DECOUPLING AND TAIL LAWS FOR SMALL-ORDER METRIC-VALUED NONLOCAL ENERGIES: A STRUCTURAL VIEW OF THE MAZ'YA–SHAPOSHNIKOVA FORMULA

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Abstract. We develop a measure-theoretic framework for small-order limits of nonlocal energies with metric-valued maps. For globally L^p maps, we consider interaction measures whose marginals have uniformly bounded L^∞ densities a_α and b_α. If these densities converge weakly-* to a_∞ and b_∞, the normalized energies converge to the L^p energy weighted by $a_\infty + b_\infty$ if and only if the interaction measures escape every bounded rectangle. For maps that are only locally L^p, we introduce target laws describing the distribution of the values sampled at infinity. Convergence of their p-distance profiles yields the limiting interaction energy for every $1 \le p < \infty$ and arbitrary Polish targets, with integrated p-Wasserstein convergence as a sufficient criterion. The arguments include the endpoint $p = 1$ and require no linear structure on the target space. Applications include directional and anisotropic Maz'ya–Shaposhnikova formulas, a Bernoulli-law interpretation of fractional perimeters, and an Abelian principle for heat-semigroup energies.

1. Introduction

For $0 < s < 1$, $1 \le p < \infty$, and a measurable map $u : \mathbb{R}^d \to \mathbb{R}$, consider the fractional interaction

$$[u]^p_{W^{s,p}(\mathbb{R}^d)} := \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|u(x) - u(y)|^p}{|x - y|^{d+sp}}\, dy\, dx.$$

The two endpoint regimes have different geometric content. In the Bourgain–Brezis–Mironescu regime $s \uparrow 1$, the interaction kernel concentrates near the diagonal and detects the local regularity of u. More precisely, for $p > 1$ and $u \in W^{1,p}(\mathbb{R}^d)$,

$$\lim_{s\uparrow 1}(1 - s)[u]^p_{W^{s,p}(\mathbb{R}^d)} = \kappa_{d,p} \int_{\mathbb{R}^d} |\nabla u|^p\, dx, \tag{BBM}$$

where $\kappa_{d,p} := \frac{1}{p} \int_{\mathbb{S}^{d-1}} |\omega \cdot e|^p\, d\mathcal{H}^{d-1}(\omega)$, $e \in \mathbb{S}^{d-1}$; see [6], as well as [3, 4, 5, 14, 21, 29, 35]. The endpoint $p = 1$ and the corresponding characterization of functions of bounded variation were established by Dávila [12]. Related metric-measure results were obtained in [25, 24, 33, 27, 36].

The small-order regime $s \downarrow 0$ has a different nature. The Maz'ya–Shaposhnikova formula [32] states that, for every $u \in \bigcup_{0<s<1} W^{s,p}(\mathbb{R}^d)$,

$$\lim_{s\downarrow 0} s[u]^p_{W^{s,p}(\mathbb{R}^d)} = \frac{2\sigma_{d-1}}{p} \int_{\mathbb{R}^d} |u(x)|^p\, dx, \qquad \sigma_{d-1} := \mathcal{H}^{d-1}(\mathbb{S}^{d-1}). \tag{MS}$$

After multiplication by s, interactions at every fixed bounded scale vanish, while the surviving mass is carried by pairs whose distance tends to infinity. At the level of the normalized integral, the cost $|u(x) - u(y)|^p$ then decouples into one-point contributions.

Ludwig [31] established anisotropic versions of the Maz'ya–Shaposhnikova formula. The corresponding asymptotics of fractional perimeters were studied by Dipierro–Figalli–Palatucci–Valdinoci [16]. Fractional Orlicz–Sobolev asymptotics on Carnot groups were obtained in [10]. A geometric interpretation in terms of volume growth at infinity was developed by Han [22] and,

in a metric-measure framework containing Carnot groups and spaces satisfying Bishop–Gromov-type estimates, by Han, Xu, Zambanini and the author [23]. Almost sharp conditions for broad classes of Euclidean kernels, including non-symmetric and non-homogeneous interactions, were recently obtained in [13]. A recent geometric characterization appears in [20].

These results motivate a product-measure formulation in which the small-order limit is determined by the escape of normalized interaction mass from bounded subsets of the product space.

More precisely, let $(\Pi_\alpha)_\alpha$ be a family of positive measures on $X \times X$, with marginals $a_\alpha\mu$ and $b_\alpha\mu$, where $a_\alpha, b_\alpha \in L^\infty_+(X,\mu)$, and write

$$\mathcal{E}_\alpha(u) := \int_{X\times X} d_Y(u(x),u(y))^p \, d\Pi_\alpha(x,y).$$

Assume that $a_\alpha \overset{*}{\rightharpoonup} a_\infty$ and $b_\alpha \overset{*}{\rightharpoonup} b_\infty$ in $L^\infty(X,\mu)$.

Our first question is then:

Given a metric space (Y,d_Y), which asymptotic condition on Π_α characterizes the decoupling

$$\mathcal{E}_\alpha(u) \longrightarrow \int_X (a_\infty + b_\infty) d_Y(u(x),y_0)^p \, d\mu(x) \quad \textit{for every } u \in L^p(X;Y)\textit{?}$$

The second is:

What replaces the limit above when the map is only locally L^p and retains nontrivial states at infinity?

The latter issue is closely related to the mass-at-infinity formalism introduced by Davoli–Fanizza–Picerni [15] in the Euclidean setting. On Riemannian manifolds, Caselli–Gennaioli [9] found an analogous phenomenon in which the small-order limit is sensitive to stochastic completeness and, in the non-conservative case, to a bounded harmonic function recording the mass that survives at infinity. The results described below provide a common measure-theoretic answer to the preious two questions.

Decoupling. Let (X,d_X) be a proper metric space, let μ be a positive Radon measure finite on bounded sets, and let (Y,d_Y,y_0) be a pointed metric space containing at least two points. Consider a net $(\Pi_\alpha)_\alpha$ of positive Borel measures on $X \times X$ whose marginals satisfy

$$(\pi_1)_\#\Pi_\alpha = a_\alpha\mu, \qquad (\pi_2)_\#\Pi_\alpha = b_\alpha\mu,$$

where

$$\sup_\alpha(\|a_\alpha\|_\infty + \|b_\alpha\|_\infty) < \infty, \qquad a_\alpha \overset{*}{\rightharpoonup} a_\infty, \qquad b_\alpha \overset{*}{\rightharpoonup} b_\infty.$$

For $u \in L^p(X;Y)$, set

$$J_{\Pi_\alpha}(u) := \int_{X\times X} d_Y(u(x),u(y))^p \, d\Pi_\alpha(x,y), \qquad M_p(u) := \int_X d_Y(u(x),y_0)^p \, d\mu(x).$$

Theorem 2.8 proves the equivalence

$$\Pi_\alpha(K\times K) \longrightarrow 0 \quad \text{for every bounded Borel set } K \subset X$$

if and only if

$$J_{\Pi_\alpha}(u) \longrightarrow \int_X (a_\infty + b_\infty) d_Y(u,y_0)^p \, d\mu \quad \text{for every } u \in L^p(X;Y).$$

We call the first property *escape from bounded rectangles.* Under the stated marginal assumptions, it is equivalent to metric-valued decoupling. Necessity is detected by maps taking only two

values; sufficiency follows from a compact-support decomposition and a uniform L^p-stability estimate. The proof uses only the metric structure of the target and includes $p = 1$.

Target laws and energy profiles at infinity. Let $\Omega \subset X$ be bounded, let $K_\varepsilon(x, \cdot)$ be a finite positive kernel (see Section 3 for the definition) from Ω to X, and write

$$m_\varepsilon(x) := K_\varepsilon(x, X).$$

For $m_\varepsilon(x) > 0$, define the probability law on Y seen from x by

$$\lambda_{\varepsilon,x} := u_\# \left(\frac{K_\varepsilon(x, \cdot)}{m_\varepsilon(x)} \right).$$

Now suppose that Y is Polish, $u|_\Omega \in L^p(\Omega; Y)$, and that, for some $\lambda_\infty \in \mathcal{P}_p(Y)$, the associated p-distance profiles converge in the sense that, for every $a \in Y$,

$$\left(\int_Y d_Y(a, z)^p \, d\lambda_{\varepsilon,x}(z) \right)^{1/p} \longrightarrow \left(\int_Y d_Y(a, z)^p \, d\lambda_\infty(z) \right)^{1/p} \quad \text{in } L^p(\Omega).$$

If, moreover, $m_\varepsilon \to m$ in $L^\infty(\Omega)$, then Theorem 3.3 gives

$$\int_\Omega \int_X d_Y(u(x), u(y))^p K_\varepsilon(x, \, dy) \, d\mu(x) \longrightarrow \int_\Omega m(x) \int_Y d_Y(u(x), z)^p \, d\lambda_\infty(z) \, d\mu(x).$$

Thus the state at infinity is represented by a probability law rather than by a scalar moment. The quadratic case in [15] is recovered from the first two moments; the profile formulation works for every $1 \le p < \infty$ and avoids algebraic expansions of the interaction cost.

A first application is a directional Maz'ya–Shaposhnikova formula. Let $\Omega \subset \mathbb{R}^d$ be bounded and let $u \in L^p_{\mathrm{loc}}(\mathbb{R}^d; Y)$. Suppose that there exists a Borel map

$$u_\infty : \mathbb{S}^{d-1} \to Y, \qquad d_Y(u_\infty(\cdot), y_0) \in L^p(\mathbb{S}^{d-1}),$$

such that,

$$\lim_{s \downarrow 0} s \int_{B_1^c} \frac{d_Y(u(y), a)^p}{|y|^{d+sp}} \, dy = \frac{1}{p} \int_{\mathbb{S}^{d-1}} d_Y(a, u_\infty(\theta))^p \, d\mathcal{H}^{d-1}(\theta) \qquad \forall a \in Y.$$

Set

$$Q_\Omega := (\mathbb{R}^d \times \mathbb{R}^d) \setminus (\Omega^c \times \Omega^c)$$

and define the localized fractional interaction by

$$\mathcal{E}^\Omega_{s,p}(u) := \iint_{Q_\Omega} \frac{d_Y(u(x), u(y))^p}{|x - y|^{d+sp}} \, dy \, dx.$$

If $\mathcal{E}^\Omega_{s_0,p}(u) < \infty$ for some $s_0 > 0$, then Theorem 3.11 yields

$$\lim_{s \downarrow 0} \frac{s}{2} \mathcal{E}^\Omega_{s,p}(u) = \frac{1}{p} \int_\Omega \int_{\mathbb{S}^{d-1}} d_Y(u(x), u_\infty(\theta))^p \, d\mathcal{H}^{d-1}(\theta) \, dx.$$

For $Y = \mathbb{R}$ and $p = 2$, this condition reduces to convergence of the first two moments at infinity and the formula agrees with [15, Theorem 2.11]. The argument also applies to norms induced by origin-symmetric convex bodies. This gives the metric-valued anisotropic formula of Proposition 3.13 and recovers Ludwig's theorem [31] when the state at infinity vanishes and the map is compactly supported.

Singular energies and fractional perimeters. If $E \subset \mathbb{R}^n$ has finite fractional perimeter in Ω at one positive order and

$$\alpha(E) := \lim_{s \downarrow 0} s \int_{E \cap B_1^c} \frac{dy}{|y|^{n+s}}$$

exists, set

$$\vartheta_E := \frac{\alpha(E)}{\sigma_{n-1}}, \qquad \lambda_\infty^E := (1-\vartheta_E)\delta_0 + \vartheta_E \delta_1.$$

Proposition 4.5 then yields

$$\lim_{s\downarrow 0} s\ \mathrm{Per}_s(E;\Omega) = \sigma_{n-1}\big[(1-\vartheta_E)|E\cap\Omega| + \vartheta_E|E^c\cap\Omega|\big].$$

This recovers the formula of Dipierro–Figalli–Palatucci–Valdinoci [16] and identifies its mass-at-infinity parameter as the parameter of a Bernoulli target law.

Heat-semigroup energies. Let $q_t(x,y)$ have marginal densities $a_t, b_t \in L^\infty_+(X,\mu)$, and set

$$J_p(t,u) := \int_X \int_X d_Y(u(x),u(y))^p q_t(x,y)\ d\mu(y)\ d\mu(x).$$

If the corresponding couplings escape from bounded rectangles as $t\to\infty$ and a_t, b_t converge weakly-* to a_∞, b_∞, then

$$J_p(t,u) \longrightarrow \int_X (a_\infty + b_\infty) d_Y(u,y_0)^p\ d\mu.$$

An Abelian argument transfers this large-time limit to the small-order asymptotics of

$$\mathscr{N}^{q,\beta}_{s,p}(u)^p := \int_0^\infty t^{-\beta sp-1} J_p(t,u)\ dt.$$

Under the short-time integrability assumption in Theorem 5.3, one obtains

$$\lim_{s\downarrow 0} s\mathscr{N}^{q,\beta}_{s,p}(u)^p = \frac{1}{\beta p}\int_X (a_\infty + b_\infty) d_Y(u,y_0)^p\ d\mu.$$

This formula determines the constants in the non-symmetric hypoelliptic setting of Buseghin–Garofalo–Tralli [8].

Moreover, on a complete connected Riemannian manifold of infinite volume, the marginals need not be constant multiples of the volume measure. If h_t is the minimal heat kernel and

$$\theta_M(x) := \lim_{t\to\infty} \int_M h_t(x,y)\ d\mu(y),$$

Corollary 5.7 gives

$$\lim_{s\downarrow 0} s\mathscr{N}^{h,1/2}_{s,p}(u)^p = \frac{4}{p}\int_M \theta_M(x) d_Y(u(x),y_0)^p\ d\mu(x).$$

This extends the large-time result of Caselli–Gennaioli [9] from scalar-valued maps and $p=2$ to arbitrary metric targets and every $1\le p<\infty$. Further applications include infinite weighted graphs and the dimension-free Baouendi–Grushin formula of Li–Wang [26].

Carnot groups. Finally, let G be a Carnot group of homogeneous dimension Q, equipped with a homogeneous norm N_G, and let σ_G be the corresponding polar measure on

$$S_G := \{h\in G : N_G(h)=1\}.$$

For $u\in L^p(G;Y)$, define

$$E_{s,p}(u) := \int_G\int_G \frac{d_Y(u(x),u(y))^p}{N_G(x^{-1}y)^{Q+sp}}\ dy\ dx.$$

If $E_{\theta,p}(u)<\infty$ for some $\theta>0$, Theorem 6.2 gives

$$\lim_{s\downarrow 0} sE_{s,p}(u) = \frac{2\sigma_G(S_G)}{p}\int_G d_Y(u(x),y_0)^p\ dx.$$

No differential structure of the target is used. In particular, the result applies to maps between Carnot groups endowed with left-invariant homogeneous distances. The scalar Carnot-group formula is recovered by taking $Y = \mathbb{R}$. Similarly, the scalar heat-semigroup formula of Garofalo–Tralli [19, Theorem 1.5] is recovered when two Haar marginals of unit mass survive at large time.

The approach developed in the present paper can be generalized to the case $s \uparrow 1$, the results will appear in [34].

Organization of the paper. Section 2 establishes the abstract two-marginal decoupling theorem and characterizes metric-valued decoupling by escape from bounded rectangles. Section 3 develops the theory of target laws and their energy profiles, and derives the Euclidean directional and anisotropic formulas. Section 4 separates short- and long-range interactions and interprets fractional-perimeter asymptotics through Bernoulli laws at infinity. Section 5 proves the Abelian semigroup principle and applies it to non-symmetric hypoelliptic semigroups, Riemannian manifolds, weighted graphs, and Baouendi–Grushin structures. Section 6 concludes with the metric-valued Maz'ya–Shaposhnikova formula on Carnot groups.

Acknowledgments: The author thanks Elisa Davoli for some comments on a preliminary version of the paper.

2. Metric-valued L^p maps, two-marginal couplings and the abstract decoupling theorem

Throughout this section (X, d_X) denotes a proper metric space[1] and μ a positive Radon measure, finite on bounded sets and not necessarily globally finite. We also fix a pointed metric space (Y, d_Y, y_0) and $1 \le p < \infty$.

Definition 2.1. *A measurable map $u : X \to Y$ belongs to $L^p(X;Y)$, relative to the chosen base point y_0, if*

$$M_p(u) := \int_X d_Y(u(x), y_0)^p \, d\mu(x) < \infty.$$

For $u, v \in L^p(X;Y)$ we set

$$d_p(u,v) := \left(\int_X d_Y(u(x), v(x))^p \, d\mu(x) \right)^{1/p}.$$

The dependence on the base point is harmless on a finite-measure source, but it matters on an infinite-measure space. We therefore keep the pointed target throughout the paper.

Lemma 2.2. *For every $u \in L^p(X;Y)$ there exists a sequence $u_j \in L^p(X;Y)$ such that*

(i) *$u_j = y_0$ outside a compact set $K_j \subset X$;*
(ii) *$d_Y(u_j(x), y_0) \le j$ for every $x \in X$;*
(iii) *$d_p(u_j, u) \to 0$.*

Proof. The finite Borel measure $d_Y(u(x), y_0)^p \mu$ is Radon. Hence for every $j \in \mathbb{N}$ one can choose a compact $K_j \subset X$ such that

$$\int_{X \setminus K_j} d_Y(u(x), y_0)^p \, d\mu < 2^{-j}.$$

[1] A metric space (X, d_X) is called proper if every closed bounded subset of X is compact.

Define

$$u_j(x) := \begin{cases} u(x), & x \in K_j \text{ and } d_Y(u(x), y_0) \le j, \\ y_0, & \text{otherwise.} \end{cases}$$

Then (i) and (ii) are immediate. Moreover,

$$\begin{aligned} d_p(u_j, u)^p &= \int_{\{u_j \neq u\}} d_Y(u(x), y_0)^p \, d\mu(x) \\ &\le \int_{X \setminus K_j} d_Y(u(x), y_0)^p \, d\mu(x) + \int_{\{d_Y(u(x), y_0) > j\}} d_Y(u(x), y_0)^p \, d\mu(x). \end{aligned}$$

The first term tends to zero by construction and the second tends to zero by dominated convergence. Hence (iii) follows. □

Definition 2.3. *Let $a, b \in L^\infty_+(X, \mu)$. A positive Borel measure Π on $X \times X$ is called an* (a, b)-marginal coupling *of μ if*

$$(\pi_1)_\# \Pi = a\mu, \qquad (\pi_2)_\# \Pi = b\mu,$$

where $\pi_1(x, y) = x$ and $\pi_2(x, y) = y$. Equivalently, for every nonnegative Borel function φ,

$$\int_{X \times X} \varphi(x) \, d\Pi(x, y) = \int_X a\varphi \, d\mu, \qquad \int_{X \times X} \varphi(y) \, d\Pi(x, y) = \int_X b\varphi \, d\mu.$$

For such a coupling and $u \in L^p(X; Y)$, define the *increment energy*

$$J_\Pi(u) := \int_{X \times X} d_Y(u(x), u(y))^p \, d\Pi(x, y).$$

The triangle inequality and the marginal identities give

$$J_\Pi(u) \le 2^{p-1}(\|a\|_\infty + \|b\|_\infty) M_p(u) < +\infty. \tag{1}$$

Definition 2.4. *Let $(\Pi_\alpha)_\alpha$ be a family of positive Borel measures on $X \times X$ indexed by a directed set*[2]*. We say that* $(\Pi_\alpha)_\alpha$ escapes from bounded rectangles *if, for every bounded Borel set $K \subset X$,*

$$\Pi_\alpha(K \times K) \longrightarrow 0, \tag{2}$$

where the limit is understood along the directed set.

Remark 2.5. Since X is proper, it is equivalent to test (2) on compact sets. Indeed, if $K \subset X$ is bounded and Borel, then $\overline{K}$ is compact and

$$0 \le \Pi_\alpha(K \times K) \le \Pi_\alpha(\overline{K} \times \overline{K}).$$

Thus, if $\Pi_\alpha(C \times C) \to 0$ for every compact $C \subset X$, then $\Pi_\alpha(K \times K) \to 0$. The converse is immediate because every compact subset of a metric space is bounded and Borel.

Lemma 2.6. *Let Π be an (a, b)-marginal coupling of μ. Then for every $u, v \in L^p(X; Y)$,*

$$|J_\Pi(u)^{1/p} - J_\Pi(v)^{1/p}| \le (\|a\|_\infty^{1/p} + \|b\|_\infty^{1/p}) d_p(u, v).$$

[2]A partially ordered set $(I, \preceq)$ is called a *directed set* if, for every $i, j \in I$, there exists $k \in I$ such that

$$i \preceq k \qquad \text{and} \qquad j \preceq k.$$

Proof. The reverse triangle inequality in $L^p(X \times X, \Pi)$ gives

$$|J_\Pi(u)^{1/p} - J_\Pi(v)^{1/p}| \leq \left(\int_{X\times X} |d_Y(u(x),u(y)) - d_Y(v(x),v(y))|^p \, d\Pi \right)^{1/p}.$$

By the metric triangle inequality,

$$|d_Y(u(x),u(y)) - d_Y(v(x),v(y))| \leq d_Y(u(x),v(x)) + d_Y(u(y),v(y)).$$

Minkowski's inequality and the marginal identities imply

$$\begin{aligned} &\left(\int_{X\times X} |d_Y(u(x),u(y)) - d_Y(v(x),v(y))|^p \, d\Pi \right)^{1/p} \\ &\leq \left(\int d_Y(u(x),v(x))^p \, d\Pi \right)^{1/p} + \left(\int d_Y(u(y),v(y))^p \, d\Pi \right)^{1/p} \\ &\leq \|a\|_\infty^{1/p} d_p(u,v) + \|b\|_\infty^{1/p} d_p(u,v). \end{aligned}$$

This concludes the proof. □

Proposition 2.7. *Let Π be an (a,b)-marginal coupling and suppose that $u : X \to Y$ satisfies $u = y_0$ outside a bounded Borel set K and $d_Y(u(x),y_0) \leq L$ for every $x \in X$. Then*

$$J_\Pi(u) = \int_X (a+b) d_Y(u,y_0)^p \, d\mu + R_\Pi(u;K), \tag{3}$$

where

$$R_\Pi(u;K) = \int_{K\times K} \left[d_Y(u(x),u(y))^p - d_Y(u(x),y_0)^p - d_Y(u(y),y_0)^p \right] d\Pi \tag{4}$$

and

$$|R_\Pi(u;K)| \leq 2^{p+1} L^p \Pi(K \times K). \tag{5}$$

Proof. Since $u = y_0$ on K^c, we get

$$d_Y(u(x),u(y))^p - d_Y(u(x),y_0)^p - d_Y(u(y),y_0)^p = 0 \quad \text{for } (x,y) \notin K \times K.$$

Hence

$$\begin{aligned} J_\Pi(u) = &\int_{X\times X} \left(d_Y(u(x),y_0)^p + d_Y(u(y),y_0)^p \right) d\Pi \\ &+ \int_{K\times K} \left[d_Y(u(x),u(y))^p - d_Y(u(x),y_0)^p - d_Y(u(y),y_0)^p \right] d\Pi. \end{aligned}$$

The first integral equals $\int_X (a+b) d_Y(u,y_0)^p \, d\mu$ by the marginal identities. On $K \times K$, the triangle inequality gives

$$d_Y(u(x),u(y))^p \leq 2^p L^p.$$

Thus the absolute value of the bracket in (4) is bounded by $2^{p+1}L^p$, which gives (5). □

Theorem 2.8. *Assume that Y contains at least two points. Let $a_\alpha, b_\alpha, a_\infty, b_\infty \in L^\infty_+(X,\mu)$, indexed by a directed set, and suppose that*

$$\sup_\alpha (\|a_\alpha\|_\infty + \|b_\alpha\|_\infty) < \infty, \qquad a_\alpha \overset{*}{\rightharpoonup} a_\infty, \qquad b_\alpha \overset{*}{\rightharpoonup} b_\infty \quad \text{in } L^\infty(X,\mu). \tag{6}$$

Here weak- convergence means convergence of the integrals against every function in $L^1(X,\mu)$. Let $(\Pi_\alpha)_\alpha$ be a family of (a_α,b_α)-marginal couplings. Then the following statements are equivalent:*

(i) *Π_α escapes from bounded rectangles.*

(ii) *For every* $u \in L^p(X;Y)$,

$$J_{\Pi_\alpha}(u) \longrightarrow \int_X (a_\infty + b_\infty) d_Y(u, y_0)^p \, d\mu. \tag{7}$$

(iii) *Fix* $y_1 \in Y \setminus \{y_0\}$. *For every bounded Borel set* $K \subset X$, *the map*

$$u_K(x) := \begin{cases} y_1, & x \in K, \\ y_0, & x \notin K, \end{cases}$$

verifies (7).

Proof. We first prove (i)⇒(ii). Let u_j be the truncations from Lemma 2.2. For fixed $j \in \mathbb{N}$, u_j equals y_0 outside a compact set K_j and $d_Y(u_j(x), y_0) \le j$ on X. By Proposition 2.7,

$$J_{\Pi_\alpha}(u_j) = \int_X (a_\alpha + b_\alpha) d_Y(u_j, y_0)^p \, d\mu + R_{\alpha,j}, \qquad |R_{\alpha,j}| \le C_{p,j} \Pi_\alpha(K_j \times K_j) \to 0.$$

Here $C_{p,j} := 2^{p+1} j^p$, and the convergence follows from Remark 2.5. Since $d_Y(u_j, y_0)^p \in L^1(X, \mu)$, weak-* convergence gives

$$J_{\Pi_\alpha}(u_j) \longrightarrow \int_X (a_\infty + b_\infty) d_Y(u_j, y_0)^p \, d\mu.$$

By Lemma 2.6 and (6), there is a constant $C > 0$ independent of α and j such that

$$|J_{\Pi_\alpha}(u)^{1/p} - J_{\Pi_\alpha}(u_j)^{1/p}| \le C d_p(u, u_j).$$

Set

$$J_\infty(v) := \int_X (a_\infty + b_\infty) d_Y(v, y_0)^p \, d\mu.$$

The reverse triangle inequality in $L^p(X, (a_\infty + b_\infty)\mu; Y)$ gives

$$|J_\infty(u)^{1/p} - J_\infty(u_j)^{1/p}| \le \|a_\infty + b_\infty\|_\infty^{1/p} d_p(u, u_j).$$

Now

$$\begin{aligned} &|J_{\Pi_\alpha}(u)^{1/p} - J_\infty(u)^{1/p}| \\ &\le |J_{\Pi_\alpha}(u)^{1/p} - J_{\Pi_\alpha}(u_j)^{1/p}| \\ &\quad + |J_\infty(u_j)^{1/p} - J_\infty(u)^{1/p}| \\ &\quad + |J_{\Pi_\alpha}(u_j)^{1/p} - J_\infty(u_j)^{1/p}|. \end{aligned}$$

Taking first the limit in α and then letting $j \to \infty$ proves

$$J_{\Pi_\alpha}(u)^{1/p} \longrightarrow J_\infty(u)^{1/p}.$$

Thus (7) holds.

The implication (ii)⇒(iii) is immediate because $\mu(K) < \infty$ for bounded K, hence $u_K \in L^p(X;Y)$.

It remains to prove (iii)⇒(i). Set

$$c := d_Y(y_0, y_1) > 0.$$

By the definition of u_K,

$$\begin{aligned} J_{\Pi_\alpha}(u_K) &= c^p \big[\Pi_\alpha(K \times K^c) + \Pi_\alpha(K^c \times K)\big] \\ &= c^p \big[\Pi_\alpha(K \times X) + \Pi_\alpha(X \times K) - 2\Pi_\alpha(K \times K)\big] \\ &= c^p \big[\int_K (a_\alpha + b_\alpha) \, d\mu - 2\Pi_\alpha(K \times K)\big]. \end{aligned}$$

On the other hand,

$$M_p(u_K) = c^p \mu(K).$$

Therefore, assumption (iii) and weak-* convergence, tested against $\mathbf{1}_K \in L^1(X,\mu)$, give

$$c^p\big[\int_K (a_\alpha + b_\alpha)\, d\mu - 2\Pi_\alpha(K\times K)\big] \longrightarrow c^p \int_K (a_\infty + b_\infty)\, d\mu.$$

It follows that

$$2c^p \Pi_\alpha(K\times K) = c^p \int_K (a_\alpha + b_\alpha)\, d\mu - J_{\Pi_\alpha}(u_K) \longrightarrow 0.$$

Since $c > 0$, we conclude that

$$\Pi_\alpha(K\times K) \longrightarrow 0,$$

and hence $(\Pi_\alpha)_\alpha$ escapes from bounded rectangles. □

3. Target tail laws, energy profiles and localized interactions

The preceding theorem concerns globally L^p maps. We now treat the regime in which the map may carry a nontrivial state at infinity and need only be L^p on a bounded region of X. Throughout this section (Y, d_Y) is assumed to be Polish[3]. For $1 \le p < \infty$, let $\mathcal{P}_p(Y)$ denote the Borel probability measures λ on Y with finite p-moment relative to y_0:

$$\int_Y d_Y(y, y_0)^p \, d\lambda(y) < \infty.$$

Let W_p be the p-Wasserstein distance

$$W_p(\lambda,\eta)^p := \inf_{\gamma\in\Gamma(\lambda,\eta)} \int_{Y\times Y} d_Y(z,w)^p \, d\gamma(z,w).$$

Here $\Gamma(\lambda,\eta)$ denotes the set of probability measures on $Y\times Y$ whose first marginal is λ and whose second marginal is η.

Let $\Omega \subset X$ be a bounded Borel set with $\mu(\Omega) > 0$. Let $(K_\varepsilon)_{\varepsilon>0}$ be a family of finite positive *Borel kernels from Ω to X*; that is, for every $\varepsilon > 0$ and every Borel set $A \subset X$, the map

$$x \longmapsto K_\varepsilon(x, A)$$

is Borel measurable, while $K_\varepsilon(x,\cdot)$ is a finite positive Borel measure on X for every $x \in \Omega$. Write

$$m_\varepsilon(x) := K_\varepsilon(x, X) < \infty.$$

Let $u : X \to Y$ be Borel measurable. On the set where $m_\varepsilon(x) > 0$ define the *normalized target law seen from x* by

$$\lambda_{\varepsilon,x} := u_\# \left(\frac{K_\varepsilon(x,\cdot)}{m_\varepsilon(x)} \right),$$

and set $\lambda_{\varepsilon,x} := \delta_{y_0}$ whenever $m_\varepsilon(x) = 0$. Thus, if $m_\varepsilon(x) > 0$, then for every Borel $\psi : Y \to [0,\infty]$,

$$\int_Y \psi(z)\, d\lambda_{\varepsilon,x}(z) = \frac{1}{m_\varepsilon(x)} \int_X \psi(u(y))\, K_\varepsilon(x,\, dy).$$

Since Y is Polish, $x \mapsto \lambda_{\varepsilon,x}$ is a Borel probability kernel (viewed as a map into $\mathcal{P}(Y)$ with its weak Borel structure); see, e.g., [2, Proposition 7.26]. In particular, the function $x \mapsto W_p(\lambda_{\varepsilon,x}, \lambda)^p$ is measurable for every fixed $\lambda \in \mathcal{P}_p(Y)$, see e.g. [1, Proposition 7.1.3].

[3]A metric space is called Polish if it is complete and separable.

Definition 3.1. *Let $u : X \to Y$ be Borel measurable with $u|_\Omega \in L^p(\Omega;Y)$ and $\lambda_{\varepsilon,x} \in \mathcal{P}_p(Y)$ for almost every $x \in \Omega$. A measure $\lambda_\infty \in \mathcal{P}_p(Y)$ is called an L^p* target tail law *for the family K_ε if*

$$\int_\Omega W_p(\lambda_{\varepsilon,x}, \lambda_\infty)^p \, d\mu(x) \longrightarrow 0 \quad \text{as } \varepsilon \to 0^+ \tag{8}$$

If, in addition,

$$\operatorname*{ess\,sup}_{\substack{x\in\Omega \\ m_\varepsilon(x)>0}} \frac{K_\varepsilon(x,B)}{m_\varepsilon(x)} \longrightarrow 0 \quad \text{as } \varepsilon \to 0^+ \qquad \forall \text{ bounded Borel } B \subset X,$$

we say that λ_∞ is a target law *at infinity.*

For $a \in Y$ set

$$P_{\varepsilon,x}(a) := \left(\int_Y d_Y(a,z)^p \, d\lambda_{\varepsilon,x}(z) \right)^{1/p}, \qquad P_\infty(a) := \left(\int_Y d_Y(a,z)^p \, d\lambda_\infty(z) \right)^{1/p}. \tag{9}$$

We say that the target laws converge to λ_∞ in *p-energy profile* on Ω if

$$P_{\varepsilon,\cdot}(a) \longrightarrow P_\infty(a) \quad \text{in } L^p(\Omega) \qquad \text{for every } a \in Y. \tag{10}$$

This condition records the family of costs tested by the energy and need not determine the limiting probability measure uniquely. Since all the functions in (9) are 1-Lipschitz in a, it is equivalent to require (10) only on a fixed countable dense subset of Y.

Lemma 3.2. *For $a \in Y$ and $\lambda, \eta \in \mathcal{P}_p(Y)$ the following inequality holds*

$$\left| \left(\int_Y d_Y(a,z)^p \, d\lambda(z) \right)^{1/p} - \left(\int_Y d_Y(a,z)^p \, d\eta(z) \right)^{1/p} \right| \le W_p(\lambda,\eta).$$

Proof. Let $\gamma \in \Gamma(\lambda,\eta)$. The reverse triangle inequality in $L^p(Y \times Y, \gamma)$ and the triangle inequality give

$$\begin{aligned} \left| \left(\int_Y d_Y(a,z)^p \, d\lambda(z) \right)^{1/p} - \left(\int_Y d_Y(a,z)^p \, d\eta(z) \right)^{1/p} \right| &\le \left(\int |d_Y(a,z) - d_Y(a,w)|^p \, d\gamma(z,w) \right)^{1/p} \\ &\le \left(\int d_Y(z,w)^p \, d\gamma(z,w) \right)^{1/p}. \end{aligned}$$

The conclusion follows taking the infimum over γ. □

Theorem 3.3. *Let $u : X \to Y$ be Borel measurable with $u|_\Omega \in L^p(\Omega;Y)$. Assume that $\lambda_{\varepsilon,x} \in \mathcal{P}_p(Y)$ a.e. $x \in \Omega$, that (10) holds for some $\lambda_\infty \in \mathcal{P}_p(Y)$, and that*

$$\sup_\varepsilon \|m_\varepsilon\|_{L^\infty(\Omega)} < \infty, \qquad m_\varepsilon \longrightarrow m \quad \text{in } L^\infty(\Omega).$$

Then

$$\begin{aligned} &\lim_{\varepsilon\to 0^+} \int_\Omega \int_X d_Y(u(x),u(y))^p \, K_\varepsilon(x,\, dy) \, d\mu(x) \\ &\qquad = \int_\Omega m(x) \int_Y d_Y(u(x),z)^p \, d\lambda_\infty(z) \, d\mu(x). \end{aligned}$$

Proof. For a.e. $x \in \Omega$, the maps $a \mapsto P_{\varepsilon,x}(a)$ and $a \mapsto P_\infty(a)$ are 1-Lipschitz. Indeed, this follows from the reverse triangle inequality in L^p applied to the functions $z \mapsto d_Y(a,z)$ and $z \mapsto d_Y(b,z)$. Let $(u_j)_j$ be a sequence of finite-valued Borel maps such that

$$\int_\Omega d_Y(u_j(x),u(x))^p \, d\mu(x) \longrightarrow 0. \tag{11}$$

Such a sequence exists because Y is separable and $u|_\Omega \in L^p(\Omega;Y)$.

If u_j takes the values $a_{j,1}, \dots, a_{j,N_j}$, then (10), applied to this finite family, gives

$$P_{\varepsilon,\cdot}(u_j(\cdot)) \longrightarrow P_\infty(u_j(\cdot)) \quad \text{in } L^p(\Omega).$$

The Lipschitz estimate therefore yields

$$\begin{aligned}&\|P_{\varepsilon,\cdot}(u(\cdot)) - P_\infty(u(\cdot))\|_{L^p(\Omega)}\\ &\quad\le 2\|d_Y(u,u_j)\|_{L^p(\Omega)} + \|P_{\varepsilon,\cdot}(u_j(\cdot)) - P_\infty(u_j(\cdot))\|_{L^p(\Omega)}.\end{aligned}$$

First letting $\varepsilon \to 0$ and then $j \to \infty$ proves

$$A_\varepsilon := P_{\varepsilon,\cdot}(u(\cdot)) \longrightarrow A_\infty := P_\infty(u(\cdot)) \quad \text{in } L^p(\Omega). \tag{12}$$

Moreover,

$$A_\infty(x) \le d_Y(u(x), y_0) + P_\infty(y_0),$$

so $A_\infty \in L^p(\Omega)$ and $\sup_\varepsilon \|A_\varepsilon\|_{L^p(\Omega)} < \infty$ for small ε.

We claim that

$$A_\varepsilon^p \longrightarrow A_\infty^p \quad \text{in } L^1(\Omega).$$

For $p = 1$ this is exactly (12). If $p > 1$, then using $|a^p - b^p| \le p|a-b|(a^{p-1} + b^{p-1})$ and the Hölder inequality we get

$$\|A_\varepsilon^p - A_\infty^p\|_{L^1} \le p\|A_\varepsilon - A_\infty\|_{L^p}\big(\|A_\varepsilon\|_{L^p}^{p-1} + \|A_\infty\|_{L^p}^{p-1}\big) \longrightarrow 0.$$

We claim that

$$\int_X d_Y(u(x),u(y))^p K_\varepsilon(x,\, dy) = m_\varepsilon(x) A_\varepsilon(x)^p. \tag{13}$$

Indeed, if $m_\varepsilon(x) = 0$ then $K_\varepsilon(x,\cdot) = 0$ and the equality trivially follows. If $m_\varepsilon(x) > 0$ then

$$\begin{aligned}\int_X d_Y(u(x),u(y))^p K_\varepsilon(x,\, dy) &= m_\varepsilon(x) \int_X d_Y(u(x),u(y))^p \frac{K_\varepsilon(x,\, dy)}{m_\varepsilon(x)}\\ &= m_\varepsilon(x) \int_Y d_Y(u(x),z)^p \; d\lambda_{\varepsilon,x}(z)\\ &= m_\varepsilon(x) A_\varepsilon(x)^p.\end{aligned}$$

Consequently,

$$\begin{aligned}&\left|\int_\Omega m_\varepsilon A_\varepsilon^p \; d\mu - \int_\Omega m A_\infty^p \; d\mu\right|\\ &\quad\le \|m_\varepsilon\|_\infty \|A_\varepsilon^p - A_\infty^p\|_{L^1} + \|m_\varepsilon - m\|_\infty \|A_\infty^p\|_{L^1} \longrightarrow 0.\end{aligned}$$

Together with (13), this concludes the proof. □

Corollary 3.4. *Under the remaining assumptions of Theorem 3.3, the integrated Wasserstein condition* (8) *implies the conclusion of that theorem.*

Proof. The condition (8) implies (10). Indeed, for every $a \in Y$, Lemma 3.2 gives

$$|P_{\varepsilon,x}(a) - P_\infty(a)| \le W_p(\lambda_{\varepsilon,x}, \lambda_\infty)$$

and the conclusion follows taking the $L^p(\Omega)$ norm. □

3.1. **The real-valued quadratic case.** If $Y = \mathbb{R}$ and $p = 2$ one can weaken the convergence of the whole target law to convergence of only three real-valued functions.

Theorem 3.5. *Let $u : X \to \mathbb{R}$ be measurable with $u|_\Omega \in L^2(\Omega)$. For finite positive kernels $K_\varepsilon(x, dy)$ define:*

$$m_\varepsilon(x) := \int_X K_\varepsilon(x, dy),$$
$$b_\varepsilon(x) := \int_X u(y) K_\varepsilon(x, dy),$$
$$c_\varepsilon(x) := \int_X u(y)^2 K_\varepsilon(x, dy),$$

whenever these quantities are finite. Assume

$$m_\varepsilon \to m \quad \text{in } L^\infty(\Omega), \qquad b_\varepsilon \to b \quad \text{in } L^2(\Omega), \qquad c_\varepsilon \to c \quad \text{in } L^1(\Omega).$$

Then

$$\int_\Omega \int_X |u(x) - u(y)|^2 K_\varepsilon(x, dy)\, d\mu(x) \longrightarrow \int_\Omega (mu^2 - 2bu + c)\, d\mu.$$

In particular, if m, b, c are constants M_0, M_1, M_2, the limit can be written as

$$M_0 \int_\Omega u^2\, d\mu - 2M_1 \int_\Omega u\, d\mu + M_2 \mu(\Omega).$$

Proof. For almost every $x \in X$ an explicit computation gives

$$\int_X |u(x) - u(y)|^2 K_\varepsilon(x, dy) = m_\varepsilon(x) u(x)^2 - 2b_\varepsilon(x) u(x) + c_\varepsilon(x).$$

The first term converges in L^1 by the L^∞ convergence of m_ε and $u^2 \in L^1(\Omega)$. The mixed term converges in L^1 by Cauchy–Schwarz, and the last term converges by assumption. □

Remark 3.6. Let $\Omega \subset \mathbb{R}^d$ be bounded, $s > 0$ and let

$$K_s(x, dy) := s\, \mathbf{1}_{\Omega^c}(y) |x - y|^{-d-2s}\, dy.$$

Following [15], for a measurable function $u : \mathbb{R}^d \to \mathbb{R}$ we denote by

$$\alpha(u) := \lim_{s \to 0^+} s \int_{B_1^c(0)} \frac{u(y)}{|y|^{d+2s}}\, dy,$$

whenever the limit exists and is finite. This is the mass at infinity introduced in [15]. With this notation, we call

$$M_0 = \alpha(1) = \frac{\mathcal{H}^{d-1}(\mathbb{S}^{d-1})}{2}, \qquad M_1 = \alpha(u), \qquad M_2 = \alpha(u^2).$$

Their scalar hypotheses are the existence and finiteness of the relevant masses at infinity. In the notation of [15, Theorem 1.3], the assumptions on $\alpha(u^+)$, $\alpha(u^-)$ and $\alpha(u^2)$ determine the three coefficients in Theorem 3.5. The resulting limiting functional has the form

$$\alpha(1) \int_\Omega u^2 - 2\alpha(u) \int_\Omega u + \alpha(u^2) |\Omega|,$$

which is exactly the constant-coefficient functional appearing in Theorem 3.5. When $u(r, \theta) \to u_\infty(\theta)$ at infinity, [15, Theorem 2.11] rewrites their limit as

$$\frac{1}{2} \int_\Omega \int_{\mathbb{S}^{d-1}} |u(x) - u_\infty(\theta)|^2\, d\mathcal{H}^{d-1}(\theta)\, dx.$$

Writing $\sigma_{d-1} := \mathcal{H}^{d-1}(\mathbb{S}^{d-1})$, this has exactly the tail-law form with

$$\lambda_\infty = (u_\infty)_\# \left(\frac{\mathcal{H}^{d-1}\lfloor_{\mathbb{S}^{d-1}}}{\sigma_{d-1}} \right), \qquad m_\infty = \frac{\sigma_{d-1}}{2}.$$

The corresponding tail law is the push-forward of normalized spherical measure under u_∞, together with total mass $\sigma_{d-1}/2$. Its quadratic energy profile depends only on the first two moments, while convergence in W_2 retains the entire probability law.

Theorem 3.3 does not apply directly to the full kernel $s\mathbf{1}_{\Omega^c}(y)|x-y|^{-d-2s}\,dy$, whose mass can blow up as x approaches $\partial\Omega$. To obtain uniformly bounded kernel masses, we introduce a fixed cutoff and estimate the bounded-range contribution separately.

3.2. **Angular tail laws and radial limits.** We now apply the energy-profile theorem to the Euclidean fractional kernel. This yields a metric-valued directional Maz'ya–Shaposhnikova formula [32] for arbitrary $1 \le p < \infty$ without requiring pointwise or uniform convergence along rays.

Throughout this subsection $d \ge 1$, $\sigma_{d-1} := \mathcal{H}^{d-1}(\mathbb{S}^{d-1})$, and $\Omega \subset \mathbb{R}^d$ is bounded. Let (Y, d_Y, y_0) be Polish. We write $u \in L^p_{\mathrm{loc}}(\mathbb{R}^d; Y)$ if $d_Y(u(\cdot), y_0) \in L^p_{\mathrm{loc}}(\mathbb{R}^d)$. Let $u_\infty : \mathbb{S}^{d-1} \to Y$ be Borel with $d_Y(u_\infty(\cdot), y_0) \in L^p(\mathbb{S}^{d-1})$. For $a \in Y$ define the Abelian p-energy profile

$$\mathcal{A}^u_{s,p}(a) := s \int_{B_1^c} \frac{d_Y(u(y), a)^p}{|y|^{d+sp}}\, dy. \tag{14}$$

Our tail assumption, denoted by EP_p, is

$$\mathcal{A}^u_{s,p}(a) \longrightarrow \frac{1}{p} \int_{\mathbb{S}^{d-1}} d_Y(a, u_\infty(\theta))^p \, d\mathcal{H}^{d-1}(\theta) \qquad \text{for every } a \in Y. \tag{15}$$

Define

$$\lambda_\infty := (u_\infty)_\# \left(\frac{\mathcal{H}^{d-1}\lfloor_{\mathbb{S}^{d-1}}}{\sigma_{d-1}} \right). \tag{16}$$

Then $\lambda_\infty \in \mathcal{P}_p(Y)$ because

$$\int_Y d_Y(z, y_0)^p \, d\lambda_\infty(z) = \frac{1}{\sigma_{d-1}} \int_{\mathbb{S}^{d-1}} d_Y(u_\infty(\theta), y_0)^p \, d\mathcal{H}^{d-1}(\theta) < \infty.$$

Condition EP_p depends on u_∞ only through the p-distance profile of the push-forward law (16). In particular, it does not by itself assert convergence along individual rays, and different spherical representatives with the same profile give the same limiting functional. Under the radial criterion below, u_∞ is also the L^p-limit of the radial slices.

Proposition 3.7. *Let $u \in L^p_{\mathrm{loc}}(\mathbb{R}^d; Y)$ and suppose that*

$$\int_{\mathbb{S}^{d-1}} d_Y(u(r\theta), u_\infty(\theta))^p \, d\mathcal{H}^{d-1}(\theta) \longrightarrow 0 \qquad (r \to \infty), \tag{17}$$

where the convergence may be understood after changing the radial representative on a null set of radii. Then (15) *holds.*

Proof. For $a \in Y$ set

$$H_a(r) := \left(\int_{\mathbb{S}^{d-1}} d_Y(a, u(r\theta))^p \, d\mathcal{H}^{d-1}(\theta) \right)^{1/p}$$

and

$$H_{\infty,a} := \left(\int_{\mathbb{S}^{d-1}} d_Y(a, u_\infty(\theta))^p \, d\mathcal{H}^{d-1}(\theta) \right)^{1/p}.$$

The reverse triangle inequality in $L^p(\mathbb{S}^{d-1})$ gives

$$|H_a(r) - H_{\infty,a}| \le \left(\int_{\mathbb{S}^{d-1}} d_Y(u(r\theta), u_\infty(\theta))^p \, d\mathcal{H}^{d-1}(\theta) \right)^{1/p},$$

and hence $H_a(r)^p \to H_{\infty,a}^p$. Moreover, $H_a(\cdot)^p \in L^1_{\mathrm{loc}}([1,\infty))$ by the local L^p assumption. The elementary fact

$$h(r) \longrightarrow L \quad \Longrightarrow \quad s\int_1^\infty h(r) r^{-1-sp}\, dr \longrightarrow \frac{L}{p}$$

for locally integrable, eventually bounded h now gives

$$\mathcal{A}^u_{s,p}(a) = s\int_1^\infty H_a(r)^p r^{-1-sp}\, dr \longrightarrow \frac{1}{p} H^p_{\infty,a},$$

which is (15). □

Remark 3.8. If $d_Y(u(\cdot), y_0) \in L^p(\mathbb{R}^d)$, then EP_p holds with $u_\infty \equiv y_0$, or equivalently with the tail law δ_{y_0}. Indeed, the reverse triangle inequality in the weighted L^p space gives

$$\begin{aligned} &\left| \mathcal{A}^u_{s,p}(a)^{1/p} - \left(\frac{\sigma_{d-1}}{p}\right)^{1/p} d_Y(a, y_0) \right| \\ &\qquad \le \left(s\int_{B_1^c} \frac{d_Y(u(y), y_0)^p}{|y|^{d+sp}}\, dy \right)^{1/p} \le s^{1/p} \| d_Y(u(\cdot), y_0) \|_{L^p(\mathbb{R}^d)}, \end{aligned}$$

which tends to zero. In particular, the condition is automatic for globally defined fractional Sobolev maps $u \in W^{s_0,p}(\mathbb{R}^d; Y)$.

Put $M_\Omega := \sup_{x\in\Omega} |x|$ and fix $R > \max\{1, 2M_\Omega\}$. For $0 < s < 1$ and $x \in \Omega$ define

$$K_{s,R}(x,\, dy) := s\, \mathbf{1}_{\{|y|>R\}} |x-y|^{-d-sp}\, dy, \tag{18}$$

and

$$m_{s,R}(x) := K_{s,R}(x, \mathbb{R}^d) = s\int_{|y|>R} \frac{dy}{|x-y|^{d+sp}}.$$

The following estimate is implicit in [15, Lemma 2.1] and [13].

Lemma 3.9. *For $\theta \in \mathbb{S}^{d-1}$ set*

$$w_{s,R,x}(\theta) := s\int_R^\infty \frac{r^{d-1}}{|r\theta - x|^{d+sp}}\, dr.$$

Then, uniformly for $x \in \Omega$ and $\theta \in \mathbb{S}^{d-1}$,

$$w_{s,R,x}(\theta) = \frac{R^{-sp}}{p} + O(s) \qquad (s \downarrow 0). \tag{19}$$

Consequently

$$m_{s,R} \longrightarrow \frac{\sigma_{d-1}}{p} \quad \text{in } L^\infty(\Omega), \tag{20}$$

and the normalized angular densities

$$g_{s,R,x}(\theta) := \frac{w_{s,R,x}(\theta)}{m_{s,R}(x)}$$

satisfy

$$\sup_{x\in\Omega,\, \theta\in\mathbb{S}^{d-1}} \left| g_{s,R,x}(\theta) - \frac{1}{\sigma_{d-1}} \right| \longrightarrow 0. \tag{21}$$

Proof. Since $R > 2M_\Omega$, for $r \geq R$ we have $|x|/r \leq 1/2$. For $0 < s \leq 1$ the functions $a \mapsto |\theta - a|^{-d-sp}$ have gradients uniformly bounded on $|a| \leq 1/2$, uniformly also in $\theta \in \mathbb{S}^{d-1}$. Hence, by the mean value theorem applied to the function $a \to |a-\theta|^{-d-sp}$ along the segment joining 0 to x/r, we get

$$\Big||\theta - x/r|^{-d-sp} - 1\Big| \leq C\frac{|x|}{r}.$$

Therefore,

$$\begin{aligned}\left|w_{s,R,x}(\theta) - s\int_R^\infty r^{-1-sp}\,dr\right| &\leq CM_\Omega s\int_R^\infty r^{-2-sp}\,dr\\ &\leq C_R s.\end{aligned}$$

and (19) follows since

$$s\int_R^\infty r^{-1-sp}\,dr = \frac{R^{-sp}}{p}.$$

Integrating in θ and using the coarea formula proves (20). Dividing (19) by the mass, which is bounded away from zero for small s, gives (21). □

Lemma 3.10. *Let $u \in L^p_{\mathrm{loc}}(\mathbb{R}^d;Y)$ satisfy* (15). *Let $\lambda_{s,R,x}$ be the normalized target law associated with* (18). *Then $\lambda_{s,R,x} \in \mathcal{P}_p(Y)$ and, for every $a \in Y$,*

$$\sup_{x\in\Omega}\left|\left(\int_Y d_Y(a,z)^p\,d\lambda_{s,R,x}(z)\right)^{1/p} - \left(\int_Y d_Y(a,z)^p\,d\lambda_\infty(z)\right)^{1/p}\right| \longrightarrow 0. \tag{22}$$

Proof. For $a \in Y$ set

$$N_{s,R,x}(a) := s\int_{|y|>R}\frac{d_Y(a,u(y))^p}{|x-y|^{d+sp}}\,dy.$$

Since $R > 1$, local L^p integrability gives

$$s\int_{1<|y|\leq R}\frac{d_Y(a,u(y))^p}{|y|^{d+sp}}\,dy \longrightarrow 0.$$

Consequently, (15) implies

$$N_{s,R,0}(a) \longrightarrow L(a) := \frac{1}{p}\int_{\mathbb{S}^{d-1}} d_Y(a,u_\infty(\theta))^p\,d\mathcal{H}^{d-1}(\theta). \tag{23}$$

We next show that translating the centre by a bounded vector does not alter this limit. Fix $A > R$. On $R < |y| \leq A$, both $|y|$ and $|x-y|$ are bounded away from zero, uniformly for $x \in \Omega$. Hence local L^p integrability gives

$$\sup_{x\in\Omega} s\int_{R<|y|\leq A} d_Y(a,u(y))^p\left||x-y|^{-d-sp} - |y|^{-d-sp}\right|\,dy \longrightarrow 0. \tag{24}$$

For $|y| > A > 2M_\Omega$, the mean value theorem yields, uniformly for $0 < s \leq 1$,

$$\Big||x-y|^{-d-sp} - |y|^{-d-sp}\Big| \leq C\frac{M_\Omega}{|y|}|y|^{-d-sp}.$$

By (23), there exists $s_a \in (0,1)$ such that $\sup_{0<s<s_a} N_{s,R,0}(a) < \infty$. It follows that

$$\limsup_{s\downarrow 0}\sup_{x\in\Omega}|N_{s,R,x}(a) - N_{s,R,0}(a)| \leq \frac{CM_\Omega}{A}\sup_{0<s<s_a} N_{s,R,0}(a). \tag{25}$$

Letting $A \to \infty$ and using (23), we obtain

$$\sup_{x\in\Omega}|N_{s,R,x}(a) - L(a)| \longrightarrow 0. \tag{26}$$

In particular, the case $a = y_0$ shows that $\lambda_{s,R,x} \in \mathcal{P}_p(Y)$ for small s. Finally,

$$\int_Y d_Y(a,z)^p \, d\lambda_{s,R,x}(z) = \frac{N_{s,R,x}(a)}{m_{s,R}(x)}.$$

Combining (26) with (20), and observing that

$$\int_Y d_Y(a,z)^p \, d\lambda_\infty(z) = \frac{1}{\sigma_{d-1}} \int_{\mathbb{S}^{d-1}} d_Y(a, u_\infty(\theta))^p \, d\mathcal{H}^{d-1}(\theta),$$

proves (22). □

For $0 < s < 1$ define the *localized fractional interaction*

$$\mathcal{E}^\Omega_{s,p}(u) := \iint_{Q_\Omega} \frac{d_Y(u(x),u(y))^p}{|x-y|^{d+sp}} \, dy \, dx, \qquad Q_\Omega = (\mathbb{R}^d \times \mathbb{R}^d) \setminus (\Omega^c \times \Omega^c).$$

Theorem 3.11. *Let $1 \le p < \infty$, let $u \in L^p_{\mathrm{loc}}(\mathbb{R}^d; Y)$, and let $u_\infty : \mathbb{S}^{d-1} \to Y$ be Borel with $d_Y(u_\infty(\cdot), y_0) \in L^p(\mathbb{S}^{d-1})$. Assume* EP_p*, namely* (15)*. Suppose that for some $s_0 \in (0,1)$,*

$$\mathcal{E}^\Omega_{s_0,p}(u) < \infty.$$

Then

$$\lim_{s \downarrow 0} \frac{s}{2} \mathcal{E}^\Omega_{s,p}(u) = \frac{1}{p} \int_\Omega \int_{\mathbb{S}^{d-1}} d_Y(u(x), u_\infty(\theta))^p \, d\mathcal{H}^{d-1}(\theta) \, dx. \tag{27}$$

Proof. Fix $R > \max\{1, 2M_\Omega\}$. By symmetry,

$$\begin{aligned}
\frac{s}{2}\mathcal{E}^\Omega_{s,p}(u) = \frac{s}{2} \iint_{\Omega\times\Omega} \frac{d_Y(u(x),u(y))^p}{|x-y|^{d+sp}} \, dy \, dx \quad (=: \mathcal{A}_s) \\
+ s \int_\Omega \int_{\Omega^c \cap B_R} \frac{d_Y(u(x),u(y))^p}{|x-y|^{d+sp}} \, dy \, dx \quad (=: \mathcal{B}_s) \\
+ s \int_\Omega \int_{\Omega^c \cap \{|y|>R\}} \frac{d_Y(u(x),u(y))^p}{|x-y|^{d+sp}} \, dy \, dx. \quad (=: \mathcal{C}_s)
\end{aligned}$$

Now, splitting at $|x-y| = 1$ and using $s < s_0$ on the short-range part,

$$\mathcal{A}_s \le \frac{s}{2}\mathcal{E}^\Omega_{s_0,p}(u) + 2^{p-1} s \, |\Omega| \int_\Omega d_Y(u(x), y_0)^p \, dx \longrightarrow 0.$$

Similarly, with $D_R := \Omega^c \cap B_R$,

$$\begin{aligned}
\mathcal{B}_s \le s\mathcal{E}^\Omega_{s_0,p}(u) + 2^{p-1} s \, |\Omega| \int_{D_R} d_Y(u(y), y_0)^p \, dy \\
+ 2^{p-1} s \, |D_R| \int_\Omega d_Y(u(x), y_0)^p \, dx \longrightarrow 0.
\end{aligned}$$

For the last term use the kernel (18). By Lemma 3.9, $m_{s,R} \to \sigma_{d-1}/p$ in $L^\infty(\Omega)$, and, in particular,

$$\sup_{0<s<s_*} \|m_{s,R}\|_{L^\infty(\Omega)} < \infty$$

for some $s_* > 0$. Lemma 3.10 gives the p-energy-profile convergence (10), in fact uniformly in x. Therefore Theorem 3.3 gives[4]

$$\begin{aligned}\lim_{s\downarrow 0} \mathcal{C}_s &= \lim_{s\downarrow 0} s \int_\Omega \int_{\{|y|>R\}} \frac{d_Y(u(x),u(y))^p}{|x-y|^{d+sp}}\, dy\, dx \\ &= \frac{\sigma_{d-1}}{p} \int_\Omega \int_Y d_Y(u(x),z)^p\, d\lambda_\infty(z)\, dx \\ &= \frac{1}{p} \int_\Omega \int_{\mathbb{S}^{d-1}} d_Y(u(x),u_\infty(\theta))^p\, d\mathcal{H}^{d-1}(\theta)\, dx.\end{aligned}$$

Combining the three contributions proves (27). □

Remark 3.12 (Comparison with [15]). For $Y = \mathbb{R}$ and $p = 2$, Theorem 3.11 gives

$$\lim_{s\downarrow 0} \frac{s}{2} \mathcal{E}^\Omega_{s,2}(u) = \frac{1}{2} \int_\Omega \int_{\mathbb{S}^{d-1}} |u(x) - u_\infty(\theta)|^2\, d\mathcal{H}^{d-1}(\theta)\, dx,$$

which is the directional formula of [15, Theorem 2.11]. More precisely, writing

$$\alpha_s(f) := s \int_{B_1^c} \frac{f(y)}{|y|^{d+2s}}\, dy,$$

we have

$$\mathcal{A}^u_{s,2}(a) = \frac{\sigma_{d-1}}{2} a^2 - 2a\alpha_s(u) + \alpha_s(u^2).$$

Consequently EP_2 is equivalent to

$$\alpha_s(u) \longrightarrow \frac{1}{2} \int_{\mathbb{S}^{d-1}} u_\infty\, d\mathcal{H}^{d-1}, \qquad \alpha_s(u^2) \longrightarrow \frac{1}{2} \int_{\mathbb{S}^{d-1}} u_\infty^2\, d\mathcal{H}^{d-1}.$$

These are precisely the first two moments entering the quadratic mass-at-infinity formula. The hypotheses of [15, Theorem 2.11], through [15, Lemma 2.9], imply these identities, but also require bounded exterior data and uniform radial convergence. Thus the present hypotheses give their pointwise directional formula and also cover tails specified only through energy profiles. For arbitrary p and Polish targets, the energy-profile condition replaces the quadratic moment expansion by the family of costs $a \mapsto d_Y(a,\cdot)^p$.

3.2.1. *Anisotropic consequences.* The preceding directional theorem is stable under replacing the Euclidean norm by a norm induced by a symmetric convex body. This yields the following anisotropic form and, as a scalar specialization, Ludwig's limit [31, Theorem 2] when the state at infinity vanishes.

Proposition 3.13. *Let $K \subset \mathbb{R}^n$ be an origin-symmetric convex body with nonempty interior, and let $\|\cdot\|_K$ be its Minkowski norm. Set*

$$S_K := \{\omega \in \mathbb{R}^n : \|\omega\|_K = 1\},$$

and let σ_K be the polar measure on S_K, characterized by

$$\int_{\mathbb{R}^n} f(z)\, dz = \int_0^\infty \int_{S_K} f(r\omega)\, d\sigma_K(\omega)\, r^{n-1}\, dr \tag{28}$$

for every nonnegative Borel function f. In particular,

$$\sigma_K(S_K) = n|K|.$$

[4]Since $R > 2M_\Omega$, we have $\{|y| > R\} \subset \Omega^c$ and, up to the null sphere ∂B_R, $\Omega^c = (\Omega^c \cap B_R) \cup \{|y| > R\}$.

Let (Y, d_Y, y_0) be a Polish pointed metric space, let $\Omega \subset \mathbb{R}^n$ be bounded and Borel, and let $u \in L^p_{\mathrm{loc}}(\mathbb{R}^n; Y)$, where $1 \le p < \infty$. Assume that there exists a Borel map

$$u_\infty : S_K \to Y, \qquad \int_{S_K} d_Y(u_\infty(\omega), y_0)^p \, d\sigma_K(\omega) < \infty,$$

such that the anisotropic energy-profile condition

$$s \int_{\{\|y\|_K > 1\}} \frac{d_Y(u(y), a)^p}{\|y\|_K^{n+sp}} \, dy \longrightarrow \frac{1}{p} \int_{S_K} d_Y(a, u_\infty(\omega))^p \, d\sigma_K(\omega) \qquad \text{for every } a \in Y \tag{29}$$

holds.

Define

$$\mathcal{E}^{\Omega,K}_{s,p}(u) := \iint_{Q_\Omega} \frac{d_Y(u(x), u(y))^p}{\|x - y\|_K^{n+sp}} \, dy \, dx, \qquad Q_\Omega := (\mathbb{R}^n \times \mathbb{R}^n) \setminus (\Omega^c \times \Omega^c).$$

If

$$\mathcal{E}^{\Omega,K}_{s_0,p}(u) < \infty \tag{30}$$

for some $s_0 \in (0,1)$, then

$$\lim_{s \downarrow 0} \frac{s}{2} \mathcal{E}^{\Omega,K}_{s,p}(u) = \frac{1}{p} \int_\Omega \int_{S_K} d_Y(u(x), u_\infty(\omega))^p \, d\sigma_K(\omega) \, dx. \tag{31}$$

Proof. Set

$$M_{\Omega,K} := \sup_{x \in \Omega} \|x\|_K$$

and fix $R > \max\{1, 2M_{\Omega,K}\}$. For $x \in \Omega$ consider the far kernel

$$K^K_{s,R}(x, \, dy) := s \, \mathbf{1}_{\{\|y\|_K > R\}} \|x - y\|_K^{-n-sp} \, dy.$$

Using the polar formula (28), its angular density is

$$w^K_{s,R,x}(\omega) := s \int_R^\infty \frac{r^{n-1}}{\|r\omega - x\|_K^{n+sp}} \, dr.$$

Since

$$\|r\omega - x\|_K = r \|\omega - x/r\|_K$$

and

$$|\|\omega - x/r\|_K - 1| \le \frac{\|x\|_K}{r} \le \frac{1}{2},$$

the function $t \mapsto t^{-n-sp}$ is uniformly Lipschitz on the relevant compact interval, uniformly for $0 < s \le 1$. Hence

$$\left| \|\omega - x/r\|_K^{-n-sp} - 1 \right| \le C \frac{\|x\|_K}{r},$$

and therefore, uniformly for $x \in \Omega$ and $\omega \in S_K$,

$$w^K_{s,R,x}(\omega) = s \int_R^\infty r^{-1-sp} \, dr + O(s) = \frac{R^{-sp}}{p} + O(s). \tag{32}$$

Consequently, if

$$m^K_{s,R}(x) := K^K_{s,R}(x, \mathbb{R}^n),$$

then

$$m^K_{s,R} \longrightarrow \frac{\sigma_K(S_K)}{p} = \frac{n|K|}{p} \qquad \text{in } L^\infty(\Omega), \tag{33}$$

and the corresponding normalized angular densities converge uniformly to

$$\frac{1}{\sigma_K(S_K)} = \frac{1}{n|K|}.$$

Define

$$\lambda_\infty^K := (u_\infty)_\# \left(\frac{\sigma_K}{n|K|} \right).$$

The assumed p-integrability of u_∞ gives $\lambda_\infty^K \in \mathcal{P}_p(Y)$. Let $\lambda_{s,R,x}^K$ denote the normalized target law of $K_{s,R}^K(x,\cdot)$. We claim that, for every $a \in Y$,

$$\sup_{x\in\Omega} \left| \left(\int_Y d_Y(a,z)^p \, d\lambda_{s,R,x}^K(z) \right)^{1/p} - \left(\int_Y d_Y(a,z)^p \, d\lambda_\infty^K(z) \right)^{1/p} \right| \longrightarrow 0. \tag{34}$$

Indeed, set

$$N_{s,R,x}^K(a) := s \int_{\{\|y\|_K > R\}} \frac{d_Y(a,u(y))^p}{\|x-y\|_K^{n+sp}} \, dy.$$

Condition (29) and local L^p integrability on the fixed annulus $1 < \|y\|_K \le R$ give

$$N_{s,R,0}^K(a) \longrightarrow \frac{1}{p} \int_{S_K} d_Y(a, u_\infty(\omega))^p \, d\sigma_K(\omega).$$

On every fixed annulus $R < \|y\|_K \le A$, the difference between the kernels centred at x and at the origin contributes $o(1)$, uniformly in $x \in \Omega$. For $\|y\|_K > A > 2M_{\Omega,K}$, the triangle inequality for $\|\cdot\|_K$ and the mean value theorem give

$$\left| \|x-y\|_K^{-n-sp} - \|y\|_K^{-n-sp} \right| \le C \frac{M_{\Omega,K}}{\|y\|_K} \|y\|_K^{-n-sp}.$$

The same cutoff argument as in the proof of Lemma 3.10 therefore yields

$$\sup_{x\in\Omega} \left| N_{s,R,x}^K(a) - N_{s,R,0}^K(a) \right| \longrightarrow 0.$$

After division by $m_{s,R}^K(x)$ and use of (33), this proves (34); the case $a = y_0$ also shows that $\lambda_{s,R,x}^K \in \mathcal{P}_p(Y)$ for small s. By symmetry of $\|\cdot\|_K$,

$$\frac{s}{2} \mathcal{E}_{s,p}^{\Omega,K}(u) = \frac{s}{2} \iint_{\Omega\times\Omega} \frac{d_Y(u(x),u(y))^p}{\|x-y\|_K^{n+sp}} \, dy \, dx$$

$$+ s \int_\Omega \int_{\Omega^c \cap \{\|y\|_K < R\}} \frac{d_Y(u(x),u(y))^p}{\|x-y\|_K^{n+sp}} \, dy \, dx + s \int_\Omega \int_{\{\|y\|_K > R\}} \frac{d_Y(u(x),u(y))^p}{\|x-y\|_K^{n+sp}} \, dy \, dx.$$

The first two terms tend to zero. Indeed, on $\|x-y\|_K < 1$ this follows from (30), while on $\|x-y\|_K \ge 1$ both variables range in a bounded set and local L^p integrability provides an s-independent integrable bound. For the far term, Theorem 3.3, (33), and (34) give

$$\begin{aligned} \lim_{s\downarrow 0} s \int_\Omega \int_{\{\|y\|_K > R\}} \frac{d_Y(u(x),u(y))^p}{\|x-y\|_K^{n+sp}} \, dy \, dx \\ &= \frac{n|K|}{p} \int_\Omega \int_Y d_Y(u(x),z)^p \, d\lambda_\infty^K(z) \, dx \\ &= \frac{1}{p} \int_\Omega \int_{S_K} d_Y(u(x), u_\infty(\omega))^p \, d\sigma_K(\omega) \, dx. \end{aligned}$$

This proves (31). □

Corollary 3.14. *Let $(Y, d_Y, y_0) = (\mathbb{R}, |\cdot|, 0)$. Under the assumptions of Proposition 3.13 the following formula holds:*

$$\lim_{s\downarrow 0} \frac{s}{2} \iint_{Q_\Omega} \frac{|u(x)-u(y)|^p}{\|x-y\|_K^{n+sp}} \, dy \, dx = \frac{1}{p} \int_\Omega \int_{S_K} |u(x) - u_\infty(\omega)|^p \, d\sigma_K(\omega) \, dx. \tag{35}$$

Remark 3.15. If $u_\infty \equiv 0$, then (35) reduces to

$$\lim_{s\downarrow 0} \frac{s}{2} \iint_{Q_\Omega} \frac{|u(x)-u(y)|^p}{\|x-y\|_K^{n+sp}}\, dy\, dx = \frac{n|K|}{p} \int_\Omega |u(x)|^p\, dx,$$

since $\sigma_K(S_K) = n|K|$. When u is compactly supported in Ω, the formula is Ludwig's anisotropic Maz'ya–Shaposhnikova theorem [31, Theorem 2]. The localized statement also allows nontrivial exterior data and retains the p-distance profile of the anisotropic distribution at infinity.

4. Abstract small-order nonlocal energies

We formulate the short-range/far-range decomposition for singular nonlocal energies. Let $\varepsilon \downarrow 0$ be a parameter and let Λ_ε be a positive Borel measure on $X \times X$. Define

$$\mathcal{E}_\varepsilon(u) := \int_{X\times X} d_Y(u(x),u(y))^p\, d\Lambda_\varepsilon(x,y).$$

We assume that

$$\Lambda_\varepsilon = \Lambda_\varepsilon^{\mathrm{near}} + \Lambda_\varepsilon^{\mathrm{far}} \tag{36}$$

with both terms positive, and let $c_\varepsilon > 0$ be a normalization factor chosen so that the rescaled energies $c_\varepsilon \mathcal{E}_\varepsilon$ admit a nontrivial limit as $\varepsilon \downarrow 0$.

Proposition 4.1. *Let $u \in L^p(X;Y)$. Assume:*

(A1)

$$c_\varepsilon \int_{X\times X} d_Y(u(x),u(y))^p\, d\Lambda_\varepsilon^{\mathrm{near}}(x,y) \longrightarrow 0; \tag{37}$$

(A2) *$c_\varepsilon \Lambda_\varepsilon^{\mathrm{far}}$ has marginals $a_\varepsilon \mu$ and $b_\varepsilon \mu$, where $a_\varepsilon, b_\varepsilon, a_\infty, b_\infty \in L^\infty_+(X,\mu)$ and*

$$\sup_\varepsilon (\|a_\varepsilon\|_\infty + \|b_\varepsilon\|_\infty) < \infty, \qquad a_\varepsilon \overset{*}{\rightharpoonup} a_\infty, \qquad b_\varepsilon \overset{*}{\rightharpoonup} b_\infty \quad \text{in } L^\infty(X,\mu);$$

(A3) *$c_\varepsilon \Lambda_\varepsilon^{\mathrm{far}}$ escapes from bounded rectangles.*

Then

$$c_\varepsilon \mathcal{E}_\varepsilon(u) \longrightarrow \int_X (a_\infty + b_\infty) d_Y(u,y_0)^p\, d\mu. \tag{38}$$

Proof. By (36),

$$c_\varepsilon \mathcal{E}_\varepsilon(u) = c_\varepsilon \int d_Y(u(x),u(y))^p\, d\Lambda_\varepsilon^{\mathrm{near}} + \int d_Y(u(x),u(y))^p\, d(c_\varepsilon \Lambda_\varepsilon^{\mathrm{far}}).$$

The first term tends to zero by (37). The second tends to the right-hand side of (38) by Theorem 2.8. This proves (38). □

Besides $u \in L^p(X;Y)$, the only regularity assumption is (37); belonging to a specified Sobolev space is not required. Fractional Sobolev regularity at one positive order is sufficient for a large class of kernels.

4.1. **A general fractional sufficient condition.** Let $k : X \times X \to [0,\infty]$ be a Borel function independent of s; no symmetry is required. For $0 < s < 1$ define the functional

$$[u]^p_{s,p;k,<1} := \int_{\{0<d_X(x,y)<1\}} \frac{d_Y(u(x),u(y))^p}{d_X(x,y)^{sp}} k(x,y)\, d\mu(y)\, d\mu(x). \tag{39}$$

Proposition 4.2. *Assume that for some $0 < \theta < 1$,*

$$[u]_{\theta,p;k,<1} < \infty. \tag{40}$$

Then

$$\lim_{s\downarrow 0} s[u]^p_{s,p;k,<1} = 0.$$

Proof. For $0 < s < \theta$ and $0 < d_X(x,y) < 1$,

$$d_X(x,y)^{-sp} \le d_X(x,y)^{-\theta p}.$$

Therefore

$$0 \le s[u]^p_{s,p;k,<1} \le s[u]^p_{\theta,p;k,<1} \longrightarrow 0.$$

□

Remark 4.3. Proposition 4.2 is stated under the local assumption (40), since only the interaction near the diagonal is relevant for (37). In many standard situations one may impose the stronger global assumption

$$u \in L^p(X;Y), \qquad \int_{X\times X} \frac{d_Y(u(x),u(y))^p}{d_X(x,y)^{\theta p}}\, k(x,y)\, d\mu(y)\, d\mu(x) < \infty.$$

This hypothesis implies (40), and hence the conclusion of Proposition 4.2. For the usual Gagliardo-type kernels on spaces with suitable volume growth, the contribution of $d_X(x,y) \ge 1$ is controlled by the L^p-norm of u. In that setting the global formulation is equivalent, up to the L^p requirement, to the local assumption above.

Remark 4.4. A common metric-measure choice, for $x \ne y$, is

$$k_X(x,y) := \frac{1}{2}\left(\frac{1}{\mu(B(x, d_X(x,y)))} + \frac{1}{\mu(B(y, d_X(x,y)))}\right),$$

and one may set $k_X(x,x) = 0$. This formula is used whenever the ball measures are positive and finite. On a Q-Ahlfors regular space this kernel is comparable to $d_X(x,y)^{-Q}$, so (39) is comparable with the usual Gagliardo kernel $d_X(x,y)^{-Q-sp}$.

4.2. **Application: fractional perimeters as Bernoulli tail laws.** For characteristic functions, the target law is a Bernoulli law. The following proposition identifies its parameter with the mass at infinity.

Proposition 4.5. *Let $\Omega \subset \mathbb{R}^n$ be bounded and open and let $E \subset \mathbb{R}^n$ be measurable. For $0 < s < 1$ write*

$$L_s(A,B) := \int_A \int_B \frac{1}{|x-y|^{n+s}}\, dy\, dx$$

and

$$\mathrm{Per}_s(E;\Omega) := L_s(E\cap\Omega, E^c\cap\Omega) + L_s(E\cap\Omega, E^c\cap\Omega^c) + L_s(E\cap\Omega^c, E^c\cap\Omega).$$

Assume that

$$\mathrm{Per}_{s_0}(E;\Omega) < \infty$$

for some $s_0 \in (0,1)$ and that the limit

$$\alpha(E) := \lim_{s\downarrow 0} s \int_{E\cap B_1^c} \frac{dy}{|y|^{n+s}} \tag{41}$$

exists. Set

$$\sigma_{n-1} := \mathcal{H}^{n-1}(S^{n-1}), \qquad \vartheta_E := \frac{\alpha(E)}{\sigma_{n-1}} \in [0,1].$$

Then

$$\lim_{s\downarrow 0} s\, \mathrm{Per}_s(E;\Omega) = \sigma_{n-1}\left[(1-\vartheta_E)|E\cap\Omega| + \vartheta_E|E^c\cap\Omega|\right]. \tag{42}$$

Equivalently, if

$$\lambda^E_\infty := (1-\vartheta_E)\delta_0 + \vartheta_E\delta_1, \tag{43}$$

then

$$\lim_{s\downarrow 0} s\ \mathrm{Per}_s(E;\Omega) = \sigma_{n-1}\int_\Omega \int_{\{0,1\}} |\mathbf{1}_E(x) - z|\ d\lambda_\infty^E(z)\ dx. \tag{44}$$

In particular, the mass at infinity appearing in the fractional-perimeter formula of Dipierro–Figalli–Palatucci–Valdinoci [16, Theorem 2.5] *is precisely the parameter of a Bernoulli target law at infinity.*

Proof. Observe first that

$$\iint_{Q_\Omega} \frac{|\mathbf{1}_E(x) - \mathbf{1}_E(y)|}{|x-y|^{n+s}}\ dy\ dx = 2\ \mathrm{Per}_s(E;\Omega). \tag{45}$$

Fix $R > 1$ so large that $R > 2\sup_{x\in\Omega}|x|$. For $x \in \Omega$ consider the kernel

$$K_{s,R}(x,\, dy) := s\,\mathbf{1}_{\{|y|>R\}}|x-y|^{-n-s}\ dy.$$

As in Lemma 3.9,

$$K_{s,R}(x,\mathbb{R}^n) \longrightarrow \sigma_{n-1} \tag{46}$$

uniformly in $x \in \Omega$. Since R is fixed,

$$s\int_{E\cap\{1<|y|\le R\}} |y|^{-n-s}\ dy \longrightarrow 0,$$

and hence (41) implies

$$s\int_{E\cap\{|y|>R\}} |y|^{-n-s}\ dy \longrightarrow \alpha(E).$$

Moreover, uniformly for $x \in \Omega$,

$$s\int_{\{|y|>R\}} \big||x-y|^{-n-s} - |y|^{-n-s}\big|\ dy \longrightarrow 0.$$

Indeed, for $|y| > R$ the mean value theorem and $|x| \le R/2$ give

$$\big||x-y|^{-n-s} - |y|^{-n-s}\big| \le C|x|\,|y|^{-n-s-1},$$

and the last expression is integrable with an additional factor s. Consequently,

$$\sup_{x\in\Omega}\left| s\int_{E\cap\{|y|>R\}} |x-y|^{-n-s}\ dy - \alpha(E)\right| \longrightarrow 0. \tag{47}$$

The normalized push-forward of $K_{s,R}(x,\cdot)$ under $\mathbf{1}_E$ is a Bernoulli probability measure. By (46) and (47), its parameter converges uniformly to $\vartheta_E = \alpha(E)/\sigma_{n-1}$; hence the corresponding target laws converge to (43).

It remains only to remove the far cutoff. On $|x-y| < 1$, for $0 < s < s_0$,

$$|x-y|^{-n-s} \le |x-y|^{-n-s_0},$$

and therefore the bounded-range contribution, multiplied by s, tends to zero by the assumption $\mathrm{Per}_{s_0}(E;\Omega) < \infty$. On the remaining bounded region $|x-y| \ge 1$, the same conclusion follows directly from boundedness of the integration region.

Applying Theorem 3.3 with the target $Y = \{0,1\}$, and using (45), gives (44). Finally,

$$\int_{\{0,1\}} |\mathbf{1}_E(x) - z|\ d\lambda_\infty^E(z) = (1-\vartheta_E)\mathbf{1}_E(x) + \vartheta_E\mathbf{1}_{E^c}(x),$$

which yields (42). □

5. Escaping transition kernels and semigroup Besov energies

We now specialize the abstract coupling theorem to transition kernels. Let

$$q : (0,\infty) \times X \times X \to [0,\infty)$$

be jointly Borel measurable, and write $q_t(x,y) := q(t,x,y)$. For every $t > 0$, define the measure

$$d\Pi_t(x,y) := q_t(x,y)\ d\mu(x)\ d\mu(y). \tag{48}$$

For $t \geq 1$, define the marginal densities

$$a_t(x) := \int_X q_t(x,y)\ d\mu(y), \qquad b_t(y) := \int_X q_t(x,y)\ d\mu(x),$$

and assume that $a_t, b_t \in L^\infty_+(X,\mu)$. Then Π_t is an (a_t,b_t)-marginal coupling. For $u \in L^p(X;Y)$ set

$$J_p(t,u) := \int_X \int_X d_Y(u(x),u(y))^p q_t(x,y)\ d\mu(y)\ d\mu(x).$$

Corollary 5.1. *Assume that $a_t, b_t, a_\infty, b_\infty \in L^\infty_+(X,\mu)$ and*

$$\sup_{t\geq 1}(\|a_t\|_\infty + \|b_t\|_\infty) < \infty, \qquad a_t \overset{*}{\rightharpoonup} a_\infty, \qquad b_t \overset{*}{\rightharpoonup} b_\infty \quad \text{in } L^\infty(X,\mu),$$

and, for every bounded $K \subset X$,

$$\int_K \int_K q_t(x,y)\ d\mu(y)\ d\mu(x) \longrightarrow 0.$$

Then for every $u \in L^p(X;Y)$,

$$J_p(t,u) \longrightarrow \int_X (a_\infty + b_\infty) d_Y(u,y_0)^p\ d\mu \qquad (t \to \infty).$$

Proof. Apply Theorem 2.8 to the measures (48). □

The following semigroup-type Besov functional converts the large-time behaviour of $J_p(t,u)$ into a small-order asymptotic problem.

Definition 5.2. *For $s > 0$ and $\beta > 0$ define*

$$\mathscr{N}^{q,\beta}_{s,p}(u)^p := \int_0^\infty t^{-\beta sp-1} J_p(t,u)\ dt, \tag{49}$$

whenever the right-hand side is finite.

Theorem 5.3. *Assume the hypotheses of Corollary 5.1. Let $u \in L^p(X;Y)$ and suppose that*

$$\lim_{s\downarrow 0} s \int_0^1 t^{-\beta sp-1} J_p(t,u)\ dt = 0. \tag{50}$$

Then

$$\lim_{s\downarrow 0} s\mathscr{N}^{q,\beta}_{s,p}(u)^p = \frac{1}{\beta p}\int_X (a_\infty + b_\infty) d_Y(u,y_0)^p\ d\mu. \tag{51}$$

Since only the limit as $s \downarrow 0$ is considered, it is enough to assume that the integral defining $\mathscr{N}^{q,\beta}_{s,p}(u)$ is finite for all sufficiently small $s > 0$.

Proof. Split the integral at $t = 1$. The contribution from $(0,1)$ tends to zero by (50). On $(1,\infty)$ define

$$d\eta_s(t) := \beta sp\, t^{-\beta sp-1} \mathbf{1}_{(1,\infty)}(t)\ dt.$$

Then η_s is a probability measure because

$$\beta sp \int_1^\infty t^{-\beta sp-1}\ dt = 1.$$

For every $T > 1$,

$$\eta_s((1,T)) = 1 - T^{-\beta sp} \longrightarrow 0,$$

so η_s escapes to $+\infty$. By Corollary 5.1,

$$J_p(t,u) \longrightarrow L := \int_X (a_\infty + b_\infty) d_Y(u, y_0)^p \, d\mu.$$

The marginal bound and (1) give

$$0 \leq J_p(t,u) \leq 2^{p-1}(\|a_t\|_\infty + \|b_t\|_\infty) M_p(u),$$

so $J_p(t,u)$ is bounded for $t \geq 1$. Splitting at a fixed $T > 1$, using uniform convergence to L on (T,∞) and $\eta_s((1,T)) \to 0$, yields

$$\int_1^\infty J_p(t,u) \, d\eta_s(t) \longrightarrow L.$$

Finally,

$$\begin{aligned} s \int_1^\infty t^{-\beta sp-1} J_p(t,u) \, dt &= \frac{1}{\beta p} \int_1^\infty J_p(t,u) \, d\eta_s(t) \\ &\longrightarrow \frac{L}{\beta p}. \end{aligned}$$

Adding the short-time contribution proves (51). □

Proposition 5.4. *If*

$$\int_0^1 t^{-\beta s_0 p-1} J_p(t,u) \, dt < \infty$$

for some $s_0 > 0$, then (50) *holds. Moreover,*

$$\int_1^\infty t^{-\beta sp-1} J_p(t,u) \, dt$$

is finite for every $s > 0$.

Proof. For $0 < s < s_0$ and $0 < t < 1$, $t^{-\beta sp-1} \leq t^{-\beta s_0 p-1}$. Hence

$$0 \leq s \int_0^1 t^{-\beta sp-1} J_p(t,u) \, dt \leq s \int_0^1 t^{-\beta s_0 p-1} J_p(t,u) \, dt \longrightarrow 0.$$

For $t \geq 1$, the uniform L^∞ bound on the marginal densities gives $J_p(t,u) \leq C M_p(u)$, and

$$\int_1^\infty t^{-\beta sp-1} \, dt = \frac{1}{\beta sp} < \infty.$$

□

5.1. **Applications of the Abel principle.** We apply Theorem 5.3 to the non-symmetric hypoelliptic semigroups of Buseghin–Garofalo–Tralli [8], Riemannian heat semigroups, weighted graphs, and symmetric conservative ultracontractive semigroups, including the Baouendi–Grushin heat flow.

5.1.1. *Buseghin–Garofalo–Tralli and the two marginal masses.* Theorem 5.3 recovers the result of Buseghin, Garofalo, and Tralli [8], including the change of constant caused by the drift.

Let $Q = Q^* \geq 0$ and $B \in \mathbb{R}^{N\times N}$, where B^* denotes transpose, and let

$$\mathscr{A}f(X) = \operatorname{tr}(Q\nabla^2 f(X)) + \langle BX, \nabla f(X)\rangle.$$

This is the Hörmander-type operator considered in [8]. We assume the hypoellipticity condition used in [8], namely

$$K(t) := \frac{1}{t}\int_0^t e^{sB}Qe^{sB^*}\, ds > 0 \qquad \text{for every } t > 0,$$

together with the assumption

$$\operatorname{tr} B \geq 0. \tag{52}$$

Let $p(X, Z, t)$ be the heat kernel of the corresponding semigroup $P_t^{\mathscr{A}}$. Two identities recorded in [8, (2.1)–(2.2)] are

$$\int_{\mathbb{R}^N} p(X, Z, t)\, dZ = 1, \qquad \int_{\mathbb{R}^N} p(X, Z, t)\, dX = e^{-t\operatorname{tr} B}. \tag{53}$$

Thus

$$a_t = 1, \qquad b_t = e^{-t\operatorname{tr} B}.$$

The kernel has the Gaussian form

$$p(X, Z, t) = \frac{c_N}{V(t)} \exp\left(-\frac{m_t(X, Z)^2}{4t}\right),$$

where $V(t)$ is the volume function introduced in that paper. In particular,

$$0 \leq p(X, Z, t) \leq \frac{c_N}{V(t)}.$$

The large-time volume estimate proved in [8, Proposition 2.3] implies

$$V(t) \longrightarrow \infty.$$

Therefore, if $K \subset \mathbb{R}^N$ is bounded,

$$\int_K \int_K p(X, Z, t)\, dZ\, dX \leq \frac{c_N|K|^2}{V(t)} \longrightarrow 0.$$

Thus the transition couplings escape from bounded rectangles.

For a pointed metric target (Y, d_Y, y_0) and $u \in L^p(\mathbb{R}^N; Y)$ define

$$J_p^{\mathscr{A}}(t, u) := \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} d_Y(u(X), u(Z))^p p(X, Z, t)\, dZ\, dX.$$

By Corollary 5.1,

$$J_p^{\mathscr{A}}(t, u) \longrightarrow \begin{cases} 2M_p(u), & \operatorname{tr} B = 0, \\ M_p(u), & \operatorname{tr} B > 0. \end{cases}$$

Indeed, if $\operatorname{tr} B = 0$ then $a_\infty = b_\infty = 1$, while if $\operatorname{tr} B > 0$ then $a_\infty = 1$ and $b_\infty = 0$.

Define the metric-valued semigroup Besov functional

$$\mathscr{N}_{s,p}^{\mathscr{A}}(u)^p := \int_0^\infty t^{-sp/2-1} J_p^{\mathscr{A}}(t, u)\, dt.$$

This is (49) with $\beta = 1/2$.

Corollary 5.5. *Let $1 \le p < \infty$, assume the preceding hypoellipticity condition, and assume (52). Let $u \in L^p(\mathbb{R}^N; Y)$ and suppose that*

$$\mathscr{N}^{\mathscr{A}}_{s_0,p}(u) < \infty \tag{54}$$

for some $s_0 \in (0,1)$. Then

$$\lim_{s\downarrow 0} s\mathscr{N}^{\mathscr{A}}_{s,p}(u)^p = \begin{cases} \dfrac{4}{p} M_p(u), & \operatorname{tr} B = 0, \\ \dfrac{2}{p} M_p(u), & \operatorname{tr} B > 0. \end{cases} \tag{55}$$

For $Y = \mathbb{R}$, $y_0 = 0$, this recovers [8, Theorem 1.1].

Proof. By (54), the integral over $(0,1)$ at order s_0 is finite. Hence Proposition 5.4 gives the short-time condition (50). The far-time integral is finite for every $s > 0$ because, by (53),

$$J^{\mathscr{A}}_p(t,u) \le 2^{p-1}(1 + e^{-t \operatorname{tr} B}) M_p(u) \le 2^p M_p(u).$$

Apply Theorem 5.3 with $\beta = 1/2$. If $\operatorname{tr} B = 0$, then $a_\infty + b_\infty = 2$, and

$$\frac{a_\infty + b_\infty}{\beta p} = \frac{2}{p/2} = \frac{4}{p}.$$

If $\operatorname{tr} B > 0$, then $a_\infty + b_\infty = 1$, and the constant is $2/p$. This proves (55). □

Remark 5.6. For $\beta = 1/2$, the Abelian factor is $2/p$. If $\operatorname{tr} B = 0$, both limiting marginal densities equal one; if $\operatorname{tr} B > 0$, the second tends to zero. This gives $4/p$ and $2/p$, respectively.

5.2. **Riemannian heat semigroup.**

Corollary 5.7. *Let (M, g) be a complete, connected Riemannian manifold with*

$$\mu(M) = +\infty,$$

and let $h_t(x,y)$ be its minimal heat kernel. Set

$$m_t(x) := \int_M h_t(x,y)\, d\mu(y)$$

and

$$\theta_M(x) := \lim_{t\to\infty} m_t(x).$$

The latter limit exists for every $x \in M$ and defines a bounded harmonic function with values in $[0,1]$.

Let (Y, d_Y, y_0) be a pointed metric space, let $1 \le p < \infty$, and let $u \in L^p(M;Y)$. Define

$$J_p(t,u) := \int_M \int_M d_Y(u(x),u(y))^p h_t(x,y)\, d\mu(y)\, d\mu(x)$$

and

$$\mathscr{N}^{h,1/2}_{s,p}(u)^p := \int_0^\infty t^{-sp/2-1} J_p(t,u)\, dt.$$

Assume that

$$\mathscr{N}^{h,1/2}_{s_0,p}(u) < \infty$$

for some $s_0 > 0$. Then

$$\lim_{s\downarrow 0} s\, \mathscr{N}^{h,1/2}_{s,p}(u)^p = \frac{4}{p} \int_M \theta_M(x) d_Y(u(x), y_0)^p\, d\mu(x). \tag{56}$$

Proof. For $t > 0$ define

$$d\Pi_t(x, y) := h_t(x, y)\ d\mu(x)\ d\mu(y).$$

The symmetry and sub-Markov property of the heat kernel imply that both marginals of Π_t are $m_t\mu$ and $0 \le m_t \le 1$. Moreover, $m_t(x) \to \theta_M(x)$ for every $x \in M$; see [9, Theorem 1.1(ii)]. Dominated convergence therefore gives

$$m_t \overset{*}{\rightharpoonup} \theta_M \quad \text{in } L^\infty(M, \mu).$$

Let $P_t = e^{-t\Delta}$ be the minimal heat semigroup on $L^2(M, \mu)$. For every bounded Borel set $K \subset M$, completeness and Hopf–Rinow imply $\mu(K) < \infty$, and

$$\Pi_t(K \times K) = \langle \mathbf{1}_K, P_t \mathbf{1}_K \rangle_{L^2(M,\mu)}.$$

By the spectral theorem, P_t converges strongly as $t \to \infty$ to the orthogonal projection onto $\ker \Delta$. Every function in this kernel has zero Dirichlet energy and is therefore constant on the connected manifold. Since $\mu(M) = \infty$, the only such L^2 function is zero. Hence

$$\Pi_t(K \times K) \longrightarrow 0.$$

Corollary 5.1 now gives

$$J_p(t, u) \longrightarrow 2 \int_M \theta_M(x) d_Y(u(x), y_0)^p\ d\mu(x). \tag{57}$$

The assumption at order s_0 and Proposition 5.4 give the short-time condition (50). Applying Theorem 5.3 with $\beta = 1/2$ and limiting marginal densities $a_\infty = b_\infty = \theta_M$ proves (56). □

Remark 5.8. The result in [9] is formulated for scalar-valued functions and $p = 2$. Corollary 5.7 holds for $1 \le p < \infty$ and metric-valued maps.

5.2.1. *Weighted graphs.* For weighted graphs, large-time escape follows from the spectral theorem without imposing volume doubling, a Poincaré inequality, or pointwise heat-kernel bounds. We refer to [28] for the analytic and probabilistic background on graph Laplacians and heat semigroups.

Let $G = (V, b, m)$ be a connected, locally finite weighted graph, endowed with the combinatorial distance, and let L be the nonnegative self-adjoint Friedrichs realization associated with the closed canonical Dirichlet form. Write $P_t = e^{-tL}$ and denote by $p_t(x, y)$ its heat kernel with respect to m. Thus

$$P_t f(x) = \sum_{y \in V} p_t(x, y) f(y) m(y).$$

We say that G is stochastically complete if

$$\sum_{y \in V} p_t(x, y) m(y) = 1 \qquad (x \in V,\ t > 0).$$

Corollary 5.9. *Assume that G is connected, locally finite and stochastically complete, and that $m(V) = +\infty$. Let (Y, d_Y, y_0) be a pointed metric space, $1 \le p < \infty$, and $\beta > 0$. For $u \in \ell^p(V, m; Y)$ set*

$$J_p^G(t, u) := \sum_{x,y \in V} d_Y(u(x), u(y))^p p_t(x, y) m(x) m(y)$$

and

$$\mathscr{N}_{s,p}^{G,\beta}(u)^p := \int_0^\infty t^{-\beta sp - 1} J_p^G(t, u)\ dt.$$

If $\mathscr{N}^{G,\beta}_{s_0,p}(u) < \infty$ *for some* $s_0 > 0$*, then*

$$\lim_{s\downarrow 0} s\,\mathscr{N}^{G,\beta}_{s,p}(u)^p = \frac{2}{\beta p}\sum_{x\in V} d_Y(u(x),y_0)^p m(x). \tag{58}$$

Proof. Symmetry of the graph Dirichlet form and stochastic completeness imply that both marginals of

$$\Pi_t(x,y) := p_t(x,y)m(x)m(y)$$

are equal to m. It remains to verify escape from bounded rectangles. Since G is locally finite, bounded sets for the combinatorial distance are finite. For such a set $K \subset V$,

$$\Pi_t(K\times K) = \langle \mathbf{1}_K, P_t\mathbf{1}_K\rangle_{\ell^2(V,m)}.$$

By the spectral theorem, P_t converges strongly in $\ell^2(V,m)$, as $t\to\infty$, to the orthogonal projection onto $\ker L$. If $f\in\ker L$, then its Dirichlet energy vanishes, hence f is constant on V by connectedness. Since $m(V)=+\infty$, no nonzero constant belongs to $\ell^2(V,m)$, and therefore $\ker L=\{0\}$. It follows that $\Pi_t(K\times K)\to 0$. Corollary 5.1 gives

$$J^G_p(t,u) \longrightarrow 2\sum_{x\in V} d_Y(u(x),y_0)^p m(x),$$

and Theorem 5.3, together with Proposition 5.4, yields (58). □

Remark 5.10. For a finite connected weighted graph, the heat semigroup converges instead to the projection onto constants. If $M = m(V)$, the same Abelian argument gives

$$\lim_{s\downarrow 0} s\,\mathscr{N}^{G,\beta}_{s,p}(u)^p = \frac{1}{\beta p M}\sum_{x,y\in V} d_Y(u(x),u(y))^p m(x)m(y).$$

Moreover, on graphs with anomalous diffusion and walk dimension d_w, the spatially natural choice is $\beta = 1/d_w$; formula (58) then becomes

$$\lim_{s\downarrow 0} s\,\mathscr{N}^{G,1/d_w}_{s,p}(u)^p = \frac{2d_w}{p}\sum_{x\in V} d_Y(u(x),y_0)^p m(x).$$

This specialization applies whenever a walk dimension d_w is specified by two-sided sub-Gaussian heat-kernel estimates; see, e.g., [30].

5.2.2. *Symmetric conservative ultracontractive semigroups and the Baouendi–Grushin example.* The following result applies to the Baouendi–Grushin heat semigroup.

Corollary 5.11. *Assume that the kernels* q_t*,* $t>0$*, are symmetric,*

$$q_t(x,y) = q_t(y,x),$$

and conservative,

$$\int_X q_t(x,y)\ d\mu(y) = 1 \qquad \text{for a.e. } x\in X.$$

Let

$$\mathsf{P}_t f(x) := \int_X q_t(x,y) f(y)\ d\mu(y),$$

and suppose that there exist $C,\gamma>0$ *such that*

$$\|\mathsf{P}_t f\|_{L^\infty(X)} \le C t^{-\gamma}\|f\|_{L^1(X)} \qquad (t\ge 1).$$

Then the two marginals of

$$d\Pi_t(x,y) = q_t(x,y)\ d\mu(x)\ d\mu(y)$$

are both equal to μ*, and* Π_t *escapes from bounded rectangles.*

Consequently, for every pointed metric space (Y, d_Y, y_0), $1 \le p < \infty$, and $u \in L^p(X;Y)$ such that

$$\mathscr{N}^{q,1/2}_{s_0,p}(u) < \infty$$

for some $s_0 > 0$, one has

$$\lim_{s\downarrow 0} s\, \mathscr{N}^{q,1/2}_{s,p}(u)^p = \frac{4}{p}\, M_p(u).$$

Proof. Conservativeness gives the first marginal of Π_t, while symmetry gives the second one. Thus $a_t = b_t = 1$. If $K \subset X$ is bounded, then

$$\Pi_t(K \times K) = \int_K \mathsf{P}_t \mathbf{1}_K(x)\; d\mu(x) \le Ct^{-\gamma}\mu(K)^2 \longrightarrow 0.$$

Hence Corollary 5.1 applies and gives

$$J_p(t,u) \longrightarrow 2M_p(u) \qquad (t \to \infty).$$

The short-time assumption in Theorem 5.3 follows from Proposition 5.4. Taking $\beta = 1/2$ in (51) yields

$$\frac{a_\infty + b_\infty}{\beta p} = \frac{2}{p/2} = \frac{4}{p}.$$

□

We apply Corollary 5.11 to the heat kernel of the generally non-hypoelliptic Baouendi–Grushin operator. This recovers the asymptotic formula in [26]; see [26, Section 2] for background.

Corollary 5.12. *Let $\alpha \ge 0$, let $k, m \ge 1$, $n = k + m$, and consider*

$$G^n_\alpha = \mathbb{R}^k \times \mathbb{R}^m$$

with the Baouendi–Grushin operator

$$\Delta_\alpha = \Delta_x + |x|^{2\alpha}\Delta_y.$$

Let $(H_t)_{t\ge 0}$ be its heat semigroup and let $h_t(g, g')$ be the corresponding heat kernel with respect to Lebesgue measure. Let (Y, d_Y, y_0) be a pointed metric space and $1 \le p < \infty$. For $u \in L^p(G^n_\alpha; Y)$ define

$$J^\alpha_p(t,u) := \int_{G^n_\alpha}\int_{G^n_\alpha} d_Y(u(g), u(g'))^p h_t(g,g')\; dg'\; dg$$

and

$$\mathscr{N}^\alpha_{s,p}(u)^p := \int_0^\infty t^{-sp/2-1} J^\alpha_p(t,u)\; dt.$$

If

$$\mathscr{N}^\alpha_{s_0,p}(u) < \infty$$

for some $s_0 > 0$, then

$$\lim_{s\downarrow 0} s\, \mathscr{N}^\alpha_{s,p}(u)^p = \frac{4}{p}\int_{G^n_\alpha} d_Y(u(g), y_0)^p\; dg.$$

For $Y = \mathbb{R}$, this is the Maz'ya–Shaposhnikova formula of Li and Wang [26, Theorem 2.3].

Proof. The Grushin heat semigroup is conservative and its heat kernel is symmetric; see [26, Section 2]. If

$$Q = n + \alpha m$$

denotes the homogeneous dimension, the ultracontractive estimate [26, Lemma 2.1] gives

$$\|H_t f\|_{L^\infty(G^n_\alpha)} \le Ct^{-Q/2}\|f\|_{L^1(G^n_\alpha)}.$$

Consequently, for every bounded $K \subset G_\alpha^n$,

$$\int_K \int_K h_t(g,g')\, dg'\, dg = \int_K H_t \mathbf{1}_K(g)\, dg \le Ct^{-Q/2}|K|^2 \longrightarrow 0.$$

Thus the heat-kernel couplings have both marginals equal to Lebesgue measure and escape from bounded rectangles. The conclusion follows from Corollary 5.11. □

6. Application to Carnot groups

In this section we apply the abstract decoupling theorem to metric-valued maps on Carnot groups.

Let G be a Carnot group with dilations $(\delta_r)_{r>0}$, homogeneous dimension Q, Haar measure dx, and a continuous proper homogeneous norm N_G. For some basic facts on fractional Sobolev spaces on Carnot groups we refer, for instance, to [10]. Set

$$B_G(e,r) := \{h \in G : N_G(h) < r\}, \qquad S_G := \{h \in G : N_G(h) = 1\}.$$

We use the polar decomposition

$$\int_G f(h)\, dh = \int_0^\infty \int_{S_G} f(\delta_r \omega)\, d\sigma_G(\omega)\, r^{Q-1}\, dr,$$

where σ_G is the corresponding polar measure on S_G. In particular,

$$\sigma_G(S_G) = Q|B_G(e,1)|.$$

Let (Y, d_Y, y_0) be a pointed metric space and let $1 \le p < \infty$. For $u \in L^p(G;Y)$ define

$$\mathcal{E}_{s,p}(u) := \int_G \int_G \frac{d_Y(u(x),u(y))^p}{N_G(x^{-1}y)^{Q+sp}}\, dy\, dx$$

and

$$F_u(h) := \int_G d_Y(u(x),u(xh))^p\, dx.$$

By the change of variables $y = xh$ and unimodularity,

$$\mathcal{E}_{s,p}(u) = \int_G \frac{F_u(h)}{N_G(h)^{Q+sp}}\, dh. \tag{59}$$

The lemma is a direct consequence of Theorem 2.8.

Lemma 6.1. *Let $(\nu_\alpha)_\alpha$ be Borel probability measures on G such that $\nu_\alpha(K) \to 0$ for every compact set $K \subset G$. Then, for every $u \in L^p(G;Y)$,*

$$\int_G F_u(h)\, d\nu_\alpha(h) \longrightarrow 2M_p(u).$$

Proof. Associate with ν_α the coupling Π_α defined by

$$\int_{G\times G} \Phi(x,y)\, d\Pi_\alpha(x,y) := \int_G \int_G \Phi(x,xh)\, dx\, d\nu_\alpha(h)$$

for every nonnegative Borel function Φ. Since every Carnot group is unimodular, both marginals of Π_α are Haar measure. Moreover, if $K \subset G$ is bounded, then $\overline{K}^{-1}\overline{K}$ is compact and

$$\Pi_\alpha(K\times K) \le |K|\, \nu_\alpha(\overline{K}^{-1}\overline{K}) \longrightarrow 0.$$

Thus Π_α escapes from bounded rectangles. If Y has at least two points, Theorem 2.8 applies with $a_\alpha = b_\alpha = 1$; the one-point case is trivial. Since

$$J_{\Pi_\alpha}(u) = \int_G F_u(h)\, d\nu_\alpha(h),$$

the conclusion follows. □

Theorem 6.2. *Let $1 \le p < \infty$ and $u \in L^p(G;Y)$. Assume that, for all sufficiently small $s > 0$,*

$$\int_{\{N_G(h)<1\}} \frac{F_u(h)}{N_G(h)^{Q+sp}}\, dh < \infty$$

and that

$$\lim_{s\downarrow 0} s \int_{\{N_G(h)<1\}} \frac{F_u(h)}{N_G(h)^{Q+sp}}\, dh = 0. \tag{60}$$

Then

$$\lim_{s\downarrow 0} s\mathcal{E}_{s,p}(u) = \frac{2\sigma_G(S_G)}{p} M_p(u). \tag{61}$$

In particular, (61) *holds whenever*

$$\mathcal{E}_{\theta,p}(u) < \infty \tag{62}$$

for some $\theta > 0$.

Proof. For $s > 0$ define

$$d\nu_s(h) := \frac{sp}{\sigma_G(S_G)} \frac{\mathbf{1}_{\{N_G(h)>1\}}}{N_G(h)^{Q+sp}}\, dh. \tag{63}$$

By the polar formula,

$$\int_G d\nu_s = \frac{sp}{\sigma_G(S_G)} \sigma_G(S_G) \int_1^\infty r^{-1-sp}\, dr = 1.$$

Thus ν_s is a probability measure. Moreover, for every $R > 1$,

$$\nu_s(\{1 < N_G \le R\}) = sp \int_1^R r^{-1-sp}\, dr = 1 - R^{-sp} \longrightarrow 0$$

as $s \downarrow 0$. Since every compact subset of G is contained in some ball, $(\nu_s)_s$ escapes to infinity.

Split (59) into the regions $N_G(h) < 1$ and $N_G(h) > 1$. By (60), the normalized local contribution vanishes. Using (63) we can write

$$s \int_{\{N_G(h)>1\}} \frac{F_u(h)}{N_G(h)^{Q+sp}}\, dh = \frac{\sigma_G(S_G)}{p} \int_G F_u(h)\, d\nu_s(h).$$

Lemma 6.1 yields

$$\int_G F_u(h)\, d\nu_s(h) \longrightarrow 2M_p(u),$$

and (61) follows.

It remains to prove the last assertion. Assume that (62) holds for some $\theta > 0$. For $0 < s < \theta$ and $N_G(h) < 1$,

$$N_G(h)^{-Q-sp} \le N_G(h)^{-Q-\theta p}.$$

Consequently,

$$0 \le s \int_{\{N_G(h)<1\}} \frac{F_u(h)}{N_G(h)^{Q+sp}}\, dh \le s\mathcal{E}_{\theta,p}(u) \longrightarrow 0.$$

Thus (60) holds and the first part of the theorem applies. □

The theorem also gives a corresponding statement for maps between Carnot groups. Let H be another Carnot group endowed with a left-invariant homogeneous distance d_H, and write

$$N_H(z) := d_H(e_H, z).$$

Corollary 6.3. *Let $u : G \to H$ satisfy*

$$\int_G N_H(u(x))^p \, dx < \infty.$$

Assume

$$\lim_{s\downarrow 0} s \int_{\{N_G(h)<1\}} \int_G \frac{N_H(u(x)^{-1}u(xh))^p}{N_G(h)^{Q+sp}} \, dx \, dh = 0.$$

Then

$$\begin{aligned} \lim_{s\downarrow 0} s \int_G \int_G \frac{N_H(u(x)^{-1}u(y))^p}{N_G(x^{-1}y)^{Q+sp}} \, dy \, dx \\ = \frac{2\sigma_G(S_G)}{p} \int_G N_H(u(x))^p \, dx. \end{aligned} \tag{64}$$

Proof. Apply Theorem 6.2 with $(Y, d_Y, y_0) = (H, d_H, e_H)$. □

Corollary 6.4. *Let p_t be the heat kernel of a homogeneous sub-Laplacian on G. For a pointed metric space (Y, d_Y, y_0), $1 \le p < \infty$, and $u \in L^p(G; Y)$, set*

$$J_p^G(t, u) := \int_G \int_G d_Y(u(g), u(h))^p p_t(g^{-1}h) \, dh \, dg$$

and

$$\mathscr{N}_{s,p}^{G,h}(u)^p := \int_0^\infty t^{-sp/2-1} J_p^G(t, u) \, dt.$$

If $\mathscr{N}_{s_0,p}^{G,h}(u) < \infty$ for some $s_0 > 0$, then

$$\lim_{s\downarrow 0} s\mathscr{N}_{s,p}^{G,h}(u)^p = \frac{4}{p} \int_G d_Y(u(g), y_0)^p \, dg. \tag{65}$$

Proof. Let us start recalling that

$$\int_G p_t(h) \, dh = 1 \tag{66}$$

and

$$p_t(h) = t^{-Q/2} p_1(\delta_{t^{-1/2}} h). \tag{67}$$

Define

$$d\Pi_t(g, h) := p_t(g^{-1}h) \, dg \, dh.$$

The heat kernel is conservative and symmetric. Since the Haar measure on a Carnot group is unimodular[5] and (66), we have that for every $g, h \in G$,

$$\int_G p_t(g^{-1}h) \, dh = 1, \qquad \int_G p_t(g^{-1}h) \, dg = 1.$$

Thus both marginals of Π_t are Haar measure. The time-one heat kernel is bounded. The scaling property (67) gives

$$\|p_t\|_{L^\infty(G)} = t^{-Q/2} \|p_1\|_{L^\infty(G)}.$$

Consequently, for every bounded Borel set $K \subset G$,

$$\Pi_t(K \times K) \le |K|^2 \|p_t\|_{L^\infty(G)} = |K|^2 t^{-Q/2} \|p_1\|_{L^\infty(G)} \longrightarrow 0.$$

The couplings therefore escape from bounded rectangles, and Corollary 5.1 yields

$$J_p^G(t, u) \longrightarrow 2M_p(u).$$

The positive-order assumption implies the short-time condition by Proposition 5.4. Theorem 5.3 with $\beta = 1/2$ then gives (65). □

[5]A locally compact group G is called unimodular if its left and right Haar measures coincide. Equivalently, its Haar measure is invariant under both left and right translations

Remark 6.5 (Comparison with [19]). For scalar-valued maps, Corollary 6.4 coincides with [19, Theorem 1.5] by Garofalo and Tralli. With their notation,

$$\mathcal{N}^{\mathbb{G}}_{s,p}(f)^p := \int_0^\infty t^{-sp/2-1} \int_{\mathbb{G}} P_t(|f - f(g)|^p)(g)\ dg\ dt,$$

and for $1 \le p < \infty$ they prove

$$\lim_{s\downarrow 0} s\,\mathcal{N}^{\mathbb{G}}_{s,p}(f)^p = \frac{4}{p}\|f\|^p_{L^p(\mathbb{G})}$$

for f belonging to one positive heat-Besov class. A central ingredient of their proof is an integral decoupling property of the Carnot heat kernel. In the language of the present paper, that decoupling property says that the measures

$$d\Pi_t(g,h) = p_t(g^{-1}h)\ dg\ dh$$

escape every bounded rectangle as $t \to \infty$. Conservation and self-adjointness give two Haar marginals of mass one. Hence Theorem 2.8 yields $J_p(t,f) \to 2\|f\|_p^p$, and the Abel factor associated with $t^{-sp/2-1}$ is $2/p$, producing the constant $4/p$.

Use of generative AI

OpenAI's ChatGPT SOL was used for language editing and exploratory discussion. The author checked the proofs and references and is responsible for the contents of the manuscript.

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