GLOBAL MINIMALITY OF LAMELLAE FOR ISOTROPIC OHTA–KAWASAKI AND YUKAWA ENERGIES

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Abstract. We establish a sharp comparison principle for the sharp-interface Ohta–Kawasaki energy and its Yukawa-screened counterpart in dimensions $2 \le d \le 7$, for $m = 0$, and under periodic and zero-flux Neumann boundary conditions. Whenever the finite optimal stripe width is compatible with the domain, we prove global minimality of periodic lamellae and characterize all minimizers. The proof combines radial comparison kernels, a sharp Fourier bound, and rigidity for a directional translation inequality in BV. The kernels are positive measures up to dimension five; in dimensions six and seven, distributional lifting and monotonicity of the averaged translation defect yield the same sharp comparison. In the Appendix, we also prove that no finite positive measure with the prescribed Fourier multiplier exists in dimensions $d \ge 6$, and explain why the argument does not directly extend to $d \ge 8$.

AI usage statement. The central ideas of the proof were produced by OpenAI's ChatGPT-6 Astra. Specifically, the model proposed the construction of the auxiliary measure (Section 5), the comparison inequality (Section 6), and the reduction of the Neumann case to the periodic one (Section 6). The model was given a document (Ideas.pdf) by the author summarizing his previous unsuccessful approaches to the problem and why they failed, together with the preprint [8]. It was asked to treat a simplified test case, after which the author requested a global comparison against all admissible competitors. The author verified all arguments, rewrote and completed the proofs, structured the paper, wrote the introduction, the discussion of the literature and the overview of the proof (Subsection 3.3), and Appendix A. LLM assistance was also used in drafting the text. A detailed account of the original interaction together with Ideas.pdf is available as an ancillary file accompanying the arXiv version of this paper. The extension to $d = 6, 7$ arose from a later, informal exchange with the same model, prompted by the author's question whether positivity of the measure could be relaxed to a controlled amount of negativity. The model then observed that it suffices for the pairing with the geometric defect to be non-negative. The author takes full responsibility for the content of this paper.

1. Introduction

The competition between interfacial energy and nonlocal repulsion is a fundamental mechanism underlying pattern formation and symmetry breaking. Interfacial energy favours the formation of large domains, whereas repulsive interactions favour their subdivision into smaller regions. The balance between these effects can select a characteristic length scale and give rise to ordered patterns, including droplets, cylinders, and lamellae. The formation of layered patterns selects a preferred direction, even when the interfacial energy and the interaction kernel are isotropic. A central example is the Ohta–Kawasaki model, introduced in [46] to describe microphase separation in diblock copolymer melts.

A diblock copolymer consists of two chemically distinct polymer blocks joined by a covalent bond. The incompatibility of the two blocks favours segregation, while their connectivity prevents separation into macroscopically large pure phases. The relative length of the two blocks determines the morphology. For symmetric copolymers, lamellar phases of alternating layers are observed experimentally and predicted by mean-field theory, whereas asymmetric compositions lead to cylindrical, spherical or gyroid structures. The Ohta–Kawasaki functional describes this mechanism through a local phase-separation energy coupled to a nonlocal term of Coulomb type (see the derivation from a mean-field description in [12]).

In the standard formulation, on a domain $\Omega \subset \mathbb{R}^d$, the diffuse-interface energy takes the form

$$\int_\Omega \left(\frac{\varepsilon}{2} |\nabla u|^2 + \frac{1}{\varepsilon} W(u) \right) dx + \gamma \big\langle u - m, (-\Delta)^{-1}(u - m) \big\rangle_{L^2(\Omega)}.$$

Here u is an order parameter describing the local composition, W is a suitable double-well potential, ε measures the thickness of the transition layers, and

$$m = \frac{1}{|\Omega|}\int_\Omega u\,dx$$

is the prescribed average composition, which reflects the relative length of the two blocks; the symmetric case corresponds to $m = 0$. In the sharp-interface description, the order parameter takes values in $\{-1, 1\}$ and the local contribution becomes an interfacial energy. Absorbing the surface-tension constant into $J > 0$, the corresponding functional is

$$J|Du|(\Omega) + \gamma\big\langle u - m, (-\Delta)^{-1}(u - m)\big\rangle_{L^2(\Omega)}, \qquad u \in BV(\Omega; \{-1, 1\}).$$

A closely related class of models is obtained by replacing the Coulomb interaction with a screened Coulomb, or Yukawa, interaction. At the operator level, this amounts to replacing the inverse Laplacian by

$$R_\lambda = (-\Delta + \lambda^2)^{-1}, \qquad \lambda > 0.$$

Screened repulsive interactions are particularly relevant to charged colloidal suspensions and models of protein solutions. In these systems, short-range attraction can compete with electrostatic repulsion screened by the surrounding medium, leading to aggregation and spatially modulated structures (see [54, 7]).

The present paper studies the sharp-interface energy with zero preferred composition, namely

$$\mathcal{E}_{\lambda,J,\gamma}(u) = J|Du|(\Omega) + \gamma\langle u, R_\lambda u\rangle_{L^2(\Omega)},$$

on $\Omega = \mathbb{T}^d_L$ with periodic boundary conditions, or on $\Omega = (0, L)^d$ with homogeneous Neumann boundary conditions. For $\lambda = 0$, we work with zero-average configurations and interpret R_0 as the inverse Laplacian on that subspace. For $\lambda > 0$, we retain the constant mode of the resolvent and impose no average constraint: zero is the preferred composition, while the actual average of u is free to vary. The constant mode then contributes with a term penalizing an imbalance between the two phases without forbidding it (see Corollary 6.3). In the compatible case, minimizers are lamellae and therefore automatically have zero average. Our interest is in the exact selection of lamellar configurations: periodically alternating parallel layers whose width is determined by the competition between the two terms.

1.1. **Overview of the literature.** Early rigorous results on periodic pattern selection were obtained in one dimension. Müller [44] proved periodicity of minimizers for a related singularly perturbed variational problem. For the diblock copolymer functional, Ren and Wei [47, 48] studied local minimizers and the emergence of increasingly fine periodic oscillations in singular limits. We also refer to [24] for the proof of periodicity of minimizers for a one-dimensional liquid drop model.

In higher dimensions, the difficulty is of a different nature. The isotropy of the interfacial energy and interaction kernel supplies no preferred orientation in the ambient space. Proving that minimizers are lamellar requires excluding arbitrary multidimensional competitors, including configurations with curved or rearranged layers. A fundamental result of Alberti, Choksi, and Otto [4] shows that the energy of minimizers of the sharp-interface Ohta–Kawasaki functional is approximately uniformly distributed over subdomains that are large compared with the intrinsic length scale (see also [52]).

Concerning critical points and local minimality, Ren and Wei [49] studied wriggled lamellar solutions and their stability. Choksi and Sternberg [13] derived the first and second variations of the sharp-interface nonlocal isoperimetric functional. Acerbi, Fusco, and Morini [2] proved that strict positivity of the second variation implies quantitative local minimality with respect to L^1 perturbations (see also [36] for the Neumann case and [23] for related anisotropic local and global minimality results for a lamella). Cristoferi [17] constructed locally minimizing periodic configurations near strictly stable periodic constant-mean-curvature hypersurfaces, and Acerbi, Chen, and Choi [1] obtained further criteria for the local minimality of lamellae.

Global minimality has been established in particular geometric or parameter regimes. Sternberg and Topaloglu [53] proved that, on the flat two-dimensional torus, a single lamellar region is globally minimizing for sufficiently weak nonlocal interaction and an appropriate range of volume fractions. Morini and Sternberg [43] proved that minimizers in sufficiently thin domains are striped, with transitions in the optimal number of layers as the interaction strength varies. In both cases, the domain geometry or the closeness to the local isoperimetric problem allows a complete description of minimizers.

A complementary literature concerns the small-mass and dilute regimes. The geometry of individual droplets and the minimality of balls at small mass are studied in [14, 38, 39, 35, 6, 22, 9]. Effective energies, droplet distributions, and confinement are instead analyzed in [45, 10, 11, 31, 32, 40, 41, 3, 25].

An analogous competition arises in spin systems with short-range ferromagnetic and long-range antiferromagnetic interactions, where lamellar patterns are expected to minimize the energy in suitable regimes near the ferromagnetic transition, that is, when J approaches a critical value J_c. For these systems and related mean-field models, reflection positivity and localization have led to periodic-ordering results [27, 28], asymptotic optimality of stripes [29], and exact striped ground states near the transition for sufficiently rapidly decaying interactions [30].

In the continuum setting, the main line of results is based on a blow-down near the critical interaction strength. As $J \to J_c^-$, the optimal period diverges. After rescaling lengths by this period, the energy Γ-converges to a limiting functional whose structure encodes the competition between the local and the nonlocal terms. For energies with an ℓ^1-anisotropic perimeter and power-law interactions with $p > 2d$, Goldman and Runa [33] identified such a limiting functional and showed that its only finite-energy states are striped. Daneri and Runa [19] then transferred this rigidity back to the original functional for J close to J_c and $p \geq d+2$. For the relevant exponents the limiting functional penalizes corners infinitely, which forces interfaces to be flat and aligned with the preferred directions. This yields exact periodic stripe formation (see also [37]). Related results extend to diffuse-interface models [18]. For Yukawa-type interactions with ℓ^1 anisotropy [20], the corresponding limiting penalty acts as a hard-core constraint between neighbouring interfaces. For the screened energy considered here, the critical value is $J_c = \gamma\lambda^{-3}$. If h_* denotes the optimal stripe width, the relation $J = \gamma j_\lambda(h_*)$, with j_λ defined in (3.5), gives $\lambda h_* \to \infty$ if and only if $J/J_c \to 1^-$. Thus the analogous near-critical regime corresponds to stripe widths much larger than the screening length. Finally, Daneri and Runa [21] obtained the first rigorous proof of spontaneous lamellar pattern formation for fully isotropic continuum energies. For power-law kernels with exponent $p \geq d+3$, the limiting functional penalizes the curvature of interfaces infinitely, so its finite-energy states are lamellar. As a consequence, for J sufficiently close to J_c, minimizers are close in L^1, after rescaling, to periodic lamellae, and a stability argument around stripes then upgrades this closeness to exact global minimality of periodic lamellae. In the isotropic screened case considered here, the near-critical regime is expected to be accessible, at least in spirit, by the methods of [21].

Despite this progress, to the best of our knowledge, a sharp lower bound by the optimal stripe energy density has remained unavailable for the isotropic Ohta–Kawasaki and Yukawa energies throughout the parameter range admitting a finite optimal stripe width. The question addressed here is whether such a comparison can be established without a near-critical assumption or a thin-domain reduction, and whether its equality cases can be classified when the optimal width is compatible with the domain.

1.2. **Overview of the results and proof strategy.** In the present paper we establish a sharp lower bound for the screened and unscreened energies in dimensions $2 \leq d \leq 7$, under periodic and zero-flux Neumann boundary conditions. The comparison holds for every admissible binary BV configuration and yields the global minimality of periodic lamellae whenever their optimal width is compatible with the domain. We also characterize all equality cases, obtaining a complete description of the global minimizers under the corresponding compatibility condition.

The relevant length scale is the width h_* that minimizes the energy density among periodic stripes. It is characterized by

$$J = \gamma j_\lambda(h_*),$$

where j_λ is defined in (3.5). In the unscreened case, this gives the explicit scale

$$h_* = \left(\frac{12J}{\gamma}\right)^{1/3}.$$

In the screened case, a finite optimal width exists precisely when $0 < J/\gamma < \lambda^{-3}$, and it is unique. Our main estimate is

$$\mathcal{E}^{\mathrm{b}}_{\lambda,J,\gamma}(u) \geq \gamma\sigma_\lambda(h_*)L^d, \qquad \mathrm{b} \in \{\mathrm{per}, N\},$$

where $\gamma\sigma_\lambda(h_*)$ is the optimal stripe energy density, given by (3.6). Thus the value obtained by optimizing the one-dimensional stripe family provides a lower bound for the full multidimensional problem.

On the torus $\mathbb{T}^d_L$, the lower bound is attained if and only if there exists $a \in \mathbb{Z}^d \setminus \{0\}$ such that $|a| = L/(2h_*)$. Whenever this condition holds, all global minimizers are of the form

$$u(x) = \operatorname{sign}\cos\left(\frac{2\pi}{L} a \cdot x + \varphi\right), \qquad |a| = \frac{L}{2h_*}, \quad \varphi \in \mathbb{R}.$$

The admissible directions therefore include oblique stripes, according to the geometry of the frequency lattice. For the Neumann problem on $Q_L = (0, L)^d$, equality holds precisely when $n = L/h_*$ is a positive integer, and the complete class of global minimizers is

$$u(x) = \pm \operatorname{sign}\cos\left(\frac{\pi n x_j}{L}\right), \qquad j \in \{1, \dots, d\}.$$

The boundary phase intervals have width $h_*/2$, while consecutive interfaces are separated by h_*. In the screened case, no constraint on the average is imposed: attainment of the sharp bound itself forces the two phases to have equal volume. The precise statements are given in Theorems 3.1 and 3.2.

The bound comes from an auxiliary radial measure or distribution μ, which acts as a *dual certificate*. It is constructed so that the two competing terms of the energy can be compared without loss along stripes. The unscreened case $\lambda = 0$ is treated directly by the same approach. See Subsection 3.3 for a detailed explanation.

The comparison starts from the BV translation inequality (see [26, 42]), which bounds translation differences by the directional variation. In dimensions $d \leq 5$, averaging this inequality against a positive measure μ bounds the averaged translations by the perimeter, with coefficient given by the first absolute directional moment of μ. In dimensions six and seven, μ is instead a distribution. The comparison remains valid because the spherical average of the translation defect is nonnegative and nondecreasing, and the distribution has a nonnegative action on this class of radial functions. A one-sided Fourier bound $\widehat{\mu} \leq \widehat{G}_\lambda$ provides the corresponding comparison with the nonlocal term. Together, these yield an exact identity: the energy excess over $\gamma\, \sigma_\lambda(h_*) L^d$ is the sum of a geometric defect and a spectral defect, both nonnegative (see (6.4)).

The properties of μ are guided by requiring both defects to vanish on the optimal periodic stripes. The translation defect vanishes for translations of length at most the stripe width, motivating the support condition $\operatorname{supp}\mu \subset \overline{B}_{h_*}$. Stripes have Fourier support on the odd harmonics of their period, so vanishing of the spectral defect requires $\widehat{\mu} = \widehat{G}_\lambda$ at those frequencies. We first construct the one- and two-dimensional profiles and then obtain the higher-dimensional objects through the Fourier dimension walk, preserving the radial Fourier multiplier (Proposition 5.1). In this respect, the method resembles the Delsarte and Cohn–Elkies bounds [15, 16, 55]: constraints in physical and Fourier space produce a bound attained by the conjectured optimizer. Here the certificate is compactly supported and satisfies a one-sided Fourier bound, while positivity is required only through its action on the geometric defect. In the equality case, vanishing of the geometric defect forces complete parallel interfaces separated by at least h_*. The vanishing spectral defect then fixes every layer width to be exactly h_*. Even reflection transfers the comparison and its equality cases to the Neumann problem, where the reflection symmetries determine the stripe direction and phase.

The dimensional restriction reflects the regularity of these comparison kernels. The one-dimensional profile has a derivative jump at the edge of its support, and each lift by two dimensions introduces one radial derivative. Along the odd-dimensional chain, this produces a jump in the density in dimension three and a surface measure in dimension five. Positive finite measures with the prescribed Fourier multiplier cannot exist in dimensions $d \geq 6$, as shown in Appendix A. Nevertheless, one further lift preserves nonnegativity of the pairing with the geometric defect, due to the nonnegativity and monotonicity of its spherical average. This extends the comparison to $d \leq 7$. Further lifts introduce second and higher radial derivatives, whose contributions are not controlled by these properties alone. Extending the argument to higher dimensions would likely require additional quantitative information on the geometric defect, beyond nonnegativity and monotonicity, possibly combined with a different construction of the comparison measure. We do not pursue this here.

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2. Notations and preliminaries

Geometric notation. The symbols $|x|$ and $x \cdot y$ denote the Euclidean norm and scalar product, and $e_1, \dots, e_d$ are the coordinate unit vectors. For $r > 0$ and $x \in \mathbb{R}^d$, we set

$$B_r(x) := \{y \in \mathbb{R}^d : |y - x| < r\}, \qquad B_r := B_r(0), \qquad \mathbb{S}^{d-1} := \partial B_1.$$

We write B_r^d when the ambient dimension needs to be specified. For $e \in \mathbb{S}^{d-1}$, the orthogonal complement of e is $e^\perp := \{y \in \mathbb{R}^d : y \cdot e = 0\}$.

We denote by $\mathcal{L}^d$ the Lebesgue measure on $\mathbb{R}^d$ and by $\mathcal{H}^k$ the k-dimensional Hausdorff measure, normalized to agree with k-dimensional Lebesgue measure on affine k-planes. For a measurable set E, we write $|E| := \mathcal{L}^d(E)$, $\mathbf{1}_E$ for its characteristic function, and $E \triangle F := (E \setminus F) \cup (F \setminus E)$ for symmetric difference. Equalities between measurable sets or functions are understood up to Lebesgue-null sets unless otherwise stated. We set $\omega_{d-1} := \mathcal{H}^{d-1}(\mathbb{S}^{d-1})$. Given a set $E \subset \mathbb{R}^d$, co$E$ indicates its convex hull in $\mathbb{R}^d$.

For $\ell > 0$, the flat torus $\mathbb{T}^d_\ell := \mathbb{R}^d/(\ell\mathbb{Z})^d$ is endowed with the measure induced by Lebesgue measure, so $|\mathbb{T}^d_\ell| = \ell^d$. A function on $\mathbb{T}^d_\ell$ is identified with its ℓ-periodic lift to $\mathbb{R}^d$, and we use the same symbol for both. Integration over the torus means integration over a fundamental cell, for instance $[0,\ell)^d$. Translations on the torus are understood through this periodic lift. If Ω is a measurable domain or a torus with $0 < |\Omega| < \infty$, the average of $u \in L^1(\Omega)$ is

$$\overline{u} := \frac{1}{|\Omega|} \int_\Omega u \, \mathrm{d}x.$$

The domain of the average will always be clear from the context. The pairing $\langle u, v \rangle$ denotes $\int_\Omega uv \, \mathrm{d}x$ for real-valued L^2 functions. We use the usual Sobolev space $H^1(\Omega)$ of L^2 functions with first weak derivatives in L^2; on a torus, functions and test functions are periodic.

Measures, restrictions, and projections. For a Radon measure μ and a Borel set A, its restriction is $(\mu \llcorner A)(B) := \mu(A \cap B)$. The support $\operatorname{supp}\mu$ of a positive measure is the set of points whose every open neighborhood has positive μ-measure. For a Borel map $P : \mathbb{R}^d \to \mathbb{R}^k$, the pushforward $P_\#\mu$ is defined by $P_\#\mu(A) := \mu(P^{-1}(A))$. Equivalently,

$$\int_{\mathbb{R}^k} f(y) \, \mathrm{d}(P_\#\mu)(y) = \int_{\mathbb{R}^d} f(Px) \, \mathrm{d}\mu(x)$$

for every nonnegative Borel function f and every positive measure μ. When P is a coordinate projection, $P_\#\mu$ is also called the corresponding marginal of μ.

A positive measure is *radial* if $R_\#\mu = \mu$ for every orthogonal transformation R, and *even* if $\mu(-A) = \mu(A)$ for every Borel set A. Radial measures are even.

Functions of bounded variation and perimeter. For an open set $\Omega \subset \mathbb{R}^d$, the space $BV(\Omega)$ consists of the functions $u \in L^1(\Omega)$ whose distributional gradient Du is a finite $\mathbb{R}^d$-valued Radon measure. Thus

$$\int_\Omega u \operatorname{div}\phi \, \mathrm{d}x = -\int_\Omega \phi \cdot \mathrm{d}Du \qquad \text{for every } \phi \in C^1_c(\Omega;\mathbb{R}^d).$$

We denote by $|Du|$ its total variation measure and equip $BV(\Omega)$ with the norm $\|u\|_{L^1(\Omega)} + |Du|(\Omega)$. The notation BV_{loc} means that these conditions hold locally. On a torus, the definition uses smooth periodic test fields. In particular, the variation on a torus includes any jump across the identified faces of a fundamental cell. On an open cube, $|Du|(Q_L)$ is the relative variation inside the cube and contains no boundary contribution from its faces.

On the bounded domains considered here, a measurable set E has finite perimeter in Ω if $\mathbf{1}_E \in BV(\Omega)$, and $P(E;\Omega) := |D\mathbf{1}_E|(\Omega)$. We write $BV(\Omega;\{-1,1\})$ for the BV functions taking only the values -1 and 1 almost everywhere. For such a function, with $E := \{u = 1\}$, one has

$$u = 2\mathbf{1}_E - 1, \qquad Du = 2D\mathbf{1}_E, \qquad |Du|(\Omega) = 2P(E;\Omega).$$

If $\partial^* E$ denotes the reduced boundary of E, the perimeter measure satisfies $|D\mathbf{1}_E| = \mathcal{H}^{d-1}\llcorner(\partial^* E \cap \Omega)$.

When a specific representative of a measurable set in $\mathbb{R}^d$ is needed, its density-one representative is

$$E^{(1)} := \left\{ x \in \mathbb{R}^d : \lim_{r \downarrow 0} \frac{|E \cap B_r(x)|}{|B_r|} = 1 \right\}.$$

Fourier conventions. On $T = \mathbb{T}^d_\ell$, with volume $V = \ell^d$, the frequency lattice is $\mathcal{K}_\ell := 2\pi\mathbb{Z}^d/\ell$. For $U \in L^2(T)$ we use the normalized coefficients

$$\widehat{U}(q) := \frac{1}{V}\int_T U(x)e^{-iq\cdot x}\,\mathrm{d}x, \qquad q \in \mathcal{K}_\ell.$$

With this convention,

$$U = \sum_{q\in\mathcal{K}_\ell} \widehat{U}(q)e^{iq\cdot x} \quad \text{in } L^2(T), \qquad \int_T |U|^2\,\mathrm{d}x = V\sum_{q\in\mathcal{K}_\ell} |\widehat{U}(q)|^2.$$

In particular, $\widehat{U}(0) = \overline{U}$, so zero mean is equivalent to the absence of the constant Fourier mode. For real-valued U, one has $\widehat{U}(-q) = \overline{\widehat{U}(q)}$.

For a finite measure μ on $\mathbb{R}^d$, we instead use the un-normalized Euclidean Fourier transform

$$\widehat{\mu}(\xi) := \int_{\mathbb{R}^d} e^{-i\xi\cdot z}\,\mathrm{d}\mu(z), \qquad \xi \in \mathbb{R}^d.$$

If $d\mu = k(z)\,\mathrm{d}z$, we also write $\widehat{k} = \widehat{\mu}$. For an even positive measure, $\widehat{\mu}$ is real-valued; for a radial measure it depends only on $|\xi|$. If $P : \mathbb{R}^d \to \mathbb{R}^k$ is a coordinate projection, then

$$\widehat{P_\#\mu}(\eta) = \widehat{\mu}(P^{\mathsf{T}}\eta), \qquad \eta \in \mathbb{R}^k,$$

where $P^{\mathsf{T}}\eta$ inserts zero entries in the omitted coordinates. This identity follows directly from the definition of pushforward.

For a finite positive measure μ and a periodic function $U \in L^2(T)$, convolution is defined on the periodic lift by

$$(\mu * U)(x) := \int_{\mathbb{R}^d} U(x - z)\,\mathrm{d}\mu(z).$$

It is well-defined almost everywhere and satisfies

$$\|\mu * U\|_{L^2(T)} \le \mu(\mathbb{R}^d)\|U\|_{L^2(T)}, \qquad \widehat{\mu * U}(q) = \widehat{\mu}(q)\widehat{U}(q).$$

If μ is even, $U(x - z)$ may equivalently be replaced by $U(x + z)$. For real-valued U and even positive μ, Parseval's identity therefore gives

$$\langle U, \mu * U\rangle = V\sum_{q\in\mathcal{K}_\ell} \widehat{\mu}(q)|\widehat{U}(q)|^2.$$

One-dimensional functions and striped sets. We call a binary function on $\mathbb{R}^d$ *one-dimensional* or *striped* if there exist $\nu \in \mathbb{S}^{d-1}$ and $g \in BV_{\mathrm{loc}}(\mathbb{R}; \{-1, 1\})$ such that

$$U(x) = g(x\cdot\nu) \qquad \text{for almost every } x \in \mathbb{R}^d.$$

For a function on a torus this definition refers to its periodic lift. The positive phase is then $\{x : x\cdot\nu \in I\}$, where $I := \{t : g(t) = 1\}$. If g is nonconstant, its jumps describe complete parallel interfaces, and the regions between consecutive interfaces are the layers or stripes. The distance between consecutive interfaces is the layer width. Constant functions are included as degenerate one-dimensional functions.

By *periodic stripes of width* $h > 0$ we mean the profiles

$$U_{h,\nu,\varphi}(x) := \operatorname{sign}\cos\Big(\frac{\pi}{h}x\cdot\nu + \varphi\Big), \qquad \nu \in \mathbb{S}^{d-1}, \quad \varphi \in \mathbb{R},$$

or their positive-phase sets. Their interfaces are

$$x\cdot\nu = h\left(k + \frac{1}{2} - \frac{\varphi}{\pi}\right), \qquad k \in \mathbb{Z}.$$

Both phases have layer width h, whereas the full period in the normal direction is $2h$: translation by $h\nu$ exchanges the phases, and translation by $2h\nu$ preserves them. Such a profile defines a function on $\mathbb{T}^d_\ell$ precisely when

$$\ell\nu_i \in 2h\mathbb{Z} \quad (i = 1, \dots, d), \qquad \text{equivalently} \qquad \frac{\ell\nu}{2h} \in \mathbb{Z}^d.$$

We will use the word "width" always referring to the distance between consecutive interfaces in the full profile.

3. Main results

3.1. **The functionals and the admissible classes.** Let $Q_L := (0,L)^d$ and $\mathbb{T}^d_L := \mathbb{R}^d/(L\mathbb{Z})^d$, with $L>0$.

For $\lambda \geq 0$ and either $\Omega = Q_L$ or $\Omega = \mathbb{T}^d_L$, we define the admissible classes

$$\mathcal{A}_\lambda(\Omega) = \begin{cases} BV(\Omega;\{-1,1\}), & \lambda > 0, \\ \{u \in BV(\Omega;\{-1,1\}) : \int_\Omega u = 0\}, & \lambda = 0. \end{cases} \tag{3.1}$$

For $\lambda > 0$, let R^{per}_λ and R^N_λ denote the inverses of the operator $(-\Delta + \lambda^2)$ with periodic and zero-flux Neumann boundary conditions, respectively. We remark that in the Neumann case, $w = R^N_\lambda u$ means

$$\int_{Q_L} (\nabla w \cdot \nabla \varphi + \lambda^2 w\varphi)\,\mathrm{d}x = \int_{Q_L} u\varphi\,\mathrm{d}x \qquad \text{for every } \varphi \in H^1(Q_L). \tag{3.2}$$

The definition for R^{per}_λ uses periodic H^1 functions and test functions instead.

When $\lambda = 0$, the inverse is defined only if the average of u on Ω is zero, and the solution is normalized imposing $\int_\Omega w = 0$. On the other hand, if $\lambda > 0$, then we get $\overline{w} = \overline{u}/\lambda^2$, and no additional normalization is imposed.

For $\mathrm{b} \in \{\mathrm{per}, N\}$ then the screened and unscreened Ohta-Kawasaki energies are

$$N^{\mathrm{b}}_\lambda(u) = \langle u, R^{\mathrm{b}}_\lambda u\rangle, \qquad \mathcal{E}^{\mathrm{b}}_{\lambda,J,\gamma}(u) = J|Du|(\Omega) + \gamma N^{\mathrm{b}}_\lambda(u), \qquad J,\gamma > 0. \tag{3.3}$$

In particular, it holds that

$$N^{\mathrm{b}}_\lambda(u) = \int_\Omega \left(|\nabla w|^2 + \lambda^2 w^2\right)\mathrm{d}x. \tag{3.4}$$

For $h > 0$ we define the following quantities (they will be motivated by the computations in Section 4)

$$j_\lambda(h) := \begin{cases} \dfrac{\sinh(\lambda h) - \lambda h}{\lambda^3(1+\cosh(\lambda h))}, & \lambda > 0, \\ \dfrac{h^3}{12}, & \lambda = 0, \end{cases} \tag{3.5}$$

$$\sigma_\lambda(h) := \begin{cases} \dfrac{\cosh(\lambda h) - 1}{\lambda^2(1+\cosh(\lambda h))}, & \lambda > 0, \\ \dfrac{h^2}{4}, & \lambda = 0. \end{cases} \tag{3.6}$$

Assuming $\gamma = 1$, the quantity $j_\lambda(h)$ represents the parameter J in (3.3) such that periodic stripes of width h are optimal among periodic striped competitors and $\sigma_\lambda(h)$ is the energy density associated to such optimal periodic stripes.

3.2. **Periodic and Neumann minimizers.** We now present the main results of this paper, both in the screened and unscreened case as well as in the periodic and Neumann case.

Theorem 3.1 (Periodic comparison and equality cases)**.** *Let $2 \leq d \leq 7$, $\lambda \geq 0$, and $L, h, \gamma > 0$. Set $J = \gamma j_\lambda(h)$. Then every $u \in \mathcal{A}_\lambda(\mathbb{T}^d_L)$ satisfies*

$$\mathcal{E}^{\mathrm{per}}_{\lambda,J,\gamma}(u) \geq \gamma\sigma_\lambda(h) L^d. \tag{3.7}$$

Equality holds if and only if

$$u(x) = \operatorname{sign}\cos\left(\frac{2\pi}{L} a\cdot x + \varphi\right), \qquad a \in \mathbb{Z}^d \setminus \{0\}, \quad |a| = \frac{L}{2h}, \quad \varphi \in \mathbb{R}. \tag{3.8}$$

Consequently, if such an integer vector exists, the minimum equals $\gamma\sigma_\lambda(h)L^d$, and (3.8) is the complete class of global minimizers.

Theorem 3.2 (Neumann comparison and equality cases)**.** *Let $2 \leq d \leq 7$, $\lambda \geq 0$, and $L, h, \gamma > 0$. Set $J = \gamma j_\lambda(h)$. Then every $u \in \mathcal{A}_\lambda(Q_L)$ satisfies*

$$\mathcal{E}^{N}_{\lambda,J,\gamma}(u) \geq \gamma\sigma_\lambda(h) L^d. \tag{3.9}$$

Equality is possible if and only if $n = L/h$ is a positive integer. In this case its complete class is

$$u(x) = \pm \operatorname{sign} \cos\Big(\frac{\pi n x_j}{L}\Big), \qquad j \in \{1, \dots, d\}. \tag{3.10}$$

These functions are exactly the global minimizers.

Remark 3.3. For $\lambda > 0$, neither theorem assumes zero mean. Attainment of the sharp comparison bound forces it (see Corollary 6.3). On the other hand, for $\lambda = 0$, zero mean is necessary for the inverse Laplacian to exist.

Remark 3.4. The condition $L/(2h) \in \mathbb{N}$ is sufficient for coordinate-aligned periodic stripes, however, being the problem isotropic, on a multidimensional torus the lattice condition in (3.8) also allows oblique stripes.

3.3. **Ideas of the proof.** In this subsection we present the main ideas behind inequalities (3.7) and (3.9).

Consider first $U \in BV(T; \{-1, 1\})$ on a torus T of volume V. Write $DU = \theta |DU|$ and introduce

$$\mathscr{T}_U(z) := \int_T |U(x+z) - U(x)| \, \mathrm{d}x, \qquad \mathscr{A}_U(z) := \int_T |z \cdot \theta(x)| \, \mathrm{d}|DU|(x). \tag{3.11}$$

The directional translation inequality (see Lemma 6.1) gives

$$\mathscr{T}_U(z) \le \mathscr{A}_U(z). \tag{3.12}$$

Heuristically, the right-hand side counts every interface crossing along a segment, while the left-hand side only records whether the two endpoints lie in different phases. Two crossings may return to the original phase, and this creates a strict loss in (3.12).

Suppose now that μ is an even, positive, finite measure on $\mathbb{R}^d$ such that

$$\mu(\mathbb{R}^d) = \sigma_\lambda(h), \qquad \int_{\mathbb{R}^d} |z \cdot \nu| \, \mathrm{d}\mu(z) = j_\lambda(h) \quad \text{for every } \nu \in \mathbb{S}^{d-1}. \tag{3.13}$$

The identity $ab = 1 - |a - b|$ for $a, b \in \{-1, 1\}$ gives $\int_T U(x)U(x+z) \, \mathrm{d}x = V - \mathscr{T}_U(z)$. Integrating against μ, and then integrating (3.12) against μ and using (3.13), we obtain

$$\langle U, \mu * U \rangle = \sigma_\lambda(h) V - \int_{\mathbb{R}^d} \mathscr{T}_U(z) \, \mathrm{d}\mu(z) \ge \sigma_\lambda(h) V - j_\lambda(h) |DU|(T), \tag{3.14}$$

where the convolution is evaluated on the periodic extension of U.

For a one-dimensional striped set with layer width at least h, we have $\mathscr{T}_U(z) = \mathscr{A}_U(z)$ for every $|z| \le h$. Assume that the support of μ is $\overline{B_h}$ and that its Lebesgue density is positive on B_h. Equality in (3.14) then forces $\mathscr{T}_U(z) = \mathscr{A}_U(z)$ for a.e. $|z| < h$, hence for all $|z| < h$ by continuity of both sides in z. By the rigidity statement (Lemma 6.4), this happens if and only if U is a striped set with layer width at least h.

To control the left-hand side of (3.14) by the energy, we need $N_\lambda^{\mathrm{per}}(U) \ge \langle U, \mu * U \rangle$. A sufficient condition is

$$\widehat{\mu}(\xi) \le \frac{1}{\lambda^2 + |\xi|^2}. \tag{3.15}$$

For $\lambda = 0$ this is imposed only for $\xi \neq 0$, and the zero-mean constraint removes the constant mode.

Combining (3.15) with (3.14), we get

$$J|DU|(T) + \gamma N_\lambda^{\mathrm{per}}(U) \ge \gamma \sigma_\lambda(h) V + \big(J - \gamma j_\lambda(h)\big) |DU|(T).$$

We choose $h = h_*$ so that $\gamma j_\lambda(h_*) = J$. The perimeter terms then cancel, and the energy is bounded below by $\gamma \sigma_\lambda(h_*) V$. More precisely, the energy excess over $\gamma \sigma_\lambda(h_*) V$ equals the sum of a geometric defect, $\gamma \int (\mathscr{A}_U - \mathscr{T}_U) \, \mathrm{d}\mu$, and a spectral defect, $\gamma\big(N_\lambda^{\mathrm{per}}(U) - \langle U, \mu * U \rangle\big)$. Both are nonnegative (see (6.4)).

For the periodic striped set with width h and normal ν, the nonzero Fourier frequencies are the odd multiples of $(\pi/h)\nu$. The spectral defect vanishes on this configuration only if

$$\widehat{\mu}((2k+1)\pi\nu/h) = \frac{1}{\lambda^2 + ((2k+1)\pi/h)^2}, \qquad k \in \mathbb{Z}. \tag{3.16}$$

The measure μ is thus designed to give equality for the candidate periodic striped set and a lower bound for every other competitor.

We now sketch how to construct μ, starting from the one-dimensional case. For $\lambda > 0$ the whole-line fundamental solution is

$$G_\lambda(t) = \frac{e^{-\lambda|t|}}{2\lambda}, \qquad (-\partial_t^2 + \lambda^2)G_\lambda = \delta_0.$$

We look for a symmetric finite difference

$$k_1(t) = aG_\lambda(t) - \frac{b}{2}\big(G_\lambda(t-h) + G_\lambda(t+h)\big). \tag{3.17}$$

For $t \geq h$, we have $k_1(t) = \big(a - b\cosh(\lambda h)\big)G_\lambda(t)$, and by evenness the same holds for $t \leq -h$. Hence compact support in $[-h, h]$ requires

$$a = b\cosh(\lambda h).$$

At an odd stripe frequency $\xi = (2k+1)\pi/h$ we have $\cos(\xi h) = -1$, so

$$\widehat{k}_1(\xi) = \frac{a - b\cos(\xi h)}{\xi^2 + \lambda^2} = \frac{a+b}{\xi^2+\lambda^2}.$$

Matching $\widehat{G}_\lambda(\xi)$ therefore requires $a + b = 1$. Writing $C_{\lambda,h} := \cosh(\lambda h)$, these two conditions give

$$a = \frac{C_{\lambda,h}}{1 + C_{\lambda,h}}, \qquad b = \frac{1}{1 + C_{\lambda,h}}. \tag{3.18}$$

For $0 \leq t < h$, substituting these values gives

$$k_1(t) = \frac{C_{\lambda,h}e^{-\lambda t} - e^{-\lambda h}\cosh(\lambda t)}{2\lambda(1 + C_{\lambda,h})} = \frac{\sinh(\lambda(h-t))}{2\lambda(1+C_{\lambda,h})}.$$

Therefore

$$k_1(t) = \frac{\sinh(\lambda(h - |t|))}{2\lambda(1 + C_{\lambda,h})}\,\mathbf{1}_{\{|t|<h\}}, \tag{3.19}$$

which is positive on $(-h, h)$, and

$$m_{\lambda,h}(\xi) := \widehat{k}_1(\xi) = \frac{C_{\lambda,h} - \cos(h|\xi|)}{(1 + C_{\lambda,h})(\lambda^2 + |\xi|^2)}. \tag{3.20}$$

Since $C_{\lambda,h} - \cos(h|\xi|) \leq 1 + C_{\lambda,h}$, with equality if and only if $h|\xi| \in (2\mathbb{Z}+1)\pi$, the multiplier $m_{\lambda,h}$ satisfies (3.15), with equality exactly at the odd stripe frequencies (3.16). In particular, the inequality is strict at $\xi = 0$. A direct computation gives

$$\int_{\mathbb{R}} k_1 \,\mathrm{d}t = \frac{C_{\lambda,h} - 1}{\lambda^2(1 + C_{\lambda,h})}, \qquad \int_{\mathbb{R}} |t|\, k_1(t)\,\mathrm{d}t = \frac{\sinh(\lambda h) - \lambda h}{\lambda^3(1 + C_{\lambda,h})},$$

and these are the quantities $\sigma_\lambda(h)$ and $j_\lambda(h)$ in (3.13). In dimension one, the measure $\mathrm{d}\mu := k_1(t)\,\mathrm{d}t$ therefore has all the required properties.

The key observation is that a striped set only sees the one-dimensional marginal of a kernel. Suppose a radial measure μ on $\mathbb{R}^d$ projects onto every line to $k_1(t)\,\mathrm{d}t$. Then its mass and its directional first moment are those of k_1. By the projection-slice theorem, $\widehat{\mu}(\xi) = m_{\lambda,h}(|\xi|)$. Hence (3.13), (3.15) and (3.16) are inherited automatically, and the only nontrivial requirement in higher dimensions is *positivity* (or, for $d = 6, 7$, the weaker substitute below).

Projection through two transverse dimensions links radial densities in dimensions d and $d+2$:

$$k_d(r) = 2\pi \int_r^h s\, k_{d+2}(s)\,\mathrm{d}s, \qquad k_{d+2}(r) = -\frac{k_d'(r)}{2\pi r} \quad \text{for } 0 < r < h. \tag{3.21}$$

In particular, (3.19) gives the three-dimensional density

$$k_3(r) = \frac{\cosh(\lambda(h-r))}{4\pi(1 + C_{\lambda,h})r}\,\mathbf{1}_{\{0<r<h\}}. \tag{3.22}$$

Unlike k_1, which vanishes continuously at $|t| = h$, the density k_3 has a jump at $r = h$. Each lift by two dimensions introduces one radial derivative. In dimension five, the distributional identity $k_5 = -k_3'/(2\pi r)$ therefore produces a positive surface measure on ∂B_h accounting for this jump. Positive finite measures with the prescribed multiplier cannot exist in dimensions $d \geq 6$, as proved in Appendix A. This describes the main mechanism of the comparison in dimensions $d \leq 5$.

For $d = 6, 7$, we continue the Fourier walk in the sense of distributions and exploit additional properties of the geometric defect. Its spherical average

$$\psi_U(r) := \fint_{\mathbb{S}^{d-1}} \big(\mathscr{A}_U(re) - \mathscr{T}_U(re)\big)\, d\mathcal{H}^{d-1}(e)$$

is nonnegative, nondecreasing and Lipschitz on $[0, \infty)$, and belongs to $C^1((0, \infty))$ (Lemma 6.1). The weak form of the dimension walk gives

$$\big\langle \mu^{(d)}_{\lambda,h}, \psi_U(|\cdot|)\big\rangle = \int_{\mathbb{R}^{d-2}} \left[\psi_U(|z|) + \frac{|z|}{d-2}\psi_U'(|z|)\right] d\mu^{(d-2)}_{\lambda,h}(z) \geq 0.$$

Thus, positivity of the lower-dimensional measure, together with monotonicity of the averaged defect, ensures that the geometric contribution remains nonnegative in dimensions six and seven. The Fourier multiplier, mass and radial first-moment identity are preserved by the lift, so the same sharp comparison follows.

Moreover, a vanishing geometric contribution forces $\psi_U(r) = 0$ for $0 < r < h$, because the lower-dimensional measure has strictly positive absolutely continuous density. Non-negativity and continuity of the original defect then imply $\mathscr{A}_U(z) = \mathscr{T}_U(z)$ for every $|z| < h$. Consequently, the same rigidity argument applies. The construction and the extended pairings are established in Proposition 5.1.

Finally, for $\lambda = 0$ the same construction applies directly. The corresponding kernel and multiplier are

$$k_1(t) = \tfrac{1}{4}\,(h - |t|)_+, \qquad m_{\lambda,h}(\xi) = \frac{1 - \cos(h|\xi|)}{2|\xi|^2},$$

with $\sigma_0(h) = h^2/4$ and $j_0(h) = h^3/12$, so that $h_* = (12J/\gamma)^{1/3}$.

4. Continuous optimization of one-dimensional periodic stripes

We now focus on the optimization in one dimension and the derivation of the relevant quantities in (3.5) and (3.6). Although the results in this section are already known, we present them for completeness.

4.1. **The periodic cell problem.** For each $h > 0$, let $s_h(x) := \operatorname{sign}\cos(\pi x/h)$ on the one-dimensional torus $\mathbb{T}^1_{2h}$. Define $w_{\lambda,h} := R^{\mathrm{per}}_\lambda s_h$ as its periodic potential, normalized to have zero mean when $\lambda = 0$. We define the *periodic stripe energy density* to be the energy of one full period divided by its length:

$$e_{\lambda,J,\gamma}(h) := \frac{1}{2h}\mathcal{E}^{\mathrm{per}}_{\lambda,J,\gamma}(s_h) = \frac{J}{2h}|Ds_h|(\mathbb{T}^1_{2h}) + \frac{\gamma}{2h}\int_{-h}^{h} s_h(x) w_{\lambda,h}(x)\,\mathrm{d}x. \tag{4.1}$$

For $\lambda > 0$, w is even and anti-periodic, that is

$$w(-x) = w(x), \qquad w(x + h) = -w(x).$$

On the positive layer $[-h/2, h/2]$, the equation and boundary values are

$$-w'' + \lambda^2 w = 1, \qquad w'(0) = 0, \qquad w(\pm h/2) = 0.$$

Solving the boundary-value problem gives the solution

$$p_{\lambda,h}(x) = \frac{1}{\lambda^2}\left(1 - \frac{\cosh(\lambda x)}{\cosh(\lambda h/2)}\right), \qquad |x| \leq h/2. \tag{4.2}$$

We extend it by setting $w(x + kh) = (-1)^k p_{\lambda,h}(x)$. Notice that the interface values are zero, and the derivatives match because

$$p'_{\lambda,h}(h/2) = -\frac{\tanh(\lambda h/2)}{\lambda} = -p'_{\lambda,h}(-h/2).$$

Thus, the extension is C^1 and solves the equation distributionally. Moreover, anti-periodicity gives zero mean.

Using the fact that $s_h w$ has period h, the nonlocal energy per unit length is given by

$$n_\lambda(h) := \frac{1}{h}\int_{-h/2}^{h/2} p_{\lambda,h}(x)\,\mathrm{d}x = \frac{1}{\lambda^2} - \frac{1}{\lambda^2 h\cosh(\lambda h/2)}\frac{2\sinh(\lambda h/2)}{\lambda} = \frac{1}{\lambda^2} - \frac{2}{\lambda^3 h}\tanh(\lambda h/2). \tag{4.3}$$

A similar computation for $\lambda = 0$ gives

$$p_{0,h}(x) = \frac{h^2}{8} - \frac{x^2}{2}.$$

The same antiperiodic extension is C^1, has zero mean, and solves $-w'' = s_h$. Direct integration gives

$$n_0(h) = \frac{1}{h}\int_{-h/2}^{h/2}\left(\frac{h^2}{8} - \frac{x^2}{2}\right)\mathrm{d}x = \frac{1}{h}\left(\frac{h^3}{8} - \frac{h^3}{24}\right) = \frac{h^2}{12}. \tag{4.4}$$

There are two jumps of size two in each cell of length $2h$. The total energy density is therefore

$$e_{\lambda,J,\gamma}(h) = \frac{2J}{h} + \gamma n_\lambda(h). \tag{4.5}$$

4.2. **Continuous minimization.** For $\lambda = 0$,

$$e'_{0,J,\gamma}(h) = -\frac{2J}{h^2} + \frac{\gamma h}{6}, \qquad e''_{0,J,\gamma}(h) = \frac{4J}{h^3} + \frac{\gamma}{6} > 0.$$

The energy diverges at both 0 and $+\infty$. Hence, its unique global minimizer is given by

$$h_* = \left(\frac{12J}{\gamma}\right)^{1/3}, \qquad e_{0,J,\gamma}(h_*) = \frac{\gamma h_*^2}{4}. \tag{4.6}$$

For $\lambda > 0$, define for brevity $a := \lambda h/2$. Differentiating (4.3) yields

$$n'_\lambda(h) = \frac{2}{\lambda^3 h^2}\tanh a - \frac{1}{\lambda^2 h}\operatorname{sech}^2 a.$$

Consequently,

$$e'_{\lambda,J,\gamma}(h) = \frac{2\gamma}{h^2}\left[\frac{\tanh a - a\operatorname{sech}^2 a}{\lambda^3} - \frac{J}{\gamma}\right] = \frac{2\gamma}{h^2}\left[j_\lambda(h) - \frac{J}{\gamma}\right]. \tag{4.7}$$

Above we used the expression for $j_\lambda(h)$ in (3.5), since $\sinh(2a)/(1+\cosh(2a)) = \tanh a$ and $2a/(1+\cosh(2a)) = a\operatorname{sech}^2 a$. Moreover,

$$j'_\lambda(h) = \frac{h}{2\lambda}\operatorname{sech}^2 a\,\tanh a > 0, \qquad \lim_{h\to 0^+} j_\lambda(h) = 0, \quad \lim_{h\to+\infty} j_\lambda(h) = \lambda^{-3}. \tag{4.8}$$

Thus a finite continuous optimal width exists exactly when $0 < J/\gamma < \lambda^{-3}$. It is unique and is characterized by

$$J = \gamma j_\lambda(h_*). \tag{4.9}$$

Indeed, the derivative in (4.7) is negative before h_* and positive after it, proving global minimality within the continuous periodic stripe family. At this width h_* we have

$$e_{\lambda,J,\gamma}(h_*) = \frac{\gamma}{\lambda^2} - \frac{2\gamma}{\lambda^3 h_*}\left(\frac{\lambda h_*}{2}\right)\operatorname{sech}^2(\lambda h_*/2) = \frac{\gamma}{\lambda^2}\tanh^2(\lambda h_*/2) = \gamma\sigma_\lambda(h_*). \tag{4.10}$$

If $J/\gamma \geq \lambda^{-3}$, the energy is strictly decreasing at every finite width and converges to γ/λ^2 as $h \to \infty$; and there is no minimizer in the stripe family.

4.3. **Neumann cells and compatibility with the domain.** On $(0, L)$, let $n = L/h \in \mathbb{N}$ and set

$$u_n(x) := \operatorname{sign}\cos(\pi n x/L).$$

The interfaces are at $(k + \frac{1}{2})h$, $k = 0, \ldots, n-1$. Notice also that the endpoint phase intervals have length $h/2$, whereas all interior intervals have length h. There are exactly n interior jumps, giving $|Du_n|((0, L)) = 2n$.

Restrict the potential constructed above to $(0, L)$. By evenness and anti-periodicity of w, we have

$$w'(0) = 0, \qquad w'(L) = w'(nh) = (-1)^n w'(0) = 0.$$

Thus the restriction of w to $(0, L)$ solves

$$-w'' + \lambda^2 w = u_n \quad \text{in } (0, L), \qquad w'(0) = w'(L) = 0.$$

Moreover, evenness and anti-periodicity imply $w(h - x) = -w(x)$, and hence

$$\int_0^h w(x)\,dx = 0.$$

Consequently,

$$\int_0^L w(x)\,dx = \sum_{k=0}^{n-1} \int_{kh}^{(k+1)h} w(x)\,dx = \sum_{k=0}^{n-1} (-1)^k \int_0^h w(x)\,dx = 0.$$

In particular, when $\lambda = 0$, this restriction satisfies the required mean-zero normalization.

Finally, both s_h and w are anti-periodic with period h, so their product is h-periodic. Since $u_n = s_h|_{(0,L)}$ and $L = nh$, we obtain

$$\frac{1}{L}\int_0^L u_n(x)w(x)\,dx = \frac{1}{h}\int_0^h s_h(x)w(x)\,dx = \frac{1}{h}\int_{-h/2}^{h/2} p_{\lambda,h}(x)\,dx = n_\lambda(h).$$

Hence,

$$\frac{\mathcal{E}^N_{\lambda,J,\gamma}(u_n)}{L} = \frac{2J}{h} + \gamma n_\lambda(h). \tag{4.11}$$

Thus, for every h with $L/h \in \mathbb{N}$, the Neumann stripe $u_{L/h}$ has the same energy per unit length as the periodic stripe of width h. In particular, if $L/h_* \in \mathbb{N}$, then u_{L/h_*} attains the optimal periodic stripe energy density $e_{\lambda,J,\gamma}(h_*)$.

5. Construction of the comparison measure

In this section we construct the radial comparison measures and distributions used to derive the sharp comparison inequality. We begin with the one- and two-dimensional profiles and obtain the higher-dimensional objects through the Fourier dimension walk. We then establish their Fourier transforms and moment identities, positivity as measures up to dimension five, and non-negativity of their action on nonnegative, nondecreasing radial test functions in dimensions six and seven.

Proposition 5.1 (Radial comparison measures and distributions)**.** *Let $h > 0$, $\lambda \geq 0$, and $1 \leq d \leq 7$. There is a unique compactly supported distribution $\mu^{(d)}_{\lambda,h}$ on $\mathbb{R}^d$ with Fourier transform*

$$\widehat{\mu^{(d)}_{\lambda,h}}(\xi) = \frac{\cosh(\lambda h) - \cos(h|\xi|)}{(1+\cosh(\lambda h))(\lambda^2 + |\xi|^2)}. \tag{5.1}$$

The distribution is invariant under orthogonal transformations and supported in $\overline{B}_h$. At $\lambda = 0$ the right-hand side is understood by continuity at $\xi = 0$, where its value is $h^2/4$.

For $1 \leq d \leq 5$, $\mu^{(d)}_{\lambda,h}$ is a finite positive measure whose absolutely continuous part has strictly positive density almost everywhere in B_h. Its mass and first absolute directional moments satisfy

$$\mu^{(d)}_{\lambda,h}(\mathbb{R}^d) = \sigma_\lambda(h), \qquad \int_{\mathbb{R}^d} |z\cdot\nu|\,d\mu^{(d)}_{\lambda,h}(z) = j_\lambda(h) \quad (\nu \in \mathbb{S}^{d-1}). \tag{5.2}$$

For $d \in \{6,7\}$, $\mu^{(d)}_{\lambda,h}$ has distributional order at most one. Its action on smooth radial tests extends to radial functions $\psi(z) = \psi_0(|z|)$ with

$$\psi_0 \in \mathrm{Lip}([0,h+\delta]) \cap C^1((0,h+\delta)) \quad \text{for some } \delta > 0,$$

by the pairing formula

$$\langle \mu^{(d)}_{\lambda,h}, \psi\rangle := \int_{\mathbb{R}^{d-2}} \left[\psi_0(|z|) + \frac{|z|}{d-2}\psi_0'(|z|)\right] d\mu^{(d-2)}_{\lambda,h}(z). \tag{5.3}$$

The bracket at $z = 0$ is assigned the value $\psi_0(0)$. In particular, this pairing is nonnegative whenever ψ_0 is nonnegative and nondecreasing on $[0,h]$. The mass and generalized first absolute directional moments in dimensions $d \in \{6,7\}$ satisfy

$$\langle \mu^{(d)}_{\lambda,h}, 1\rangle = \sigma_\lambda(h), \qquad c_d\langle \mu^{(d)}_{\lambda,h}, |\cdot|\rangle = j_\lambda(h), \qquad c_d := \frac{\Gamma(d/2)}{\sqrt{\pi}\,\Gamma((d+1)/2)}.$$

Above, the first pairing uses a smooth cut-off $\chi \in C^\infty_c(\mathbb{R}^d)$ equal to one on a neighborhood of $\overline{B}_h$, while the radial pairing with $|\cdot|$ is defined by (5.3).

Proof. Set $C_{\lambda,h} := \cosh(\lambda h)$ and $A_{\lambda,h} := [8\pi^2(1 + C_{\lambda,h})]^{-1}$. We first construct the one- and two-dimensional measures and then obtain the higher-dimensional distributions by the two-step Fourier dimension walk.

We shall use the following observation. Let μ_d be a compactly supported radial distribution on $\mathbb{R}^d$, and let $\pi_d : \mathbb{R}^d \to \mathbb{R}$ be the coordinate projection $\pi_d(z) = z_1$. If $(\pi_d)_{\#}\mu_d = \mu_1$, then

$$\widehat{\mu_d}(\eta, 0, \dots, 0) = \langle \mu_d, e^{-i\eta\pi_d(\cdot)} \rangle = \langle \mu_1, e^{-i\eta\,\cdot} \rangle = \widehat{\mu_1}(\eta), \qquad \eta \in \mathbb{R}.$$

Here, compact support allows the distributions to act on smooth functions without compact support, by inserting a cutoff equal to one near their support. Moreover, orthogonal invariance of μ_d implies orthogonal invariance of its Fourier transform. Thus, for every $\xi \in \mathbb{R}^d$,

$$\widehat{\mu_d}(\xi) = \widehat{\mu_d}(|\xi|, 0, \dots, 0) = \widehat{\mu_1}(|\xi|).$$

Consequently, a compactly supported radial distribution with one-dimensional marginal μ_1 satisfies (5.1) whenever μ_1 does.

Step 1: the one-dimensional profile. For $\lambda > 0$, define

$$k^{(1)}_{\lambda,h}(t) := \frac{\sinh(\lambda(h - |t|))}{2\lambda(1 + C_{\lambda,h})} \mathbf{1}_{\{|t|<h\}}, \qquad d\mu^{(1)}_{\lambda,h}(t) := k^{(1)}_{\lambda,h}(t)\, dt. \tag{5.4}$$

For $\lambda = 0$, use its continuous limit

$$k^{(1)}_{0,h}(t) = \frac{(h - |t|)_+}{4}. \tag{5.5}$$

These are finite positive even densities. Taking Fourier transforms with the convention $e^{-it\xi}$ proves (5.1) in dimension one (see Subsection 3.3 for the computations). At $\lambda = \xi = 0$, the value follows by continuity (or by integration of (5.5)).

Step 2: the two-dimensional profile. Define $d\mu^{(2)}_{\lambda,h}(z) = k^{(2)}_{\lambda,h}(|z|)\, dz$, where

$$k^{(2)}_{\lambda,h}(r) := \frac{1}{2\pi(1 + C_{\lambda,h})} \int_r^h \frac{\cosh(\lambda(h - s))}{\sqrt{s^2 - r^2}}\, ds, \qquad 0 < r < h, \tag{5.6}$$

and the density is zero for $r \geq h$. It is strictly positive on $(0, h)$. For $|t| < h$, Tonelli's theorem and the change of variables $r^2 = t^2 + (s^2 - t^2)v$ yield

$$\begin{aligned} 2\int_{|t|}^h \frac{r k^{(2)}_{\lambda,h}(r)}{\sqrt{r^2 - t^2}}\, dr &= \frac{1}{\pi(1 + C_{\lambda,h})} \int_{|t|}^h \cosh(\lambda(h - s)) \left[\int_{|t|}^s \frac{r\, dr}{\sqrt{s^2 - r^2}\sqrt{r^2 - t^2}} \right] ds \\ &= \frac{1}{2(1 + C_{\lambda,h})} \int_{|t|}^h \cosh(\lambda(h - s))\, ds = k^{(1)}_{\lambda,h}(t). \end{aligned}$$

Thus the one-dimensional marginal of $\mu^{(2)}_{\lambda,h}$ is $\mu^{(1)}_{\lambda,h}$. In particular, $\mu^{(2)}_{\lambda,h}$ is finite. Its rotational invariance, together with the preceding observation, proves (5.1) in dimension two. At $\lambda = 0$,

$$k^{(2)}_{0,h}(r) = \frac{1}{4\pi} \operatorname{arcosh}(h/r) = \frac{1}{4\pi} \log \frac{h + \sqrt{h^2 - r^2}}{r}.$$

Step 3: the Fourier walk and positivity through dimension five. For inverse Fourier transforms of a common radial multiplier, the distributional dimension walk is

$$k^{(d+2)}_{\lambda,h}(r) = -\frac{1}{2\pi r}\, \partial_r k^{(d)}_{\lambda,h}(r). \tag{5.7}$$

We use the radial distributional interpretation of this identity from [34, Theorem 4.1]. Starting from dimensions one and two, this constructs the chains $1 \to 3 \to 5 \to 7$ and $2 \to 4 \to 6$. At each step the resulting distribution is orthogonally invariant, has the Fourier transform (5.1), and is supported in $\overline{B}_h$. The last assertion also follows from the radial test identity below, since differentiation does not enlarge the radial support. Uniqueness follows from injectivity of the Fourier transform on tempered distributions.

The first odd lift gives

$$k^{(3)}_{\lambda,h}(r) = \frac{\cosh(\lambda(h - r))}{4\pi(1 + C_{\lambda,h})r} \mathbf{1}_{\{0<r<h\}}.$$

The profile in dimension one is continuous at h, so this step has no surface term. In the next lift, its three-dimensional successor has a jump at h, and thus its distributional derivative gives

$$d\mu^{(5)}_{\lambda,h}(r) = A_{\lambda,h}\left[\frac{\lambda\sinh(\lambda(h-r))}{r^2} + \frac{\cosh(\lambda(h-r))}{r^3}\right]\mathbf{1}_{\{0<r<h\}}\,dz + \frac{A_{\lambda,h}}{h^2}\,d\mathcal{H}^4\llcorner\partial B_h.$$

Both measures are positive, with strictly positive absolutely continuous densities. Their origin singularities are respectively $O(r^{-1})$ and $O(r^{-3})$, which are integrable in their respective dimensions; the surface part is finite.

For the even lift, integration by parts in (5.6) gives

$$k^{(2)}_{\lambda,h}(r) = \frac{1}{2\pi(1+C_{\lambda,h})}\left[\operatorname{arcosh}(h/r) + \lambda\int_r^h \sinh(\lambda(h-s))\operatorname{arcosh}(s/r)\,ds\right].$$

Differentiation on $(0,h)$ and (5.7) yield

$$k^{(4)}_{\lambda,h}(r) = \frac{1}{4\pi^2(1+C_{\lambda,h})r^2}\left[\frac{h}{\sqrt{h^2-r^2}} + \lambda\int_r^h \frac{s\sinh(\lambda(h-s))}{\sqrt{s^2-r^2}}\,ds\right], \qquad 0<r<h. \tag{5.8}$$

The two-dimensional profile tends to zero at h, and its derivative is integrable near h, so this last lift has no surface term. The expression in (5.8) is strictly positive, is $O(r^{-2})$ near zero, and is $O((h-r)^{-1/2})$ near h. Thus it defines a finite positive measure in dimension four. These calculations prove all the positivity and density assertions through dimension five.

Step 4: radial pairings in dimensions six and seven. For a smooth compactly supported radial test $\psi(z) = \psi_0(|z|)$, the weak form of (5.7) is

$$\langle \mu^{(d+2)}_{\lambda,h}, \psi\rangle = -\frac{\omega_{d+1}}{2\pi}\langle \partial_r k^{(d)}_{\lambda,h}, r^d\psi_0\rangle = \frac{\omega_{d+1}}{2\pi}\langle k^{(d)}_{\lambda,h}, (r^d\psi_0)'\rangle = \left\langle \mu^{(d)}_{\lambda,h}, \psi_0(|\cdot|) + \frac{|\cdot|}{d}\psi_0'(|\cdot|)\right\rangle, \tag{5.9}$$

where we used $\omega_{d+1} = 2\pi\omega_{d-1}/d$. For $d = 4, 5$, the second equality above is justified using a cut-off approximation on $(\varepsilon, 2\varepsilon)$: since $k^{(d)}_{\lambda,h}(r) = O(r^{2-d})$ close to zero, the error is bounded by

$$\frac{C}{\varepsilon}\int_\varepsilon^{2\varepsilon} r^d |k^{(d)}_{\lambda,h}(r)|\,|\psi_0(r)|\,dr \le C\varepsilon^2\|\psi_0\|_\infty.$$

Consequently (5.9) gives (5.3). The resulting expression is bounded by a constant times the C^1 norm of the test function on a fixed neighborhood of $\overline{B}_h$. This implies that these distributions have order at most one.

To extend the radial pairing, let ψ_0 belong to the class in the statement. Its derivative is bounded, and $r\psi_0'(r) \to 0$ as $r \to 0^+$. Choose smooth cut-off functions χ_ε equal to zero on $[0,\varepsilon]$ and to one on $[2\varepsilon,\infty)$, with $|\chi_\varepsilon'| \le C/\varepsilon$. The profiles

$$\psi_0(0) + \chi_\varepsilon(r)(\psi_0(r) - \psi_0(0))$$

converge to ψ_0 in the norm $\|\psi_0\|_\infty + \|r\psi_0'\|_\infty$ on a neighborhood of $[0,h]$. They are constant near zero and may then be approximated smoothly away from zero in the same norm. A fixed outer cut-off makes the corresponding radial tests compactly supported. Since $\mu^{(d)}_{\lambda,h}$ is finite for $d = 4, 5$, we get convergence of their pairings to the right-hand side of (5.3) establishing the assertion.

Step 5: mass and first absolute directional moments. Recall that for $1 \le d \le 5$, the line marginal of $\mu^{(d)}_{\lambda,h}$ is given by $\mu^{(1)}_{\lambda,h}$. Rotational invariance makes this true in every direction. For $\lambda > 0$, integration of (5.4) therefore gives

$$\mu^{(d)}_{\lambda,h}(\mathbb{R}^d) = \frac{1}{\lambda(1+C_{\lambda,h})}\int_0^h \sinh(\lambda(h-t))\,dt = \frac{C_{\lambda,h}-1}{\lambda^2(1+C_{\lambda,h})} = \sigma_\lambda(h),$$

$$\int_{\mathbb{R}^d} |z\cdot\nu|\,d\mu^{(d)}_{\lambda,h}(z) = \frac{1}{\lambda(1+C_{\lambda,h})}\int_0^h t\sinh(\lambda(h-t))\,dt = \frac{\sinh(\lambda h) - \lambda h}{\lambda^3(1+C_{\lambda,h})} = j_\lambda(h).$$

For $\lambda = 0$, (5.5) gives

$$\int_{\mathbb{R}} k^{(1)}_{0,h}(t)\,dt = \frac{h^2}{4}, \qquad \int_{\mathbb{R}} |t| k^{(1)}_{0,h}(t)\,dt = \frac{h^3}{12}.$$

It remains to check the lifted moments. For $d = 4, 5$, formula (5.9) with a cut-off equal to one near the support preserves the mass. Moreover, using the extended radial pairing,

$$\langle \mu_{\lambda,h}^{(d+2)}, |\cdot| \rangle = \frac{d+1}{d} \int_{\mathbb{R}^d} |z| \, d\mu_{\lambda,h}^{(d)}(z).$$

Since

$$c_d = \frac{1}{\omega_{d-1}} \int_{\mathbb{S}^{d-1}} |\theta_1| \, d\mathcal{H}^{d-1}(\theta), \qquad c_{d+2} = \frac{d}{d+1} c_d,$$

radiality and (5.2) imply

$$c_{d+2} \langle \mu_{\lambda,h}^{(d+2)}, |\cdot| \rangle = c_d \int_{\mathbb{R}^d} |z| \, d\mu_{\lambda,h}^{(d)}(z) = j_\lambda(h).$$

This completes the proof. □

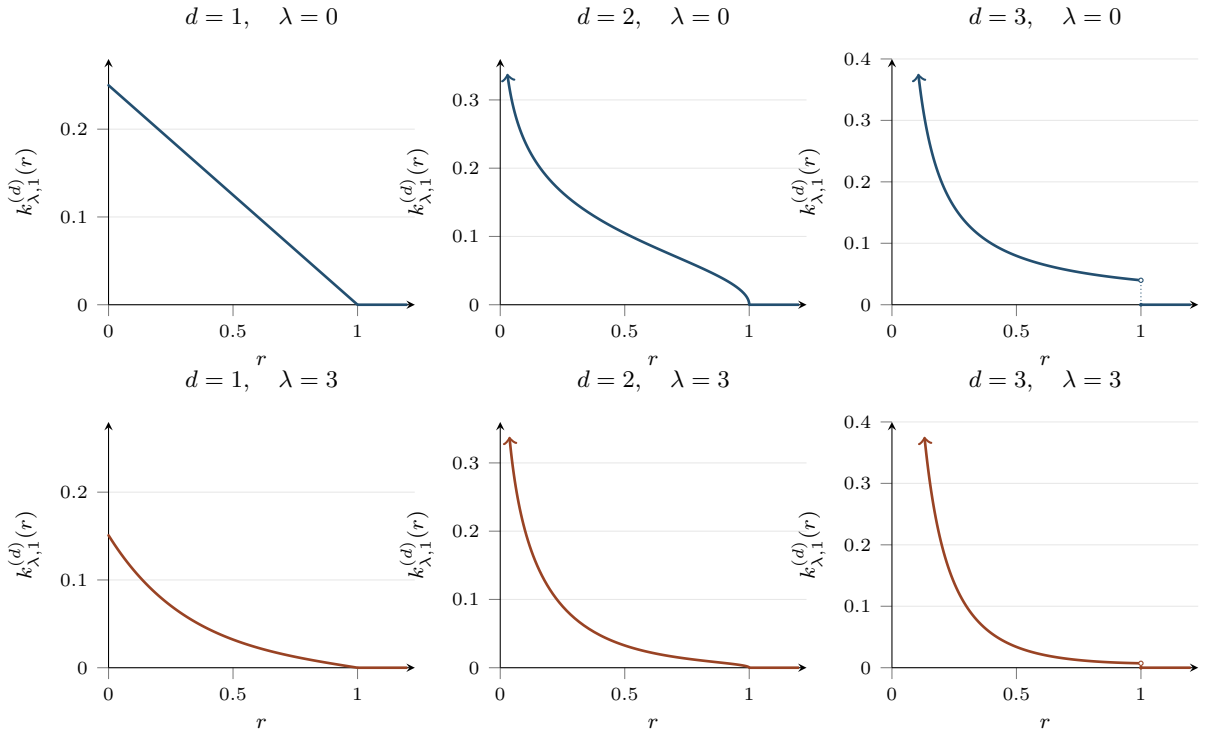


FIGURE 1. Radial profiles of the Lebesgue densities $k_{\lambda,1}^{(d)}(r)$ of the comparison measures in Proposition 5.1, with $h = 1$ and $r = |z|$. The first row corresponds to $\lambda = 0$ and the second to $\lambda = 3$; the columns correspond to $d = 1, 2, 3$. Arrows indicate divergence as $r \to 0^+$ for $d = 2, 3$. All densities vanish for $r > 1$. For $d = 3$, the open circle marks the positive left limit at $r = 1$.

6. The sharp estimate, rigidity, and proofs of the theorems

In this section we prove the additional intermediate results needed to prove Theorems 3.1 and 3.2.

6.1. The directional translation inequality.

We now state and prove a general result for BV function on the torus. A similar statement also appears in [42, Lemma 3.1] and [26, Proposition 11].

Lemma 6.1 (Translation control and radial geometric defect)**.** *Let T be a periodic torus and let $U \in BV(T)$. Write $DU = \theta|DU|$ and define $\mathscr{T}_U, \mathscr{A}_U$ as in (3.11). Then, for every $z \in \mathbb{R}^d$, we have*

$$\mathscr{T}_U(z) \le \mathscr{A}_U(z). \tag{6.1}$$

In particular, the function $\mathscr{D}_U(z) := \mathscr{A}_U(z) - \mathscr{T}_U(z)$ *is nonnegative and Lipschitz on* $\mathbb{R}^d$. *For every* $e \in \mathbb{S}^{d-1}$, *the map* $r \mapsto \mathscr{D}_U(re)$ *is nondecreasing on* $[0,\infty)$. *Consequently, its spherical average*

$$\psi_U(r) := \fint_{\mathbb{S}^{d-1}} \mathscr{D}_U(re)\, d\mathcal{H}^{d-1}(e)$$

is nonnegative, nondecreasing and Lipschitz on $[0,\infty)$, *with* $\psi_U(0) = 0$ *and*

$$\mathrm{Lip}(\psi_U) \le 2c_d |DU|(T), \qquad c_d := \fint_{\mathbb{S}^{d-1}} |e_1|\, d\mathcal{H}^{d-1}(e).$$

Finally, if $d \ge 2$ *and* $U \in BV(T;\{-1,1\})$, *then* $\psi_U \in C^1((0,\infty))$.

Proof. In the proof, periodic functions and measures are identified with their periodic extensions to $\mathbb{R}^d$. For smooth periodic U, the fundamental theorem of calculus gives

$$|U(x+z) - U(x)| \le \int_0^1 |z \cdot \nabla U(x+tz)|\, dt.$$

Integration over the torus and translation invariance prove (6.1). Given a general BV function, let $U_\varepsilon = \rho_\varepsilon * U$ be a periodic mollification. Then

$$\int_T |z \cdot \nabla U_\varepsilon|\, dx \le |z \cdot DU|(T) = \mathscr{A}_U(z),$$

whereas $|\mathscr{T}_{U_\varepsilon}(z) - \mathscr{T}_U(z)| \le 2\|U_\varepsilon - U\|_{L^1(T)}$. Passing to the limit proves (6.1).

The triangle inequality and translation invariance imply

$$|\mathscr{T}_U(z) - \mathscr{T}_U(z')| \le \mathscr{T}_U(z-z') \le \mathscr{A}_U(z-z') \le |z-z'|\,|DU|(T).$$

Together with

$$|\mathscr{A}_U(z) - \mathscr{A}_U(z')| \le |z-z'|\,|DU|(T),$$

this proves that $\mathscr{D}_U$ is Lipschitz.

Fix $e \in \mathbb{S}^{d-1}$ and $0 \le r \le s$. The same argument gives

$$\mathscr{T}_U(se) - \mathscr{T}_U(re) \le \mathscr{T}_U((s-r)e) \le (s-r)|D_eU|(T).$$

Since $\mathscr{A}_U(re) = r|D_eU|(T)$, it follows that

$$0 \le \mathscr{D}_U(se) - \mathscr{D}_U(re) \le 2(s-r)|D_eU|(T).$$

Averaging over e proves monotonicity of ψ_U and the Lipschitz bound. Indeed, the polar decomposition and rotational invariance give

$$\fint_{\mathbb{S}^{d-1}} |D_eU|(T)\, d\mathcal{H}^{d-1}(e) = \int_T \fint_{\mathbb{S}^{d-1}} |e \cdot \theta(x)|\, d\mathcal{H}^{d-1}(e)\, d|DU|(x) = c_d|DU|(T).$$

It remains to prove C^1 regularity of ψ_U when $U \in BV(T;\{-1,1\})$. For $z \in \mathbb{R}^d$ and $r > 0$ set

$$\mathcal{C}(z) := \int_T U(x+z)U(x)\, dx, \qquad \text{and} \qquad \overline{\mathcal{C}}(r) := \fint_{\mathbb{S}^{d-1}} \mathcal{C}(re)\, d\mathcal{H}^{d-1}(e).$$

Observe that using the identity $|U(x+z) - U(x)| = 1 - U(x+z)U(x)$ we get for every $r > 0$

$$\psi_U(r) = c_d r|DU|(T) - |T| + \overline{\mathcal{C}}(r).$$

Thus, it is enough to prove that $\overline{\mathcal{C}}'(r) \in C((0,\infty))$. Assume temporarily that $\mathcal{C}$ is smooth. Then, using the divergence theorem we would have

$$\frac{d}{dr} \fint_{\mathbb{S}^{d-1}} \mathcal{C}(re)\, d\mathcal{H}^{d-1}(e) = \frac{1}{\omega_{d-1} r^{d-1}} \int_{\partial B_r} \nabla \mathcal{C} \cdot \nu_{B_r}\, d\mathcal{H}^{d-1}(x) = \frac{1}{\omega_{d-1} r^{d-1}} \int_{B_r} \Delta \mathcal{C}(z)\, dz. \tag{6.2}$$

That is,

$$\overline{\mathcal{C}}'(r) = \frac{(\Delta \mathcal{C})(B_r)}{\omega_{d-1} r^{d-1}} \qquad \text{for } r > 0.$$

Therefore, if we prove that $r \mapsto (\Delta\mathcal{C})(B_r)$ is continuous on $(0,\infty)$, we can justify equality (6.2) using approximations with a mollifier and the identity above would imply the continuity of $\overline{\mathcal{C}}'$. To this aim, let us set $\check{U}(x) := U(-x)$. We have $\mathcal{C} = U * \check{U}$ and thus $\Delta\mathcal{C} = \sum_{i=1}^d D_iU * D_i\check{U}$. In particular, $\Delta\mathcal{C}$ is a signed finite measure on T. Moreover, being $U \in BV(T;\{-1,1\})$, we have that $|DU|$ vanishes on sets of zero

$\mathcal{H}^{d-1}$-measure. In particular, for every $r>0$ and $x\neq y$ we get $|DU|(\partial B_r(x)\cap\partial B_r(y))=0$. This means that $|DU|(\partial B_r(x))>0$ only for a countable number of centers $x\in\mathbb{R}^d$, a $|DU|$-null set. Thus, for every $r>0$,

$$\int_T |DU|(\partial B_r(x))\,d|DU|(x)=0,$$

giving $|\Delta\mathcal{C}|(\partial B_r)=0$. This implies the continuity of $r\mapsto(\Delta\mathcal{C})(B_r)$ on $(0,\infty)$. Let now ρ_ε be a nonnegative, radial standard mollifier and set $\mathcal{C}_\varepsilon:=\rho_\varepsilon*\mathcal{C}$, and $\overline{\mathcal{C}}_\varepsilon$ its spherical average. Since $\mathcal{C}$ is Lipschitz, $\overline{\mathcal{C}}_\varepsilon$ converges uniformly to $\overline{\mathcal{C}}$. Using (6.2) since $\mathcal{C}_\varepsilon$ is smooth,

$$\overline{\mathcal{C}}_\varepsilon'(r)=\frac{1}{\omega_{d-1}r^{d-1}}\int_{B_r}(\rho_\varepsilon*(\Delta\mathcal{C}))(x)\,dx.$$

For every $r>0$, by dominated convergence we have that $(\rho_\varepsilon*(\Delta\mathcal{C}))(B_r)$ converges to $(\Delta\mathcal{C})(B_r)$ since ∂B_r is $|\Delta\mathcal{C}|$-negligible.

Fix $a\in(0,\infty)$. For $r\in(a,\infty)$, integrating from a to r and passing to the limit in $\varepsilon\to0$ by dominated convergence yields

$$\overline{\mathcal{C}}(r)-\overline{\mathcal{C}}(a)=\int_a^r\frac{(\Delta\mathcal{C})(B_t)}{\omega_{d-1}t^{d-1}}\,dt.$$

Hence $\overline{\mathcal{C}}\in C^1((0,\infty))$. □

6.2. **The global comparison.** We are ready to show the sharp global comparison inequality which will entail Theorems 3.1 and 3.2.

Proposition 6.2 (Sharp comparison on the torus)**.** *Let $1\le d\le 7$, $\lambda\ge0$, $h>0$, and let $T=\mathbb{T}^d_\ell$ have volume $V=\ell^d$. For every $U\in\mathcal{A}_\lambda(T)$,*

$$N^{\mathrm{per}}_\lambda(U)+j_\lambda(h)|DU|(T)\ge\sigma_\lambda(h)V. \tag{6.3}$$

More precisely, let $\mu^{(d)}_{\lambda,h}$ be given by Proposition 5.1, and set

$$\psi_U(r):=\fint_{\mathbb{S}^{d-1}}\mathscr{D}_U(re)\,d\mathcal{H}^{d-1}(e).$$

Then,

$$N^{\mathrm{per}}_\lambda(U)+j_\lambda(h)|DU|(T)-\sigma_\lambda(h)V=\langle\mu^{(d)}_{\lambda,h},\psi_U(|\cdot|)\rangle+V\sum_{q\in(2\pi/\ell)\mathbb{Z}^d}\frac{1+\cos(h|q|)}{(1+\cosh(\lambda h))(\lambda^2+|q|^2)}|\widehat{U}(q)|^2. \tag{6.4}$$

For $\lambda=0$, the sum omits $q=0$. Both terms on the right-hand side are nonnegative.

For $d\le5$, the first term agrees with

$$\int_{\mathbb{R}^d}\mathscr{D}_U(z)\,d\mu^{(d)}_{\lambda,h}(z).$$

For $d=6,7$, it is the extended radial pairing defined in Proposition 5.1.

Proof. Write $\mu:=\mu^{(d)}_{\lambda,h}$ and define

$$Q_\mu(U):=V\sum_{q\in(2\pi/\ell)\mathbb{Z}^d}\widehat{\mu}(q)|\widehat{U}(q)|^2.$$

This series converges absolutely. Indeed, $\widehat{\mu}$ has the same radial profile as the Fourier transform of the finite positive one-dimensional measure by (5.1), and therefore $|\widehat{\mu}(q)|\le\sigma_\lambda(h)$.

We first identify $Q_\mu(U)$ with a radial pairing. Set as before

$$\mathcal{C}_U(z):=\int_T U(x+z)U(x)\,dx,\qquad\overline{\mathcal{C}}_U(r):=\fint_{\mathbb{S}^{d-1}}\mathcal{C}_U(re)\,d\mathcal{H}^{d-1}(e).$$

Since U takes values in $\{-1,1\}$,

$$\psi_U(r)=c_dr|DU|(T)-V+\overline{\mathcal{C}}_U(r). \tag{6.5}$$

Reasoning as in the proof of Lemma 6.1 we have that $\mathcal{C}_U$ is Lipschitz. More precisely, for every $e\in\mathbb{S}^{d-1}$

$$|\mathcal{C}_U(z+re)-\mathcal{C}_U(z)|\le|r|\,|D_eU|(T). \tag{6.6}$$

Moreover, by Lemma 6.1 and (6.5), $\overline{\mathcal{C}}_U \in C^1((0,\infty))$ when $d = 6, 7$.

Let ρ_ε be a nonnegative radial smooth mollifier on $\mathbb{R}^d$. Set $\mathcal{C}_{U,\varepsilon} := \rho_\varepsilon * \mathcal{C}_U$, using the periodic extension of $\mathcal{C}_U$. Denote its spherical average again by $\overline{\mathcal{C}}_{U,\varepsilon}$. Orthogonal invariance of μ and the Fourier expansion of the smooth periodic function $\mathcal{C}_{U,\varepsilon}$ give

$$\big\langle \mu, \overline{\mathcal{C}}_{U,\varepsilon}(|\cdot|)\big\rangle = \langle \mu, \mathcal{C}_{U,\varepsilon}\rangle = V \sum_{q \in (2\pi/\ell)\mathbb{Z}^d} \widehat{\mu}(q)\widehat{\rho}_\varepsilon(q)|\widehat{U}(q)|^2.$$

Since $|\widehat{\rho}_\varepsilon| \le 1$ and $\widehat{\rho}_\varepsilon(q) \to 1$ as $\varepsilon \to 0^+$, dominated convergence shows that the right-hand side tends to $Q_\mu(U)$.

We now pass to the limit on the left-hand side. If $d \le 5$, this follows from uniform convergence of the spherical averages and finiteness of μ. For $d = 6, 7$, radial convolution commutes with spherical averaging. Consequently, $\overline{\mathcal{C}}_{U,\varepsilon}$ converges to $\overline{\mathcal{C}}_U$ in $C^1_{\mathrm{loc}}((0,\infty))$, because the radial function $z \mapsto \overline{\mathcal{C}}_U(|z|)$ is C^1 away from the origin. Furthermore, convolution preserves (6.6), so

$$|\overline{\mathcal{C}}_{U,\varepsilon}(r)| \le V, \qquad |\overline{\mathcal{C}}'_{U,\varepsilon}(r)| \le c_d |DU|(T).$$

The extended pairing formula therefore gives

$$\big\langle \mu, \overline{\mathcal{C}}_{U,\varepsilon}(|\cdot|)\big\rangle = \int_{\mathbb{R}^{d-2}} \left[\overline{\mathcal{C}}_{U,\varepsilon}(|z|) + \frac{|z|}{d-2}\overline{\mathcal{C}}'_{U,\varepsilon}(|z|)\right] d\mu^{(d-2)}_{\lambda,h}(z).$$

The preceding bounds permit dominated convergence against the finite measure $\mu^{(d-2)}_{\lambda,h}$. We have therefore proved, in every dimension $1 \le d \le 7$, that

$$Q_\mu(U) = \big\langle \mu, \overline{\mathcal{C}}_U(|\cdot|)\big\rangle. \tag{6.7}$$

By the mass and radial first-moment identities in Proposition 5.1,

$$\langle \mu, 1\rangle = \sigma_\lambda(h), \qquad c_d\langle \mu, |\cdot|\rangle = j_\lambda(h).$$

Combining these identities with (6.5) and (6.7) yields

$$Q_\mu(U) + j_\lambda(h)|DU|(T) - \sigma_\lambda(h)V = \big\langle \mu, \psi_U(|\cdot|)\big\rangle \ge 0. \tag{6.8}$$

Indeed, for $d \le 5$ this follows from positivity of μ and nonnegativity of ψ_U. For $d = 6, 7$, Lemma 6.1 shows that ψ_U is nonnegative, nondecreasing, Lipschitz, and C^1 away from zero, so the positivity assertion of Proposition 5.1 applies. For $d \le 5$, rotational invariance also gives

$$\big\langle \mu, \psi_U(|\cdot|)\big\rangle = \int_{\mathbb{R}^d} \mathscr{D}_U(z)\, d\mu(z).$$

Finally, for $\lambda > 0$,

$$N^{\mathrm{per}}_\lambda(U) = V \sum_{q \in (2\pi/\ell)\mathbb{Z}^d} \frac{|\widehat{U}(q)|^2}{\lambda^2 + |q|^2}.$$

For $\lambda = 0$, the same formula holds with $q \neq 0$, and $\widehat{U}(0) = 0$ by the admissibility condition. By (5.1),

$$\frac{1}{\lambda^2 + |q|^2} - \widehat{\mu}(q) = \frac{1 + \cos(h|q|)}{(1 + \cosh(\lambda h))(\lambda^2 + |q|^2)} \ge 0, \tag{6.9}$$

with $q \neq 0$ when $\lambda = 0$. Subtracting $Q_\mu(U)$ from the Fourier representation of $N^{\mathrm{per}}_\lambda(U)$ and using (6.8) proves (6.4) and (6.3). □

The next result entails that if $\lambda > 0$ then equality forces $\overline{U} = 0$.

Corollary 6.3 (The screened constant mode)**.** *If $\lambda > 0$, then*

$$N^{\mathrm{per}}_\lambda(U) + j_\lambda(h)|DU|(T) - \sigma_\lambda(h)V \ge \frac{V}{\lambda^2}\operatorname{sech}^2\left(\frac{\lambda h}{2}\right)|\overline{U}|^2. \tag{6.10}$$

In particular, equality in (6.3) *forces $\overline{U} = 0$.*

Proof. Retaining the $q = 0$ term in (6.4), its coefficient is

$$\frac{2}{(1+\cosh(\lambda h))\lambda^2} = \lambda^{-2}\operatorname{sech}^2\left(\frac{\lambda h}{2}\right) > 0.$$

□

6.3. **Rigidity of zero translation defect.** In this subsection, we prove that if $U \in BV(\mathbb{T}^d_\ell;\{-1,1\})$ and its translation defect $\mathscr{D}_U$ vanishes in a neighborhood of the origin, then U is one-dimensional.

Lemma 6.4 (Zero translation defect implies parallel layers)**.** *Let $U \in BV(\mathbb{T}^d_\ell;\{-1,1\})$ be non-constant up to $\mathcal{L}^d$-null sets, and let $h > 0$. Suppose*

$$\mathscr{D}_U(z) = 0 \qquad \text{for } \mathcal{L}^d\text{-almost every } |z| < h. \tag{6.11}$$

For the periodic lift to $\mathbb{R}^d$, let E_+ and E_- be the density-one representatives of $\{U = 1\}$ and $\{U = -1\}$, respectively. Then E_+ and E_- are open, and their common boundary is a locally finite family of complete parallel affine hyperplanes. Consecutive planes have distance at least h. In particular,

$$U(x) = g(\nu \cdot x) \quad \text{almost everywhere} \tag{6.12}$$

for some unit vector ν and $g \in BV_{\mathrm{loc}}(\mathbb{R};\{-1,1\})$ whose values alternate across the intervals.

Proof. Set $T := \mathbb{T}^d_\ell$, and also denote by U its periodic lift to $\mathbb{R}^d$. By continuity of $\mathscr{D}_U$ given by Lemma 6.1, we get that (6.11) holds for every $|z| < h$.

Step 1: Short segments contain at most one jump. Fix $e \in \mathbb{S}^{d-1}$ and $r \in (0,h)$. By (6.11),

$$\mathscr{T}_U(re) = \mathscr{A}_U(re) = r\int_T |e\cdot\theta|\, d|DU| = r\,|D_eU|(T).$$

By the slicing theorem for BV functions (see [5, section 3.11]), for $\mathcal{H}^{d-1}$-almost every $y \in e^\perp$, the function

$$f_y(s) := U(y + se)$$

belongs to $BV_{\mathrm{loc}}(\mathbb{R};\{-1,1\})$. It is therefore piecewise constant, with locally finitely many jumps, each of size two. Define

$$\Lambda_{e,r}(y+se) := \frac{1}{2}|Df_y|((s,s+r)).$$

By definition $\Lambda_{e,r}(x)$ counts the jumps along the open segment from x to $x + re$, and

$$|U(x+re) - U(x)| \le 2\Lambda_{e,r}(x) \qquad \text{for almost every } x.$$

Let $F := [0,\ell)^d$ be a half-open fundamental cell. Slicing and Tonelli's theorem give

$$\begin{aligned} 2\int_F \Lambda_{e,r}(x)\,dx &= \int_{e^\perp}\int_{\mathbb{R}} \mathbf{1}_F(y+se)\int_{(s,s+r)} d|Df_y|(t)\,ds\,d\mathcal{H}^{d-1}(y) \\ &= \int_0^r \int_{\mathbb{R}^d} \mathbf{1}_F(z-\tau e)\,d|D_eU|(z)\,d\tau = \int_0^r |D_eU|(F+\tau e)\,d\tau \\ &= r\,|D_eU|(T), \end{aligned}$$

where the last equality follows from periodicity, since every translate of F is a fundamental cell for the periodic measure $|D_eU|$. Therefore,

$$\int_F \big(2\Lambda_{e,r}(x) - |U(x+re)-U(x)|\big)\,dx = \mathscr{A}_U(re) - \mathscr{T}_U(re) = \mathscr{D}_U(re) = 0.$$

Since the integrand is nonnegative, we infer that

$$2\Lambda_{e,r}(x) = |U(x+re)-U(x)| \le 2 \qquad \text{for } \mathcal{L}^d\text{-almost every } x.$$

Hence $\Lambda_{e,r} \le 1$ $\mathcal{L}^d$-almost everywhere. By periodicity and Fubini's theorem, for $\mathcal{H}^{d-1}$-almost every $y \in e^\perp$ and $\mathcal{H}^1$-almost every $s \in \mathbb{R}$, the restriction $t \mapsto U(y+te)$ has at most one jump in $(s, s+r)$.

Step 2: Local half-space structure. Let $B \subset \mathbb{R}^d$ be any open ball with $\operatorname{diam} B < h$, and choose $r \in (\operatorname{diam} B, h)$. For every direction e, Step 1 implies that $\mathcal{H}^{d-1}$-almost every line parallel to e contains at most one jump inside B. Thus both phases have interval sections in B, modulo one-dimensional null sets. Applying Lemma 6.5 to $B \cap \{U = 1\}$, we conclude that U is either constant almost everywhere in B, or

$$U = 2\mathbf{1}_H - 1 \qquad \text{almost everywhere in } B$$

for a half-space H whose boundary meets B. In the nonconstant case, the density-one representatives satisfy

$$E_+ \cap B = B \cap \operatorname{int} H, \qquad E_- \cap B = B \cap \operatorname{int}(H^c).$$

In the constant case, one representative fills B and the other is empty. In particular, setting $\Gamma := \operatorname{supp}|DU|$, we infer that for every $p \in \Gamma$ and every $0 < \rho < h/2$, there is an affine hyperplane P_p through p such that $\Gamma \cap B_\rho(p) = P_p \cap B_\rho(p)$. Since balls of diameter less than h cover $\mathbb{R}^d$, the local description shows that E_+ and E_- are open and that $\partial E_+ = \partial E_- = \Gamma$.

Step 3: Complete parallel hyperplanes and their spacing. The local supporting hyperplane is unique and locally constant along Γ. Indeed, if $q \in \Gamma \cap B_\rho(p)$, then $q \in P_p$, so $P_p \cap B_\rho(p) \cap B_\rho(q)$ is a relatively open neighborhood of q in P_p. It is contained in P_q, hence $P_p = P_q$. Consequently, each connected component C of Γ lies in a single affine hyperplane P. The local description makes C relatively open in P. It is also relatively closed in P, because connected components of the closed set Γ are closed. Since P is connected, $C = P$. Thus Γ is a union of complete affine hyperplanes. Distinct such hyperplanes are disjoint, since they belong to different connected components of Γ, and hence parallel.

Their mutual distances are at least h. Indeed, if two planes had distance $a < h$, a ball centered halfway between them, with radius in $(a/2, h/2)$, would meet both. This contradicts Step 2, since that ball has diameter less than h and can contain at most one planar interface. The family of planes is therefore locally finite.

Step 4: One-dimensionality. Since U is nonconstant, Γ is nonempty. Choose a common unit normal ν and write

$$\Gamma = \bigcup_{t \in \mathcal{S}} \{x \in \mathbb{R}^d : x \cdot \nu = t\},$$

where $\mathcal{S} \subset \mathbb{R}$ is closed and its distinct points have distance at least h. For every connected component I of $\mathbb{R} \setminus \mathcal{S}$, the corresponding slab

$$\{x \in \mathbb{R}^d : x \cdot \nu \in I\}$$

is connected and has zero variation measure. Hence U is constant almost everywhere on that slab. Define the function g on I to be this constant value. Then, by definition

$$U(x) = g(x \cdot \nu) \qquad \mathcal{L}^d\text{-almost everywhere.}$$

The local half-space description shows that the values switch across every interface, so the phases alternate. Since $\mathcal{S}$ is locally finite, g has only finitely many jumps on each bounded interval, and therefore $g \in BV_{\mathrm{loc}}(\mathbb{R}; \{-1, 1\})$. □

The following result was used in the proof of Lemma 6.4. Although the result is probably standard, we could not find anywhere a rigorous proof, hence we sketch its proof.

Lemma 6.5 (Euclidean monotone sets)**.** *Let $B \subset \mathbb{R}^d$ be an open ball and let $A \subset B$ be measurable. Suppose that, for almost every direction $e \in \mathbb{S}^{d-1}$ and for $\mathcal{H}^{d-1}$-almost every $y \in e^\perp$, both*

$$A \cap (y + \mathbb{R}e) \qquad \text{and} \qquad (B \setminus A) \cap (y + \mathbb{R}e)$$

agree, up to $\mathcal{H}^1$-null sets, with intervals, possibly empty. Then there exists a half-space $H \subset \mathbb{R}^d$ such that

$$|A \triangle (B \cap H)| = 0,$$

allowing also the cases $|A| = 0$ and $|B \setminus A| = 0$.

Proof. If $|A| = 0$ or $|B \setminus A| = 0$, the conclusion is immediate.

We first prove that A agrees almost everywhere with a convex set. The hypothesis concerning the sections of A, together with Fubini's theorem and the Blaschke–Petkantschin formula, implies

$$(1-t)x + ty \in A \quad \text{for almost every } (x, y, t) \in A \times A \times (0, 1). \tag{6.13}$$

Here and below, almost every refers to the appropriate product Lebesgue measure. To justify this implication, parametrize pairs of distinct points by their common line and their two coordinates on that line. On almost every such line ℓ, the section $A \cap \ell$ agrees almost everywhere with an interval I_ℓ. For almost every pair of endpoints in $A \cap \ell$, both endpoints belong to I_ℓ and are distinct. Their interpolating segment lies in I_ℓ, and its points belong to A for almost every interpolation parameter. The Blaschke–Petkantschin formula then transfers this assertion to almost every pair in $A \times A$ (see [51, Section 7.2]).

We now iterate (6.13). For $k \geq 1$, consider

$$\Delta_k^\circ = \left\{ (\lambda_1, \dots, \lambda_k) : \lambda_i > 0,\ \sum_{i=1}^k \lambda_i < 1 \right\}, \qquad \lambda_0 = 1 - \sum_{i=1}^k \lambda_i.$$

Then

$$\sum_{i=0}^k \lambda_i x_i \in A \quad \text{for almost every } (x_0, \dots, x_k, \lambda) \in A^{k+1} \times \Delta_k^\circ. \tag{6.14}$$

We prove this by induction on k. The case $k = 1$ is (6.13). For the induction step, write

$$\sum_{i=0}^k \lambda_i x_i = (1-t)z + t x_k, \qquad z = \sum_{i=0}^{k-1} \Lambda_i x_i,$$

where $t = \lambda_k$ and $\Lambda_i = \lambda_i/(1-t)$. By the induction hypothesis, $z \in A$ for almost every choice of the preceding vertices and coefficients.

For each fixed $\Lambda \in \Delta_{k-1}^\circ$, the map

$$(x_0, \dots, x_k, t) \longmapsto \left(\sum_{i=0}^{k-1} \Lambda_i x_i,\ x_k,\ t \right)$$

is a surjective affine map. Its inverse image of a Lebesgue-null set is therefore null. Consequently, the exceptional set in (6.13), applied to (z, x_k, t), remains null after this substitution. Fubini's theorem and the smooth change of coefficient variables $(\Lambda, t) \mapsto \lambda$ prove (6.14). Taking $k = d$ and applying Fubini, we obtain

$$\left| \mathrm{co}\{x_0, \dots, x_d\} \setminus A \right| = 0 \quad \text{for almost every } (x_0, \dots, x_d) \in A^{d+1}. \tag{6.15}$$

Indeed, for affinely independent vertices the barycentric map from Δ_d° onto the interior of their simplex is an invertible affine map. For affinely dependent vertices, the simplex has zero d-dimensional measure.

Let $E := A^{(1)} \cap B$, and $K := \mathrm{co}\, E$. By the Lebesgue density theorem, $|A \setminus K| = 0$. We show that every simplex with vertices in E is contained in A up to a null set. Fix affinely independent $p_0, \dots, p_d \in E$ and set $S := \mathrm{co}\{p_0, \dots, p_d\}$. Every sufficiently small product neighborhood of $(p_0, \dots, p_d)$ meets A^{d+1} in a set of positive measure. We can therefore choose tuples $(x_0^{(n)}, \dots, x_d^{(n)})$ satisfying (6.15), with $x_i^{(n)} \to p_i$ for every i. Set $S_n = \mathrm{co}\{x_0^{(n)}, \dots, x_d^{(n)}\}$. Every point of $\operatorname{int} S$ belongs to S_n for all sufficiently large n. Since $|\partial S| = 0$, Fatou's lemma gives

$$|S \setminus A| \leq \liminf_{n \to \infty} |S_n \setminus A| = 0.$$

Choose a countable dense subset $D \subset E$. By Carathéodory's theorem, $\mathrm{co}\, D$ is a countable union of simplices with vertices in D. Hence $|(\mathrm{co}\, D) \setminus A| = 0$. Moreover, we have $\overline{\mathrm{co}\, D} = \overline{K}$. Convex sets with the same closure have the same interior, and the boundary of a convex set is Lebesgue-null. Thus $|K \setminus \mathrm{co}\, D| = 0$, and consequently $|K \setminus A| = 0$. We thus have proved $|A \triangle K| = 0$. Notice also that $K \subset B$, since B is convex.

Applying the same argument to $B \setminus A$ yields a convex set $K' \subset B$ such that $|(B \setminus A) \triangle K'| = 0$. Both K and K' have nonempty interior, because both have positive measure. Their interiors are disjoint, since $|K \cap K'| = 0$. If one of them has empty interior, it is null and we choose H disjoint from B or containing B. Otherwise, the separation theorem for convex sets [50, Theorem 11.3] gives a half-space H with $\operatorname{int} K \subset H$ and $\operatorname{int} K' \subset \mathbb{R}^d \setminus H$. Since $|B \setminus (K \cup K')| = 0$ and convex sets have null boundary, we conclude $|A \triangle (B \cap H)| = 0$. □

6.4. **Proof of the periodic theorem.** We are finally ready to prove Theorem 3.1.

Proof of Theorem 3.1. Multiplying (6.3) by γ and using $J = \gamma j_\lambda(h)$ proves (3.7).

Suppose now that equality holds in (3.7). Then both nonnegative terms in (6.4) vanish. In particular,

$$\left\langle \mu_{\lambda,h}^{(d)}, \psi_U(|\cdot|) \right\rangle = 0.$$

For $d \leq 5$, positivity of the Lebesgue density of $\mu_{\lambda,h}^{(d)}$ in B_h, together with continuity and non-negativity of ψ_U, implies $\psi_U(r) = 0$ for every $0 < r < h$. For $d = 6, 7$, since ψ_U is non-negative and non-decreasing,

the pairing in (5.3) is non-negative, hence, the vanishing of the pairing implies $\psi_0 = 0$ on $[0, h]$ and $\psi_0' = 0$ on $(0, h]$.

Thus, in every dimension under consideration,

$$\fint_{\mathbb{S}^{d-1}} \mathscr{D}_U(re)\, d\mathcal{H}^{d-1}(e) = \psi_U(r) = 0 \qquad (0 < r < h).$$

Since $\mathscr{D}_U$ is continuous and nonnegative, its restriction to each sphere ∂B_r must vanish identically. Together with $\mathscr{D}_U(0) = 0$, this yields

$$\mathscr{D}_U(z) = 0 \qquad \text{for every } |z| < h.$$

The function U has zero mean. Indeed, this is imposed for $\lambda = 0$, and follows from Corollary 6.3 for $\lambda > 0$. It is therefore non-constant. Lemma 6.4 yields $U(x) = g(\nu \cdot x)$ with parallel interfaces spaced at least h apart for some $\nu \in \mathbb{S}^{d-1}$.

Tangential translation invariance implies that every nonzero Fourier frequency with $\widehat{U}(q) \neq 0$ is parallel to ν. To verify this, for each $v \perp \nu$ one has $U(\cdot + v) = U$, hence $e^{iq\cdot v}\widehat{U}(q) = \widehat{U}(q)$. Varying v gives $q \perp \nu^{\perp}$. Vanishing of the spectral term in (6.4) further implies that if $\widehat{U}(q) \neq 0$ for $q \neq 0$, then $h|q| \in (2\mathbb{N} - 1)\pi$.

Since such q's are parallel to ν, we get that $e^{iq\cdot(h\nu)} = -1$. The zero Fourier coefficient vanishes, so uniqueness of Fourier series gives

$$U(x + h\nu) = -U(x) \quad \text{almost everywhere}, \qquad g(t + h) = -g(t) \quad \text{almost everywhere}. \tag{6.16}$$

Thus, (6.16) exchanges the two phases and leaves their interface family invariant. That is, if S is the set of jump points of g, then $S + h = S$. Choose $t_0 \in S$. The set $t_0 + h\mathbb{Z}$ is contained in S. No additional jump point can lie between two consecutive points of this grid, because distinct points have distance at least h. Hence $S = t_0 + h\mathbb{Z}$. The phases alternate, so g is a square wave with layer width exactly h. Its period group is exactly $2h\mathbb{Z}$. Using the fact that U is L-periodic in each coordinate, we infer that for every $i = 1, \dots, d$,

$$U(x + Le_i) = g\big((x + Le_i) \cdot \nu\big) = g(x \cdot \nu + L\nu_i).$$

Therefore, $g(t + L\nu_i) = g(t)$ for $\mathcal{L}^1$-almost every t. So $L\nu_i$ must be a period of g, which in turn means $L\nu_i = 2h\, a_i$ for some $a_i \in \mathbb{Z}$. Since $|\nu| = 1$,

$$|a| = \frac{L}{2h}|\nu| = \frac{L}{2h} > 0,$$

so $a \neq 0$ and the phase of the square wave gives exactly (3.8).

Conversely, let U have the form (3.8). Notice that by symmetry it has zero mean and variation $|DU|(T) = 2V/h$. Its Fourier support consists of odd multiples of $2\pi a/L$. In particular, for every $q \in (2\pi/L)\mathbb{Z}^d$ with $q \neq 0$ such that $\widehat{U}(q) \neq 0$, since $|a| = L/2h$, we have $h|q| \in (2\mathbb{N} - 1)\pi$. Thus the spectral term in (6.4) vanishes. If $|z| \leq h$, then $|z \cdot \nu| \leq h$, and direct translation of the square wave gives

$$\mathscr{T}_U(z) = \frac{2V}{h}|z \cdot \nu| = \mathscr{A}_U(z).$$

This implies that the geometric term in (6.4) also vanishes since the measure is supported in $\overline{B}_h$. This proves equality and completes the classification. □

6.5. **Even reflection and the Neumann functional.** Before proving Theorem 3.2 we need a preliminary result.

Lemma 6.6 (Exact reflection identities)**.** *Let $u \in \mathcal{A}_\lambda(Q_L)$. Define its even reflection on $(-L, L)^d$ by*

$$U(x) := u(|x_1|, \dots, |x_d|),$$

and extend it with period $2L$ in every coordinate. Then

$$|DU|(\mathbb{T}^d_{2L}) = 2^d |Du|(Q_L), \qquad \int_{\mathbb{T}^d_{2L}} U = 2^d \int_{Q_L} u, \tag{6.17}$$

and

$$N^{\mathrm{per}}_\lambda(U) = 2^d N^N_\lambda(u). \tag{6.18}$$

Proof. We first observe that on each open reflected copy of Q_L, reflection is an isometry and preserves total variation. Across a coordinate face $x_i = 0$, the two one-sided traces agree by evenness. Moreover, the same holds across the periodic identification $x_i = -L \sim L$. Therefore, there is no additional jump term on these faces (see e.g. [5, Theorem 3.84]). This entails $U \in BV(\mathbb{T}^d_{2L}; \{-1, 1\})$ and (6.17).

To prove (6.18) let $v = R^N_\lambda u$ on Q_L (with mean-zero normalization if $\lambda = 0$), and let W be its even reflection. Since again the traces match along the faces, we get $W \in H^1(\mathbb{T}^d_{2L})$. For a smooth periodic test function ψ defined on $(-L, L)^d$, set

$$\varphi(y) := \sum_{\varepsilon \in \{-1,1\}^d} \psi(\varepsilon_1 y_1, \dots, \varepsilon_d y_d), \qquad y \in Q_L.$$

Clearly $\varphi \in H^1(Q_L)$ and satisfies

$$\partial_i \varphi(y) = \sum_{\varepsilon \in \{-1,1\}^d} \varepsilon_i (\partial_i \psi)(\varepsilon y).$$

We now use φ in (3.2). Since $\partial_i W(\varepsilon y) = \varepsilon_i \partial_i v(y)$,

$$\int_{\mathbb{T}^d_{2L}} (\nabla W \cdot \nabla \psi + \lambda^2 W \psi)\, \mathrm{d}x = \int_{\mathbb{T}^d_{2L}} U \psi \, \mathrm{d}x.$$

By density, the identity holds for every $\psi \in H^1(\mathbb{T}^d_{2L})$. For $\lambda > 0$ this characterizes $R^{\mathrm{per}}_\lambda U$. For $\lambda = 0$, U and the reflected potential both have zero mean, and the zero-mean solution is unique. Hence the reflected potential equals $R^{\mathrm{per}}_\lambda U$. Therefore, using a change of variables,

$$N^{\mathrm{per}}_\lambda(U) = \int_{\mathbb{T}^d_{2L}} UW \, \mathrm{d}x = 2^d \int_{Q_L} uv \, \mathrm{d}x = 2^d N^N_\lambda(u).$$

This concludes the proof. □

6.6. **Proof of the Neumann theorem.** We now prove Theorem 3.2, it will follow as a consequence of Theorem 3.1.

Proof of Theorem 3.2. Let $u \in \mathcal{A}_\lambda(Q_L)$, and like in Lemma 6.6, define its even extension by $U(x) = u(|x_1|, \dots, |x_d|)$ for $x \in (-L, L)^d$ followed by $2L$-periodic extension in each coordinate. By Lemma 6.6,

$$|DU|(\mathbb{T}^d_{2L}) = 2^d |Du|(Q_L), \qquad N^{\mathrm{per}}_\lambda(U) = 2^d N^N_\lambda(u).$$

This gives that $\mathcal{E}^{\mathrm{per}}_{\lambda,J,\gamma}(U) = 2^d \mathcal{E}^N_{\lambda,J,\gamma}(u)$. Moreover, $\overline{U} = \overline{u}$ by (6.17). In particular, when $\lambda = 0$, the mean-zero constraint is preserved, so U is an admissible periodic competitor. Since $J = \gamma j_\lambda(h)$, Proposition 6.2 on $\mathbb{T}^d_{2L}$ gives $\mathcal{E}^{\mathrm{per}}_{\lambda,J,\gamma}(U) \geq \gamma \sigma_\lambda(h)(2L)^d$. Dividing by 2^d we obtain

$$\mathcal{E}^N_{\lambda,J,\gamma}(u) \geq \gamma \sigma_\lambda(h) L^d,$$

which is (3.9). For $\lambda > 0$, applying Corollary 6.3 to U gives

$$\mathcal{E}^N_{\lambda,J,\gamma}(u) - \gamma \sigma_\lambda(h) L^d \geq \frac{\gamma L^d}{\lambda^2} \operatorname{sech}^2\left(\frac{\lambda h}{2}\right) |\overline{u}|^2.$$

Thus equality forces $\overline{u} = 0$ also in the screened case.

Suppose now that equality holds in (3.9), that is

$$\mathcal{E}^{\mathrm{per}}_{\lambda,J,\gamma}(U) = \gamma \sigma_\lambda(h)(2L)^d.$$

By Theorem 3.1 applied with domain size $2L$, we infer that there exist $a \in \mathbb{Z}^d \setminus \{0\}$ and $\varphi \in \mathbb{R}$ such that

$$U(x) = \operatorname{sign} \cos(q \cdot x + \varphi), \qquad q = \frac{\pi}{L} a, \qquad |a| = \frac{L}{h}.$$

In particular, $q \in \pi \mathbb{Z}^d / L$ and $|q| = \pi / h$.

For each $i \in \{1, \dots, d\}$, let us now define $R_i(x_1, \dots, x_d) := (x_1, \dots, -x_i, \dots, x_d)$. By construction of the even extension, we have that $U(R_i x) = U(x)$ $\mathcal{L}^d$-almost everywhere.

The interfaces of the periodic lift of U, $\Gamma := \operatorname{supp} |DU|$ in $\mathbb{R}^d$, are given by

$$\Gamma = \bigcup_{k \in \mathbb{Z}} \left\{ x \in \mathbb{R}^d : q \cdot x + \varphi = \frac{\pi}{2} + k\pi \right\}.$$

In particular, Γ is a family of complete parallel hyperplanes with normal q. The identity $U \circ R_i = U$ implies $R_i\Gamma = \Gamma$. On the other hand, reflecting these hyperplanes gives a family with normal R_iq. Since the interface family has a unique normal line, R_iq must be parallel to q. This implies $R_iq = \pm q$ for every $i = 1, \dots, d$.

Suppose now that $q_i \neq 0$ and $q_k \neq 0$ for distinct indices i and k. Reflection in the i-th coordinate changes q_i to $-q_i$, so $R_iq \neq q$. It leaves q_k unchanged, so $R_iq \neq -q$. This is a contradiction. Therefore q has exactly one nonzero coordinate.

Since $q \in (\pi/L)\mathbb{Z}^d$, there exist $j \in \{1, \dots, d\}$ and $n \in \mathbb{N}$ with $n \geq 1$ such that $q = \pm\frac{\pi n}{L}e_j$. Thus, $nh = L$ by the condition on $|q|$. In particular, $L/h \in \mathbb{N}$.

Replacing φ by $-\varphi$ if necessary, we may choose the positive orientation and write

$$U(x) = \operatorname{sign}\cos\left(\frac{\pi n x_j}{L} + \varphi\right).$$

Set $\kappa := \pi n/L$. We now show that $\varphi \in \pi\mathbb{Z}$. The fundamental Fourier terms of the square wave are

$$\operatorname{sign}\cos(\kappa x_j + \varphi) = \frac{2}{\pi}e^{i\varphi}e^{i\kappa x_j} + \frac{2}{\pi}e^{-i\varphi}e^{-i\kappa x_j} + (\text{harmonics of order} \geq 3),$$

with the expansion understood in L^2. Thus, with the normalized Fourier coefficients on the torus,

$$\widehat{U}(\kappa e_j) = \frac{2}{\pi}e^{i\varphi}, \qquad \widehat{U}(-\kappa e_j) = \frac{2}{\pi}e^{-i\varphi}.$$

Reflection in the j-th coordinate interchanges these two frequencies. Since $U \circ R_j = U$, their coefficients must agree, giving $e^{i\varphi} = e^{-i\varphi}$. It follows that $\sin\varphi = 0$, or equivalently, $\varphi \in \pi\mathbb{Z}$. Therefore,

$$U(x) = \pm\operatorname{sign}\cos\left(\frac{\pi n x_j}{L}\right).$$

Restricting U to Q_L gives exactly (3.10).

Conversely, suppose now $n = L/h \in \mathbb{N}$, and let

$$u(x) = \varepsilon\operatorname{sign}\cos\left(\frac{\pi n x_j}{L}\right), \qquad \varepsilon \in \{-1, 1\}.$$

This function belongs to $BV(Q_L; \{-1, 1\})$ and it has zero mean. Thus, u is admissible for every $\lambda \geq 0$, including $\lambda = 0$.

Its even periodic extension on $\mathbb{T}^d_{2L}$ is given by

$$U(x) = \varepsilon\operatorname{sign}\cos\left(\frac{\pi n x_j}{L}\right).$$

Indeed, this expression is even in every coordinate and $2L$-periodic. Its fundamental frequency is $q = (\pi n/L)e_j$, with $|q| = \pi/h$. It therefore belongs to the periodic equality class in Theorem 3.1. This implies that

$$\mathcal{E}^{\mathrm{per}}_{\lambda,J,\gamma}(U) = \gamma\sigma_\lambda(h)(2L)^d.$$

Applying the reflection identities of Lemma 6.6 once more yields

$$\mathcal{E}^{N}_{\lambda,J,\gamma}(u) = 2^{-d}\mathcal{E}^{\mathrm{per}}_{\lambda,J,\gamma}(U) = \gamma\sigma_\lambda(h)L^d.$$

These profiles attain the lower bound and are therefore global minimizers. □

Appendix A. Limits of positivity in higher dimensions

The comparison distributions constructed in Proposition 5.1 are finite positive measures in dimensions $d \leq 5$. In dimensions six and seven, their action remains non-negative on non-negative, non-decreasing radial test functions, which is sufficient for the sharp comparison inequality. Nevertheless, positivity as a measure cannot hold in any dimension $d \geq 6$.

More precisely, consider the radial multiplier

$$m_{\lambda,h}(\xi) = \frac{\cosh(\lambda h) - \cos(h|\xi|)}{(1 + \cosh(\lambda h))(\lambda^2 + |\xi|^2)}, \tag{A.1}$$

with the value at $\lambda = 0$ and $\xi = 0$ understood by continuity.

Proposition A.1. *For every $h > 0$, $\lambda \geq 0$, and $d \geq 6$, there is no finite positive Borel measure on $\mathbb{R}^d$ whose Fourier transform equals (A.1) at every real frequency.*

Before proving the Proposition, we need a preliminary lemma.

Lemma A.2. *Let η be a finite positive rotation-invariant measure on $\mathbb{R}^6$. Away from the origin, its projection onto $\mathbb{R}^4$ has a radial density $g\colon (0,\infty)\to[0,+\infty)$ satisfying*

$$g(r)\le \frac{2\eta(\mathbb{R}^6)}{\pi^2 r^4} \qquad \text{for } r>0. \tag{A.2}$$

In particular, g is bounded on every annulus bounded away from the origin. It also has a nonincreasing radial representative.

Proof. By rotation invariance, η decomposes into normalized surface measures on spheres, weighted by the distribution of its mass with respect to the radius, together with a possible atom at the origin. Let ν denote this distribution on $(0,\infty)$.

The radius-s sphere in $\mathbb{R}^6$ is parameterized, away from a surface-null subset, by

$$(y,\vartheta)\longmapsto \left(y,\sqrt{s^2-|y|^2}\cos\vartheta,\sqrt{s^2-|y|^2}\sin\vartheta\right),\quad y\in B^4_s,\quad 0\le\vartheta<2\pi.$$

Its surface element is given by $s\,dy\,d\vartheta$. Since $\mathcal{H}^5(\partial B^6_s)=\pi^3 s^5$, the projection of normalized surface measure has density

$$\frac{2\pi s}{\pi^3 s^5}\mathbf{1}_{\{|y|<s\}}=\frac{2}{\pi^2 s^4}\mathbf{1}_{\{|y|<s\}}.$$

Hence, for $r>0$

$$g(r)=\int_{(r,\infty)}\frac{2}{\pi^2 s^4}\,d\nu(s)\le\frac{2\nu((0,\infty))}{\pi^2 r^4}\le\frac{2\eta(\mathbb{R}^6)}{\pi^2 r^4} \tag{A.3}$$

The quantity on the left hand side above is non-increasing in r by definition. This proves (A.2). □

Proof of Proposition A.1. Suppose first that a finite positive measure η on $\mathbb{R}^6$ has Fourier transform $m_{\lambda,h}$. Since $m_{\lambda,h}$ is radial, uniqueness of Fourier transforms of finite measures implies that η is rotation invariant.

We now project η onto $\mathbb{R}^4$. Its Fourier transform is the restriction of $m_{\lambda,h}$ to $\mathbb{R}^4$. By Proposition 5.1, this is also the Fourier transform of $\mu^{(4)}_{\lambda,h}$, so the two projected measures must coincide. The four-dimensional density in (5.8) is

$$k^{(4)}_{\lambda,h}(r)=2\int_0^{\sqrt{h^2-r^2}} b_{\lambda,h}\left(\sqrt{r^2+t^2}\right)\,dt+\frac{2A_{\lambda,h}}{h\sqrt{h^2-r^2}},\qquad 0<r<h. \tag{A.4}$$

Its integral term is nonnegative. Therefore, recalling $A_{\lambda,h}=1/(8\pi^2(1+C_{\lambda,h}))$, with $C_{\lambda,h}=\cosh(\lambda h)$, we infer

$$k^{(4)}_{\lambda,h}(r)\ge\frac{2A_{\lambda,h}}{h\sqrt{h^2-r^2}}\longrightarrow+\infty\qquad\text{as } r\to h.$$

This contradicts Lemma A.2, which bounds the projected density uniformly on $h/2<r<h$.

This proves nonexistence in dimension six. If a positive finite measure in dimension $d>6$ had transform $m_{\lambda,h}$, its projection onto a six-dimensional subspace would provide the excluded positive measure there. The conclusion thus follows for every $d\ge 6$. □

Remark A.3 (Further dimensional lifts)**.** Proposition A.1 concerns positivity as a measure and does not preclude the use of distributional comparison kernels. Indeed, for $d=4,5$, the identity

$$\left\langle \mu^{(d+2)}_{\lambda,h},\psi_0(|\cdot|)\right\rangle=\int_{\mathbb{R}^d}\left[\psi_0(|z|)+\frac{|z|}{d}\psi_0'(|z|)\right]\,d\mu^{(d)}_{\lambda,h}(z)$$

shows that non-negativity and monotonicity of the radial profile suffice to obtain a non-negative pairing in dimensions six and seven.

A further lift introduces a second derivative. For smooth radial tests, applying the same identity twice would give

$$\left\langle \mu^{(d+4)}_{\lambda,h},\psi_0(|\cdot|)\right\rangle=\int_{\mathbb{R}^d}\left[\psi_0(|z|)+\frac{2d+3}{d(d+2)}|z|\psi_0'(|z|)+\frac{|z|^2}{d(d+2)}\psi_0''(|z|)\right]\,d\mu^{(d)}_{\lambda,h}(z),\qquad d=4,5.$$

Here $\mu^{(d+4)}_{\lambda,h}$ denotes the inverse Fourier transform of (A.1) in dimension $d+4$, defined by continuing the distributional Fourier walk. Thus dimensions eight and nine involve second derivatives of the radial test, and subsequent lifts introduce higher derivatives.

Non-negativity and monotonicity of ψ_0 alone do not guarantee a non-negative pairing in this formula. The properties of the geometric defect established above therefore no longer yield positivity directly. Whether its additional geometric structure can be used to extend the sharp comparison to dimensions $d \geq 8$ remains open.

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