A NOTE ON THE COMPARISON PRINCIPLE FOR DEGENERATE SUB-ELLIPTIC EQUATIONS

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Abstract. We show that the comparison principle holds for smooth sub- and super-solutions of a class of degenerate sub-elliptic equations that includes the sub-elliptic ∞-Laplacian, when the horizontal gradient does not vanish at the maximal points of their difference. The equations are defined by an arbitrary family of smooth vector fields; in particular, our results apply to equations defined with respect to a set of Hörmander vector fields and in Carnot groups. The proofs have been formalized and verified in the Lean proof assistant.

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and let $X_1, \dots, X_m$, $m \leq n$, be smooth vector fields on $\mathbb{R}^n$ satisfying Hörmander's rank condition

$$\dim \big(\mathtt{Lie}(X_1, \dots, X_m)(x)\big) = n \tag{1.1}$$

at every point $x \in \mathbb{R}^n$. The main example we have in mind is that of a Carnot group $\mathbb{G} = (\mathbb{R}^n, *)$ of step $r \geq 1$, that is, a connected and simply connected nilpotent Lie group whose Lie algebra $\mathfrak{g}$ of left-invariant vector fields admits a stratification $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_r$ with $[\mathfrak{g}_1, \mathfrak{g}_j] = \mathfrak{g}_{j+1}$ for $j < r$ and $[\mathfrak{g}_1, \mathfrak{g}_r] = \{0\}$, with $\{X_1, \dots, X_m\}$ a basis of the first layer $\mathfrak{g}_1$; these vector fields are left invariant with respect to $\mathbb{G}$, homogeneous of degree one with respect to its dilations, and satisfy (1.1). We refer to [9] for the structure and properties of Carnot groups. The case $m = n$ corresponds to the Euclidean setting $X_i = \partial_{x_i}$; the relevant case is $m < n$. The simplest non-abelian example is the Heisenberg group $\mathbb{H}^1 = \mathbb{R}^3$ with

$$X_1 = \partial_{x_1} - \tfrac{1}{2}\, x_2\, \partial_{x_3}, \qquad X_2 = \partial_{x_2} + \tfrac{1}{2}\, x_1\, \partial_{x_3}, \qquad [X_1, X_2] = \partial_{x_3}.$$

However, no group structure is used in this note: our results hold, and are proved, for an arbitrary Hörmander family of smooth vector fields, which we fix from now on.

We consider the equation

$$\mathcal{L}u := -\operatorname{Tr}\big(a(\mathfrak{X}u)\mathfrak{X}\mathfrak{X}u\big) = -\sum_{i,j=1}^{m} a_{i,j}(\mathfrak{X}u) X_j X_i u = 0 \qquad \text{in } \Omega, \tag{1.2}$$

where $\mathfrak{X}u = (X_1 u, \dots, X_m u)$ is the sub-elliptic (horizontal) gradient, $\mathfrak{X}\mathfrak{X}u = (X_j X_i u)_{i,j}$ is the sub-elliptic second derivative matrix, and $a_{i,j} \in C(\mathbb{R}^m \setminus \{0\})$ are such that the $(m \times m)$ matrix $a(\xi) = (a_{i,j}(\xi))_{i,j}$ is symmetric and positive semi-definite for every $\xi \neq 0$ and satisfies the ellipticity condition

$$\mathcal{E}(\xi) := \big\langle a(\xi)\, \xi, \xi \big\rangle > 0 \qquad \forall\, \xi \in \mathbb{R}^m \setminus \{0\}. \tag{1.3}$$

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Furthermore, we assume that a is homogeneous, i.e. there exists a constant $\beta \in \mathbb{R}$ such that

$$a(t\xi) = t^\beta\, a(\xi) \qquad \forall\ t > 0,\ \xi \in \mathbb{R}^m \setminus \{0\}. \tag{1.4}$$

Since $a(\xi)$ is symmetric, the equation (1.2) can also be written as

$$\mathcal{L}u = -\operatorname{Tr}\big(a(\mathfrak{X}u)\mathfrak{X}\mathfrak{X}^\star u\big) = 0,$$

where $\mathfrak{X}\mathfrak{X}^\star u$ is the symmetrized sub-elliptic second derivative; see Section 2. The matrix a need not be defined at $\xi = 0$; this allows normalized operators such as the normalized p-Laplacian. Condition (1.3) asks for ellipticity in one direction only: the matrix $a(\xi)$ may be degenerate in all the directions orthogonal to ξ, as it is for the ∞-Laplacian, for which $a(\xi) = \xi \otimes \xi$ has rank one.

Remark 1.1. (a) By (1.4), $a(\xi) = |\xi|^\beta a(\xi/|\xi|)$ and $\mathcal{E}(\xi) = |\xi|^{2+\beta}\mathcal{E}(\xi/|\xi|)$ for $\xi \neq 0$, so that a is determined by its restriction to the unit sphere, and $\mathcal{E}(\xi) \geq a_0|\xi|^{2+\beta}$ for some $a_0 > 0$ (Lemma 2.1). The equation is degenerate when $\beta > 0$, since then $a(\xi) \to 0$ as $\xi \to 0$, and singular when $\beta < 0$. If a extends continuously to $\xi = 0$, then $\beta \geq 0$, and $\beta = 0$ only if a is constant. Only $t > 0$ enters (1.4); in particular, (1.4) holds if $a(t\xi) = |t|^\beta a(\xi)$ for all $t \in \mathbb{R} \setminus \{0\}$, but a need not be even (see the examples below).

(b) Only points at which the horizontal gradient does not vanish enter the notions of sub- and super-solution used below (Definition 1.2), and at such points $\mathcal{L}u = |\mathfrak{X}u|^\beta\, \mathcal{L}_0 u$, where $\mathcal{L}_0 u := -\operatorname{Tr}\big(a(\mathfrak{X}u/|\mathfrak{X}u|)\mathfrak{X}\mathfrak{X}u\big)$ is the operator of the 0-homogeneous matrix $a(\xi/|\xi|)$. Hence these notions do not change if a is multiplied by a positive function of ξ; for instance, replacing $a(\xi)$ by $|\xi|^\gamma a(\xi)$ replaces β by $\beta + \gamma$, and one may always normalize $\beta = 0$. Consequently, all the results below also hold for the matrices $\phi(\xi)\, a(\xi)$, where $\phi \in C(\mathbb{R}^m \setminus \{0\})$ is positive and a satisfies (1.3) and (1.4); the value of β plays no role in the statements.

(c) Condition (1.4) excludes matrices that change shape along rays. For instance, for the mean curvature type matrix $a(\xi) = \mathbb{I}_m - \xi \otimes \xi/(1+|\xi|^2)$, whose eigenvalues are $1/(1+|\xi|^2)$ in the direction of ξ and 1 in the orthogonal directions, $a(t\xi)$ is not a multiple of $a(\xi)$ when $m \geq 2$ and $t \neq 1$, so that not even the normalization in (b) makes it homogeneous.

(d) In the Euclidean case, (1.3) is the ellipticity condition of Barles and Busca [6] in the direction of the gradient, and (1.3), (1.4) imply their condition for small gradients, $\mathcal{E}(\xi) \geq c\,|\xi|^2 \operatorname{Tr} a(\xi)$ for all $\xi \neq 0$, since both sides are $|\xi|^{2+\beta}$ times a positive continuous function of $\xi/|\xi|$. On the other hand, a is only assumed to be continuous away from the origin, not locally Lipschitz continuous.

We consider smooth sub- and super-solutions in the following classical sense.

Definition 1.2. Let $U \subset \mathbb{R}^n$ be open. A function $u \in C^2(U)$ is a (classical) sub-solution (resp. super-solution) of (1.2) in U if $\mathcal{L}u(x) \leq 0$ (resp. $\mathcal{L}u(x) \geq 0$) at every point $x \in U$ such that $\mathfrak{X}u(x) \neq 0$; we then write $\mathcal{L}u \leq 0$ (resp. $\mathcal{L}u \geq 0$) in U.

No condition is imposed at the horizontal critical points of u, where a need not be defined. When a extends continuously to $\xi = 0$, as for the sub-Laplacian and the ∞-Laplacian, Definition 1.2 is equivalent to requiring the inequality at every point of U. Indeed, the inequality extends by continuity to the critical points which are limits of non-critical points, and on the interior of the set of critical points u is locally constant, by (1.1), so that $\mathcal{L}u = 0$ there.

The equation (1.2) encompasses a large class of degenerate equations. The sub-Laplacian $\Delta_{\mathfrak{X}} u = \operatorname{Tr}(\mathfrak{X}\mathfrak{X}u) = \sum_{i=1}^m X_i^2 u$ corresponds to $a(\xi) \equiv \mathbb{I}_m$ and $\beta = 0$. Among non-linear equations, one of the most important cases is the sub-elliptic ∞-Laplacian

$$\Delta_{\mathfrak{X},\infty} u = \big\langle \mathfrak{X}\mathfrak{X}^\star u\, \mathfrak{X}u, \mathfrak{X}u \big\rangle,$$

which corresponds to $a(\xi) = \xi \otimes \xi$ and $\beta = 2$; the uniqueness of viscosity solutions of its Dirichlet problem was proved by Bieske [7] in the Heisenberg group and by Wang [16] in Carnot groups, and it also follows from comparison with cones [12, 1]. More generally, $a(\xi) = \nabla f(\xi) \otimes \nabla f(\xi)$ gives the equation

$$\big\langle \mathfrak{X}\mathfrak{X}^\star u \, \nabla f(\mathfrak{X}u), \nabla f(\mathfrak{X}u) \big\rangle = 0,$$

studied by C. Wang [16] in the context of absolute minimizers of $\|f(\mathfrak{X}u)\|_{L^\infty}$ for convex functions f, positive and positively homogeneous of degree $k \geq 1$ (see [16] for the precise hypotheses); here $\beta = 2(k-1)$, convexity plays no role in (1.3) and (1.4), and a is not even if f is not. The normalized p-Laplacian

$$\Delta^N_{\mathfrak{X},p} u = \Delta_{\mathfrak{X}} u + (p-2)|\mathfrak{X}u|^{-2} \Delta_{\mathfrak{X},\infty} u,$$

for $1 < p < \infty$, corresponds to $a(\xi) = \mathbb{I}_m + (p-2)\, \xi \otimes \xi / |\xi|^2$, which is discontinuous at the origin, and $\beta = 0$, while the p-Laplacian

$$\sum_{j=1}^m X_j \big(|\mathfrak{X}u|^{p-2} X_j u \big) = |\mathfrak{X}u|^{p-2} \Delta^N_{\mathfrak{X},p} u,$$

with $\beta = p-2$, has the same sub- and super-solutions (Remark 1.1(b)). Further examples are the tilted ∞-Laplacian $\big\langle \mathfrak{X}\mathfrak{X}^\star u \, T\mathfrak{X}u, T\mathfrak{X}u \big\rangle = 0$, where T is an $m \times m$ matrix with positive definite symmetric part, which corresponds to $a(\xi) = T\xi \otimes T\xi$ and $\beta = 2$ and, for non-symmetric T, is an operator of rank one that is not of Aronsson type, since $T\xi$ is then not parallel to a gradient field; and the orthotropic p-Laplacian

$$\sum_i X_i \big(|X_i u|^{p-2} X_i u \big) = (p-1) \sum_i |X_i u|^{p-2} X_i X_i u,$$

for $p \geq 2$, which corresponds to $a(\xi) = \operatorname{diag}(|\xi_1|^{p-2}, \dots, |\xi_m|^{p-2})$ and $\beta = p-2$, and for $p = 4$ is, up to a factor, the diagonal ∞-Laplacian $\sum_i (X_i u)^2 X_i X_i u$. Finally, first-order terms $H(\mathfrak{X}u)$, with $H \in C(\mathbb{R}^m \setminus \{0\})$ positively homogeneous of degree $\beta + 1$, can be added with obvious changes in the statements and proofs below, as for the biased ∞-Laplacian $\Delta_{\mathfrak{X},\infty} u + \gamma |\mathfrak{X}u|^3 = 0$ of the biased tug-of-war [15] and the ∞-Laplacian with drift $\Delta_{\mathfrak{X},\infty} u + \big\langle b, \mathfrak{X}u \big\rangle |\mathfrak{X}u|^2 = 0$; for simplicity, we restrict ourselves to (1.2).

Our main result is the comparison principle for smooth sub- and super-solutions in the non-degenerate case.

Theorem 1.3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and let $\mathcal{L}$ be as in (1.2), with $a_{i,j} \in C(\mathbb{R}^m \setminus \{0\})$, $a(\xi)$ symmetric and positive semi-definite, satisfying (1.3) and (1.4) for some $\beta \in \mathbb{R}$. Let $u, v \in C^2(\Omega) \cap C(\bar\Omega)$ be such that $\mathcal{L}u \leq 0 \leq \mathcal{L}v$ in Ω and $u \leq v$ on $\partial\Omega$. If $\mathfrak{X}u(x) \neq 0$ for every $x \in \operatorname{argmax}_\Omega (u - v)$, then $u \leq v$ in Ω.*

At every point of $\operatorname{argmax}_\Omega (u - v)$ we have $\nabla u = \nabla v$, hence $\mathfrak{X}u = \mathfrak{X}v$; so the hypothesis of Theorem 1.3 can equivalently be imposed on v. In particular, we obtain the following.

Corollary 1.4. *Let Ω and $\mathcal{L}$ be as in Theorem 1.3, and let $u, v \in C^2(\Omega) \cap C(\bar\Omega)$ with $\mathcal{L}u \leq 0 \leq \mathcal{L}v$ in Ω and $u \leq v$ on $\partial\Omega$. If v, or u, has no horizontal critical points in Ω, then $u \leq v$ in Ω. Consequently, the Dirichlet problem for (1.2) has at most one solution $u \in C^2(\Omega) \cap C(\bar\Omega)$ without horizontal critical points, and no other solution in $C^2(\Omega) \cap C(\bar\Omega)$ with the same boundary values.*

For instance, any function v with $\mathfrak{X}v \equiv \lambda$ for some constant $\lambda \in \mathbb{R}^m \setminus \{0\}$ and $\mathfrak{X}\mathfrak{X}v \equiv 0$ is a solution of (1.2) without horizontal critical points; in a Carnot group, in exponential coordinates, these are the horizontal affine functions $v(x) = \big\langle \lambda, x_1 \big\rangle + c$, where $x_1 \in \mathbb{R}^m$ denotes the first-layer coordinates and $c \in \mathbb{R}$. Hence every sub-solution (resp. super-solution) $u \in C^2(\Omega) \cap C(\bar\Omega)$ which lies below (resp. above) such a v on $\partial\Omega$ lies below (resp.

above) v in Ω. In the absence of comparison with cones, this is the natural replacement of comparison with affine functions.

We remark that the comparison principle is non-trivial even in the Euclidean case, since we need to overcome the case where the gradient vanishes. We follow the first step of the argument of Barles and Busca [6] (see also [2]): a small perturbation $u \mapsto h_\lambda(u)$ together with (1.3) turns the sub-solution into a strict sub-solution at the points where its horizontal gradient does not vanish, and this gives the comparison principle when the gradient does not vanish at the maximal points of $u - v$, which is Theorem 1.3. The homogeneity (1.4) makes the effect of the perturbation an exact identity (Lemma 3.2), which is why continuity of a away from the origin suffices. Remarkably, this step uses no structure of the vector fields beyond the chain rule and the identity (2.4) below: this is why Theorem 1.3 holds for an arbitrary smooth Hörmander family, and in fact condition (1.1) itself enters the proof only through the discussion of Definition 1.2.

In the Euclidean case, Barles and Busca remove the non-degeneracy assumption of Theorem 1.3 by a second step: one considers the translates $u(\cdot + h) - v$, and either the gradient of $u(\cdot + h)$ does not vanish at the maximal points for some small h, so that the first step applies to the sub-solution $u(\cdot + h)$, or the maximum of $u(\cdot + h) - v$ does not depend on h near some h_0, and then the strong maximum principle forces u and v to be constant near the maximal points, contradicting the boundary inequality. On a non-abelian Carnot group this second step does not go through as it stands, because the two properties of the Euclidean translation that it uses are split between two different operations: left translations $x \mapsto h * x$ preserve the equation, since the vector fields are left-invariant, but admit no first-order expansion in terms of the horizontal gradient, while right translations $x \mapsto x * h$, for which $u(x * h) = u(x) + \langle \mathfrak{X}u(x), \pi_1(h) \rangle + o(\|h\|)$ with π_1 the projection onto the first-layer coordinates, do not preserve sub-solutions in general. The following question is therefore left open.

Question 1.5. Let $\mathbb{G}$ be a Carnot group of step $r \geq 2$, let $X_1, \dots, X_m$ be a basis of the first layer of its Lie algebra, and let $\mathcal{L}$ be as in Theorem 1.3. Is it true that $u \leq v$ in Ω for all $u, v \in C^2(\Omega) \cap C(\bar{\Omega})$ with $\mathcal{L}u \leq 0 \leq \mathcal{L}v$ in Ω and $u \leq v$ on $\partial\Omega$, without any assumption on $\operatorname{argmax}_\Omega(u - v)$?

For some particular equations of the class, such as the ∞-Laplacian, the p-Laplacian and the Aronsson equations of convex functions, uniqueness results on Carnot groups are known by other methods [7, 16, 12, 8]; our argument does not rely on approximation by solutions of p-Laplace equations nor on a variational structure. For linear equations defined by arbitrary Hörmander vector fields, the comparison principle follows from the maximum principle of Bony [10]. The strong maximum principle, which enters the Euclidean second step above, was proved for quasilinear degenerate elliptic equations defined by Hörmander vector fields by Capogna and Zhou [11], and for fully nonlinear equations by Bardi and Goffi [3]; we also refer to [5] and [4] for related results.

Our results are limited to smooth sub- and super-solutions. In the Euclidean case and in the case of Carnot groups, arbitrary viscosity sub-solutions and super-solutions can be approximated, respectively, by semi-convex sub-solutions and semi-concave super-solutions (see [13, 2, 16, 8]), which suggests that the arguments can be extended to viscosity solutions. This extension will be addressed in a forthcoming paper.

Theorem 1.3 and Corollary 1.4, both the comparison and the uniqueness assertion of the corollary, have been formalized in Lean 4 on top of the Mathlib library, precisely in the general setting of this note: the Lean statements are for an arbitrary family of smooth vector fields satisfying (1.1), with no group structure. The formal proofs follow Section 3 step by step and are checked by the Lean kernel using only the standard axioms. A companion note describes the formalization and states every Lean definition and theorem, so that they can be compared with the statements of this paper. See [14].

The paper is organized as follows. Section 2 contains notation and the consequences of (1.3) and (1.4) that we use. In Section 3 we prove Theorem 1.3 and Corollary 1.4.

2. Notations and Preliminaries

For a function $u : \Omega \to \mathbb{R}$, we denote the sub-elliptic gradient and second derivative by

$$\mathfrak{X}u = (X_1 u, \dots, X_m u) \quad \text{and} \quad \mathfrak{X}\mathfrak{X}u = (X_j X_i u)_{i,j}.$$

The second derivative matrix is not symmetric in general, hence we denote the symmetrized second derivative by

$$\mathfrak{X}\mathfrak{X}^\star u = \frac{1}{2}\left(\mathfrak{X}\mathfrak{X}u + (\mathfrak{X}\mathfrak{X}u)^T\right) = \frac{1}{2}\big(X_i X_j u + X_j X_i u\big)_{i,j}.$$

The standard Euclidean scalar product on $\mathbb{R}^k$ is denoted by $\langle \cdot, \cdot \rangle$, the Euclidean vector fields by ∂_{x_i}, $\nabla u = (\partial_{x_1} u, \dots, \partial_{x_n} u)$ is the Euclidean gradient, DF is the Jacobian matrix of a vector function $F : \Omega \to \mathbb{R}^n$ and $D^2 u = D(\nabla u) = (\partial_{x_i}\partial_{x_j} u)_{i,j}$ is the Euclidean Hessian. For $x \in \mathbb{R}^n$ and $E \subset \mathbb{R}^n$, $\operatorname{dist}(x, E) = \inf\{|x - y| : y \in E\}$ denotes the Euclidean distance.

For symmetric matrices $A, B \in \mathbb{R}^{k\times k}$, we write $A \le B$ if $\langle A\xi, \xi\rangle \le \langle B\xi, \xi\rangle$ for every $\xi \in \mathbb{R}^k$, and $|A| = \sqrt{\operatorname{Tr}(A^T A)}$. If $A, Z \ge 0$, then $\operatorname{Tr}(AZ) = |\sqrt{Z}\sqrt{A}|^2 \ge 0$; hence

$$\operatorname{Tr}(AY) \le \operatorname{Tr}(AX) \qquad \forall\ A \ge 0,\ Y \le X. \tag{2.1}$$

We record the consequences of (1.3) and (1.4) that we use.

Lemma 2.1. *Assume (1.3) and (1.4). Then for all $t > 0$, $\xi \neq 0$ and every $X \in \mathbb{R}^{m\times m}$,*

$$\mathcal{E}(t\xi) = t^{2+\beta}\,\mathcal{E}(\xi) \qquad \text{and} \qquad \operatorname{Tr}\big(a(t\xi)X\big) = t^\beta\ \operatorname{Tr}\big(a(\xi)X\big). \tag{2.2}$$

Moreover,

$$\mathcal{E}(\xi) \ge a_0\,|\xi|^{2+\beta} \quad \forall\ \xi \in \mathbb{R}^m \setminus \{0\}, \qquad \text{where} \quad a_0 := \min_{\zeta\in\mathbb{S}^{m-1}} \big\langle a(\zeta)\zeta, \zeta\big\rangle > 0. \tag{2.3}$$

Proof. The identities (2.2) follow at once from (1.4). The function $\mathcal{E}$ is continuous, and positive on the compact set $\mathbb{S}^{m-1}$ by (1.3), so that $a_0 > 0$. Finally, applying (2.2) to $\xi/|\xi|$ with $t = |\xi|$, we get $\mathcal{E}(\xi) = |\xi|^{2+\beta}\mathcal{E}(\xi/|\xi|) \ge a_0|\xi|^{2+\beta}$. □

There is a matrix $\sigma : \mathbb{R}^n \to \mathbb{R}^{n\times m}$, $\sigma(x) = [\sigma^1(x), \dots, \sigma^m(x)]$ with $\sigma^j : \mathbb{R}^n \to \mathbb{R}^n$ smooth, such that $X_j = \sigma^j(x)\cdot\nabla$; since the vector fields are non-zero, $\sigma(x) \neq 0$ for every x. Hence, for any $u \in C^2$,

$$\mathfrak{X}u = \sigma(x)^T \nabla u \quad \text{and} \quad \mathfrak{X}\mathfrak{X}^\star u = \sigma(x)^T D^2 u\,\sigma(x) + \mathcal{M}(x, \nabla u), \tag{2.4}$$

where $\mathcal{M}(x, \zeta) \in \mathbb{R}^{m\times m}$ is the matrix with entries

$$\mathcal{M}(x,\zeta)_{i,j} = \frac{1}{2}\Big(\big\langle D\sigma^j(x)\,\sigma^i(x), \zeta\big\rangle + \big\langle D\sigma^i(x)\,\sigma^j(x), \zeta\big\rangle\Big); \tag{2.5}$$

note that the map $\zeta \mapsto \mathcal{M}(x, \zeta)$ is linear. Indeed $X_j X_i u = \big\langle D^2 u\,\sigma^i, \sigma^j\big\rangle + \big\langle D\sigma^i\,\sigma^j, \nabla u\big\rangle$. In particular, if $u, w \in C^2$ satisfy $\nabla u(x) = \nabla w(x)$ and $D^2 u(x) \le D^2 w(x)$ at a point x, then $\mathfrak{X}u(x) = \mathfrak{X}w(x)$ and $\mathfrak{X}\mathfrak{X}^\star u(x) \le \mathfrak{X}\mathfrak{X}^\star w(x)$.

For any function $f : \Omega \to \mathbb{R}$, we denote the set of maximal points by

$$\operatorname{argmax}_\Omega(f) = \big\{x \in \Omega : f(x) = \sup_\Omega f\big\}, \tag{2.6}$$

which may be empty. If f_k, f are continuous, $f_k \to f$ uniformly, and $x_k \to x \in \Omega$ for some $x_k \in \operatorname{argmax}_\Omega(f_k)$, then $x \in \operatorname{argmax}_\Omega(f)$. Also, $\operatorname{argmax}_\Omega(f + c) = \operatorname{argmax}_\Omega(f)$ if c is constant.

3. Comparison principle

In this section we prove Theorem 1.3 and Corollary 1.4. Throughout, $\mathcal{L}$ is as in (1.2) with $a_{i,j} \in C(\mathbb{R}^m \setminus \{0\})$, $a(\xi)$ symmetric and positive semi-definite, satisfying (1.3) and (1.4) for some $\beta \in \mathbb{R}$, and $\Omega \subset \mathbb{R}^n$ is a bounded open set, not necessarily connected. We follow the first step of the approach of Barles and Busca [6], who studied the Euclidean case.

We begin with the following elementary observation.

Lemma 3.1. *Let $U \subset \mathbb{R}^n$ be open, let $u, v \in C^2(U)$, and let $x_0 \in U$ be a local maximum point of $u - v$ with $\mathfrak{X}u(x_0) \neq 0$. Then $\mathfrak{X}v(x_0) = \mathfrak{X}u(x_0) =: \xi_0$, $\mathfrak{X}\mathfrak{X}^\star u(x_0) \leq \mathfrak{X}\mathfrak{X}^\star v(x_0)$ and $\mathcal{L}u(x_0) \geq \mathcal{L}v(x_0)$. In particular, it is impossible that $\mathcal{L}u(x_0) < 0 \leq \mathcal{L}v(x_0)$ or $\mathcal{L}u(x_0) \leq 0 < \mathcal{L}v(x_0)$.*

Consequently, if $u, v \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfy $\mathcal{L}u < 0 \leq \mathcal{L}v$ (resp. $\mathcal{L}u \leq 0 < \mathcal{L}v$) in Ω, $u \leq v$ on $\partial\Omega$, and $\mathfrak{X}u \neq 0$ on $\operatorname{argmax}_\Omega(u - v)$, then $u \leq v$ in Ω.

Proof. The interior maximality at x_0 implies $\nabla u(x_0) = \nabla v(x_0)$ and $D^2u(x_0) \leq D^2v(x_0)$, which together with (2.4) yields $\mathfrak{X}u(x_0) = \mathfrak{X}v(x_0) = \xi_0$ and $\mathfrak{X}\mathfrak{X}^\star u(x_0) \leq \mathfrak{X}\mathfrak{X}^\star v(x_0)$. Using (2.1) with $a(\xi_0) \geq 0$, we have

$$0 \leq \operatorname{Tr}\Big(a(\xi_0)\big(\mathfrak{X}\mathfrak{X}^\star v(x_0) - \mathfrak{X}\mathfrak{X}^\star u(x_0)\big)\Big) = -\mathcal{L}v(x_0) + \mathcal{L}u(x_0).$$

For the second assertion, we proceed by contradiction: if $u(x) > v(x)$ for some $x \in \Omega$, then, since $u - v \leq 0$ on $\partial\Omega$, the maximum of $u - v$ over $\bar{\Omega}$ is positive and attained at some $x_0 \in \operatorname{argmax}_\Omega(u - v)$; there $\mathfrak{X}u(x_0) = \mathfrak{X}v(x_0) \neq 0$, so the inequalities $\mathcal{L}u(x_0) < 0 \leq \mathcal{L}v(x_0)$ (resp. $\mathcal{L}u(x_0) \leq 0 < \mathcal{L}v(x_0)$) hold, which contradicts the first part. □

The next goal is to relax the strict inequality in Lemma 3.1 to $\mathcal{L}u \leq 0 \leq \mathcal{L}v$. This is difficult due to the degeneracy of the equation. Given a sub-solution u, the strategy is to construct small perturbations $u_\lambda = h_\lambda(u)$, with $h_\lambda \in C^2(\mathbb{R})$ and $\lambda > 0$ small, which are strict sub-solutions at the points where $\mathfrak{X}u$ does not vanish. The construction uses the ellipticity (1.3) and the homogeneity condition (1.4); we have the following technical lemma.

Lemma 3.2. *Let $\beta \in \mathbb{R}$ be as in (1.4), let $h \in C^2(\mathbb{R})$, let $w \in C^2(U)$ for an open set U, and let $x \in U$ with $\mathfrak{X}w(x) \neq 0$ and $h'(w(x)) > 0$. Then*

$$\mathcal{L}(h \circ w)(x) = -h'(w(x))^\beta \Big[h''(w(x))\,\mathcal{E}\big(\mathfrak{X}w(x)\big) - h'(w(x))\,\mathcal{L}w(x)\Big]. \tag{3.1}$$

Proof. We have $\mathfrak{X}(h \circ w) = h'(w)\mathfrak{X}w$ and $\mathfrak{X}\mathfrak{X}(h \circ w) = h'(w)\mathfrak{X}\mathfrak{X}w + h''(w)\mathfrak{X}w \otimes \mathfrak{X}w$. Hence, with $t = h'(w(x)) > 0$ and $\xi = \mathfrak{X}w(x)$, using (1.4) we have, at x,

$$\begin{aligned}\mathcal{L}(h \circ w)(x) &= -t \operatorname{Tr}\big(a(t\xi)\mathfrak{X}\mathfrak{X}w(x)\big) - h''(w(x))\big\langle a(t\xi)\xi, \xi\big\rangle \\ &= -t^\beta\Big[h''(w(x))\big\langle a(\xi)\xi, \xi\big\rangle + t \operatorname{Tr}\big(a(\xi)\mathfrak{X}\mathfrak{X}w(x)\big)\Big],\end{aligned}$$

which is (3.1). □

If $a(t\xi) = |t|^\beta a(\xi)$ for all $t \neq 0$, the same computation shows that (3.1) holds, with $|h'(w(x))|^\beta$ in place of $h'(w(x))^\beta$, whenever $h'(w(x)) \neq 0$. Using Lemma 3.2 together with the ellipticity (1.3), we show the following.

Proposition 3.3. *Let $u, v \in C^2(\Omega) \cap C(\bar{\Omega})$ be such that $\mathcal{L}u \leq 0 \leq \mathcal{L}v$ in Ω and $u \leq v$ on $\partial\Omega$. If $\mathfrak{X}u$ (equivalently, $\mathfrak{X}v$) does not vanish at any point of $\operatorname{argmax}_\Omega(u - v)$, then $u \leq v$ in Ω.*

Proof. We proceed by contradiction and assume that $M_0 := \max_{\bar{\Omega}}(u - v) > 0$. Since $u - v \leq 0$ on $\partial\Omega$, the set $E := \operatorname{argmax}_\Omega(u - v)$ is a non-empty compact subset of Ω. By

assumption and continuity of $\mathfrak{X}u$, $\theta := \min_E |\mathfrak{X}u| > 0$, and there is an open set U with $E \subset U \subset\subset \Omega$ and $|\mathfrak{X}u| \geq \theta/2$ on U.

For $\lambda > 0$ let

$$h_\lambda(s) = s + \lambda(s - u_0)^2, \qquad u_0 = \min_{\bar\Omega} u, \qquad u_\lambda := h_\lambda \circ u \in C^2(\Omega) \cap C(\bar\Omega).$$

(Any $h_\lambda \in C^2(\mathbb{R})$ with $h_\lambda(s) \geq s$, $h_\lambda'(s) > 0$ and $h_\lambda''(s) > 0$ for s in the range of u, and $h_\lambda \to \mathrm{id}$ uniformly there as $\lambda \to 0^+$, would do.) On the range of u we have $h_\lambda' \geq 1$ and $h_\lambda'' = 2\lambda$; moreover $u \leq u_\lambda$ and

$$\|u_\lambda - u\|_{L^\infty(\Omega)} \leq 4\lambda \|u\|^2_{L^\infty(\Omega)}. \tag{3.2}$$

Let $E_\lambda := \mathrm{argmax}_{\bar\Omega}(u_\lambda - v)$. Since $\max_{\bar\Omega}(u_\lambda - v) \geq M_0$, while $u_\lambda - v \leq 4\lambda\|u\|^2_{L^\infty}$ on $\partial\Omega$, we have $E_\lambda \subset \Omega$ whenever $4\lambda\|u\|^2_{L^\infty} < M_0$. Moreover $\sup_{x\in E_\lambda} \mathrm{dist}(x, E) \to 0$ as $\lambda \to 0^+$: otherwise there are $\lambda_k \to 0$ and $x_k \in E_{\lambda_k}$ with $\mathrm{dist}(x_k, E) \geq \varepsilon_0 > 0$, and a subsequence converges to some $\bar x \in \bar\Omega$ with $(u - v)(\bar x) = \lim_k (u_{\lambda_k} - v)(x_k) \geq M_0$ by (3.2); thus $\bar x \in E$, a contradiction. Fix $\lambda > 0$ so small that $E_\lambda \subset U$, and let $x_\lambda \in E_\lambda$.

Since $x_\lambda \in U$, we have $|\mathfrak{X}u(x_\lambda)| \geq \theta/2$, hence $\mathfrak{X}u_\lambda(x_\lambda) = h_\lambda'(u(x_\lambda))\mathfrak{X}u(x_\lambda) \neq 0$. As x_λ is an interior maximum point of $u_\lambda - v$, Lemma 3.1 gives $\mathfrak{X}v(x_\lambda) = \mathfrak{X}u_\lambda(x_\lambda) \neq 0$ and

$$\mathcal{L}u_\lambda(x_\lambda) \geq \mathcal{L}v(x_\lambda) \geq 0, \tag{3.3}$$

where the last inequality holds because v is a super-solution and $\mathfrak{X}v(x_\lambda) \neq 0$. On the other hand, u is a sub-solution and $\mathfrak{X}u(x_\lambda) \neq 0$, so that $\mathcal{L}u(x_\lambda) \leq 0$, and Lemma 3.2 with $t := h_\lambda'(u(x_\lambda)) \geq 1$ and $\xi := \mathfrak{X}u(x_\lambda)$ gives

$$\mathcal{L}u_\lambda(x_\lambda) = -t^\beta \Big[2\lambda\,\mathcal{E}(\xi) - t\,\mathcal{L}u(x_\lambda)\Big] \leq -2\lambda\, t^\beta\, \mathcal{E}(\xi) \leq -2\lambda\, a_0\, t^\beta\, |\xi|^{2+\beta} < 0, \tag{3.4}$$

by (2.3). This contradicts (3.3), and the proof is complete. (Since $\mathfrak{X}u = \mathfrak{X}v$ on E, the hypothesis on $\mathfrak{X}v$ is the same; alternatively, one can perturb v by $v_\lambda = v - \lambda(v - v_1)^2$, $v_1 = \max_{\bar\Omega} v$, for which $h_\lambda' \geq 1$ and $h_\lambda'' = -2\lambda$ on the range of v, so that $\mathcal{L}v_\lambda > 0$ at the points where $\mathfrak{X}v \neq 0$, by Lemma 3.2.) □

Proof of Theorem 1.3 and Corollary 1.4. Theorem 1.3 is a restatement of Proposition 3.3. As for Corollary 1.4, if v (resp. u) has no horizontal critical points, then $\mathfrak{X}u = \mathfrak{X}v \neq 0$ at every point of $\mathrm{argmax}_\Omega(u - v)$, and Proposition 3.3 applies. If u and v are solutions with $u = v$ on $\partial\Omega$ and v has no horizontal critical points, then $u \leq v$; exchanging the roles of u and v, so that v is now the sub-solution and $\mathfrak{X}v \neq 0$ on $\mathrm{argmax}_\Omega(v - u)$, we also get $v \leq u$. □

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