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THE VARIATIONAL FORMULATION OF THE FOLLOW-THE-LEADER MODEL

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ABSTRACT. We provide a variational formulation of a finite-dimensional Follow-the-Leader system through a minimizing-movement scheme based on a discrete transport distance with nonlinear mobility. Despite the degeneracy of the kinetic action at maximal density, we prove finiteness of the distance, C^1 regularity of geodesics, and suitable Euler–Lagrange relations. For a decreasing external potential, we show that the JKO scheme yields an approximate implicit Euler discretization of the Follow-the-Leader dynamics. Consequently, as $\tau \to 0^+$, the piecewise affine interpolations converge, up to a subsequence, to a C^1 solution of the continuous Follow-the-Leader system. This complements the many-particle convergence theory for the discrete nonlinear-mobility transport distance by establishing the continuous-time limit at fixed number of particles.

1. INTRODUCTION

Systems of Follow-the-Leader (FTL) type are among the classic microscopic descriptions of one-dimensional traffic flow. In their original form, the dynamics of each vehicle is determined by the state of the vehicle immediately ahead. In formulas, having denoted with $x_0(t), \dots, x_N(t)$ the (time-dependent) positions of $N+1$ vehicles, and assuming $x_i(t) < x_{i+1}(t)$ for all $i \in \{0, \dots, N-1\}$, a prototype FTL system reads

$$\begin{aligned} \dot{x}_i(t) &= \theta\left(\frac{1}{N(x_{i+1}(t) - x_i(t))}\right) \qquad i \in \{0, \dots, N-1\}, \\ \dot{x}_N(t) &= \theta(0). \end{aligned} \tag{1.1}$$

The map θ computes the speed of vehicles and is a function of a quantity with physical dimension of a one-dimensional density. Here, θ is assumed to be decreasing, and the law for the right-most vehicle x_N prescribes the maximal speed (as there is no vehicle ahead of x_N). Nonlinear FTL systems have been studied since the early developments of mathematical traffic theory; see, for instance, [18].

The typical macroscopic counterpart of a FTL system is the classical Lighthill–Whitham–Richards (LWR) model

$$\partial_t \rho + \partial_x \left(\rho \theta(\rho)\right) = 0, \tag{1.2}$$

which describes the evolution of the vehicle density ρ through a scalar conservation law [23, 28]. The connection between (1.1) and (1.2) has been extensively investigated. In particular, first-order FTL systems such as (1.1) can be interpreted as discrete Lagrangian approximations of nonlinear scalar conservation laws of the form (1.2), and rigorous many-particle limits were established in [12, 9]; see also [20, 19] for an interpretation of the FTL dynamics as a numerical approximation of the LWR model via classical monotone finite-difference schemes away from vacuum, and recent [8] for the case in presence of vacuum. In these results, the main task is to solve the Cauchy problem for (1.2) using a many-particle approximation ($N \to +\infty$) via a discrete density defined from (1.1). Said approach has been extended to similar models in [10, 11, 13, 3, 27, 25, 5].

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A different perspective on a certain class of evolution equations is provided by the variational theory of gradient flows. The seminal minimizing-movement construction of Jordan, Kinderlehrer and Otto [21] showed that the linear Fokker–Planck equation can be recovered as the continuous-time limit of an implicit variational scheme based on the quadratic Wasserstein distance, see also [26]. This viewpoint has subsequently developed into a general metric theory of gradient flows in spaces of probability measures; we refer to [2] and the references therein. In the classical geodesic interpretation of the Wasserstein distance in [4], the transport cost corresponds to a linear mobility. More precisely, computing the 2-Wasserstein distance between two probability densities ρ_1 and ρ_2 consists in connecting them via a geodesic solving a linear continuity equation

$$\partial_t \rho + \operatorname{div}(\rho v) = 0, \qquad 0 < t < 1, \tag{1.3}$$

and by minimizing, with respect to v, the kinetic action

$$\int_0^1 \int_{\mathbb{R}^d} \rho |v|^2 \, \mathrm{d}x \, \mathrm{d}t. \tag{1.4}$$

In (1.3) and in (1.4), the "mobility" is the ρ-depending function $m(\rho) = \rho$. Framing (1.2) in this context requires $m(\rho) = \rho\theta(\rho)$, that is, m is nonlinear and, as it turns out (most of the times) in traffic flow theory, *degenerate* at some positive density, the most typical example being $m(\rho) = \rho(R - \rho)$ with R being some "maximal" density at which motion is inhibited. To incorporate nonlinear or degenerate mobilities, several generalized dynamical transport distances have been introduced, notably in [15, 24]. These distances, denoted by d_m, are computed by minimizing the action

$$\int_0^1 \int_{\mathbb{R}^d} m(\rho) |v|^2 \, \mathrm{d}x \, \mathrm{d}t$$

with a general mobility function $\rho \mapsto m(\rho)$, to replace the standard kinetic action (1.4) used in [4], and by solving the nonlinear continuity equation

$$\partial_t \rho + \operatorname{div}(m(\rho) v) = 0, \qquad 0 < t < 1,$$

instead of (1.3). Consequently, a whole class of nonlinear PDEs featuring the formal structure

$$\partial_t \rho + \operatorname{div}\left(m(\rho) \nabla \frac{\delta F}{\delta \rho} \right) = 0,$$

with F a suitable functional and $\delta F / \delta \rho$ being a formal functional derivative in the spirit of [26], has the potential of being interpreted as a gradient flow of F with respect to the "mobility-dependent" distance defined in [15]. However, such a task is made complicated by the fact that the target equation has the structure of a nonlinear conservation law, and it therefore needs a concept of *entropy solution* in Kružkov's sense, see [22], in order to obtain uniqueness. Detecting a time-discrete counterpart of Kružkov's entropy condition at the level of the JKO scheme is currently under study.

The relation between scalar conservation laws, FTL systems and gradient-flow structures has already been observed from a discrete particle viewpoint. In [7], scalar conservation laws are discussed from the perspective of gradient flows and the FTL system is interpreted as a gradient flow on a non-flat metric at the discrete level. Related deterministic particle schemes have also been used to construct solutions of transport and aggregation–diffusion equations with nonlinear mobility; see, for example, [17, 6]. These works highlight that the geometry perceived by a particle is naturally modified by the local discrete density and suggest that nonlinear-mobility transport metrics are well suited to the Lagrangian description of such systems.

A systematic discrete optimal-transport formulation was introduced more recently by Di Marino, Portinale and Radici in [14]. There, generalized Wasserstein distances with nonlinear mobility on the real line are discretized by suitable finite-dimensional distances on cones of ordered particles. The discrete metrics are proved to converge, as the number of particles N tends to infinity, to the corresponding continuum transport distance. Moreover, for every fixed time step $\tau > 0$, the

associated discrete minimizing-movement schemes converge to the continuum JKO scheme. Thus, [14] rigorously develops the spatial, or many-particle, side of the relation between nonlinear-mobility optimal transport and deterministic particle approximations.

The aim of this paper is to investigate the complementary time-continuous side of this picture. We keep the number of particles fixed and ask whether the finite-dimensional JKO scheme generated by the discrete transport distance defined in [14] converges, as $\tau \to 0^+$, to the corresponding Follow-the-Leader dynamics. In other words, instead of starting from the FTL ordinary differential equations and using them to approximate a continuum equation, we start from the discrete variational geometry and recover the FTL system itself as a minimizing movement. This provides a rigorous finite-dimensional variational formulation of the dynamics and complements the limit $N \to +\infty$ studied in [14].

More precisely, for a fixed integer $N \geq 1$ and a maximal density $R > 0$, we work with $N+1$ particles on the closed cone

$$\mathcal{K} = \left\{ x = (x_0, \dots, x_N) \in \mathbb{R}^{N+1} : x_{i+1} - x_i \geq \frac{1}{NR} \text{ for all } i \in \{0, \dots, N-1\} \right\}.$$

The discrete densities are given by the $N+1$ maps $R_i \colon \mathcal{K} \to [0, R]$ defined by

$$\begin{aligned} R_i(x) &= \frac{1}{N(x_{i+1} - x_i)}, \qquad i \in \{0, \dots, N-1\}, \\ R_N(x) &= 0. \end{aligned}$$

Note that $x_{i+1} - x_i = \frac{1}{NR}$ is equivalent to the "congested" density $R_i(x) = R$. In the linear mobility case, given $x = (x_0, \dots, x_N)$ and $y = (y_0, \dots, y_N)$ in $\mathcal{K}$, minimizing the discrete action

$$\frac{1}{N+1} \sum_{i=0}^{N} |\xi_i|^2 \tag{1.5}$$

with respect to all vectors $\xi = (\xi_0, \dots, \xi_N) \in \mathbb{R}^{N+1}$ such that

$$\dot{x}_i(t) = \xi_i, \qquad 0 < t < 1$$

easily provides Euclidean distance between x and y, which is somehow the discrete counterpart of the 2-Wasserstein distance between the two empirical measures of x and y respectively. We now sketch the same procedure in the nonlinear mobility framework. We consider a *velocity* function $\theta \in C^1([0, R])$ satisfying

$$\theta'(\rho) \leq 0 \quad \text{for all } \rho \in [0, R], \qquad \theta(\rho) > 0 \quad \text{for all } \rho \in [0, R), \qquad \theta(R) = 0. \tag{1.6}$$

The discrete action (1.5) is replaced by

$$\frac{1}{N+1} \sum_{i=0}^{N} \frac{1}{\theta(R_i(x))} |\xi_i|^2$$

and we still impose the set of ODEs $\dot{x}_i(t) = \xi_i(t)$. Hence, the weight $1/\theta(R_i)$ encodes a dependence on the spatial configuration $x(t)$ in the metric structure. Such a metric structure is, in this case, "one-sided", because the velocity ξ_i is only tempered by the proximity of the right-neighbor x_{i+1} to x_i. When a pair of particles x_i and x_{i+1} approaches the minimal admissible distance $1/(NR)$, the velocity map θ degenerates and, consequently, the weight $1/\theta(R_i(x))$ blows up. This degeneracy/blow up of the action near the boundary of the convex cone $\mathcal{K}$ is precisely what encodes congestion, but it also creates the main analytical difficulty: the associated variational distance d_θ, whose precise expression is given in (2.4), is defined on a closed cone with a singular kinetic cost, and standard smooth finite-dimensional Riemannian arguments cannot be applied directly.

Our first task is therefore to understand the geometry generated by this action. Although the discrete distance was introduced in [14] as an extended pseudodistance, we show that it is finite

between every pair of configurations in $\mathcal{K}$, and hence it defines a genuine distance on the whole closed cone. We then study constant-speed geodesics. By combining localized first variations on the non-congested sets with one-dimensional BV arguments at the interfaces with the congested region, we derive Euler–Lagrange relations for suitable normalized velocities and prove that every minimizing geodesic is actually of class C^1. The analysis also yields a sign-monotonicity principle for these normalized velocities, which is crucial in the minimizing-movement argument.

We fix a *potential* $V \in C^1(\mathbb{R})$ satisfying

$$V'(x) < 0 \quad \text{for all } x \in \mathbb{R}, \tag{1.7}$$

and assume that there exist constants $C, D \geq 0$ and $s \in (0,2)$ such that

$$V(x) \geq -C|x|^s - D \quad \text{for all } x \in \mathbb{R}. \tag{1.8}$$

For a given starting point $x \in \mathcal{K}$ and a time step $\tau > 0$, we consider the discrete JKO problem

$$x_\tau \in \operatorname*{argmin}_{y \in \mathcal{K}} \left\{ \frac{1}{2\tau} d_\theta^2(x,y) + \mathcal{V}_N(y) \right\},$$

with

$$\mathcal{V}_N(y) = \frac{1}{N+1} \sum_{i=0}^{N} V(y_i).$$

A key result of our analysis is that the minimizing geodesic associated with one JKO step immediately leaves the congested boundary: even if the initial configuration belongs to $\partial\mathcal{K}$, for every positive geodesic time the discrete densities R_i are strictly smaller than R. This interior property allows us to use the full Euler–Lagrange system and to identify the terminal optimality condition

$$\alpha_{\tau,i}(1) = -\tau V'(u_{\tau,i}(1)), \qquad \alpha_{\tau,i} = \frac{\dot{u}_{\tau,i}}{\theta(R_i(u_\tau))}.$$

The monotonicity assumption $V' < 0$ then implies positivity of the normalized velocities and provides the estimates needed to iterate the construction.

We can now state our main result. Fix an initial configuration $x^0 \in \mathcal{K}$. For every $\tau > 0$, set $x_\tau^0 := x^0$ and choose recursively

$$x_\tau^m \in \operatorname*{argmin}_{y \in \mathcal{K}} \left\{ \frac{1}{2\tau} d_\theta^2(x_\tau^{m-1}, y) + \mathcal{V}_N(y) \right\} \quad \text{for all } m \geq 1. \tag{1.9}$$

Define the piecewise affine interpolation $x_\tau \colon [0,+\infty) \to \mathcal{K}$ by

$$x_\tau(t) := \sum_{m=0}^{+\infty} \frac{(t - m\tau)(x_\tau^{m+1} - x_\tau^m) + \tau x_\tau^m}{\tau} \mathbf{1}_{[m\tau,(m+1)\tau)}(t) \quad \text{for all } t \in [0,+\infty). \tag{1.10}$$

Theorem 1.1. *Let $N \geq 1$ be an integer and let $R > 0$. Assume that $\theta \in C^1([0,R])$ satisfies (1.6) and that $V \in C^1(\mathbb{R})$ satisfies (1.7)–(1.8). Let $x^0 \in \mathcal{K}$ and let x_τ be the affine interpolations defined by (1.9)–(1.10). Then, there exists a function $x \in C^1([0,+\infty);\mathcal{K})$ such that, for all $T > 0$,*

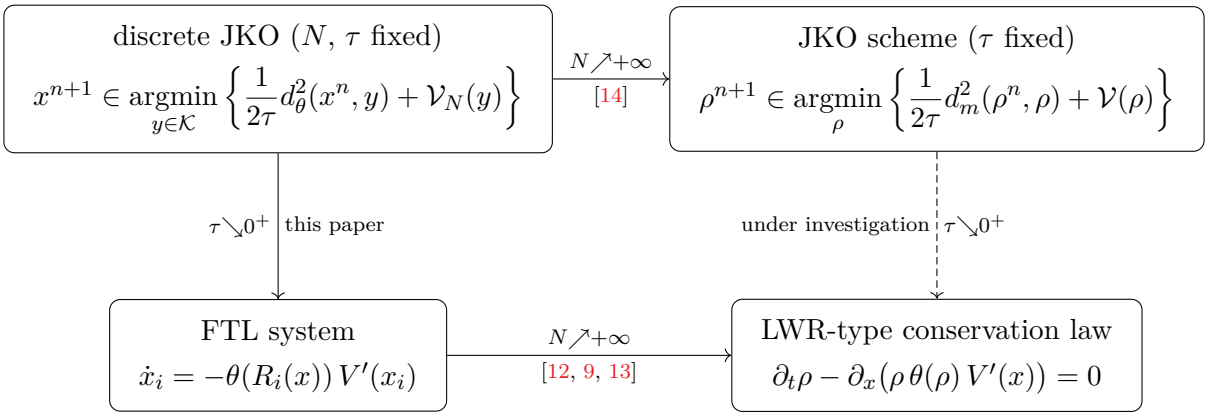
$$\begin{aligned} &x_\tau \to x && \text{strongly in } C^0([0,T];\mathcal{K}) \text{ as } \tau \to 0^+, \\ &\dot{x}_\tau \to \dot{x} && \text{weakly* in } L^\infty((0,T);\mathbb{R}^{N+1}) \text{ as } \tau \to 0^+. \end{aligned}$$

Moreover, x satisfies the Follow-the-Leader system

$$\begin{cases} \dot{x}_i(t) = -\theta(R_i(x(t)))V'(x_i(t)) & \text{for } t \in [0,+\infty) \text{ and } i \in \{0,\dots,N\}, \\ x(0) = x^0. \end{cases} \tag{1.11}$$

The proof does not rely only on an abstract minimizing-movement theorem. Instead, the regularity and Euler–Lagrange equations of the discrete geodesics allow us to show that every JKO step satisfies the implicit Euler discretization of the FTL system up to a defect of order $O(\tau)$. Uniform estimates on finite time intervals then yield compactness, and every vanishing sequence of time steps admits a subsequence whose piecewise affine interpolations converge to a C^1 solution of (1.11). Since such a solution is unique (see Lemma 4.4), the whole family converges as $\tau \to 0^+$. In the particular case $V(x) = -x$, this reduces exactly to the standard FTL (1.1) with the leading particle moving at the free velocity $\theta(0)$.

The result therefore completes, at fixed N, the time-continuous part of the discrete variational picture initiated in [14]: the latter establishes convergence of the discrete transport geometry and of the JKO steps in the many-particle limit $N \to +\infty$ for fixed τ, whereas here we keep N fixed and let $\tau \to 0^+$, recovering the microscopic FTL dynamics. This identifies the FTL system as the continuous-time minimizing movement associated with a discrete nonlinear-mobility transport geometry and clarifies the variational mechanism underlying the particle model.



The paper is organized as follows. In Section 2 we introduce the discrete variational distance d_θ and study the regularity and the Euler–Lagrange equations of its geodesics. In Section 3 we analyze a single minimizing-movement step, prove the strict interior property, and derive the terminal optimality conditions. Finally, in Section 4 we iterate the JKO scheme, establish uniform estimates and an approximate implicit Euler relation, and prove Theorem 1.1 by passing to the continuous-time limit.

2. Framework and preliminary results

2.1. **A discrete variational distance.** We begin by recalling the framework for the *discrete variational distance* introduced in [14, Section 2.2]. Let $N \in \mathbb{N}$ and $R > 0$ be fixed. We define the closed cone $\mathcal{K} \subset \mathbb{R}^{N+1}$ of *admissible particle configurations* as

$$\mathcal{K} := \left\{ x = (x_0, \dots, x_N) \in \mathbb{R}^{N+1} : x_{i+1} - x_i \geq \frac{1}{NR} \text{ for all } i \in \{0, \dots, N-1\} \right\}.$$

Here $N+1$ is the *number of particles*, while R represents the *maximum discrete density*. From now on, we omit the explicit dependence on N and R from the notation, as both parameters remain fixed throughout the paper.

For all $i \in \{0, \dots, N\}$ we define the *discrete densities* $R_i \colon \mathcal{K} \to [0, R]$ as

$$\begin{aligned} R_i(x) &:= \frac{1}{N(x_{i+1} - x_i)} && \text{for all } x \in \mathcal{K} \text{ and } i \in \{0, \dots, N-1\}, \\ R_N(x) &:= 0 && \text{for all } x \in \mathcal{K}. \end{aligned}$$

For future reference, we note that a direct computation yields

$$\partial_i R_i(x) = N R_i(x)^2 \qquad \text{for all } x \in \mathcal{K} \text{ and } i \in \{0, \dots, N-1\},$$
$$\partial_{i+1} R_i(x) = -N R_i(x)^2 \qquad \text{for all } x \in \mathcal{K} \text{ and } i \in \{0, \dots, N-1\}.$$

In particular, the following uniform bounds follow

$$|\partial_i R_i(x)| \le N R^2 \qquad \text{for all } x \in \mathcal{K} \text{ and } i \in \{0, \dots, N-1\},$$
$$|\partial_{i+1} R_i(x)| \le N R^2 \qquad \text{for all } x \in \mathcal{K} \text{ and } i \in \{0, \dots, N-1\}.$$

We fix a *velocity function* $\theta \in C^1([0,R])$ satisfying

$$\theta'(\rho) \le 0 \quad \text{for all } \rho \in [0,R], \qquad \theta(\rho) > 0 \quad \text{for all } \rho \in [0,R), \qquad \theta(R) = 0. \tag{2.1}$$

These assumptions on θ are satisfied throughout the paper. Note that θ attains its maximum at $\rho = 0$, that is,

$$\|\theta\|_{L^\infty((0,R))} = \theta(0).$$

Therefore, the convention $R_N(x) = 0$ reflects the *Follow-the-Leader* dynamics: the velocity of the leading particle is unaffected by the particles behind it.

We then define the function $\phi_\theta \colon [0,R] \times \mathbb{R} \to [0,+\infty]$ for all $(\rho, j) \in [0,R] \times \mathbb{R}$ as

$$\phi_\theta(\rho, j) \coloneqq \begin{cases} \dfrac{|j|^2}{\theta(\rho)} & \text{if } \rho < R, \\ 0 & \text{if } \rho = R \text{ and } j = 0, \\ +\infty & \text{otherwise,} \end{cases}$$

and the *discrete action functional* $\Phi_\theta \colon \mathcal{K} \times \mathbb{R}^{N+1} \to [0,+\infty]$ as

$$\Phi_\theta(x,\xi) \coloneqq \frac{1}{N+1} \sum_{i=0}^{N} \phi_\theta(R_i(x), \xi_i) \quad \text{for all } (x,\xi) \in \mathcal{K} \times \mathbb{R}^{N+1}. \tag{2.2}$$

By construction, the function $\Phi_\theta \colon \mathcal{K} \times \mathbb{R}^{N+1} \to [0,+\infty]$ is lower semicontinuous, convex in the second variable, and satisfies

$$\Phi_\theta(x,\xi) \ge \frac{|\xi|^2}{(N+1)\theta(0)} \quad \text{for all } (x,\xi) \in \mathcal{K} \times \mathbb{R}^{N+1}. \tag{2.3}$$

Finally, we define the *discrete variational distance* $d_\theta \colon \mathcal{K}^2 \to [0,+\infty]$ for all $x, y \in \mathcal{K}$ as

$$d_\theta(x,y) \coloneqq \inf \left\{ \int_0^1 \Phi_\theta(v(t), \dot v(t))\, \mathrm{d}t : v \in W^{1,2}((0,1);\mathcal{K}),\ v(0) = x,\ v(1) = y \right\}^{\frac{1}{2}}. \tag{2.4}$$

From (2.3) we obtain the lower bound

$$d_\theta(x,y)^2 \ge \frac{|x-y|^2}{(N+1)\theta(0)} \quad \text{for all } x, y \in \mathcal{K}. \tag{2.5}$$

d_θ is introduced in [14] as a pseudo-distance that may attain the value $+\infty$. The following lemma shows that $d_\theta(x,y) < +\infty$ for all $x, y \in \mathcal{K}$, and hence d_θ is a genuine distance. The proof relies on the construction of an explicit curve $v \in W^{1,2}((0,1);\mathcal{K})$ satisfying $v(0) = x$ and $v(1) = y$ for which the action has finite integral, namely

$$\int_0^1 \Phi_\theta(v(t), \dot v(t))\, \mathrm{d}t < +\infty.$$

Since Φ_θ may only blow up when $\theta(R_i(v(t))) = 0$ for some $i \in \{0, \dots, N\}$, which occurs if two consecutive components of $v(t)$ differ exactly by $1/(NR)$, our strategy is to increasingly add positive integer terms to each component of x and y, suitably interpolate the resulting two vectors on the

interval $t \in [1/3, 2/3]$, and then progressively reconnect to the initial and final points x and y on the intervals $[0, 1/3]$ and $[2/3, 1]$, respectively, via piecewise linear interpolation.

Lemma 2.1. *For all $x, y \in \mathcal{K}$ we have*

$$d_\theta(x, y) < +\infty.$$

Proof. Let $x, y \in \mathcal{K}$ be fixed. For all $i \in \{1, \dots, N\}$ we define

$$x^i := x + \sum_{k=0}^{i-1} (N-k)\mathrm{e}_{N-k}, \qquad y^i := y + \sum_{k=0}^{i-1} (N-k)\mathrm{e}_{N-k}.$$

In the above expressions, e_j denotes a vector of the canonical basis of $\mathbb{R}^{N+1}$ for $j \in \{0, \dots, N\}$. By convention, for $i = 0$ we set $x^0 := x$ and $y^0 := y$. We consider the curve

$$v(s) := \begin{cases} (3Ns - i)(x^{i+1} - x^i) + x^i & s \in \left[\frac{i}{3N}, \frac{i+1}{3N}\right], \ i \in \{0, \dots, N-1\}, \\ (3s-1)(y^N - x^N) + x^N & s \in \left[\frac{1}{3}, \frac{2}{3}\right], \\ (3Ns - 3N + i + 1)(y^i - y^{i+1}) + y^{i+1} & s \in \left[\frac{3N-i-1}{3N}, \frac{3N-i}{3N}\right], \ i \in \{0, \dots, N-1\}. \end{cases}$$

By construction, $v(0) = x$, $v(1) = y$, and $v \in \mathrm{Lip}([0,1]; \mathcal{K})$. Moreover,

$$\begin{aligned} R_{N-i}(v(s)) &\le \tfrac{R}{1+NR} && \text{for all } s \in \left[\tfrac{i}{3N}, \tfrac{i+1}{3N}\right], \ i \in \{0, \dots, N-1\}, \\ R_{N-i}(v(s)) &\le \tfrac{R}{1+NR} && \text{for all } s \in \left[\tfrac{3N-i-1}{3N}, \tfrac{3N-i}{3N}\right], \ i \in \{0, \dots, N-1\}, \\ R_i(v(s)) &\le \tfrac{R}{1+NR} && \text{for all } s \in \left[\tfrac{1}{3}, \tfrac{2}{3}\right], \ i \in \{0, \dots, N-1\}. \end{aligned}$$

Then, by recalling that

$$\sum_{i=0}^{N-1} (N-i)^2 = \sum_{j=1}^{N} j^2 = \frac{N(N+1)(2N+1)}{6},$$

we deduce the following estimate

$$\begin{aligned} d_\theta(x, y)^2 &\le \int_0^1 \Phi_\theta(v(t), \dot v(t))\, \mathrm{d}t \\ &= \frac{1}{N+1} \sum_{i=0}^{N-1} \int_{\frac{i}{3N}}^{\frac{i+1}{3N}} \phi_\theta(R_{N-i}(v(t)), 3N(N-i))\, \mathrm{d}t + \int_{\frac{1}{3}}^{\frac{2}{3}} \Phi_\theta(v(t), 3(y-x))\, \mathrm{d}t \\ &\quad + \frac{1}{N+1} \sum_{i=0}^{N-1} \int_{\frac{3N-i-1}{3N}}^{\frac{3N-i}{3N}} \phi_\theta(R_{N-i}(v(t)), 3N(N-i))\, \mathrm{d}t \\ &\le \frac{N^2(2N+1) + 3|y-x|^2}{\theta\left(\frac{R}{1+NR}\right)} < +\infty, \end{aligned}$$

which concludes the proof. □

As a consequence of Lemma 2.1, we obtain that d_θ is a distance on $\mathcal{K}$. For the reader's convenience, we briefly recall the proof below.

Proposition 2.2. *The function d_θ is a distance on $\mathcal{K}$.*

Proof. By Lemma 2.1, the discrete variational distance $d_\theta(x, y)$ is well-defined for all $x, y \in \mathcal{K}$. Moreover, it is straightforward to see that $d_\theta(x, y) \ge 0$ for all $x, y \in \mathcal{K}$ and $d_\theta(x, y) = 0$ if and only if $x = y$. Indeed, the fact that $\phi_\theta(R, 0) = 0$ by definition ensures that constant curves are admissible in (2.4). Symmetry follows by defining, for every $v \in W^{1,2}((0,1); \mathcal{K})$ with $v(0) = x$ and $v(1) = y$,

the curve $w(t) = v(1-t)$. Clearly, w has the same regularity as v and satisfies $w(0) = y$, $w(1) = x$, and

$$\int_0^1 \Phi_\theta(w(t), \dot w(t))\,\mathrm{d}t = \int_0^1 \Phi_\theta(v(t), \dot v(t))\,\mathrm{d}t,$$

which implies $d_\theta(x,y) = d_\theta(y,x)$.

Finally, let us show the triangle inequality. Fix $x, y, z \in \mathcal{K}$. For all $\varepsilon > 0$, let $u_\varepsilon, v_\varepsilon \in W^{1,2}((0,1);\mathcal{K})$ be almost optimal curves for the distances $d_\theta(x,y)$ and $d_\theta(y,z)$, respectively, that is, satisfying $u_\varepsilon(0) = x$, $u_\varepsilon(1) = v_\varepsilon(0) = y$, $v_\varepsilon(1) = z$, and

$$\int_0^1 \Phi_\theta(u_\varepsilon(t), \dot u_\varepsilon(t))\,\mathrm{d}t \le (d_\theta(x,y)+\varepsilon)^2, \qquad \int_0^1 \Phi_\theta(v_\varepsilon(t), \dot v_\varepsilon(t))\,\mathrm{d}t \le (d_\theta(y,z)+\varepsilon)^2.$$

We set

$$\lambda_\varepsilon := \frac{d_\theta(x,y)+\varepsilon}{d_\theta(x,y)+d_\theta(y,z)+2\varepsilon} \in (0,1),$$

and we define $w_\varepsilon \in W^{1,2}((0,1);\mathcal{K})$ as

$$w_\varepsilon(s) := \begin{cases} u_\varepsilon\left(\frac{s}{\lambda_\varepsilon}\right) & \text{if } s \in [0,\lambda_\varepsilon], \\ v_\varepsilon\left(\frac{s-\lambda_\varepsilon}{1-\lambda_\varepsilon}\right) & \text{if } s \in [\lambda_\varepsilon, 1]. \end{cases}$$

Then $w_\varepsilon(0) = x$ and $w_\varepsilon(1) = z$. Moreover, by the change of variables formula,

$$\begin{aligned} d_\theta(x,z)^2 &\le \int_0^1 \Phi_\theta(w_\varepsilon(t), \dot w_\varepsilon(t))\,\mathrm{d}t \\ &= \lambda_\varepsilon \int_0^1 \Phi_\theta\left(u_\varepsilon(t), \frac{1}{\lambda_\varepsilon}\dot u_\varepsilon(t)\right)\mathrm{d}t + (1-\lambda_\varepsilon)\int_0^1 \Phi_\theta\left(v_\varepsilon(t), \frac{1}{1-\lambda_\varepsilon}\dot v_\varepsilon(t)\right)\mathrm{d}t \\ &\le \frac{(d_\theta(x,y)+\varepsilon)^2}{\lambda_\varepsilon} + \frac{(d_\theta(y,z)+\varepsilon)^2}{1-\lambda_\varepsilon} = (d_\theta(x,y)+d_\theta(y,z)+2\varepsilon)^2. \end{aligned}$$

Sending $\varepsilon \to 0^+$, we deduce the triangle inequality. □

We now recall the following result, proved in [14, Lemma 2.12], which establishes the existence of *constant-speed* geodesics for the distance d_θ.

Lemma 2.3. *For all $x, y \in \mathcal{K}$ there exists $u \in W^{1,2}((0,1);\mathcal{K})$, with $u(0) = x$ and $u(1) = y$, such that*

$$d_\theta(x,y)^2 = \int_0^1 \Phi_\theta(u(t), \dot u(t))\,\mathrm{d}t.$$

Moreover, if

$$u \in \operatorname{argmin}\left\{\int_0^1 \Phi_\theta(v(t), \dot v(t))\,\mathrm{d}t : v \in W^{1,2}((0,1);\mathcal{K}),\ v(0) = x,\ v(1) = y\right\},$$

then

$$\Phi_\theta(u(s), \dot u(s)) = d_\theta(x,y)^2 \quad \text{for a.e. } s \in (0,1). \tag{2.6}$$

2.2. **Regularity of geodesics.** In this section, we prove two further properties of geodesics:

- geodesics are in $C^1([0,1];\mathcal{K})$;
- geodesics satisfy a suitable system of Euler–Lagrange equations.

Fix $x, y \in \mathcal{K}$ and a minimizing curve

$$u \in \operatorname{argmin}\left\{\int_0^1 \Phi_\theta(v(t), \dot v(t))\,\mathrm{d}t : v \in W^{1,2}((0,1);\mathcal{K}),\ v(0) = x,\ v(1) = y\right\}.$$

For all $i \in \{0, \dots, N\}$ we define the sets

$$A_i := \{t \in (0,1) : R_i(u(t)) < R\}, \qquad C_i := \{t \in (0,1) : R_i(u(t)) = R\} = (0,1) \setminus A_i. \tag{2.7}$$

Since $u \in C^0([0,1];\mathcal{K})$, the sets $A_i \subseteq (0,1)$ are open for all $i \in \{0,\dots,N\}$. Moreover, we note that

$$A_N = (0,1), \qquad C_N = \emptyset.$$

To prove that $u \in C^1([0,1];\mathcal{K})$, our strategy relies on splitting $(0,1)$ into A_i and C_i. On each open set A_i, we derive the Euler–Lagrange equations associated with the minimality of u to deduce local C^1 regularity. Conversely, on the sets C_i, we show that the i-th and $(i+1)$-th components of u have zero velocity. By showing that these two pieces of information match continuously at the interface $\partial_{(0,1)}A_i = \partial_{(0,1)}C_i$, we conclude the global C^1 regularity of u. We begin with the following lemma.

Lemma 2.4. *We have $u \in \mathrm{Lip}([0,1];\mathcal{K})$ and*

$$\dot{u}_i(s) = \dot{u}_{i+1}(s) = 0 \quad \textit{for a.e. } s \in C_i \textit{ and all } i \in \{0,\dots,N-1\}.$$

Proof. From (2.6) in Lemma 2.3 we obtain

$$|\dot{u}_i(s)|^2 \le (N+1)\theta(R_i(u(s)))d_\theta(x,y)^2 \quad \text{for a.e. } s \in (0,1) \text{ and all } i \in \{0,\dots,N\}.$$

Hence, $u \in \mathrm{Lip}([0,1];\mathcal{K})$ and

$$\dot{u}_i(s) = 0 \quad \text{for a.e. } s \in C_i \text{ and all } i \in \{0,\dots,N-1\}.$$

For all $i \in \{0,\dots,N-1\}$ we define the functions $\gamma_i\colon [0,1] \to \mathbb{R}$ as

$$\gamma_i(s) \coloneqq R_i(u(s)) \quad \text{for all } s \in [0,1].$$

Then $\gamma_i \in \mathrm{Lip}([0,1])$ and $\gamma_i \equiv R$ on C_i. Thus

$$0 = \dot{\gamma}_i(s) = NR^2(\dot{u}_i(s) - \dot{u}_{i+1}(s)) = -NR^2\dot{u}_{i+1}(s) \quad \text{for a.e. } s \in C_i \text{ and all } i \in \{0,\dots,N-1\},$$

which implies $\dot{u}_{i+1} \equiv 0$ almost everywhere on C_i for all $i \in \{0,\dots,N-1\}$. □

We now proceed to derive the Euler–Lagrange equations for the geodesic u on the sets A_i. To this end, it is convenient to introduce the following auxiliary functions. For all $i \in \{0,\dots,N\}$ we define the functions $\alpha_i\colon A_i \to \mathbb{R}$ and $\beta_i\colon [0,1] \to \mathbb{R}$

$$\alpha_i(s) \coloneqq \frac{\dot{u}_i(s)}{\theta(R_i(u(s)))} \qquad \text{for a.e. } s \in A_i, \tag{2.8}$$

$$\beta_i(s) \coloneqq -\frac{N}{2}\theta'(R_i(u(s)))R_i(u(s))^2 \qquad \text{for all } s \in [0,1]. \tag{2.9}$$

In view of Lemma 2.4, for all $i \in \{0,\dots,N\}$ we have $\alpha_i \in L^\infty_{\mathrm{loc}}(A_i)$ and $\beta_i \in C^0([0,1])$. Moreover, we point out that

$$\alpha_N = \frac{\dot{u}_N}{\theta(0)} \in L^\infty((0,1)), \qquad \beta_N \equiv 0 \quad \text{in } [0,1]. \tag{2.10}$$

Before deriving the Euler–Lagrange equations satisfied by the geodesic u, we recall two preliminary results concerning one-dimensional BV functions that will be needed later. We begin with the following lemma.

Lemma 2.5. *Let $A \subset \mathbb{R}$ be an open set and let $f, g \in L^1_{\mathrm{loc}}(A)$ satisfy*

$$\int_A f(t)\dot{\varphi}(t)\,\mathrm{d}t + \int_A g(t)\varphi(t)\,\mathrm{d}t \ge 0 \quad \textit{for all non-negative functions } \varphi \in C^\infty_c(A).$$

Then, $f \in \mathrm{BV}_{\mathrm{loc}}(A)$ and there exists a non-negative Radon measure $\mu \in \mathcal{M}^+(A)$ such that

$$Df = g\mathcal{L}^1 - \mu \quad \textit{in } \mathcal{M}(A).$$

Proof. The conclusion follows immediately from the Riesz representation theorem for non-negative linear functionals (see, e.g., [16, Theorem 1.39]) and from the definition of BV functions. □

The next result concerns the existence of left-continuous and right-continuous representatives of one-dimensional $\mathrm{BV}_{\mathrm{loc}}$ functions. A proof can be found in [1, Theorem 3.28], where it is shown for the case of an interval. This can be easily extended to a general open set $A \subset \mathbb{R}$ by recalling that any open subset of $\mathbb{R}$ can be written as a countable union of pairwise disjoint open intervals.

Lemma 2.6. *Let $A \subset \mathbb{R}$ be an open set and let $f \in \mathrm{BV}_{\mathrm{loc}}(A)$. Then, there exist two functions $f^{\pm}\colon A \to \mathbb{R}$, with f^- left-continuous in A and f^+ right-continuous in A, satisfying*

$$f(s) = f^{\pm}(s) \quad \text{for a.e. } s \in A, \qquad f^+(s) - f^-(s) = Df(\{s\}) \quad \text{for all } s \in A.$$

In particular,

$$f^-(s) = \lim_{h\to 0^+} \frac{1}{h}\int_{s-h}^{s} f(t)\,\mathrm{d}t \quad \text{for all } s \in A, \qquad f^+(s) = \lim_{h\to 0^+} \frac{1}{h}\int_{s}^{s+h} f(t)\,\mathrm{d}t \quad \text{for all } s \in A.$$

Remark 2.7. As a consequence of Lemma 2.6, we deduce that $f \in \mathrm{BV}_{\mathrm{loc}}(A)$ admits a continuous representative in A if and only if $Df(\{s\}) = 0$ for all $s \in A$. For this representative, for every interval (s_1, s_2) with $s_1 < s_2$ compactly contained in A, we have

$$f(s_2) - f(s_1) = Df([s_1, s_2]).$$

We are now in a position to state the Euler–Lagrange equations satisfied by the geodesic u on the open sets A_i.

Proposition 2.8. *We have*

$$\alpha_0 \in \mathrm{Lip}_{\mathrm{loc}}(A_0),$$
$$\alpha_i \in \mathrm{Lip}_{\mathrm{loc}}(A_{i-1} \cap A_i) \cap \mathrm{BV}_{\mathrm{loc}}(A_i) \quad \text{for all } i \in \{1, \dots, N\},$$

and the following Euler–Lagrange equations hold

$$\dot\alpha_0 = \alpha_0^2\beta_0 \qquad \text{a.e. in } A_0, \tag{2.11}$$
$$\dot\alpha_i = \alpha_i^2\beta_i - \alpha_{i-1}^2\beta_{i-1} \qquad \text{a.e. in } A_{i-1}\cap A_i \text{ for all } i \in \{1,\dots,N\}. \tag{2.12}$$

Moreover, there exists a non-negative Radon measure $\mu_i \in \mathcal{M}^+(A_i)$ for all $i \in \{1, \dots, N\}$ such that

$$D\alpha_i = \alpha_i^2\beta_i\mathcal{L}^1 - \mu_i \quad \text{in } \mathcal{M}(A_i) \text{ for all } i \in \{1,\dots,N\}. \tag{2.13}$$

Proof. The proof is divided into two steps: we first derive the Euler–Lagrange equation (2.11) for α_0. Then, we consider α_i for a generic $i \in \{1, \dots, N\}$ and we deduce (2.12) and (2.13).

Step 1: Case $i = 0$.
Let $\varphi \in C_c^\infty(A_0)$ be fixed. We observe that α_0 is uniformly bounded on $\mathrm{supp}(\varphi)$. Moreover, still due to the compactness of $\mathrm{supp}(\varphi)$, there exists $\varepsilon_\varphi > 0$ such that for all $\varepsilon \in (-\varepsilon_\varphi, \varepsilon_\varphi)$ we have

$$R_j(u(s) + \varepsilon\varphi(s)\mathrm{e}_0) \le R \quad \text{for all } s \in [0,1] \text{ and } j \in \{0,\dots,N\}.$$

Thus, $v := u + \varepsilon\varphi\mathrm{e}_0 \in W^{1,2}((0,1);\mathcal{K})$ and satisfies $v(0) = u(0)$ and $v(1) = u(1)$. Hence, the minimality condition on u implies

$$\int_0^1 \Phi_\theta(u(t), \dot u(t))\,\mathrm{d}t \le \int_0^1 \Phi_\theta(u(t) + \varepsilon\varphi(t)\mathrm{e}_0, \dot u(t) + \varepsilon\dot\varphi(t)\mathrm{e}_0)\,\mathrm{d}t \quad \text{for all } \varepsilon \in (-\varepsilon_\varphi, \varepsilon_\varphi).$$

We also observe that $\Phi_\theta(u(t) + \varepsilon\varphi(t)\mathrm{e}_0, \dot u(t) + \varepsilon\dot\varphi(t)\mathrm{e}_0)$ perturbs $\Phi_\theta(u(t), \dot u(t))$ only on the term $i = 0$ in the sum (2.2), and only on the set A_0. Thus

$$\int_{A_0} \frac{|\dot u_0(t)|^2}{\theta(R_0(u(t)))}\,\mathrm{d}t \le \int_{A_0} \frac{|\dot u_0(t) + \varepsilon\dot\varphi(t)|^2}{\theta(R_0(u(t) + \varepsilon\varphi(t)\mathrm{e}_0))}\,\mathrm{d}t \quad \text{for all } \varepsilon \in (-\varepsilon_\varphi, \varepsilon_\varphi).$$

By dividing by ε and sending $\varepsilon \to 0$ for a positive and a negative ε, respectively, thanks to the dominated convergence theorem we conclude that for all $\varphi \in C_c^\infty(A_0)$

$$\int_{A_0} \alpha_0(t)\dot\varphi(t)\,\mathrm{d}t + \int_{A_0} \alpha_0(t)^2\beta_0(t)\varphi(t)\,\mathrm{d}t = 0.$$

Since $\alpha_0^2\beta_0 \in L^\infty_{\mathrm{loc}}(A_0)$, we obtain $\alpha_0 \in \mathrm{Lip}_{\mathrm{loc}}(A_0)$ and that (2.11) holds.

Step 2: Case $i \in \{1,\dots,N\}$.
Let $\varphi \in C_c^\infty(A_{i-1}\cap A_i)$ be fixed. As in Step 1, there exists $\varepsilon_\varphi > 0$ such that for all $\varepsilon \in (-\varepsilon_\varphi, \varepsilon_\varphi)$ we have

$$R_j(u(s) + \varepsilon\varphi(s)\mathrm{e}_i) \le R \quad \text{for all } s \in [0,1] \text{ and } j \in \{0,\dots,N\}.$$

Hence, the minimality condition on u implies

$$\int_0^1 \Phi_\theta(u(t),\dot u(t))\,\mathrm{d}t \le \int_0^1 \Phi_\theta(u(t)+\varepsilon\varphi(t)\mathrm{e}_i, \dot u(t)+\varepsilon\dot\varphi(t)\mathrm{e}_i)\,\mathrm{d}t \quad \text{for all } \varepsilon \in (-\varepsilon_\varphi,\varepsilon_\varphi).$$

In this case, $\Phi_\theta(u(t)+\varepsilon\varphi(t)\mathrm{e}_i, \dot u(t)+\varepsilon\dot\varphi(t)\mathrm{e}_i)$ perturbs $\Phi_\theta(u(t),\dot u(t))$ only on the terms $i-1$ and i in the sum (2.2), and only on the set $A_{i-1}\cap A_i$. Thus,

$$\begin{aligned}
&\int_{A_{i-1}\cap A_i} \left(\frac{|\dot u_{i-1}(t)|^2}{\theta(R_{i-1}(u(t)))} + \frac{|\dot u_i(t)|^2}{\theta(R_i(u(t)))}\right)\mathrm{d}t \\
&\le \int_{A_{i-1}\cap A_i} \left(\frac{|\dot u_{i-1}(t)|^2}{\theta(R_{i-1}(u(t)+\varepsilon\varphi(t)\mathrm{e}_i))} + \frac{|\dot u_i(t)+\varepsilon\dot\varphi(t)|^2}{\theta(R_i(u(t)+\varepsilon\varphi(t)\mathrm{e}_i))}\right)\mathrm{d}t \quad \text{for all } \varepsilon \in (-\varepsilon_\varphi,\varepsilon_\varphi).
\end{aligned}$$

As in Step 1, by dividing by ε and letting $\varepsilon \to 0$ for a positive and a negative ε, respectively, thanks to the dominated convergence theorem we conclude that for all $\varphi \in C_c^\infty(A_{i-1}\cap A_i)$

$$\int_{A_{i-1}\cap A_i} \alpha_i(t)\dot\varphi(t)\,\mathrm{d}t + \int_{A_{i-1}\cap A_i} \alpha_i(t)^2\beta_i(t)\varphi(t)\,\mathrm{d}t - \int_{A_{i-1}\cap A_i} \alpha_{i-1}(t)^2\beta_{i-1}(t)\varphi(t)\,\mathrm{d}t = 0.$$

Since $\alpha_i^2\beta_i - \alpha_{i-1}^2\beta_{i-1} \in L^\infty_{\mathrm{loc}}(A_{i-1}\cap A_i)$, we deduce that $\alpha_i \in \mathrm{Lip}_{\mathrm{loc}}(A_{i-1}\cap A_i)$ and that (2.12) holds.

Let us prove (2.13). We now fix a non-negative function $\varphi \in C_c^\infty(A_i)$. Then, there exists $\varepsilon_\varphi > 0$ such that for all $\varepsilon \in (0,\varepsilon_\varphi)$ we have

$$R_j(u(s) + \varepsilon\varphi(s)\mathrm{e}_i) \le R \quad \text{for all } s \in [0,1] \text{ and } j \in \{0,\dots,N\}.$$

Hence, the minimality condition on u implies

$$\int_0^1 \Phi_\theta(u(t),\dot u(t))\,\mathrm{d}t \le \int_0^1 \Phi_\theta(u(t)+\varepsilon\varphi(t)\mathrm{e}_i, \dot u(t)+\varepsilon\dot\varphi(t)\mathrm{e}_i)\,\mathrm{d}t \quad \text{for all } \varepsilon \in (0,\varepsilon_\varphi).$$

In this case, $\Phi_\theta(u(t)+\varepsilon\varphi(t)\mathrm{e}_i, \dot u(t)+\varepsilon\dot\varphi(t)\mathrm{e}_i)$ perturbs $\Phi_\theta(u(t),\dot u(t))$ only on the terms $i-1$ and i in the sum (2.2), and only on the set A_i. Thus,

$$\begin{aligned}
&\int_{A_i} \left(\frac{|\dot u_{i-1}(t)|^2}{\theta(R_{i-1}(u(t)))} + \frac{|\dot u_i(t)|^2}{\theta(R_i(u(t)))}\right)\mathrm{d}t \\
&\le \int_{A_i} \left(\frac{|\dot u_{i-1}(t)|^2}{\theta(R_{i-1}(u(t)+\varepsilon\varphi(t)\mathrm{e}_i))} + \frac{|\dot u_i(t)+\varepsilon\dot\varphi(t)|^2}{\theta(R_i(u(t)+\varepsilon\varphi(t)\mathrm{e}_i))}\right)\mathrm{d}t \quad \text{for all } \varepsilon \in (0,\varepsilon_\varphi).
\end{aligned}$$

Since $\varphi \ge 0$ and $\varepsilon > 0$, we have $\theta(R_{i-1}(u(t))) \le \theta(R_{i-1}(u(t)+\varepsilon\varphi(t)\mathrm{e}_i))$. Hence,

$$\int_{A_i} \frac{|\dot u_i(t)|^2}{\theta(R_i(u(t)))}\,\mathrm{d}t \le \int_{A_i} \frac{|\dot u_i(t)+\varepsilon\dot\varphi(t)|^2}{\theta(R_i(u(t)+\varepsilon\varphi(t)\mathrm{e}_i))}\,\mathrm{d}t \quad \text{for all } \varepsilon \in (0,\varepsilon_\varphi).$$

By dividing by ε and letting $\varepsilon \to 0^+$, thanks to the dominated convergence theorem we conclude that for all non-negative functions $\varphi \in C_c^\infty(A_i)$ we have

$$\int_{A_i} \alpha_i(t)\dot\varphi(t)\,\mathrm{d}t + \int_{A_i} \alpha_i(t)^2\beta_i(t)\varphi(t)\,\mathrm{d}t \ge 0.$$

In view of the fact that $\alpha_i^2\beta_i \in L^\infty_{\mathrm{loc}}(A_i)$, we can apply Lemma 2.5 to deduce that $\alpha_i \in \mathrm{BV}_{\mathrm{loc}}(A_i)$ and that there exists a non-negative Radon measure $\mu_i \in \mathcal{M}^+(A_i)$ such that (2.13) holds. □

Remark 2.9. In view of (2.10), we deduce that $\alpha_N \in \mathrm{Lip}_{\mathrm{loc}}(A_{N-1}) \cap \mathrm{BV}_{\mathrm{loc}}((0,1))$ and that it satisfies the following Euler–Lagrange equations

$$\dot\alpha_N = -\alpha_{N-1}^2\beta_{N-1} \quad \text{a.e. in } A_{N-1}, \qquad D\alpha_N = -\mu_N \quad \text{in } \mathcal{M}((0,1)).$$

By combining Lemma 2.4 and Proposition 2.8, we can finally deduce that every geodesic u belongs to $C^1([0,1];\mathcal{K})$.

Proposition 2.10. *We have that $u \in C^1([0,1];\mathcal{K})$. Moreover*

$$\dot u_i(s) = \dot u_{i+1}(s) = 0 \quad \textit{for all } s \in C_i \textit{ and all } i \in \{0,\dots,N-1\}, \tag{2.14}$$

and

$$\Phi_\theta(u(s),\dot u(s)) \le d_\theta(x,y)^2 \quad \textit{for all } s \in [0,1]. \tag{2.15}$$

Proof. The proof of the regularity of u proceeds by induction on $i \in \{0,\dots,N\}$.

We begin by showing the regularity of the first component, u_0. Since $\dot u_0$ is already regular on A_0, we extend this function by setting $\dot u_0 = 0$ outside A_0. This extension defines the same geodesic curve by virtue of Lemma 2.4, and we prove that it is continuous in $[0,1]$.

Then, for all $i \in \{1,\dots,N\}$, we proceed inductively: assuming the continuity of the velocity of the $(i-1)$-th component, we use the Euler–Lagrange equations to deduce the continuity of the velocity of the i-th component. Specifically, we utilize the Euler–Lagrange equation (2.12) to establish regularity of $\dot u_i$ on the sets $A_{i-1} \cap A_i$. Then, we extend this function by setting $\dot u_i = 0$ outside $A_{i-1} \cap A_i$. This extension defines the same geodesic curve in virtue of Lemma 2.4, and we employ the Euler–Lagrange equation (2.13) to show that it is continuous in $[0,1]$. By repeating this argument through all components, we conclude that the geodesic u belongs to $C^1([0,1];\mathcal{K})$.

Step 1: Case $i = 0$.
Lemma 2.4 implies $u_0 \in \mathrm{Lip}([0,1])$ and

$$\dot u_0(s) = 0 \quad \text{for a.e. } s \in C_0.$$

Moreover, since

$$\dot u_0(s) = \theta(R_0(u(s)))\alpha_0(s) \quad \text{for a.e. } s \in A_0,$$

Proposition 2.8 implies $\dot u_0 \in C^0(A_0)$ and

$$\dot u_0(s) = \theta(R_0(u(s)))\alpha_0(s) \quad \text{for all } s \in A_0.$$

Consequently, Lemma 2.3 implies

$$|\dot u_0(s)|^2 \le (N+1)\theta(R_0(u(s)))d_\theta(x,y)^2 \quad \text{for all } s \in A_0. \tag{2.16}$$

Let us define $v_0 \colon (0,1) \to \mathbb{R}$ as

$$v_0(s) := \begin{cases} \dot u_0(s) & \text{if } s \in A_0, \\ 0 & \text{if } s \in C_0. \end{cases}$$

We have that $v_0 = \dot u_0$ a.e. in $(0,1)$. Moreover, we claim that $v_0 \in C^0((0,1))$. Indeed, due to (2.16), for all $s_0 \in \partial A_0 \cap (0,1)$ there exists

$$\lim_{s \in A_0,\, s \to s_0} \dot u_0(s) = 0.$$

Thus, $\dot u_0 \in C^0((0,1))$ and from Lemma 2.3 we obtain

$$|\dot u_0(s)|^2 \le (N+1)\theta(R_0(u(s)))d_\theta(x,y)^2 \quad \text{for all } s \in (0,1). \tag{2.17}$$

Finally, we claim that $\dot u_0 \in C^0([0,1])$. At $s = 0$, if $R_0(u(0)) = R$, then (2.17) implies

$$\lim_{s \to 0^+} \dot u_0(s) = 0.$$

If, on the other hand, $R_0(u(0)) < R$, then there exists $c \in (0,1)$ such that $\alpha_0 \in \mathrm{Lip}_{\mathrm{loc}}((0,2c))$ and

$$\dot\alpha_0(s) = \alpha_0(s)^2\beta_0(s) \quad \text{for a.e. } s \in (0,2c).$$

Thus, for all $s \in (0,c)$ we have

$$\alpha_0(c) - \alpha_0(s) = \int_s^c \alpha_0(t)^2 \beta_0(t)\,\mathrm{d}t.$$

Since $\alpha_0^2\beta_0 \in L^\infty((0,c))$, there exists

$$\lim_{s\to 0^+} \alpha_0(s) = \alpha_0(c) - \int_0^c \alpha_0(t)^2\beta_0(t)\,\mathrm{d}t,$$

which implies the existence of the following limit

$$\lim_{s\to 0^+} \dot{u}_0(s) = \lim_{s\to 0^+} \theta(R_0(u(s)))\alpha_0(s) = \theta(R_0(u(0)))\left(\alpha_0(c) - \int_0^c \alpha_0(t)^2\beta_0(t)\,\mathrm{d}t\right).$$

We can argue in a similar way for $s = 1$, and thus $\dot{u}_0 \in C^0([0,1])$.

Step 2: Case $i \in \{1,\dots,N\}$.
By induction, we assume that $\dot{u}_{i-1} \in C^0([0,1])$. By Lemma 2.4 we have that $u_i \in \mathrm{Lip}([0,1])$ and

$$\dot{u}_i(s) = 0 \quad \text{for a.e. } s \in C_{i-1} \cup C_i.$$

Moreover, since

$$\dot{u}_i(s) = \theta(R_i(u(s)))\alpha_i(s) \quad \text{for a.e. } s \in A_i,$$

Proposition 2.8 gives $\dot{u}_i \in C^0(A_{i-1}\cap A_i) \cap \mathrm{BV}_{\mathrm{loc}}(A_i)$. Let $\alpha_i^\pm \colon A_i \to \mathbb{R}$ be the left-continuous and right-continuous representatives of α_i given by Lemma 2.6 and define $\dot{u}_i^\pm \colon A_i \to \mathbb{R}$ as

$$\dot{u}_i^\pm(s) = \theta(R_i(u(s)))\alpha_i^\pm(s) \quad \text{for all } s \in A_i. \tag{2.18}$$

We have that

$$\dot{u}_i^\pm(s) = \dot{u}_i(s) \qquad \text{for all } s \in A_{i-1}\cap A_i, \tag{2.19}$$

$$\dot{u}_i^-(s) - \dot{u}_i^+(s) = \theta(R_i(u(s)))\mu_i(\{s\}) \geq 0 \qquad \text{for all } s \in A_i, \tag{2.20}$$

where μ_i comes from the statement of Proposition 2.8. Let us fix

$$s_0 \in A_i \setminus A_{i-1} = C_{i-1} \cap A_i.$$

Since $s_0 \in C_{i-1}$, the distance $u_i(s_0) - u_{i-1}(s_0)$ is the least possible. Hence, for $h > 0$ sufficiently small we have

$$\begin{aligned} 0 &\leq \frac{u_i(s_0+h) - u_i(s_0) - u_{i-1}(s_0+h) + u_{i-1}(s_0)}{h} \\ &= \frac{1}{h}\int_{s_0}^{s_0+h} \dot{u}_i(s)\,\mathrm{d}s - \frac{u_{i-1}(s_0+h) - u_{i-1}(s_0)}{h}, \end{aligned}$$

and similarly

$$\begin{aligned} 0 &\geq \frac{u_i(s_0-h) - u_i(s_0) - u_{i-1}(s_0-h) + u_{i-1}(s_0)}{-h} \\ &= \frac{1}{h}\int_{s_0-h}^{s_0} \dot{u}_i(s)\,\mathrm{d}s - \frac{u_{i-1}(s_0-h) - u_{i-1}(s_0)}{-h}. \end{aligned}$$

By sending $h \to 0^+$ and using that $u_{i-1} \in C^1([0,1])$ and $\dot{u}_{i-1}(s_0) = 0$, we obtain

$$\dot{u}_i^+(s_0) \geq 0 \geq \dot{u}_i^-(s_0).$$

Hence, by (2.19)–(2.20) we have

$$\dot{u}_i^+(s) = \dot{u}_i^-(s) = 0 \quad \text{for all } s \in C_{i-1}\cap A_i,$$

which implies $\dot{u}_i \in C^0(A_i)$. In particular, Lemma 2.3 implies

$$|\dot{u}_i(s)|^2 \leq (N+1)\theta(R_i(u(s)))d_\theta(x,y)^2 \quad \text{for all } s \in A_i. \tag{2.21}$$

Similarly to Step 1, we define $v_i\colon (0,1)\to\mathbb{R}$ as

$$v_i(s) := \begin{cases} \dot u_i(s) & \text{if } s\in A_i,\\ 0 & \text{if } s\in C_i. \end{cases}$$

We have $v_i=\dot u_i$ a.e. on $(0,1)$ and by arguing as for v_0 and using (2.21) we obtain $v_i\in C^0((0,1))$. Hence $u_i\in C^1((0,1))$ and

$$|\dot u_i(s)|^2 \le (N+1)\theta(R_i(u(s)))d_\theta(x,y)^2 \quad \text{for all } s\in(0,1). \tag{2.22}$$

Moreover,

$$\dot u_i(s)=0 \quad \text{for all } s\in (C_{i-1}\cap A_i)\cup C_i = C_{i-1}\cup C_i.$$

Finally, we claim that $u_i\in C^1([0,1])$. At $s=0$, if $R_i(u(0))=R$, by (2.22) we have

$$\lim_{s\to 0^+}\dot u_i(s)=0.$$

On the other hand, if $R_i(u(0))<R$, then there exists $c\in(0,1)$ such that $\alpha_i\in \mathrm{BV}_{\mathrm{loc}}((0,2c))\cap C^0((0,2c))$ and there exists a non-negative Radon measure $\mu_i\in\mathcal{M}^+((0,2c))$ such that

$$D\alpha_i=\alpha_i^2\beta_i\mathcal{L}^1-\mu_i \quad \text{in } \mathcal{M}((0,2c)).$$

Thus, for all $s\in(0,c)$ we have

$$\alpha_i(c)-\alpha_i(s)=\int_s^c \alpha_i(t)^2\beta_i(t)\,\mathrm{d}t-\mu_i([s,c]).$$

Since $\alpha_i^2\beta_i\in L^\infty((0,c))$ and $\mu_i\in\mathcal{M}^+((0,2c))$, we deduce that there exists

$$\lim_{s\to0^+}\alpha_i(s)=\alpha_i(c)-\int_0^c\alpha_i(t)^2\beta_i(t)\,\mathrm{d}t+\mu_i((0,c])\in(-\infty,+\infty],$$

which implies the existence of the following limit

$$\lim_{s\to0^+}\dot u_i(s)=\theta(R_i(u(0)))\left(\alpha_i(c)-\int_0^c\alpha_i(t)^2\beta_i(t)\,\mathrm{d}t+\mu_i((0,c])\right)\in(-\infty,+\infty].$$

On the other hand, we have that $\dot u_i\in L^\infty((0,1))$, and so

$$\lim_{s\to0^+}\dot u_i(s)\in\mathbb{R}.$$

We can argue in a similar way for $s=1$, and recover $\dot u_i\in C^0([0,1])$.

Finally, for each $s\in[0,1]$, choose $s_n\to s$ from the full-measure set in (2.6). The C^1 regularity of u and the lower semicontinuity of Φ_θ give (2.15). □

Remark 2.11. Since $u\in C^1([0,1];\mathcal{K})$, we can improve the regularity of α_i established in Proposition 2.8. Indeed, in view of (2.18), we deduce that $\alpha_i\in C^0(A_i)$ for all $i\in\{1,\dots,N\}$. In particular, $\alpha_N=\dot u_N/\theta(0)\in C^0([0,1])$.

We conclude this section by showing some additional properties of $\alpha_i\in\mathrm{BV}_{\mathrm{loc}}(A_i)\cap C^0(A_i)$, which follow from the Euler–Lagrange equations (2.13). In particular, we prove that any function $\alpha_i\in\mathrm{BV}_{\mathrm{loc}}(A_i)\cap C^0(A_i)$ satisfying (2.13) exhibits a *sign monotonicity property* on every connected component of A_i. More precisely, within each connected component, the zero set of α_i is an interval, and α_i can change sign only from positive to negative across this set, with either sign region possibly empty.

Lemma 2.12. *Let $a_1,a_2\in\mathbb{R}$ with $a_1<a_2$. Let $f\in\mathrm{BV}_{\mathrm{loc}}((a_1,a_2))\cap C^0((a_1,a_2))$ and $g\in L^1_{\mathrm{loc}}((a_1,a_2))$ satisfy*

$$Df\le f^2g\mathcal{L}^1 \quad \text{in } \mathcal{M}((a_1,a_2)).$$

Assume that

$$\{t\in(a_1,a_2): f(t)=0\}\neq\emptyset.$$

Then, there exist $b_1, b_2 \in \mathbb{R}$ *with*

$$a_1 \le b_1 \le b_2 \le a_2$$

such that

$$f(s) > 0 \quad \textit{for all } s \in (a_1, b_1), \qquad f(s) = 0 \quad \textit{for all } s \in (b_1, b_2), \qquad f(s) < 0 \quad \textit{for all } s \in (b_2, a_2).$$

Proof. We define

$$b_1 := \inf\{t \in (a_1, a_2) : f(t) = 0\} \in [a_1, a_2), \qquad b_2 := \sup\{t \in (a_1, a_2) : f(t) = 0\} \in (a_1, a_2].$$

We start by showing that

$$f(s) = 0 \quad \text{for all } s \in (b_1, b_2).$$

By contradiction, assume that there exists $s_0 \in (b_1, b_2)$ such that

$$f(s_0) \neq 0.$$

Since f is continuous in (a_1, a_2), we can consider the biggest interval $(c_1, c_2) \subseteq (b_1, b_2)$ such that $s_0 \in (c_1, c_2)$ and $f(s) \neq 0$ for all $s \in (c_1, c_2)$. Then

$$a_1 < c_1 < c_2 < a_2, \qquad f(c_1) = f(c_2) = 0.$$

By assumptions, we have

$$\frac{1}{f^2} Df \le g\mathcal{L}^1 \quad \text{in } \mathcal{M}((c_1, c_2)). \tag{2.23}$$

Note that the function $1/f \in \mathrm{BV}_{\mathrm{loc}}((c_1, c_2))$ and

$$D\left(\frac{1}{f}\right) = -\frac{1}{f^2} Df \quad \text{in } \mathcal{M}((c_1, c_2)).$$

Thus, by (2.23) and Remark 2.7,

$$-\frac{1}{f(s_2)} + \frac{1}{f(s_1)} \le \int_{s_1}^{s_2} g(t)\,\mathrm{d}t \quad \text{for all } s_1, s_2 \in (c_1, c_2) \text{ with } s_1 < s_2,$$

which implies

$$f(s_2) - f(s_1) \le f(s_1) f(s_2) \int_{s_1}^{s_2} g(t)\,\mathrm{d}t \quad \text{for all } s_1, s_2 \in (c_1, c_2) \text{ with } s_1 < s_2,$$

since f has constant sign in (c_1, c_2). By sending $s_1 \to c_1^+$ we get

$$f(s_2) \le 0 \quad \text{for all } s_2 \in (c_1, c_2),$$

while by sending $s_2 \to c_2^-$ we deduce

$$-f(s_1) \le 0 \quad \text{for all } s_1 \in (c_1, c_2).$$

Thus $f(s) = 0$ for all $s \in (c_1, c_2)$, which contradicts $f(s_0) \neq 0$. This proves the claim.

Assume now that $b_1 > a_1$. Then $f(b_1) = 0$ and by arguing as before we have

$$f(s_2) - f(s_1) \le f(s_1) f(s_2) \int_{s_1}^{s_2} g(t)\,\mathrm{d}t \quad \text{for all } s_1, s_2 \in (a_1, b_1) \text{ with } s_1 < s_2.$$

By sending $s_2 \to b_1^-$ and using that $f(s) \neq 0$ for all $s \in (a_1, b_1)$ we get

$$-f(s_1) < 0 \quad \text{for all } s_1 \in (a_1, b_1).$$

Similarly, if $b_2 < a_2$, then $f(b_2) = 0$ and we deduce

$$f(s_2) - f(s_1) \le f(s_1) f(s_2) \int_{s_1}^{s_2} g(t)\,\mathrm{d}t \quad \text{for all } s_1, s_2 \in (b_2, a_2) \text{ with } s_1 < s_2.$$

By sending $s_1 \to b_2^+$ and using that $f(s) \neq 0$ for all $s \in (b_2, a_2)$ we obtain

$$f(s_2) < 0 \quad \text{for all } s_2 \in (b_2, a_2),$$

which concludes the proof. □

As a consequence, we derive the following result, which can be applied to α_0.

Corollary 2.13. *Let $a_1, a_2 \in \mathbb{R}$ with $a_1 < a_2$. Let $f \in \mathrm{AC}_{\mathrm{loc}}((a_1,a_2))$ and $g \in L^1_{\mathrm{loc}}((a_1,a_2))$ satisfy*

$$\dot f = f^2 g \quad \text{a.e. in } (a_1,a_2).$$

Then, either

$$f(s) = 0 \quad \text{for all } s \in (a_1,a_2)$$

or

$$f(s) \neq 0 \quad \text{for all } s \in (a_1,a_2).$$

Proof. Assume that $\{t \in (a_1,a_2) : f(t) = 0\} \neq \emptyset$. Applying Lemma 2.12 to both pairs (f,g) and $(-f,-g)$, we deduce that $f(s) = 0$ for all $s \in (a_1,a_2)$. □

3. The minimization problem for the discrete JKO scheme

In this section, we analyze a single step of the discrete JKO scheme associated with a decreasing *potential* V and the *discrete variational distance* d_θ. The limiting dynamics are given by the continuous *Follow-the-Leader* system

$$\begin{cases} \dot x_i(t) = -\theta(R_i(x(t)))V'(x_i(t)) & \text{for } t \in [0,+\infty) \text{ and } i \in \{0,\dots,N\}, \\ x(0) = x^0, \end{cases} \tag{3.1}$$

where $x^0 \in \mathcal{K}$ is a fixed initial state. We recall the assumptions on the potential V and investigate the properties of a single JKO step for a fixed time step $\tau > 0$ and a given initial point $x \in \mathcal{K}$.

We fix a *potential* $V \in C^1(\mathbb{R})$ satisfying

$$V'(x) < 0 \quad \text{for all } x \in \mathbb{R}, \tag{3.2}$$

and assume that there exist constants $C, D \geq 0$ and $s \in (0,2)$ such that

$$V(x) \geq -C|x|^s - D \quad \text{for all } x \in \mathbb{R}. \tag{3.3}$$

These assumptions on V are satisfied throughout this paper. Let us fix a *starting point* $x \in \mathcal{K}$ and a *time step* $\tau > 0$. A single JKO step consists of finding a target point $x_\tau \in \mathcal{K}$ that minimizes

$$\mathcal{K} \ni y \mapsto \frac{1}{2\tau} d_\theta(x,y)^2 + \frac{1}{N+1}\sum_{i=0}^N V(y_i). \tag{3.4}$$

The lower bound (3.3) ensures that the quantity in (3.4) is bounded from below, allowing for the existence of minimizers. The next lemma, which is proved in [14, Proposition 2.17(i)], establishes the well-posedness of this minimization problem.

Lemma 3.1. *There exists a solution $x_\tau \in \mathcal{K}$ of the JKO step*

$$x_\tau \in \operatorname{argmin}\left\{ \frac{1}{2\tau} d_\theta(x,y)^2 + \frac{1}{N+1}\sum_{i=0}^N V(y_i) : y \in \mathcal{K} \right\}. \tag{3.5}$$

Observe that the JKO step (3.5) involves a double minimization: one with respect to the target configuration $y \in \mathcal{K}$, and the other associated with the definition (2.4) of the discrete distance $d_\theta(x,y)$. In the next result, we show that the minimum problem (3.5) can be equivalently reformulated as a single *joint* minimization over curves $v \in W^{1,2}((0,1);\mathcal{K})$ with fixed initial datum $v(0) = x$ and a free endpoint $v(1)$ penalized by the potential V.

Lemma 3.2. *Let $x_\tau \in \mathcal{K}$ be a solution of the JKO step (3.5) and let $u_\tau \in W^{1,2}((0,1);\mathcal{K})$ be a solution of*

$$u_\tau \in \operatorname{argmin}\left\{ \int_0^1 \Phi_\theta(v(t),\dot v(t))\,\mathrm{d}t : v \in W^{1,2}((0,1);\mathcal{K}),\ v(0) = x,\ v(1) = x_\tau \right\}. \tag{3.6}$$

Then u_τ is a solution of the joint minimization problem

$$u_\tau \in \operatorname{argmin}\left\{\frac{1}{2\tau}\int_0^1 \Phi_\theta(v(t),\dot v(t))\,\mathrm{d}t + \frac{1}{N+1}\sum_{i=0}^N V(v_i(1)) : v\in W^{1,2}((0,1);\mathcal{K}),\ v(0)=x\right\}. \quad (3.7)$$

Conversely, if u_τ is a solution of the joint minimization problem (3.7), then $x_\tau := u_\tau(1)$ is a solution of the JKO step (3.5).

Proof. Let $x_\tau\in\mathcal{K}$ be a solution of (3.5) and let $u_\tau\in W^{1,2}((0,1);\mathcal{K})$ be a solution of (3.6). Fix $v\in W^{1,2}((0,1);\mathcal{K})$ with $v(0)=x$ and consider

$$w\in\operatorname{argmin}\left\{\int_0^1 \Phi_\theta(u(t),\dot u(t))\,\mathrm{d}t : u\in W^{1,2}((0,1);\mathcal{K}),\ u(0)=x,\ u(1)=v(1)\right\}.$$

Then, (2.6) in Lemma 2.3 implies

$$\begin{aligned}
&\frac{1}{2\tau}\int_0^1 \Phi_\theta(u_\tau(t),\dot u_\tau(t))\,\mathrm{d}t + \frac{1}{N+1}\sum_{i=0}^N V(u_{\tau,i}(1))\\
&= \frac{1}{2\tau}d_\theta(x,x_\tau)^2 + \frac{1}{N+1}\sum_{i=0}^N V(x_{\tau,i})\\
&\le \frac{1}{2\tau}d_\theta(x,v(1))^2 + \frac{1}{N+1}\sum_{i=0}^N V(v_i(1))\\
&= \frac{1}{2\tau}\int_0^1 \Phi_\theta(w(t),\dot w(t))\,\mathrm{d}t + \frac{1}{N+1}\sum_{i=0}^N V(v_i(1))\\
&\le \frac{1}{2\tau}\int_0^1 \Phi_\theta(v(t),\dot v(t))\,\mathrm{d}t + \frac{1}{N+1}\sum_{i=0}^N V(v_i(1)),
\end{aligned}$$

and the assertion follows by taking the inf on the right hand side with respect to $v\in W^{1,2}((0,1);\mathcal{K})$ with $v(0)=x$.

Let u_τ be a solution of (3.7). Fix $y\in\mathcal{K}$ and consider

$$w\in\operatorname{argmin}\left\{\int_0^1 \Phi_\theta(u(t),\dot u(t))\,\mathrm{d}t : u\in W^{1,2}((0,1);\mathcal{K}),\ u(0)=x,\ u(1)=y\right\}.$$

Then, Lemma 2.3 implies

$$\begin{aligned}
&\frac{1}{2\tau}d_\theta(x,u_\tau(1))^2 + \frac{1}{N+1}\sum_{i=0}^N V(u_{\tau,i}(1))\\
&\le \frac{1}{2\tau}\int_0^1 \Phi_\theta(u_\tau(t),\dot u_\tau(t))\,\mathrm{d}t + \frac{1}{N+1}\sum_{i=0}^N V(u_{\tau,i}(1))\\
&\le \frac{1}{2\tau}\int_0^1 \Phi_\theta(w(t),\dot w(t))\,\mathrm{d}t + \frac{1}{N+1}\sum_{i=0}^N V(w_i(1))\\
&= \frac{1}{2\tau}d_\theta(x,y)^2 + \frac{1}{N+1}\sum_{i=0}^N V(y_i),
\end{aligned}$$

and the assertion follows by taking the inf with respect to y on the right hand side. □

In view of Lemma 3.2, studying the properties of a solution x_τ of the JKO step (3.5) reduces to investigating the properties of a geodesic u_τ of the joint minimization problem (3.7). Therefore, we

fix a geodesic

$$u_\tau \in \operatorname{argmin}\left\{\frac{1}{2\tau}\int_0^1 \Phi_\theta(v(t),\dot v(t))\,\mathrm{d}t + \frac{1}{N+1}\sum_{i=0}^{N} V(v_i(1)) : v\in W^{1,2}((0,1);\mathcal{K}),\, v(0)=x\right\}.$$

Since Lemma 3.2 implies that u_τ is also a geodesic for the distance $d_\theta(x,u_\tau(1))$, Proposition 2.10 ensures that $u_\tau\in C^1([0,1];\mathcal{K})$.

Associated with the geodesic u_τ, for all $i\in\{0,\dots,N\}$ let $A_{\tau,i}, C_{\tau,i}\subset(0,1)$ be the sets defined in (2.7), and let $\alpha_{\tau,i}\colon A_{\tau,i}\to\mathbb{R}$ and $\beta_{\tau,i}\colon[0,1]\to\mathbb{R}$ be the functions defined in (2.8) and (2.9), respectively. Our next goal is to show that, under the monotonicity assumption (3.2) on the potential V, the path $u_\tau(s)$ lies in the interior of the cone $\mathcal{K}$ for all $s\in(0,1]$, namely

$$R_i(u_\tau(s)) < R \quad \text{for all } s\in(0,1] \text{ and all } i\in\{0,\dots,N-1\}.$$

Consequently, the Euler–Lagrange equations (2.11)–(2.12) hold on the entire interval $(0,1]$. Moreover, we derive a Neumann-type boundary condition at $s=1$ for the functions $\alpha_{\tau,i}$. These two facts will be crucial for obtaining uniform estimates in τ for the JKO scheme, allowing us to pass to the time-continuous limit as $\tau\to0$.

Proposition 3.3. *We have*

$$R_i(u_\tau(s)) < R \quad \textit{for all } s\in(0,1] \textit{ and } i\in\{0,\dots,N-1\}. \tag{3.8}$$

In particular, $\alpha_{\tau,i}\in\mathrm{Lip}_{\mathrm{loc}}((0,1))$ *for all* $i\in\{0,\dots,N\}$ *and the following limit exists*

$$\alpha_{\tau,i}(1) \coloneqq \lim_{s\to1^-}\alpha_{\tau,i}(s) = -\tau V'(u_{\tau,i}(1)) \quad \textit{for all } i\in\{0,\dots,N\}. \tag{3.9}$$

Moreover, we have

$$\alpha_{\tau,i}(s) > 0 \quad \textit{for all } s\in(0,1] \textit{ and all } i\in\{0,\dots,N\}. \tag{3.10}$$

Proof. We proceed by backward induction on i, starting from the last component $i=N$ and moving down to $i=0$.

For a fixed index $i\in\{0,\dots,N\}$, the idea is to combine the regularity obtained in Proposition 2.10 with the monotonicity condition (3.2) on the potential V to deduce (3.10) for i via the sign monotonicity condition established in Lemma 2.12. Consequently, we obtain that $\dot u_{\tau,i}(s)>0$ for all $s\in(0,1]$ and, in view of (2.14) in Proposition 2.10, for $i\geq1$ we conclude that $C_{\tau,i-1}=\emptyset$, which yields (3.8) for $i-1$. This last condition allows us to iterate the procedure and proceed by backward induction to the case $i-1$.

To carry out this strategy, we proceed by perturbing the minimization problem (3.7) with a suitable test function in order to derive the Neumann-type condition (3.9) at the endpoint of the geodesic u_τ. To this end, for each $\eta\in(0,1)$, we define $\varphi_\eta\colon[0,1]\to[0,1]$ as

$$\varphi_\eta(s) \coloneqq \begin{cases} 0 & \text{for } s\in[0,1-\eta),\\ \dfrac{s-1+\eta}{\eta} & \text{for } s\in[1-\eta,1].\end{cases}$$

Step 1: Case $i=N$.
For clarity, we divide this step into three substeps.

Step 1.1: Proof of (3.10) for $i=N$.
Proposition 2.10 implies that $\alpha_{\tau,N}=\dot u_{\tau,N}/\theta(0)\in C^0([0,1])$. By exploiting the non-negativity of φ_η, for all $\varepsilon>0$ we have

$$R_j(u_\tau(s)+\varepsilon\varphi_\eta(s)\mathrm{e}_N)\leq R \quad \text{for all } s\in[0,1] \text{ and } j\in\{0,\dots,N\}.$$

Thus, $v := u_\tau + \varepsilon\varphi_\eta \mathrm{e}_N \in W^{1,2}((0,1);\mathcal{K})$ and satisfies $v(0) = u_\tau(0)$. Hence, for all $\varepsilon > 0$ the minimality condition on u_τ implies

$$\begin{aligned}&\frac{1}{2\tau}\int_0^1 \Phi_\theta(u_\tau(t),\dot u_\tau(t))\,\mathrm{d}t\\ &\le \frac{1}{2\tau}\int_0^1 \Phi_\theta(u_\tau(t)+\varepsilon\varphi_\eta(t)\mathrm{e}_N,\dot u_\tau(t)+\varepsilon\dot\varphi_\eta(t)\mathrm{e}_N)\,\mathrm{d}t + \frac{V(u_{\tau,N}(1)+\varepsilon)-V(u_{\tau,N}(1))}{N+1}.\end{aligned}$$

We observe that $\Phi_\theta(u_\tau(t)+\varepsilon\varphi_\eta(t)\mathrm{e}_N,\dot u_\tau(t)+\varepsilon\dot\varphi_\eta(t)\mathrm{e}_N)$ perturbs $\Phi_\theta(u_\tau(t),\dot u_\tau(t))$ only in the terms $i = N-1$ and $i = N$ in the sum (2.2), and only on the set $(1-\eta,1)$. Thus, we obtain

$$\begin{aligned}&\frac{1}{2\tau}\int_{1-\eta}^1 \left(\frac{|\dot u_{\tau,N-1}(t)|^2}{\theta(R_{N-1}(u_\tau(t)))}+\frac{|\dot u_{\tau,N}(t)|^2}{\theta(0)}\right)\mathrm{d}t\\ &\le \frac{1}{2\tau}\int_{1-\eta}^1 \left(\frac{|\dot u_{\tau,N-1}(t)|^2}{\theta(R_{N-1}(u_\tau(t)+\varepsilon\varphi_\eta(t)\mathrm{e}_N))}+\frac{|\dot u_{\tau,N}(t)+\varepsilon\dot\varphi_\eta(t)|^2}{\theta(0)}\right)\mathrm{d}t\\ &\quad + V(u_{\tau,N}(1)+\varepsilon)-V(u_{\tau,N}(1)).\end{aligned}$$

We have $\theta(R_{N-1}(u_\tau(t))) \le \theta(R_{N-1}(u_\tau(t)+\varepsilon\varphi_\eta(t)\mathrm{e}_N))$, since $\varphi_\eta \ge 0$ and $\varepsilon > 0$. Therefore,

$$\frac{1}{2\tau}\int_{1-\eta}^1 \frac{|\dot u_{\tau,N}(t)|^2}{\theta(0)}\,\mathrm{d}t \le \frac{1}{2\tau}\int_{1-\eta}^1 \frac{|\dot u_{\tau,N}(t)+\varepsilon\dot\varphi_\eta(t)|^2}{\theta(0)}\,\mathrm{d}t + V(u_{\tau,N}(1)+\varepsilon)-V(u_{\tau,N}(1)).$$

Expanding the square, cancelling common terms, and dividing by $\varepsilon > 0$, we obtain

$$-\frac{1}{\tau}\int_{1-\eta}^1 \alpha_{\tau,N}(t)\dot\varphi_\eta(t)\,\mathrm{d}t \le \frac{V(u_{\tau,N}(1)+\varepsilon)-V(u_{\tau,N}(1))}{\varepsilon} + \frac{\varepsilon}{2\tau}\int_{1-\eta}^1 \frac{|\dot\varphi_\eta(t)|^2}{\theta(0)}\,\mathrm{d}t.$$

Passing to the limit as $\varepsilon \to 0^+$ and multiplying by $\tau > 0$, we get

$$-\frac{1}{\eta}\int_{1-\eta}^1 \alpha_{\tau,N}(t)\,\mathrm{d}t = -\int_{1-\eta}^1 \alpha_{\tau,N}(t)\dot\varphi_\eta(t)\,\mathrm{d}t \le \tau V'(u_{\tau,N}(1)).$$

Since $\alpha_{\tau,N} \in C^0([0,1])$, we deduce that

$$\alpha_{\tau,N}(1) = \lim_{\eta\to 0^+}\frac{1}{\eta}\int_{1-\eta}^1 \alpha_{\tau,N}(t)\,\mathrm{d}t \ge -\tau V'(u_{\tau,N}(1)) > 0. \tag{3.11}$$

In view of Proposition 2.8, see also Remark 2.9, the function $\alpha_{\tau,N}$ satisfies

$$D\alpha_{\tau,N} = -\mu_{\tau,N} \quad \text{in } \mathcal{M}((0,1)),$$

where $\mu_{\tau,N} \in \mathcal{M}^+((0,1))$ is a non-negative Radon measure. Consequently, $\alpha_{\tau,N}$ is non-increasing and the inequality (3.11) implies

$$\alpha_{\tau,N}(s) > 0 \quad \text{for all } s \in (0,1]. \tag{3.12}$$

Step 1.2: Proof of (3.8) *for* $i = N-1$. In view of (3.12), we have

$$\dot u_{\tau,N}(s) > 0 \quad \text{for all } s \in (0,1].$$

Therefore, (2.14) in Proposition 2.10 implies

$$R_{N-1}(u_\tau(s)) < R \quad \text{for all } s \in (0,1).$$

It remains to show that $R_{N-1}(u_\tau(1)) < R$. By contradiction, assume that

$$R_{N-1}(u_\tau(1)) = R.$$

Thus, (2.15) implies that $\dot u_{\tau,N-1}(1) = 0$. Moreover

$$u_{\tau,N}(s) - u_{\tau,N-1}(s) > \frac{1}{NR} = u_{\tau,N}(1) - u_{\tau,N-1}(1) \quad \text{for all } s \in (0,1).$$

In particular,

$$\frac{u_{\tau,N-1}(1)-u_{\tau,N-1}(s)}{1-s} > \frac{u_{\tau,N}(1)-u_{\tau,N}(s)}{1-s} \quad \text{for all } s\in(0,1).$$

Since $u_\tau \in C^1([0,1];\mathcal{K})$, passing to the limit as $s\to 1^-$ yields

$$0=\dot u_{\tau,N-1}(1)\geq \dot u_{\tau,N}(1)>0,$$

which leads to a contradiction. Consequently,

$$R_{N-1}(u_\tau(s))<R \quad \text{for all } s\in(0,1]. \tag{3.13}$$

Step 1.3: Proof of the Neumann-type condition (3.9) for $i=N$.
In view of (3.13), there exists $\varepsilon_\eta>0$ such that for all $\varepsilon\in(0,\varepsilon_\eta)$ we have

$$R_j(u_\tau(s)-\varepsilon\varphi_\eta(s)\mathrm{e}_N)\leq R \quad \text{for all } s\in[0,1] \text{ and } j\in\{0,\dots,N\}.$$

Therefore, arguing as before, for all $\varepsilon\in(0,\varepsilon_\eta)$ the minimality of u_τ implies

$$\begin{aligned}&\frac{1}{\tau}\int_{1-\eta}^1 \alpha_{\tau,N}(t)\dot\varphi_\eta(t)\,\mathrm{d}t+\frac{1}{2\tau\varepsilon}\int_{1-\eta}^1\left(\frac{|\dot u_{\tau,N-1}(t)|^2}{\theta(R_{N-1}(u_\tau(t)))}-\frac{|\dot u_{\tau,N-1}(t)|^2}{\theta(R_{N-1}(u_\tau(t)-\varepsilon\varphi_\eta(t)\mathrm{e}_N))}\right)\mathrm{d}t\\&\leq\frac{V(u_{\tau,N}(1)-\varepsilon)-V(u_{\tau,N}(1))}{\varepsilon}+\frac{\varepsilon}{2\tau}\int_{1-\eta}^1\frac{|\dot\varphi_\eta(t)|^2}{\theta(0)}\,\mathrm{d}t.\end{aligned}$$

Thanks to the dominated convergence theorem, we can pass to the limit as $\varepsilon\to 0^+$ in the above inequality, and after multiplying by $\tau>0$, we obtain

$$\frac{1}{\eta}\int_{1-\eta}^1\alpha_{\tau,N}(t)\,\mathrm{d}t-\int_{1-\eta}^1\alpha_{\tau,N-1}^2(t)\beta_{\tau,N-1}(t)\varphi_\eta(t)\,\mathrm{d}t\leq-\tau V'(u_{\tau,N}(1)).$$

By (3.13), the function $\alpha_{\tau,N-1}^2\beta_{\tau,N-1}$ is locally bounded near $s=1$. Hence, since $\alpha_{\tau,N}\in C^0([0,1])$, taking the limit as $\eta\to 0^+$ yields

$$\alpha_{\tau,N}(1)\leq-\tau V'(u_{\tau,N}(1)).$$

Combining this inequality with (3.11), we conclude that

$$\alpha_{\tau,N}(1)=-\tau V'(u_{\tau,N}(1)).$$

Step 2: Case $i\in\{1,\dots,N-1\}$.
As before, we divide this step into three substeps.

Step 2.1: Proof of (3.10) for i.
By backward induction (starting from Step 1.2), we have

$$R_i(u_\tau(s))<R \quad \text{for all } s\in(0,1]. \tag{3.14}$$

In particular, $A_{\tau,i}=(0,1)$ and $\alpha_{\tau,i}=\dot u_{\tau,i}/\theta(R_i(u_\tau))\in C^0((0,1))$ by Proposition 2.10. Moreover, in view of Proposition 2.8, $\alpha_{\tau,i}\in \mathrm{BV}_{\mathrm{loc}}((0,1))$ and there exists a non-negative Radon measure $\mu_{\tau,i}\in\mathcal{M}^+((0,1))$ such that

$$D\alpha_{\tau,i}=\alpha_{\tau,i}^2\beta_{\tau,i}\mathcal{L}^1-\mu_{\tau,i} \quad \text{in } \mathcal{M}((0,1)).$$

Let $c\in(0,1)$ be fixed. As observed in Remark 2.7, since $\alpha_{\tau,i}\in\mathrm{BV}_{\mathrm{loc}}((0,1))$, for all $s\in(c,1)$ we can write

$$\alpha_{\tau,i}(s)=\alpha_{\tau,i}(c)+\int_c^s\alpha_{\tau,i}(t)^2\beta_{\tau,i}(t)\,\mathrm{d}t-\mu_{\tau,i}([c,s]).$$

Since $\alpha_{\tau,i}^2\beta_{\tau,i}$ is locally bounded near $s=1$ by (3.14), the following limit exists

$$\alpha_{\tau,i}(1):=\lim_{s\to1^-}\alpha_{\tau,i}(s)=\alpha_{\tau,i}(c)+\int_c^1\alpha_{\tau,i}(t)^2\beta_{\tau,i}(t)\,\mathrm{d}t-\mu_{\tau,i}([c,1))\in[-\infty,+\infty).$$

Moreover, such a limit is finite since $\alpha_{\tau,i}$ is locally bounded near $s=1$. Thus, we have

$$\alpha_{\tau,i}(1) = \lim_{s\to 1^-} \alpha_{\tau,i}(s) \in \mathbb{R}. \tag{3.15}$$

In particular, $\alpha_{\tau,i} \in C^0((0,1])$.

By (3.14) and the fact that φ_η is non-negative, we can find $\varepsilon_\eta > 0$ such that for all $\varepsilon \in (0,\varepsilon_\eta)$

$$R_j(u_\tau(s) + \varepsilon\varphi_\eta(s)\mathrm{e}_i) \le R \quad \text{for all } s\in[0,1] \text{ and } j \in \{0,\dots,N\}.$$

Arguing as in Step 1.1, for all $\varepsilon \in (0,\varepsilon_\eta)$ the minimality of u_τ implies

$$\begin{aligned}
&\frac{1}{2\tau}\int_0^1 \Phi_\theta(u_\tau(t),\dot u_\tau(t))\,\mathrm{d}t \\
&\le \frac{1}{2\tau}\int_0^1 \Phi_\theta(u_\tau(t)+\varepsilon\varphi_\eta(t)\mathrm{e}_i, \dot u_\tau(t)+\varepsilon\dot\varphi_\eta(t)\mathrm{e}_i)\,\mathrm{d}t + \frac{V(u_{\tau,i}(1)+\varepsilon)-V(u_{\tau,i}(1))}{N+1}.
\end{aligned}$$

We observe that $\Phi_\theta(u_\tau(t)+\varepsilon\varphi_\eta(t)\mathrm{e}_i, \dot u_\tau(t)+\varepsilon\dot\varphi_\eta(t)\mathrm{e}_i)$ perturbs $\Phi_\theta(u_\tau(t),\dot u_\tau(t))$ only in the terms $i-1$ and i in the sum (2.2), and only on the set $(1-\eta,1)$. Thus, we obtain

$$\begin{aligned}
&\frac{1}{2\tau}\int_{1-\eta}^1 \left(\frac{|\dot u_{\tau,i-1}(t)|^2}{\theta(R_{i-1}(u_\tau(t)))} + \frac{|\dot u_{\tau,i}(t)|^2}{\theta(R_i(u_\tau(t)))}\right)\mathrm{d}t \\
&\le \frac{1}{2\tau}\int_{1-\eta}^1 \left(\frac{|\dot u_{\tau,i-1}(t)|^2}{\theta(R_{i-1}(u_\tau(t)+\varepsilon\varphi_\eta(t)\mathrm{e}_i))} + \frac{|\dot u_{\tau,i}(t)+\varepsilon\dot\varphi_\eta(t)|^2}{\theta(R_i(u_\tau(t)+\varepsilon\varphi_\eta(t)\mathrm{e}_i))}\right)\mathrm{d}t \\
&\quad + V(u_{\tau,i}(1)+\varepsilon) - V(u_{\tau,i}(1)).
\end{aligned}$$

Since $\varphi_\eta \ge 0$ and $\varepsilon > 0$, we have $\theta(R_{i-1}(u_\tau(t))) \le \theta(R_{i-1}(u_\tau(t)+\varepsilon\varphi_\eta(t)\mathrm{e}_i))$. Therefore,

$$\frac{1}{2\tau}\int_{1-\eta}^1 \frac{|\dot u_{\tau,i}(t)|^2}{\theta(R_i(u_\tau(t)))}\,\mathrm{d}t \le \frac{1}{2\tau}\int_{1-\eta}^1 \frac{|\dot u_{\tau,i}(t)+\varepsilon\dot\varphi_\eta(t)|^2}{\theta(R_i(u_\tau(t)+\varepsilon\varphi_\eta(t)\mathrm{e}_i))}\,\mathrm{d}t + V(u_{\tau,i}(1)+\varepsilon) - V(u_{\tau,i}(1)).$$

By expanding the square $|\dot u_{\tau,i}(t)+\varepsilon\dot\varphi_\eta(t)|^2$ and dividing by $\varepsilon>0$, we obtain

$$\begin{aligned}
&-\frac{1}{\tau}\int_{1-\eta}^1 \frac{\dot u_{\tau,i}(t)}{\theta(R_i(u_\tau(t)+\varepsilon\varphi_\eta(t)\mathrm{e}_i))}\dot\varphi_\eta(t)\,\mathrm{d}t + \frac{1}{2\tau\varepsilon}\int_{1-\eta}^1 \left(\frac{|\dot u_{\tau,i}(t)|^2}{\theta(R_i(u_\tau(t)))} - \frac{|\dot u_{\tau,i}(t)|^2}{\theta(R_i(u_\tau(t)+\varepsilon\varphi_\eta(t)\mathrm{e}_i))}\right)\mathrm{d}t \\
&\le \frac{V(u_{\tau,i}(1)+\varepsilon)-V(u_{\tau,i}(1))}{\varepsilon} + \frac{\varepsilon}{2\tau}\int_{1-\eta}^1 \frac{|\dot\varphi_\eta(t)|^2}{\theta(R_i(u_\tau(t)+\varepsilon\varphi_\eta(t)\mathrm{e}_i))}\,\mathrm{d}t.
\end{aligned}$$

Passing to the limit as $\varepsilon\to 0^+$ and multiplying by $\tau>0$, the dominated convergence theorem yields

$$-\frac{1}{\eta}\int_{1-\eta}^1 \alpha_{\tau,i}(t)\,\mathrm{d}t - \int_{1-\eta}^1 \alpha_{\tau,i}(t)^2\beta_{\tau,i}(t)\varphi_\eta(t)\,\mathrm{d}t \le \tau V'(u_{\tau,i}(1)).$$

Since $\alpha_{\tau,i}\in C^0((0,1])$ by (3.15), taking the limit as $\eta\to 0^+$ yields

$$\alpha_{\tau,i}(1) \ge -\tau V'(u_{\tau,i}(1)) > 0. \tag{3.16}$$

By Proposition 2.8, the function $\alpha_{\tau,i} \in \mathrm{BV}_{\mathrm{loc}}((0,1))$ satisfies

$$D\alpha_{\tau,i} = \alpha_{\tau,i}^2\beta_{\tau,i}\mathcal{L}^1 - \mu_{\tau,i} \quad \text{in } \mathcal{M}((0,1)),$$

where $\mu_{\tau,i}$ is a non-negative Radon measure. Hence, we can apply Lemma 2.12 to the pair $(\alpha_{\tau,i},\beta_{\tau,i})$. In particular, in view of (3.16), the sign monotonicity condition implies

$$\alpha_{\tau,i}(s) > 0 \quad \text{for all } s\in(0,1]. \tag{3.17}$$

Step 2.2: Proof of (3.8) *for* $i-1$.
Thanks to (3.17), we know that

$$\dot u_{\tau,i}(s) > 0 \quad \text{for all } s\in(0,1],$$

which yields

$$R_{i-1}(u_\tau(s)) < R \quad \text{for all } s \in (0,1), \tag{3.18}$$

in view of (2.14) in Proposition 2.10. To show that (3.18) holds also at $s = 1$, we can argue exactly as in Step 1.2. By contradiction, assuming that $R_{i-1}(u_\tau(1)) = R$, identity (2.15) implies $\dot{u}_{\tau,i-1}(1) = 0$. Then, adapting the argument of Step 1.2, we have

$$\frac{u_{\tau,i-1}(1) - u_{\tau,i-1}(s)}{1-s} > \frac{u_{\tau,i}(1) - u_{\tau,i}(s)}{1-s} \quad \text{for all } s \in (0,1).$$

Since $u_\tau \in C^1([0,1];\mathcal{K})$, passing to the limit as $s \to 1^-$ gives

$$0 = \dot{u}_{\tau,i-1}(1) \geq \dot{u}_{\tau,i}(1) > 0,$$

which leads to a contradiction. Consequently,

$$R_{i-1}(u_\tau(s)) < R \quad \text{for all } s \in (0,1]. \tag{3.19}$$

Step 2.3: Proof of the Neumann condition (3.9) for i.
In view of (3.19), there exists $\varepsilon_\eta > 0$ such that for all $\varepsilon \in (0, \varepsilon_\eta)$ we have

$$R_j(u_\tau(s) - \varepsilon\varphi_\eta(s)\mathrm{e}_i) \leq R \quad \text{for all } s \in [0,1] \text{ and } j \in \{0,\dots,N\}.$$

Therefore, arguing as in Step 1.3, for all $\varepsilon \in (0, \varepsilon_\eta)$ the minimality of u_τ implies

$$\begin{aligned}
&\frac{1}{\tau}\int_{1-\eta}^1 \frac{\dot{u}_{\tau,i}(t)}{\theta(R_i(u_\tau(t) - \varepsilon\varphi_\eta(t)\mathrm{e}_i))}\dot{\varphi}_\eta(t)\,\mathrm{d}t + \frac{1}{2\tau\varepsilon}\int_{1-\eta}^1 \left(\frac{|\dot{u}_{\tau,i}(t)|^2}{\theta(R_i(u_\tau(t)))} - \frac{|\dot{u}_{\tau,i}(t)|^2}{\theta(R_i(u_\tau(t) - \varepsilon\varphi_\eta(t)\mathrm{e}_i))}\right)\mathrm{d}t \\
&\quad + \frac{1}{2\tau\varepsilon}\int_{1-\eta}^1 \left(\frac{|\dot{u}_{\tau,i-1}(t)|^2}{\theta(R_{i-1}(u_\tau(t)))} - \frac{|\dot{u}_{\tau,i-1}(t)|^2}{\theta(R_{i-1}(u_\tau(t) - \varepsilon\varphi_\eta(t)\mathrm{e}_i))}\right)\mathrm{d}t \\
&\leq \frac{V(u_{\tau,i}(1) - \varepsilon) - V(u_{\tau,i}(1))}{\varepsilon} + \frac{\varepsilon}{2\tau}\int_{1-\eta}^1 \frac{|\dot{\varphi}_\eta(t)|^2}{\theta(R_i(u_\tau(t) - \varepsilon\varphi_\eta(t)\mathrm{e}_i))}\,\mathrm{d}t.
\end{aligned}$$

Passing to the limit as $\varepsilon \to 0^+$ by the dominated convergence theorem, and multiplying by $\tau > 0$ we obtain

$$\frac{1}{\eta}\int_{1-\eta}^1 \alpha_{\tau,i}(t)\,\mathrm{d}t + \int_{1-\eta}^1 [\alpha_{\tau,i}^2(t)\beta_{\tau,i}(t) - \alpha_{\tau,i-1}^2(t)\beta_{\tau,i-1}(t)]\varphi_\eta(t)\,\mathrm{d}t \leq -\tau V'(u_{\tau,i}(1)).$$

Since $\alpha_{\tau,i}^2\beta_{\tau,i} - \alpha_{\tau,i-1}^2\beta_{\tau,i-1}$ is locally bounded near $s = 1$ by (3.14) and (3.19), and $\alpha_{\tau,i} \in C^0((0,1])$ by (3.15), taking the limit as $\eta \to 0^+$ yields

$$\alpha_{\tau,i}(1) \leq -\tau V'(u_{\tau,i}(1)).$$

Combining this inequality with (3.16), we conclude that

$$\alpha_{\tau,i}(1) = -\tau V'(u_{\tau,i}(1)).$$

Step 3: Case $i = 0$.
We only need to prove (3.9) and (3.10) for $i = 0$. Indeed, in view of Step 2.2 (or Step 1.2 when $N = 1$), we have

$$R_0(u_\tau(s)) < R \quad \text{for all } s \in (0,1]. \tag{3.20}$$

In particular, $A_{\tau,0} = (0,1)$ and $\alpha_{\tau,0} = \dot{u}_{\tau,0}/\theta(R_0(u_\tau)) \in C^0((0,1))$. Moreover, (2.11) implies

$$\dot{\alpha}_{\tau,0} = \alpha_{\tau,0}^2\beta_{\tau,0} \quad \text{in } (0,1). \tag{3.21}$$

Let $c \in (0,1)$ be fixed. For all $s \in (c,1)$, we can write

$$\alpha_{\tau,0}(s) = \alpha_{\tau,0}(c) + \int_c^s \alpha_{\tau,0}(t)^2\beta_{\tau,0}(t)\,\mathrm{d}t.$$

Therefore, since $\alpha_{\tau,0}^2\beta_{\tau,0}$ is locally bounded near $s=1$ by (3.20), the following limit exists

$$\alpha_{\tau,0}(1) \coloneqq \lim_{s\to 1^-} \alpha_{\tau,0}(s) = \alpha_{\tau,0}(c) + \int_c^1 \alpha_{\tau,0}(t)^2\beta_{\tau,0}(t)\,\mathrm{d}t \in \mathbb{R}, \tag{3.22}$$

which yields $\alpha_{\tau,0} \in C^0((0,1])$.

By (3.20), there exists $\varepsilon_\eta > 0$ such that for all $\varepsilon \in (-\varepsilon_\eta, \varepsilon_\eta)$ we have

$$R_j(u_\tau(s) + \varepsilon\varphi_\eta(s)\mathrm{e}_0) \le R \quad \text{for all } s \in [0,1] \text{ and } j \in \{0,\dots,N\}.$$

Thus, as in the previous steps, for all $\varepsilon \in (-\varepsilon_\eta, \varepsilon_\eta)$ the minimality of u_τ yields

$$\begin{aligned}&\frac{1}{2\tau}\int_0^1 \Phi_\theta(u_\tau(t), \dot u_\tau(t))\,\mathrm{d}t\\ &\le \frac{1}{2\tau}\int_0^1 \Phi_\theta(u_\tau(t) + \varepsilon\varphi_\eta(t)\mathrm{e}_0, \dot u_\tau(t) + \varepsilon\dot\varphi_\eta(t)\mathrm{e}_0)\,\mathrm{d}t + \frac{V(u_{\tau,0}(1)+\varepsilon) - V(u_{\tau,0}(1))}{N+1}.\end{aligned}$$

Note that $\Phi_\theta(u_\tau(t) + \varepsilon\varphi_\eta(t)\mathrm{e}_0, \dot u_\tau(t) + \varepsilon\dot\varphi_\eta(t)\mathrm{e}_0)$ perturbs $\Phi_\theta(u_\tau(t), \dot u_\tau(t))$ only on the term $i=0$ in the sum (2.2), and only on the set $(1-\eta, 1)$. Hence,

$$\frac{1}{2\tau}\int_{1-\eta}^1 \frac{|\dot u_{\tau,0}(t)|^2}{\theta(R_0(u_\tau(t)))}\,\mathrm{d}t \le \frac{1}{2\tau}\int_{1-\eta}^1 \frac{|\dot u_{\tau,0}(t) + \varepsilon\dot\varphi_\eta(t)|^2}{\theta(R_0(u_\tau(t) + \varepsilon\varphi_\eta(t)\mathrm{e}_0))}\,\mathrm{d}t + V(u_{\tau,0}(1)+\varepsilon) - V(u_{\tau,0}(1)).$$

By dividing by ε and sending $\varepsilon \to 0$ for a positive and a negative ε, respectively, thanks to the dominated convergence theorem, we conclude that

$$-\frac{1}{\eta}\int_{1-\eta}^1 \alpha_{\tau,0}(t)\,\mathrm{d}t - \int_{1-\eta}^1 \alpha_{\tau,0}(t)^2\beta_{\tau,0}(t)\varphi_\eta(t)\,\mathrm{d}t = \tau V'(u_{\tau,0}(1)).$$

Since $\alpha_{\tau,0} \in C^0((0,1])$ by (3.22), taking the limit as $\eta \to 0^+$, we derive

$$\alpha_{\tau,0}(1) = -\tau V'(u_{\tau,0}(1)) > 0.$$

Finally, since $\alpha_{\tau,0}$ satisfies (3.21), by Corollary 2.13 we deduce that

$$\alpha_{\tau,0}(s) > 0 \quad \text{for all } s \in (0,1],$$

which concludes the proof. □

Remark 3.4. Since the starting point $x \in \mathcal{K}$ is arbitrary, we may have

$$R_i(u_\tau(0)) = R_i(x) = R \quad \text{for some } i \in \{0,\dots,N-1\}.$$

Hence, the statement of Proposition 3.3 cannot be extended to $s=0$ in a straightforward manner. Nevertheless, it is worth pointing out that, even if the starting point lies on the boundary, the target point $x_\tau = u_\tau(1)$ always belongs to the interior of the cone $\mathcal{K}$, in view of (3.8).

4. Convergence of the JKO scheme to the continuous Follow-the-Leader system

In this section, we introduce the JKO scheme. We fix an *initial state* $x^0 \in \mathcal{K}$ and a *time step* $\tau > 0$. Setting $x_\tau^0 \coloneqq x^0$, for every $m \in \mathbb{N}$ we recursively define

$$u_\tau^m \in \operatorname{argmin}\left\{\frac{1}{2\tau}\int_0^1 \Phi_\theta(v(t), \dot v(t))\,\mathrm{d}t + \frac{1}{N+1}\sum_{i=0}^N V(v_i(1)) : v \in W^{1,2}((0,1);\mathcal{K}),\ v(0) = x_\tau^{m-1}\right\},$$

$$x_\tau^m \coloneqq u_\tau^m(1).$$

We note that, in view of Lemma 3.2, for all $m \in \mathbb{N}$ the point x_τ^m is a solution to the JKO step with starting point x_τ^{m-1}, i.e.,

$$x_\tau^m \in \operatorname{argmin}\left\{\frac{1}{2\tau}d_\theta(x_\tau^{m-1}, y)^2 + \frac{1}{N+1}\sum_{i=0}^N V(y_i) : y \in \mathcal{K}\right\}. \tag{4.1}$$

The piecewise affine interpolation $x_\tau\colon [0,+\infty) \to \mathcal{K}$ defined in (1.10) is given by

$$x_\tau(t) \coloneqq \sum_{m=0}^{+\infty} \frac{(t-m\tau)(x_\tau^{m+1}-x_\tau^m)+\tau x_\tau^m}{\tau} \mathbf{1}_{[m\tau,(m+1)\tau)}(t) \quad \text{for all } t \in [0,+\infty), \tag{4.2}$$

and the piecewise constant interpolation $\overline{x}_\tau\colon [0,+\infty) \to \mathcal{K}$ is defined by

$$\overline{x}_\tau(t) \coloneqq \sum_{m=0}^{+\infty} x_\tau^{m+1} \mathbf{1}_{[m\tau,(m+1)\tau)}(t) \quad \text{for all } t \in [0,+\infty).$$

We begin by establishing the uniform estimates that will allow us to pass to the continuous-time limit in the JKO scheme and prove Theorem 1.1.

For all $i \in \{0,\dots,N\}$, $\tau > 0$, and $m \in \mathbb{N}$, let $\alpha_{\tau,i}^m$ and $\beta_{\tau,i}^m$ be the functions defined in (2.8) and (2.9), respectively, associated with the geodesic u_τ^m. As a consequence of Proposition 3.3, for all $i \in \{0,\dots,N\}$ and $m \in \mathbb{N}$ we have $\alpha_{\tau,i}^m \in C^0((0,1])$. Thus, applying Proposition 2.8 as well, we deduce that $\alpha_{\tau,i}^m \in C^1((0,1])$ and that the following Euler–Lagrange system holds:

$$\begin{cases} \dot\alpha_{\tau,0}^m(s) = (\alpha_{\tau,0}^m(s))^2 \beta_{\tau,0}^m(s) & \text{for all } s \in (0,1], \\ \dot\alpha_{\tau,i}^m(s) = (\alpha_{\tau,i}^m(s))^2 \beta_{\tau,i}^m(s) - (\alpha_{\tau,i-1}^m(s))^2 \beta_{\tau,i-1}^m(s) & \text{for all } s \in (0,1] \text{ and } i \in \{1,\dots,N\}, \\ \alpha_{\tau,i}^m(s) > 0 & \text{for all } s \in (0,1] \text{ and } i \in \{0,\dots,N\}, \\ \alpha_{\tau,i}^m(1) = -\tau V'(u_{\tau,i}^m(1)) & \text{for all } i \in \{0,\dots,N\}. \end{cases} \tag{4.3}$$

Moreover, for all $i \in \{0,\dots,N\}$ and $m \in \mathbb{N}$ we have

$$\|\beta_{\tau,i}^m\|_{L^\infty((0,1))} \le \frac{\|\theta'\|_{L^\infty((0,R))} N R^2}{2}. \tag{4.4}$$

To pass to the limit as $\tau \to 0^+$, we must show that the functions $\alpha_{\tau,i}^m$ for $i \in \{0,\dots,N\}$ admit a uniform bound with respect to τ. To this end, for all *time horizons* $T > 0$, we define

$$M_T \coloneqq \left\lceil \frac{T}{\tau} \right\rceil.$$

We begin by establishing a uniform bound for x_τ^m for all $0 < \tau \le 1$ and $m \in \{0,\dots,M_T\}$.

Lemma 4.1. *Let $T > 0$. There exists a constant $C_T > 0$ such that*

$$|x_\tau^m| \le C_T \quad \textit{for all } 0 < \tau \le 1 \textit{ and } m \in \{0,\dots,M_T\}. \tag{4.5}$$

Proof. The case $m = 0$ is immediate. Fix $0 < \tau \le 1$ and $m \in \{1,\dots,M_T\}$. By (2.5) and (4.1), it follows that

$$|x_\tau^{m-1} - x_\tau^m|^2 \le (N+1)\theta(0) d_\theta(x_\tau^{m-1}, x_\tau^m)^2 \le 2\theta(0)\tau \sum_{i=0}^N [V(x_{\tau,i}^{m-1}) - V(x_{\tau,i}^m)].$$

Hence,

$$|x^0 - x_\tau^m|^2 \le \left(\sum_{k=1}^m |x_\tau^{k-1} - x_\tau^k| \right)^2 \le m \sum_{k=1}^m |x_\tau^{k-1} - x_\tau^k|^2 \le 2\theta(0)(T+1) \sum_{i=0}^N [V(x_i^0) - V(x_{\tau,i}^m)],$$

since

$$m\tau \le M_T \tau \le T + 1.$$

In view of (3.3), it follows that

$$|x^0 - x_\tau^m|^2 \le 2\theta(0)(T+1) \left(\sum_{i=0}^N V(x_i^0) + C \sum_{i=0}^N |x_{\tau,i}^m|^s + (N+1)D \right).$$

By Jensen's inequality

$$\left(\frac{1}{N+1}\sum_{i=0}^{N}|x_{\tau,i}^m|^s\right)^{\frac{2}{s}} \le \frac{1}{N+1}\sum_{i=0}^{N}|x_{\tau,i}^m|^2 = \frac{|x_\tau^m|^2}{N+1}.$$

Thus

$$\sum_{i=0}^{N}|x_{\tau,i}^m|^s \le (N+1)^{1-\frac{s}{2}}|x_\tau^m|^s.$$

Moreover, by Young's inequality

$$|x^0 - x_\tau^m|^2 = |x^0|^2 + |x_\tau^m|^2 - 2x^0\cdot x_\tau^m \ge \frac{1}{2}|x_\tau^m|^2 - |x^0|^2.$$

Therefore,

$$|x_\tau^m|^2 \le 2|x^0|^2 + 4\theta(0)(T+1)\left(\sum_{i=0}^{N}V(x_i^0) + C(N+1)^{1-\frac{s}{2}}|x_\tau^m|^s + (N+1)D\right).$$

Since $s\in(0,2)$, the above inequality implies the existence of a constant $C_T>0$, independent of $0<\tau\le 1$ and $m\in\{0,\dots,M_T\}$, such that the uniform bound (4.5) holds. □

For all $T>0$, let $C_T>0$ be the constant provided by Lemma 4.1. To obtain a uniform estimate with respect to τ for the functions $\alpha_{\tau,i}^m$ for all $i\in\{0,\dots,N\}$ and $m\in\{1,\dots,M_T\}$, we require τ to satisfy the bound

$$0<\tau\le\min\left\{1,\frac{1}{2\|\theta'\|_{L^\infty((0,R))}\|V'\|_{L^\infty((-C_T,C_T))}NR^2}\right\}. \tag{4.6}$$

In the following lemma, using the Euler–Lagrange system (4.3), we show that the functions $\alpha_{\tau,i}^m$ are also well-defined at $s=0$ (even if the initial point x^0 lies on the boundary $\partial\mathcal{K}$), and we establish a uniform estimate for $\alpha_{\tau,i}^m$ for all $i\in\{0,\dots,N\}$ and $m\in\{1,\dots,M_T\}$.

Lemma 4.2. *Let $T>0$ and assume (4.6). For all $i\in\{0,\dots,N\}$ and $m\in\{1,\dots,M_T\}$ we have $\alpha_{\tau,i}^m\in C^1([0,1])$ and*

$$0\le\alpha_{\tau,i}^m(s)\le 2\|V'\|_{L^\infty((-C_T,C_T))}\tau \quad \textit{for all } s\in[0,1]. \tag{4.7}$$

Proof. We fix $m\in\{1,\dots,M_T\}$ and we proceed by induction on $i\in\{0,\dots,N\}$.

Step 1: Case $i=0$.
In view of the Euler–Lagrange system (4.3), the function $\alpha_{\tau,0}^m$ satisfies

$$\dot\alpha_{\tau,0}^m = (\alpha_{\tau,0}^m)^2\beta_{\tau,0}^m \ge 0 \quad \text{in } (0,1]. \tag{4.8}$$

Thus, $\alpha_{\tau,0}^m$ is non-decreasing in $(0,1]$, which, together with (3.10) in Proposition 3.3, implies

$$0<\alpha_{\tau,0}^m(s)\le\alpha_{\tau,0}^m(1) = -\tau V'(x_{\tau,0}^m)\le 2\|V'\|_{L^\infty((-C_T,C_T))}\tau \quad \text{for all } s\in(0,1],$$

thanks to (4.5) in Lemma 4.1. In particular, $\alpha_{\tau,0}^m\in L^\infty((0,1))$. Hence, by exploiting (4.8), the function $\alpha_{\tau,0}^m$ can be extended to $s=0$, yielding $\alpha_{\tau,0}^m\in C^1([0,1])$ and (4.7) for $i=0$.

Step 2: Case $i\in\{1,\dots,N\}$.
By induction, we may assume

$$0\le\alpha_{\tau,i-1}^m(s)\le 2\|V'\|_{L^\infty((-C_T,C_T))}\tau \quad \text{for all } s\in[0,1].$$

Then, by exploiting (4.3), (4.4), and (4.6) we have

$$\begin{aligned}\dot\alpha_{\tau,i}^m &\ge -(\alpha_{\tau,i-1}^m)^2\beta_{\tau,i-1}^m\\ &\ge -4\|V'\|_{L^\infty((-C_T,C_T))}^2\|\beta_{\tau,i-1}^m\|_{L^\infty((0,1))}\tau^2\end{aligned}$$

$$\geq -2\|\theta'\|_{L^\infty((0,R))}\|V'\|^2_{L^\infty((-C_T,C_T))}NR^2\tau^2 \geq -\|V'\|_{L^\infty((-C_T,C_T))}\tau \quad \text{in } (0,1].$$

Integrating the above inequality over $[s,1]$ for all $s\in(0,1)$, we get

$$\alpha^m_{\tau,i}(1)-\alpha^m_{\tau,i}(s)\geq -\|V'\|_{L^\infty((-C_T,C_T))}\tau \quad \text{for all } s\in(0,1],$$

which yields

$$0<\alpha^m_{\tau,i}(s)\leq -\tau V'(x^m_{\tau,i})+\|V'\|_{L^\infty((-C_T,C_T))}\tau\leq 2\|V'\|_{L^\infty((-C_T,C_T))}\tau \quad \text{for all } s\in(0,1],$$

thanks to (3.10) and (4.5). In particular, $\alpha^m_{\tau,i}\in L^\infty((0,1))$. Hence, by exploiting the Euler–Lagrange equation

$$\dot\alpha^m_{\tau,i}=(\alpha^m_{\tau,i})^2\beta^m_{\tau,i}-(\alpha^m_{\tau,i-1})^2\beta^m_{\tau,i-1} \quad \text{in } (0,1],$$

the function $\alpha^m_{\tau,i}$ can be extended to $s=0$, yielding $\alpha^m_{\tau,i}\in C^1([0,1])$ and (4.7) for i. □

To pass to the limit as $\tau\to 0^+$ in the interpolations x_τ of the JKO scheme defined in (4.2), we would ideally want the sequence $(x^m_\tau)_{m\in\mathbb{N}}$ to satisfy an implicit Euler scheme of the form

$$\frac{x^m_{\tau,i}-x^{m-1}_{\tau,i}}{\tau}=-\theta(R_i(x^m_\tau))V'(x^m_{\tau,i}) \quad \text{for all } i\in\{0,\dots,N\} \text{ and } m\in\mathbb{N}, \tag{4.9}$$

which is the time-discrete approximation of the Follow-the-Leader system (3.1). Unfortunately, this is not exactly the case. However, we can use the uniform estimates from Lemma 4.2 to show that (4.9) holds up to an error term that vanishes as $\tau\to 0^+$.

Lemma 4.3. *Let $T>0$ and assume (4.6). For all $i\in\{0,\dots,N\}$ and $m\in\{1,\dots,M_T\}$ we have*

$$\frac{x^m_{\tau,i}-x^{m-1}_{\tau,i}}{\tau}=-\theta(R_i(x^m_\tau))V'(x^m_{\tau,i})+\zeta^m_{\tau,i}, \tag{4.10}$$

where the remainder

$$\zeta^m_{\tau,i}:=\frac{u^m_{\tau,i}(1)-u^m_{\tau,i}(0)-\dot u^m_{\tau,i}(1)}{\tau} \tag{4.11}$$

satisfies

$$|\zeta^m_{\tau,i}|\leq 2\theta(0)\|\theta'\|_{L^\infty((0,R))}\|V'\|^2_{L^\infty((-C_T,C_T))}NR^2\tau. \tag{4.12}$$

Proof. We fix $i\in\{0,\dots,N\}$ and $m\in\{1,\dots,M_T\}$. Since $u^m_\tau(1)=x^m_\tau$, we can rephrase the Neumann-type boundary condition $\alpha^m_{\tau,i}(1)=-\tau V'(x^m_{\tau,i})$ as

$$\dot u^m_{\tau,i}(1)=-\tau\theta(R_i(x^m_\tau))V'(x^m_{\tau,i}).$$

Thus, by using also $u^m_\tau(0)=x^{m-1}_\tau$, we can write

$$\frac{x^m_{\tau,i}-x^{m-1}_{\tau,i}}{\tau}=-\theta(R_i(x^m_\tau))V'(x^m_{\tau,i})+\zeta^m_{\tau,i},$$

where

$$\zeta^m_{\tau,i}:=\frac{u^m_{\tau,i}(1)-u^m_{\tau,i}(0)-\dot u^m_{\tau,i}(1)}{\tau}.$$

It remains to prove (4.12). For $i=N$ we have

$$\dot u^m_{\tau,N}=\theta(0)\alpha^m_{\tau,N} \quad \text{in } [0,1],$$

and since $\alpha^m_{\tau,N}\in C^1([0,1])$, we conclude that $\dot u^m_{\tau,N}\in C^1([0,1])$ and, by exploiting (4.3),

$$\ddot u^m_{\tau,N}=\theta(0)\dot\alpha^m_{\tau,N}=-\theta(0)(\alpha^m_{\tau,N-1})^2\beta^m_{\tau,N-1} \quad \text{in } [0,1].$$

From (4.4) and (4.7), it follows that

$$\|\ddot u^m_{\tau,N}\|_{L^\infty((0,1))}\leq 2\theta(0)\|\theta'\|_{L^\infty((0,R))}\|V'\|^2_{L^\infty((-C_T,C_T))}NR^2\tau^2.$$

For $i\in\{0,\dots,N-1\}$, since

$$\dot u^m_{\tau,i}=\theta(R_i(u^m_\tau))\alpha^m_{\tau,i} \quad \text{in } [0,1],$$

and $\alpha^m_{\tau,i} \in C^1([0,1])$, we conclude that $\dot{u}^m_{\tau,i} \in C^1([0,1])$ and, in view of (4.3),

$$\begin{aligned}\ddot{u}^m_{\tau,i} &= N\theta'(R_i(u^m_\tau))R_i(u^m_\tau)^2[\dot{u}^m_{\tau,i} - \dot{u}^m_{\tau,i+1}]\alpha^m_{\tau,i} + \theta(R_i(u^m_\tau))\dot{\alpha}^m_{\tau,i}\\ &= -\theta(R_i(u^m_\tau))(\alpha^m_{\tau,i})^2\beta^m_{\tau,i} + 2\theta(R_{i+1}(u^m_\tau))\alpha^m_{\tau,i}\alpha^m_{\tau,i+1}\beta^m_{\tau,i} - \theta(R_i(u^m_\tau))(\alpha^m_{\tau,i-1})^2\beta^m_{\tau,i-1} \quad \text{in } [0,1],\end{aligned}$$

where the terms involving $i-1$ are omitted when $i=0$. From (4.4) and (4.7), it follows that

$$\begin{aligned}\ddot{u}^m_{\tau,i} &\geq -\theta(R_i(u^m_\tau))(\alpha^m_{\tau,i})^2\beta^m_{\tau,i} - \theta(R_i(u^m_\tau))(\alpha^m_{\tau,i-1})^2\beta^m_{\tau,i-1}\\ &\geq -4\theta(0)\|\theta'\|_{L^\infty((0,R))}\|V'\|^2_{L^\infty((-C_T,C_T))}NR^2\tau^2 \quad \text{in } [0,1],\end{aligned}$$

and

$$\ddot{u}^m_{\tau,i} \leq 2\theta(R_{i+1}(u^m_\tau))\alpha^m_{\tau,i}\alpha^m_{\tau,i+1}\beta^m_{\tau,i} \leq 4\theta(0)\|\theta'\|_{L^\infty((0,R))}\|V'\|^2_{L^\infty((-C_T,C_T))}NR^2\tau^2 \quad \text{in } [0,1].$$

Thus, for all $i \in \{0,\dots,N\}$ we get the estimate

$$\|\ddot{u}^m_{\tau,i}\|_{L^\infty((0,1))} \leq 4\theta(0)\|\theta'\|_{L^\infty((0,R))}\|V'\|^2_{L^\infty((-C_T,C_T))}NR^2\tau^2.$$

Therefore, by Taylor's formula, we infer

$$|u^m_{\tau,i}(1) - u^m_{\tau,i}(0) - \dot{u}^m_{\tau,i}(1)| \leq \frac{\|\ddot{u}^m_{\tau,i}\|_{L^\infty((0,1))}}{2} \leq 2\theta(0)\|\theta'\|_{L^\infty((0,R))}\|V'\|^2_{L^\infty((-C_T,C_T))}NR^2\tau^2,$$

which implies (4.12). □

We are now in a position to prove Theorem 1.1, the main result of this paper: as $\tau \to 0^+$, the interpolations x_τ of the JKO scheme defined in (4.2) converge to a solution $x \in C^1([0,+\infty);\mathcal{K})$ of the Follow-the-Leader system (3.1). Before that, we need the following lemma proving uniqueness of solutions for (3.1).

Lemma 4.4. *Let $N \geq 1$ be an integer and let $R > 0$. Assume that $\theta \in C^1([0,R])$ satisfies (2.1) and that $V \in C^1(\mathbb{R})$ satisfies (3.2)–(3.3). For a given initial condition $x^0 \in \mathcal{K}$, the Follow-the-Leader system (3.1) has at most one solution $x \in C^1([0,+\infty);\mathcal{K})$.*

Proof. In the particular case in which $V' \in W^{2,\infty}(\mathbb{R})$, the statement follows from [13, Proposition 2.2]. However, the set of assumptions (3.2)–(3.3) may allow for lower regularity for V. We then define

$$a(x) := -V'(x) \quad \text{for } x \in \mathbb{R},$$

and

$$H(x) := \int_0^x \frac{1}{a(y)}\mathrm{d}y \quad \text{for } x \in \mathbb{R}.$$

The assumption (3.2) implies $H \in C^1(\mathbb{R})$. Moreover, H is strictly increasing and therefore invertible on its image and there holds, for all $x \in \mathbb{R}$,

$$(H^{-1})'(H(x)) = a(x).$$

Let $x \in C^1([0,+\infty);\mathcal{K})$ be a solution to (3.1). For all $i \in \{0,\dots,N\}$, we set

$$z_i(t) := H(x_i(t)) \quad \text{for } t \in [0,+\infty).$$

Clearly, z_i is differentiable in time and satisfies, for all $t \in [0,\infty)$,

$$\dot{z}_i(t) = H'(x_i(t))a(x_i(t))\theta(R_i(x(t))) = \theta\left(\frac{1}{N(H^{-1}(z_{i+1}(t)) - H^{-1}(z_i(t)))}\right) \tag{4.13}$$

for all $i \in \{0,\dots,N-1\}$ and

$$\dot{z}_N(t) = \theta(0). \tag{4.14}$$

Since H^{-1} is C^1, the system (4.13)–(4.14) for a given initial datum has a unique solution on $[0,T)$ for some maximal time $T > 0$, due to classical Cauchy-Lipschitz theory. Now let $x^1, x^2 \in C^1([0,+\infty);\mathcal{K})$ be two solutions of (3.1) with the same initial condition x^0. Then, $z^1_i = H(x^1_i)$ and $z^2_i = H(x^2_i)$ are both defined on $[0,+\infty)$ and solve the transformed system (4.13)–(4.14) with the same initial

conditions $H(x_i^0)$. By uniqueness, they coincide on $[0,+\infty)$. Since H is invertible, this yields $x^1 = x^2$. □

Proof of Theorem 1.1. Let $T > 0$ and take $\tau > 0$ satisfying (4.6). We note that $x_\tau \in \mathrm{Lip}([0,T];\mathcal{K})$. Moreover, by (4.5), we have

$$\|x_\tau\|_{L^\infty((0,T);\mathcal{K})} \le \sup_{m\in\{0,\dots,M_T\}} |x_\tau^m| \le C_T,$$

while, by (4.7) and the identity

$$\dot u_{\tau,i}^m = \theta(R_i(u_\tau^m))\alpha_{\tau,i}^m \quad \text{for all } i \in \{0,\dots,N\} \text{ and } m \in \{1,\dots,M_T\},$$

we get

$$\begin{aligned}\|\dot x_{\tau,i}\|_{L^\infty((0,T))} &= \sup_{m\in\{1,\dots,M_T\}} \left|\frac{x_{\tau,i}^m - x_{\tau,i}^{m-1}}{\tau}\right| \\ &= \sup_{m\in\{1,\dots,M_T\}} \left|\frac{u_{\tau,i}^m(1) - u_{\tau,i}^m(0)}{\tau}\right| \\ &\le \sup_{m\in\{1,\dots,M_T\}} \frac{\|\dot u_{\tau,i}^m\|_{L^\infty((0,1))}}{\tau} \le 2\theta(0)\|V'\|_{L^\infty((-C_T,C_T))}. \end{aligned} \tag{4.15}$$

Hence, by the Ascoli–Arzelà theorem and a diagonal argument, there is a sequence $(\tau_k)_{k\in\mathbb{N}} \subset (0,1)$ with $\tau_k \to 0^+$ as $k \to +\infty$ and a function $x \in \mathrm{Lip}_{\mathrm{loc}}([0,+\infty);\mathcal{K})$ such that, for all $T > 0$,

$$\begin{aligned} x_{\tau_k} &\to x && \text{strongly in } C^0([0,T];\mathcal{K}) \text{ as } k \to +\infty, \\ \dot x_{\tau_k} &\to \dot x && \text{weakly* in } L^\infty((0,T);\mathbb{R}^{N+1}) \text{ as } k \to +\infty. \end{aligned}$$

In particular, $x(0) = x^0$, since $x_{\tau_k}(0) = x^0$ for all $k \in \mathbb{N}$.

Let us define $\overline{\zeta}_\tau \colon [0,+\infty) \to \mathbb{R}^{N+1}$ as

$$\overline{\zeta}_{\tau,i}(t) \coloneqq \sum_{m=0}^{+\infty} \zeta_{\tau,i}^{m+1} \mathbf{1}_{[m\tau,(m+1)\tau)}(t) \quad \text{for all } i \in \{0,\dots,N\} \text{ and } t \in [0,+\infty),$$

where $\zeta_{\tau,i}^m$ is the remainder term defined in (4.11). Fix $\varphi \in C_c^\infty((0,+\infty))$ and $T_\varphi > 0$ such that $\mathrm{supp}(\varphi) \subset (0,T_\varphi)$. For all $k \in \mathbb{N}$ sufficiently large, we can rephrase (4.10) in Lemma 4.3 as

$$\dot x_{\tau_k,i}(t) = -\theta(R_i(\overline{x}_{\tau_k}(t)))V'(\overline{x}_{\tau_k,i}(t)) + \overline{\zeta}_{\tau_k,i}(t) \quad \text{for all } i \in \{0,\dots,N\} \text{ and a.e. } t \in (0,T_\varphi),$$

which yields

$$\int_0^\infty \left[\dot x_{\tau_k,i}(t) + \theta(R_i(\overline{x}_{\tau_k}(t)))V'(\overline{x}_{\tau_k,i}(t))\right]\varphi(t)\,\mathrm{d}t = \int_0^\infty \overline{\zeta}_{\tau_k,i}(t)\varphi(t)\,\mathrm{d}t \quad \text{for all } i \in \{0,\dots,N\}.$$

By (4.15), for all $k \in \mathbb{N}$ sufficiently large we have

$$|\overline{x}_{\tau_k}(t) - x_{\tau_k}(t)| \le \tau_k\|\dot x_{\tau_k}\|_{L^\infty((0,T_\varphi))} \le 2\sqrt{N+1}\theta(0)\|V'\|_{L^\infty((-C_{T_\varphi},C_{T_\varphi}))}\tau_k \quad \text{for all } t \in [0,T_\varphi].$$

Thus, for all $i \in \{0,\dots,N\}$ and $k \in \mathbb{N}$ sufficiently large

$$\begin{aligned} \theta(R_i(\overline{x}_{\tau_k}(t))) &\to \theta(R_i(x(t))) && \text{for all } t \in [0,T_\varphi] \text{ as } k \to +\infty, \\ V'(\overline{x}_{\tau_k,i}(t)) &\to V'(x_i(t)) && \text{for all } t \in [0,T_\varphi] \text{ as } k \to +\infty, \end{aligned}$$

and

$$\begin{aligned} |\theta(R_i(\overline{x}_{\tau_k}(t)))| &\le \|\theta\|_{L^\infty((0,R))} && \text{for all } t \in [0,T_\varphi], \\ |V'(\overline{x}_{\tau_k,i}(t))| &\le \|V'\|_{L^\infty((-C_{T_\varphi},C_{T_\varphi}))} && \text{for all } t \in [0,T_\varphi]. \end{aligned}$$

Moreover, in view of (4.12), for all $i \in \{0, \dots, N\}$ and $k \in \mathbb{N}$ sufficiently large

$$|\overline{\zeta}_{\tau_k,i}(t)| \leq 2\theta(0)\|\theta'\|_{L^\infty((0,R))}\|V'\|^2_{L^\infty((-C_{T_\varphi},C_{T_\varphi}))}NR^2\tau_k \quad \text{for all } t \in [0, T_\varphi].$$

Hence, the dominated convergence theorem yields

$$\int_0^\infty \left[\dot{x}_i(t) + \theta(R_i(x(t)))V'(x_i(t))\right]\varphi(t)\,\mathrm{d}t = 0 \quad \text{for all } i \in \{0, \dots, N\}.$$

Since $\varphi \in C_c^\infty((0, +\infty))$ is arbitrary, we deduce that

$$\dot{x}_i(t) = -\theta(R_i(x(t)))V'(x_i(t)) \quad \text{for a.e. } t \in (0, +\infty).$$

Finally, observing that $\theta(R_i(x))V'(x_i) \in C^0([0, +\infty))$ for all $i \in \{0, \dots, N\}$, we conclude that $x \in C^1([0, +\infty); \mathcal{K})$ and it is a solution to the Follow-the-Leader system (3.1). From Lemma 4.4 we know that (3.1) has at most one solution in said space. Therefore, the whole family x_τ satisfies, as $\tau \to 0^+$, the same convergence properties as x_{τ_k} as $k \to +\infty$. □

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