CLASSIFICATION OF BOUNDARY BLOW-UPS FOR THE MUMFORD–SHAH FUNCTIONAL WITH DIRICHLET BOUNDARY CONDITIONS

FRANCESCO DEANGELIS

Abstract. We classify boundary blow-ups of minimizers of the Mumford–Shah functional on bounded planar domains with Dirichlet boundary conditions. At every point where the discontinuity set meets the boundary of the domain, we prove that the discontinuity set of each blow-up is the tangent line to the boundary at that point. This settles the conjecture formulated by the author in 2024.

1. Introduction

The Mumford–Shah functional was introduced in [MS89] as a variational model for image segmentation. Its two unknowns are a function and a set across which the function may be discontinuous. The competition between the Dirichlet energy of the function and the length of this set gives rise to a free discontinuity problem: the set is itself determined by minimization. In this paper we consider the functional with prescribed Dirichlet boundary conditions and study the geometry of the discontinuity set near the boundary.

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with boundary of class $C^{1,\gamma}$, and let $g \in C^{1,\gamma}(\partial\Omega)$ for some $\gamma \in (0,1)$. An admissible pair consists of a closed set $K \subset \overline{\Omega}$ and a function $u \in W^{1,2}(\Omega \setminus K)$ whose trace agrees with g on $\partial\Omega \setminus K$. We consider minimizers of

$$E(K,u) = \int_{\Omega\setminus K} |\nabla u|^2 + \mathcal{H}^1(K) \tag{1.1}$$

among all such pairs, where $\mathcal{H}^1$ denotes one-dimensional Hausdorff measure. The first term penalizes spatial variation of u away from K, while the second measures the length of K. In particular, the length term includes $K \cap \partial\Omega$, whereas the boundary datum is imposed only on $\partial\Omega \setminus K$. Thus the choice of K also determines the portion of the boundary on which u must attain the prescribed values. Existence of minimizers for this Dirichlet problem was established by Carriero and Leaci [CL90].

This problem also arises in the variational approach to brittle fracture under antiplane deformation [FM98, BFM08]. Here u represents the displacement, g the prescribed boundary displacement, and K the fracture set. The two terms in the functional account for stored elastic energy and the energy required to create the fracture.

We call K the discontinuity set. As in [Dea24], we work with normalized minimizers, meaning that

$$\mathcal{H}^1(K \cap B_\rho(x)) > 0 \qquad \text{for every } x \in K \text{ and } \rho > 0,$$

where $B_\rho(x)$ is the open Euclidean ball of radius ρ centered at x. This condition removes portions of the closed set that are invisible to its length measure. No smoothness of K is part of the admissibility conditions. The regularity problem is to determine its local geometry from minimality.

At interior points, the Mumford–Shah conjecture describes K as a locally finite union of C^1 arcs, with possible free endpoints and triple junctions at angles of $120°$. The author proved this conjecture in [Dea26]. A central tool in the interior theory is to magnify a minimizer around a point and classify the resulting limits, called blow-ups. Epsilon-regularity theorems then turn sufficiently good approximation by one of these limiting configurations into regularity at smaller scales. For

Key words and phrases. Mumford–Shah functional, Dirichlet boundary conditions, generalized minimizers, boundary regularity, blow-down.

contributions to the regularity and classification theory, see [DGCL89, DMMS92, Dav96, Bon96, AP97, AFP97, AFP99, Leg99, BD01, DL02, Lem11, LF13, DLF13, BL14, DPF14, Lem15, LM18, AM19, DLFG21, Fis22, Zha25, LL26]. A comprehensive account is given in [DLF25].

At points of $K \cap \partial\Omega$, the prescribed boundary values enter the limiting problem. The corresponding classification is the subject of the present paper.

Boundary blow-ups. Fix $x \in K \cap \partial\Omega$. For $r > 0$, consider the rescaled domain, set, and function

$$\Omega_{x,r} = \frac{\Omega - x}{r}, \qquad K_{x,r} = \frac{K - x}{r}, \qquad v_{x,r}(y) = \frac{u(x + ry) - g(x)}{\sqrt{r}}.$$

The exponent $1/2$ is chosen so that, under this rescaling, both the Dirichlet energy and the length are multiplied by r^{-1}. Let $T_x\partial\Omega$ denote the tangent space to $\partial\Omega$ at x, and let H_x be the half-plane bounded by $T_x\partial\Omega$ on the interior side of Ω. As $r \downarrow 0$, the rescaled domains converge locally to H_x. The subtraction of $g(x)$ and the regularity of g ensure that the rescaled boundary data tend locally to zero.

A boundary blow-up at x is a limiting pair (K_0, u_0) obtained along a sequence $r_j \downarrow 0$. The compactness theory of [Dea24] gives local Hausdorff convergence of the sets K_{x,r_j} and convergence of the functions on complementary components, after subtraction of componentwise additive constants when necessary. The precise convergence is recalled in Section 2. After a rotation, we may take $H_x = U := \{y_2 > 0\}$, so that $\partial U = \mathbb{R} \times \{0\}$. The limiting set K_0 is a closed subset of $\overline{U}$ containing the origin, and u_0 is locally Sobolev on $U \setminus K_0$, with zero trace on $\partial U \setminus K_0$. On components joined to the lower half-plane through $\partial U \setminus K_0$, this trace fixes the additive normalization.

These limiting pairs are generalized minimizers of the half-plane problem in the sense of [Dea24, Definition 2.5]; the precise definition is recalled in Section 2.

Previous results and the main theorem. The boundary regularity theorem of [Dea24] shows that if $x \in K \cap \partial\Omega$ and K is sufficiently close to the tangent line $x + T_x\partial\Omega$ to $\partial\Omega$ at x in a small ball centered at x, then K is a $C^{1,\alpha}$ graph in a smaller ball, for some $\alpha \in (0, 1)$. This makes the classification of boundary blow-ups the natural next step. The same work conjectured that every such blow-up has discontinuity set $T_x\partial\Omega$ and established partial classifications, including the cases in which the limiting Dirichlet energy vanishes or the limiting discontinuity set is connected.

Our classification result, Theorem 2.1, settles this conjecture by classifying all normalized generalized minimizers (K, u) on U whose discontinuity set meets ∂U. We prove that, for some $c \in \mathbb{R}$,

$$K = \partial U, \qquad u \equiv c \quad \text{in } U.$$

In particular, for every minimizer of (1.1) and every $x \in K \cap \partial\Omega$, each boundary blow-up has discontinuity set $T_x\partial\Omega$. This identifies the limiting geometry at every such point and supplies the classification required by the boundary epsilon-regularity theory.

Global regularity for the Dirichlet problem. Combining this classification with the boundary epsilon-regularity theorem of [Dea24] and the solution of the Mumford–Shah conjecture [Dea26] gives a complete description of the discontinuity set, including its behavior at the boundary. The conclusion also holds for the full Mumford–Shah functional: for $f \in L^\infty(\Omega)$ and $\lambda \geq 0$, set

$$E_\lambda(K, u) = \int_{\Omega \setminus K} \left(|\nabla u|^2 + \lambda |u - f|^2 \right) \mathrm{d}x + \mathcal{H}^1(K). \tag{1.2}$$

Here f is the fidelity datum, while g remains the prescribed boundary datum.

Theorem 1.1 (Global regularity for the Dirichlet problem)**.** *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with boundary of class $C^{1,\gamma}$, let $g \in C^{1,\gamma}(\partial\Omega)$ for some $\gamma \in (0, 1)$, and let $f \in L^\infty(\Omega)$ and $\lambda \geq 0$. Let (K, u) be a normalized minimizer of (1.2) among admissible pairs with $u = g$ on $\partial\Omega \setminus K$. Then K is a finite union of regular C^1 arcs, with the following local structure:*

- *(i) At every point of $K \cap \Omega$, the set K is locally a single regular arc, a regular arc ending at that point, or three regular arcs meeting only at that point at angles of $120°$.*
- *(ii) At every $x \in K \cap \partial\Omega$, the set K is locally a single $C^{1,\alpha}$ arc containing x in its interior, for some $\alpha \in (0, 1)$, and its tangent line at x is $x + T_x\partial\Omega$.*

In particular, the free endpoints and triple junctions are finite in number and lie in Ω.

Proof. We first explain the inclusion of the fidelity term when $\lambda > 0$. By truncation and minimality, $|u| \le M := \max\{\|f\|_\infty, \|g\|_\infty\}$. By truncation, it suffices to consider competitors whose functions take values in $[-M, M]$. Since f and these functions are bounded, the difference of the fidelity terms in a comparison supported in $B_r(x)$ is $O(r^2)$. The rescaled energy is obtained by dividing by r, so this contributes an $O(r)$ error, which vanishes as $r \downarrow 0$. The boundary compactness and epsilon-regularity proofs of [Dea24] adapt to include these lower-order contributions. In particular, boundary blow-ups are generalized minimizers of the same homogeneous half-plane problem.

For $x \in K \cap \partial\Omega$, Theorem 2.1 shows that every boundary blow-up has discontinuity set $T_x\partial\Omega$. The boundary epsilon-regularity theorem [Dea24, Theorem 1.1], with the extension just described, gives a single $C^{1,\alpha}$ arc containing x in its interior, with tangent line $x + T_x\partial\Omega$ at x.

The interior description follows from [Dea26, Theorem 1.1]. These local descriptions give neighborhoods containing no free endpoints or triple junctions, except possibly their centers in the interior case. Compactness of $K \subset \overline{\Omega}$ then gives finitely many such points and a finite cover by regular arcs, proving the global conclusion. □

For homogeneous Neumann boundary conditions, the analogous global conclusion follows by combining [Dea26, Theorem 1.1] with the boundary regularity result of David and Léger [DL02, Section 13]; see also [Dav05, Section 79]. For a bounded domain with C^1 boundary and a bounded fidelity datum, the closure of the discontinuity set is a finite union of C^1 arcs with the same interior structure, and the arcs that reach the boundary end there orthogonally. Thus the interior and boundary results together give the complete regularity description for both the Dirichlet and homogeneous Neumann problems.

Proof strategy. For a generalized minimizer (K, u) on U, translate a point of $K \cap \partial U$ to the origin and consider

$$K_R = \frac{K}{R}, \qquad u_R(y) = \frac{u(Ry)}{\sqrt{R}}.$$

A limit of these pairs along a sequence $R \to \infty$, with the same convergence and additive normalizations as above, is called a blow-down. Boundary epsilon-regularity gives rigidity at infinity: if a blow-down has discontinuity set ∂U, then $K = \partial U$ (Theorem 3.2). A monotonicity argument provides the circle-intersection criterion of Proposition 4.1. This criterion yields the classification when $\mathbb{R}^2 \setminus K$ is disconnected (Theorem 5.1).

It remains to treat the case in which $\mathbb{R}^2 \setminus K$ is connected. Normal domain variations then define a nonnegative Radon measure μ on ∂U, and a linear energy bound gives finite total mass (Lemma 6.1). Away from K, this measure has density equal to the square of the normal derivative of u. Identifying ∂U with $\mathbb{R}$, the measures associated with the rescaled pairs satisfy $\mu_R(A) = \mu(RA)$ for Borel sets $A \subset \mathbb{R}$. Their mass therefore tends to zero on every compact boundary interval away from the origin as $R \to \infty$.

For a blow-down (K_∞, u_∞), the density formula and harmonic compactness yield zero Neumann data on boundary intervals disjoint from K_∞. Together with the inherited zero Dirichlet data, unique continuation would give $u_\infty = 0$ on $U \setminus K_\infty$ if $\mathbb{R}^2 \setminus K_\infty$ were connected. The classification of elementary generalized minimizers [Dea24, Theorem 2.13] would then give $K_\infty = \partial U$, a contradiction. Hence $\mathbb{R}^2 \setminus K_\infty$ is disconnected for every blow-down (Lemma 6.2). The classification in that case gives a flat blow-down, and rigidity at infinity concludes the proof.

Section 2 fixes the variational setting and states the main theorem. Sections 3–5 establish rigidity at infinity, the circle-intersection criterion, and the classification when $\mathbb{R}^2 \setminus K$ is disconnected. Section 6 constructs the measure on the boundary and completes the proof.

2. Definitions and main result

The half-plane problem. We set $U = \{x_2 > 0\}$ and recall the definition of generalized minimizer from [Dea24, Definition 2.5]. An admissible pair consists of a closed set $K \subset \overline{U}$ and a function

$u \in W^{1,2}_{\mathrm{loc}}(U \setminus K)$, with zero trace on $\partial U \setminus K$, such that the local energy

$$E(K,u;O) := \int_{O\cap(U\setminus K)} |\nabla u|^2 \,\mathrm{d}x + \mathcal{H}^1(K \cap O)$$

is finite for every bounded open set $O \subset \mathbb{R}^2$.

We call an admissible pair (K,u) a *generalized minimizer* on U if

$$E(K,u;O) \le E(J,v;O)$$

for every bounded open set $O \subset \mathbb{R}^2$ and every admissible pair (J,v) that agrees with (K,u) outside O and preserves the separation of exterior points. More precisely, any two points in $\mathbb{R}^2 \setminus (O \cup K)$ belonging to distinct connected components of $\mathbb{R}^2 \setminus K$ must also belong to distinct connected components of $\mathbb{R}^2 \setminus J$.

As for the original minimizers, all generalized minimizers are normalized: $\mathcal{H}^1(K \cap B_r(z)) > 0$ for every $z \in K$ and $r > 0$. Moreover, K is $\mathcal{H}^1$-rectifiable, by the interior argument in [DLF25, Appendix D, Step 1] and the inclusion $K \cap \partial U \subset \partial U$. We write $K_{x,r} = (K - x)/r$ and $u_{x,r}(y) = r^{-1/2}u(x + ry)$.

A generalized minimizer is called *elementary* if

$$\int_{U\setminus K} |\nabla u|^2 \,\mathrm{d}x = 0.$$

This is the terminology of [Dea24, Definition 2.12].

Rescaling and limiting pairs. For every $x \in \partial U$, the rescalings defined above preserve the half-plane variational problem and satisfy

$$E(K_{x,r}, u_{x,r}; O) = \frac{1}{r} E(K,u; x + rO).$$

A *blow-up* is a subsequential limit along scales $r_j \downarrow 0$; a *blow-down* is a subsequential limit along scales $r_j \uparrow \infty$. Throughout the paper, convergence of pairs is understood in the sense of [Dea24, Proposition 2.3 and Theorem 2.9]. In particular, if $(K_j, u_j) \to (K_\infty, u_\infty)$, then $K_j \to K_\infty$ locally in the Hausdorff distance, and the functions converge in C^∞_{loc} on each connected component of $U \setminus K_\infty$, after subtracting suitable componentwise additive constants when necessary. On components connected to the lower half-plane through $\partial U \setminus K_\infty$, no constants need to be subtracted, since the zero Dirichlet condition already fixes the normalization. On these components, the convergence also holds locally in C^1 up to $\partial U \setminus K_\infty$.

Main result. The classification in Theorem 2.1 below follows from Lemma 6.1 and Theorems 5.1 and 3.2.

Theorem 2.1 (Classification of generalized minimizers meeting the boundary)**.** *Let (K,u) be a normalized generalized minimizer on $U = \{x_2 > 0\}$. If $K \cap \partial U \neq \emptyset$, then*

$$K = \partial U, \qquad u \equiv c \quad \text{in } U$$

for some $c \in \mathbb{R}$.

3. Rigidity at infinity

First we remark that the epsilon-regularity theorem presented in [Dea24, Theorem 1.1] is true also for generalized minimizers. Its proof is a simple adaptation of the proof given there. The corresponding statement is the following.

Theorem 3.1 (Epsilon-regularity for generalized minimizers)**.** *There exist constants $\eta, \varepsilon, \alpha \in (0,1)$ with the following property. Assume*

(i) (K,u) is a generalized minimizer on $U = \{x_2 > 0\}$ and $x \in K \cap \partial U$,
(ii) $r^{-1}\mathrm{dist}_H(K \cap \overline{B}_{2r}(x), \partial U \cap \overline{B}_{2r}(x)) + r^{\frac{1}{2}} < \varepsilon$.

Then $K \cap B_{\eta r}(x)$ is the graph of a $C^{1,\alpha}$ function.

As a consequence of the epsilon-regularity theorem, we obtain the following rigidity result.

Theorem 3.2 (Rigidity at infinity)**.** *Let (K,u) be a generalized minimizer on $U=\{x_2>0\}$ with $0\in K$. Suppose that $r_n\uparrow\infty$ and $K_{0,r_n}\to\partial U$ locally in the Hausdorff distance as $n\to\infty$. Then $K=\partial U$.*

Proof. Let η,ε,α be given by Theorem 3.1 and set $r=\frac{1}{4}\varepsilon^2$. Define $(K_n,u_n):=(K_{0,r_n},u_{0,r_n})$. Since $K_n\to\partial U$ locally in the Hausdorff distance, we have

$$r^{-1}\mathrm{dist}_H(K_n\cap\overline{B}_{2r},\partial U\cap\overline{B}_{2r})+r^{\frac{1}{2}}<\varepsilon$$

for sufficiently large $n\in\mathbb{N}$. Moreover, (K_n,u_n) is a generalized minimizer on U and $0\in K_n\cap\partial U$ for every $n\in\mathbb{N}$. Therefore, by Theorem 3.1, $K_n\cap B_{\eta r}$ is the graph of a $C^{1,\alpha}$ function. This implies that $K\cap B_{\eta r r_n}$ is the graph of a $C^{1,\alpha}$ function for sufficiently large $n\in\mathbb{N}$, and hence K is connected. Therefore, by [Dea24, Proposition 2.15], $K=\partial U$. □

4. A circle-intersection criterion

Proposition 4.1. *Let (K,u) be a generalized minimizer on $U=\{x_2>0\}$ with $0\in K$. Suppose that there exists $R>0$ such that $\mathcal{H}^0(K\cap\partial B_r)\geq 2$ for all $r>R$. Then $K=\partial U$.*

Proof of Proposition 4.1. Let $r_n\uparrow\infty$ as $n\to\infty$ and define $(K_n,u_n):=(K_{0,r_n},u_{0,r_n})$ for all $n\in\mathbb{N}$. By [Dea24, Theorem 2.9], there is (K_∞,u_∞) generalized minimizer on U such that, up to subsequences, (K_n,u_n) converges to (K_∞,u_∞) in the sense of the theorem.

Then, by Lemma 4.2, (K_∞,u_∞) is elementary. Therefore, since $0\in K_\infty\cap\partial U$, $K_\infty=\partial U$ by [Dea24, Theorem 2.13]. In particular, $K_{0,r_n}\to\partial U$ locally in the Hausdorff distance and Theorem 3.2 implies that $K=\partial U$. □

We now prove the auxiliary lemma used above.

Lemma 4.2. *Let (K,u) satisfy the assumptions of Proposition 4.1. Let $r_j\uparrow\infty$ and suppose that the rescalings (K_{0,r_j},u_{0,r_j}) converge to (K_∞,u_∞) in the sense of* [Dea24, Theorem 2.9]*. Then (K_∞,u_∞) is elementary.*

Proof. Define, for $r>0$,

$$F(r):=\frac{2}{r}\int_{B_r\cap U\setminus K}|\nabla u|^2+\frac{1}{r}\mathcal{H}^1(K\cap B_r).$$

By [Dea24, Proposition 2.10], F is nondecreasing on (R,∞). The standard energy upper bound (cf. [Dea24, Proposition 8.1]) gives $l_\infty:=\lim_{r\to\infty}F(r)<\infty$. Moreover, the proof of [Dea24, Proposition 2.10] yields

$$F'(\rho)\geq\frac{1}{\rho}\int_{\partial B_\rho\cap U\setminus K}|\partial_\nu u|^2\,\mathrm{d}\mathcal{H}^1 \tag{4.1}$$

for a.e. $\rho>R$.

Set $K_j=K_{0,r_j}$ and $u_j=u_{0,r_j}$, and fix $0<a<b$. For j large enough that $ar_j>R$, integrating (4.1) between ar_j and br_j (the singular part of the distributional derivative of F is nonnegative), then applying the coarea formula [AFP00, Section 2.12, equation (2.74)] and rescaling, gives

$$\int_{(B_b\setminus\overline{B}_a)\cap(U\setminus K_j)}\frac{1}{|x|}\left|\frac{x}{|x|}\cdot\nabla u_j(x)\right|^2\,\mathrm{d}x\leq F(br_j)-F(ar_j)\longrightarrow 0. \tag{4.2}$$

Local uniform convergence of the gradients away from K_∞ [Dea24, Proposition 2.3], together with Fatou's lemma and the arbitrariness of a,b, therefore gives

$$\frac{x}{|x|}\cdot\nabla u_\infty(x)=0\qquad\text{for a.e. } x\in U\setminus K_\infty.$$

Integrating by parts, using the weak Euler–Lagrange equation and the zero Dirichlet condition on $\partial U\setminus K_\infty$, we obtain, for a.e. $\rho>0$,

$$\int_{B_\rho\cap(U\setminus K_\infty)}|\nabla u_\infty|^2=\int_{\partial B_\rho\cap(U\setminus K_\infty)}u_\infty\,\partial_\nu u_\infty\,\mathrm{d}\mathcal{H}^1=0.$$

Thus (K_∞,u_∞) is elementary. □

5. Disconnected complement

Theorem 5.1. *Let (K,u) be a generalized minimizer on $U=\{x_2>0\}$ with $0\in K$. Suppose that $\mathbb{R}^2\setminus K$ is not connected. Then $K=\partial U$.*

Proof. Let Ω_* be the connected component of $\mathbb{R}^2\setminus K$ containing the lower half-plane. Since $\mathbb{R}^2\setminus K$ is not connected, there exists C connected component of $\mathbb{R}^2\setminus K$ different from Ω_*. The argument in the proof of [DLF25, Lemma 4.3.1] also applies here and shows that C is unbounded. In particular, since $C\neq\emptyset$, there is $R>0$ such that $C\cap B_r\neq\emptyset$ for every $r>R$. Since C is connected and unbounded, we must have $C\cap\partial B_r\neq\emptyset$ for all $r>R$. Observe that we cannot have $\mathcal{H}^0(K\cap\partial B_r)\leq 1$ for some $r>R$, otherwise ∂B_r connects C and Ω_*. Hence $\mathcal{H}^0(K\cap\partial B_r)\geq 2$ for all $r>R$. We conclude by Proposition 4.1. □

6. Proof of the main theorem

We prove Theorem 2.1 using the results above. The key step is to represent the first variation under normal domain variations by a finite nonnegative measure on ∂U. Under rescaling, its mass tends to zero on compact boundary intervals away from the origin. Compactness and the classification when $\mathbb{R}^2\setminus K$ is disconnected then give a flat blow-down, to which rigidity at infinity applies.

Lemma 6.1 (A finite measure on the boundary)**.** *Let (K,u) be a normalized generalized minimizer on U such that $\mathbb{R}^2\setminus K$ is connected. There is a finite nonnegative Radon measure μ on $\mathbb{R}$ such that*

$$\int_{U\setminus K} T_u : D(e_2\varphi)\,\mathrm{d}x+\int_K(\tau\otimes\tau):D(e_2\varphi)\,\mathrm{d}\mathcal{H}^1=\int_{\mathbb{R}}\varphi(s,0)\,\mathrm{d}\mu(s) \tag{6.1}$$

for every $\varphi\in C^1_c(\mathbb{R}^2)$, where $T_u=|\nabla u|^2I-2\nabla u\otimes\nabla u$ and τ is an approximate unit tangent to K. On each open interval $I\subset\mathbb{R}$ with $I\times\{0\}\subset\partial U\setminus K$,

$$\mu\llcorner I=|\partial_2 u(s,0)|^2\,\mathrm{d}s. \tag{6.2}$$

For $r>0$, let μ_r be the measure associated with the rescaled pair $(K_{0,r},u_{0,r})$ through (6.1). Then

$$\mu_r(A)=\mu(rA)\qquad\text{for every Borel set } A\subset\mathbb{R}. \tag{6.3}$$

Proof. The standard half-ball comparison (cf. [Dea24, Proposition 8.1]) gives

$$E(K,u;B_R)\leq(\pi+2)R\qquad(R>0). \tag{6.4}$$

Extend u by zero to the lower half-plane and denote the extension by $\widehat{u}$. The zero trace gives $\widehat{u}\in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^2\setminus K)$. Since $\mathbb{R}^2\setminus K$ is connected, the separation condition in the definition of generalized minimizer is vacuous.

For $\varphi\in C^1_c(\mathbb{R}^2)$ with $\varphi(s,0)\geq 0$, set

$$X=\varphi e_2,\qquad \Phi_t=\mathrm{Id}+tX.$$

For $|t|\|DX\|_\infty<1$, the map Φ_t is a C^1 diffeomorphism of $\mathbb{R}^2$, and $\Phi_t(\overline{U})\subset\overline{U}$ for $t\geq 0$. The pair

$$K_t=\Phi_t(K),\qquad w_t=(\widehat{u}\circ\Phi_t^{-1})|_U$$

is admissible. The transported zero extension vanishes in the lower half-plane.

Changing variables in the Dirichlet integral and applying the area formula [AFP00, Theorem 2.91] to the length term, minimality shows that the left-hand side of (6.1), denoted by $A(\varphi)$, is nonnegative. If $\varphi|_{\partial U}=0$, applying this inequality to φ and $-\varphi$ gives $A(\varphi)=0$. By linearity, A therefore depends only on the restriction of φ to ∂U. By the Riesz representation theorem [AFP00, Corollary 1.55], the induced positive linear functional on $C^1_c(\mathbb{R})$ is represented by a locally finite nonnegative Radon measure μ.

To prove finiteness on the whole line, take $0\leq\chi\in C^1_c(B_2)$ equal to one on B_1 and set $\chi_R(x)=\chi(x/R)$. Applying (6.1) with $\varphi=\chi_R$ and using (6.4), we obtain

$$\mu([-R,R])\leq\frac{C}{R}\left(\int_{B_{2R}\cap(U\setminus K)}|\nabla u|^2\,\mathrm{d}x+\mathcal{H}^1(K\cap B_{2R})\right)\leq C. \tag{6.5}$$

Letting $R \uparrow \infty$ gives $\mu(\mathbb{R}) < \infty$.

On a boundary interval $I \times \{0\} \subset \partial U \setminus K$, harmonic boundary regularity and $u = 0$ give $\partial_1 u = 0$. Hence

$$(T_u(-e_2)) \cdot e_2 = |\partial_2 u|^2 \quad \text{on } I \times \{0\}.$$

Using $\operatorname{div} T_u = 0$, integration by parts in (6.1), with test functions supported near $I \times \{0\}$ and away from K, yields (6.2). Finally, inserting $\varphi(\cdot/r)$ into (6.1) and changing variables in the bulk and surface terms proves (6.3); the critical factor $r^{-1/2}$ in $u_{0,r}$ leaves no additional factor in front of $\mu(rA)$. □

Lemma 6.2 (Separation in a blow-down)**.** *Let (K, u) be a normalized generalized minimizer on U with $0 \in K$ such that $\mathbb{R}^2 \setminus K$ is connected. For every blow-down (K_∞, u_∞) obtained from* [Dea24, Theorem 2.9]*, $\mathbb{R}^2 \setminus K_\infty$ is disconnected.*

Proof. Let $R_j \uparrow \infty$ give the blow-down, and write $K_j = K_{0,R_j}$ and $u_j = u_{0,R_j}$. The limit is a normalized generalized minimizer, $0 \in K_\infty$, and the functions converge without subtracting constants on components accessible from the lower half-plane, as in [Dea24, Proposition 2.3 and Theorem 2.9].

Let $I \subset \mathbb{R}$ be a compact interval such that $I \times \{0\} \subset \partial U \setminus K_\infty$ and set $d = \operatorname{dist}(I, 0) > 0$. A neighborhood of $I \times \{0\}$ is disjoint from K_j for large j. The measure of Lemma 6.1 satisfies

$$\int_I |\partial_2 u_j(s, 0)|^2 \, \mathrm{d}s = \mu(R_j I) \leq \mu(\{s : |s| \geq d R_j\}) \longrightarrow 0. \tag{6.6}$$

Here the convergence follows from $\mu(\mathbb{R}) < \infty$. By [Dea24, Proposition 2.3(i) and Theorem 2.9], $u_j \to u_\infty$ in C^1 up to $I \times \{0\}$. Combining this with (6.6) and the zero Dirichlet condition gives

$$u_\infty = \partial_2 u_\infty = 0 \quad \text{on } I \times \{0\}.$$

Let Ω_* be the component of $\mathbb{R}^2 \setminus K_\infty$ containing the lower half-plane, and let V be a component of $U \setminus K_\infty$ contained in Ω_*. A path in Ω_* from V to the lower half-plane first meets ∂U at a point outside K_∞. A small ball there is disjoint from K_∞ and its upper half belongs to V. Thus V reaches an interval of the preceding kind. After shrinking the ball, both u_∞ and $\partial_2 u_\infty$ vanish on its diameter. The extension of u_∞ by zero to the lower half-ball is therefore weakly harmonic in the whole ball. Since it vanishes on the lower half-ball, unique continuation gives $u_\infty = 0$ on the upper half-ball and then throughout V.

If $\mathbb{R}^2 \setminus K_\infty$ were connected, every upper component would be of this kind, so $u_\infty = 0$ on $U \setminus K_\infty$. Thus (K_∞, u_∞) is elementary. Since $0 \in K_\infty \cap \partial U$, [Dea24, Theorem 2.13] gives $K_\infty = \partial U$, contradicting the connectedness of $\mathbb{R}^2 \setminus K_\infty$. □

Proof of Theorem 2.1. Translate a point of $K \cap \partial U$ to 0. To prove $K = \partial U$, suppose by contradiction that $K \neq \partial U$. Then Theorem 5.1 implies that $\mathbb{R}^2 \setminus K$ is connected. Extract a blow-down (K_∞, u_∞) using [Dea24, Theorem 2.9]. By Lemma 6.2, $\mathbb{R}^2 \setminus K_\infty$ is disconnected. Since $0 \in K_\infty$, Theorem 5.1 gives $K_\infty = \partial U$. Therefore $K_{0,R_j} \to \partial U$ locally in the Hausdorff distance for some $R_j \uparrow \infty$, and Theorem 3.2 yields $K = \partial U$, a contradiction. Since $K = \partial U$ is connected, [Dea24, Proposition 2.15] gives $\nabla u = 0$ in U, and hence u is constant. □

Acknowledgements

Funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany's Excellence Strategy EXC 2044/2 – 390685587, Mathematics Münster: Dynamics–Geometry–Structure.

The author used ChatGPT Astra to complete the proof of Theorem 2.1, building on his previous progress on the Dirichlet boundary problem. The author had also established further partial results, omitted here because they follow as corollaries of this result. The finiteness argument for the measure in Lemma 6.1, the separation argument for blow-downs in Lemma 6.2, and the concluding proof of Theorem 2.1 were developed with the assistance of ChatGPT Astra. These are the arguments in Section 6; they build on the rigidity and topological results established earlier in the manuscript and on the author's previous work [Dea24]. ChatGPT was also used during the

subsequent revision and simplification of the mathematical arguments and their exposition. The author has revised the manuscript and takes full responsibility for its mathematical content and references.

References

[AFP97] L. Ambrosio, N. Fusco, and D. Pallara, *Partial regularity of free discontinuity sets, II*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **24** (1997), no. 1, 39–62. Numdam.

[AFP99] L. Ambrosio, N. Fusco, and D. Pallara, *Higher regularity of solutions of free discontinuity problems*, Differential Integral Equations **12** (1999), no. 4, 499–520. doi:10.57262/die/1367267005.

[AFP00] L. Ambrosio, N. Fusco, and D. Pallara, *Functions of Bounded Variation and Free Discontinuity Problems*, Oxford University Press, 2000. doi:10.1093/oso/9780198502456.001.0001.

[AP97] L. Ambrosio and D. Pallara, *Partial regularity of free discontinuity sets, I*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **24** (1997), no. 1, 1–38. Numdam.

[AM19] J. Andersson and H. Mikayelyan, *Regularity up to the crack-tip for the Mumford–Shah problem*, preprint (2015), revised 2019. arXiv:1512.05094v5.

[Bon96] A. Bonnet, *On the regularity of edges in image segmentation*, Ann. Inst. H. Poincaré Anal. Non Linéaire **13** (1996), no. 4, 485–528. doi:10.1016/S0294-1449(16)30111-1.

[BD01] A. Bonnet and G. David, *Cracktip is a global Mumford–Shah minimizer*, Astérisque **274** (2001), vi+259 pp. doi:10.24033/ast.512.

[BFM08] B. Bourdin, G. A. Francfort, and J.-J. Marigo, *The variational approach to fracture*, J. Elasticity **91** (2008), no. 1–3, 5–148. doi:10.1007/s10659-007-9107-3.

[BL14] D. Bucur and S. Luckhaus, *Monotonicity formula and regularity for general free discontinuity problems*, Arch. Ration. Mech. Anal. **211** (2014), no. 2, 489–511. doi:10.1007/s00205-013-0671-3.

[CL90] M. Carriero and A. Leaci, *Existence theorem for a Dirichlet problem with free discontinuity set*, Nonlinear Anal. **15** (1990), no. 7, 661–677. doi:10.1016/0362-546X(90)90006-3.

[DMMS92] G. Dal Maso, J.-M. Morel, and S. Solimini, *A variational method in image segmentation: existence and approximation results*, Acta Math. **168** (1992), no. 1–2, 89–151. doi:10.1007/BF02392977.

[Dav96] G. David, *C^1-arcs for minimizers of the Mumford–Shah functional*, SIAM J. Appl. Math. **56** (1996), no. 3, 783–888. doi:10.1137/S0036139994276070.

[Dav05] G. David, *Singular Sets of Minimizers for the Mumford–Shah Functional*, Progress in Mathematics **233**, Birkhäuser, Basel, 2005. doi:10.1007/b137039.

[DL02] G. David and J.-C. Léger, *Monotonicity and separation for the Mumford–Shah problem*, Ann. Inst. H. Poincaré Anal. Non Linéaire **19** (2002), no. 5, 631–682. doi:10.1016/S0294-1449(02)00097-5.

[Dea24] F. Deangelis, *Boundary regularity for the Mumford–Shah functional with Dirichlet boundary conditions*, preprint (2024), arXiv:2404.07162v1.

[Dea26] F. Deangelis, *Solution of the Mumford–Shah conjecture*, preprint (2026), arXiv:2609.26732v2.

[DGCL89] E. De Giorgi, M. Carriero, and A. Leaci, *Existence theorem for a minimum problem with free discontinuity set*, Arch. Rational Mech. Anal. **108** (1989), no. 3, 195–218. doi:10.1007/BF01052971.

[LF13] C. De Lellis and M. Focardi, *Density lower bound estimates for local minimizers of the 2d Mumford–Shah energy*, Manuscripta Math. **142** (2013), no. 1–2, 215–232. doi:10.1007/s00229-012-0601-y.

[DLF13] C. De Lellis and M. Focardi, *Higher integrability of the gradient for minimizers of the 2d Mumford–Shah energy*, J. Math. Pures Appl. (9) **100** (2013), no. 3, 391–409. doi:10.1016/j.matpur.2013.01.006.

[DLF25] C. De Lellis and M. Focardi, *The Regularity Theory for the Mumford–Shah Functional on the Plane*, EMS Monographs in Mathematics, EMS Press, 2025. doi:10.4171/EMM/13. Numbered references refer to the author manuscript, arXiv:2308.14660v2.

[DLFG21] C. De Lellis, M. Focardi, and S. Ghinassi, *Endpoint regularity for 2d Mumford–Shah minimizers: on a theorem of Andersson and Mikayelyan*, J. Math. Pures Appl. (9) **155** (2021), 83–110. doi:10.1016/j.matpur.2021.05.001.

[DPF14] G. De Philippis and A. Figalli, *Higher integrability for minimizers of the Mumford–Shah functional*, Arch. Ration. Mech. Anal. **213** (2014), no. 2, 491–502. doi:10.1007/s00205-014-0729-x.

[Fis22] J. Fischer, *A monotonicity formula for minimizers of the Mumford–Shah functional in 2d and a sharp lower bound on the energy density*, preprint (2022). arXiv:2203.13177v1.

[FM98] G. A. Francfort and J.-J. Marigo, *Revisiting brittle fracture as an energy minimization problem*, J. Mech. Phys. Solids **46** (1998), no. 8, 1319–1342. doi:10.1016/S0022-5096(98)00034-9.

[LL26] C. Labourie and A. Lemenant, *Finite number of traces for Mumford–Shah minimizers in dimension 2*, C. R. Math. **364** (2026), 301–310. doi:10.5802/crmath.826.

[Leg99] J.-C. Léger, *Flatness and finiteness in the Mumford–Shah problem*, J. Math. Pures Appl. (9) **78** (1999), no. 4, 431–459. doi:10.1016/S0021-7824(99)00019-7.

[Lem11] A. Lemenant, *Regularity of the singular set for Mumford–Shah minimizers in $\mathbb{R}^3$ near a minimal cone*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **10** (2011), no. 3, 561–609. doi:10.2422/2036-2145.2011.3.03.

[Lem15] A. Lemenant, *A rigidity result for global Mumford–Shah minimizers in dimension three*, J. Math. Pures Appl. (9) **103** (2015), no. 4, 1003–1023. doi:10.1016/j.matpur.2014.10.002.

[LM18] A. Lemenant and H. Mikayelyan, *Stationarity of the crack-front for the Mumford–Shah problem in 3D*, J. Math. Anal. Appl. **462** (2018), no. 2, 1555–1569. doi:10.1016/j.jmaa.2018.02.055.

[MS89] D. Mumford and J. Shah, *Optimal approximations by piecewise smooth functions and associated variational problems*, Comm. Pure Appl. Math. **42** (1989), no. 5, 577–685. doi:10.1002/cpa.3160420503.

[Zha25] Y. R.-Y. Zhang, *A growth estimate for the planar Mumford–Shah minimizers at a tip point: An alternative proof of David–Léger*, Ann. Fenn. Math. **50** (2025), no. 1, 145–156. doi:10.54330/afm.160044.

Applied Mathematics Münster, University of Münster, Einsteinstrasse 62, 48149 Münster, Germany
Email address: francesco.deangelis@uni-muenster.de