one has $[s^J]O_n(s) = 0$ if and only if $r$ is odd. Among the coefficients of $O_n(s)$ in degrees at least $n$, no two consecutive ones vanish, and every three consecutive ones include a zero.

**Proposition 4.10.** *If $n$ is odd and $\Phi_L = \mathrm{id}$, then $n = 5$. After interchanging $N_1, N_2$ if necessary, their cohomology algebras, over either $\mathbb{Q}$ or $\mathbb{F}_2$, are exterior on generators of degrees $(5,7)$ and $(5,11)$, respectively, and their quotients by $H$ have dimensions eleven and fifteen, respectively.*

*Proof.* In case (i) of Lemma 4.9, write $a_i = 2r_i$. Then $r_1 + r_2 = (n+1)/2$ and $r_i \geq 1$. Lemma 4.5 gives

$$Q_i(s) = [r_i]_{s^2}(1 - s^n).$$

One has $[s^r]Q_i(s) = 0$ for $2r_i - 1 \leq r \leq n-1$. By (4.17), the intersection of the two shifted intervals is

$$[C_0 + r_{\max} - 1,\, C_0 + n - 1 - r_{\max}] \cap \mathbb{Z}, \qquad r_{\max} = \max(r_1, r_2).$$

It has $n - 2r_{\max} + 1 \geq 2$ integers. Every such integer $J$ is at least $n$. By (4.19),

$$[s^J]O_n(s) = \sum_{i=1}^{2} \varepsilon_{F_i} [s^{J-h_i}]Q_i(s) = 0.$$

Thus $O_n$ would have two consecutive zero coefficients, contrary to (4.20).

In case (ii), both $a_i$ are odd and both $b_i$ are even. Now

$$Q_i(s) = [b_i/2]_{s^2}(1 - s^{a_i}), \qquad [s^r]Q_i(s) = (-1)^r \quad (a_i - 1 \leq r \leq b_i - 1).$$

The endpoint differences of $\mathcal{I}_1, \mathcal{I}_2$ are $b_1 - a_1 = ad - a - d$ and $b_2 - a_2 = a$. Since their centers agree, their intersection has exactly

$$\min(ad - a - d, a) + 1 = \min\big((a-1)(d-1), a+1\big)$$

integers. This number is at least four unless $(a,d) = (3,2)$: $a + 1 \geq 4$, and the positive even number $(a-1)(d-1)$ equals two only for that pair. If the intersection has at least four integers, choose $J$ in it with $[s^J]O_n(s) = 0$. Since $J \geq n$, also $[s^J]T_n(s) = 0$. Put $u_i = \varepsilon_{F_i}[s^{J-h_i}]Q_i(s) \in \{1, -1\}$. By (4.19), all other local rank contributions vanish in degree $J$, so their traces vanish as well. Since $\kappa_{F_2} = -\kappa_{F_1}$, the ordinary trace equation and $\kappa_{F_1}$ times the twisted trace equation give

$$u_1 + u_2 = 0, \qquad u_1 - u_2 = 0,$$

which is impossible for $u_i \in \{1, -1\}$.

The remaining pair is $(a,d) = (3,2)$. It gives $n = 5$, $(a_1, b_1) = (3,4)$ and $(a_2, b_2) = (3,6)$; thus $\dim F_1 = 11$ and $\dim F_2 = 15$. □

4.6. **The case $n = 5$.**

**Proposition 4.11.** *If $n = 5$, then $\Phi_L = j$.*

*Proof.* Suppose $n = 5$ and $\Phi_L = \mathrm{id}$. By Proposition 4.10, the fixed manifolds $N_1, N_2$ have odd-generator degrees $(5, 7)$ and $(5, 11)$. Put $F_i = N_i/H$, and let $c$ be the generator of $H^2(BH; \mathbb{F}_2)$. By (4.13), the first nonzero transgression for the free $H$-action is $c^{k_i}$, with

$$\beta_{\mathbb{F}_2}(F_i) = 2k_i, \qquad k_1 \in \{3, 4\}, \qquad k_2 \in \{3, 6\}.$$

Equation (4.14) gives $\beta_{\mathbb{F}_2}(U) = 12$. The two disjoint $F_i$ are half-period fixed components, so Smith's inequality gives $2k_1 + 2k_2 \le 12$. It follows that $k_1 = k_2 = 3$. In particular, the degree-five generator of $N_2$ transgresses to $c^3$ for the $H$-action.

By Proposition 2.4, $K_0 = H \times \langle \widehat{\rho} \rangle$ acts freely on $N_2$. Its base ring is

$$R = H^*(BK_0; \mathbb{F}_2) = \mathbb{F}_2[c, s], \qquad |c| = 2, \quad |s| = 1.$$

Here $c$ is pulled back from $BH$, and $s$ is the nonzero class of $H^1(B\langle \widehat{\rho} \rangle; \mathbb{F}_2)$, pulled back to $BK_0$. The action on the fibre cohomology is trivial, since all its nonzero groups are one dimensional over $\mathbb{F}_2$. Let $\alpha, \beta$ be the generators of $H^*(N_2; \mathbb{F}_2)$, with $|\alpha| = 5$ and $|\beta| = 11$. The Borel spectral sequence begins with $R \otimes \Lambda(\alpha, \beta)$. Naturality with the $H$-Borel spectral sequence gives

$$d_6\alpha = f_6(c, s), \qquad f_6(c, 0) = c^3.$$

By bidegrees and multiplicativity,

$$d_6\beta = 0, \qquad d_6(\alpha\beta) = f_6\beta.$$

Since $R$ is a domain and $f_6 \neq 0$,

$$E_7 = R/(f_6) \otimes \Lambda(\beta).$$

The next possible nonzero differential is $d_{12}\beta$. Its target has total degree at least twelve, so $E_\infty^{10,0} = (R/(f_6))^{10}$. Therefore

$$\ker\big(R^{10} \longrightarrow H^{10}_{K_0}(N_2; \mathbb{F}_2)\big) = f_6 R^4. \tag{4.21}$$

By Proposition 2.4, the characteristic homomorphism for the ambient quotient $P/K_0 = \mathbb{P}_{\mathbb{R}}(TM)$ satisfies

$$K_5(c, s^2) = s^{10} + c^2 s^6 + c^4 s^2 = 0.$$

Restriction to $N_2$ and (4.21) show that $f_6$ divides this polynomial. Set $S = s^2$. Homogeneity gives $f_6(c, s) = F(c, S)$, where $F$ has ordinary degree three and $F(c, 0) = c^3$. The quotient has cohomological degree four, so it too is a polynomial in $c, S$. The divisibility therefore holds in $\mathbb{F}_2[c, S]$, where

$$K_5(c, S) = S(c^2 + cS + S^2)^2.$$

The quadratic factor is irreducible in $\mathbb{F}_2[c, S]$. The only homogeneous cubic divisor of $K_5(c, S)$ is therefore $S(c^2 + cS + S^2)$, which vanishes at $S = 0$. This contradicts $F(c, 0) = c^3$. □

*Proof of Theorem 4.8.* If $\Phi_L = \mathrm{id}$, Proposition 4.10 gives $n = 5$, contrary to Proposition 4.11. Hence $\Phi_L = j$. □

## 5. The regular index and reversal characters

By Theorems 4.1 and 4.8, $\Phi_L = j$. Let $U_m$, $m \in \mathbb{Z}_{\ge 1}$, denote the regular critical manifold represented at length $mL$; write $U = U_1$ when a representation is required. Put

$$I = i(U_1), \qquad N = 2n - 1, \qquad D = I + N, \qquad \mu = D/2.$$

For a reversal-invariant represented component $F$, put $\sigma_F = \varepsilon_F(-1)^{h_F}$. Lemma 3.2 gives $\mu \in \mathbb{Z}_{>0}$. If $F$ is represented at length $\ell$ and $m \ge 1$, let $F^m$ denote the connected critical component at length $m\ell$ containing its $m$th iterates. Its dimension may exceed that of $F$.

5.1. **Iteration of the orientation character.**

**Lemma 5.1.** *Let $F$ be a reversal-invariant represented component, let $m \geq 1$ be odd, and put $G = F^m$. Then $G$ is reversal invariant and $\sigma_G = \sigma_F$.*

*Proof.* The iteration map $I_m : F \to G$ commutes with reversal. Hence the connected critical components $G$ and $\rho(G)$ both contain $I_m(F)$ and are equal. Pull all bundles back to $F$. On $I_m^* \nu_G^-$, translation by $1/m$ in the loop parameter defines a bundle automorphism $T$ with $T^m = \mathrm{id}$ and $\rho T \rho = T^{-1}$. We choose Hilbert metrics invariant under these actions. The Hessian commutes with $T$, so its negative spectral bundle has the real isotypic decomposition for $\langle T \rangle$; its finite-group spectral projectors are continuous bundle maps. This is the cyclic index-form decomposition of [16, Proposition 3.2.8 and Theorem 3.2.9, pp. 90–91].

Iteration identifies the $T$-fixed variation space with the original variation space and multiplies the index form by $m^2$. The image of $\nu_F^-$ is therefore a maximal negative subbundle of this fixed variation space. Its projection onto the $T$-fixed negative spectral bundle is an equivariant isomorphism: the projection is injective because its kernel lies in a nonnegative subspace, and both bundles have rank $i(F)$.

Since $m$ is odd, the remaining real isotypic bundles are indexed by conjugate pairs $e^{\pm i\theta}$, $0 < \theta < \pi$. On such a bundle $E_\theta$, the map

$$J_\theta = \frac{T - \cos\theta \ \mathrm{id}}{\sin\theta}$$

satisfies $J_\theta^2 = -\mathrm{id}$ and $\rho J_\theta = -J_\theta \rho$. Thus $E_\theta$ has its complex orientation and reversal is complex antilinear. If its complex rank is $a_\theta$, its real orientation character is $(-1)^{a_\theta}$. Since $o_G$ is trivial and $F$ is connected, pullback identifies $H^0(G; o_G)$ with $H^0(F; I_m^* o_G)$ equivariantly. Taking determinant lines of the above splitting gives

$$\varepsilon_G = \varepsilon_F (-1)^{\sum_\theta a_\theta} = \varepsilon_F (-1)^{(i(G) - i(F))/2}.$$

Since $h_G - h_F = (i(G) - i(F))/2$, this proves $\sigma_G = \sigma_F$. The decompositions take place in the negative bundles; additional kernel directions at the iterate do not enter them. □

**Lemma 5.2.** *For every $m \geq 1$,*

$$i(U_m) = mI + (m-1)N = mD - N, \qquad \varepsilon_{U_m} = \varepsilon_U^m.$$

*Proof.* The index formula is Lemma 3.4 with $k = 0$. Identify each represented regular manifold with $U = SM$. For $v \in U$, put $c_v(t) = p\varphi_t(v)$. For a real function $f \in H^1([0, mL])$, the index form on $f\dot{c}_v$ is $\int_0^{mL} |f'|^2 \, dt$. Thus only normal variations contribute to the negative bundle. Fix an orientation of $M$ and orient

$$\xi_v = v^\perp \subset T_{p(v)}M$$

by placing $v$ before a positive basis of $\xi_v$. Its rank is $N$. Reversal induces the identity on the underlying subspace $\xi_v = \xi_{-v}$ and reverses this orientation.

Since $\varphi_L = \mathrm{id}$ on $U$, every normal Jacobi field along $c_v$ returns with its initial value and covariant derivative after $L$, and hence after $mL$. Let $\mathcal{P}_m$ and $\mathcal{H}_m$ be the Hilbert bundles of periodic and fixed-endpoint normal $H^1$ fields on $[0, mL]$, respectively, and let $q_m$ be the index form. The energy Hessian on $[0, 1]$, transported by linear reparametrization, equals $mL\, q_m$. For $a \in \xi_v$, let $J_a$ be the Jacobi field with initial data $(a, 0)$. It lies in the kernel of the periodic form. The decomposition

$$V = (V - J_{V(0)}) + J_{V(0)}$$

splits $q_m$ into its restriction to $\mathcal{H}_m$ and this kernel subbundle. It commutes with reversal, which sends $V(t)$ to $V(mL - t)$ along $c_{-v}$. Thus the fixed-endpoint and periodic negative bundles are equivariantly isomorphic.

Let $\mathcal{W}_m \subset \mathcal{H}_m$ consist of the fields vanishing at every $jL$, $0 \leq j \leq m$. Restriction to the $m$ intervals identifies its form with the direct sum of $m$ copies of the fixed-endpoint form at period $L$. Its negative

bundle is $(\nu_U^-)^{\oplus m}$. Its kernel is $K_m = \xi^{\oplus m}$: on interval $j$, the corresponding Jacobi field has initial data $(0, w_j)$ and final derivative $w_j$. Evaluation at the interior division points gives an exact sequence

$$0 \longrightarrow \mathcal{W}_m \longrightarrow \mathcal{H}_m \xrightarrow{\mathrm{ev}} E_m \longrightarrow 0, \qquad E_m = \xi^{\oplus(m-1)}.$$

For $w = (w_1, \dots, w_m) \in K_m$ and $\mathrm{ev}(V) = x = (x_1, \dots, x_{m-1})$, integration by parts gives

$$q_m(w, V) = \sum_{j=1}^{m-1} \langle w_j - w_{j+1}, x_j \rangle = \langle \Delta w, x \rangle, \qquad (\Delta w)_j = w_j - w_{j+1}.$$

The map $\Delta : K_m \to E_m$ is onto and has diagonal kernel $\xi$. Reversal sends $w_j$ to $-w_{m+1-j}$ and $x_j$ to $x_{m-j}$; hence $\Delta$ is equivariant.

Equip the field bundles with their reversal-invariant $H^1$ metrics and $E_m$ with the product metric. The map $s = \mathrm{ev}^*(\mathrm{ev}\,\mathrm{ev}^*)^{-1}$ is a continuous equivariant right inverse of evaluation. Put $\mathcal{A}_m = K_m^\perp \cap \mathcal{W}_m$. The forms $q_m|_{\mathcal{W}_m}$ are represented by a norm-continuous family of self-adjoint operators of the form identity plus compact. Their kernel bundle $K_m$ has constant rank $mN$, so the restriction of $q_m$ to $\mathcal{A}_m$ is invertible. Compactness of $U$ gives a uniform spectral gap at zero on $\mathcal{A}_m$. There is therefore a unique continuous equivariant map $B : E_m \to \mathcal{A}_m$ such that

$$q_m(Bx, a) = q_m(sx, a) \qquad (a \in \mathcal{A}_m).$$

Put $s_0 = s - B$. Then $s_0(E_m)$ is $q_m$-orthogonal to $\mathcal{A}_m$ and still projects isomorphically to $E_m$.

Choose a reversal-invariant complement $K_m'$ of $\ker \Delta$ in $K_m$, and identify it with $E_m$ by $\Delta$. On $K_m' \oplus s_0(E_m)$ the form is

$$(k, x) \longmapsto 2\langle k, x \rangle + \langle Sx, x \rangle,$$

where $S$ is a continuous self-adjoint equivariant endomorphism of $E_m$. With $k' = k + Sx/2$, the form is $2\langle k', x \rangle$. The graph $k = -Sx/2 - x$ is a maximal negative subbundle, equivariantly isomorphic to $E_m$. The diagonal kernel of $\Delta$ pairs to zero with every summand. Together with the negative bundle of $\mathcal{A}_m$, this graph gives a negative subbundle of rank $mI + (m-1)N = i(U_m)$. Projection onto the periodic negative spectral bundle is continuous and reversal equivariant. By the projection argument in the proof of Lemma 5.1, it is an isomorphism on this subbundle. Consequently

$$\nu_{U_m}^- \simeq (\nu_U^-)^{\oplus m} \oplus \xi^{\oplus(m-1)}$$

as reversal-equivariant bundles over $U$.

Reversal reverses the order of the summands in each part. On each copy of the first part it induces the original reversal on $\nu_U^-$; on each copy of the second it induces the identity $\xi_v \to \xi_{-v}$. Its orientation character is therefore

$$\varepsilon_U^m (-1)^{Im(m-1)/2} (-1)^{m-1+N(m-1)(m-2)/2}.$$

Since $I$ and $N$ are odd, the exponent apart from $\varepsilon_U^m$ is congruent to $m(m-1)$ modulo two. It is even, so $\varepsilon_{U_m} = \varepsilon_U^m$. □

**Proposition 5.3.** *Let $F$ be a reversal-invariant underlying critical manifold, and put $q = q_F$. Then $q$ is odd and $\Phi_{L/q}|_{P(F)} = j|_{P(F)}$. Represent $F$ at its generic prime period $L/q$. For $m \in \mathbb{Z}_{\geq 1}$, let $G = F^m$ be represented at length $mL/q$. Then $G$ is reversal invariant and*

$$\kappa_G = (-1)^m, \qquad \sigma_G = (-1)^{n+(n+1)m}. \tag{5.1}$$

*Thus $\sigma_G = \kappa_G$ when $n$ is even, whereas $\sigma_G = -1$ when $n$ is odd. There is exactly one represented component $C$ of index one; every geodesic in $C$ is prime, and $\varepsilon_C = \kappa_C = -1$.*

*Proof.* The flow circle $\mathbb{R}/L\mathbb{Z}$ preserves $F$, since it commutes with the return map and is connected. Its action on $F$ is smooth and locally free. By the principal orbit theorem [7, Chapter IV, Theorem 3.1, p. 179], it has a dense principal stratum. Since the circle is abelian, the stabilizers on this stratum are one finite subgroup $K$. Every element of $K$ fixes the dense stratum and hence $F$; conversely, the

kernel of the action is contained in every stabilizer. Thus $K$ is the kernel. Put $q = |K|$ and $\tau = L/q$. If $F$ is represented at length $\ell$, then $\ell \in \tau\mathbb{Z}_{>0}$ and

$$F \subset \mathrm{Fix}(\varphi_\tau) \subset \mathrm{Fix}(\varphi_\ell).$$

The component of $\mathrm{Fix}(\varphi_\tau)$ containing $F$ is a connected subset of $\mathrm{Fix}(\varphi_\ell)$, so it equals $F$. At this generic prime representation, Lemma 4.3 gives

$$\Phi_\tau|_{P(F)} = \kappa_F \,\mathrm{id}, \qquad \kappa_F \in \{1, -1\}.$$

Since $q\tau = L$ and $\Phi_L = j$, one has $\kappa_F^q = -1$. Thus $q$ is odd and $\kappa_F = -1$.

Every represented component has positive odd index by Lemma 3.2, and $H^0(B_F; \mathbb{Q}) = \mathbb{Q}$. Thus the coefficient of $s^0$ in the dimension series of Proposition 3.3 counts the index-one represented components. This coefficient is one, so there is exactly one such component $C$; uniqueness makes it reversal invariant. The coefficients of $s^0$ in the ordinary and twisted trace series are $-1$ and $1$, respectively. Since reversal acts trivially on $H^0(B_C; \mathbb{Q})$, they give $\varepsilon_C = -1$ and $\varepsilon_C\kappa_C = 1$. Hence $\kappa_C = -1$.

Bott's iteration formula gives $i(\gamma^m) \geq i(\gamma)$ for every closed geodesic $\gamma$ and every $m \geq 1$ [16, Theorem 3.2.9, p. 91]. Since $C$ is the unique index-one represented component and every nonconstant critical index is positive, the minimal-index argument of [33, p. 952] implies that every geodesic in $C$ is prime. Its generic stabilizer order $q_C$ is odd, and its $q_C$th iterate lies in $U$. Lemma 5.1 gives $\sigma_U = \sigma_C = -1$.

By Lemma 5.2,

$$h_{U_m} = m\mu - n, \qquad \sigma_{U_m} = (-1)^n\big(\varepsilon_U(-1)^\mu\big)^m.$$

The equality $\sigma_U = -1$ gives $\varepsilon_U(-1)^\mu = (-1)^{n+1}$. The iteration map from $F$ commutes with reversal, so $G$ is reversal invariant. By Lemma 4.3, the return $\Phi_{mL/q}$ is a constant scalar on $P(G)$. Its restriction to $P(F)$ is $(-1)^m$, so $\kappa_G = (-1)^m$. The $q$th iterate of $G$ lies in $U_m$. Since $q$ is odd, Lemma 5.1 gives $\sigma_G = \sigma_{U_m}$ and proves (5.1). □

5.2. **The regular index.**

**Theorem 5.4.** *There is an integer $r \geq 0$ such that*

$$I = 4nr + 1, \qquad D = 2n(2r + 1), \qquad \varepsilon_U = -1. \tag{5.2}$$

*Proof.* The fixed sets at lengths $\ell$ and $\ell + L$ coincide. Their components have the same orbit quotients, and their indices differ by $D$, by Lemma 3.4. For a reversal-invariant component with generic period $L/q$, write $\ell = mL/q$. Addition of $L$ replaces $m$ by $m + q$. Proposition 5.3 and the oddness of $q$ show that its ordinary local reversal trace is multiplied by

$$\delta = (-1)^{n+1+\mu},$$

and its twisted trace by $-\delta$. An exchanged pair contributes zero to both traces.

The smooth circle action on the compact manifold $U$ has finitely many orbit types [7, Chapter IV, Proposition 1.2, p. 173]. Its stabilizers are finite, so only finitely many represented lengths occur in $(0, L]$. Each corresponding fixed set is a compact submanifold and has finitely many components. Let $\mathcal{C}_0$ be the finite set of reversal-invariant represented components at these lengths, and put

$$A(s) = \sum_{F \in \mathcal{C}_0} \varepsilon_F s^{h_F} Q_F(s), \qquad B(s) = \sum_{F \in \mathcal{C}_0} \varepsilon_F \kappa_F s^{h_F} Q_F(s).$$

Every positive represented length has a unique expression $\ell_0 + kL$, with $0 < \ell_0 \leq L$ and $k \geq 0$. Thus

$$O_n(s) = \frac{A(s)}{1 - \delta s^\mu}, \qquad T_n(s) = \frac{B(s)}{1 + \delta s^\mu}. \tag{5.3}$$

For $F \in C_0$, the free reversal involution on $B_F$ has Lefschetz number $Q_F(1) = 0$. Hence $A(1) = B(1) = 0$. Since

$$\lim_{s\to 1} T_n(s) = \lim_{s\to 1} \frac{1-s^{n+1}}{1-s^2} = \frac{n+1}{2} \neq 0,$$

the second identity in (5.3) excludes $\delta = 1$. Therefore $\delta = -1$.

Recall the polynomial

$$Q_n(s) = \frac{(1-s^n)(1-s^{n+1})}{1-s^2}$$

of (3.6). One of $n, n+1$ is even, so this quotient is a polynomial. It is relatively prime to $1+s^n$ in $\mathbb{Q}[s]$. Indeed, if $z^n = -1$ and $z^2 \neq 1$, then $Q_n(z) = 2/(1-z) \neq 0$. The only remaining possible common root is $z = -1$ when $n$ is odd, and then $Q_n(-1) = n+1 \neq 0$. Since $O_n = -Q_n/(1+s^n)$, the first identity in (5.3) gives

$$1 + s^n \mid 1 + s^\mu.$$

Evaluation at $s = \exp(\pi \mathrm{i}/n)$ yields $\mu = n(2r+1)$ for an integer $r$. Since $\mu > 0$, one has $r \geq 0$. It follows that $D = 2n(2r+1)$ and $I = D - (2n-1) = 4nr + 1$. Finally, $\varepsilon_U(-1)^\mu = (-1)^{n+1}$ and $\mu \equiv n \pmod 2$ give $\varepsilon_U = -1$. □

Put $t = 2r+1$, so $D = 2nt$.

**Corollary 5.5.** *All prime geodesic periods are equal if and only if $r = 0$.*

*Proof.* By Proposition 5.3, the unique index-one component $C$ consists of prime geodesics. If $r = 0$, then $I = 1$, so $C = U_1$ and every geodesic has prime period $L$. Conversely, if all prime periods are equal, their common value is $L$; hence $C = U_1$, and (5.2) gives $r = 0$. □

**Corollary 5.6.** *Among reversal-invariant represented components $F$ with $\kappa_F = +1$, the least index is $2n+1$, and there is exactly one component at that index. It is $C^2$ and has the same underlying critical manifold as $C$. Every effective Bott weight of $C$ is strictly less than $1/2$.*

*Proof.* The global traces satisfy

$$O_n(s) + T_n(s) = \frac{2s^n T_n(s)}{1+s^n}.$$

This series vanishes in degrees below $n$ and has coefficient 2 in degree $n$. Components with $\kappa_F = -1$ cancel in the sum, and exchanged pairs contribute zero. Hence there are reversal-invariant components with $\kappa_F = 1$. Let $j_0$ be their least half-index, and let $a$ be the number having half-index $j_0$. By (5.1), each has $\varepsilon_F = (-1)^{n+j_0}$, so

$$[s^{j_0}](O_n + T_n) = 2(-1)^{n+j_0} a \neq 0.$$

Consequently $j_0 = n$ and $a = 1$.

The component $C^2$ is reversal invariant and satisfies $\kappa_{C^2} = 1$. Let $\beta_1, \dots, \beta_k$ be the effective Bott weights of $C$ at its prime period. Lemma 3.4 gives

$$i(C^2) = 2n + 1 - 2\sum_{\nu=1}^{k} \lfloor 2\beta_\nu \rfloor \leq 2n+1.$$

The preceding bound forces equality. Every summand is therefore zero, so $0 < \beta_\nu < 1/2$. In particular, $2\beta_\nu \notin \mathbb{Z}$ for every $\nu$, and the nullity formula in Lemma 3.4 gives $\dim C^2 = \dim C$. The inclusion $C \subset C^2$ is open by equality of dimensions and closed by compactness. Since $C^2$ is connected, the two underlying critical manifolds coincide. □

**Corollary 5.7.** *Suppose $r > 0$. The index-one component $C$ is proper. If $\dim C = 2d_C + 1$, then*

$$b_2(B_C) = 2, \qquad d_C \geq 3,$$

*and there is no critical component of index three.*

*Proof.* The connected quotient $B_C$ has a free reversal involution and is therefore not a point. Its symplectic class is nonzero and negative under reversal. Since $n \geq 2$, Proposition 3.3 gives

$$[s]D_n = 2, \qquad [s]O_n = [s]T_n = 0.$$

Thus $1 \leq b_2(B_C) \leq 2$. If $b_2(B_C) = 1$, its degree-two class would contribute ordinary trace $+1$ and twisted trace $-1$, because $\varepsilon_C = \kappa_C = -1$. The remaining rank one would come from $H^0(B_F; \mathbb{Q})$ for a single component $F$ of index three. This component must be reversal invariant, since an exchanged pair contributes rank two. The two trace equations force $\varepsilon_F = \kappa_F = -1$. As $h_F = 1$, this gives $\sigma_F = 1$, whereas Proposition 5.3 gives $\sigma_F = -1$ whenever $\kappa_F = -1$. Therefore $b_2(B_C) = 2$ and

$$\operatorname{tr}(\rho^* | H^2(B_C; \mathbb{Q})) = 0.$$

The rank in half-degree one is exhausted, so there is no component of index three.

Poincaré duality excludes $d_C = 1$. If $d_C = 2$, the reversal traces in degrees zero, two, and four would be $1, 0, 1$, respectively, giving Lefschetz number two. This contradicts freeness, so $d_C \geq 3$. Finally, $r > 0$ gives $I > 1$, and hence $C \neq U$. □

5.3. **Even Jordan chains for the rotation class.** Write

$$X = (\Lambda M)_{S^1}, \qquad \mathcal{M} = M \times BS^1.$$

For a positive regular value $a$ of the energy $E$, put

$$X_a = \left(E^{-1}((-\infty, a])\right)_{S^1}, \qquad H_a = H^*(X_a, \mathcal{M}; \mathcal{L}).$$

Here $\mathcal{L}$ is the normalized sign local system. For a graded rational vector space $V$ with reversal involution $\rho^*$, write

$$V^{k,\pm} = \ker(\rho^* \mp \mathrm{id} : V^k \to V^k), \qquad V^{\pm} = \bigoplus_k V^{k,\pm}.$$

Multiplication by the rotation class $u \in H^2(BS^1; \mathbb{Q})$ on relative equivariant cohomology satisfies $\rho^*(ux) = -u\rho^* x$.

**Lemma 5.8.** *The module $H_a$ is finite dimensional, is concentrated in odd degrees, and is annihilated by a power of $u$. Moreover*

$$H_a^- = uH_a^+, \qquad \ker(u : H_a \longrightarrow H_a) \subset H_a^-. \tag{5.4}$$

*Thus $u : H_a^+ \to H_a^-$ is an isomorphism, with degree shift two.*

*Proof.* There are finitely many critical components below $a$. Since $\nu_F^-$ is $S^1$-orientable and $\mathcal{L}|_{F_{S^1}} \cong \mathbb{Q}$ by Lemma 3.2 and Proposition 3.3, equivariant Thom theory [13, §1.5 and §2.1–2.2, pp. 91, 94–97] gives the local group $H^{*-i(F)}(B_F; \mathbb{Q})$. Since $i(F)$ is odd and $H^{\mathrm{odd}}(B_F; \mathbb{Q}) = 0$, the sublevel groups are finite dimensional and concentrated in odd degrees. The cohomology sequences of pairs of sublevels give surjective restriction maps. Multiplication by $u$ is nilpotent because it raises degree.

For each integer $k$, the degreewise stabilization in the proof of Proposition 3.3 identifies $H^k(X, \mathcal{M}; \mathcal{L})$ with $H_b^k$ by restriction for all sufficiently large regular values $b$. Composing with the epimorphism $H_b^k \to H_a^k$ for such a $b > a$ shows that

$$H^*(X, \mathcal{M}; \mathcal{L}) \longrightarrow H_a$$

is surjective. This restriction is reversal equivariant and $\mathbb{Q}[u]$-linear.

By the reversal-equivariant $\mathbb{Q}[u]$-linear isomorphism (3.4), the global relative group is $H^{*-1}(\mathcal{M}; \mathbb{Q})$. On $H^*(\mathcal{M}; \mathbb{Q}) = \mathbb{Q}[h, u]/(h^{n+1})$, reversal fixes $h$ and negates $u$. Every negative eigenvector is therefore $u$ times a positive eigenvector. For $x \in H_a^-$, choose a global lift $\xi$ and replace it by $(\xi - \rho^*\xi)/2$. Write $\xi = u\eta$ with $\eta$ positive. Restriction gives $x = u(\eta|_{X_a})$. Since $uH_a^+ \subset H_a^-$, it follows that $H_a^- = uH_a^+$.

One has $\operatorname{tr}(\rho^* | H_a) = 0$. Indeed, trace is additive on the finite Morse filtration. An invariant component contributes $\varepsilon_F \kappa_F Q_F(1) = 0$, while an exchanged pair contributes zero. Thus $\dim H_a^+ = \dim H_a^-$. The surjective map $u : H_a^+ \to H_a^-$ is therefore an isomorphism. Since $u$ interchanges the two eigenspaces, its kernel is contained in $H_a^-$. □

**Theorem 5.9.** *Every represented critical component is invariant under reversal. Its characters and dimension satisfy*

$$\varepsilon_F = \kappa_F, \qquad \dim F \equiv 3 \pmod 4,$$

*and its generic stabilizer order is odd. On $A_F = H^*(B_F;\mathbb{Q})$, let $u$ act by multiplication by $u_F$. This action induces isomorphisms*

$$u : A_F^{2j,+} \xrightarrow{\simeq} A_F^{2j+2,-} \qquad (j \geq 0). \tag{5.5}$$

*The module $A_F$ is a sum of even-length $u$-Jordan chains with positive initial vectors. In particular,*

$$Q_F(s) = \frac{1-s}{1+s} P_F(s). \tag{5.6}$$

*If $n$ is even, every critical index is congruent to one modulo four.*

*Proof.* Let regular values $a < b$ enclose exactly one critical energy. Odd-degree concentration gives an exact sequence of graded reversal-equivariant $\mathbb{Q}[u]$-modules

$$0 \longrightarrow V \longrightarrow H_b \longrightarrow H_a \longrightarrow 0, \qquad V = H^*(X_b, X_a; \mathcal{L}).$$

If $x \in V^-$, Lemma 5.8 writes $x = uy$ with $y \in H_b^+$. The image of $y$ in $H_a$ is positive and is killed by $u$; the same lemma forces that image to vanish. Thus $y \in V$. The inclusion in $H_b$ also gives

$$V^- = uV^+, \qquad \ker(u|V) \subset V^-. \tag{5.7}$$

The Thom isomorphism splits $V$ into the groups $H^{*-i(F)}(B_F;\mathbb{Q})$, each preserved by $u$. Suppose $\rho(F) \neq F$, and let $x$ be the Thom image of $1 \in H^0(B_F;\mathbb{Q})$. Then $x - \rho^* x$ is a nonzero vector in $V^-$. Its projection to the $F$ summand cannot be divisible by $u$, since a preimage would lie in $H^{-2}(B_F;\mathbb{Q}) = 0$. This contradicts (5.7). Thus every component is invariant. For an invariant component, the Thom image of 1 cannot be negative for the same reason. Hence $\varepsilon_F \kappa_F = 1$. Each component summand is now reversal stable, and reversal acts trivially on its twisted Thom line. Consequently (5.7) descends to $A_F$ and gives (5.5).

The nilpotent operator $v = u^2$ preserves $A_F^+$. Choose homogeneous Jordan chains for $v$ there. A chain $x, vx, \dots, v^{a-1}x$, together with its image under the isomorphism $u : A_F^+ \to A_F^-$, gives the chain

$$x, ux, u^2x, \dots, u^{2a-1}x.$$

These even-length chains form a basis of $A_F$. Put $J_F(s) = \sum_{j\geq 0} \dim_{\mathbb{Q}} A_F^{2j,+} s^j$. Then $P_F = (1+s)J_F$ and $Q_F = (1-s)J_F$, giving (5.6).

Write $\dim F = 2d_F + 1$. Its quotient top class is killed by $u$, so it is negative. Reversal is antisymplectic and has sign $(-1)^{d_F}$ on that class. Hence $d_F$ is odd. Proposition 5.3 now gives odd generic stabilizer order for every component. If $n$ is even, the same proposition gives $\varepsilon_F(-1)^{h_F} = \kappa_F$; since $\varepsilon_F = \kappa_F$, the integer $h_F$ is even. □

**Corollary 5.10.** *Let $R_\pm(s)$ be the sum of $s^{h_F} P_F(s)$ over all represented components with $\varepsilon_F = \kappa_F = \pm 1$. Then*

$$R_-(s) = \frac{D_n(s)}{1+s^n} = \frac{[n]_s[n+1]_s}{1-s^{2n}}, \qquad R_+(s) = s^n R_-(s).$$

*Equivalently, if $J_F = P_F/(1+s)$ and $J_\pm$ are the corresponding sums of $s^{h_F} J_F$, then*

$$J_-(s) = \frac{J_U(s)}{1-s^{2n}}, \qquad J_+(s) = s^n J_-(s), \qquad J_U(s) = \frac{[n]_s[n+1]_s}{1+s}. \tag{5.8}$$

*Each coefficient in these formal series is a finite sum of nonnegative integers.*

*Proof.* Theorem 5.9 gives

$$R_+ + R_- = D_n, \qquad \frac{1-s}{1+s}(R_+ - R_-) = O_n.$$

Substitution of the expressions in Proposition 3.3 gives the first two identities. Dividing by $1+s$ proves the remaining ones. For the regular quotient, Lemma 4.5 and the cohomology of $P$ give $P_U = [n]_s[n+1]_s$, so its positive-eigenspace polynomial is the stated $J_U$. □

The polynomial $J_U$ factors as

$$J_U(s) = \begin{cases} [n/2]_{s^2}[n+1]_s, & n \text{ even}, \\ [n]_s[(n+1)/2]_{s^2}, & n \text{ odd}. \end{cases}$$

Its coefficients are positive precisely in degrees $0, \dots, 2n-2$.

For $b \in \mathbb{Z}_{\geq 0}$, put

$$I_b^- = [2nb, 2nb + 2n - 2] \cap \mathbb{Z}, \qquad I_b^+ = n + I_b^-.$$

These are the *coefficient intervals* of $J_-$ and $J_+$, respectively. For $0 \leq j \leq 2n-2$,

$$[s^{2nb+j}]J_- = [s^{n+2nb+j}]J_+ = [s^j]J_U.$$

Moreover, $[s^{2nb+2n-1}]J_- = [s^{n+2nb+2n-1}]J_+ = 0$.

**Corollary 5.11.** *For each component $F$, the polynomial $J_F$ has nonnegative integer coefficients, is palindromic of degree $d_F - 1$, and has constant and leading coefficients one. Its coefficient in every even degree between zero and $d_F - 1$ is positive.*

*For a summand $s^{h_F}J_F$ of $J_-$, the integer $h_F$ is even; for a summand of $J_+$, the integer $h_F - n$ is even. The support of each regular summand in* (5.8) *is a coefficient interval. The indices $b$ of these intervals satisfy*

$$b \equiv \frac{t-1}{2} \pmod{t} \quad \text{in } J_-, \qquad b \equiv -1 \pmod{t} \quad \text{in } J_+.$$

*Proof.* By Theorem 5.9, $J_F(s) = \sum_{j \geq 0} \dim_{\mathbb{Q}} A_F^{2j,+} s^j$, so its coefficients are nonnegative integers. Poincaré duality makes $P_F$ palindromic of degree $d_F$. Since $P_F = (1+s)J_F$, it makes $J_F$ palindromic of degree $d_F - 1$. Connectedness gives its constant coefficient one, and hence also its leading coefficient. By Lemma 3.1, $u_F^{2j} \neq 0$ whenever $2j \leq d_F$. Since reversal fixes $u_F^{2j}$, the coefficient of $s^{2j}$ in $J_F$ is positive.

Let $\ell$ be the represented length of $F$, and write $\ell = mL/q_F$ with $m \in \mathbb{Z}_{\geq 1}$. Apply Proposition 5.3 to the generic prime representation of its underlying manifold. Together with $\varepsilon_F = \kappa_F$, it gives

$$\kappa_F = (-1)^m, \qquad (-1)^{h_F} = (-1)^{n(m+1)}.$$

Thus $h_F$ is even when $\kappa_F = -1$, and $h_F \equiv n \pmod 2$ when $\kappa_F = 1$. Finally Lemma 5.2 gives $h_{U_m} = n(tm-1)$. For $m = 2j+1$, the support of $s^{h_{U_m}}J_U$ is $I_{tj+(t-1)/2}^-$; for $m = 2j$, $j \geq 1$, it is $I_{tj-1}^+$. □

5.4. **Floors and nullities.** For $x \in \mathbb{R}$, write $\{x\} = x - \lfloor x \rfloor$ for its fractional part.

**Corollary 5.12.** *Every component represented at its generic prime period has index congruent to one modulo four.*

*Proof.* At represented length $L/q_F$, Proposition 5.3 gives $\kappa_F = \sigma_F = -1$, while Theorem 5.9 gives $\varepsilon_F = \kappa_F$. Thus $(-1)^{h_F} = 1$. □

**Proposition 5.13.** *Let $F$ be represented at its generic prime period $L/q$, and let $\beta_1, \dots, \beta_k \in (0,1)$ be its effective Bott weights. Each weight is rational with odd reduced denominator dividing $q$, and*

$$\sum_{\nu=1}^{k} \lfloor m\beta_\nu \rfloor \in 2\mathbb{Z} \qquad (m \geq 1). \tag{5.9}$$

*For every integer $e > 1$, the number of weights with exact reduced denominator $e$ is even. In particular $k$ is even, and the nullity increase at every iterate is divisible by four.*

*Proof.* The $q$th return derivative is the identity, so Lemma 3.4 gives $q\beta_\nu \in \mathbb{Z}$ for every $\nu$. Since $0 < \beta_\nu < 1$, its reduced denominator divides $q$, which is odd by Theorem 5.9. Put

$$h_m = h_{F^m}, \qquad S(m) = \sum_{\nu=1}^{k} \lfloor m\beta_\nu \rfloor.$$

Theorem 5.9 and (5.1) give

$$(-1)^{h_m} = (-1)^{n(m+1)},$$

whereas (3.8) gives

$$h_m = mh_F + n(m-1) - S(m).$$

Since $h_F$ is even by Corollary 5.12,

$$S(m) \equiv n(m-1) - n(m+1) = 0 \pmod 2.$$

The floor formula is valid also when some $m\beta_\nu$ is integral.

Let $z(m) = \#\{\nu \,:\, m\beta_\nu \in \mathbb{Z}\}$. By Lemma 3.4 and Theorem 5.9,

$$2z(m) = \dim F^m - \dim F \in 4\mathbb{Z}.$$

For each $e \mid q$, let $c_e$ count the weights with reduced denominator $e$; thus $c_1 = 0$. For every divisor $d$ of $q$,

$$z(d) = \sum_{e \mid d} c_e \in 2\mathbb{Z}.$$

Induction over the divisors of $q$ gives

$$c_d = z(d) - \sum_{\substack{e \mid d \\ e < d}} c_e \in 2\mathbb{Z}.$$

No other denominator occurs. Hence every reduced denominator occurs with even multiplicity, $k = z(q)$ is even, and every nullity increase $2z(m)$ is divisible by four. □

**Lemma 5.14.** *Let $F$ be represented at its generic prime period $L/q$, and let $\beta_1, \dots, \beta_k$ be its effective Bott weights. Put $B = \sum_{\nu=1}^{k} \beta_\nu$. The limit*

$$\Delta_F = \lim_{m \to \infty} \frac{i(F^m)}{m}$$

*exists and satisfies*

$$\Delta_F = 2h_F + 2n - 2B, \qquad q\Delta_F = D = 2nt. \tag{5.10}$$

*Proof.* Dividing (3.8) by $m$ and using $\lfloor m\beta_\nu \rfloor / m \to \beta_\nu$ gives

$$\Delta_F = 2h_F + 2n - 2B.$$

At $m = q$, one has $F^q = U_1$ and $q\beta_\nu \in \mathbb{Z}$ for every $\nu$. Thus (3.8) gives

$$I + (2n-1) = q(2h_F + 2n - 2B) = q\Delta_F = D.$$

Theorem 5.4 identifies $D$ with $2nt$. □

## 6. Prime-power fixed-point sets and large critical manifolds

Let $T$ be the torus generated by $H$ and $\Phi$ on $P$. For an underlying critical manifold $F \subset U$, put $P(F) = \pi^{-1}(F)$.

### 6.1. **Degrees along a chain of cyclic subgroups.**

The degree bounds below hold for both $\Phi_L = \mathrm{id}$ and $\Phi_L = j$.

**Lemma 6.1.** *Let $K \subset T$ be cyclic of order $p^e$, where $p$ is prime and $e \in \mathbb{Z}_{\geq 1}$. Assume that lifted reversal normalizes $K$ and that $P^K$ is nonempty. For $s \in \{0, \dots, e\}$, let $K_s$ be its subgroup of order $p^s$, and put $N_s = P^{K_s}$, with $N_0 = P$. Then $N_s$ is connected, and there are positive integers $a_s < b_s$, with $a_s + b_s$ odd, such that*

$$H^*(N_s; \mathbb{Q}) = \Lambda_{\mathbb{Q}}(\alpha_{2a_s-1}, \beta_{2b_s-1}).$$

*The degrees satisfy*

$$a_s \leq a_{s-1}, \qquad b_s \leq b_{s-1} \qquad (1 \leq s \leq e). \tag{6.1}$$

*In particular, $a_e \leq n$ and $b_e \leq n+1$.*

*Proof.* Since $P^K \subset N_s$, every $N_s$ is nonempty. For $s \geq 1$, Lemma 4.4 gives a connected manifold $N_s = P(F_s)$ with $\beta_{\mathbb{F}_p}(N_s) = 4$. Since $K_s$ is characteristic in $K$, the manifolds $N_s$ and $F_s$ are reversal invariant. The Poincaré-duality and Euler-characteristic argument in the proof of Lemma 4.4, with rational coefficients, gives $\beta_{\mathbb{Q}}(N_s) \geq 4$. The universal coefficient theorem then gives $\beta_{\mathbb{Q}}(N_s) = 4$ and shows that $H^*(N_s;\mathbb{Z})$ has no $p$-primary torsion. Lemma 4.5 gives the rational algebra in the statement. The mod-$p$ cohomology has the same four nonzero degrees. The squares of the odd generators vanish by degree, and their product is nonzero by Poincaré duality. Hence $H^*(N_s;\mathbb{F}_p) = \Lambda_{\mathbb{F}_p}(\alpha_{2a_s-1}, \beta_{2b_s-1})$. For $N_0 = P$, these assertions hold with $a_0 = n$ and $b_0 = n+1$ by Lemma 2.3.

For $1 \leq s \leq e$, the group $K_s/K_{s-1} \cong \mathbb{Z}/p\mathbb{Z}$ acts on $N_{s-1}$ with fixed set $N_s$. Both are compact smooth manifolds, hence finitistic, with finitely generated integral cohomology. Apply [7, Chapter VII, Theorems 8.1(1) and 9.1(1), pp. 410 and 416] for $p = 2$ and odd $p$, respectively. Connectedness and the exterior ring on two generators of distinct positive odd degrees exclude the other alternatives. The product alternative gives a labelling $u, v$ of the generator degrees of $N_s$ with $u \leq 2a_{s-1} - 1$ and $v \leq 2b_{s-1} - 1$. Hence

$$2a_s - 1 = \min(u,v) \leq 2a_{s-1} - 1, \qquad 2b_s - 1 = \max(u,v) \leq 2b_{s-1} - 1.$$

Iteration gives the last two inequalities. □

6.2. **Restriction to an iterated component.** Let $F \subsetneq U$ be represented at its generic prime period $L/q$. Write $\dim F = 2d+1$ and $k = 2n-1-d > 0$, and let $\beta_1, \dots, \beta_k$ be its effective Bott weights. For $m \in \mathbb{Z}_{\geq 1}$, put

$$S(m) = \sum_{\nu=1}^{k} \lfloor m\beta_\nu \rfloor, \qquad T_F(m) = S(m) - mh_F.$$

The integer $h_F$ and all values $S(m)$ are even by Corollary 5.12 and Proposition 5.13. For an underlying critical manifold $Y$, the $T$-action on $P(Y)$ is locally free, by the independence of the infinitesimal generators in the proof of Lemma 4.5; its orbit space is $B_Y$. Let

$$r_Y : P(Y)_T \longrightarrow B_Y, \qquad p_Y : P(Y)_T \longrightarrow BT$$

be the orbit map and the Borel projection. Let $K_Y$ be the finite kernel of the action. The map $P(Y)_T \to P(Y)_{T/K_Y}$ has rationally acyclic homotopy fibre $BK_Y$. For a $T$-invariant metric on $P(Y)$, the effective $T/K_Y$-action has an orbit map $P(Y)_{T/K_Y} \to B_Y$ inducing a rational cohomology isomorphism [3, §§2.5–2.6, pp. 1360–1361]. Hence $r_Y^*$ is an isomorphism. Define the characteristic homomorphism by

$$\chi_Y = (r_Y^*)^{-1} p_Y^* : H^*(BT;\mathbb{Q}) \longrightarrow H^*(B_Y;\mathbb{Q}).$$

**Lemma 6.2.** *Suppose $\chi_F$ is onto. For $m \geq 1$, put $G = F^m$, represented at length $mL/q$. Then restriction*

$$H^*(B_G;\mathbb{Q}) \longrightarrow H^*(B_F;\mathbb{Q})$$

*is onto and commutes with reversal. Consequently $[s^j]J_G \geq [s^j]J_F$ for every $j \geq 0$. The homomorphism $\chi_F$ is onto if $H^*(P(F);\mathbb{Q})$ is exterior on two odd generators.*

*Proof.* The $T$-equivariant inclusion $P(F) \subset P(G)$ induces a map $\iota : B_F \to B_G$. Naturality of the Borel projection and the orbit map gives

$$\chi_F = \iota^* \chi_G.$$

Since $\chi_F$ is onto, so is $\iota^*$. By Theorem 5.9, both components are reversal invariant, and $\iota^*\rho^* = \rho^*\iota^*$. Applying $(1+\rho^*)/2$ to a preimage gives surjectivity on each $+1$-eigenspace. Their rank polynomials are $J_G$ and $J_F$ by the same theorem.

If $H^*(P(F);\mathbb{Q})$ is exterior on two odd generators, then $\beta_{\mathbb{Q}}(P(F)) = 4$, and Lemma 4.5 gives surjectivity of $\chi_F$. □

**Proposition 6.3.** *Assume that $\chi_F$ is onto and that every coefficient of $J_F$ in degrees $0, \dots, d-1$ is positive. For every $m \geq 1$, the least nonnegative residue of $T_F(m)$ modulo $2n$ avoids $1, \dots, d$.*

*Proof.* Fix $m \geq 1$ and put $G = F^m$, represented at length $mL/q$. By (5.1) and Theorem 5.9, $\varepsilon_G = \kappa_G = (-1)^m$. Equation (3.8) gives

$$h_G = mh_F + n(m-1) - S(m) = n(m-1) - T_F(m).$$

Let $a \in \{0, \dots, 2n-1\}$ be the residue of $T_F(m)$ modulo $2n$, and suppose $1 \leq a \leq d$. Then

$$g = h_G + a - 1 \geq h_G \geq 0, \qquad g \equiv n(m-1) - 1 \pmod{2n}.$$

For odd $m$, $g \equiv -1 \pmod{2n}$, so its coefficient in $J_-$ is zero. For even $m$, $g \equiv n-1 \pmod{2n}$, so its coefficient in $J_+$ is zero. Both assertions follow from (5.8), since $\deg J_U = 2n-2$. On the other hand, $s^{h_G} J_G$ is a summand of the corresponding series and Lemma 6.2 gives

$$[s^{a-1}]J_G \geq [s^{a-1}]J_F > 0.$$

The nonnegativity of all summands gives a contradiction. □

**Theorem 6.4.** *Under the hypotheses of Proposition 6.3, suppose $d \geq n$. Then*

$$X(m) = \sum_{\nu=1}^{k} \{m\beta_\nu\} \in 2\mathbb{Z} \qquad (m \geq 1).$$

*In particular, for $B = \sum_\nu \beta_\nu$ there is an integer $\ell \geq 0$ such that*

$$0 < B < k, \qquad B \in 2\mathbb{Z}, \qquad h_F = B + 2n\ell, \qquad \Delta_F = 2n(2\ell+1), \qquad q(2\ell+1) = t. \tag{6.2}$$

*Thus $q \mid t$.*

*Proof.* Here $k = 2n - 1 - d < n$. By Proposition 6.3, the least nonnegative residue of $T_F(m)$ modulo $2n$ lies in $\{0, 2n-k, \dots, 2n-1\}$. Let $\tau(m) \in [-k, 0] \cap \mathbb{Z}$ be the representative of $T_F(m)$ modulo $2n$, and put

$$c(m) = \sum_\nu \lfloor 2\{m\beta_\nu\} \rfloor, \qquad 0 \leq c(m) \leq k.$$

The identity $T_F(2m) = 2T_F(m) + c(m)$ gives

$$\tau(2m) - 2\tau(m) - c(m) \in 2n\mathbb{Z} \cap [-2k, 2k] = \{0\}.$$

Since $X(2m) = 2X(m) - c(m)$, the function $Y(m) = \tau(m) + X(m)$ satisfies

$$Y(2m) = 2Y(m), \qquad |Y(m)| \leq k.$$

Thus $2^j |Y(m)| = |Y(2^j m)| \leq k$ for every $j \in \mathbb{Z}_{\geq 0}$, and $Y(m) = 0$. The integer $T_F(m)$ is even, so $\tau(m)$ and $X(m) = -\tau(m)$ are even.

At $m = 1$, $X(1) = B$ and $T_F(1) = -h_F$. Hence $h_F - B \in 2n\mathbb{Z}$; write $h_F = B + 2n\ell$. The inequalities $0 < \beta_\nu < 1$ give $0 < B < k$. Lemma 5.14 yields

$$\Delta_F = 2h_F + 2n - 2B = 2n(2\ell+1), \qquad q(2\ell+1) = t > 0.$$

Therefore $\ell \geq 0$. □

6.3. **Subgroups of the lifted geodesic flow.** Since $\Phi_L = j$ and $L$ is the least period of $\varphi$, the lifted flow has least period $2L$. For an odd positive integer $a$, put

$$C_a = \langle j\Phi_{L/a} \rangle \subset T.$$

Since $j\Phi_{L/a} = \Phi_{2L(a+1)/(2a)}$ and $\gcd\big((a+1)/2, a\big) = 1$, $C_a$ is the subgroup of order $a$ of the lifted flow circle. Lifted reversal inverts it by Lemma 4.2.

Let $F$ be a connected component of $U^{\varphi_{L/a}}$, represented at length $L/a$. Theorem 5.9 and Lemma 4.3 give

$$\Phi_{L/a}|_{P(F)} = \kappa_F \,\mathrm{id}, \qquad \kappa_F \in \{1, -1\}.$$

Taking the $a$th power gives $\kappa_F^a = -1$, so $\kappa_F = -1$. Consequently

$$P^{C_a} = \pi^{-1}\big(U^{\varphi_{L/a}}\big).$$

If $a$ is a prime power and this set is nonempty, Lemma 4.4 makes it connected. In particular, when the generic denominator $q$ of $F$ is a prime power, Proposition 5.3 gives $P(F) = P^{C_q}$.

If $F \subset G$, every time map fixing $G$ pointwise also fixes $F$. The kernels of the geodesic circle action on $F$ and $G$ have orders $q_F$ and $q_G$ by Proposition 5.3. Hence

$$q_G \mid q_F. \tag{6.3}$$

For a prime $p$ and $v \in \mathbb{Z}_{\geq 1}$, write $p^v \| q_F$ if $p^v \mid q_F$ and $p^{v+1} \nmid q_F$. For such a divisor, put

$$G_{p^v}(F) = \pi(P^{C_{p^v}}).$$

Since $L/p^v = (q_F/p^v)(L/q_F)$, the manifold $F$ lies in $U^{\varphi_{L/p^v}}$. The fixed-set identity and Lemma 4.4 show that $G_{p^v}(F)$ is a connected critical manifold containing $F$. It is proper, since $p^v > 1$ and $L$ is the least common period. Its geodesic circle kernel contains the subgroup of order $p^v$; therefore

$$p^v \mid q_{G_{p^v}(F)} \mid q_F. \tag{6.4}$$

**Lemma 6.5.** *Let $F$ be proper, with generic denominator $q \mid t$, and put $d = (\dim F - 1)/2$ and $k = 2n-1-d$. Let $\beta_1, \dots, \beta_k$ be the effective Bott weights of $F$ at its generic prime period. Let $m \in \mathbb{Z}_{\geq 1}$ be nonresonant, that is, $m\beta_\nu \notin \mathbb{Z}$ for every $\nu$. The support of the summand $s^{h_{F^m}} J_{F^m}$ is contained in one coefficient interval of its character series in* (5.8)*. If $c_m$ is the least integer in that interval, then*

$$h_{F^m} - c_m = X(m) := \sum_{\nu=1}^{k} \{m\beta_\nu\} \in 2\mathbb{Z} \cap (0, k),$$
$$h_{F^m} + d - 1 - c_m \leq 2n - 4.$$

*For $m = 2j$, the interval is $I^+_{tj/q-1}$.*

*Proof.* Put $K = t/q$, an odd positive integer, and $B = \sum_{\nu=1}^{k} \beta_\nu$. Equation (5.10) gives

$$h_F - B = n(K-1) \in 2n\mathbb{Z}.$$

Corollary 5.12 and Proposition 5.13 give $h_F, S(m), k \in 2\mathbb{Z}$. Thus $B \in 2\mathbb{Z}$ and $X(m) = mB - S(m) \in 2\mathbb{Z}$. Nonresonance gives $0 < X(m) < k$, hence $2 \leq X(m) \leq k-2$.

Lemma 3.4 gives $\dim F^m = \dim F$ at a nonresonant order. The connected closed submanifolds $F \subset F^m$ therefore coincide as underlying critical manifolds. Hence $J_{F^m} = J_F$. Put $h_m = h_{F^m}$. Equation (3.8) gives

$$h_m = n(Km - 1) + X(m).$$

One has $\varepsilon_{F^m} = \kappa_{F^m} = (-1)^m$ by Theorem 5.9 and (5.1). For odd $m$, the initial degree is $2nb + X(m)$ with $b = (Km-1)/2 \geq 0$. For $m = 2j$, it is $n + 2nb + X(m)$ with $b = Kj - 1 \geq 0$. By Corollary 5.11, $\deg J_F = d - 1$, and

$$\operatorname{supp}(s^{h_m} J_F) \subset [c_m + 2, c_m + 2n - 4] \cap \mathbb{Z},$$

where $c_m = 2nb$ for odd $m$ and $c_m = n + 2nb$ for even $m$. Indeed, $X(m) + d - 1 \leq (k-2) + (d-1) = 2n - 4$. □

**Theorem 6.6.** *Let $F$ be proper, write $q = q_F$, and put $d = (\dim F - 1)/2$. If $d \geq n$, at least one of the following holds:*

*(1) $n$ is odd, $d = n$, and for some full prime-power divisor $p^v \| q$ one has*

$$F = \pi(P^{C_{p^v}}), \qquad H^*(P(F); \mathbb{Q}) = \Lambda_{\mathbb{Q}}(\alpha_1, \beta_{2n+1}),$$
$$P_F(s) = [n+1]_s, \qquad J_F(s) = 1 + s^2 + \cdots + s^{n-1};$$

*(2) its generic denominator $q$ divides $t$, and the conclusions of Lemma 6.5 hold.*

*In particular, every proper critical manifold of dimension greater than $2n+1$ has generic denominator dividing $t$.*

*Proof.* By the proof of Proposition 5.3, $q = 1$ would give $F = U$; hence $q > 1$. For each full prime-power divisor $p^v \| q$, put $G = G_{p^v}(F)$. Lemmas 4.5 and 6.1 give

$$P_G(s) = [a]_s[b]_s, \qquad 1 \le a < b \le n+1, \qquad a+b \text{ odd}, \qquad d_G = a+b-2 \ge d.$$

If one such $G$ has $a = 1$, then $n \le d \le d_G = b-1 \le n$. Thus $d = d_G = n$ and $F = G$, by the equal-dimension argument in the proof of Lemma 6.5. Hence $b = n+1$, and $n$ is odd by Theorem 5.9. Lemma 4.5 gives the first alternative and its polynomials.

Suppose that $a \ge 2$ for every $G_{p^v}(F)$. Write $\{a, b\} = \{e, o\}$, where $e$ is even and $o$ is odd. Then $e \ge 2$, $o \ge 3$, and

$$J_G(s) = [e/2]_{s^2}[o]_s.$$

Since $o \ge 3$, the intervals $[2j, 2j+o-1]$, for integers $0 \le j < e/2$, cover $[0, e+o-3]$. Thus every coefficient in degrees $0, \dots, e+o-3 = d_G - 1$ is positive. The characteristic homomorphism is onto by Lemma 4.5, and $d_G \ge n$. Theorem 6.4 therefore gives $q_G \mid t$. By (6.4), $p^v \mid t$ for every full prime-power divisor of $q$. Hence $q \mid t$, and Lemma 6.5 applies. □

**Proposition 6.7.** *Suppose $n$ is even and $F$ is proper with $d_F = n-1$. If its generic denominator $q_F$ does not divide $t$, then for every full prime-power divisor $p^v \| q_F$ with $p^v \nmid t$ one has $F = G_{p^v}(F) = \pi(P^{C_{p^v}})$. In particular, $H^*(P(F); \mathbb{Q})$ is exterior on two odd generators, and its total rational Betti number is four.*

*Proof.* Since $q_F \nmid t$, there is a full prime-power divisor $p^v \| q_F$ with $p^v \nmid t$. Fix any such divisor and put $G = G_{p^v}(F)$. If $F \ne G$, their connectedness and the dimension congruence in Theorem 5.9 give $d_G \ge n+1$. Theorem 6.6 then gives $q_G \mid t$, contrary to $p^v \mid q_G$. Thus $F = G$, and Lemma 6.1 gives the asserted cohomology. □

## 7. Iteration and period arithmetic

Let $r \in \mathbb{Z}_{\ge 0}$ be as in Theorem 5.4, and put $I = 4nr+1$ and $t = 2r+1$. Let $F$ be a proper underlying critical manifold, represented at its generic prime period $L/q_F$. Put

$$\dim F = 2d_F + 1, \qquad h_F = \frac{i(F)-1}{2}, \qquad k_F = 2n-1-d_F.$$

By Theorem 5.9 and Corollary 5.12, the integers $q_F, d_F$ are odd and $h_F$ is even. Let $\beta_1, \dots, \beta_{k_F} \in (0,1)$ be the effective Bott weights of $F$. Since $F$ is proper and $d_F \ge 1$, $0 < k_F \le 2n-2$. Proposition 5.13 gives $k_F \in 2\mathbb{Z}$ and odd reduced denominators dividing $q_F$.

For $m \ge 1$, put $G_m = F^m$, represented at length $mL/q_F$, and set

$$\begin{aligned} S_F(m) &= \sum_{\nu=1}^{k_F} \lfloor m\beta_\nu \rfloor, \qquad X_F(m) = \sum_{\nu=1}^{k_F} \{m\beta_\nu\}, \\ z_F(m) &= \#\{\nu \,:\, m\beta_\nu \in \mathbb{Z}\}, \qquad h_F(m) = \frac{i(F^m)-1}{2}. \end{aligned} \tag{7.1}$$

Lemma 3.4 gives

$$h_F(m) = m(h_F+n) - n - S_F(m), \qquad \dim G_m = 2d_F + 1 + 2z_F(m). \tag{7.2}$$

An order $m$ is *nonresonant* if $z_F(m) = 0$. Then $G_m = F$ as underlying manifolds, by the equal-dimension argument in the proof of Lemma 6.5. Both $S_F(m)$ and $z_F(m)$ are even by Proposition 5.13.

Corollary 5.10 gives

$$J_-(s) = \frac{J_U(s)}{1-s^{2n}}, \qquad J_+(s) = s^n J_-(s), \qquad J_U(s) = \frac{[n]_s[n+1]_s}{1+s}. \tag{7.3}$$

Write $I_b^-$ and $I_b^+ = n + I_b^-$, $b \in \mathbb{Z}_{\geq 0}$, for the coefficient intervals defined after Corollary 5.10.

For $a \in \mathbb{Z}$, put $c_a = [s^a]J_U$. In particular $c_{-1} = 0$. For $0 \leq a < n$, equation (7.3) gives $c_a + c_{a-1} = a+1$, hence $c_a = \lfloor a/2 \rfloor + 1$. By Corollary 5.11, $J_U$ is palindromic of degree $2n-2$. Together with $c_{2n-1} = 0$, this gives

$$[s^{2j+1}]J_U(s) = \min\{j+1, n-j-1\}, \qquad 0 \leq j \leq n-1. \tag{7.4}$$

In particular,

$$[s]J_U = [s^{2n-3}]J_U = 1, \qquad [s^{2j}]J_U = [s^{2j+1}]J_U \quad (0 \leq 2j+1 < n).$$

Equation (5.1) and Theorem 5.9 give $\varepsilon_{G_m} = \kappa_{G_m} = (-1)^m$. For a condition $A$, put $\mathbf{1}_A = 1$ if $A$ holds and $\mathbf{1}_A = 0$ otherwise. Define the initial residue by

$$v_F(m) \equiv h_F(m) - n\mathbf{1}_{2|m} \pmod{2n}, \qquad 0 \leq v_F(m) < 2n. \tag{7.5}$$

Equation (7.2) gives $v_F(m) \in 2\mathbb{Z}$. Since $J_{G_m}(0) = 1$ and every summand in Corollary 5.10 has nonnegative coefficients, there is a unique $b \in \mathbb{Z}_{\geq 0}$ such that

$$h_F(m) - n\mathbf{1}_{2|m} = 2nb + v_F(m), \qquad 0 \leq v_F(m) \leq 2n-2.$$

Thus $h_F(m) \in I_b^-$ for odd $m$ and $h_F(m) \in I_b^+$ for even $m$.

Lemma 5.14 and (7.2) give

$$\begin{aligned} \Delta_F &= 2h_F + 2n - 2\sum_\nu \beta_\nu, \qquad q_F\Delta_F = 2nt, \\ h_F(m) &= \frac{nmt}{q_F} - n + X_F(m). \end{aligned} \tag{7.6}$$

These identities hold at every order. For the regular component, set $k_U = 0$ and $S_U(m) = X_U(m) = z_U(m) = 0$ for every $m \geq 1$. Let $Y$ be a component represented at length $mL/q_Y$. Its representation at $L/q_Y$ and its $m$th iterate both have underlying manifold $Y$, by the proof of Proposition 5.3. Hence $z_Y(m) = 0$ by Lemma 3.4.

### 7.1. **A range of residues and divisibility.**

**Proposition 7.1.** *Let $F$ be proper. Suppose that the residues at all its nonresonant orders lie in an interval $[a, b] \subset [0, 2n)$ such that*

$$2b - a < 2n, \qquad b - 2a + k_F < 2n. \tag{7.7}$$

*Then there exist $W \in 2\mathbb{Z}$ and $\ell \in \mathbb{Z}_{\geq 0}$ such that*

$$W = \sum_\nu \beta_\nu, \qquad 0 < W < k_F, \qquad h_F = W + 2n\ell, \qquad q_F(2\ell+1) = t. \tag{7.8}$$

*At every nonresonant order $m$,*

$$X_F(m) = v_F(m). \tag{7.9}$$

*In particular $q_F \mid t$. At every nonresonant even order $2j$,*

$$\operatorname{supp}\big(s^{h_F(2j)}J_F\big) \subset I^+_{tj/q_F - 1}. \tag{7.10}$$

*Proof.* Put $k = k_F$, $v = v_F$, $X = X_F$, and

$$c(m) = \sum_{\nu=1}^{k} \lfloor 2\{m\beta_\nu\} \rfloor.$$

Then $0 \leq c(m) \leq k$ and

$$X(2m) = 2X(m) - c(m), \qquad h_F(2m) = 2h_F(m) + n - c(m).$$

Equation (7.5) gives

$$v(2m) \equiv 2v(m) - c(m) \pmod{2n}. \tag{7.11}$$

Since every reduced weight denominator is odd, $2^j m$ is nonresonant for every nonresonant $m$ and every $j \geq 0$. At such an order, (7.7) gives

$$v(2m) - 2v(m) + c(m) \in 2n\mathbb{Z} \cap [a - 2b, b - 2a + k] \subset 2n\mathbb{Z} \cap (-2n, 2n) = \{0\}.$$

Thus $E(m) = v(m) - X(m)$ satisfies $E(2m) = 2E(m)$. The bounds $0 \leq v(m) < 2n$ and $0 < X(m) < k < 2n$ give $|E(m)| < 2n$ at every nonresonant order. The boundedness argument in the proof of Theorem 6.4, applied along the nonresonant orders $2^j m$, gives $E(m) = 0$.

Order one is nonresonant. Therefore $W = X(1) = v(1)$ is even and $0 < W < k$. Equation (7.5) gives $h_F - W = 2n\ell$ with $\ell \in \mathbb{Z}$. Equation (7.6) yields

$$2nt = q_F\big(2h_F + 2n - 2W\big) = 2nq_F(2\ell + 1).$$

Since $t > 0$, one has $\ell \geq 0$ and $q_F \mid t$. Lemma 6.5 gives (7.10). □

**Corollary 7.2.** *Let $n \geq 3$, and let $F$ be a proper seven-dimensional critical manifold. If*

$$b_2(B_F) \geq \left\lceil \frac{n}{3} \right\rceil + 1,$$

*then $q_F \mid t$. Equations (7.8)–(7.10) hold for $F$.*

*Proof.* Put $c = b_2(B_F) - 1 \geq \lceil n/3 \rceil$. Corollary 5.11 and $P_F = (1 + s)J_F$ give

$$J_F(s) = 1 + cs + s^2, \qquad k_F = 2n - 4.$$

At a nonresonant order, write $v_F(m) = 2j$, with $0 \leq j \leq n-1$. The summand is $s^{h_F(m)} J_F$. Nonnegativity of all other summands and (7.4) give

$$c \leq [s^{2j+1}] J_U(s) = \min\{j + 1, n - j - 1\}.$$

Thus every nonresonant residue belongs to $[a, b]$, where

$$a = 2c - 2, \qquad b = 2n - 2c - 2.$$

Applying the preceding coefficient inequality at $m = 1$ gives $c \leq \lfloor n/2 \rfloor$. Since $c \geq 1$, one has $0 \leq a \leq b < 2n$. Moreover,

$$2b - a = b - 2a + k_F = 4n - 6c - 2 \leq 2n - 2 < 2n.$$

Proposition 7.1 applies. □

7.2. **Addition of orders, including resonances.**

**Proposition 7.3.** *Let $F$ be a proper underlying critical manifold with $b_2(B_F) \geq 3$. Assume that, for every $l \geq 1$, there is $b \in \mathbb{Z}_{\geq 0}$ such that $\operatorname{supp}(s^{h_F(l) - n\mathbf{1}_{2|l}} J_{G_l}) \subset I_b^-$. If $m$ is nonresonant and $a = v_F(m)$ satisfies*

$$2n - 2d_F - 6 < a < d_F + 5, \tag{7.12}$$

*then $X_F(m) = v_F(m)$.*

*Proof.* Put $d = d_F$, $k = k_F$, $q = q_F$, $B = \sum_\nu \beta_\nu$, $v = v_F$, $S = S_F$, $X = X_F$, and $z = z_F$. Equations (7.2) and (7.5) give

$$v(l) \equiv lh_F - S(l) \pmod{2n}. \tag{7.13}$$

Since every weight denominator divides $q$,

$$S(l + q) = S(l) + qB.$$

By (7.6) and the oddness of $q$ and $t$,

$$q(h_F - B) = n(t - q) \in 2n\mathbb{Z}.$$

Thus $v, X, z$ induce functions on $\mathbb{Z}/q\mathbb{Z}$, denoted by the same letters, with

$$v(0) = v(q) = 0, \qquad X(0) = X(q) = 0, \qquad z(0) = z(q) = k.$$

By (7.2) and Corollary 5.11, $J_{G_l}$ has degree $d+z(l)-1$ and nonzero constant and leading coefficients. Support containment gives

$$0 \le v(l) \le 2n - 2 - (d + z(l) - 1) = k - z(l). \tag{7.14}$$

If $z(l) = 0$, then $G_l$ has underlying manifold $F$. The identity $P_F = (1 + s)J_F$ from Theorem 5.9 and palindromicity from Corollary 5.11 give $J_F = 1$ and $b_2(B_F) = 1$ if $d = 1$. Thus $d \ge 3$, and

$$[s]J_F = [s^{d-2}]J_F = b_2(B_F) - 1 \ge 2.$$

If $v(l) = 0$, coefficient comparison gives $[s]J_F \le [s]J_U = 1$. If $v(l) = k$, it gives $[s^{d-2}]J_F \le [s^{k+d-2}]J_U = [s^{2n-3}]J_U = 1$. Both inequalities contradict $b_2(B_F) \ge 3$. Thus $v(l) \ne 0, k$. Since $v(l)$ is even,

$$2 \le v(l) \le k - 2 \qquad \text{if } z(l) = 0.$$

If $z(l) > 0$, then $z(l) \ge 2$ by Proposition 5.13. Together with (7.14), these inequalities give

$$v(l) \le k - 2, \qquad v(l) + z(l) \ge 2 \qquad (l \in \mathbb{Z}/q\mathbb{Z}). \tag{7.15}$$

For $l \ge 1$, put

$$c(l, m) = \sum_{\nu=1}^{k} \lfloor \{l\beta_\nu\} + \{m\beta_\nu\} \rfloor .$$

Since $q\beta_\nu \in \mathbb{Z}$, one has $c(l + q, m) = c(l, m)$. Thus $c(\cdot, m)$ also defines a function on $\mathbb{Z}/q\mathbb{Z}$. The normal Jacobi path in the proof of Lemma 3.4 has finite-order endpoint in $\mathrm{Sp}(4n-2, \mathbb{R})$, and index and normal nullity $i(G_j)$ and $2d + 2z(j)$ at order $j \ge 1$. Since $m$ is nonresonant, $z(m) = z(\gcd(l, m)) = 0$. The floor identity $S(l + m) = S(l) + S(m) + c(l, m)$ and (7.2) give

$$i(G_{l+m}) - i(G_l) - i(G_m) = 2n - 1 - 2c(l, m).$$

The endpoint has $4n-2$ eigenvalues on the unit circle, counted with algebraic multiplicity. Long–Zhu's addition inequalities [21, Theorem 2.4, inequalities (2.54)–(2.55)] therefore give

$$z(l + m) \le c(l, m) \le k - z(l). \tag{7.16}$$

Let $A = v(l) + a - c(l, m)$. The floor identity and (7.13) imply $A - v(l + m) \in 2n\mathbb{Z}$. Equations (7.15) and (7.16) give

$$A - v(l + m) \ge a - k + v(l) + z(l) - v(l + m) \ge a - 2k + 4,$$
$$A - v(l + m) \le a + v(l) - z(l + m) - v(l + m) \le a + k - 4.$$

Since $k = 2n - 1 - d$, condition (7.12) is equivalent to

$$2k - 2n - 4 < a < 2n - k + 4.$$

Consequently

$$A - v(l + m) \in 2n\mathbb{Z} \cap (-2n, 2n) = \{0\},$$

and

$$v(l + m) = v(l) + v(m) - c(l, m).$$

The identity $X(l + m) = X(l) + X(m) - c(l, m)$ now gives, for $E = v - X$,

$$E(l + m) - E(l) = E(m) \qquad (l \in \mathbb{Z}/q\mathbb{Z}).$$

Translation by $m$ permutes $\mathbb{Z}/q\mathbb{Z}$, so

$$qE(m) = \sum_{l\in\mathbb{Z}/q\mathbb{Z}} \big(E(l+m) - E(l)\big) = 0.$$

Thus $X_F(m) = v_F(m)$. □

**Corollary 7.4.** *Under the hypotheses of Proposition 7.3, let $m$ be a nonresonant order satisfying* (7.12)*, and let $b$ index the coefficient interval containing the support of its summand. Its represented period is*

$$\frac{mL}{q_F} = \begin{cases} (2b+1)L/t, & m \text{ odd}, \\ 2(b+1)L/t, & m \text{ even}. \end{cases} \tag{7.17}$$

*In particular, for $m = 2j$ one has $b = tj/q_F - 1$. If $m = 1$ satisfies the residue condition, then $q_F \mid t$.*

*Proof.* By Proposition 7.3 and (7.6),

$$\begin{aligned} 2nb &= h_F(m) - n\mathbf{1}_{2|m} - v_F(m) \\ &= n\left(\frac{mt}{q_F} - 1 - \mathbf{1}_{2|m}\right). \end{aligned}$$

This gives (7.17) and, for $m = 2j$, $b = tj/q_F - 1$. At $m = 1$ it gives $t/q_F = 2b + 1$. □

7.3. **A lower bound for the mean index.** Let $C \subset U$ be the underlying manifold of the unique component of index one in Proposition 5.3.

**Lemma 7.5.** *If $F \neq C$ is a proper underlying critical manifold, then*

$$\Delta_F > d_F + 3. \tag{7.18}$$

*Proof.* Proposition 5.13 gives $z_F(2) = 0$, so the underlying manifold of $G_2$ is $F$. By Corollary 5.6, the underlying manifold of $C^2$ is $C$. Since $F \neq C$, the represented components $G_2$ and $C^2$ are distinct, and $G_2$ has coefficient character $+1$ by Theorem 5.9 and (5.1). Corollaries 5.6 and 5.11 give $h_F(2) > n$ and $h_F(2) \equiv n \pmod 2$; hence $h_F(2) \geq n + 2$. Since $k_F > 0$ and $0 < \{2\beta_\nu\} < 1$ for every $\nu$, $0 < X_F(2) < k_F$. Equation (7.6) gives

$$\Delta_F = h_F(2) + n - X_F(2) > 2n + 2 - k_F = d_F + 3.$$

□

## 8. Differences of adjacent coefficients

For an underlying critical manifold $F$, put

$$D_F(x) = \sum_{j\geq 0} \big([s^{2j}]J_F(s) - [s^{2j+1}]J_F(s)\big)x^j. \tag{8.1}$$

Let $\boldsymbol{D}$ be the set of underlying critical manifolds $F$ such that $D_F \notin \mathbb{Z}_{\geq 0}[x]$. Degrees of represented summands are measured from 0 in $J_-$, from $n$ in $J_+$, or from the initial degree of a specified coefficient interval. Coefficients outside the support of a Laurent polynomial are zero.

By Corollary 5.11, $J_F$ is palindromic of even degree $d_F - 1$, with nonnegative integer coefficients and constant and leading coefficients one. Its coefficient in every even degree between zero and $d_F - 1$ is positive. The initial and final degrees of every represented summand, relative to its character origin or the initial degree of any coefficient interval of the same character series, are even.

For a represented component with underlying manifold $F$ and length $mL/q_F$, one has $z_F(m) = 0$ and coefficient character $(-1)^m$, as in Section 7.

For $\alpha \in 2\mathbb{Z}$, put $Q(s) = s^\alpha J_F(s) \in \mathbb{Z}[s, s^{-1}]$. If $[s^{2j+1}]Q > 0$, then $\alpha \leq 2j$ and $2j + 2 \leq \alpha + d_F - 1$, since both endpoints are even. Corollary 5.11 therefore gives

$$[s^{2j+1}]Q > 0 \quad\Longrightarrow\quad [s^{2j}]Q > 0, \quad [s^{2j+2}]Q > 0 \qquad (j \in \mathbb{Z}). \tag{8.2}$$

Consequently, if the coefficients of the chosen character series in two adjacent degrees are both one, the unique summand nonzero in the odd degree is also the unique summand nonzero in the even degree.

8.1. **Inverse images with total Betti number four.**

**Lemma 8.1.** *If $\beta_{\mathbb{Q}}(P(F)) = 4$, then $D_F \in \mathbb{Z}_{\geq 0}[x]$. More precisely, choose integers $a \geq 1$ and $b \geq 0$ such that*

$$P_F(s) = [2a]_s[2b+1]_s,$$

*as in Lemma 4.5. Then*

$$J_F(s) = [a]_{s^2}[2b+1]_s, \qquad D_F(x) = x^b[a]_x. \tag{8.3}$$

*Proof.* With $[0]_s = 0$, the identity $P_F = (1+s)J_F$ in Corollary 5.10 gives

$$J_F(s) = [a]_{s^2}\big([b+1]_{s^2} + s[b]_{s^2}\big).$$

Hence

$$D_F(x) = [a]_x\big([b+1]_x - [b]_x\big) = x^b[a]_x.$$

□

**Lemma 8.2.** *For every underlying critical manifold $F$, $\beta_{\mathbb{Q}}(P(F)) \in 4\mathbb{Z}_{>0}$. If $F \in \mathcal{D}$, then $F$ is proper, $\beta_{\mathbb{Q}}(P(F)) \geq 8$, and $q_F$ has at least two distinct odd prime divisors.*

*Proof.* Put $N = P(F)$. As in the proof of Lemma 4.4, $N$ is closed, connected and oriented, and its free $H$-action gives $\chi(N) = 0$. Theorem 5.9 gives $\dim N = 2d_F + 2 \in 4\mathbb{Z}$. Since $d_F + 1$ is even, Poincaré duality [11, Theorem 3.30] gives

$$\beta_{\mathbb{Q}}(N) = 2\sum_{j \text{ odd}} b_j(N) = 4\sum_{\substack{j < d_F+1 \\ j \text{ odd}}} b_j(N) \in 4\mathbb{Z}_{>0}.$$

If $F \in \mathcal{D}$, Lemma 8.1 excludes the value four. Since $\beta_{\mathbb{Q}}(P) = 4$ by Lemma 2.3, one has $F \neq U$ and $q_F > 1$. Theorem 5.9 makes $q_F$ odd. If $q_F$ were a prime power, the fixed-set identity preceding (6.3) would give $P(F) = P^{C_{q_F}}$. Lifted reversal normalizes $C_{q_F}$, so Lemma 6.1 would give $\beta_{\mathbb{Q}}(P(F)) = 4$, a contradiction. □

**Lemma 8.3.** *Suppose that $F$ is proper, $q_F \nmid t$, and $\beta_{\mathbb{Q}}(P(F)) \neq 4$. Then*

$$d_F \leq \begin{cases} n-3, & n \text{ even}, \\ n-2, & n \text{ odd}. \end{cases} \tag{8.4}$$

*Proof.* If $d_F \geq n$, Theorem 6.6 gives either $q_F \mid t$ or $\beta_{\mathbb{Q}}(P(F)) = 4$, contrary to the hypotheses. Thus $d_F < n$. For even $n$, Proposition 6.7 also excludes $d_F = n-1$. Since $d_F$ is odd by Theorem 5.9, the stated bounds follow. □

8.2. **Support in two coefficient intervals.** Since $\deg J_F = d_F - 1 \leq 2n-2$, the support of a represented summand meets at most two coefficient intervals. If it meets two, they are consecutive. Relative to the initial degree of the later interval, write the summand as $s^\alpha J_F(s)$, where

$$\alpha \in 2\mathbb{Z}_{<0}, \qquad E = \alpha + d_F - 1 \in 2\mathbb{Z}_{\geq 0}.$$

**Lemma 8.4.** *Every represented summand with underlying manifold $F$ such that $\beta_{\mathbb{Q}}(P(F)) = 4$ is supported in one coefficient interval.*

*Proof.* By Lemma 8.1, one has $J_F(s) = [a]_{s^2}[2b+1]_s$, with $a \geq 1$ and $b \geq 0$. If $b \geq 1$, the coefficient calculation in the proof of Theorem 6.6 gives

$$[s^i]J_F > 0 \qquad (0 \leq i \leq 2a + 2b - 2).$$

If its support met two coefficient intervals, it would have a positive coefficient in the intervening degree, whereas the corresponding character series has coefficient zero there.

If $b = 0$, then $J_F = [a]_{s^2}$. Suppose the support of $Q = s^\alpha J_F$ meets two coefficient intervals, with degrees relative to the initial degree of the later one. Since $\alpha < 0 \leq E$, one has $[s^0]Q = 1$ and $[s]Q = 0$. But $[s^0]J_U = [s]J_U = 1$, contrary to the consequence of (8.2). □

**Proposition 8.5.** *If the support of a represented summand with underlying manifold $F$ meets two coefficient intervals, then $q_F \nmid t$, $\beta_{\mathbb{Q}}(P(F)) \geq 8$, and $d_F$ satisfies* (8.4)*. Relative to the initial degree of the later interval, its final degree $E$ is at most $n-6$ for even $n$, and at most $n-5$ for odd $n$.*

*Proof.* Corollary 5.11 gives $F \neq U$. Its represented order is nonresonant, so Lemma 6.5 gives $q_F \nmid t$.

Lemma 8.4 gives $\beta_{\mathbb{Q}}(P(F)) \neq 4$. Lemma 8.2 gives $\beta_{\mathbb{Q}}(P(F)) \in 4\mathbb{Z}_{>0}$, so this Betti number is at least eight. Lemma 8.3 gives (8.4). Since $\alpha \leq -2$,

$$E = \alpha + d_F - 1 \leq d_F - 3 \leq \begin{cases} n-6, & n \text{ even}, \\ n-5, & n \text{ odd}. \end{cases}$$

□

**Lemma 8.6.** *If the support of a represented summand with underlying manifold $F$ meets two coefficient intervals, then*

$$\alpha \leq -4, \qquad E \geq 2, \qquad d_F \geq 7.$$

*In particular* $\dim F \geq 15$*. Moreover, $E \leq n-8$ for even $n$, and $E \leq n-7$ for odd $n$.*

*Proof.* Write $Q = s^\alpha J_F$ in degrees relative to the initial degree of the later interval. This interval is not the first, since $\alpha < 0$. The coefficients of the corresponding character series in degrees $-3, -2, 0, 1$ are all one, by (7.4) and Corollary 5.11. The constant and leading coefficients of $J_F$ give

$$([s^{-3}]Q, [s^{-2}]Q) = (0,1) \quad \text{if } \alpha = -2,$$
$$([s^0]Q, [s]Q) = (1,0) \quad \text{if } E = 0.$$

Since the character-series coefficients in each pair are one, (8.2) excludes both possibilities. Thus $\alpha \leq -4$ and $E \geq 2$. Hence

$$d_F - 1 = E - \alpha \geq 6, \qquad E \leq d_F - 5.$$

Proposition 8.5 gives the two upper bounds for $E$. □

**Corollary 8.7.** *If $2 \leq n \leq 8$, every represented summand is supported in one coefficient interval.*

*Proof.* A represented summand whose support met two coefficient intervals would have $d_F \geq 7$ by Lemma 8.6, whereas Proposition 8.5 gives $d_F \leq 5$ for $2 \leq n \leq 8$. □

8.3. **Negative coefficients of $D_F$.** A coefficient interval is *regular* if it contains the support of a regular summand. By Corollary 5.11, this summand has polynomial $J_U$ relative to the interval's initial degree. Nonnegativity forces every other summand to vanish in this interval. A coefficient interval with no regular summand is *nonregular*.

**Lemma 8.8.** *Let $n \geq 3$ and fix a nonregular coefficient interval. Exactly one summand has a nonzero constant coefficient in degrees relative to the initial degree of the interval. Its constant and linear coefficients are one, and its final degree $E$ is even and satisfies*

$$2 \leq E < n-1. \tag{8.5}$$

*Every other summand whose support meets the interval has initial degree at least two.*

*Proof.* Take degrees relative to the initial degree of the interval. The coefficients are nonnegative integers and $[s^0]J_U = 1$, so there is a unique summand $A_0$ with $[s^0]A_0 = 1$. If the support of a summand $A = s^a J_F$ meets the interval and $a \leq 0$, then $a \leq 0 \leq a + d_F - 1$. Since both endpoints are even, Corollary 5.11 gives $[s^0]A > 0$, whence $A = A_0$. Thus every other summand whose support meets the interval has positive even initial degree, hence at least two. By (8.2), every summand with a positive coefficient at one also has a positive coefficient at zero. Hence

$$[s]A_0 = [s]J_U = 1.$$

Write $A_0 = s^{a_0} J_{F_0}$, where $a_0 \le 0$ is even, and put $d_0 = d_{F_0}$. Its final degree $E = a_0 + d_0 - 1$ is even and satisfies $E \ge 2$. The underlying critical manifold $F_0$ is proper because the interval is nonregular.

Suppose $a_0 = 0$. If $d_0 \ge n$, Theorem 6.6 gives either $q_{F_0} \mid t$ or $J_{F_0}(s) = 1 + s^2 + \cdots + s^{n-1}$. The first alternative gives $a_0 \ge 2$ by Lemma 6.5; the second contradicts $[s]A_0 = 1$. Thus $E = d_0 - 1 < n - 1$. If $a_0 < 0$, Proposition 8.5 gives $E < n - 1$. □

**Theorem 8.9.** *Let $n \ge 3$ and fix a nonregular coefficient interval of $J_-$ or $J_+$. Relative to its initial degree, there is a summand $s^a J_F$ supported in $[0, 2n-2]$, with $F \in \mathcal{D}$ and $a \in 2\mathbb{Z}$ satisfying*

$$2 \le a < n - 1. \tag{8.6}$$

*More precisely, for some even $E < n - 1$ and some $j \in \mathbb{Z}_{\ge 0}$,*

$$a + 2j = E, \qquad [s^{2j+1}]J_F > [s^{2j}]J_F.$$

*Proof.* Let $A_0$ and $E$ be as in Lemma 8.8. Write $A_\lambda$, $\lambda \in \Lambda$, for the summands in the same character series, with degrees relative to the initial degree of the chosen interval; let $0 \in \Lambda$ be the index of $A_0$. Then

$$[s^E]A_0 = 1, \qquad [s^{E+1}]A_0 = 0.$$

Since $E$ is even and $E + 1 < n$, equation (7.4) and the coefficient equalities following it give $[s^E]J_U = [s^{E+1}]J_U$. The coefficient sums are finite by Corollary 5.10; hence

$$\sum_{\lambda \in \Lambda \setminus \{0\}} \left( [s^E]A_\lambda - [s^{E+1}]A_\lambda \right) = -1.$$

Choose $\lambda \ne 0$ with $[s^{E+1}]A_\lambda > [s^E]A_\lambda$, and write $A_\lambda = s^a J_F$. Its positive coefficient at $E + 1$ and its even initial degree give $a \le E$. Since $A_\lambda \ne A_0$ and its support meets the chosen interval, Lemma 8.8 gives

$$2 \le a \le E < n - 1.$$

The integer $j = (E - a)/2$ is nonnegative and satisfies $[s^{2j+1}]J_F > [s^{2j}]J_F$.

If its support met the next coefficient interval, Proposition 8.5 would give $d_F - 1 < n - 1$, whereas its final degree would satisfy

$$2n \le a + d_F - 1 < (n - 1) + (n - 1) = 2n - 2.$$

Thus its support is contained in the chosen interval. □

8.4. **A denominator not dividing $t$.**

**Lemma 8.10.** *Let $F$ be proper. There exists a proper underlying critical manifold $G \supseteq F$ of maximal dimension among the proper underlying critical manifolds containing $F$. It satisfies:*

1. *$q_G \mid q_F$. For every prime $p \mid q_G$, $P(G) = P^{C_p}$, and $\beta_{\mathbb{Q}}(P(G)) = 4$.*
2. *The order 2 is nonresonant for $G$.*
3. *If $q_G \mid t$, the summand of $G^2$ is supported in the nonregular coefficient interval $b = t/q_G - 1$ of $J_+$.*
4. *If $Y$ is an underlying critical manifold and $j \in \mathbb{Z}_{\ge 1}$ satisfies*

$$z_Y(2j) = 0, \qquad \frac{2jL}{q_Y} = \frac{2L}{q_G},$$

*then $Y = G$ and the summands corresponding to $Y^{2j}$ and $G^2$ coincide.*

*Proof.* The set of dimensions of proper critical manifolds containing $F$ is nonempty and bounded. Choose $G$ of maximal dimension. Equation (6.3) gives $q_G \mid q_F$. Since $G$ is proper, $q_G > 1$; it is odd by Theorem 5.9.

For $p \mid q_G$, the subgroup $C_p \subset C_{q_G}$ fixes $P(G)$ and is normalized by lifted reversal. Put $G_p = \pi(P^{C_p})$. By Lemmas 4.4 and 6.1, the fixed manifold $P^{C_p} = P(G_p)$ is connected and satisfies $\beta_{\mathbb{Q}}(P(G_p)) = 4$. The manifold $G_p$ is proper, since $\varphi_{L/p} \neq \mathrm{id}_U$. The closed connected inclusion $G \subseteq G_p$ and maximality give $\dim G_p = \dim G$ and $G_p = G$. Hence $P(G) = P^{C_p}$. Proposition 5.13 gives $z_G(2) = 0$.

If $q_G \mid t$, Lemma 6.5 gives support containment for the summand of $G^2$ in coefficient interval $b = t/q_G - 1$ of $J_+$. One has $0 \leq b < t - 1$, whereas the first regular coefficient interval of $J_+$ has index $t - 1$ by Corollary 5.11. Thus interval $b$ is nonregular.

Under the hypotheses of (4), equality of periods gives $q_Y = jq_G$. Choose a prime $p \mid q_G$. Then $C_p \subset C_{q_Y}$ fixes $P(Y)$, so

$$P(Y) \subseteq P^{C_p} = P(G), \qquad Y \subseteq G.$$

By $z_Y(2j) = 0$ and (2), both $Y$ and $G$ are connected components of $U^{\varphi_{2L/q_G}}$. Their inclusion gives $Y = G$. Thus $q_Y = q_G$ and $j = 1$, so the represented summands coincide. □

**Theorem 8.11.** *Let $n \geq 3$. If the prime periods are unequal, there is an $F \in \mathcal{D}$ with $q_F \nmid t$. Every such $F$ satisfies*

$$\beta_{\mathbb{Q}}(P(F)) \geq 8, \qquad d_F \leq \begin{cases} n-3, & n \text{ even}, \\ n-2, & n \text{ odd}, \end{cases}$$

*and its denominator has at least two distinct odd prime divisors.*

*Proof.* By Corollary 5.5, $r > 0$. The first regular coefficient interval of $J_-$ has index $r$ by Corollary 5.11. Thus its interval of index zero is nonregular, and Theorem 8.9 gives $\mathcal{D} \neq \emptyset$.

Suppose that $q_F \mid t$ for every $F \in \mathcal{D}$. Every member of $\mathcal{D}$ is proper by Lemma 8.2. Choose $F \in \mathcal{D}$, and let $G \supseteq F$ be as in Lemma 8.10. Then $q_G \mid q_F \mid t$, and the summand of $G^2$ is supported in the nonregular coefficient interval $b = t/q_G - 1$ of $J_+$. By Theorem 8.9, this interval also contains the support of a summand corresponding to $Y^{2j}$, where $Y \in \mathcal{D}$, $j \geq 1$, and $z_Y(2j) = 0$. Since $q_Y \mid t$, Lemma 6.5 places this nonzero summand in coefficient interval $tj/q_Y - 1$. The coefficient intervals are disjoint, so

$$\frac{tj}{q_Y} - 1 = \frac{t}{q_G} - 1, \qquad q_Y = jq_G, \qquad \frac{2jL}{q_Y} = \frac{2L}{q_G}.$$

Lemma 8.10(4) gives $Y = G$. But $\beta_{\mathbb{Q}}(P(G)) = 4$, so Lemma 8.1 gives $G \notin \mathcal{D}$, contrary to $Y \in \mathcal{D}$.

Thus there is an $F \in \mathcal{D}$ with $q_F \nmid t$. Lemma 8.2 gives its Betti number and the prime-divisor condition. Lemma 8.3 gives its dimension bound. Together with Proposition 2.1, Theorem 5.4, and Corollary 5.5, these conclusions prove Theorem B. □

**Corollary 8.12.** *If $n \geq 3$ and the prime periods are unequal, there exist $F \in \mathcal{D}$ and an odd $m \geq 1$ such that*

$$z_F(m) = 0, \qquad i(F) \leq i(F^m) < 2n - 1.$$

*Proof.* The coefficient interval of index zero in $J_-$ is nonregular by Corollaries 5.5 and 5.11. Let $F^m$ be the represented component whose summand is given by Theorem 8.9 in that interval. Then $F \in \mathcal{D}$, $z_F(m) = 0$, and $m$ is odd. Its initial degree $a$ satisfies $a < n - 1$, so

$$i(F^m) = 2a + 1 < 2n - 1.$$

For a generic prime geodesic $c \in F$, the index inequality [33, p. 952] gives

$$i(F) = \mathrm{ind}(c) \leq \mathrm{ind}(c^m) = i(F^m) < 2n - 1.$$

□

## 9. Equality of prime periods

9.1. **A cumulative parity comparison.** Let $\mathcal{F}$ be the set of underlying critical manifolds $F$ such that

$$J_F(s) = [2]_s[d_F - 1]_s = 1 + 2\sum_{j=1}^{d_F-2} s^j + s^{d_F-1}, \qquad d_F \geq 3. \tag{9.1}$$

For $F \in \mathcal{F}$, the integer $d_F$ is odd by Theorem 5.9. Since $P_F = (1+s)J_F$ by Corollary 5.10, one has $b_2(B_F) = 1 + [s]J_F = 3$. Moreover,

$$D_F(x) = -1 + x^{(d_F-1)/2}. \tag{9.2}$$

Thus $\mathcal{F} \subseteq \mathcal{D}$. For $F \in \mathcal{F}$, one has $[s^j]J_F > 0$ for $0 \leq j \leq d_F - 1$. The support of a represented summand with underlying manifold $F$ therefore lies in one coefficient interval, since the intervening coefficient of the character series is zero.

Put $R = \lfloor n/2 \rfloor - 1$. For the regular polynomial $P_U = [n]_s[n+1]_s$, equation (8.3) gives

$$D_U(x) = x^{R+1}[[n/2]]_x. \tag{9.3}$$

For an underlying critical manifold $F$ and $a \in 2\mathbb{Z}$, set

$$C_R(a,F) = \sum_{u=0}^{R} \big([s^{2u-a}]J_F - [s^{2u+1-a}]J_F\big) = \sum_{u=0}^{R} [x^u]\big(x^{a/2}D_F(x)\big). \tag{9.4}$$

Coefficients outside the support are zero; for $a < 0$, the last expression is a coefficient sum of a Laurent polynomial. Thus $F \notin \mathcal{D}$ implies $C_R(a,F) \geq 0$.

Fix a coefficient interval. Write $A_\lambda = s^{a_\lambda}J_{F_\lambda}$, $\lambda \in \Lambda$, for the represented summands whose supports meet it, with degrees relative to its initial degree. Thus $a_\lambda \in 2\mathbb{Z}$. The index $0 \in \Lambda$ corresponds to the unique summand with nonzero constant coefficient. Each coefficient is a finite sum of nonnegative integers by Corollary 5.10. Since the interval has finitely many degrees, $\Lambda$ is finite, and

$$\sum_{\lambda\in\Lambda} C_R(a_\lambda, F_\lambda) = \sum_{u=0}^{R} [x^u]D_U(x) = 0.$$

**Lemma 9.1.** *Let $n \geq 3$ and fix a nonregular coefficient interval. Suppose that every summand $s^aJ_F$ with $F \in \mathcal{D}$ whose support meets this interval satisfies $F \in \mathcal{F}$. Then there is a summand supported in this interval whose underlying manifold $F$ belongs to $\mathcal{F}$ and whose relative initial degree $a \in 2\mathbb{Z}$ satisfies*

$$\begin{cases} n - d_F + 1 \leq a \leq n-2, & n \text{ even}, \\ n - d_F \leq a \leq n-3, & n \text{ odd}. \end{cases} \tag{9.5}$$

*Proof.* Let $A_0 = s^{a_0}J_{F_0}$ be the summand in Lemma 8.8, and let $E$ be its final degree. If $F_0 \in \mathcal{F}$, its summand is supported in the interval, so $a_0 = 0$ and $[s]A_0 = 2$, contrary to that lemma. Thus $F_0 \notin \mathcal{F}$, and the hypothesis gives $F_0 \notin \mathcal{D}$.

The integer $E$ is even and satisfies $2 \leq E < n-1$, so $0 \leq E/2 \leq R$. Since $E = a_0 + d_{F_0} - 1$ and the leading coefficient of $J_{F_0}$ is one,

$$[x^{E/2}]\big(x^{a_0/2}D_{F_0}(x)\big) = 1, \qquad C_R(a_0,F_0) \geq 1.$$

Consequently,

$$\sum_{\lambda\neq 0} C_R(a_\lambda, F_\lambda) = -C_R(a_0,F_0) \leq -1.$$

Choose $\lambda \neq 0$ with $C_R(a_\lambda, F_\lambda) < 0$, and put $a = a_\lambda$ and $F = F_\lambda$. Then $F \in \mathcal{D}$, so $F \in \mathcal{F}$. Its summand is supported in the interval, and hence $a \geq 0$. Equation (9.2) gives

$$C_R(a,F) = -\mathbf{1}_{a/2\leq R} + \mathbf{1}_{a/2+(d_F-1)/2\leq R} < 0.$$

Therefore

$$\frac{a}{2} \le R < \frac{a + d_F - 1}{2}, \qquad 2R - d_F + 3 \le a \le 2R,$$

where the last inequalities follow because $a$ and $d_F - 1$ are even. For even $n$, one has $2R = n - 2$; for odd $n$, one has $2R = n - 3$. These are the two bounds in (9.5). □

9.2. **A sufficient condition for equality of periods.**

**Theorem 9.2.** *Let $n \ge 3$. Suppose every $F \in \mathcal{D}$ with $q_F \nmid t$ satisfies:*

*(1) $F \in \mathcal{F}$;*

*(2) for every $\ell \in \mathbb{Z}_{\ge 1}$, the represented component $G_\ell = F^\ell$ satisfies*

$$\operatorname{supp}\left(s^{h_F(\ell) - n\mathbf{1}_{2|\ell}} J_{G_\ell}\right) \subset I_b^- \quad \text{for some } b \in \mathbb{Z}_{\ge 0};$$

*(3) $d_F$ satisfies*

$$d_F \ge \begin{cases} n - 5, & n \text{ even}, \\ n - 4, & n \text{ odd}. \end{cases} \tag{9.6}$$

*Then all prime periods are equal.*

*Proof.* Suppose the prime periods are unequal. Then $r > 0$ by Corollary 5.5. Every $F \in \mathcal{D}$ is proper by Lemma 8.2.

For such an $F$ with $q_F \nmid t$, let $m$ be nonresonant and let $b$ index the coefficient interval containing the support of its summand. Its relative initial degree is

$$a = h_F(m) - n\mathbf{1}_{2|m} - 2nb = v_F(m)$$

by (7.5). The interval (9.5) is contained in (7.12). Indeed, when $n$ is even,

$$(n - d_F + 1) - (2n - 2d_F - 6) = (d_F + 5) - (n - 2) = d_F - n + 7 \ge 2.$$

When $n$ is odd,

$$(n - d_F) - (2n - 2d_F - 6) = d_F - n + 6 \ge 2,$$
$$(d_F + 5) - (n - 3) = d_F - n + 8 \ge 4.$$

Since $b_2(B_F) = 3$, condition (2) and Corollary 7.4 give (7.17) at every nonresonant order whose initial degree satisfies (9.5).

Suppose $q_K \mid t$ for some proper underlying critical manifold $K$, and choose $G \supseteq K$ as in Lemma 8.10. Then $q_G \mid q_K \mid t$, $\beta_{\mathbb{Q}}(P(G)) = 4$, and the summand of $G^2$ is supported in the nonregular coefficient interval of $J_+$

$$b = \frac{t}{q_G} - 1.$$

Every summand whose support meets this interval has a proper underlying critical manifold, since regular summands are supported in regular coefficient intervals by Corollary 5.11. Let $Y^{2j}$ be its represented critical component, with $j \ge 1$ and $z_Y(2j) = 0$, as in Section 7. If $q_Y \mid t$, the support of the corresponding summand is contained in coefficient interval $tj/q_Y - 1$ by Lemma 6.5. Its support meets interval $b$, and the coefficient intervals are disjoint. Hence

$$\frac{tj}{q_Y} - 1 = \frac{t}{q_G} - 1, \qquad \frac{2jL}{q_Y} = \frac{2L}{q_G}.$$

Lemma 8.10(4) gives $Y = G$ and equality of the represented summands.

By Lemma 8.1, $G \notin \mathcal{D}$. Thus every summand whose support meets this interval and whose underlying manifold $F$ belongs to $\mathcal{D}$ satisfies $q_F \nmid t$, and hence $F \in \mathcal{F}$. Lemma 9.1 gives a represented

component $F^m$ whose summand is supported in this interval with initial degree in (9.5). Here $z_F(m) = 0$, and $m$ is even because the character is positive. Corollary 7.4 gives

$$\frac{mL}{q_F} = \frac{2(b+1)L}{t} = \frac{2L}{q_G}.$$

Lemma 8.10(4), with $j = m/2$, implies $F = G$, contrary to $F \in \mathcal{D}$ and $G \notin \mathcal{D}$. Therefore $q_K \nmid t$ for every proper underlying critical manifold $K$.

The coefficient interval of index zero in $J_-$ is nonregular, since $r > 0$ and the first regular interval has index $r$ by Corollary 5.11. Every $F \in \mathcal{D}$ is proper, so $q_F \nmid t$ and $F \in \mathcal{F}$. Lemma 9.1 gives a represented component $F^m$ whose summand is supported in this interval at an initial degree $a$ satisfying (9.5). The order $m$ is nonresonant and odd. Corollary 7.4 gives

$$\frac{mL}{q_F} = \frac{L}{t}, \qquad q_F = mt. \tag{9.7}$$

Corollary 5.7 gives $b_2(B_C) = 2$, whereas $b_2(B_F) = 3$. Thus $F \neq C$. Lemma 7.5 and (7.6), together with the evenness of $a$ and both bounds in (7.12), give

$$\frac{2n}{m} = \Delta_F > d_F + 3, \qquad 2n - 2d_F - 4 \leq a \leq d_F + 3, \qquad 2n \leq 3d_F + 7,$$
$$m = \frac{2n}{\Delta_F} < \frac{2n}{d_F+3} \leq \frac{3d_F+7}{d_F+3} = 3 - \frac{2}{d_F+3} < 3.$$

Since $m$ is a positive odd integer, $m = 1$. Equation (9.7) now gives $q_F = t$, contrary to $q_K \nmid t$ for every proper underlying critical manifold $K$. □

**Corollary 9.3.** *If $n \geq 3$ and the prime periods are unequal, there is an $F \in \mathcal{D}$ with $q_F \nmid t$ for which at least one of the following holds:*

*(1) $J_F(s) \neq [2]_s[d_F - 1]_s$;*

*(2) $d_F \leq n - 7$ if $n$ is even, or $d_F \leq n - 6$ if $n$ is odd;*

*(3) for some $\ell \in \mathbb{Z}_{\geq 1}$, the support of the summand corresponding to $F^\ell$ meets two coefficient intervals.*

*Proof.* By Theorem 5.9 and Corollary 5.11, every $F \in \mathcal{D}$ has odd $d_F \geq 3$. If no $F \in \mathcal{D}$ with $q_F \nmid t$ satisfied any of the alternatives, exclusion of (2) would give (9.6). Every represented summand meets at most two coefficient intervals, by the support bound in Section 8. Thus exclusion of (3) gives condition (2) of Theorem 9.2. All hypotheses of Theorem 9.2 would then hold, contrary to the inequality of prime periods. □

9.3. **The range $3 \leq n \leq 8$.**

**Corollary 9.4.** *If $3 \leq n \leq 8$, all prime periods are equal.*

*Proof.* Suppose that the prime periods are unequal. By Theorem 8.11, there is an $F \in \mathcal{D}$ with $q_F \nmid t$, and every such $F$ satisfies (8.4). By Lemma 8.2 and Theorem 5.9, $F$ is proper and $d_F$ is odd. Corollary 5.11 gives $J_F = 1$ when $d_F = 1$. Thus $d_F \geq 3$. If $d_F = 3$, Corollary 5.11 and $F \in \mathcal{D}$ give

$$J_F(s) = 1 + \lambda s + s^2, \qquad D_F(x) = 1 - \lambda + x, \qquad \lambda \in \mathbb{Z}_{\geq 2}. \tag{9.8}$$

For $n = 3, 4$, the dimension bound gives $d_F \leq 1$, a contradiction.

For $n = 5, 6$, it gives $d_F = 3$. Equation (9.8) and the identity $P_F = (1 + s)J_F$ give $b_2(B_F) = \lambda + 1 \geq 3 = \lceil n/3 \rceil + 1$. Corollary 7.2 then gives $q_F \mid t$, a contradiction.

Let $n = 7$ or 8, and let $F \in \mathcal{D}$ with $q_F \nmid t$. The dimension bound and the oddness of $d_F$ give $d_F \in \{3, 5\}$. If $d_F = 5$, Corollary 5.11 gives

$$J_F(s) = 1 + \lambda s + \mu s^2 + \lambda s^3 + s^4, \qquad \lambda, \mu \in \mathbb{Z}, \quad \lambda \geq 0, \quad \mu \geq 1.$$

Since $D_F(x) = (1-\lambda) + (\mu-\lambda)x + x^2$ has a negative coefficient, $\lambda \geq 2$ and $b_2(B_F) = \lambda + 1 \geq 3$. By Corollary 8.7, the summand of every represented component $F^\ell$, $\ell \geq 1$, is supported in one coefficient interval. The coefficient comparison preceding (7.15) gives, at every nonresonant order $m$,

$$2 \leq v_F(m) \leq k_F - 2 = 2n - 8.$$

Put $A = 2$ and $B = 2n - 8$. Then $[A, B] \subset [0, 2n)$ and

$$2B - A = B - 2A + k_F = 4n - 18 < 2n.$$

Proposition 7.1 gives $q_F \mid t$, a contradiction.

Therefore $d_F = 3$. If the integer $\lambda$ in (9.8) satisfies $\lambda \geq 3$, then $b_2(B_F) = \lambda + 1 \geq 4 = \lceil n/3 \rceil + 1$, so Corollary 7.2 again gives $q_F \mid t$. Consequently every $F \in \boldsymbol{D}$ with $q_F \nmid t$ satisfies

$$d_F = 3, \qquad J_F(s) = 1 + 2s + s^2 = [2]_s[d_F - 1]_s.$$

The summands corresponding to $F^\ell$, for all $\ell \geq 1$, are supported in one coefficient interval each, by Corollary 8.7. Moreover, $d_F = n - 4$ for $n = 7$ and $d_F = n - 5$ for $n = 8$. Theorem 9.2 gives equality of prime periods, a contradiction. □

**Proposition 9.5.** *If $n = 2$, all prime periods are equal.*

*Proof.* Suppose $r > 0$. Corollary 5.7 gives a proper critical manifold $C$ with $d_C \geq 3$. Hence

$$7 \leq \dim C = 2d_C + 1 \leq \dim SM = 7.$$

The connected closed inclusion $C \subset SM$ is therefore an equality, as in the proof of Lemma 6.5, contrary to properness. Thus $r = 0$, and Corollary 5.5 gives equality of prime periods. □

*Proof of Theorem A.* Proposition 2.1 makes $M$ simply connected and gives (2.1). Proposition 9.5 applies when $n = 2$, and Corollary 9.4 applies when $3 \leq n \leq 8$. □

*Proof of Corollary 1.1.* The case $n = 2$ follows from Theorem A. For $n \geq 3$, unequal prime periods would give, by Theorem B, an underlying critical manifold $F$ whose generic denominator $q_F$ has at least two distinct odd prime divisors. Since $q_F$ is the stabilizer order of a generic unit tangent vector in $F$, this contradicts the hypothesis. All prime periods are therefore equal, and their common value is $L$ by its minimality. □

*Remark* 9.6. If $F \in \boldsymbol{\mathcal{F}}$ has dimension seven, the interval in (7.12) is $2n - 12 < a < 8$, and

$$(2n - 12, 8) \cap 2\mathbb{Z} = \begin{cases} \{4, 6\}, & n = 7, \\ \{6\}, & n = 8, \\ \emptyset, & n = 9. \end{cases}$$

At $n = 9$, both expressions in (7.7) equal $4n - 18 = 2n$ for $d_F = 5$ and the interval $[2, 2n - 8]$. For $n = 9$, if the support of a represented summand with underlying manifold $F$ meets two coefficient intervals, let $\alpha$ and $E$ be its initial and final degrees relative to the initial degree of the later interval. Proposition 8.5 and Lemma 8.6 give $d_F = 7$, $\alpha = -4$, and $E = 2$.

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