AN ARONSON-BÉNILAN / LI-YAU ESTIMATE IN THE JKO SCHEME IN SMALL DIMENSION

FANCH COUDREUSE

ABSTRACT. We derive an Aronson-Bénilan / Li-Yau estimate in the JKO scheme associated to the porous-medium, heat, and fast-diffusion equations, in dimensions 1 and 2, and on simple domains (cubes, quarter-space, half-spaces, whole space, and the torus). Our method is based on a maximum principle for the determinant of the Hessian of Brenier potentials, iterated as a one-step improvement along the scheme. As a consequence, we obtain local L^∞ bounds on the density, uniform in the time step, consistent with the continuous-time result. As a byproduct, we rigorously derive the optimality conditions in the fast-diffusion case, filling a gap in the literature.

1. INTRODUCTION

This paper focuses on the JKO scheme approximation to the porous-medium, heat and fast-diffusion equations

$$\partial_t \rho = \Delta \rho^m$$

on a convex domain Ω, or in the torus $\mathbb{T}^d$, in the space of probability measures with finite second moment, with Neumann boundary condition and given initial data. The different names for this equation refer to the different regimes for the values of m:

- The case $m > 1$ corresponds to the porous-medium equation. It exhibits slow-diffusion-type behavior, and a free-boundary can appear if one starts from a compactly supported initial datum. We refer to the monograph [42] by Vazquez for the general theory of this equation.
- The linear case $m = 1$ corresponds to the classical heat equation, where strong smoothing effects occur, and solutions are instantaneously strictly positive.
- Finally, the case $0 < m < 1$ corresponds to the so-called fast-diffusion regime. Solutions are again strictly positive for positive time. In this setting, one can distinguish further regimes for the parameter m. In our analysis, two of them will be relevant:
  - The regime $m > m_c^1 := 1 - \frac{2}{d}$ to ensure well-posedness in the space of probability measures.
  - The regime $m > m_c^2 := 1 - \frac{2}{d+2}$ corresponding to existence of solutions in the space of probability measures with finite second moment.

  In both cases, one can check using Barenblatt profiles (corresponding to Dirac initial datum) that this threshold is sharp. We refer to the survey [6] by Bonforte and Figalli for more results on this equation.

Date: January 2026.

Note that all these equations are particular instances of the general filtration equation $\partial_t \rho = \Delta \phi(\rho)$.

Since the seminal work of Jordan, Kinderlehrer and Otto [26], it is now well understood that a large class of diffusion equations posed on the space of probability measures can be interpreted as the gradient flow on the Wasserstein space. While this can be made rigorous using the general theory of gradient flows in metric spaces developed by Ambrosio, Gigli and Savaré in [1], another classical approach to tackle this interpretation is to use the JKO (Jordan-Kinderlehrer-Otto) scheme, which can be seen as an implicit Euler scheme in the Wasserstein space. This scheme takes the following form: given a time step τ and a functional $\mathscr{E}$ over the space of probability measures with finite second moment, one constructs a sequence $(\rho_k^\tau)_{k\geq 0}$ by fixing $\rho_0 \in \mathcal{P}_2(\Omega)$ and iterating

$$\rho_k^\tau \in \operatorname{argmin}_{\eta \in \mathcal{P}_2(\Omega)} \mathscr{E}[\eta] + \frac{1}{2\tau} \mathrm{W}_2^2(\eta, \rho_{\mathrm{k}}^\tau)$$

Then one expects that the curve obtained by interpolation of the values of the sequence converges to a weak solution to the equation

$$\partial_t \rho = \nabla \cdot \left(\rho \nabla \frac{\delta \mathscr{E}}{\delta \rho}[\rho] \right)$$

with ρ_0 as initial datum, and suitable Neumann boundary conditions, where $\frac{\delta \mathscr{E}}{\delta \rho}$ is the first variation of $\mathscr{E}$ with respect to linear perturbation of the measures. This convergence can be made rigorous using the general theory of [1], or by [39, Chapter 8], provided that the scheme exists, the initial datum is of finite energy (i.e. $\mathscr{E}[\rho_0] < +\infty$), and the functional $\mathscr{E}$ admits λ-convexity with respect to Wasserstein geodesics.

For our setting of interest, the functional $\mathscr{E}$ is taken to be of the form $\mathscr{E}_m[\rho] = \int_\Omega f_m(\rho)\, \mathrm{d}\mathcal{L}^d$ for $\rho \ll \mathcal{L}^d$, where $f_m(z) = \frac{1}{m-1} z^m$ for $m \neq 1$, and $f_1(z) = z \log z$. This functional falls into the general theory of geodesically convex functionals provided that $m \geq m_c^{\mathrm{geo}} = 1 - \frac{1}{d}$, in which case f_m satisfies the McCann conditions [36] [39, Section 7.3.2].

A natural question in the study of the JKO scheme is whether the qualitative and quantitative properties known for the continuous flow can be recovered at the discrete level, uniformly in the time step τ. Such results are desirable for at least two reasons: they provide robustness of the scheme in recovering the behavior of the continuous in time equation, and they can be leveraged to improve convergence rates. Several properties of the continuous flow have been investigated in this direction: Lipschitz and continuity estimates [29] [20] [11], BV and Sobolev estimates with applications to $L_t^2 H_x^2$-convergence [38] [17] [18] [41], or comparison principle and L^1-contraction [24] [34] [23].

One particularly desirable estimate is the Aronson-Bénilan / Li-Yau estimate. It states that, for $m > m_c^1$, and for any positive solution ρ to $\partial_t \rho = \Delta \rho^m$, the pressure variable —defined by $p = \frac{m}{m-1} \rho^{m-1}$ for $m \neq 1$, and $p = \log \rho$ for $m = 1$ —, satisfies the sub-harmonic lower bound:

$$\Delta p \geq -\frac{\alpha_{d,m}}{t} \qquad \alpha_{d,m} = \frac{d}{d(m-1)+2} \tag{1.1}$$

In the context of the heat equation, this inequality was proved by Li and Yau [31] [45] and bears their name. We note that in this context, on $\mathbb{R}^d$ or $\mathbb{T}^d$, Hamilton [22] derived the stronger semi-convexity estimate $D^2 \log \rho \succeq -\frac{1}{2t}$, called the Li-Yau-Hamilton matrix inequality. The remaining cases were tackled by Aronson and Bénilan [2] [3] (see [42, Chapter 9] for a review of this estimate in the context of the porous-medium equation). Subsequently, this estimate has been extended

to more general frameworks: smooth manifolds [33], filtration equations [16], with extension to L^p-version, eventually with a source term [4], or to the Keller-Segel model in [19]. This estimate is fundamental, as it is the cornerstone for deriving L^1-L^∞-regularization effects for the equation (see Lemma 2.12), and for studying the regularity of the solution [3] [10] and its free-boundary [8].

The first study of such an inequality in the JKO scheme was performed by P.W. Lee [30] for the heat equation in the torus, where he proved that a version of the Li-Yau-Hamilton inequality holds at the level of the JKO scheme, at least for regular initial data. This was then extended by the author in [15], still in the torus, for a more general class of equations of granular-medium type: $\partial_t \rho = \Delta \rho + \nabla \cdot (\rho[\nabla V + \nabla W * \rho])$, under no assumptions on the initial datum; this estimate was then used to derive $L^2_{t,loc} H^2_x$-strong convergence of the scheme. In this paper, we take a first step toward a proof of an Aronson-Bénilan estimate in the JKO scheme by focusing on small dimension and simple domains.

1.1. **Main Result.** We consider an iteration of the JKO scheme starting from some measure $\rho_0 \in \mathcal{P}_2(\Omega)$:

$$\rho^\tau_{k+1} \in \underset{\eta \in \mathcal{P}_{2,ac}(\Omega)}{\arg\min}\ \mathscr{E}_m[\eta] + \frac{1}{2\tau} W_2^2(\eta, \rho^\tau_k)$$

assuming $m > m_c^1$, and additionally $m > m_c^2$ if Ω is unbounded. Although the super-linear case $m \geq 1$ is well understood and can be found in most references to the topic [26] [37] [39], the case $m < 1$, to the best of our knowledge, has not been fully treated in the literature, due to the non super-linear behavior of the function f_m used to define the entropy. We therefore devote some time in Section 3 to fill this gap, proving existence, uniqueness, and deriving optimality conditions in bounded domains.

Our main result, namely the Aronson-Bénilan / Li-Yau estimate in the JKO scheme, can then be stated. Defining the discrete pressure variable $p^\tau_k := \frac{m}{m-1}(\rho^\tau_k)^{m-1}$ for $m \neq 1$, and $p^\tau_k = \log \rho^\tau_k$, it takes the following form:

Theorem 1.1 (Aronson-Bénilan in JKO Scheme)**.** *Suppose that Ω is either: the torus, a cube, a quarter space, a half-space, or the whole space, in dimension $d = 1$ or 2. Then for all $k \geq 1$, $u^\tau_k := \tau p^\tau_k + \frac{1}{2}|\cdot|^2$ is convex finite on Ω, and there exists a universal sequence $(X_k)_{k\geq 1}$ valued in $[0,1]$, depending only on m, d, such that, in the Monge-Ampère sense*

$$\det\big(D^2 u^\tau_k\big)^{\frac{1}{d}} \geq 1 - X_k \tag{1.2}$$

Furthermore, as $k \to +\infty$ we have

$$X_k \sim \frac{1}{d(m-1)+2} \cdot \frac{1}{k} \tag{1.3}$$

By "the Monge-Ampère sense", we mean that the inequality should be interpreted as a lower bound on the Monge-Ampère measure associated with u^τ_k, see Definition 2.9 for an introduction to this object.

Interestingly, this JKO version of the estimate is slightly stronger than what one would expect translating the classical estimate; indeed, by the AM-GM inequality, one has that, for C^2 functions, $\frac{1}{d}\Delta u \geq \det\big(D^2 u\big)^{1/d}$, hence formally the lower bound on the determinant can be translated into $\Delta p^\tau_k \geq -d\tau^{-1} X_k$ (this can be made rigorous using viscosity solutions, see Lemma 2.11). Letting

$\tau \to 0$, and $k\tau \simeq t$, we recover the Aronson-Bénilan / Li-Yau estimate using the asymptotic of $(X_k)_{k\geq 1}$ (we refer to 5.1 for a precise statement). On the other hand, a linearization of the determinant as $\tau \to 0$ (expecting p_k^τ to converge to the pressure of the continuous equation), shows that this estimate is, asymptotically, not better than the Aronson-Bénilan.

Combining this result with the AM-GM like inequality of Lemma 2.11, and the L^1-L^∞ regularization Lemma 2.12, we obtain the following immediate corollary:

Corollary 1.2 (Local uniform L^∞-bounds on the JKO)**.** *For $\mathbb{B}_r = B_r(x) \subset \Omega$ with r small enough, there exists a constant $M = M(\Omega, t_0, \tau_0, m, r)$ such that for all $\tau \leq \tau_0$, $t \geq t_0$ one has*

$$||\rho_k^\tau||_{L^\infty(\mathbb{B}_r)} \leq M$$

This matches the classical L^∞-regularization effects for the porous-medium, heat, and fast-diffusion equations in the regime $m > m_c^1$.

The restriction to small dimension and simple domains stems from our strategy of proof, based on a maximum principle argument: this strategy yields an algebraic matrix inequality involving the Hessian of u_k^τ at the maximum point. Only in dimension 1 or 2 can this inequality be used to derive a lower bound of the determinant of the Hessian. On the other hand, the simple domain assumption is there to be able to handle boundary maximum points. This is handled through an analysis of the behavior of the transport map on the boundary of a cube. An extension to a broader class of domains and to higher dimension would necessitate new ideas.

1.2. Structure of the paper.

- In Section 2, we recall basic results in the theory of optimal transport, functionals over probability measures, and Monge-Ampère measures that will be used in the proof.
- In Section 3, we study the one-step JKO problem, in particular in the regime $m < 1$, and prove existence, uniqueness and optimality conditions for minimizers.
- In Section 4, we show a one-step improvement of Monge-Ampère lower bound on simple domains, under regularity of the initial datum.
- In Section 5, we complete the proof of the Aronson-Bénilan estimate.
- Finally in Appendix A, we show how to obtain L^1-L^∞ regularization effects on general domains under sub-harmonic assumptions.

1.3. Acknowledgment.

This work was supported by the European Union via the ERC AdG 101054420 EYAWKAJKOS. The author would like to thank Filippo Santambrogio for suggesting the problem, and for his valuable help in some technical parts of the proof. The author is also grateful to Ivan Gentil for valuable discussions and feedbacks during the preparation of this work.

2. Preliminaries

We recall here some basic results in optimal transport, and functionals over the space of probability measures, and Monge-Ampère measures. We refer to the monographs by Villani [44, 43] or Santambrogio [39] for further references. Throughout, Ω denotes a convex domain, that is a convex subset with non-empty interior, eventually unbounded, of $\mathbb{R}^d$, or the torus $\mathbb{T}^d$, and in both cases, $d(x, y)$ is the classical distance on Ω (Euclidean on subsets of $\mathbb{R}^d$, and quotient distance on $\mathbb{T}^d$). By

a slight abuse of notations, we shall make no distinction between an absolutely continuous measure and its density with respect to Lebesgue. Similarly, we shall always confuse classes of functions (resp. measures) on the torus and the corresponding class of periodic functions (resp. $\mathbb{Z}^d$-translation invariant measures).

2.1. **The Wasserstein distance.** Let $\mathcal{P}_2(\Omega)$ be the set of all positive measures on Ω with finite mass and finite second moment $M_2[\mu] := \int_\Omega |x|^2 \,\mathrm{d}\mu\,(x) < +\infty$.

Definition 2.1 (Wasserstein distance of order 2)**.** Let $\mu, \nu \in \mathcal{P}_2(\Omega)$. A transport plan between μ and ν is a probability measure γ on $\Omega \times \Omega$ with first and second marginals given by μ and ν. The set of all transport plans between μ and ν will be denoted by $\Pi(\mu, \nu)$. The *Wasserstein distance of order* 2 between μ and ν is defined as

$$W_2(\mu,\nu)^2 = \min_{\gamma\in\Pi(\mu,\nu)} \int_{\Omega\times\Omega} d(x,y)^2 \,\mathrm{d}\gamma\,(x,y) \tag{2.1}$$

The fact that this is a genuine minimum follows from the direct method in the calculus of variations, and any minimizer is called an optimal transport plan between μ and ν. It is well-known that W_2 is a metric, which metrizes the narrow convergence together with convergence of the second moment, and we shall say that $\mu_n \to \mu$ in $\mathbb{W}_2(\Omega)$ when $W_2(\mu_n, \mu) \to 0$.

A fundamental result in the theory is the so-called Kantorovich duality.

Theorem 2.2 (Kantorovich duality)**.** *Let $\mu, \nu \in \mathcal{P}_2(\Omega)$. Then one has*

$$\frac{1}{2}W_2(\mu,\nu)^2 = \sup\left\{ \int_\Omega \psi \,\mathrm{d}\mu + \int_\Omega \phi \,\mathrm{d}\nu \;\middle|\; \psi(x) + \phi(y) \le \frac{1}{2}d(x,y)^2 \right\} \tag{2.2}$$

Moreover, the supremum is attained at a pair (not necessarily unique) (ψ, ϕ) of c-conjugate functions, i.e. satisfying

$$\psi(x) = \phi^c(x) = \inf_{y\in\Omega}\left\{\frac{1}{2}d(x,y)^2 - \phi(y)\right\} \qquad \phi(y) = \psi^c(y) = \inf_{x\in\Omega}\left\{\frac{1}{2}d(x,y)^2 - \psi(x)\right\}$$

Furthermore, if γ is an optimal transport plan between μ and ν, then the inequality $\psi(x) + \phi(y) \le \frac{1}{2}d(x,y)^2$ is an equality γ-a.e. Such a pair is called a pair of Kantorovich potentials from μ to ν.

The transformation $\psi \mapsto \psi^c$ is usually called the c-transform, and functions that are c-transform of another function are called c-concave functions. It is easy to check that if ψ is c-concave, then $\psi = \psi^{cc}$. In the particular setting we are working with, c-concavity is equivalent to upper semi-continuity and 1-semi-concavity. It is worth noticing that, in the case of the torus, by periodicity of the involved functions, one can rewrite the c-transform as $\psi^c(x) = \inf_{y\in\mathbb{R}^d}\{\frac{1}{2}|x-y|^2 - \psi(y)\}$, which allows to treat both cases with the same definition. The functions $u := \frac{1}{2}|\cdot|^2 - \psi$ and $v := \frac{1}{2}|\cdot|^2 - \phi$ are usually called the Brenier potentials from ρ to μ. Those are convex functions, satisfying $u = v^*$ and $v = u^*$.

2.2. **Brenier's theorem and Caffarelli's regularity.** The existence of Kantorovich potentials is the first step in the proof of Brenier's theorem, stating that under absolute continuity assumptions on the densities, the optimal transport plan is in fact an optimal transport map. While this theorem was originally proved by Brenier [7] in the Euclidean case, it was extended to Riemannian manifolds, including the torus, by McCann in [35]. The case of the torus can ,in fact, be studied independently by identifying probability measures on $\mathbb{T}^d$ with periodic measures on $\mathbb{R}^d$. This has been done by Cordero - Erausquin in [14] (in French, see section 1.3.2 of [39] for an English version).

Theorem 2.3 (Brenier - Cordero - McCann)**.** *Let $\mu, \nu \in \mathcal{P}_2(\Omega)$, and (ψ, ϕ) be a pair of Kantorovich potentials from μ to ν.*

(1) *If $\mu \ll \mathcal{L}^d$, then ψ is twice differentiable μ-a.e. And, defining the (μ-a.e. defined) map $T(x) := x - \nabla\psi(x)$, one has $T_{\#}\mu = \nu$, and $(\mathrm{id}, \mathrm{T})_{\#}\mu$ is the unique optimal transport plan from μ to ν. We call T the optimal transport map from μ to ν.*

(2) *If we also have $\nu \ll \mathcal{L}^d$, and if S is the optimal transport map from ν to μ, then we have $T \circ S = \mathrm{id}$ (resp. $S \circ T = \mathrm{id}$) ν-a.e. (resp. μ-a.e.). Furthermore, the Monge-Ampère equation holds μ-a.e.*

$$\nu(T(x)) \det(DT(x)) = \mu(x) \tag{2.3}$$

Notice that, in the periodic case, the map T satisfies $T(x + n) = T(x) + n$ for all $n \in \mathbb{Z}^d$, and therefore defines a map from $\mathbb{T}^d$ to itself.

In general, the optimal potentials are only locally Lipschitz on the support of the measures, and their gradients are of locally bounded variation (using the semi-concavity assumption). In computations however, it is sometimes necessary to assume higher regularity of those functions. This type of regularity can be obtained using the celebrated regularity theory for the Monge-Ampère equation developed by Caffarelli [9] (we refer to the book [21] for an introduction to this deep subject), which roughly states that the transport map is one derivative more regular than the densities. Although originally developed in the Euclidean setting, this can be extended to the torus under the same assumptions as shown by Cordero-Erausquin in [14] (see also [32]).

Unfortunately, Caffarelli's original theory assumes strong regularity of the boundary on the domain, which the cube does not satisfy. On the other hand, using a reflection-type argument, it was proved by Jhaveri in dimension 2 in [25], and extended by Chen, Liu, and Wang in [12] to other dimension, that one can still obtain some regularity in this case.

Theorem 2.4 (Caffarelli's regularity in torus and cubes)**.** *Suppose $\Omega = \mathbb{T}^d$ or $\Omega = Q = [0, 1]^d$. Let (ψ, ϕ) be a pair of Kantorovich potentials between two absolutely continuous densities μ, ν, and suppose that there exists $\varepsilon > 0$ such that $\varepsilon \leq \nu, \mu \leq \varepsilon^{-1}$ a.e.. Then*

(1) *There exists $\beta > 0$ depending only on ε such that $\psi, \phi \in C^{1,\beta}(\Omega)$. Furthermore, ψ, ϕ are uniformly 1-concave.*

(2) *If for some $\alpha \in (0, 1)$ we have $\mu, \nu \in C^{k,\alpha}(\Omega)$ with $k = 0, 1$, then $\psi, \phi \in C^{k+2,\alpha}(\Omega)$, and the Monge-Ampère equation holds in the classical sense.*

In the torus, one can remove the constraints on k, that is, if μ, ν are $C^{k,\alpha}(\mathbb{T}^d)$ for some $k \geq 0$, then the Kantorovich potentials are of class $C^{k+2,\alpha}(\mathbb{T}^d)$. On the other hand, in the cube, the $C^{3,\alpha}(Q)$

regularity for is sharp, as shown by a counterexample for higher regularity constructed by Jhaveri [25]. It is worth mentioning that the subject of finding optimal regularity for the transport map in rough domains is a vast topic that has received considerable attention in recent years.

2.3. **Entropy functional.** We introduce the following family of convex functions for $m > 0$.

$$f_m(t) := \begin{cases} \frac{1}{m-1} t^m & m \neq 1 \\ t \log t & m = 1 \end{cases}$$

whose Legendre transform is given by

$$f_m^*(s) = \begin{cases} c_m \, [s]_+^{\frac{m}{m-1}} & \text{if } m > 1 \\ e^{s-1} & \text{if } m = 1 \\ \begin{cases} c_m \, (-s)^{\frac{m}{m-1}} & s < 0 \\ +\infty & s \geq 0 \end{cases} & \text{if } m < 1 \end{cases}$$

where $c_m := |m-1|^{\frac{1}{m-1}} [m^{\frac{1}{1-m}} + m^{\frac{m}{1-m}}] > 0$ for $m \neq 1$, and $[s]_+ = \max(s, 0)$ is the positive part of s.

For a probability measure $\rho \in \mathcal{P}_2(\Omega)$, we write the Lebesgue decomposition of ρ with respect to the Lebesgue measure on Ω as $\rho = \rho^{ac} \cdot \mathcal{L}^d + \rho^{\perp}$.

Definition 2.5 (m-entropy)**.** Let $\rho \in \mathcal{P}_2(\Omega)$. The m-entropy of ρ is defined as follows

$$\text{for } m \geq 1 \text{ by } \mathscr{E}_m[\rho] = \begin{cases} \int_\Omega f_m(\rho^{ac}) \, \mathrm{d}\mathcal{L}^d & \text{if } \rho^{\perp} = 0 \\ +\infty & \text{else} \end{cases} \tag{2.4}$$

$$\text{for } m < 1 \text{ by } \mathscr{E}_m[\rho] = \int_\Omega f_m(\rho^{ac}) \, \mathrm{d}\mathcal{L}^d \tag{2.5}$$

The reason for the apparent asymmetry in the definition lies in the different behavior of f_m at $+\infty$. Indeed, for $m \geq 1$, f_m is super-linear (i.e. $t^{-1} f_m(t) \to +\infty$). Whereas for $m < 1$, we have $\lim_{t \to +\infty} t^{-1} f_m(t) = 0$. By standard considerations in the theory of local functionals ([39, Section 7]), in order to ensure some lower semi-continuity property, one needs to take into account the singular part.

Proposition 2.6 (Lower semi-continuity of entropy)**.** *Suppose that $\rho_n \to \rho$ narrowly in $\mathcal{P}_2(\Omega)$. Then we have $\mathscr{E}_m[\rho] \leq \liminf_n \mathscr{E}_m[\rho_n]$ provided that one of the following conditions holds*

1. *Ω is bounded.*
2. *$m \geq 1$.*
3. *Ω is unbounded, $m > m_c^2$ and the sequence has uniformly bounded second moment, i.e. $\sup_n M_2[\rho_n] < +\infty$.*

Proof. For Ω bounded this follows from [39, Proposition 7.7], and the case $m > 1$ follows by positivity of f_m and [39, Remark 7.8].

For the unbounded case, this follows from a simple adaptation of the argument of [40, Proposition 2.1] or [27, Section 2]. We first need to find a continuous function b such that $(1+|x|)^{-2}|b(x)| \to 0$ as $+\infty$, and such that $f_m^*(-b)$ is integrable. Then, using Jensen's inequality, we have $f_m(t) + f_m^*(-b) + tb \geq 0$, and we deduce that

$$\begin{aligned}\mathscr{E}_m[\rho] &= -\int_\Omega b\,\mathrm{d}\rho - \int_\Omega f_m^*(-b)\,\mathrm{d}\mathcal{L}^d + \int_\Omega f_m(\rho^{ac})\,\mathrm{d}\mathcal{L}^d + \int_\Omega b\,\mathrm{d}\rho + \int_\Omega f^*(-b)\,\mathrm{d}\mathcal{L}^d \\ &= -\int_\Omega b\,\mathrm{d}\rho - \int_\Omega f_m^*(-b)\,\mathrm{d}\mathcal{L}^d + \sup_{K\Subset\Omega} \mathscr{E}_m[\rho \llcorner K] + \int_K b\,\mathrm{d}\rho + \int_K f^*(-b)\,\mathrm{d}\mathcal{L}^d\end{aligned}$$

where we used the positivity given by Jensen's inequality to transform the three last terms into the supremum.

By the previous result on bounded domain, and continuity of b, each functionals in the supremum are l.s.c. for the narrow convergence, and the supremum is therefore itself l.s.c. Furthermore, if a sequence converges narrowly and admits uniformly bounded second moment, then the integral of this sequence against b converges as b admits sub-quadratic growth.

Thus, if one can find such a b, then the l.s.c. result follows.

(1) For $m = 1$, we can simply take $b(x) = |x|$.

(2) For $0 < m < 1$, we want to take $b(x) = 1 + |x|^q$ for some $0 \leq q < 2$. Then $f_m^*(-b) = c_m(1+|x|^q)^{\frac{m}{m-1}}$ and the integrability hypothesis is true provided that $\frac{qm}{m-1} < -d$, i.e. $q > \frac{(1-m)d}{m}$. There exists such a q if and only if $2 > \frac{(1-m)d}{m}$, i.e. $m > m_c^2$. □

Finally, we will need the following moment estimate for the entropy, which will be used to derive coercivity in the one-step JKO scheme problem.

Proposition 2.7 (Lower bound on entropy)**.** *Suppose $m > m_c^2$. Then there exists a constant $C(m,d) \in \mathbb{R}$ such that for all $\rho \in \mathcal{P}_2(\mathbb{R}^d)$ we have*

$$\mathscr{E}_m[\rho] \geq \begin{cases} C(m,d) M_2[\rho]^{\frac{d}{2}(1-m)} & m \neq 1 \\ C(1,d) - \frac{d}{2}\log M_2[\rho] & m = 1 \end{cases}$$

Moreover $C(m,d)$ can be obtained by solving the variational problem

$$C(m,d) = \inf_{\rho\in\mathcal{P}_2(\mathbb{R}^d), M_2[\rho]=1} \mathscr{E}_m[\rho]$$

Proof. We consider the following optimization problem for $m > m_c^2$:

$$E_m(M) = \inf_{\rho\in\mathcal{P}_2(\mathbb{R}^d), M_2[\rho]=M} \mathscr{E}_m[\rho]$$

which is finite, as $\mathscr{E}_m[\rho] \geq 0$ for $m > 1$, and using Fenchel's inequality $f_m(t) \geq -(1+|x|^2)t + f_m^*(-(1+|x|^2))$ for $m \leq 1$. By approximation, we can restrict the optimization to $\rho \ll \mathcal{L}^d$. If ρ is admissible, then $\eta = M^{\frac{d}{2}}\rho(\sqrt{M}\cdot)$ satisfies $\eta \in \mathcal{P}_2(\mathbb{R}^d)$, $M_2[\eta] = 1$, $\mathscr{E}_m[\eta] = M^{\frac{d}{2}(m-1)}\mathscr{E}_m[\rho]$ for $m \neq 1$, and $\mathscr{E}_1[\eta] = \mathscr{E}_1[\rho] + \frac{d}{2}\log M$ for $m = 1$. This rescaling implies that

$$E_m(M) = \begin{cases} E_m(1) M^{\frac{d}{2}(1-m)} & m \neq 1 \\ E_m(1) - \frac{d}{2}\log M & m = 1 \end{cases}$$

and we conclude using $\mathscr{E}_m[\rho] \geq E_m(M_2[\rho])$ for any $\rho \in \mathcal{P}_2(\mathbb{R}^d)$. □

Remark 2.8. One can prove that optimizers for $E_m(M)$ exist and are Gaussian for $m = 1$, and of Barenblatt form for $m \neq 1$, i.e. of the form $(A - B|x|^2)_+^{\frac{1}{m-1}}$ for $m > 1$ and $(A + B|x|^2)^{\frac{1}{m-1}}$ for $m < 1$ with $A, B > 0$. Furthermore, one can then express $E_m(1)$ in terms of Gamma-type functions (for example, $E_1(1) = \frac{d}{2}\log(2\pi d)$, attained for the standard Gaussian).

2.4. **Monge-Ampère measure.** The Monge-Ampère measure is a weak extension of the non-linear second order quantity $\det\big(D^2 u\big)$, to an arbitrary convex function u. It allows one to define such a notion even if the function u is not regular, and is the cornerstone of the regularity theory of the Monge-Ampère equation. A detailed introduction can be found in [21] or [28].

Definition 2.9 (Monge-Ampère measure)**.** Let Ω be a convex domain of $\mathbb{R}^d$, $u : \Omega \to \mathbb{R}$ a convex function. The Monge-Ampère measure of u is the measure on $\mathrm{int}(\Omega)$ defined by

$$\mu_u(E) := |\partial u[E]| \qquad \partial u[E] = \bigcup_{x \in E} \partial u(x)$$

The fact that this defines a genuine Borel measure is a non-trivial fact in the theory, see [21, Theorem 2.3] for a proof. If u is of class C^2 (or even merely $C^{1,1}$), then this measure coincides with $\det\big(D^2 u\big) \cdot \mathcal{L}^d$. As is usual with weak notions, this measure admits better stability than the non-linear object $\det\big(D^2 u\big)$.

Proposition 2.10 (Stability of Monge-Ampère measure, Proposition 2.6 [21])**.** *Suppose that $u_n \to u$ locally uniformly on $\mathrm{int}(\Omega)$. Then $\mu_{u_n} \rightharpoonup \mu_u$ in the weak-$*$ topology (in duality with $C_c(\mathrm{int}(\Omega))$).*

We will say that an inequality of the form $\det\big(D^2 u\big) \geq \lambda$ holds in the Monge-Ampère sense if one has $\mu_u \geq \lambda \cdot \mathcal{L}^d$ in the sense of measures. We shall also sometimes write $\det\big(D^2 u\big)^{1/d} \geq \lambda$ as a shorthand for $\det\big(D^2 u\big) \geq \lambda^d$. The AM-GM inequality $\Delta u \geq d \cdot \det\big(D^2 u\big)^{1/d}$ can be extended to obtain a sub-harmonic bound from a Monge-Ampère lower bound.

Lemma 2.11 (Sub-harmonicity from Monge-Ampère lower bound)**.** *Suppose that $\det\big(D^2 u\big)^{1/d} \geq \lambda$ in the Monge-Ampère sense. Then $\Delta u \geq d \cdot \lambda$ in the viscosity/weak sense.*

Proof. By [28, Proposition 7.7], the inequality $\det\big(D^2 u\big) \geq \lambda^d$ also holds in the viscosity sense in $\mathrm{int}(\Omega)$. Furthermore, by [28, Theorem 7.2], convexity of u implies that we have $\lambda_{min}(D^2 u) \geq 0$ in the viscosity sense. We argue that this implies the asserted result. Indeed, fix $\psi \in C^2(\mathbb{R}^d)$ such that $\psi - u$ admits a local minimum at some point $x_0 \in \mathrm{int}(\Omega)$, then combining the two viscosity inequalities, we have that $D^2\psi(x_0)$ is symmetric semi-definite positive, and $\det\big(D^2\psi(x_0)\big) \geq \lambda^d$. But by the AM-GM inequality, for any symmetric semi-definite positive matrix N, we have $\mathrm{Tr}\, N \geq d \cdot \det(N)^{1/d}$, therefore we have

$$\Delta\psi(x_0) = \mathrm{Tr}\, D^2\psi(x_0) \geq d \cdot \det\big(D^2\psi(x_0)\big)^{\frac{1}{d}} \geq d \cdot \lambda$$

concluding the proof by the arbitrariness of ψ and x_0. □

2.5. **A L^1-L^∞-regularization Lemma.** Doing approximation, we shall need a L^1-L^∞ regularization Lemma for functions admitting some sub-harmonic lower bound. We define the function h_m by

$$h_m(z) := \begin{cases} z^{m-1} & m > 1 \\ \log z & m = 1 \\ -z^{m-1} & m < 1 \end{cases}$$

Lemma 2.12 (L^1-L^∞ regularization effects)**.** *Suppose $m > m_c^1$. Let $\mathbb{B}_2 = B_{2r}(x_0)$ be some ball, and let $\mathbb{B}_1 = B_r(x_0)$. Then if $g \in L^1_+(\mathbb{B}_2)$, with $||g||_{L^1(\mathbb{B}_2)} \leq 1$, is such that $\Delta h_m(g) \geq -K$ weakly on $\mathbb{B}_2$ (assuming $g > 0$ if $m \leq 1$). Then there exists constants $r_*(d, m, K)$ and $M(r, d, m, K)$ such that if $r < r_*$ then $g \in L^\infty(\mathbb{B}_1)$ with $||g||_{L^\infty(\mathbb{B}_1)} \leq M$.*

Proof. The case $m > 1$ follows from [42, Lemma A.3]. In the case $m_c^1 < m \leq 1$, h_m is increasing convex. Let $y \in \mathbb{B}_1$, then $B_r(y) \subset \mathbb{B}_2$, and using the sub-harmonicity of $h_m(g) + \frac{K}{2}|x-y|^2$ we have

$$\begin{aligned} h_m(g(y)) &\leq \fint_{B_r(y)} h_m(g)\,\mathrm{d}\mathcal{L}^d + \frac{K}{2}\fint_{B_r(0)} |x|^2\,\mathrm{d}x \\ &= \fint_{\mathbb{B}_1} h_m(g)\,\mathrm{d}\mathcal{L}^d + c_d K r^2 \end{aligned}$$

On the other hand, using concavity of h_m and its monotony we can bound

$$\fint_{B_r(y)} h_m(g)\,\mathrm{d}\mathcal{L}^d \leq h_m\left(\fint_{B_r(y)} g\,\mathrm{d}\mathcal{L}^d\right) \leq h_m\left(\frac{1}{r^d\omega_d}||g||_{L^1(\mathbb{B}_2)}\right)$$

where ω_d is the volume of $B_1(0)$. Combining the two bounds, and using that $||g||_{L^1(\mathbb{B}_2} \leq 1$, we obtain

$$\begin{cases} \log g(y) \leq -\log \omega_d r^d + c_d K r^2 & m = 1 \\ -g^{m-1}(y) \leq -\omega_d^{1-m} r^{d(1-m)} + c_d K r^2 & m < 1 \end{cases}$$

In the first case, taking the exponential give the asserted $L^\infty(\mathbb{B}_1)$-bound. On the other hand, to get a L^∞-bound in the second case, we need to ensure that the right-hand-side is negative. This is true provided that $r^{2+d(m-1)} \leq C(d, m)K^{-1}$ for some constant $C(d, m)$, which concludes. □

3. The JKO Scheme

The JKO scheme consists of iterating the following minimization problem:

$$\rho \in \operatorname*{arg\,min}_{\eta \in \mathcal{P}_2(\Omega)} \mathscr{E}_m[\eta] + \frac{1}{2\tau} W_2^2(\eta, \mu) \tag{3.1}$$

where ,throughout this section, we assume that $m > m_c^1$, and additionaly that $m > m_c^2$ if Ω is unbounded. As explained in the introduction, to handle the case $m < 1$, one would like to prove that minimizers are always absolutely continuous, unique, and characterize such minimizers by their optimality conditions, at least on bounded domains.

For unbounded domain, we will focus on domains admitting some boundary regularity: we shall say that an convex domain $\Omega \subset \mathbb{R}^d$ is volume-regular if there exists a constant $V > 0$ such that $|\Omega \cap B_r(x)| \geq V r^d$ for all $x \in \Omega$, $r \leq \mathrm{diam}(\Omega)$. Any bounded convex domain is in fact volume regular, but for unbounded domain, one need to ensure for instance some uniform-Lipschitz regularity of the boundary to ensure that this is the case (note that convex domains has locally Lipschitz boundary,

but the Lipschitz constant might blow up far from the origin for unbounded domains). We will see that this condition is enough to ensure uniqueness and absolutely continuity of minimizers.

3.1. **Existence and Qualitative properties.** Existence readily follows from the direct method using the results of the previous section. Uniqueness of minimizers is more involved: for $m \geq 1$ it follows immediately from the strict convexity of $\mathscr{E}_m$ and convexity of $W_2^2(\cdot, \mu)$. On the other hand, for $m < 1$, the entropy is no longer strictly convex, but it admits a weaker form of strict convexity, which, combined with optimality conditions, still gives uniqueness of minimizers.

Proposition 3.1 (Existence of minimizers)**.** *For any $\mu \in \mathcal{P}_2(\Omega)$, and additionally assume Ω to be volume-regular if $m < 0$. Then there exists a unique minimizer for the one-step JKO problem starting from μ. We will denote by $Q_m^\tau[\mu]$ this unique minimizer.*

Proof of Proposition 3.1. We divide the proof into existence and uniqueness.

- **Existence**: In bounded domains, this follows immediately from l.s.c. of the entropy and Wasserstein distance ([39, Proposition 7.4]) and from compactness of sequences of probability measures for narrow convergence on such sets. We now focus on the case where Ω is unbounded (and therefore $m > m_c^2$). Let $\rho \in \mathcal{P}_2(\Omega)$. Integrating the inequality $\frac{1}{2}|x-y|^2 \geq \frac{1}{4}|x|^2 - \frac{1}{4}|y|^2$ against any optimal transport plan from ρ to μ, we obtain $\frac{1}{2\tau}W_2^2(\rho,\mu) \geq \frac{1}{4\tau}M_2[\rho] - \frac{1}{4\tau}M_2[\mu]$. Combined with Proposition 2.7 this gives
$$\mathscr{E}_m[\rho] + \frac{1}{2\tau}W_2^2(\rho,\mu) \geq \begin{cases} \frac{1}{4\tau}M_2[\rho] + C(m,d)M_2[\rho]^{\frac{d}{2}(1-m)} - \frac{1}{4\tau}M_2[\mu] & m \neq 1 \\ \frac{1}{4\tau}M_2[\rho] - \frac{d}{2}\log M_2[\rho] - C(1,d) - \frac{1}{4\tau}M_2[\mu] & m = 1 \end{cases} \tag{3.2}$$
This implies that any minimizing sequence has uniformly bounded second moment, which gives narrow compactness of such sequences by Prokhorov's theorem. Combining this with Proposition 2.6 we can again use the direct method to conclude.
- **Uniqueness**: For $m \geq 1$, this follows immediately from strict convexity of $\mathscr{E}_m$ and convexity of $W_2^2(\cdot,\mu)$. For $m < 1$, we only have the following weaker version of strict convexity: if ρ, η are such that $\mathscr{E}_m[t\rho + (1-t)\eta] = t\mathscr{E}_m[\rho] + (1-t)\mathscr{E}_m[\eta]$ for some $t \in (0,1)$, then $\rho^{ac} = \eta^{ac}$. In particular, uniqueness holds if there exists at least one absolutely continuous minimizer. As this is the case if Ω is volume-regular by Corollary 3.6, we can conclude on the uniqueness. □

3.2. **Optimality conditions.** Before deriving the optimality conditions, let's prove a qualitative behavior in the case $m \leq 1$.

Proposition 3.2 (Positivity and integrability of optimizers)**.** *Let $\mu \in \mathcal{P}_2(\Omega)$, ρ a minimizer for the one-step JKO problem starting from μ. If $m \leq 1$, then $\rho^{ac} > 0$ a.e. and $f_m'(\rho^{ac}) \in L^1_{loc}(\Omega)$*

Proof. We closely follow the proof of [39, Lemma 8.6] with minor modifications. Let $\xi \in \mathcal{P}_{2,ac}(\Omega)$ be a constant density if Ω bounded, and in the case of Ω unbounded, proportional to $e^{-|x|^2}$ for $m = 1$, and to $(1+|x|^2)^{\frac{1}{m-1}}$ for $m < 1$. It satisfies $\mathscr{E}_m[\xi] < +\infty$ and $|f_m'(\xi)| \leq A(1+|x|^2)$ for some constant $A > 0$.

Let ρ be a minimizer, with absolutely continuous part g, and set $\rho_\varepsilon = \varepsilon\xi + (1-\varepsilon)\rho$ whose absolute continuous part is $g_\varepsilon = \varepsilon\xi + (1-\varepsilon)g$. We will first prove that $g > 0$ a.e., then we will derive the integrability of $f_m'(g)\xi$, which will conclude.

- Using convexity of $W_2^2(\cdot,\mu)$ and optimality of ρ, if $\Sigma = \{g = 0\}$ we have

$$-\int_\Sigma f_m(\varepsilon\xi)\,\mathrm{d}\mathcal{L}^d + \int_{\Omega\setminus\Sigma}(f_m(g) - f_m(g_\varepsilon))\,\mathrm{d}\rho \le \frac{\varepsilon}{2\tau}(W_2^2(\xi,\mu) - W_2^2(\rho,\mu))$$

  By convexity of f_m, we have $f_m(g) - f_m(g_\varepsilon) \ge f_m'(g_\varepsilon)(g - g_\varepsilon) = \varepsilon f_m'(g_\varepsilon)(g-\xi)$. However, using the monotonicity of f_m', we have

$$f_m'(g_\varepsilon)(g-\xi) = \frac{1}{1-\varepsilon}f_m'(g_\varepsilon)(g_\varepsilon - \xi) \ge \frac{1}{1-\varepsilon}f_m'(\xi)(g_\varepsilon-\xi) = f_m'(\xi)(g-\xi)$$

  By hypothesis $|f_m'(\xi)| \le A(1+|x|^2)$, which implies that $f_m'(\xi)(g-\xi) \in L^1(\Omega)$. Integrating, we obtain

$$-\int_\Sigma f_m(\varepsilon\xi)\,\mathrm{d}\mathcal{L}^d \le \varepsilon\left(\int_\Omega f_m'(\xi)(\xi-g)\,\mathrm{d}\mathcal{L}^d + \frac{1}{2\tau}W_2^2(\xi,\mu) - \frac{1}{2\tau}W_2^2(\rho,\mu)\right)$$

  Therefore $\varepsilon^{-1}\int_\Sigma f_m(\varepsilon\xi)\,\mathrm{d}\mathcal{L}^d$ is bounded from below.

  - For $m = 1$, this is equal to $\log(\varepsilon)\int_\Sigma \xi\,\mathrm{d}\mathcal{L}^d + \varepsilon\int_\Sigma \xi\log(\xi)\,\mathrm{d}\mathcal{L}^d$ which converges to $-\infty$ if $|\Sigma| \neq 0$ as $\xi > 0$.
  - For $m < 1$, this is equal to $\frac{m}{m-1}\varepsilon^{m-1}\int_\Sigma \xi^m\,\mathrm{d}\mathcal{L}^d$ which again converges to $-\infty$ if $|\Sigma| \neq 0$ as $\xi > 0$.

  Hence we have $|\Sigma| \neq 0$ and $g > 0$ a.e.

- Rewriting the previous inequalities using $g > 0$ a.e. we had

$$\int_\Omega f_m'(g_\varepsilon)(g-\xi)\,\mathrm{d}\mathcal{L}^d \le \frac{1}{2\tau}W_2^2(\xi,\mu) - \frac{1}{2\tau}W_2^2(\rho,\mu)$$
$$f_m'(g_\varepsilon)(g-\xi) \ge f_m'(\xi)(g-\xi) \in L^1(\Omega)$$

  Applying Fatou's lemma to the positive part, and dominated convergence to the negative part, using the second inequality. We can pass to the limit $\varepsilon \to 0$ in both inequalities, which provides the integrability of $f_m'(g)(g-\xi)$. Finally, as $f_m'(g)g = mf_m(g) \in L^1(\Omega)$ by $-\infty < \mathscr{E}_m[\rho] < +\infty$, we conclude on the integrability of $f_m'(g)\xi$. □

Remark 3.3. In bounded domains, one shall be able, using the results of [5, Appendix B], to derive that $\rho^{ac} \ge \delta(m,\tau,\Omega) > 0$. This is due to the fact that f_m satisfies the lower Inada condition $f_m'(0) = -\infty$. Similarly, for $m \ge 1$, one can obtain $\rho^{ac} \le M(m,\tau,\Omega)$ by the upper Inada condition $f_m'(+\infty) = +\infty$.

In the super-linear case $m \ge 1$, the optimality conditions in the JKO scheme are pretty well understood. We refer for example to [39, Section 7.4.1] or to [24]. For $m < 1$, we will follows ideas used by Khanh and Santambrogio in the context of q-moment measure in [27] to show that a minimizer can't have a singular part.

Theorem 3.4 (Optimality conditions)**.** *Let Ω bounded, and $\mu \in \mathcal{P}_2(\Omega)$. Let ρ be a minimizer of the one-step JKO from μ. Then there exists a tuple of Kantorovitch potentials (ψ, ϕ) from ρ to μ such that:*

(1) *$m = 1$: We have*
$$\tau \log \rho = -\psi$$

(2) *$m > 1$: We have*
$$\tau \frac{m}{m-1} \rho^{m-1} = [-\psi]_+$$
where $[z]_+ = \max(z, 0)$. In particular, $\tau f'_m(\rho) + \frac{1}{2}|x|^2 = \max(\frac{1}{2}|x|^2 - \psi, \frac{1}{2}|x|^2)$ is convex.

(3) *$m_c^1 < m < 1$: ρ is absolutely continuous, $\psi < 0$, and we have*
$$\tau \frac{m}{m-1} \rho^{m-1} = -\psi$$

Proof. The only new result is the last case. We divide into several steps:

- **Step 1 - Directional derivative inequality:** We say that a measure $\chi \in \mathcal{P}_2(\Omega)$ is admissible if $f'_m(\rho^{ac})\chi^{ac} \in L^1(\Omega)$ and $\mathscr{E}_m[\chi] < +\infty$. Note that if χ^{ac} is bounded, then χ is admissible by Proposition 3.2. Fix such a χ, and let $\rho_\varepsilon := \varepsilon\chi + (1-\varepsilon)\rho$. Then by dominated convergence theorem, $\mathscr{E}_m[\rho_\varepsilon]$ is differentiable at 0 with
$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}_{|\varepsilon=0} \mathscr{E}_m[\rho_\varepsilon] = \int_\Omega f'_m(\rho^{ac})(\chi^{ac} - \rho^{ac}) \,\mathrm{d}\mathcal{L}^d$$
On the other hand, since ρ is supported on Ω, then by [39, Proposition 7.17-7.18], the Kantorovich potentials (ψ, ϕ) from ρ to μ are unique up to translation, and $W_2^2(\rho_\varepsilon, \mu)$ is differentiable at 0 with
$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}_{|\varepsilon=0} \frac{1}{2} W_2^2(\rho_\varepsilon, \mu) = \int_\Omega \psi \,\mathrm{d}[\chi - \rho]$$
Since, by optimality of ρ we have $\frac{\mathrm{d}}{\mathrm{d}\varepsilon}_{|\varepsilon=0} \tau\mathscr{E}_m[\rho_\varepsilon] + \frac{1}{2}W_2^2(\rho_\varepsilon, \mu) \geq 0$, and dividing the derivative of the Wasserstein distance into the absolutely continuous and singular part, we deduce that
$$\int_\Omega (\tau f'_m(\rho^{ac}) + \psi)(\rho^{ac} - \chi^{ac}) \,\mathrm{d}\mathcal{L}^d + \int_\Omega \psi \,\mathrm{d}[\chi^\perp - \rho^\perp] \geq 0 \tag{3.3}$$

- **Step 2 - Pointwise optimality condition:** Define $C := \operatorname{essinf}(\tau f'_m(\rho^{ac}) + \psi)$ and $C' := \inf \psi$, well defined by continuity of ψ. If we take $\chi^\perp = \rho^\perp$ in 3.3, and using the argument of [39, Proposition 7.20], we get that $\tau f'_m(\rho^{ac}) + \psi = C$ a.e. on Ω. On the other hand, if we let $\chi^{ac} = \rho^{ac}$, then taking $\chi^\perp$ concentrated on the set $\{\psi = C'\}$ shows that $\rho^\perp$ is concentrated on this set. Therefore we have
$$\begin{cases} \tau f'_m(\rho^{ac}) + \psi = C & \text{a.e.} \\ \psi = C' & \rho^\perp \text{ a.e.} \end{cases}$$

- **Step 3 - Equality of constants under existence of singular part:** Suppose $\rho^\perp \neq 0$, let $t := \rho^{ac}(\Omega)$ and $s := \rho^\perp(\Omega)$, so that $t + s = 1$. Let $a, b \geq 0$ such that $at + bs = 1$, then the measure $\chi = a\rho^{ac} \cdot \mathcal{L}^d + b\rho^\perp$ is admissible. Putting this measure into equation 4.3

we get $Ct(a-1)+C's(b-1)\geq 0$. As $s>0$, solving for b gives $b=\frac{1-at}{s}$ provided that $0\leq a\leq 1/t$. Replacing b by this value, and s by $1-s$ we obtain

$$Ct(a-1)+C's(b-1)=(a-1)t(C-C')\geq 0$$

Since $1/t>1$, $a-1$ can take both positive and negative value, therefore we must have $C=C'$. In particular, if $\rho^\perp$ is non-zero, the conditions become

$$\begin{cases}\tau f'_m(\rho^{ac})+\psi=C & \text{a.e.}\\ \psi=C & \rho^\perp \text{ a.e.}\end{cases}$$

- **Step 4 - Absolute continuity of ρ:** Suppose that $\rho^\perp\neq 0$, then $C=C'$. Let x_0 be a minimum for ψ, then by Lemma 3.5 below, we have

$$\tau f'_m(\rho^{ac}(x))=C-\psi(x)\geq -\frac{1}{2}|x-x_0|^2$$

Hence

$$\rho^{ac}(x)\geq\left(\frac{|m-1|}{\tau m}\right)^{\frac{1}{m-1}}|x-x_0|^{\frac{2}{m-1}}$$

As convex domains satisfy interior cone condition, we can find a cone

$$C(\nu,\theta,h)=\{tv,t\leq h,v\in\mathbb{S}^d,|v\cdot\nu|\leq\theta\}$$

such that $x+C(\nu,\theta,h)\subset\Omega$. Indeed, consider a ball $\overline{B}_F(x_0,r)\subset\Omega$, then the set $\mathrm{conv}(\mathrm{x},\overline{\mathrm{B}}_\mathrm{F}(\mathrm{x}_0,\mathrm{r}))$ contains such a set. Integrating the previous inequality over this set we have:

$$\begin{aligned}1\geq\int_{x_0+C(\nu,\theta,h)}\rho^{ac}\,\mathrm{d}\mathcal{L}^d&\geq\left(\frac{|m-1|}{\tau m}\right)^{\frac{1}{m-1}}\int_{C(\nu,\theta,h)}|x|^{\frac{2}{m-1}}\,\mathrm{d}x\\&=\left(\frac{|m-1|}{\tau m}\right)^{\frac{1}{m-1}}\mathcal{H}^{d-1}(\Sigma_{\beta,\nu})\int_0^h r^{d-1+\frac{2}{m-1}}\,\mathrm{d}r\end{aligned}$$

where $\Sigma_{\nu,\beta}:=\{v\in\mathbb{S}^d,|v\cdot\theta|\leq\theta\}$. The last term being infinite as $m>m_c^1$, we get a contradiction. Therefore ρ is absolutely continuous, concluding the proof up to replacing ψ by $\psi-C$. □

Lemma 3.5 (Quadratic deviation from minimum)**.** *Let Ω be bounded convex, let (ψ,ϕ) be a pair of Kantorovitch potentials between two measures ρ,η on Ω. Let $x_0\in\Omega$ be a minimum point of ψ (which exists by continuity), then for all $x\in\Omega$ we have*

$$\psi(x)\leq\psi(x_0)+\frac{1}{2}|x-x_0|^2 \tag{3.4}$$

Proof. Define $u(x):=\frac{1}{2}|x|^2-\psi$. By Kantorovich duality, we have $u(x)=\max_{y\in\Omega}x\cdot y-u^*(y)$, hence $\partial u[x_0]$ is non-empty and contains a point in Ω. Let p be such a point. By the sub-differential inequality we have

$$u(x)\geq u(x_0)+p\cdot(x-x_0)$$

But as $u(x)\leq\frac{1}{2}|x|^2-\psi(x_0)=\frac{1}{2}|x|^2-\frac{1}{2}|x_0|^2+u(x_0)$ we get

$$\frac{1}{2}|x|^2-\frac{1}{2}|x_0|^2+p\cdot(x_0-x)=\frac{1}{2}(2p-x_0-x)\cdot(x_0-x)\geq 0$$

for all $x \in \Omega$. In particular, if we take $x = p$, we get $-|p - x_0|^2 \geq 0$, therefore $p = x_0$. We deduce that

$$\begin{aligned}\psi(x) &= \frac{1}{2}|x|^2 - u(x) \leq \frac{1}{2}|x|^2 - u(x_0) - x_0 \cdot (x - x_0) \\ &= \psi(x_0) + \frac{1}{2}|x|^2 - \frac{1}{2}|x_0|^2 - x_0 \cdot (x - x_0) = \psi(x_0) + \frac{1}{2}|x - x_0|^2\end{aligned}$$

Therefore ψ deviates at most quadratically from its minimum. □

As a first consequence of the optimality conditions, and by approximation, one can derive that there exists at least an optimizers are always absolutely continuous, even in the unbounded case.

Corollary 3.6 (Absolute continuity in unbounded domains)**.** *Suppose Ω is a volume-regular domain, $m_c^2 < m < 1$. Then there exists a least one absolutely continuous minimizer for the one-step JKO problem.*

Proof. In the bounded case, this follows immediately from the optimality conditions. We now consider Ω unbounded volume-regular domain (with constant V), and set $\Omega_N := \Omega \cap B_N(0)$, which is, for N large enough, volume-regular with constant $\frac{1}{2}V$. We approximate μ by a sequence $\mu_N \in \mathcal{P}_2(\Omega_N)$ converging in $\mathbb{W}_2(\Omega)$ to μ, and we let $(\rho_N)_N$ be the corresponding sequence of minimizers. By Proposition 3.8, up to subsequence, $\rho_N \to \rho$ in $\mathbb{W}_2(\Omega)$ where ρ is a minimizer of the one-step JKO scheme starting from μ.

By the optimality conditions, for each N, there exists a Kantorovitch potential ψ_N such that $\tau \frac{m}{m-1} \rho_N^{m-1} = -\psi_N$. We argue that this implies uniform L^∞-bound on ρ_N. Indeed, consider x_0 maximum point of ρ_N, or equivalently, minimum point of ψ_N. By Lemma 3.5, for all $x \in \Omega_N$, $\tau \frac{m}{m-1} \rho_N^{m-1}(x_0) \leq \tau \frac{m}{m-1} \rho_N^{m-1}(x) - \frac{1}{2}|x - x_0|^2$. Integrating over $B(x_0, r) \cap \Omega_N$ for $r \leq \mathrm{diam}(\Omega_N)$ and using monotony and convexity of $-t^{m-1}$, as in the proof of Lemma 2.12, we obtain

$$\begin{aligned}\tau \frac{m}{m-1} ||\rho_N||_\infty^{m-1} &\leq \tau \frac{m}{m-1} \frac{1}{|\Omega_N \cap B(x_0, r)|^{m-1}} - c_d \frac{r^{d+2}}{|\Omega_N \cap B(x_0, r)|} \\ &\leq c_1(m, \tau, V) r^{d(1-m)} - c_2(m, V) r^2\end{aligned}$$

taking r small enough, depending only on m, τ, V, the right-hand-side is negative, and we can inverse the relation to obtain $||\rho_N||_{L^\infty(\Omega_N)} \leq M(\tau, m, V)$ uniformly in N. Combining this with the $\mathbb{W}_2(\Omega)$-convergence implies that the limit ρ is absolutely continuous (and even $L^\infty(\Omega)$). □

Finally, we have the following propagation of upper and lower bound. It follows either by a maximum principle type argument, in the spirit of [39, Proposition 7.32], or by the comparison principle, either using the L^1-contraction principle obtained by Jacobs, Kin, and Tong in [24] in the super-linear case, or by the general theory developed by Léger and Sylvestre in [34], which can be adapted to the non super-linear case once uniqueness is ensured.

Proposition 3.7 (Proposition of upper and lower bound)**.** *Suppose that $\varepsilon \leq \mu \leq \varepsilon^{-1}$, then the same holds for $Q_m^\tau[\mu]$.*

3.3. **Stability of the JKO scheme.** In this section, we show a simple stability result when both the domain, and the initial data, are approximated. More precisely, consider the following framework:

- We let $(\Omega_n)_{n\geq 0}$ be a non-decreasing sequence of convex domains of $\mathbb{R}^d$ (resp. $\Omega_n = \mathbb{T}^d$ for all $n \geq 0$), and we set $\Omega = \bigcup_{n\geq 0} \Omega_n$.
- For each $n \geq 0$, we let $\mu_n \in \mathcal{P}_2(\Omega_n)$, and we consider $\mu \in \mathcal{P}_2(\Omega)$. Furthermore we assume that $\mu_n \to \mu$ in $\mathbb{W}_2(\Omega)$. We let ρ_n be a minimizer for the one-step JKO problem starting from μ_n on the domain Ω_n.

We then have the following stability result, which follows by a simple Γ-convergence argument.

Proposition 3.8 (Stability)**.** *Under the above framework, then up to subsequence we have $\rho_n \to \rho$ in $\mathbb{W}_2(\Omega)$ as $n \to +\infty$, where ρ is a minimizer of the one-step JKO problem starting from μ on the domain Ω.*

Proof. We will first prove that the convergence holds narrowly by a Γ-convergence argument. Then using again this Γ-convergence, we will improve this narrow convergence to full $\mathbb{W}_2(\Omega)$-convergence.

- **Γ-convergence of JKO functional**: We define the functional:
$$\mathscr{J}_n[\eta] = \begin{cases} \mathscr{E}_m[\eta] + \frac{1}{2\tau} W_2^2(\eta, \mu_n) & \text{if } \eta \in \mathcal{P}_2(\Omega_n) \\ +\infty & \text{else} \end{cases}$$
We shall prove that $(\mathscr{J}_n)_{n\geq 0}$ Γ-converges to $\mathscr{J}[\eta] = \mathscr{E}_m[\eta] + \frac{1}{2\tau} W_2^2(\eta, \mu)$ for the narrow convergence in $\mathcal{P}_2(\Omega)$.
  (1) $\Gamma - \liminf$*:* If $\eta_n \rightharpoonup \eta$ and $\sup_n \mathscr{J}_n[\eta_n] < +\infty$, then if Ω is bounded, we can use l.s.c of the entropy and joint l.s.c of the W_2-distance to conclude. On unbounded Ω, we use inequality 3.2 obtained during the proof of Proposition 3.1 to derive uniform upper bound on $(M_2[\rho_n])_n$ (using that $(M_2[\mu_n])_n$ is uniformly bounded by $\mathbb{W}_2(\Omega)$ convergence), and then conclude using joint l.s.c of W_2 and the l.s.c result of Proposition 2.6.
  (2) $\Gamma - \limsup$*:* Fix $\eta \in \mathcal{P}_2(\Omega)$ with $\mathscr{J}[\eta] < +\infty$, and define for n large enough (such that $\eta(\Omega_n) \neq 0$) the measure $\eta_n = \eta(\Omega_n)^{-1} \eta \llcorner \Omega_n$, which converges to η in $\mathbb{W}_2(\Omega)$. As W_2 is continuous for this convergence, we have $W_2^2(\eta_n, \mu_n) \to W_2^2(\eta, \mu)$, and monotone convergence provides $\mathscr{E}_m[\eta_n] \to \mathscr{E}_m[\eta]$.
- **Narrow convergence of** $(\rho_n)_{n\geq 0}$: We argue that the sequence $(\rho_n)_{n\geq 0}$ is precompact for the narrow convergence of the sequence of minimizers for this convergence. In bounded domain, this is immediate. In unbounded domains, fix some $\eta_0 \in \mathcal{P}_2(\Omega_0)$ with finite $\mathscr{E}_m[\eta_0]$, then $\sup_n \mathscr{J}_n[\rho_n] \leq \mathscr{E}_m[\eta_0] + \frac{1}{2\tau}(M_2[\eta_0] + \sup_n M_2[\mu_n]) < +\infty$. Therefore using again inequality 3.2 we deduce that $\sup_n M_2[\rho_n] < +\infty$, providing tightness, and hence precompactness, of the sequence of minimizers. By Γ-convergence, the sequence ρ_n is converging narrowly, up to subsequence, to ρ minimizer of the One-Step JKO problem starting from μ (we will now assume that we did this extraction).
- **Upgrading to $\mathbb{W}_2(\Omega)$-convergence**: Using that the $\Gamma - \liminf$ and $\Gamma - \limsup$ inequalities must be equalities for the sequence $(\rho_n)_{n\geq 0}$, we deduce that we have convergence of the

transport distance: $W_2^2(\rho_n,\mu_n)\to W_2^2(\rho,\mu)$. We argue that such convergence is enough to deduce $\mathbb{W}_2(\Omega)$ convergence of $(\rho_n)_n$ toward ρ. To do so, let γ_n be an optimal transport plan between ρ_n and μ_n. By standard stability results in optimal transport theory, γ_n is converging narrowly to γ, optimal transport plan between ρ and μ. Next we use that $|x|^2=2|y|^2+2|x-y|^2-|x-2y|^2$ which gives, after integrating against γ_n,

$$M_2[\rho_n]=2M_2[\mu_n]+2W_2^2(\rho_n,\mu_n)-\int_{\Omega\times\Omega}|x-2y|^2\,\mathrm{d}\gamma_n$$

Now using narrow convergence of γ_n, we have $\int_{\Omega\times\Omega}|x-2y|^2\,\mathrm{d}\gamma\le\liminf_n\int_{\Omega\times\Omega}|x-2y|^2\,\mathrm{d}\gamma_n$. Therefore by $\mathbb{W}_2(\Omega)$-convergence of $(\mu_n)_n$ and convergence of $W_2^2(\rho_n,\mu_n)$ we obtain

$$\limsup_n M_2[\rho_n]\ge 2M_2[\mu]+2W_2^2(\rho,\mu)-\int_{\Omega\times\Omega}|x-2y|^2\,\mathrm{d}\gamma=M_2[\rho]$$

which, combined with the l.s.c. of the second moment for the narrow convergence gives $M_2[\rho_n]\to M_2[\rho]$, i.e. $\rho_n\to\rho$ in $\mathbb{W}_2(\Omega)$. □

More interestingly, when the domain Ω_N are bounded, one can say a bit more. Let $u_N:=\tau f_m'(\rho_N)+\frac12|x|^2$, which, by optimality, is a Brenier's potential from ρ_N to μ_N. We also let $u:=\tau f_m'(\rho)+\frac12|x|^2$, assuming that Ω is volume-regular to ensure absolute continuity of minimizers in the case $m<1$.

Proposition 3.9 (Convergence of potentials)**.** *We have $u_N\to u$ locally uniformly on Ω. In the sense that if $Q\subset\mathrm{int}(\Omega)$ is bounded, then $Q\subset\Omega_N$ for any N large enough, and $u_N\to u$ uniformly on Q.*

Proof. Let $\mathbb{B}_2=B_{2r}(x)\subset\Omega$, contained in all the Ω_N for N large enough. By optimality condition, u_N is convex, which implies that $\Delta f_m'(\rho_N)\ge-\frac{d}{\tau}$. Using Lemma 2.12 to ρ_N on $Q_{3r}(x)$, we deduce that $(\rho_N)_N$ is uniformly bounded on $\mathbb{B}_1=B_r(x)$ (as $\rho_N(Q_r(x))\le1$).

(1) If $m>1$, then as $f_m'(\rho_N)=\frac{m}{m-1}\rho_N^{m-1}$ is non-negative, we have $u_N\ge\frac12|x|^2$, i.e. u_N is uniformly lower-bounded on $\mathbb{B}_1$. It is also uniformly upper bounded on this set by the uniform upper bound on ρ_N. Therefore the sequence $(u_N)_{N\ge0}$ is uniformly bounded on $\mathbb{B}_1$. Since the sequence is convex, it converges, up to subsequence, locally uniformly on $\mathbb{B}_1$, to another convex function u. But as $\rho_N=\frac{m-1}{\tau m}\left(u_N-\frac12|x|^2\right)^{\frac{1}{m-1}}$, the limiting function must be equal to u.

(2) If $m\le1$, the upper bound only provides an uniform upper bound on $(u_N)_{N\ge0}$ on $\mathbb{B}_1$. We argue that this sequence must also be bounded from below on $\mathbb{B}_{1/3}=B_{r/3}(x)$. Indeed, suppose that this is not the case, then we can find $x_N\in\mathbb{B}_{1/3}$ such that $u_N(x_N)\to-\infty$. Let $y\in\mathbb{B}_{1/3}$, and define $z_N=2y-x_N\in B_r(x)$, then we have $y=\frac12(z_N+x_N)$, hence

$$u_N(y)\le\frac12u_N(x_N)+\frac12u_N(z_N)\le\frac12u_N(x_N)+\frac12\sup_{N,\mathbb{B}_1}u_N$$

Therefore $u_N\to-\infty$ uniformly on $\mathbb{B}_{1/3}$. Exploiting the relation between u_N and ρ_N, this shows that $\rho_N\to0$ uniformly on $\mathbb{B}_{1/3}$, hence $\rho=0$ on this set. But as $\rho>0$ a.e. this is absurd. Hence the sequence is uniformly lower bounded on $\mathbb{B}_{1/3}$, and converges, up to subsequence, locally uniformly, to a convex function, which again must be equal to u.

Therefore for all $x \in \mathrm{int}(\Omega)$, u_N converges uniformly to u in a neighborhood of x, which concludes. □

4. One-Step Improvement of Monge-Ampère lower bound

The first step toward our result is to show that if μ is regular enough and satisfies a Monge-Ampère lower bound, then after one step of the JKO scheme, this lower bound will be improved.

Theorem 4.1 (One-Step Improvement of Monge-Ampère lower bound)**.** *Suppose that Ω is a cube or the torus in dimension $d \in \{1, 2\}$. Let $\mu \in \mathcal{P}_{ac}(\Omega) \cap C^2(\Omega)$, strictly positive, and such that $u_m[\mu] = \frac{1}{2}|x|^2 + \tau f'_m(\mu)$ is convex. Then the same holds for $\rho = Q^\tau_m[\rho]$.*

Furthermore, suppose that $\det\big(D^2 u_m[\mu]\big)^{\frac{1}{d}} \geq \lambda \geq 0$, and additionally in the case of a cube, that $\nabla\mu(x) \cdot n = 0$ for all $x \in \partial Q \setminus \mathcal{C}$, and $\nabla\eta(x) = 0$ for all $x \in \mathcal{C}$ (see Section 4.1 for the definition of $\mathcal{C}$). Then one of the following holds:

(1) $\det\big(D^2 u_m[\rho]\big)^{\frac{1}{d}} \geq 1$.

(2) *There exists $\Lambda > 0$ such that $\det\big(D^2 u_m[\rho]\big)^{\frac{1}{d}} \geq \Lambda$ and*

$$1 + \frac{1}{\Lambda^{d(m-1)+1}} - \frac{1}{\Lambda^{dm}} \geq \lambda \tag{4.1}$$

Proof. As μ is $C^2(\Omega)$ and strictly positive, there exists $\varepsilon > 0$ such that $\varepsilon \leq \mu \leq \varepsilon^{-1}$. Under this regularity assumption, if $\rho = Q^\tau_m[\mu]$, we have the following:

(1) We have $\varepsilon \leq \rho \leq \varepsilon^{-1}$, and if (ψ, ϕ) is a pair of Kantorovich potentials from ρ to μ, then $\tau f'_m(\rho) = -\psi$. In particular, $u_m[\rho]$ is a Brenier potential from ρ to μ, and therefore convex.

(2) ρ, ψ are of class $C^{3,\beta}(\Omega)$ for some $\beta \in (0, 1)$ and $u_m[\rho]$ is strictly convex, in particular, $\det\big(D^2 u_m[\rho]\big) > 0$ on Ω.

The first point is a consequence of 3.7. The second one follows from a bootstrap argument: the bound away from 0 and $+\infty$ for ρ, μ provides $\psi \in C^{1,\beta}(\Omega)$ for some $\beta < 1$, and the corresponding Brenier potential is strictly convex, which in turn gives $\rho \in C^{1,\beta}(\Omega)$ by the optimality condition and the uniform bounds on ρ. Combining this with $\mu \in C^2(\Omega) \subset C^{1,\beta}(\Omega)$, we deduce that ψ, and hence ρ, are of class $C^{3,\beta}$.

In particular, under these regularity assumptions, using that $u_m[\rho]$ is a Brenier potential, the following Monge-Ampère equation holds in the classical sense:

$$\det\big(D^2 u_m[\rho]\big) = \frac{\rho}{\mu(\nabla u_m[\rho])}$$

We shall argue at a minimum point for $J := \det\big(D^2 u_m[\rho]\big)$, and let Λ^d be the minimum of J, which is strictly positive by strict convexity of $u_m[\rho]$. We will first treat the case of an interior minimum, and then treat the case where the minimum is attained at a boundary point.

For simplicity, we shall write u for $u_m[\rho]$ and v for $u_m[\rho]$. We also let $p := \tau f'_m(\rho)$ and $q := \tau f'_m(\mu)$, so that $u = \frac{1}{2}|x|^2 + p$, $v = \frac{1}{2}|x|^2 + q$. Furthermore, applying f'_m to the Monge-Ampère equation

gives

$$\begin{cases} J^{m-1} = \det\big(D^2u\big)^{m-1} = \frac{p}{q(\nabla u)} & m \neq 1 \\ \log J = \log\det\big(D^2u\big) = p - q(\nabla u) & m = 1 \end{cases} \tag{4.2}$$

Let x_0 be a minimizer of J, and suppose that x_0 is in the interior of Ω. From now on, all computations shall be performed at this particular point x_0.

- **Step 1: Second-order optimality conditions at x_0**: Taking the Hessian of the Monge-Ampère equation, and letting $R := D^2u\, D^2q(\nabla u)\, D^2u + D^2\nabla u \cdot \nabla q(\nabla u)$, we get

$$D^2p = \begin{cases} J^{m-1}R + (m-1)J^{m-2}\nabla J \cdot D^2u\, \nabla q(\nabla u) + q(\nabla u)\, D^2(J^{m-1}) & m \neq 1 \\ R + D^2(\log J) & m = 1 \end{cases}$$

  where $D^2\nabla u \cdot \nabla q(\nabla u) = \sum_i D^2(\partial_i u) \cdot \partial_i q(\nabla u)$.

  For a symmetric matrix A, we shall write $A \succeq 0$ if A is positive semi-definite. We argue that for each choice of m, the last term in each respective case is positive semi-definite. Indeed:

  - For $m > 1$, x_0 is also a minimum of J^{m-1}, hence $D^2J^{m-1} \succeq 0$, and since $q(\nabla u) \geq 0$ (by $q = \frac{\tau m}{m-1}\mu^{m-1} \geq 0$), we have $q(\nabla u)D^2J^{m-1} \succeq 0$.
  - For $m = 1$, x_0 is also a minimum of $\log J$, hence $D^2 \log J \succeq 0$.
  - For $m < 1$, x_0 is now a maximum of J^{m-1}, hence $D^2J^{m-1} \preceq 0$. Since $q = \frac{\tau m}{m-1}\mu^{m-1}$ and $m - 1 < 0$, we have $q \leq 0$, therefore $q(\nabla u)D^2J^{m-1} \succeq 0$.

  Using the first-order optimality condition $\nabla J = 0$, we get the inequalities

$$\begin{cases} D^2p \succeq J^{m-1}[D^2uD^2q(\nabla u)D^2u + D^2\nabla u \cdot \nabla q(\nabla u)] & m \neq 1 \\ D^2p \succeq D^2uD^2q(\nabla u)D^2u + D^2\nabla u \cdot \nabla q(\nabla u) & m = 1 \end{cases}$$

  To eliminate the third-order term, we shall use that, by first-order optimality, $0 = J^{-1}\nabla J = \nabla \log J = \operatorname{Tr}\big[D^2u\big]^{-1} D^2\nabla u$. Since $D^2u \succeq 0$, taking the trace against $[D^2u]^{-1}$ in the previous inequalities gives

$$\operatorname{Tr} D^2p[D^2u]^{-1} \geq J^{m-1} \operatorname{Tr} D^2q(\nabla u)D^2u \tag{4.3}$$

  We shall next exploit this algebraic matrix inequality.

- **Step 2: Exploiting inequality 4.3**: Replacing D^2p by $D^2u - \mathrm{I_d}$, D^2q by $D^2v - \mathrm{I_d}$, and J by Λ^d, we obtain

$$1 - \frac{1}{d}\operatorname{Tr}\big[D^2u\big]^{-1} + \frac{\Lambda^{d(m-1)}}{d}\operatorname{Tr} D^2u \geq \frac{\Lambda^{d(m-1)}}{d}\operatorname{Tr} D^2v(\nabla u)D^2u$$

  Next, we use the classical AM-GM inequality for matrices, which gives $\frac{1}{d}\operatorname{Tr} AB \geq \det(A)^{\frac{1}{d}} \det(B)^{\frac{1}{d}}$ for positive symmetric matrices A, B. This gives $\frac{1}{d}\operatorname{Tr} D^2v(\nabla u)D^2u \geq \det\big(D^2v(\nabla u)\big)^{\frac{1}{d}} \det\big(D^2u\big)^{\frac{1}{d}}$, which can be bounded from below by $\lambda \cdot \Lambda$, using $\det\big(D^2v(\nabla u)\big)^{1/d} \geq \lambda$ by assumption and the definition of Λ. Therefore, we get

$$1 - \frac{1}{d}\operatorname{Tr}\big[D^2u\big]^{-1} + \frac{\Lambda^{d(m-1)}}{d}\operatorname{Tr} D^2u \geq \Lambda^{d(m-1)+1} \cdot \lambda \tag{4.4}$$

We now distinguish between the case $d = 1$ and $d = 2$.

- **Step 3: Dimension one**: In dimension one, we have $D^2u = u''$ and $\det\big(D^2u\big) = u'' = \Lambda$. Hence we obtain
$$1 - \frac{1}{\Lambda} + \Lambda^{d(m-1)+1} \geq \Lambda^{d(m-1)+1} \cdot \lambda$$
and the result follows by dividing by $\Lambda^{d(m-1)+1}$.

- **Step 4: Dimension two**: In dimension two, we exploit the equality $\operatorname{Tr} A^{-1} = \frac{\operatorname{Tr} A}{\det A}$, valid for any invertible symmetric matrix A (and specific to this dimension), which follows by diagonalization. We obtain
$$1 + \frac{1}{d} \operatorname{Tr} D^2u \left(\Lambda^{d(m-1)} - \frac{1}{\Lambda^d} \right) \geq \Lambda^{d(m-1)+1} \cdot \lambda$$
If $\Lambda \geq 1$, there is nothing to do, otherwise we have $\Lambda^{d(m-1)} - \frac{1}{\Lambda^d} \leq 0$, and we can again use the AM-GM inequality, which gives $\frac{1}{d} \operatorname{Tr} D^2u \geq \det\big(D^2u\big)^{\frac{1}{d}} = \Lambda$ to get
$$1 + \Lambda^{d(m-1)+1} - \frac{1}{\Lambda^{d-1}} \geq \Lambda^{d(m-1)+1} \cdot \lambda$$
and we conclude again by dividing by $\Lambda^{d(m-1)+1}$.

This conclude the proof in case of interior minimum point. □

Remark 4.2. The reason for restricting to small dimension is apparent in the proof: one needs to control the determinant from below using both the trace of the Hessian and its inverse. This is possible only for dimension at most 2, and in larger dimension, it is possible to construct examples of matrices $A, B \succ 0$ with $\det(A) \geq \lambda$ and $\det(B)$ arbitrarily small, such that, upon replacing all $D^2v(\nabla u)$ by A, and D^2u by B, the algebraic inequality 4.4 holds true. The typical case is, in dimension 3, to take $B = \operatorname{Diag}(\varepsilon, \varepsilon, \varepsilon^{\mathrm{r}})$ for some well chosen r, and $A = c\mathrm{I}_{\mathrm{d}}$ for some constant $c > 0$.

On the other hand, we expect D^2u to be close to identity (as we expect $D^2f'_m(\rho)$ to be of order 1), and then one should in principle be able to linearize the inequality. It is, however, unclear that such a linearization can be done as the order-one estimate on D^2u_m is only formal, or strongly depends on μ, which would limit its usefulness for treating the case of general initial data when iterating the estimate.

This analysis assumed that x_0 interior point; on the cube, however, the minimum can be attained at the boundary. For PDEs, this is typically handled using the Hopf lemma. Here, however, we shall need a more involved analysis. On the other hand, a careful inspection of the proof shows that it suffices to prove that even if x_0 is at the boundary, we have $\nabla J(x_0) = 0$, and $\operatorname{Tr}\big[D^2u(x_0)\big]^{-1} D^2J(x_0) \geq 0$, thereby relaxing the full $D^2J(x_0) \succeq 0$ hypothesis.

4.1. **Treating the boundary.** The previous analysis was conducted assuming the minimum was attained in the interior of the domain. In order to tackle the general case, we shall take care of a boundary minimimum point.

We let $\Omega = Q$ be a cube in dimension one or two. We denote by $\mathcal{C}$ the set of corners of Q, and we call face of Q the closures of any connected components of $\partial Q \setminus \mathcal{C}$ (in dimension 2). We say that a point is in the interior of a face if it belongs to a face, and is not a corner. If x is such a point,

we let F_x be the unique face such that $x \in F_x$. We also consider a convex function $h : \mathbb{R}^d \to \mathbb{R}$ such that $Q = \{h \leq 0\}$ with equality at the boundary, h of class $C^1(\mathbb{R}^d \setminus \mathcal{C})$ and $\nabla h(x) = n(x)$ the outward pointing normal at any point x in the interior of a face. We shall note that n is in fact constant on any face F, and $n \perp (F - F)$.

We let $N_x Q$ be the tangent cone of Q at a point $x \in Q$, which is defined as the set of all $v \in \mathbb{R}^d$ such that for some sequence $y_n \in Q$ and $t_n > 0$ converging to 0 one has $y_n = x + t_n v + o(t_n)$. As Q is convex, this coincides with the set of admissible directions, i.e. $v \in \mathbb{R}^d$ such that $x + tv \in \Omega$ for all t small enough. Using Taylor's expansion, we easily see that if Φ is a C^1-diffeomorphism of Ω, then $D\Phi(x) N_x \Omega = N_{\Phi(x)} \Omega$.

We recall the following optimality condition result at boundary points:

Lemma 4.3 (Boundary optimality condition)**.** *Let and suppose that $x \in \partial Q$ is a minimum (resp. maximum) point of a function f. Then if f is C^1 near x:*

1. *For all $v \in N_x Q$, we have $\nabla f(x) \cdot v \geq 0$ (resp. ≤ 0). (in other word, $\nabla f(x)$ is in the polar cone of $N_x Q$).*
2. *If f is C^2, then for all $v \in N_x Q$ is such that $\nabla f(x) \cdot v = 0$. Then $D^2 f(x)[v, v] \geq 0$ (resp. ≤ 0).*
3. *If $x \in \partial Q \setminus \mathcal{C}$, there exists $\lambda \leq 0$ (resp. ≥ 0) such that $\nabla f(x) = \lambda \cdot n(x)$.*

Proof. We shall only consider the minimum case. The first two points follows by a Taylor expansion up to first and second order, as $0 \leq f(x + tv) - f(x) = t \nabla f(x) \cdot v + t^2 \frac{1}{2} D^2 f(x)[v, v] + o(t^2)$ for all $t > 0$ small enough. For the last point, we use $N_x Q = \{v \in \mathbb{R}^d, v \cdot n(x) \leq 0\}$ □

The reason of using a cube instead of a general domain lies in the following result on the behavior of any optimal transport map on the boundary of such a set. It would be interesting to see if such a result also holds on polygonal domains, using for example the recent regularity theory for general convex domain developed in [13], but this would need the adaptation of the tangent cone argument to a non-C^1-setting.

Proposition 4.4 (Behavior of transport on ∂Q)**.** *Let ν, μ be two probability measures with strictly positive densities of class $C^{0,\alpha}(Q)$ for some $\alpha < 1$, and let T be the optimal transport map from ν to μ. Then*

1. *T is the identity when restricted to the set of corners.*
2. *In dimension 2, if F is a face of Q, then $T(F) = F$.*

Proof. By Caffarelli's regularity on the cube 2.4, T is a $C^{1,\alpha}$-diffeomorphism of Q, therefore $DT(x)(N_x Q) = N_{T(x)} Q$ for any $x \in Q$. We argue that this implies that corner are sent to corner, interior face point to interior face point, and interior point to interior point. A simple way to see this is to look at the largest dimension of a subspace contained $N_x Q$, which completely characterize corners, interior faces, and interior point, which is an algebraic invariant by linear invertible map (such as $DT(x)$).

Now, for the first point, we notice that, by bijectivity of T, $T(\mathcal{C}) = \mathcal{C}$. By [39, Theorem 1.38] the support of the optimal transport plan from ν to μ, equal to $\{(x, T(x)), x \in Q\}$, is c-cyclically monotone. By restriction, the same holds for $\{(x, T(x)), x \in \mathcal{C}\}$. But then, using [39, Theorem 1.49], T is the optimal transport map from $\sum_{c\in\mathcal{C}} \delta_c$ to itself, i.e. it must be the identity on $\mathcal{C}$.

Let's finally prove the last point. Let F be a face, and write it down $F = [c_1, c_2]$ for two corners $c_1, c_2 \in \mathcal{C}$. Then we already now that $T(]c_1, c_2[) \subset \partial Q \setminus \mathcal{C}$. As T is continuous, $T(]c_1, c_2[)$ is connected, therefore it is fully contained in a face G. But then $T([c_1, c_2])$ is also contained in G, and contains $[T(c_1), T(c_2)] = [c_1, c_2] = F$ by connectedness. Hence $F \subset T(F) \subset G$, which forces $F = G$ and concludes the proof. □

We now have all the ingredients needed to finish the proof.

Ending proof of Theorem 4.1. As above, we let $\Lambda := \min_{x\in Q} \det\big(D^2u\big)^{1/d}$, and x_0 some minimizer. The goal is to prove, if x_0 is not an interior point, then at x_0 the inequality 4.3 still holds true. We let $T = \nabla u$ be the optimal transport map from ρ to μ, which satisfies the hypothesis of Proposition 4.4. We divide the reasoning into several steps.

- **Step 1: Gradient at corners**: First notice that $x + \tau\nabla p = \nabla u = T(x)$ for all $x \in Q$. Since $T(x) = x$ at corners, we must have $\nabla p = 0$ on the corners. Taking the gradient of equation 4.2 we get
$$\begin{cases} (m-1)J^{m-2}\nabla J = \frac{\nabla p}{q(\nabla u)} - JD^2u\nabla q(\nabla u) & m \neq 1 \\ J^{-1}\nabla J = \nabla p - D^2u\nabla q(\nabla u) & m = 1 \end{cases}$$
Applying this at corner point we get, for $c_m = m - 1$ for $m \neq 1$ and 1 else $c_m J^{m-3}\nabla J = -D^2u\nabla q$. Since, by assumption, we have $\nabla q = 0$ at corners, we deduce that $\nabla J(x) = 0$.

- **Step 2: Gradient at interior face points**: The function $h(\nabla u)$ is maximized at points of ∂Q. Furthermore, since $\nabla u(x) \in \partial Q \setminus \mathcal{C}$ for any $x \in \partial Q \setminus \mathcal{C}$, $h(\nabla u)$ is C^1 near any point in the interior of a face. By Lemma 4.3, there must be a function $x \to \lambda(x) \geq 0$ such that
$$\nabla[h(\nabla u)] = D^2u\nabla \cdot h(\nabla u) = D^2u \cdot n = \lambda(x) \cdot n(x)$$
at any $x \in \partial Q \setminus \mathcal{C}$. Furthermore, since $x \cdot n = 0$ for any such x, we also have $T(x) \cdot n = 0$. Since $\nabla p(x) = T(x) - x$ we deduce that $\nabla p(x) \cdot n = 0$. Putting this into the Monge-Ampère equation we obtain on $\partial Q \setminus \mathcal{C}$
$$\begin{aligned} c_m J^{m-3}\nabla J(x) \cdot n(x) &= -D^2u\nabla q(\nabla u(x)) \cdot n(x) = -\nabla q(\nabla u(x)) \cdot (D^2u(x) \cdot n(x)) \\ &= -\lambda(x)\nabla q(\nabla u(x)) \cdot n(x) = \lambda(x)\nabla q(\nabla u(x)) \cdot n(\nabla u(x)) = 0 \end{aligned}$$
by assumption, were we use that $\nabla u(x)$ is on the same face as x. Hence we obtain $\nabla J \cdot n = 0$ on $\partial Q \setminus \mathcal{C}$.

- **Step 3: The case of an interior face minimum point**: Consider $d = 2$ and assume that $x_0 \in \partial Q \setminus \mathcal{C}$. By Lemma 4.3, there exists $\lambda \leq 0$ such that $\nabla J(x_0) = \lambda n(x_0)$. But by the previous discussion, we have $\nabla J(x_0) \cdot n(x_0) = 0$. Hence $\lambda = 0$ and we deduce that $\nabla J(x_0) = 0$. Using again Lemma 4.3, we obtain $D^2J(x_0)[v, v] \geq 0$ for all $v \in N_{x_0}Q$. This is then also true for all $v \in -N_{x_0}Q$ (as $D^2J(x_0)[-v, -v] = D^2J(x_0)[v, v]$). As $N_{x_0}Q \cup (-N_{x_0}Q) = \mathbb{R}^d$, we deduce that $D^2J(x_0) \succeq 0$ and $\nabla J(x_0) = 0$. Therefore we can deduce the inequality 4.3 as in the interior point case and we conclude.

- **Step 4: The case of a corner minimum point**: We notice that in order to derive the inequality 4.3 we only need $\nabla J(x_0) = 0$ and $\mathrm{Tr}\big[D^2u(x_0)\big]^{-1}D^2J \geq 0$. The first point being proved in Step 1, we shall prove the second. By Lemma 4.3, for all $v \in N_{x_0}Q$ we get $D^2J(x_0)[v,v] \geq 0$. If $d = 1$, this implies that $D^2J = J'' \geq 0$ and the reasoning is done. If $d = 2$, we can, up to rotation, assume that $x = (0,0)$. Notice that $\nabla u(t,0) = (s(t),0)$ for some $s(t)$. In particular, $\partial_2 u(t,0) = 0$, hence $\partial_{12}u(t,0) = 0$. Letting $t \to 0$ gives $\partial_{12}u(0,0) = 0$. Therefore D^2u is diagonal in the canonical basis. But since $e_1, e_2 \in N_xQ$ we have
$$\mathrm{Tr}\big[D^2u(x_0)\big]^{-1}D^2J(x_0) = \frac{J_{11}(x_0)}{u_{11}(x_0)} + \frac{J_{22}(x_0)}{u_{22}(x_0)} \geq 0$$
which concludes the proof. □

5. Proof of the main Theorem

We let Ω being either the torus, a cube, a quarter-space a half-space or the whole space in dimension 1 or 2 (which are all volume-regular). We assume that $m > m_c^1 = 1 - \frac{2}{d}$ if Ω is bounded, and $m > m_c^2 =$ if Ω is unbounded.

We introduce the function
$$F_{d,m}[X] := \frac{1}{(1-X)^{d(m-1)+2}} - \frac{1}{(1-X)^{d(m-1)+1}} = \frac{X}{(1-X)^{d(m-1)+2}}$$
We shall note that this function has the following properties:

- $F_{d,m}$ is increasing, and define a bijection from $[0,1)$ to $[0,+\infty)$.
- $F_{d,m}[X] \geq X$ for all $X \in [0,1)$.

In particular, we can define a sequence $(X_k)_{k\geq 1}$ of $[0,1]$ by $X_1 = 1$, and $X_{k+1} = F_{d,m}^{-1}[X_k]$.

We can now prove the main theorem of the paper: the asymptotic Aronson-Bénilan estimate in the JKO scheme.

Theorem 5.1 (Asymptotic Aronson-Bénilan estimate)**.** *Let $\rho_0 \in \mathcal{P}_2(\Omega)$, consider the iteration of the JKO scheme starting from ρ_0: $\rho_{k+1}^\tau = Q_m^\tau[\rho_k^\tau]$. We let $(\rho_t^\tau)_{t\geq 0}$ be the piecewise constant interpolation of the values of the $(\rho_k^\tau)_{k\geq 0}$, i.e. $\rho_t^\tau = \rho_k^\tau$ on $[k\tau,(k+1)\tau)$.*

(1) *The function $u_k^\tau := \tau f_m'(\rho_k^\tau) + \frac{1}{2}|x|^2$ is convex for all $k \geq 1$.*

(2) *For all $k \geq 1$ we have the inequality, in the Monge-Ampère sense, for the sequence $(X_k)_{k\geq 1}$ defined above:*
$$\det\big(D^2u_k\big)^{\frac{1}{d}} \geq 1 - X_k \tag{5.1}$$
and we have $X_k \sim \frac{1}{(d(m-1)+2)k}$ as $k \to +\infty$.

(3) *The following asymptotic Aronson-Bénilan estimate holds: for all $t_0 > 0$, and $\varepsilon > 0$ there exists δ such that for all $\tau < \delta$, $t \geq t_0$ we have:*
$$\Delta f_m'(\rho_t^\tau) \geq -(1+\varepsilon)\frac{d}{d(m-1)+2} \cdot \frac{1}{t} \tag{5.2}$$

Proof. We divide into several steps.

- **Bounded domain, regular initial data**: We first prove points (1) and (2) under strong regularity assumption on the initial data, on the torus and on the cube. More precisely, we assume that ρ_0 is of class $C^2(\Omega)$ and positive. We check that ρ_k^τ satisfies the hypothesis of Theorem 4.1:

  (1) By propagation of lower and upper bound, there exists some $\varepsilon > 0$ such that $\varepsilon \leq \rho_k^\tau \leq \varepsilon^{-1}$ for all $k \geq 0$.

  (2) By the optimality conditions, $u_k^\tau = \tau f_m'(\rho_k^\tau) + \frac{1}{2}|x|^2$ is convex for all $k \geq 1$.

  (3) Therefore $f_m'(\rho_k^\tau)$ is Lipschitz, which combined with the upper and lower bounds, implies that ρ_k^τ is itself Lipschitz.

  (4) If ρ_k^τ is of class $C^2(\Omega)$, then as ρ_{k+1}^τ is Lipschitz, Caffarelli's regularity implies that the Kantorovitch potentials are $C^2(\Omega)$, which in turn implies that ρ_{k+1}^τ is $C^2(\Omega)$ by optimality conditions. Since ρ_0 is $C^2(\Omega)$, we can propagate this regularity: ρ_k^τ is $C^2(\Omega)$ for all $k \geq 0$.

  (5) Finally, if Ω is the cube, then by optimality conditions, we have for $k \geq 1$, $\tau \nabla f_m'(\rho_k^\tau) = T_k - \mathrm{id}$ where T_k is the optimal transport map from ρ_k^τ to ρ_{k-1}^τ. Then if x is a corner, we obtain $\nabla f_m'(\rho_k^\tau)(x) = 0$ since $T_k(x) = x$, and if x is in the interior of a face F, then $T_k(x)$ is in the interior of the same face, and we have $\nabla f_m'(\rho_k^\tau)(x) \cdot n(x) = T_k(x) \cdot n_F - x \cdot n_F = 0$ as $n \perp (F - F)$.

  Therefore we can iterate the One-Step improvement of Monge-Ampère lower bound starting from ρ_1^τ.

  Let Λ_k is the infimum of $\det\big(D^2 u_k^\tau\big)^{1/d}$, then either $\Lambda_{k+1} \geq 1$, or $F_{d,m}[1 - \Lambda_{k+1}] \leq 1 - \Lambda_k$. We argue that $\Lambda_k \geq 1 - X_k$ for all $k \geq 0$. Indeed, this is true for $k = 1$ as $\Lambda_1 \geq 0$ by convexity of u_1^τ. Suppose that this is true for some $k \geq 1$, then using the One-Step improvement 4.1, we either have $\Lambda_{k+1} \geq 1$, which gives $\Lambda_{k+1} \geq 1 - X_{k+1}$ as $X_{k+1} \geq 0$, or, by algebraic manipulation of the inequality appearing in the Theorem, $F_{d,m}[1 - \Lambda_{k+1}] \leq 1 - \Lambda_k \leq X_k = F_{d,m}[X_{k+1}]$, which yields $1 - \Lambda_{k+1} \leq X_{k+1}$ as $F_{d,m}$ is increasing, and we conclude by induction.

- **General case** We now approximate the domain, if unbounded, by an increasing sequence of cubes $(\Omega_N)_{N \geq 0}$, and the initial datum by $C^2(\Omega_N)$ positive initial datum. Iterating Proposition 3.8, the iterates are converging in $\mathbb{W}_2(\Omega_N)$, and by Proposition 3.9, the convex functions $(u_{k,N}^\tau)_{N \geq 0}$ are converging locally uniformly to u_k^τ. Using the stability of the Monge-Ampère measure, this concludes the general case.

- **Asymptotic for** $(X_k)_{k \geq 0}$ By $F_{d,m}[X] \geq X$, we deduce that the sequence $(X_k)_{k \geq 0}$ is decreasing, hence converging. And one easily see that the only fixed point of $F_{d,m}$ is 0. To obtain the asymptotic, we observe that one can linearize $F_{d,m}$ around 0 as $F_{d,m}[X] = X(1 + \alpha X + o(X))$, for $\alpha = d(m-1) + 2$. Therefore we have

$$X_k = X_{k+1}(1 + \alpha X_{k+1} + o(X_{k+1}))$$

  as $k \to +\infty$. Hence

$$\frac{1}{X_k} = \frac{1}{X_{k+1}} \cdot \frac{1}{1 + \alpha X_{k+1} + o(X_{k+1})} = \frac{1}{X_{k+1}} - \alpha + o(1)$$

we deduce that $\frac{1}{X_{k+1}} - \frac{1}{X_k} \to \alpha$ as $k \to +\infty$. Applying Cesàro lemma, we deduce that $\frac{1}{kX_{k+1}} \to \alpha$, i.e. $X_{k+1} \sim \frac{1}{\alpha k}$.

- **Asymptotic Aronson-Bénilan** Using the AM-GM inequality for Monge-Ampère measure 2.11. We have $\Delta f'_m(\rho_k^\tau) \geq -\frac{dX_k}{\tau}$. Fix $\varepsilon > 0$, for all $k \geq k_0$ large enough, we have $X_k \leq \frac{1+\varepsilon}{\alpha(k+1)}$, hence if $t_0 \geq (k_0+1)\tau$, i.e. $\tau \leq \frac{k_0+1}{t_0}$, then for $t \geq t_0$ we deduce that

$$\Delta f'_m(\rho_t^\tau) \geq -d\frac{1+\varepsilon}{\alpha\tau(k+1)} \geq -(1+\varepsilon)\frac{d}{d(m-1)+2} \cdot \frac{1}{t}$$

□

References

[1] Luigi Ambrosio, Nicola Gigli, and Giusuppe Savare. *Gradient Flows: In metric spaces and in the space of probability measures*. Lectures in Mathematics. ETH Zürich. Birkhäuser Basel, 2005.

[2] D. G. Aronson. Regularity Properties of Flows Through Porous Media. *SIAM Journal on Applied Mathematics*, 17(2):461–467, 1969.

[3] D. G. Aronson and P. Bénilan. Régularité des solutions de l'équation des milieux poreux dans $\mathbb{R}^\kappa$. *CR Acad. Sci. Paris Sér. AB*, 288(2):103–105, 1979.

[4] Giulia Bevilacqua, Benoit Perthame, and Markus Schmidtchen. The aronson–bénilan estimate in lebesgue spaces. *Annales de l'Institut Henri Poincaré C, Analyse non linéaire*, 40:259–286, 07 2022.

[5] Adrien Blanchet and Guillaume Carlier. Optimal transport and Cournot-Nash equilibria. *Mathematics of Operations Research*, 41(1):125–145, July 2015.

[6] Matteo Bonforte and Alessio Figalli. The Cauchy–Dirichlet problem for the fast diffusion equation on bounded domains. *Nonlinear Analysis*, 239:113394, 2024.

[7] Yann Brenier. Polar factorization and monotone rearrangement of vector-valued functions. *Communications on Pure and Applied Mathematics*, 44:375–417, 1991.

[8] Luis. A. Cafarelli and Avner Friedman. Regularity of the Free Boundary of a Gas Flow in an n-dimensional Porous Medium. *Indiana University Mathematics Journal*, 29(3):361–391, 1980.

[9] Luis Caffarelli. Some regularity properties of solutions of Monge-Ampère equation. *Communications on Pure and Applied Mathematics*, 44:965–969, 1991.

[10] Luis A. Caffarelli and Avner Friedman. Continuity of the Density of a Gas Flow in a Porous Medium. *Transactions of the American Mathematical Society*, 252:99–113, 1979.

[11] Thibault Caillet and Filippo Santambrogio. Fisher information and continuity estimates for nonlinear but 1-homogeneous diffusive PDEs (via the JKO scheme). *Bulletin of the Hellenic Mathematical Society*, 2025. cvgmt preprint.

[12] Shibing Chen, Jiakun Liu, and Xu-Jia Wang. Regularity of optimal mapping between hypercubes. *Advanced Nonlinear Studies*, 23, 09 2023.

[13] Tristan C. Collins and Freid Tong. Boundary regularity of optimal transport maps on convex domains, 2025.

[14] Dario Cordero-Erausquin. Sur le transport de mesures périodiques. *Comptes Rendus de l'Académie des Sciences - Series I - Mathematics*, 329(3):199–202, 1999.

[15] Fanch Coudreuse. Li-Yau-Hamilton Inequality on the JKO Scheme for the Granular-Medium Equation, 2025.

[16] M. G. Crandall and M. Pierre. Regularizing effects for $u_t = \Delta\phi(u)$. *Trans. Amer. Math. Soc.*, 274(1), 1982.

[17] Simone Di Marino and Filippo Santambrogio. JKO estimates in linear and non-linear Fokker–Planck equations, and Keller–Segel: L^p and Sobolev bounds. *Annales de l'I.H.P. Analyse non linéaire*, 39(6):1485–1517, 2022.

[18] Charles Elbar. Sobolev estimates for the Keller-Segel system and applications to the JKO scheme, 2026.

[19] Charles Elbar, Alejandro Fernández-Jiménez, and Filippo Santambrogio. A Li-Yau and Aronson-Bénilan approach for the Keller-Segel system with critical exponent, 2025.

[20] Vincent Ferrari and Filippo Santambrogio. Lipschitz estimates on the jko scheme for the fokker–planck equation on bounded convex domains. *Applied Mathematics Letters*, 112:106806, 2021.

[21] Alessio Figalli. *The Monge-Ampère Equation and Its Applications*. Zurich lectures in advanced mathematics. European Mathematical Society, 2017.

[22] Richard S Hamilton. Matrix Harnack estimate for the heat equation. *Communications in analysis and geometry*, 1(1):113–126, 1993.

[23] Mikaela Iacobelli, Francesco S. Patacchini, and Filippo Santambrogio. Weighted Ultrafast Diffusion Equations: From Well-Posedness to Long-Time Behaviour. *Archive for Rational Mechanics and Analysis*, 232:1165–1206, December 2018.
[24] Matt Jacobs, Inwon C. Kim, and Jiajun Tong. The L^1-contraction principle in optimal transport. *Annali Scuola Normal Superiore - Classe di Scienze*, 2020.
[25] Yash Jhaveri. On the (in)stability of the identity map in optimal transportation. *Calculus of Variations and Partial Differential Equations*, 58:1–25, 2017.
[26] Richard Jordan, David Kinderlehrer, and Felix Otto. The Variational Formulation of the Fokker–Planck Equation. *SIAM Journal on Mathematical Analysis*, 29(1):1–17, 1998.
[27] Huynh Khanh and Filippo Santambrogio. q-moment measures and applications: a new approach via optimal transport. *Journal of Convex Analysis*, 28(4):1033–1052, 2021.
[28] Nam Q. Le. *Analysis of Monge–Ampère Equations*. AMS, 2024.
[29] Paul W. Y. Lee. On the Jordan–Kinderlehrer–Otto scheme. *Journal of Mathematical Analysis and Applications*, 429, 09 2015.
[30] Paul W. Y. Lee. A Harnack inequality for the Jordan-Kinderlehrer-Otto scheme. *Journal of Evolution Equations*, 18(1):143–152, 2018.
[31] Peter Li and Shing Yau. On the parabolic kernel of the Schrödinger operator. *Acta Mathematica*, 156:153–201, 07 1986.
[32] Yanyan Li and Siyuan Lu. Monge-Ampère equation with bounded periodic data. *Analysis in Theory and Applications*, 2019.
[33] Peng Lu, Lei Ni, Juan-Luis Vázquez, and Cédric Villani. Local Aronson–Bénilan estimates and entropy formulae for porous medium and fast diffusion equations on manifolds. *Journal de Mathématiques Pures et Appliquées*, 91(1):1–19, 2009.
[34] Flavien Léger and Maxime Sylvestre. A synthetic approach to comparison principles for variational problems, with applications to optimal transport, 2025.
[35] Robert McCann. Polar factorization of maps on Riemannian manifolds. *Geometric and Functional Analysis*, 11:589–608, 08 2001.
[36] Robert J. McCann. A Convexity Principle for Interacting Gases. *Advances in Mathematics*, 128:153–179, 1997.
[37] Felix Otto. The Geometry of Dissipative Evolution Equations: The Porous Medium Equation. *Comm Partial Differential Equations*, 26, 04 2000.
[38] Guido Philippis, Alpár Mészáros, Filippo Santambrogio, and Bozhidar Velichkov. BV Estimates in Optimal Transportation and Applications. *Archive for Rational Mechanics and Analysis*, 219, 03 2015.
[39] Filippo Santambrogio. *Optimal transport for applied mathematicians*. Springer, 2015.
[40] Filippo Santambrogio. Dealing with moment measures via entropy and optimal transport. *Journal of Functional Analysis*, 271(2):418–436, 2016.
[41] Filippo Santambrogio and Gayrat Toshpulatov. Strong L^2H^2 convergence of the JKO scheme for the Fokker-Planck equation. *Archive for Rational Mechanics and Analysis*, 248, 10 2024.
[42] Juan Luis Vazquez. *The Porous Medium Equation: Mathematical Theory*. Oxford Science Publications, 09 2007.
[43] Cédric Villani. *Topics in Optimal Transportation Theory*, volume 58. AMS, 01 2003.
[44] Cédric Villani. *Optimal transport – Old and New*, volume 338. Springer, 01 2008.
[45] S.-T. Yau. On the Harnack inequalities of partial differential equations. *Comm. Anal. Geom.*, 2(4), 1994.