Horizontal Kakeya maximal operators in finite Heisenberg groups: Exact exponents and applications

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Abstract

We study Kakeya maximal operators associated with horizontal lines in finite Heisenberg groups $\mathbb{H}_n(\mathbb{F}_q)$. For the operator parameterized only by projective horizontal directions, we show that projection to $\mathbb{F}_q^{2n}$ reduces the problem to the affine finite field Kakeya maximal operator, and we determine the exact $\ell^u \to \ell^v$ growth exponent for all n and all $1 \le u, v \le \infty$. We then introduce a refined-direction operator that also records the central slope of a horizontal line. In $\mathbb{H}_1(\mathbb{F}_q)$, we prove the sharp $\ell^2 \to \ell^2$ estimate

$$\|\mathcal{M}^{\mathrm{rd}}_{\mathbb{H}_1} F\|_{\ell^2(\mathcal{D}_1)} \lesssim q^{1/2} \|F\|_{\ell^2(\mathbb{H}_1(\mathbb{F}_q))},$$

deduce the exact mixed-norm exponent formula, and obtain lower bounds for horizontal Heisenberg Kakeya sets with prescribed refined directions. The argument is purely Fourier-analytic and does not use the polynomial method. An outlook toward a new approach to the affine Kakeya problem in $\mathbb{F}_q^3$ will be discussed in this paper.

1 Introduction

The classical Kakeya needle problem asks for the smallest planar region in which a unit line segment can be rotated through all directions. While the original formulation is geometric, a major modern development is the study of *Kakeya (Besicovitch) sets*: subsets of $\mathbb{R}^n$ that contain a unit line segment in every direction. Besicovitch [2] showed that such sets can have Lebesgue measure zero, revealing that *containing all directions* is compatible with extreme measure-theoretic thinness.

Despite this, Kakeya sets are widely expected to be large in a finer sense. The *Kakeya set conjecture* predicts that any Kakeya set $E \subset \mathbb{R}^n$ must have Hausdorff and Minkowski dimension n. This conjecture is a central organizing problem because it quantifies the limits of how efficiently long thin objects in many directions can overlap, and it is intertwined with major questions in harmonic analysis.

The Kakeya maximal operator. A convenient analytic formulation uses tubes at scale $\delta \in (0, 1)$. Let T denote the δ-neighbourhood of a unit line segment (so T has length 1 and cross-

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sectional radius δ). For $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $e \in \mathbb{S}^{n-1}$, define the *Kakeya maximal operator*

$$K_\delta f(e) \;:=\; \sup_{T \| e} \frac{1}{|T|} \int_T |f(\mathbf{x})|\, d\mathbf{x}, \tag{1}$$

where the supremum runs over all such tubes T with axis parallel to e. The (scale-invariant) *Kakeya maximal operator conjecture* asserts that

$$\|K_\delta f\|_{L^n(\mathbb{S}^{n-1})} \;\lesssim_{\varepsilon,n}\; \delta^{-\varepsilon}\, \|f\|_{L^n(\mathbb{R}^n)} \qquad \text{for all } \varepsilon > 0, \tag{2}$$

and is closely related (indeed essentially equivalent, after discretization) to quantitative overlap bounds for direction-separated tube families. Such bounds imply sharp lower estimates for the Minkowski/Hausdorff size of Kakeya sets.

In dimension $n = 2$, Córdoba [3] proved sharp L^2-type estimates for K_δ. In higher dimensions, Wolff's seminal work [27] obtained improved L^p bounds for Kakeya-type maximal operators, yielding strong lower bounds for the possible dimension of Kakeya sets and setting the stage for much subsequent progress. Kakeya sets also arise as obstructions in Fourier analysis; for instance, Fefferman's counterexample to the ball multiplier problem is historically linked to Besicovitch-type configurations [14]. Recent progress has clarified parts of the Euclidean picture, especially in three dimensions: Katz and Zahl [23] obtained an improved lower bound $\dim_H(E) \geq \frac{5}{2} + \varepsilon_0$ for Besicovitch sets in $\mathbb{R}^3$, and Wang and Zahl [26] proved the three-dimensional Kakeya conjecture.

The finite field model (abelian). A useful companion viewpoint replaces Euclidean space by a vector space over a finite field. Let $\mathbb{F}_q$ be the field with q elements. A set $K \subset \mathbb{F}_q^n$ is a *finite field Kakeya set* if it contains a full affine line in every direction: for every $[\mathbf{v}] \in \mathbb{P}^{n-1}(\mathbb{F}_q)$, there exists $\mathbf{y} \in \mathbb{F}_q^n$ such that

$$\{\mathbf{y} + s\mathbf{v} : s \in \mathbb{F}_q\} \subset K.$$

The finite field analogue of the Kakeya conjecture asks whether such a set must have size comparable to the whole space, i.e. $|K| \geq c_n q^n$ with c_n independent of q. Dvir's breakthrough resolution of this problem introduced the polynomial method into Kakeya theory [9]. Building on this method, Ellenberg, Oberlin, and Tao [10] proved sharp Kakeya-type maximal estimates associated to algebraic varieties over finite fields, highlighting how algebraic structure may replace multiscale harmonic analysis.

While highly effective, the polynomial method relies fundamentally on algebraic vanishing. A natural complementary objective is to develop Fourier-analytic and geometric approaches that do not rely on algebraic vanishing, and therefore, extend to settings lacking strong algebraic structure.

In this paper, we pursue this program in the model setting of the finite Heisenberg group.

Kakeya sets and maximal operators in the Heisenberg group. In the sub-Riemannian geometry of the first Heisenberg group $\mathbb{H}_1$, a natural analogue of the Kakeya problem asks how small a set can be if it contains *horizontal* unit line segments in every horizontal direction. Studying Kakeya phenomena in the Heisenberg group provides a natural test case for understanding which aspects of classical Kakeya behavior depend on Euclidean structure and which persist in a nonabelian setting with non-isotropic dilations. The interaction between the group law, anisotropic dilations, and the horizontal distribution produces genuinely different geometric and analytic phenomena, making the Heisenberg group a particularly rich environment in which to probe the limits of Euclidean Kakeya theory. Here the geometry is that determined by the canonical

generating directions (the first layer of the stratification), and size is measured in terms of Hausdorff dimension with respect to a homogeneous metric such as the Koranyi metric.

In this direction, Liu [24] introduced a notion of Heisenberg Kakeya set and proved the sharp lower bound $\dim_{\mathbb{H}}(E) \geq 3$ for Kakeya sets in $\mathbb{H}_1$, with sharpness obtained (under this definition) by the horizontal plane. A complementary analytic viewpoint is provided by Kakeya-type maximal operators: Venieri [28] showed that, in analogy with the Euclidean theory, suitable L^p bounds for a Kakeya maximal operator imply lower bounds for the Heisenberg Hausdorff dimension of (bounded) Besicovitch/Kakeya sets. More recently, Fässler, Pinamonti, and Wald [13] defined Heisenberg Kakeya maximal operators M_δ by averaging over δ-neighbourhoods of horizontal unit segments (Koranyi tubes) and proved an essentially optimal estimate in $\mathbb{H}_1$, recovering Liu's sharp dimension bound as an application. Related techniques appear in [12, 22, 29].

These developments motivate a finite field Heisenberg Kakeya problem adapted to horizontal geometry. Our aim is to develop a discrete counterpart of Heisenberg maximal-operator theory.

The finite field Heisenberg maximal operator. Let q be a prime power. We identify the nth Heisenberg group over $\mathbb{F}_q$ with

$$\mathbb{H}_n(\mathbb{F}_q) := \mathbb{F}_q^n \times \mathbb{F}_q^n \times \mathbb{F}_q$$

equipped with the group law

$$(\mathbf{x}, \mathbf{y}, t) \cdot (\mathbf{x}', \mathbf{y}', t') = \Big(\mathbf{x} + \mathbf{x}',\, \mathbf{y} + \mathbf{y}',\, t + t' + \big(\mathbf{x} \cdot \mathbf{y}' - \mathbf{y} \cdot \mathbf{x}'\big)\Big),$$

where $\mathbf{x} \cdot \mathbf{y}'$ denotes the standard dot product on $\mathbb{F}_q^n$. A *horizontal direction* is an element $[\mathbf{v}] \in \mathbb{P}^{2n-1}(\mathbb{F}_q)$, represented by a nonzero vector $\mathbf{v} = (\mathbf{a}, \mathbf{b}) \in \mathbb{F}_q^n \times \mathbb{F}_q^n$ up to scaling. For $\mathbf{p} = (\mathbf{x}_0, \mathbf{y}_0, t_0) \in \mathbb{H}_n(\mathbb{F}_q)$ and $\mathbf{v} = (\mathbf{a}, \mathbf{b}) \neq \mathbf{0}$, we define the horizontal line through $\mathbf{p}$ in direction $[\mathbf{v}]$ as the right coset

$$L_{\mathbf{p},[\mathbf{v}]} := \{\, \mathbf{p} \cdot (s\mathbf{a}, s\mathbf{b}, 0) : s \in \mathbb{F}_q \,\}.^1$$

A direct computation gives the parametrization

$$L_{\mathbf{p},[\mathbf{v}]} = \Big\{ \big(\mathbf{x}_0 + s\mathbf{a},\ \mathbf{y}_0 + s\mathbf{b},\ t_0 + (\mathbf{x}_0 \cdot \mathbf{b} - \mathbf{y}_0 \cdot \mathbf{a})s\big) :\ s \in \mathbb{F}_q \Big\}, \tag{3}$$

and hence $|L_{\mathbf{p},[\mathbf{v}]}| = q$. The term $(\mathbf{x}_0 \cdot \mathbf{b} - \mathbf{y}_0 \cdot \mathbf{a})\, s$ is the *Heisenberg twist*: it depends on the basepoint $(\mathbf{x}_0, \mathbf{y}_0)$ and couples it to the direction $[\mathbf{v}]$ via the underlying symplectic form. This twist is what distinguishes horizontal Kakeya from the standard affine Kakeya problem on $\mathbb{F}_q^{2n+1}$.

We now define two maximal operators associated with horizontal lines. They differ in how much directional information they retain, and the gap between them drives the main results of this paper.

Throughout this paper, our maximal operators are defined using sums along lines (not averages). The exponent formulas below reflect this definition.

To avoid notational conflict, we consistently use $[\mathbf{v}]$ for projective directions and the letters u, v for exponents.

[1] When the basepoint $\mathbf{p}$ is not essential, we may write $L_{[\mathbf{v}]}$ by an abuse of notation. In certain proofs, we also adopt the notation $L_{\omega,\tau}$ for horizontal lines where their defining formula is clear from context.

For $F : \mathbb{H}_n(\mathbb{F}_q) \to \mathbb{C}$, we define the associated maximal operator

$$\mathcal{M}_{\mathbb{H}_n} F([\mathbf{v}]) := \max_{\mathbf{p} \in \mathbb{H}_n(\mathbb{F}_q)} \sum_{\mathbf{x} \in L_{\mathbf{p},[\mathbf{v}]}} |F(\mathbf{x})| \qquad ([\mathbf{v}] \in \mathbb{P}^{2n-1}(\mathbb{F}_q)). \tag{4}$$

This operator remembers only the spatial direction $[\mathbf{v}] = [\mathbf{a} : \mathbf{b}]$, in particular, it forgets the slope of the line in the central variable.

To capture the full Heisenberg geometry, we introduce a finer notion.

Definition 1.1. *Let $L = L_{p,[\mathbf{v}]}$ be a horizontal line in $\mathbb{H}_n(\mathbb{F}_q)$, where $\mathbf{v} = (\mathbf{a}, \mathbf{b}) \in \mathbb{F}_q^n \times \mathbb{F}_q^n$ satisfies $(\mathbf{a}, \mathbf{b}) \neq (\mathbf{0}, \mathbf{0})$. We define the* t-slope *of L by*

$$c(L) \ := \ \mathbf{x}_0 \cdot \mathbf{b} - \mathbf{y}_0 \cdot \mathbf{a} \in \mathbb{F}_q,$$

where $\mathbf{p} = (\mathbf{x}_0, \mathbf{y}_0, t_0)$ is an arbitrary point of L, and $\cdot$ denotes the standard dot product on $\mathbb{F}_q^n$.

The quantity $c(L)$ is well-defined: if $\mathbf{p}$ is replaced by another point $\mathbf{p} \cdot (s\mathbf{a}, s\mathbf{b}, 0) \in L$, then we have the same slope.

Definition 1.2. *Define the refined direction set*

$$\mathcal{D}_n := \big\{ [\mathbf{a} : \mathbf{b} : c] \in \mathbb{P}^{2n}(\mathbb{F}_q) : \ (\mathbf{a}, \mathbf{b}) \neq (\mathbf{0}, \mathbf{0}) \big\}.$$

For a horizontal line $L = L_{\mathbf{p},[(\mathbf{a},\mathbf{b})]}$, its refined direction is

$$\mathrm{Dir}(L) := [\mathbf{a} : \mathbf{b} : c(L)] \in \mathcal{D}_n.$$

Note that $\mathcal{D}_n = \mathbb{P}^{2n}(\mathbb{F}_q) \setminus \{[0 : \cdots : 0 : 1]\}$, i.e. the projective space with the single vertical direction removed. Moreover, for every $[\mathbf{a} : \mathbf{b} : c] \in \mathcal{D}_n$, there exists a horizontal line $L \subset \mathbb{H}_n(\mathbb{F}_q)$ with $\mathrm{Dir}(L) = [\mathbf{a} : \mathbf{b} : c]$. The set $\mathcal{D}_n$ fibers over $\mathbb{P}^{2n-1}(\mathbb{F}_q)$ via the projection $[\mathbf{a} : \mathbf{b} : c] \mapsto [\mathbf{a} : \mathbf{b}]$, with each fiber having q elements.

Definition 1.3. *A set $E \subset \mathbb{H}_n(\mathbb{F}_q)$ is called a full-direction horizontal Heisenberg Kakeya set if for every $\delta \in \mathcal{D}_n$ there exists a horizontal line $L \subset E$ with* $\mathrm{Dir}(L) = \delta$.

Although $\mathbb{H}_n(\mathbb{F}_q)$ and $\mathbb{F}_q^{2n+1}$ coincide as sets, the preceding notion is *not* the same as the usual affine Kakeya property in $\mathbb{F}_q^{2n+1}$: in the Heisenberg setting one prescribes refined directions of horizontal lines, whereas affine Kakeya prescribes ambient directions of arbitrary affine lines. See Examples 11.1 and 11.2.

For $F : \mathbb{H}_n(\mathbb{F}_q) \to \mathbb{C}$, we define the full refined-direction horizontal maximal operator

$$(\mathcal{M}^{\mathrm{rd}}_{\mathbb{H}_n} F)(\omega) := \max_{\substack{L \text{ horizontal} \\ \mathrm{Dir}(L) = \omega}} \sum_{\mathbf{p} \in L} |F(\mathbf{p})|, \qquad \omega \in \mathcal{D}_n. \tag{5}$$

In particular, the refined-direction operator discriminates among horizontal lines that share the same spatial direction but have different t-slopes, so $\mathcal{M}_{\mathbb{H}_n} F([\mathbf{a} : \mathbf{b}]) = \max_{c \in \mathbb{F}_q} \mathcal{M}^{\mathrm{rd}}_{\mathbb{H}_n} F([\mathbf{a} : \mathbf{b} : c])$.

We now make the following observation. The projection $\pi(\mathbf{x}, \mathbf{y}, t) = (\mathbf{x}, \mathbf{y})$ maps each horizontal line $L_{\mathbf{p},[\mathbf{v}]}$ bijectively onto an affine line in $\mathbb{F}_q^{2n}$ with the same projective direction $[\mathbf{v}]$. By aggregating F along the t-fibers one obtains a pointwise domination $\mathcal{M}_{\mathbb{H}_n} F \leq M_{2n} G$, where M_{2n} is the affine

(abelian) Kakeya maximal operator on $\mathbb{F}_q^{2n}$ and $G(\mathbf{x},\mathbf{y}) := \|F(\mathbf{x},\mathbf{y},\cdot)\|_{\ell^u(\mathbb{F}_q)}$.[2] Direct computations reveal that sharp upper bounds for $\mathcal{M}_{\mathbb{H}_n}$ follow from existing abelian input. Note that this reduction is not reversible (Remark 2.4), since the projection forgets the t-twist. In particular, there are functions F for which the projected abelian maximal operator is of size q in every direction, whereas $\mathcal{M}_{\mathbb{H}_1}F$ remains uniformly $O(1)$. We therefore view the sharp exponent formulas for $\mathcal{M}_{\mathbb{H}_n}$ as a reference estimate: they quantify what can be proved when one insists on the projective direction parameter alone, and they motivate the refined-direction framework.

For $1 \le u, v \le \infty$, let $A_n(u,v)$ be the smallest non-negative real number such that there exists $C_{n,u,v} > 0$ with

$$\|\mathcal{M}_{\mathbb{H}_n}F\|_{\ell^v(\mathbb{P}^{2n-1}(\mathbb{F}_q))} \le C_{n,u,v}\, q^{A_n(u,v)}\, \|F\|_{\ell^u(\mathbb{H}_n(\mathbb{F}_q))} \tag{6}$$

for all $F : \mathbb{H}_n(\mathbb{F}_q) \to \mathbb{C}$.

In the first theorem, we prove the exact value of $A_1(u,v)$.

Theorem 1.4 (Exact exponent for $n = 1$)**.** *For all $1 \le u, v \le \infty$, one has*

$$A_1(u,v) = \max\Big(\frac{1}{v},\ 1-\frac{1}{u},\ 1+\frac{1}{v}-\frac{2}{u}\Big). \tag{7}$$

The proof of Theorem 1.4 is self-contained and is based on the projection-reduction framework as observed above. The core step is the TT^* argument yielding the $\ell^2 \to \ell^2$ bound (Lemma 2.5)

$$\|\mathcal{M}_2 G\|_{\ell^2(\mathbb{P}^1(\mathbb{F}_q))} \le \sqrt{2q}\, \|G\|_{\ell^2(\mathbb{F}_q^2)}.$$

Interpolating this ℓ^2 estimate with the endpoint ℓ^1 and ℓ^∞ bounds gives the full range.

This ℓ^2 estimate serves as a key building block in the proof of our main result, Theorem 1.6 below: the zero-frequency contribution of the refined-direction operator reduces, after Fourier decomposition in the central variable, to precisely this planar Kakeya estimate. Each ingredient is developed with the broader Fourier-analytic program in mind. When $n > 1$, we obtain the following extension by replacing the planar ℓ^2 input (Lemma 2.5) by Theorem 7.2 due to Ellenberg, Oberlin, and Tao in [10].

Theorem 1.5 (Exact exponent for general n)**.** *For all $1 \le u, v \le \infty$, one has*

$$A_n(u,v) = \max\Big(\frac{2n-1}{v},\ 1-\frac{1}{u},\ 1+\frac{2n-1}{v}-\frac{2n}{u}\Big). \tag{8}$$

Theorems 1.4 and 1.5 capture what can be extracted from the projective direction parameter $[\mathbf{v}]$ alone. To access the genuinely Heisenberg content of the problem, we now turn to the refined-direction operator.

Since $\mathbb{H}_1(\mathbb{F}_q)$ may be identified with $\mathbb{F}_q^3$, the results of Ellenberg, Oberlin, and Tao [10, Theorem 1.1 and Remark 1.4] imply the bounds

$$\|\mathcal{M}^{\mathrm{rd}}_{\mathbb{H}_1}F\|_{\ell^3(\mathcal{D}_1)} \lesssim q^{2/3}\, \|F\|_{\ell^3(\mathbb{H}_1(\mathbb{F}_q))}, \qquad \|\mathcal{M}^{\mathrm{rd}}_{\mathbb{H}_1}F\|_{\ell^2(\mathcal{D}_1)} \lesssim q\, \|F\|_{\ell^2(\mathbb{H}_1(\mathbb{F}_q))}.$$

The $\ell^3 \to \ell^3$ estimate is optimal (see Example 11.3). By contrast, the $\ell^2 \to \ell^2$ bound is not sharp. In the next theorem, we obtain the optimal exponent. The proof is purely Fourier-analytic and

[2] For a finite set X and $1 \le p \le \infty$, we use $\ell^p(X)$ to denote the normed vector space consisting of complex-valued functions on X, equipped with the standard ℓ^p-norm based on the counting measure.

does not use polynomial vanishing.

Theorem 1.6. *There exists an absolute constant C such that for all $F : \mathbb{H}_1(\mathbb{F}_q) \to \mathbb{C}$,*

$$\|\mathcal{M}^{\mathrm{rd}}_{\mathbb{H}_1} F\|_{\ell^2(\mathcal{D}_1)} \le C\, q^{\frac{1}{2}}\, \|F\|_{\ell^2(\mathbb{H}_1(\mathbb{F}_q))}.$$

Moreover, the exponent $q^{\frac{1}{2}}$ is sharp.

The proof begins by linearizing the maximal operator and decomposing in the central Fourier variable. The zero-frequency term reduces, after projection to $\mathbb{F}_q^2$, to the planar Kakeya maximal estimate. The nonzero frequencies are controlled by Plancherel and character orthogonality, which reduce the key ℓ^2 bound to a counting problem for an explicit quadratic map.

For $1 \le u, v \le \infty$, let $A_1^{\mathrm{rd}}(u, v)$ be the smallest non-negative real number such that there exists $C_{u,v} > 0$ with

$$\|\mathcal{M}^{\mathrm{rd}}_{\mathbb{H}_1} F\|_{\ell^v(\mathcal{D}_1)} \le C_{u,v}\, q^{A_1^{\mathrm{rd}}(u,v)}\, \|F\|_{\ell^u(\mathbb{H}_1(\mathbb{F}_q))} \qquad \text{for all } F : \mathbb{H}_1(\mathbb{F}_q) \to \mathbb{C}. \tag{9}$$

Combining the sharp ℓ^2 estimate with endpoint bounds and interpolation, we determine the exact mixed-norm exponent formula.

Theorem 1.7. *For all $1 \le u, v \le \infty$, one has*

$$A_1^{\mathrm{rd}}(u, v) = \max\Big(\frac{1}{v},\ 1 - \frac{1}{u},\ \frac{2}{v} - \frac{1}{u},\ 1 + \frac{2}{v} - \frac{3}{u}\Big). \tag{10}$$

The upper bounds in Theorem 1.7 are obtained by interpolating the sharp ℓ^2 estimate of Theorem 1.6 with endpoint bounds ($\ell^1 \to \ell^1$, $\ell^1 \to \ell^\infty$, $\ell^\infty \to \ell^\infty$) and ℓ^p embeddings on $\mathcal{D}_1$. For sharpness, each of the four terms in (10) is forced by an explicit test function: a point mass, the indicator of a single horizontal line, the indicator of two non-parallel affine lines lifted to $t = 0$, and the constant function $F \equiv 1$, respectively.

To extend Theorem 1.7 to higher rank $n \ge 2$, a natural first step is to establish the analogue of Theorem 1.6, namely, an estimate of the form

$$\ell^2\big(\mathbb{H}_n(\mathbb{F}_q)\big) \longrightarrow \ell^{2n}(\mathcal{D}_n).$$

A straightforward adaptation of the $n = 1$ Fourier/Plancherel strategy yields a bound on the scale of $q^{\frac{n}{2}}$, rather than the smaller scale $q^{\frac{2n-1}{2n}}$ suggested by the point-mass obstruction. Closing this gap therefore appears to require genuinely new input. We plan to address the case $n \ge 2$ in a forthcoming arXiv revision.

One of the main motivations for developing the refined-direction framework is its connection to the affine Kakeya problem. Let $E \subset \mathbb{F}_q^3$ be an affine Kakeya set, identified with a subset of $\mathbb{H}_1(\mathbb{F}_q)$. One may decompose the non-vertical direction set $\mathcal{D}_1 = \Omega_1 \sqcup \Omega_2$, where Ω_1 consists of those refined directions realized by a horizontal line contained in E, and $\Omega_2 := \mathcal{D}_1 \setminus \Omega_1$. Theorem 1.8 immediately gives $|E| \ge C^{-2}\, q\, |\Omega_1|$. In particular, $|\Omega_1| \gtrsim q^2$ already forces $|E| \gtrsim q^3$. It therefore suffices to handle the case $|\Omega_2| \gtrsim q^2$. For each $\omega \in \Omega_2$, the Kakeya line realizing direction ω is not horizontal: its basepoint is forced away from a prescribed affine locus in the $(\mathbf{x}, \mathbf{y})$-plane. We show (Section 11.2) that each such line carries a well-defined non-horizontality parameter $\mu \in \mathbb{F}_q^*$, and lines with a fixed value of μ can be straightened into horizontal lines by an explicit mapping, reducing each μ-slice to a horizontal Kakeya configuration controlled by Theorem 1.6.

The remaining task is to exploit the interaction between different μ-slices, where the basepoint-avoidance constraints provide additional structure not visible in the usual affine formulation. We regard this program as a concrete route toward a new, Fourier-analytic proof of the affine Kakeya theorem in $\mathbb{F}_q^3$.

We now discuss some applications. In the first consequence, we obtain quantitative lower bounds for horizontal Heisenberg Kakeya sets, both for the full and restricted families of directions.

Theorem 1.8. *Let $\Omega \subset \mathcal{D}_1$, and let $E \subset \mathbb{H}_1(\mathbb{F}_q)$. Let m be an integer with $1 \le m \le q$. Assume that for every $\omega \in \Omega$, there exists a horizontal line $L \subset \mathbb{H}_1(\mathbb{F}_q)$ such that*

$$\mathrm{Dir}(L) = \omega, \qquad \text{and} \qquad |E \cap L| \ge m.$$

For $1 \le u, v \le \infty$, we have

$$|E| \gtrsim_{u,v} m^u \, |\Omega|^{u/v} \, q^{-u\, A_1^{\mathrm{rd}}(u,v)}. \tag{11}$$

In particular, for $(u, v) = (2, 2)$, one has

$$|E| \gtrsim \frac{m^2 \, |\Omega|}{q}. \tag{12}$$

Hence, any full-direction horizontal Heisenberg Kakeya set in $\mathbb{H}_1(\mathbb{F}_q)$ is of size at least $\gtrsim q^3$.

Let $E \subset \mathbb{H}_1(\mathbb{F}_q)$, and define

$$M_E(\omega) := (\mathcal{M}_{\mathbb{H}_1}^{\mathrm{rd}} \mathbf{1}_E)(\omega) = \max_{\substack{L \text{ horizontal} \\ \mathrm{Dir}(L)=\omega}} |E \cap L|, \qquad \omega \in \mathcal{D}_1.$$

The next theorem provides higher moment bounds for refined directions.

Theorem 1.9. *For every $2 \le s < \infty$, we have*

$$\sum_{\omega \in \mathcal{D}_1} M_E(\omega)^s \lesssim_s q \, |E|^{s-1}.$$

We conclude the introduction with brief remarks on related work. In the affine setting, Theorem 1.8 is connected to the study of Furstenberg sets in [6, 7] and the references therein. Fourier-analytic properties of affine Kakeya sets have been investigated recently by Fraser [11], and sum-product type results in the Heisenberg setting can be found in [17, 18, 19, 20]. We also note that Kakeya problems have been studied over rings such as $(\mathbb{Z}/N\mathbb{Z})^n$, where directions are defined modulo units and multi-scale phenomena arise, see [21, 8, 5, 4].

Organization of the paper. Section 2 collects the basic notation and geometric facts, and proves the projection and domination principles that reduce horizontal Kakeya estimates on $\mathbb{H}_1(\mathbb{F}_q)$ to planar Kakeya estimates on $\mathbb{F}_q^2$. Sections 3–6 establish the sharp mixed-norm bounds for the horizontal Kakeya maximal operator $\mathcal{M}_{\mathbb{H}_1}$ in the case $n = 1$ (Theorem 1.4), including the matching lower bounds. Section 7 outlines the corresponding argument in higher rank, yielding the extension to general n (Theorem 1.5). Sections 8–10 are devoted to the refined-direction maximal operator on $\mathbb{H}_1(\mathbb{F}_q)$: we prove the sharp ℓ^2 estimate, determine the full mixed-norm phase diagram, and deduce the stated combinatorial consequences for Heisenberg Kakeya sets. Finally, Section 11 presents examples distinguishing the refined-direction horizontal Kakeya property from the classical affine Kakeya property, and proposes an approach toward a new Fourier-analytic proof of the affine

Kakeya theorem in $\mathbb{F}_q^3$.

2 Preliminaries

2.1 Notations and Basics for $\mathbb{H}_1(\mathbb{F}_q)$

Let q be a prime power and let $\mathbb{F}_q$ be a finite field. The group $\mathbb{H}_1(\mathbb{F}_q)$ is identified with $\mathbb{F}_q^3$ with the group law

$$(x, y, t) \cdot (x', y', t') = \big(x + x',\, y + y',\, t + t' + (xy' - yx')\big).$$

For any $\mathbf{p} = (x_0, y_0, t_0) \in \mathbb{H}_1(\mathbb{F}_q)$ and $[\mathbf{v}] \in \mathbb{P}^1(\mathbb{F}_q)$ with $\mathbf{v} = (a, b) \neq (0, 0)$, the horizontal line

$$L_{\mathbf{p},[\mathbf{v}]} = \{\, \mathbf{p} \cdot (sa, sb, 0) :\ s \in \mathbb{F}_q \,\} = \big\{(x_0 + sa,\ y_0 + sb,\ t_0 + s(x_0 b - y_0 a)) :\ s \in \mathbb{F}_q\big\}$$

has refined direction $\mathrm{Dir}(L) = [a : b : (x_0 b - y_0 a)] \in \mathcal{D}_1$, where $\mathcal{D}_1 = \mathbb{P}^2(\mathbb{F}_q) \setminus \{[0 : 0 : 1]\}$. Given $(a, b) \neq (0, 0)$, the map $(x, y) \mapsto (xb - ya)$ is a nontrivial linear functional and hence is surjective onto $\mathbb{F}_q$.

Let us denote by $\mathcal{L}$ the set of all horizontal lines in $\mathbb{H}_1(\mathbb{F}_q)$. For $[\mathbf{v}] \in \mathbb{P}^1(\mathbb{F}_q)$, $\omega \in \mathcal{D}_1$, or $\mathbf{p} \in \mathbb{F}_q^2$, denote $\mathcal{L}([\mathbf{v}])$, $\mathcal{L}(\omega)$ and $\mathcal{L}(\mathbf{p})$ the set of all lines in $\mathcal{L}$ with direction $[\mathbf{v}]$, with refined direction ω, or through point $\mathbf{p}$, respectively. Note that

$$|\mathbb{H}_1(\mathbb{F}_q)| = q^3, \quad |\mathbb{P}^1(\mathbb{F}_q)| = q + 1, \quad |\mathcal{D}_1| = q^2 + q, \quad |L_{\mathbf{p},[\mathbf{v}]}| = q.$$

It follows that

$$|\mathcal{L}| = \frac{q^3 \cdot (q + 1)}{q} = q^2(q + 1).$$

Given $[\mathbf{v}] \in \mathbb{P}^1(\mathbb{F}_q)$, one has

$$|\mathcal{L}([v])| = \frac{q^2(q + 1)}{q + 1} = q^2.$$

Given $\omega \in \mathcal{D}_1$, one has

$$|\mathcal{L}(\omega)| = \frac{q^2(q + 1)}{q^2 + q} = q.$$

Given $\mathbf{p} \in \mathbb{F}_q^2$, one has

$$|\mathcal{L}(\mathbf{p})| = \frac{q^2(q + 1)}{q^2} = q + 1.$$

Moreover, there is a natural one-to-one correspondence between $\mathcal{L}(\mathbf{p})$ and $\mathbb{P}^1(\mathbb{F}_q)$. So each line in $\mathcal{L}(\mathbf{p})$ has different directions and different refined directions.

2.2 Basic lemmas in analysis

The first lemma collects Propositions 6.11 and 6.12 of [15].

Lemma 2.1. *Let X be a finite set with $|X| = N$ and let $1 \leq r \leq s \leq \infty$. Then for every $g : X \to \mathbb{C}$, one has*

$$\|g\|_{\ell^s(X)} \leq \|g\|_{\ell^r(X)},$$

and

$$\|g\|_{\ell^r(X)} \leq N^{\frac{1}{r} - \frac{1}{s}} \|g\|_{\ell^s(X)}.$$

The second lemma is an interpolation, whose proof can be found in [16].

Lemma 2.2. *Let X and Y be finite sets, and let*

$$T : \ell^{p_0}(X) \cap \ell^{p_1}(X) \longrightarrow \ell^{r_0}(Y) \cap \ell^{r_1}(Y)$$

be a linear operator. Let $1 \le p_0, p_1, r_0, r_1 \le \infty$. Assume that the following estimates hold for all functions $f : X \to \mathbb{C}$:

$$\|Tf\|_{\ell^{r_0}(Y)} \le C_0 \|f\|_{\ell^{p_0}(X)}, \qquad \|Tf\|_{\ell^{r_1}(Y)} \le C_1 \|f\|_{\ell^{p_1}(X)}.$$

Then, for any $0 \le \theta \le 1$, we have

$$\|Tf\|_{\ell^{r}(Y)} \le C_0^{1-\theta} C_1^{\theta} \|f\|_{\ell^{p}(X)},$$

where the exponents p and r are defined by

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \qquad \frac{1}{r} = \frac{1-\theta}{r_0} + \frac{\theta}{r_1}.$$

2.3 Planar domination lemma

For $f : \mathbb{F}_q^2 \to \mathbb{C}$, define

$$\mathcal{M}_2 f([\mathbf{v}]) := \max_{\ell \| [\mathbf{v}]} \sum_{\mathbf{x} \in \ell} |f(\mathbf{x})|, \qquad [\mathbf{v}] \in \mathbb{P}^1(\mathbb{F}_q), \tag{13}$$

where the maximum is taken over all affine lines $\ell \subset \mathbb{F}_q^2$ with direction $[\mathbf{v}]$.

Lemma 2.3. *Let $1 \le u \le \infty$ and $F : \mathbb{H}_1(\mathbb{F}_q) \to \mathbb{C}$ be given. For $(x, y) \in \mathbb{F}_q^2$, define*

$$G(x, y) := \Big(\sum_{t \in \mathbb{F}_q} |F(x, y, t)|^u \Big)^{\frac{1}{u}}$$

when $1 \le u < \infty$, and

$$G(x, y) := \max_{t \in \mathbb{F}_q} |F(x, y, t)|$$

when $u = \infty$. Then

$$\|G\|_{\ell^u(\mathbb{F}_q^2)} = \|F\|_{\ell^u(\mathbb{H}_1(\mathbb{F}_q))}, \tag{14}$$

and

$$\mathcal{M}_{\mathbb{H}_1} F([\mathbf{v}]) \le \mathcal{M}_2 G([\mathbf{v}]) \tag{15}$$

for all $[\mathbf{v}] \in \mathbb{P}^1(\mathbb{F}_q)$.

Proof. The identity (14) follows by expanding both sides:

$$\|G\|_{\ell^u(\mathbb{F}_q^2)}^u = \sum_{x,y} \sum_{t} |F(x, y, t)|^u = \|F\|_{\ell^u(\mathbb{H}_1(\mathbb{F}_q))}^u.$$

Fix $[\mathbf{v}] \in \mathbb{P}^1(\mathbb{F}_q)$ and choose a representative $\mathbf{v} = (a, b) \ne (0, 0)$. Fix $\mathbf{p} = (x_0, y_0, t_0)$ and set $L = L_{\mathbf{p},[\mathbf{v}]}$. Let $\pi : \mathbb{H}_1(\mathbb{F}_q) \to \mathbb{F}_q^2$ be the projection defined by $\pi(x, y, t) = (x, y)$. By (3), the projected set $\pi(L)$ equals the affine line

$$\ell = \{(x_0 + as,\ y_0 + bs) : s \in \mathbb{F}_q\} \subset \mathbb{F}_q^2,$$

which has direction $[\mathbf{v}]$. Since $\mathbf{v} \neq 0$, the map $s \mapsto (x_0 + as, y_0 + bs)$ is a bijection from $\mathbb{F}_q$ onto ℓ. Therefore, for each $(x, y) \in \ell$, there is a unique $t = t(x, y)$ such that $(x, y, t) \in L$. Hence

$$\sum_{\mathbf{z} \in L} |F(\mathbf{z})| = \sum_{(x,y) \in \ell} |F(x, y, t(x, y))|.$$

For each fixed (x, y), one has

$$|F(x, y, t(x, y))| \leq \Big(\sum_{t \in \mathbb{F}_q} |F(x, y, t)|^u \Big)^{\frac{1}{u}} = G(x, y).$$

Summing over $(x, y) \in \ell$ implies

$$\sum_{\mathbf{z} \in L} |F(\mathbf{z})| \leq \sum_{(x,y) \in \ell} G(x, y) \leq \mathcal{M}_2 G([\mathbf{v}]).$$

Taking the maximum over $\mathbf{p}$ gives (15). The case $u = \infty$ is identical. □

Remark 2.4. *It is worth noting that the projection domination $\mathcal{M}_{\mathbb{H}_1} F \leq \mathcal{M}_2 G$ is not reversible. To see this, fix a nonsquare $\eta \in \mathbb{F}_q^*$ and define*

$$\phi : \mathbb{F}_q^2 \to \mathbb{F}_q, \qquad \phi(x, y) := x^2 + \eta y^2.$$

Let

$$E_\phi := \{(x, y, t) \in \mathbb{H}_1(\mathbb{F}_q) : \ t = \phi(x, y)\}.$$

A direct computation shows that for all directions $[\mathbf{v}]$, $\mathcal{M}_2 G([\mathbf{v}]) = q$ and $\mathcal{M}_{\mathbb{H}_1} F([\mathbf{v}]) \leq 2$.

2.4 Planar Kakeya ℓ^2 bound

Lemma 2.5. *For every function $f : \mathbb{F}_q^2 \to \mathbb{C}$, one has*

$$\|\mathcal{M}_2 f\|_{\ell^2(\mathbb{P}^1(\mathbb{F}_q))} \leq \sqrt{2}\, q^{\frac{1}{2}}\, \|f\|_{\ell^2(\mathbb{F}_q^2)}.$$

Proof. Let $\widetilde{f} := |f|$. For every affine line $\ell \subset \mathbb{F}_q^2$, one has

$$\sum_{\mathbf{x} \in \ell} |f(\mathbf{x})| = \sum_{\mathbf{x} \in \ell} \tilde{f}(\mathbf{x}).$$

Taking the maximum over all affine lines ℓ with direction $[\mathbf{v}]$ implies

$$\mathcal{M}_2 f([\mathbf{v}]) = \mathcal{M}_2 \tilde{f}([\mathbf{v}]) \qquad \text{for all } [\mathbf{v}] \in \mathbb{P}^1(\mathbb{F}_q).$$

Also,

$$\|\tilde{f}\|_{\ell^2(\mathbb{F}_q^2)} = \|f\|_{\ell^2(\mathbb{F}_q^2)}.$$

Therefore, it is enough to prove the stated inequality for $\tilde{f}$. Without loss of generality, we assume at the beginning of the proof that $f \geq 0$.

For each direction $[\mathbf{v}] \in \mathbb{P}^1(\mathbb{F}_q)$, we choose an affine line $\ell_{[\mathbf{v}]} \subset \mathbb{F}_q^2$ with direction $[\mathbf{v}]$. For this fixed

family $\{\ell_{[\mathbf{v}]}\}_{[\mathbf{v}]}$, define the linear operator $T : \ell^2(\mathbb{F}_q^2) \to \ell^2(\mathbb{P}^1(\mathbb{F}_q))$ by

$$Th([\mathbf{v}]) := \sum_{\mathbf{x}\in\ell_{[\mathbf{v}]}} h(\mathbf{x}).$$

It is enough to prove that the estimate

$$\|T\|_{\ell^2(\mathbb{F}_q^2)\to\ell^2(\mathbb{P}^1(\mathbb{F}_q))} \le \sqrt{2q}, \tag{16}$$

holds uniformly over all such line families. Here $\|T\|_{X\to Y} := \sup\{\|Th\|_Y : \ \|h\|_X = 1\}$ denotes the operator norm.

To see how this implies the conclusion, for a given function f, and each $[\mathbf{v}] \in \mathbb{P}^1(\mathbb{F}_q)$, we choose an affine line $\ell_{[\mathbf{v}]} \subset \mathbb{F}_q^2$ with direction $[\mathbf{v}]$ such that

$$\mathcal{M}_2 f([\mathbf{v}]) = \sum_{\mathbf{x}\in\ell_{[\mathbf{v}]}} f(\mathbf{x}).$$

Let T be the operator associated with this family. Since the bound $\|T\|_{\ell^2(\mathbb{F}_q^2)\to\ell^2(\mathbb{P}^1(\mathbb{F}_q))} \le \sqrt{2q}$ is uniform over all line families, it applies to this choice. Therefore, $Tf = \mathcal{M}_2 f$, and

$$\|\mathcal{M}_2 f\|_{\ell^2(\mathbb{P}^1(\mathbb{F}_q))} = \|Tf\|_{\ell^2(\mathbb{P}^1(\mathbb{F}_q))} \le \sqrt{2q}\,\|f\|_{\ell^2(\mathbb{F}_q^2)}.$$

We now prove the estimate (16). We compute the adjoint $T^* : \ell^2(\mathbb{P}^1(\mathbb{F}_q)) \to \ell^2(\mathbb{F}_q^2)$. For $g : \mathbb{P}^1(\mathbb{F}_q) \to \mathbb{C}$ and $x \in \mathbb{F}_q^2$, one has

$$T^* g(\mathbf{x}) = \sum_{[\mathbf{v}]\in\mathbb{P}^1(\mathbb{F}_q)} g([\mathbf{v}])\,\mathbf{1}_{\ell_{[\mathbf{v}]}}(\mathbf{x}).$$

Therefore, for each $[\mathbf{v}] \in \mathbb{P}^1(\mathbb{F}_q)$,

$$\begin{aligned}(TT^* g)([\mathbf{v}]) &= \sum_{x\in\ell_{[\mathbf{v}]}} T^* g(\mathbf{x}) = \sum_{x\in\ell_{[\mathbf{v}]}} \sum_{[\mathbf{v}']\in\mathbb{P}^1(\mathbb{F}_q)} g([\mathbf{v}']\,\mathbf{1}_{\ell_{[\mathbf{v}']}}(\mathbf{x}) \\ &= \sum_{[\mathbf{v}']\in\mathbb{P}^1(\mathbb{F}_q)} g([\mathbf{v}'])\,|\ell_{[\mathbf{v}]} \cap \ell_{[\mathbf{v}']}|.\end{aligned}$$

If $[\mathbf{v}'] = [\mathbf{v}]$, then $|\ell_{[\mathbf{v}]} \cap \ell_{[\mathbf{v}']}| = |\ell_{[\mathbf{v}]}| = q$. If $[\mathbf{v}'] \ne [\mathbf{v}]$, then $\ell_{[\mathbf{v}]}$ and $\ell_{[\mathbf{v}']}$ have distinct directions, hence they meet in exactly one point in $\mathbb{F}_q^2$, so $|\ell_{[\mathbf{v}]} \cap \ell_{[\mathbf{v}']}| = 1$. Thus

$$(TT^* g)([\mathbf{v}]) = q\,g([\mathbf{v}]) + \sum_{\substack{[\mathbf{v}']\in\mathbb{P}^1(\mathbb{F}_q)\\ [\mathbf{v}']\ne[\mathbf{v}]}} g([\mathbf{v}']). \tag{17}$$

Let $J : \ell^2(\mathbb{P}^1(\mathbb{F}_q)) \to \ell^2(\mathbb{P}^1(\mathbb{F}_q))$ be the rank–one operator

$$Jg([\mathbf{v}]) := \sum_{[\mathbf{v}']\in\mathbb{P}^1(\mathbb{F}_q)} g([\mathbf{v}']).$$

Then (17) may be rewritten as

$$TT^* = (q-1)I + J.$$

We now determine the eigenvalues of J. Let $\mathbf{1}$ denote the constant function $\mathbf{1}([\mathbf{v}]) := 1$ on $\mathbb{P}^1(\mathbb{F}_q)$.

Then

$$J\mathbf{1}([\mathbf{v}]) = \sum_{[\mathbf{v}']\in\mathbb{P}^1(\mathbb{F}_q)} 1 = |\mathbb{P}^1(\mathbb{F}_q)| = q+1,$$

so $\mathbf{1}$ is an eigenvector of J with eigenvalue $q+1$.

Next, consider the orthogonal complement of $\mathrm{span}\{\mathbf{1}\}$ in $\ell^2(\mathbb{P}^1(\mathbb{F}_q))$, namely

$$\Big(\mathrm{span}\{\mathbf{1}\}\Big)^{\perp} = \Big\{g : \mathbb{P}^1(\mathbb{F}_q) \to \mathbb{C} : \ \langle g, \mathbf{1}\rangle = 0\Big\}.$$

Since the inner product is $\langle g, h\rangle = \sum_{[\mathbf{v}]} g([\mathbf{v}])\overline{h([\mathbf{v}])}$, the condition $\langle g, \mathbf{1}\rangle = 0$ is equivalent to

$$\sum_{[\mathbf{v}]\in\mathbb{P}^1(\mathbb{F}_q)} g([\mathbf{v}]) = 0.$$

For such g, one has $Jg \equiv 0$, hence every $g \in (\mathrm{span}\{\mathbf{1}\})^{\perp}$ is an eigenvector of J with eigenvalue 0.

Therefore, the spectrum of J consists of the eigenvalue $q+1$ on $\mathrm{span}\{\mathbf{1}\}$ and the eigenvalue 0 on its orthogonal complement. It follows that the spectrum of $TT^* = (q-1)I + J$ consists of exactly two eigenvalues

$$(q-1)+(q+1) = 2q \qquad \text{and} \qquad (q-1)+0 = q-1.$$

Hence,

$$\|TT^*\|_{\ell^2(\mathbb{P}^1(\mathbb{F}_q))\to\ell^2(\mathbb{P}^1(\mathbb{F}_q))} = 2q.$$

Consequently,

$$\|T\|^2_{\ell^2(\mathbb{F}_q^2)\to\ell^2(\mathbb{P}^1(\mathbb{F}_q))} = \|TT^*\|_{\ell^2(\mathbb{P}^1(\mathbb{F}_q))\to\ell^2(\mathbb{P}^1(\mathbb{F}_q))} = 2q,$$

so $\|T\|_{\ell^2(\mathbb{F}_q^2)\to\ell^2(\mathbb{F}_q^2)} \le \sqrt{2q}$. This implies

$$\|\mathcal{M}_2 f\|_{\ell^2(\mathbb{P}^1(\mathbb{F}_q))} = \|Tf\|_{\ell^2(\mathbb{P}^1(\mathbb{F}_q))} \le \sqrt{2q}\,\|f\|_{\ell^2(\mathbb{F}_q^2)}.$$

This completes the proof. □

3 Upper bound for the diagonal (Theorem 1.4)

3.1 Planar endpoint bounds

Lemma 3.1. *For all $G : \mathbb{F}_q^2 \to [0,\infty)$, we have*

$$\|\mathcal{M}_2 G\|_{\ell^1(\mathbb{P}^1(\mathbb{F}_q))} \le (q+1)\,\|G\|_{\ell^1(\mathbb{F}_q^2)}, \tag{18}$$

$$\|\mathcal{M}_2 G\|_{\ell^2(\mathbb{P}^1(\mathbb{F}_q))} \le \sqrt{2}\,q^{\frac{1}{2}}\,\|G\|_{\ell^2(\mathbb{F}_q^2)}, \tag{19}$$

$$\|\mathcal{M}_2 G\|_{\ell^\infty(\mathbb{P}^1(\mathbb{F}_q))} \le q\,\|G\|_{\ell^\infty(\mathbb{F}_q^2)}. \tag{20}$$

Proof. For (18), note that for every direction class $[\mathbf{v}]$, one has

$$\mathcal{M}_2 G([\mathbf{v}]) \le \sum_{\mathbf{z}\in\mathbb{F}_q^2} G(\mathbf{z}) = \|G\|_{\ell^1(\mathbb{F}_q^2)}.$$

Summing over all $[\mathbf{v}]$ immediately implies (18). For (20), observe that each affine line in $\mathbb{F}_q^2$ contains

exactly q points, and hence

$$\sum_{\mathbf{z}\in\ell} G(\mathbf{z}) \le q\,\|G\|_{\ell^\infty(\mathbb{F}_q^2)} \qquad \text{for every line } \ell \subset \mathbb{F}_q^2.$$

Taking the maximum over all lines ℓ gives (20). Finally, (19) follows directly from Lemma 2.5. $\square$

3.2 A planar $\ell^u \to \ell^u$ theorem

Theorem 3.2. *For every $1 \le u \le \infty$ and for all $G : \mathbb{F}_q^2 \to [0,\infty)$, we have*

$$\|\mathcal{M}_2 G\|_{\ell^u(\mathbb{P}^1(\mathbb{F}_q))} \le \sqrt{2}\, q^{\tau(u)}\, \|G\|_{\ell^u(\mathbb{F}_q^2)}$$

where

$$\tau(u) := \begin{cases} \dfrac{1}{u}, & 1 \le u \le 2, \\ 1 - \dfrac{1}{u}, & 2 \le u \le \infty. \end{cases}$$

Proof. Let $G : \mathbb{F}_q^2 \to [0,\infty)$ be given. For each direction $[\mathbf{v}] \in \mathbb{P}^1(\mathbb{F}_q)$, choose an affine line $\ell_{[\mathbf{v}]} \subset \mathbb{F}_q^2$ with direction $[\mathbf{v}]$ such that

$$\mathcal{M}_2 G([\mathbf{v}]) = \sum_{\mathbf{x}\in\ell_{[\mathbf{v}]}} G(\mathbf{x}).$$

Define the linear operator T on functions $f : \mathbb{F}_q^2 \to \mathbb{C}$ by

$$(Tf)([\mathbf{v}]) := \sum_{\mathbf{x}\in\ell_{[\mathbf{v}]}} f(\mathbf{x}), \qquad [\mathbf{v}] \in \mathbb{P}^1(\mathbb{F}_q).$$

Then

$$(TG)([\mathbf{v}]) = \mathcal{M}_2 G([\mathbf{v}]) \qquad \text{for all } [\mathbf{v}] \in \mathbb{P}^1(\mathbb{F}_q). \tag{21}$$

Therefore,

$$\|\mathcal{M}_2 G\|_{\ell^u(\mathbb{P}^1(\mathbb{F}_q))} = \|TG\|_{\ell^u(\mathbb{P}^1(\mathbb{F}_q))}. \tag{22}$$

We next record three endpoint bounds for T.

The ℓ^1 bound. For every $f : \mathbb{F}_q^2 \to \mathbb{C}$,

$$\begin{aligned} \|Tf\|_{\ell^1(\mathbb{P}^1(\mathbb{F}_q))} &= \sum_{[\mathbf{v}]\in\mathbb{P}^1(\mathbb{F}_q)} \Big| \sum_{\mathbf{x}\in\ell_{[\mathbf{v}]}} f(\mathbf{x}) \Big| \le \sum_{[\mathbf{v}]} \sum_{\mathbf{x}\in\ell_{[\mathbf{v}]}} |f(\mathbf{x})| \le (q+1) \sum_{\mathbf{x}\in\mathbb{F}_q^2} |f(\mathbf{x})| \\ &= (q+1)\,\|f\|_{\ell^1(\mathbb{F}_q^2)}. \end{aligned} \tag{23}$$

The ℓ^∞ bound. For every $f : \mathbb{F}_q^2 \to \mathbb{C}$ and every $[\mathbf{v}] \in \mathbb{P}^1(\mathbb{F}_q)$,

$$|(Tf)([\mathbf{v}])| = \Big| \sum_{\mathbf{x}\in\ell_{[\mathbf{v}]}} f(\mathbf{x}) \Big| \le \sum_{\mathbf{x}\in\ell_{[\mathbf{v}]}} |f(\mathbf{x})| \le q\,\|f\|_{\ell^\infty(\mathbb{F}_q^2)},$$

since each affine line in $\mathbb{F}_q^2$ has exactly q points. Taking the maximum over $[\mathbf{v}] \in \mathbb{P}^1(\mathbb{F}_q)$ implies

$$\|Tf\|_{\ell^\infty(\mathbb{P}^1(\mathbb{F}_q))} \le q\,\|f\|_{\ell^\infty(\mathbb{F}_q^2)}. \tag{24}$$

The ℓ^2 bound. The operator T is obtained by selecting one affine line in each direction, so the proof of Lemma 2.5 applies verbatim to T. Therefore, for every $f : \mathbb{F}_q^2 \to \mathbb{C}$,

$$\|Tf\|_{\ell^2(\mathbb{P}^1(\mathbb{F}_q))} \leq \sqrt{2}\, q^{\frac{1}{2}}\, \|f\|_{\ell^2(\mathbb{F}_q^2)}. \tag{25}$$

We now interpolate the bounds to obtain the desired exponents.

Case 1: $1 \leq u \leq 2$**.** Choose $\theta \in [0,1]$ so that

$$\frac{1}{u} = (1-\theta)\cdot 1 + \theta \cdot \frac{1}{2}.$$

Then $\theta = 2\big(1-\frac{1}{u}\big)$. Apply Lemma 2.2 to the linear operator T, using the endpoint estimates (23) and (25), we obtain

$$\|Tf\|_{\ell^u(\mathbb{P}^1(\mathbb{F}_q))} \leq \big(q+1\big)^{1-\theta}\, \big(\sqrt{2}q^{\frac{1}{2}}\big)^{\theta}\, \|f\|_{\ell^u(\mathbb{F}_q^2)} \leq 2^{1-\frac{1}{u}}\, q^{1-\frac{\theta}{2}}\, \|f\|_{\ell^u(\mathbb{F}_q^2)} \leq \sqrt{2}q^{\frac{1}{u}}\, \|f\|_{\ell^u(\mathbb{F}_q^2)} \tag{26}$$

where we have used $1-\frac{\theta}{2} = \frac{1}{u}$ and $1 \leq u \leq 2$ in the last inequality. From here, we apply (26) with $f = G$ and use (22).

Case 2: $2 \leq u \leq \infty$**.** Choose $\theta \in [0,1]$ so that

$$\frac{1}{u} = (1-\theta)\cdot \frac{1}{2} + \theta \cdot 0.$$

Then $\theta = 1 - \frac{2}{u}$. Apply Lemma 2.2 to the linear operator T, using the endpoint estimates (25) and (24), we obtain

$$\|Tf\|_{\ell^u(\mathbb{P}^1(\mathbb{F}_q))} \leq \big(\sqrt{2}q^{\frac{1}{2}}\big)^{1-\theta}\, (q)^{\theta}\, \|f\|_{\ell^u(\mathbb{F}_q^2)} \leq \sqrt{2}\, q^{\frac{1-\theta}{2}+\theta}\, \|f\|_{\ell^u(\mathbb{F}_q^2)}.$$

Since $\frac{1-\theta}{2} + \theta = 1 - \frac{1}{u}$, this implies

$$\|Tf\|_{\ell^u(\mathbb{P}^1(\mathbb{F}_q))} \leq \sqrt{2}\, q^{1-\frac{1}{u}}\, \|f\|_{\ell^u(\mathbb{F}_q^2)}, \tag{27}$$

which implies

$$\|TG\|_{\ell^u(\mathbb{P}^1(\mathbb{F}_q))} \leq \sqrt{2}\, q^{1-\frac{1}{u}}\, \|G\|_{\ell^u(\mathbb{F}_q^2)}.$$

Combining the two cases gives

$$\|TG\|_{\ell^u(\mathbb{P}^1(\mathbb{F}_q))} \leq \sqrt{2}\, q^{\tau(u)}\, \|G\|_{\ell^u(\mathbb{F}_q^2)}.$$

Together with (22), this proves the theorem. □

3.3 Diagonal bounds for $\mathcal{M}_{\mathbb{H}_1}$

Theorem 3.3. *For every $1 \leq u \leq \infty$ and for all $F : \mathbb{H}_1(\mathbb{F}_q) \to \mathbb{C}$, we have*

$$\|\mathcal{M}_{\mathbb{H}_1} F\|_{\ell^u(\mathbb{P}^1(\mathbb{F}_q))} \leq \sqrt{2}\, q^{\tau(u)}\, \|F\|_{\ell^u(\mathbb{H}_1(\mathbb{F}_q))}$$

where $\tau(u)$ is as in Theorem 3.2.

Proof. Let $1 \le u < \infty$ and define G as in Lemma 2.3. By (15) and Theorem 3.2,

$$\|\mathcal{M}_{\mathbb{H}_1} F\|_{\ell^u(\mathbb{P}^1(\mathbb{F}_q))} \le \|\mathcal{M}_2 G\|_{\ell^u(\mathbb{P}^1(\mathbb{F}_q))} \le \sqrt{2}\, q^{\tau(u)}\, \|G\|_{\ell^u(\mathbb{F}_q^2)}.$$

Using (14) implies the desired bound.

If $u = \infty$, Lemma 2.3 gives $\mathcal{M}_{\mathbb{H}_1} F \le \mathcal{M}_2 G$ pointwise, and (20) gives $\|\mathcal{M}_2 G\|_{\ell^\infty(\mathbb{P}^1(\mathbb{F}_q))} \le q\|G\|_{\ell^\infty(\mathbb{H}_1(\mathbb{F}_q))}$. Therefore,

$$\|\mathcal{M}_{\mathbb{H}_1} F\|_{\ell^\infty(\mathbb{P}^1(\mathbb{F}_q))} \le q\, \|F\|_{\ell^\infty(\mathbb{H}_1(\mathbb{F}_q))},$$

which is the case $\tau(\infty) = 1$. □

4 Upper bounds away from the diagonal (Theorem 1.4)

We now prove regionwise bounds which, when assembled, provide the upper bound in Theorem 1.4.

4.1 The case $A_1(u, v) \le 1 + \frac{1}{v} - \frac{2}{u}$

Lemma 4.1. *Let $2 \le u \le \infty$ and $1 \le v \le u$. Then, for all $F : \mathbb{H}_1(\mathbb{F}_q) \to \mathbb{C}$,*

$$\|\mathcal{M}_{\mathbb{H}_1} F\|_{\ell^v(\mathbb{P}^1(\mathbb{F}_q))} \le 2\sqrt{2}\, q^{\,1+\frac{1}{v}-\frac{2}{u}}\, \|F\|_{\ell^u(\mathbb{H}_1(\mathbb{F}_q))}.$$

Proof. Since $v \le u$, Lemma 2.1 gives

$$\|\mathcal{M}_{\mathbb{H}_1} F\|_{\ell^v(\mathbb{P}^1(\mathbb{F}_q))} \le |\mathbb{P}^1(\mathbb{F}_q)|^{\frac{1}{v}-\frac{1}{u}}\, \|\mathcal{M}_{\mathbb{H}_1} F\|_{\ell^u(\mathbb{P}^1(\mathbb{F}_q))}.$$

Since $u \ge 2$, Theorem 3.3 implies

$$\|\mathcal{M}_{\mathbb{H}_1} F\|_{\ell^u(\mathbb{P}^1(\mathbb{F}_q))} \le \sqrt{2}\, q^{1-\frac{1}{u}}\, \|F\|_{\ell^u(\mathbb{H}_1(\mathbb{F}_q))}.$$

Therefore,

$$\|\mathcal{M}_{\mathbb{H}_1} F\|_{\ell^v(\mathbb{P}^1(\mathbb{F}_q))} \le \sqrt{2}\,(q+1)^{\frac{1}{v}-\frac{1}{u}}\, q^{1-\frac{1}{u}}\, \|F\|_{\ell^u(\mathbb{H}_1(\mathbb{F}_q))} \le 2\sqrt{2}\, q^{\,1+\frac{1}{v}-\frac{2}{u}}\, \|F\|_{\ell^u(\mathbb{H}_1(\mathbb{F}_q))}.$$

This completes the proof. □

4.2 The case $A_1(u, v) \le 1 - \frac{1}{u}$

Lemma 4.2. *Let $2 \le u \le v \le \infty$. Then, for all $F : \mathbb{H}_1(\mathbb{F}_q) \to \mathbb{C}$,*

$$\|\mathcal{M}_{\mathbb{H}_1} F\|_{\ell^v(\mathbb{P}^1(\mathbb{F}_q))} \le \sqrt{2}\, q^{1-\frac{1}{u}}\, \|F\|_{\ell^u(\mathbb{H}_1(\mathbb{F}_q))}.$$

Proof. Since $u \le v$, Lemma 2.1 implies $\|g\|_{\ell^v(\mathbb{P}^1(\mathbb{F}_q))} \le \|g\|_{\ell^u(\mathbb{P}^1(\mathbb{F}_q))}$ for every g on $\mathbb{P}^1(\mathbb{F}_q)$. Applying this to $g = \mathcal{M}_{\mathbb{H}_1} F$ gives

$$\|\mathcal{M}_{\mathbb{H}_1} F\|_{\ell^v(\mathbb{P}^1(\mathbb{F}_q))} \le \|\mathcal{M}_{\mathbb{H}_1} F\|_{\ell^u(\mathbb{P}^1(\mathbb{F}_q))}.$$

Since $u \ge 2$, Theorem 3.3 implies

$$\|\mathcal{M}_{\mathbb{H}_1} F\|_{\ell^u(\mathbb{P}^1(\mathbb{F}_q))} \le \sqrt{2}\, q^{1-\frac{1}{u}}\, \|F\|_{\ell^u(\mathbb{H}_1(\mathbb{F}_q))}.$$

Thus the desired bound follows. □

Lemma 4.3. *Let $1 \le u \le 2$ and $v \ge \frac{u}{u-1}$. Then, for all $F : \mathbb{H}_1(\mathbb{F}_q) \to \mathbb{C}$,*

$$\|\mathcal{M}_{\mathbb{H}_1} F\|_{\ell^v(\mathbb{P}^1(\mathbb{F}_q))} \le \sqrt{2}\, q^{1-\frac{1}{u}}\, \|F\|_{\ell^u(\mathbb{H}_1(\mathbb{F}_q))}.$$

Proof. Similarly as before, it is sufficient to prove the claimed bound for non-negative F.

For each $[\mathbf{v}] \in \mathbb{P}^1(\mathbb{F}_q)$, choose a horizontal line $L_{[\mathbf{v}]} \in \mathcal{L}([\mathbf{v}])$ such that

$$\mathcal{M}_{\mathbb{H}_1} F([\mathbf{v}]) = \sum_{\mathbf{p} \in L_{[\mathbf{v}]}} F(\mathbf{p}).$$

With this family $\{L_{[\mathbf{v}]}\}_{[\mathbf{v}]}$ of lines, we define the associated linear transformation T on functions $\mathcal{G} : \mathbb{H}_1(\mathbb{F}_q) \to \mathbb{C}$ by

$$(T\mathcal{G})([\mathbf{v}]) := \sum_{\mathbf{p} \in L_{[\mathbf{v}]}} \mathcal{G}(\mathbf{p}).$$

It is sufficient to prove that

$$\|T\mathcal{G}\|_{\ell^v(\mathbb{P}^1(\mathbb{F}_q))} \le \sqrt{2}\, q^{1-\frac{1}{u}}\, \|\mathcal{G}\|_{\ell^u(\mathbb{H}_1(\mathbb{F}_q))}.$$

Then the lemma follows by taking $\mathcal{G} = F$.

We record two bounds for T. For every $\mathcal{G} : \mathbb{H}_1(\mathbb{F}_q) \to \mathbb{C}$ and every $[\mathbf{v}]$,

$$|(T\mathcal{G})([\mathbf{v}])| \le \sum_{\mathbf{p} \in \mathbb{H}_1(\mathbb{F}_q)} |\mathcal{G}(\mathbf{p})|.$$

Therefore,

$$\|T\mathcal{G}\|_{\ell^\infty(\mathbb{P}^1(\mathbb{F}_q))} \le \|\mathcal{G}\|_{\ell^1(\mathbb{H}_1(\mathbb{F}_q))}. \tag{28}$$

Also, the diagonal estimate at $(2,2)$ applies to the chosen family of lines, so

$$\|T\mathcal{G}\|_{\ell^2(\mathbb{P}^1(\mathbb{F}_q))} \le \sqrt{2}\, q^{\frac{1}{2}}\, \|\mathcal{G}\|_{\ell^2(\mathbb{H}_1(\mathbb{F}_q))} \qquad \text{for all } \mathcal{G} : \mathbb{H}_1(\mathbb{F}_q) \to \mathbb{C}. \tag{29}$$

From (28) and (29), we apply Lemma 2.2 to obtain the desired inequality. To this end, we observe that $(p_0, r_0) = (1, \infty)$ and $(p_1, r_1) = (2, 2)$. We now choose a parameter $\theta \in [0,1]$ such that

$$\frac{1}{u} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \qquad \frac{1}{r} = \frac{1-\theta}{r_0} + \frac{\theta}{r_1}.$$

Solving this system implies

$$r = \frac{2}{\theta} = \frac{u}{u-1}, \qquad \theta = 2\Big(1 - \frac{1}{u}\Big).$$

With these values, combining (28), (29), and Lemma 2.2 gives

$$\|T\mathcal{G}\|_{\ell^v(\mathbb{P}^1(\mathbb{F}_q))} \le \|T\mathcal{G}\|_{\ell^{\frac{u}{u-1}}(\mathbb{P}^1(\mathbb{F}_q))} \le \sqrt{2}\, q^{\frac{\theta}{2}}\, \|\mathcal{G}\|_{\ell^u(\mathbb{H}_1(\mathbb{F}_q))} = \sqrt{2}\, q^{1-\frac{1}{u}}\, \|\mathcal{G}\|_{\ell^u(\mathbb{H}_1(\mathbb{F}_q))},$$

where we have used Lemma 2.1 noting that $v \ge \frac{u}{u-1}$ in the first inequality and $\frac{\theta}{2} = 1 - \frac{1}{u}$ in the last equality. □

4.3 The case $A_1(u,v) \le \frac{1}{v}$

Lemma 4.4. *Let $1 \le v \le u \le 2$. Then, for all $F : \mathbb{H}_1(\mathbb{F}_q) \to \mathbb{C}$,*

$$\|\mathcal{M}_{\mathbb{H}_1} F\|_{\ell^v(\mathbb{P}^1(\mathbb{F}_q))} \le 2\, q^{\frac{1}{v}}\, \|F\|_{\ell^u(\mathbb{H}_1(\mathbb{F}_q))}.$$

Proof. Since $v \le u$, Lemma 2.1 gives

$$\|\mathcal{M}_{\mathbb{H}_1} F\|_{\ell^v(\mathbb{P}^1(\mathbb{F}_q))} \le |\mathbb{P}^1(\mathbb{F}_q)|^{\frac{1}{v}-\frac{1}{u}}\, \|\mathcal{M}_{\mathbb{H}_1} F\|_{\ell^u(\mathbb{P}^1(\mathbb{F}_q))}.$$

Since $u \le 2$, Theorem 3.3 implies

$$\|\mathcal{M}_{\mathbb{H}_1} F\|_{\ell^u(\mathbb{P}^1(\mathbb{F}_q))} \le \sqrt{2}\, q^{\frac{1}{u}}\, \|F\|_{\ell^u(\mathbb{H}_1(\mathbb{F}_q))}.$$

Therefore,

$$\begin{aligned}\|\mathcal{M}_{\mathbb{H}_1} F\|_{\ell^v(\mathbb{P}^1(\mathbb{F}_q))} \le& \sqrt{2}\,(q+1)^{\frac{1}{v}-\frac{1}{u}} q^{\frac{1}{u}}\, \|F\|_{\ell^u(\mathbb{H}_1(\mathbb{F}_q))} \le \sqrt{2}\,(2q)^{\frac{1}{v}-\frac{1}{u}} q^{\frac{1}{u}}\, \|F\|_{\ell^u(\mathbb{H}_1(\mathbb{F}_q))}\\ \le& 2\, q^{\frac{1}{v}}\, \|F\|_{\ell^u(\mathbb{H}_1(\mathbb{F}_q))},\end{aligned}$$

where we have used the fact that $0 \le \frac{1}{v} - \frac{1}{u} \le \frac{1}{2}$ which follows from the assumption $1 \le v \le u \le 2$, in the last inequality. This completes the proof. □

Lemma 4.5. *Let $1 \le u \le 2$ and $u \le v \le \frac{u}{u-1}$. Then, for all $F : \mathbb{H}_1(\mathbb{F}_q) \to \mathbb{C}$,*

$$\|\mathcal{M}_{\mathbb{H}_1} F\|_{\ell^v(\mathbb{P}^1(\mathbb{F}_q))} \le 2\sqrt{2}\, q^{\frac{1}{v}}\, \|F\|_{\ell^u(\mathbb{H}_1(\mathbb{F}_q))}.$$

Proof. It suffices to prove the claim for non-negative F. We take the family $\{L_{p_{[\mathbf{v}]},[\mathbf{v}]}\}_{[\mathbf{v}]}$ of horizontal lines and the linear operator T as in the proof of Lemma 4.3.

We record two endpoint bounds for T. First, we deduce by (28) that

$$\|T\mathcal{G}\|_{\ell^r(\mathbb{P}^1(\mathbb{F}_q))} \le (q+1)^{\frac{1}{r}} \|T\mathcal{G}\|_{\ell^r(\mathbb{P}^1(\mathbb{F}_q))} \le (q+1)^{\frac{1}{r}}\, \|\mathcal{G}\|_{\ell^1(\mathbb{H}_1(\mathbb{F}_q))}. \tag{30}$$

Second, since $|T\mathcal{G}| \le \mathcal{M}_{\mathbb{H}_1}(|\mathcal{G}|)$ pointwise, Theorem 3.3 with $u = 2$ gives

$$\|T\mathcal{G}\|_{\ell^2(\mathbb{P}^1(\mathbb{F}_q))} \le \|\mathcal{M}_{\mathbb{H}_1}(|\mathcal{G}|)\|_{\ell^2(\mathbb{P}^1(\mathbb{F}_q))} \le \sqrt{2}\, q^{\frac{1}{2}}\, \|\mathcal{G}\|_{\ell^2(\mathbb{H}_1(\mathbb{F}_q))}. \tag{31}$$

If $u = 2$, then $v = 2$ and (31) gives the desired estimate.

Assume now that $1 \le u < 2$. Set

$$\theta := 2\Big(1 - \frac{1}{u}\Big) \in [0,1), \qquad \text{so that} \qquad \frac{1}{u} = (1-\theta)\cdot 1 + \theta \cdot \frac{1}{2}.$$

Choose $r \in [1,\infty]$ so that

$$\frac{1}{v} = (1-\theta)\cdot \frac{1}{r} + \theta \cdot \frac{1}{2}. \tag{32}$$

Such an r exists because $u \le v \le \frac{u}{u-1}$ is equivalent to

$$\frac{1}{u} \ge \frac{1}{v} \ge 1 - \frac{1}{u} = \frac{\theta}{2},$$

and hence $0 \le \frac{1}{r} \le 1$.

Apply Lemma 2.2 to the linear operator T using (30) and (31). We obtain

$$\|T\mathcal{G}\|_{\ell^v(\mathbb{P}^1(\mathbb{F}_q))} \le \left((q+1)^{\frac{1}{r}}\right)^{1-\theta} \left(\sqrt{2}\, q^{\frac{1}{2}}\right)^{\theta} \|\mathcal{G}\|_{\ell^u(\mathbb{H}_1(\mathbb{F}_q))} \le 2\sqrt{2}\, q^{\frac{1}{v}}\, \|\mathcal{G}\|_{\ell^u(\mathbb{H}_1(\mathbb{F}_q))}.$$

where we have used $q+1 \le 2q$ and (32) in the last inequality. Now the lemma follows by taking $\mathcal{G} = F$. □

5 Lower bounds (Theorem 1.4)

5.1 The case $A_1(u,v) \ge 1 + \frac{1}{v} - \frac{2}{u}$

Lemma 5.1. *For every $1 \le u, v \le \infty$, one has $A_1(u,v) \ge 1 + \frac{1}{v} - \frac{2}{u}$.*

Proof. Let

$$S := \bigcup_{[\mathbf{v}] \in \mathbb{P}^1(\mathbb{F}_q)} L_{\mathbf{0},[\mathbf{v}]}, \qquad F := \mathbf{1}_S.$$

Distinct horizontal lines through $\mathbf{0}$ intersect only at $\mathbf{0}$. Since each such line has q points and there are $q+1$ directions, we obtain

$$|S| = 1 + (q+1)(q-1) = q^2.$$

Therefore, $\|F\|_{\ell^u(\mathbb{H}_1(\mathbb{F}_q))} = |S|^{\frac{1}{u}} = q^{\frac{2}{u}}$.

For every $[\mathbf{v}] \in \mathbb{P}^1(\mathbb{F}_q)$, the line $L_{\mathbf{0},[\mathbf{v}]}$ is contained in S, hence

$$\mathcal{M}_{\mathbb{H}_1} F([\mathbf{v}]) \ge \sum_{\mathbf{x} \in L_{\mathbf{0},[\mathbf{v}]}} \mathbf{1}_S(\mathbf{x}) = q.$$

Consequently,

$$\|\mathcal{M}_{\mathbb{H}_1} F\|_{\ell^v(\mathbb{P}^1(\mathbb{F}_q))} \ge \Big(\sum_{[\mathbf{v}] \in \mathbb{P}^1(\mathbb{F}_q)} q^v \Big)^{\frac{1}{v}} = q\,(q+1)^{\frac{1}{v}} \ge q^{1+\frac{1}{v}}.$$

Therefore, $q^{A_1(u,v)} \gtrsim q^{1+\frac{1}{v}} / q^{\frac{2}{u}}$, and hence $A_1(u,v) \ge 1 + \frac{1}{v} - \frac{2}{u}$. □

5.2 The case $A_1(u,v) \ge 1 - \frac{1}{u}$

Lemma 5.2. *For every $1 \le u, v \le \infty$, one has $A_1(u,v) \ge 1 - \frac{1}{u}$.*

Proof. Let $L \subset \mathbb{H}_1(\mathbb{F}_q)$ be a fixed horizontal line, and set $F = \mathbf{1}_L$. Then $\|F\|_{\ell^u(\mathbb{H}_1(\mathbb{F}_q))} = |L|^{\frac{1}{u}} = q^{\frac{1}{u}}$. Let $[\mathbf{v}_0]$ be the direction of L. Then $\mathcal{M}_{\mathbb{H}_1} F([\mathbf{v}_0]) \ge \sum_{\mathbf{x} \in L} \mathbf{1}_L(\mathbf{x}) = q$. Therefore, $\|\mathcal{M}_{\mathbb{H}_1} F\|_{\ell^v(\mathbb{P}^1(\mathbb{F}_q))} \ge q$ for every v. Thus $q^{A_1(u,v)} \gtrsim q / q^{\frac{1}{u}}$, and hence $A_1(u,v) \ge 1 - \frac{1}{u}$. □

5.3 The case $A_1(u,v) \ge \frac{1}{v}$

Lemma 5.3. *For every $1 \le u, v \le \infty$ one has $A_1(u,v) \ge \frac{1}{v}$.*

Proof. Fix a point $\mathbf{p}_* \in \mathbb{H}_1(\mathbb{F}_q)$ and set $F = \delta_{\mathbf{x}_*}$ where $\delta_{\mathbf{p}_*} : \mathbb{H}_1(\mathbb{F}_q) \to \{0,1\}$ is defined by

$$\delta_{\mathbf{p}_*}(\mathbf{x}) = \begin{cases} 1, & \mathbf{p} = \mathbf{p}_*, \\ 0, & \mathbf{p} \neq \mathbf{p}_*. \end{cases}$$

For every direction $[\mathbf{v}] \in \mathbb{P}^1(\mathbb{F}_q)$ there is a unique horizontal line in direction $[\mathbf{v}]$ through $\mathbf{p}_*$. Therefore, $\mathcal{M}_{\mathbb{H}_1} F([\mathbf{v}]) = 1$ for all $[\mathbf{v}]$. Hence

$$\|\mathcal{M}_{\mathbb{H}_1} F\|_{\ell^v(\mathbb{P}^1(\mathbb{F}_q))} = (q+1)^{\frac{1}{v}}, \qquad \|F\|_{\ell^u(\mathbb{H}_1(\mathbb{F}_q))} = 1.$$

Thus, any admissible exponent in (6) must satisfy $q^{A_1(u,v)} \gtrsim (q+1)^{\frac{1}{v}}$, and hence $A_1(u,v) \geq \frac{1}{v}$.

□

6 Proof of Theorem 1.4 – $\mathbb{H}_1(\mathbb{F}_q)$

Proof of Theorem 1.4. It follows from Lemmas 5.3, 5.2, and 5.1 that

$$A_1(u,v) \geq \max\Big(\frac{1}{v},\ 1-\frac{1}{u},\ 1+\frac{1}{v}-\frac{2}{u}\Big).$$

It remains to prove the matching upper bound. We distinguish four regions.

(i) The region: $1 \leq v \leq u \leq 2$**.** By Lemma 4.4 one has $A_1(u,v) \leq \frac{1}{v}$. In this region $\frac{1}{u} \geq \frac{1}{2}$ and $\frac{1}{v} \geq \frac{1}{u}$, hence $1+\frac{1}{v}-\frac{2}{u} \leq \frac{1}{v}$ and $1-\frac{1}{u} \leq \frac{1}{v}$. Therefore, $\max(\frac{1}{v}, 1-\frac{1}{u}, 1+\frac{1}{v}-\frac{2}{u}) = \frac{1}{v}$, so the upper bound is sharp.

(ii) The region: $2 \leq u \leq v \leq \infty$**.** By Lemma 4.2, one has $A_1(u,v) \leq 1-\frac{1}{u}$. In this region $\frac{1}{u} \leq \frac{1}{2}$ and $\frac{1}{v} \leq \frac{1}{u}$, hence $1+\frac{1}{v}-\frac{2}{u} \leq 1-\frac{1}{u}$ and $\frac{1}{v} \leq 1-\frac{1}{u}$. Therefore, the maximum equals $1-\frac{1}{u}$, so the upper bound is sharp.

(iii) The region: $2 \leq u \leq \infty$ **and** $1 \leq v \leq u$**.** By Lemma 4.1, one has $A_1(u,v) \leq 1+\frac{1}{v}-\frac{2}{u}$. Since $\frac{1}{u} \leq \frac{1}{2}$, we have $1+\frac{1}{v}-\frac{2}{u} \geq \frac{1}{v}$ and $1+\frac{1}{v}-\frac{2}{u} \geq 1-\frac{1}{u}$. Therefore, the maximum equals $1+\frac{1}{v}-\frac{2}{u}$, so the upper bound is sharp.

(iv) The region: $1 \leq u \leq 2$ **and** $u \leq v \leq \infty$**.** We have $A_1(u,v) \leq \max(\frac{1}{v}, 1-\frac{1}{u})$. Indeed, if $v \geq \frac{u}{u-1}$, apply Lemma 4.3. If $u \leq v \leq \frac{u}{u-1}$, apply Lemma 4.5.

In this region $\frac{1}{u} \geq \frac{1}{2}$ and $\frac{1}{v} \leq \frac{1}{u}$, hence $1+\frac{1}{v}-\frac{2}{u} \leq 1-\frac{1}{u}$. Therefore, $\max(\frac{1}{v}, 1-\frac{1}{u}, 1+\frac{1}{v}-\frac{2}{u}) = \max(\frac{1}{v}, 1-\frac{1}{u})$, so the upper bound matches the claimed maximum.

Since these four regions cover all $1 \leq u, v \leq \infty$, we conclude that

$$A_1(u,v) = \max\Big(\frac{1}{v},\ 1-\frac{1}{u},\ 1+\frac{1}{v}-\frac{2}{u}\Big).$$

This completes the proof. □

7 Proof of Theorem 1.5 – $\mathbb{H}_n(\mathbb{F}_q)$

The argument is the same as that of $\mathbb{H}_1(\mathbb{F}_q)$. The proof for $\mathbb{H}_1(\mathbb{F}_q)$ is self-contained. In higher dimensions, the only new input is the diagonal estimate in $\mathbb{F}_q^{2n}$ due to Ellenberg, Oberlin, and Tao

in [10], which is an extension of Lemma 2.5. For completeness, we provide a sketch in this section.

Note that

$$\left|\mathbb{P}^{2n-1}(\mathbb{F}_q)\right| = \frac{q^{2n}-1}{q-1}, \tag{33}$$

and hence

$$q^{2n-1} \le \left|\mathbb{P}^{2n-1}(\mathbb{F}_q)\right| \le 2q^{2n-1}. \tag{34}$$

We begin by extending the projection and domination principle of Lemma 2.3 to the higher-dimensional setting.

Lemma 7.1. *Let $1 \le u \le \infty$ and let $F : \mathbb{H}_n(\mathbb{F}_q) \to \mathbb{C}$. Define*

$$G(\mathbf{x},\mathbf{y}) := \Big(\sum_{t\in\mathbb{F}_q} |F(\mathbf{x},\mathbf{y},t)|^u\Big)^{\frac{1}{u}} \quad \text{for } 1 \le u < \infty, \qquad G(\mathbf{x},\mathbf{y}) := \max_{t\in\mathbb{F}_q} |F(\mathbf{x},\mathbf{y},t)| \quad \text{for } u = \infty.$$

Then

$$\|G\|_{\ell^u(\mathbb{F}_q^{2n})} = \|F\|_{\ell^u(\mathbb{H}_n(\mathbb{F}_q))}$$

and

$$\mathcal{M}_{\mathbb{H}_n} F([\mathbf{v}]) \le \mathcal{M}_{2n} G([\mathbf{v}]) \quad \text{for all } [\mathbf{v}] \in \mathbb{P}^{2n-1}(\mathbb{F}_q),$$

where $\mathcal{M}_{2n}$ denotes the standard Kakeya maximal operator on $\mathbb{F}_q^{2n}$.

Proof. The identity of norms follows by expanding both sides. For the domination, fix $[\mathbf{v}]$ and choose a representative $\mathbf{v} = (\mathbf{a},\mathbf{b}) \neq 0$. Let $\mathbf{p} = (\mathbf{x}_0,\mathbf{y}_0,t_0) \in \mathbb{H}_n(\mathbb{F}_q)$ and set $L = L_{\mathbf{p},[\mathbf{v}]}$. The projection $\pi(\mathbf{x},\mathbf{y},t) = (\mathbf{x},\mathbf{y})$ maps L bijectively to the affine line

$$\ell = \{(\mathbf{x}_0,\mathbf{y}_0) + s(\mathbf{a},\mathbf{b}) : s \in \mathbb{F}_q\} \subset \mathbb{F}_q^{2n}.$$

Therefore,

$$\sum_{\mathbf{x}\in L} |F(\mathbf{x})| = \sum_{(\mathbf{x},\mathbf{y})\in\ell} |F(\mathbf{x},\mathbf{y},t(\mathbf{x},\mathbf{y}))| \le \sum_{(\mathbf{x},\mathbf{y})\in\ell} G(\mathbf{x},\mathbf{y}),$$

where $t(\mathbf{x},\mathbf{y})$ is the unique t such that $(\mathbf{x},\mathbf{y},t) \in L$. Taking the maximum over p gives $\mathcal{M}_{\mathbb{H}_n} F([\mathbf{v}]) \le \mathcal{M}_{2n} G([\mathbf{v}])$. □

With the domination lemma in hand, it remains to bound $\mathcal{M}_{2n}G$ in $\ell^u(\mathbb{P}^{2n-1}(\mathbb{F}_q))$. This requires a sharp diagonal estimate for the standard Kakeya maximal operator on $\mathbb{F}_q^{2n}$, which is provided by the following result of Ellenberg, Oberlin, and Tao in [10].

Theorem 7.2 (Theorem 1.3, [10])**.** *Let $d \ge 1$ and let $f : \mathbb{F}_q^d \to \mathbb{R}$. Let $\mathcal{M}_d$ be the Kakeya maximal operator from $\mathbb{F}_q^d$ into $\mathbb{R}$ defined by*

$$\mathcal{M}_d f(\omega) = \max_{\ell \| \omega} \sum_{\mathbf{z}\in\ell} |f(\mathbf{z})| \qquad (\omega \in \mathbb{P}^{d-1}(\mathbb{F}_q)),$$

where the maximum is over all affine lines $\ell \subset \mathbb{F}_q^d$ with direction ω. Then

$$\|\mathcal{M}_d f\|_{\ell^d(\mathbb{P}^{d-1}(\mathbb{F}_q))} \le C_d\, q^{\frac{d-1}{d}}\, \|f\|_{\ell^d(\mathbb{F}_q^d)},$$

where C_d depends only on d.

Lemma 7.3. *Let $n \geq 1$. For every $1 \leq u \leq \infty$ and every $F : \mathbb{H}_n(\mathbb{F}_q) \to \mathbb{C}$ one has*

$$\|\mathcal{M}_{\mathbb{H}_n} F\|_{\ell^u(\mathbb{P}^{2n-1}(\mathbb{F}_q))} \leq C_{n,u}\, q^{\tau_{2n}(u)}\, \|F\|_{\ell^u(\mathbb{H}_n(\mathbb{F}_q))},$$

where

$$\tau_{2n}(u) = \begin{cases} \frac{2n-1}{u}, & 1 \leq u \leq 2n, \\ 1 - \frac{1}{u}, & 2n \leq u \leq \infty, \end{cases}$$

and

$$C_{n,u} := \begin{cases} 2^{\,1-\theta(u)}\, C_{2n}^{\theta(u)}, & 1 \leq u \leq 2n, \\ C_{2n}^{\frac{2n}{u}}, & 2n \leq u \leq \infty, \end{cases} \qquad \theta(u) := \frac{2n}{2n-1}\Big(1 - \frac{1}{u}\Big).$$

Proof. Let G be the function on $\mathbb{F}_q^{2n}$ associated with F as in Lemma 7.1. By that lemma, we have

$$\|\mathcal{M}_{\mathbb{H}_n} F\|_{\ell^u(\mathbb{P}^{2n-1}(\mathbb{F}_q))} \leq \|\mathcal{M}_{2n} G\|_{\ell^u(\mathbb{P}^{2n-1}(\mathbb{F}_q))}.$$

Assume first that $1 \leq u \leq 2n$. By Theorem 7.2, one has

$$\|\mathcal{M}_{2n} G\|_{\ell^{2n}(\mathbb{P}^{2n-1}(\mathbb{F}_q))} \leq C_{2n}\, q^{\frac{2n-1}{2n}}\, \|G\|_{\ell^{2n}(\mathbb{F}_q^{2n})}.$$

For each direction $[\mathbf{v}] \in \mathbb{P}^{2n-1}(\mathbb{F}_q)$, the maximal operator satisfies

$$\mathcal{M}_{2n} G([\mathbf{v}]) \leq \sum_{\mathbf{x} \in \mathbb{F}_q^{2n}} |G(\mathbf{x})|.$$

Summing over all directions implies

$$\|\mathcal{M}_{2n} G\|_{\ell^1(\mathbb{P}^{2n-1}(\mathbb{F}_q))} \leq \big|\mathbb{P}^{2n-1}(\mathbb{F}_q)\big|\, \|G\|_{\ell^1(\mathbb{F}_q^{2n})} \leq 2q^{2n-1}\, \|G\|_{\ell^1(\mathbb{F}_q^{2n})}.$$

Interpolating between the trivial ℓ^1 bound and the sharp ℓ^{2n} estimate of Ellenberg–Oberlin–Tao via Riesz–Thorin, we obtain for every $1 \leq u \leq 2n$,

$$\|\mathcal{M}_{2n} G\|_{\ell^u(\mathbb{P}^{2n-1}(\mathbb{F}_q))} \leq C_{n,u}\, q^{\frac{2n-1}{u}}\, \|G\|_{\ell^u(\mathbb{F}_q^{2n})},$$

where $C_{n,u} = 2^{\,1-\theta(u)}\, C_{2n}^{\theta(u)}$ with $\theta(u) := \frac{2n}{2n-1}\Big(1 - \frac{1}{u}\Big)$.

Assume now that $2n \leq u \leq \infty$. The trivial bound $\|\mathcal{M}_{2n} G\|_{\ell^\infty(\mathbb{P}^{2n-1}(\mathbb{F}_q))} \leq q\|G\|_{\ell^\infty(\mathbb{F}_q^{2n})}$ and interpolation with the case $u = 2n$ provide

$$\|\mathcal{M}_{2n} G\|_{\ell^u(\mathbb{P}^{2n-1}(\mathbb{F}_q))} \leq C_{n,u}\, q^{1-\frac{1}{u}}\, \|G\|_{\ell^u(\mathbb{F}_q^{2n})},$$

where $C_{n,u} = C_{2n}^{\frac{2n}{u}}$.

Therefore, $\|\mathcal{M}_{\mathbb{H}_n} F\|_{\ell^u(\mathbb{P}^{2n-1}(\mathbb{F}_q))} \leq C_{n,u} q^{1-\frac{1}{u}} \|F\|_{\ell^u(\mathbb{H}_n(\mathbb{F}_q))}$. □

The remaining arguments mirror those in Section 4, with $|\mathbb{P}^1(\mathbb{F}_q)| = q+1$ replaced by $|\mathbb{P}^{2n-1}(\mathbb{F}_q)|$.

7.1 The case $A_n(u,v) \le \frac{2n-1}{v}$

Lemma 7.4. *Let $1 \le v \le u \le 2n$. Then*

$$A_n(u,v) \le \frac{2n-1}{v}.$$

Proof. Since $\mathbb{P}^{2n-1}(\mathbb{F}_q)$ is a finite set, the standard ℓ^p–embedding implies

$$\|\mathcal{M}_{\mathbb{H}_n} F\|_{\ell^v(\mathbb{P}^{2n-1}(\mathbb{F}_q))} \le \left|\mathbb{P}^{2n-1}(\mathbb{F}_q)\right|^{\frac{1}{v}-\frac{1}{u}} \|\mathcal{M}_{\mathbb{H}_n} F\|_{\ell^u(\mathbb{P}^{2n-1}(\mathbb{F}_q))}. \tag{35}$$

Since $u \le 2n$, Lemma 7.3 gives

$$\|\mathcal{M}_{\mathbb{H}_n} F\|_{\ell^u(\mathbb{P}^{2n-1}(\mathbb{F}_q))} \le C_{n,u}\, q^{\frac{2n-1}{u}} \|F\|_{\ell^u(\mathbb{H}_n(\mathbb{F}_q))}.$$

Combining these estimates with (34) completes the proof. □

Lemma 7.5. *Let $1 < u < 2n$ and $u \le v \le (2n-1)\frac{u}{u-1}$. Then*

$$A_n(u,v) \le \frac{2n-1}{v}.$$

Proof. Let $\widetilde{F} := |F|$, the absolute value of F. Fix, for each $[\mathbf{v}] \in \mathbb{P}^{2n-1}(\mathbb{F}_q)$, a horizontal line $L_{[\mathbf{v}]}$ such that

$$\mathcal{M}_{\mathbb{H}_n}\widetilde{F}([\mathbf{v}]) = \sum_{\mathbf{x}\in L_{[\mathbf{v}]}} \widetilde{F}(\mathbf{x}).$$

Define the linear operator T on functions $\mathcal{G} : \mathbb{H}_n(\mathbb{F}_q) \to \mathbb{C}$ by

$$(T\mathcal{G})([\mathbf{v}]) = \sum_{\mathbf{x}\in L_{[\mathbf{v}]}} \mathcal{G}(\mathbf{x}).$$

Then $\|\mathcal{M}_{\mathbb{H}_n} F\|_{\ell^v(\mathbb{P}^{2n-1}(\mathbb{F}_q))} = \|T\widetilde{F}\|_{\ell^v(\mathbb{P}^{2n-1}(\mathbb{F}_q))}$ and $\|\widetilde{F}\|_{\ell^u(\mathbb{H}_n(\mathbb{F}_q))} = \|F\|_{\ell^u(\mathbb{H}_n(\mathbb{F}_q))}$.

Choose $\theta \in (0,1)$ and $r \in [1,\infty]$ such that

$$\frac{1}{u} = (1-\theta)\cdot 1 + \theta\cdot\frac{1}{2n}, \qquad \frac{1}{v} = (1-\theta)\cdot\frac{1}{r} + \theta\cdot\frac{1}{2n}. \tag{36}$$

Solving the first identity implies

$$\theta = \frac{2n}{2n-1}\left(1-\frac{1}{u}\right).$$

The assumption $u \le v \le (2n-1)\frac{u}{u-1}$ is equivalent to

$$0 \le \frac{1}{r} \le 1,$$

and therefore such a choice of r is possible.

We record two endpoint bounds for T. For every $\mathcal{G}$ and every $[\mathbf{v}]$ one has $|(T\mathcal{G})([\mathbf{v}])| \le \|\mathcal{G}\|_{\ell^1(\mathbb{H}_n(\mathbb{F}_q))}$. Therefore,

$$\|T\mathcal{G}\|_{\ell^r(\mathbb{P}^{2n-1}(\mathbb{F}_q))} \le \left|\mathbb{P}^{2n-1}(\mathbb{F}_q)\right|^{\frac{1}{r}} \|\mathcal{G}\|_{\ell^1(\mathbb{H}_n(\mathbb{F}_q))}.$$

Also, $|T\mathcal{G}| \le \mathcal{M}_{\mathbb{H}_n}(|\mathcal{G}|)$ pointwise. Lemma 7.3 at $u = 2n$ gives

$$\|T\mathcal{G}\|_{\ell^{2n}(\mathbb{P}^{2n-1}(\mathbb{F}_q))} \le C_{2n}\, q^{\frac{2n-1}{2n}}\, \|\mathcal{G}\|_{\ell^{2n}(\mathbb{H}_n(\mathbb{F}_q))}.$$

Apply Lemma 2.2 to T with these two endpoints and using (36), we obtain

$$\|T\mathcal{G}\|_{\ell^{v}(\mathbb{P}^{2n-1}(\mathbb{F}_q))} \le C_{n,u,v}\, q^{(2n-1)\left(\frac{1-\theta}{r}+\frac{\theta}{2n}\right)}\, \|\mathcal{G}\|_{\ell^{u}(\mathbb{H}_n(\mathbb{F}_q))},$$

where $C_{n,u,v} := 2^{\frac{1}{v}-\frac{\theta(n,u)}{2n}} C_{2n}^{\theta(n,u)}$ and $\theta(n,u) := \frac{2n}{2n-1}\left(1-\frac{1}{u}\right)$ Since $\frac{1-\theta}{r}+\frac{\theta}{2n} = \frac{1}{v}$, this becomes

$$\|T\widetilde{F}\|_{\ell^{v}(\mathbb{P}^{2n-1}(\mathbb{F}_q))} \le C_{n,u,v}\, q^{\frac{2n-1}{v}}\, \|\widetilde{F}\|_{\ell^{u}(\mathbb{H}_n(\mathbb{F}_q))},$$

which implies the claim. □

7.2 The case $A_n(u,v) \le 1 - \frac{1}{u}$

Lemma 7.6. *Let $2n \le u \le v \le \infty$. Then*

$$A_n(u,v) \le 1 - \frac{1}{u}.$$

Proof. Since $\mathbb{P}^{2n-1}(\mathbb{F}_q)$ is a finite set and $u \le v$, the monotonicity of ℓ^p norms implies that

$$\|\mathcal{M}_{\mathbb{H}_n} F\|_{\ell^{v}(\mathbb{P}^{2n-1}(\mathbb{F}_q))} \le \|\mathcal{M}_{\mathbb{H}_n} F\|_{\ell^{u}(\mathbb{P}^{2n-1}(\mathbb{F}_q))}.$$

Moreover, by Lemma 7.3, we have

$$\|\mathcal{M}_{\mathbb{H}_n} F\|_{\ell^{u}(\mathbb{P}^{2n-1}(\mathbb{F}_q))} \le C_{n,u} q^{1-\frac{1}{u}} \|F\|_{\ell^{u}(\mathbb{H}_n(\mathbb{F}_q))}.$$

This proves the claim. □

Lemma 7.7. *Let $1 \le u \le 2n$ and $v \ge (2n-1)\frac{u}{u-1}$. Then*

$$A_n(u,v) \le 1 - \frac{1}{u}.$$

Proof. If $u = 1$, then $\mathcal{M}_{\mathbb{H}_n} F([\mathbf{v}]) \le \|F\|_{\ell^1(\mathbb{H}_n(\mathbb{F}_q))}$ for all $[\mathbf{v}]$. Therefore, $\|\mathcal{M}_{\mathbb{H}_n} F\|_{\ell^{v}(\mathbb{P}^{2n-1}(\mathbb{F}_q))} \le |\mathbb{P}^{2n-1}(\mathbb{F}_q)|^{\frac{1}{v}} \|F\|_{\ell^1(\mathbb{F}_q^{2n})}$. Since $|\mathbb{P}^{2n-1}(\mathbb{F}_q)|^{\frac{1}{v}} \le 2^{\frac{1}{v}} q^{\frac{2n-1}{v}}$, this bound is stronger than $q^{1-\frac{1}{u}}$ when $u = 1$.

Assume that $1 < u \le 2n$. By the linearization argument in Lemma 7.5, it suffices to prove the bound for the associated linear operator T. We use the endpoint estimates

$$\|T\mathcal{G}\|_{\ell^{\infty}(\mathbb{P}^{2n-1}(\mathbb{F}_q))} \le \|\mathcal{G}\|_{\ell^{1}(\mathbb{H}_n(\mathbb{F}_q))}, \qquad \|T\mathcal{G}\|_{\ell^{2n}(\mathbb{P}^{2n-1}(\mathbb{F}_q))} \le C_{2n}\, q^{\frac{2n-1}{2n}}\, \|\mathcal{G}\|_{\ell^{2n}(\mathbb{H}_n(\mathbb{F}_q))}.$$

Let $\theta \in (0,1]$ satisfy $\frac{1}{u} = (1-\theta)\cdot 1 + \theta \cdot \frac{1}{2n}$. Then $\theta = \frac{2n}{2n-1}\left(1-\frac{1}{u}\right)$. Interpolation implies

$$\|T\mathcal{G}\|_{\ell^{\frac{2n}{\theta}}(\mathbb{P}^{2n-1}(\mathbb{F}_q))} \le C_{n,u}\, q^{\frac{(2n-1)\theta}{2n}}\, \|\mathcal{G}\|_{\ell^{u}(\mathbb{H}_n(\mathbb{F}_q))} = C_{n,u}\, q^{1-\frac{1}{u}}\, \|\mathcal{G}\|_{\ell^{u}(\mathbb{H}_n(\mathbb{F}_q))}.$$

Since $2n/\theta = (2n-1)\frac{u}{u-1}$ and $v \ge (2n-1)\frac{u}{u-1}$, one has

$$\|T\mathcal{G}\|_{\ell^{v}(\mathbb{P}^{2n-1}(\mathbb{F}_q))} \le \|T\mathcal{G}\|_{\ell^{(2n-1)u/(u-1)}(\mathbb{P}^{2n-1}(\mathbb{F}_q))} \le C_{n,u}\, q^{1-\frac{1}{u}}\, \|\mathcal{G}\|_{\ell^{u}(\mathbb{H}_n(\mathbb{F}_q))}.$$

This implies the claim. □

7.3 The case $A_n(u,v) \le 1 + \frac{2n-1}{v} - \frac{2n}{u}$

Lemma 7.8. *Let $2n \le u \le \infty$ and $1 \le v \le u$. Then*

$$A_n(u,v) \le 1 + \frac{2n-1}{v} - \frac{2n}{u}.$$

Proof. It follows from (35) and Lemma 7.3 that

$$\begin{aligned}\|\mathcal{M}_{\mathbb{H}_n} F\|_{\ell^v(\mathbb{P}^{2n-1}(\mathbb{F}_q))} &\le \left|\mathbb{P}^{2n-1}(\mathbb{F}_q)\right|^{\frac{1}{v}-\frac{1}{u}} \|\mathcal{M}_{\mathbb{H}_n} F\|_{\ell^u(\mathbb{P}^{2n-1}(\mathbb{F}_q))} \\ &\le \left|\mathbb{P}^{2n-1}(\mathbb{F}_q)\right|^{\frac{1}{v}-\frac{1}{u}} \left(C_{n,u} q^{1-\frac{1}{u}} \|F\|_{\ell^u(\mathbb{H}_n(\mathbb{F}_q))} \right).\end{aligned}$$

Using (34) implies

$$\|\mathcal{M}_{\mathbb{H}_n} F\|_{\ell^v(\mathbb{P}^{2n-1}(\mathbb{F}_q))} \le C_{n,u,v}\, q^{(2n-1)(\frac{1}{v}-\frac{1}{u})}\, q^{1-\frac{1}{u}}\, \|F\|_{\ell^u(\mathbb{H}_n(\mathbb{F}_q))} = C_{n,u,v}\, q^{1+\frac{2n-1}{v}-\frac{2n}{u}}\, \|F\|_{\ell^u(\mathbb{H}_n(\mathbb{F}_q))}.$$

This proves the claim. □

The lower bounds follow from the same three test functions as in Section 5. We include the proofs for completeness.

7.4 The case $A_n(u,v) \ge \frac{2n-1}{v}$

Lemma 7.9. *For every $1 \le u, v \le \infty$ one has $A_n(u,v) \ge \frac{2n-1}{v}$.*

Proof. Fix $\mathbf{p}_* \in \mathbb{H}_n(\mathbb{F}_q)$ and set $F = \delta_{\mathbf{p}_*}$. For every direction $[\mathbf{v}] \in \mathbb{P}^{2n-1}(\mathbb{F}_q)$ there is a unique horizontal line in direction $[\mathbf{v}]$ through $\mathbf{p}_*$. Therefore, $\mathcal{M}_{\mathbb{H}_n} F([\mathbf{v}]) = 1$ for all $[\mathbf{v}]$. Together with (34), we obtain

$$\|\mathcal{M}_{\mathbb{H}_n} F\|_{\ell^v(\mathbb{P}^{2n-1}(\mathbb{F}_q))} = \left|\mathbb{P}^{2n-1}(\mathbb{F}_q)\right|^{\frac{1}{v}} \ge q^{\frac{2n-1}{v}}, \qquad \|F\|_{\ell^u(\mathbb{H}_n(\mathbb{F}_q))} = 1.$$

By the definition of $A_n(u,v)$, we obtain $A_n(u,v) \ge \frac{2n-1}{v}$. □

7.5 The case $A_n(u,v) \ge 1 - \frac{1}{u}$

Lemma 7.10. *For every $1 \le u, v \le \infty$ one has $A_n(u,v) \ge 1 - \frac{1}{u}$.*

Proof. Let $L \subset \mathbb{H}_n(\mathbb{F}_q)$ be a fixed horizontal line and set $F = \mathbf{1}_L$. Then $\|F\|_{\ell^u(\mathbb{H}_n(\mathbb{F}_q))} = |L|^{\frac{1}{u}} = q^{\frac{1}{u}}$. Let $[\mathbf{v}_0]$ be the direction of L. For this direction one may choose the basepoint p so that $L = L_{\mathbf{p},\mathbf{v}_0}$. Therefore, $\mathcal{M}_{\mathbb{H}_n} F([\mathbf{v}_0]) \ge \sum_{\mathbf{x} \in L} \mathbf{1}_L(\mathbf{x}) = q$, and hence

$$\|\mathcal{M}_{\mathbb{H}_n} F\|_{\ell^v(\mathbb{P}^{2n-1}(\mathbb{F}_q))} \ge q$$

for every $1 \le v \le \infty$. This forces $A_n(u,v) \ge 1 - \frac{1}{u}$. □

7.6 The case $A_n(u,v) \ge 1 + \frac{2n-1}{v} - \frac{2n}{u}$

Lemma 7.11. *For every $1 \le u, v \le \infty$ one has $A_n(u,v) \ge 1 + \frac{2n-1}{v} - \frac{2n}{u}$.*

Proof. Let

$$S := \bigcup_{[\mathbf{v}]\in\mathbb{P}^{2n-1}(\mathbb{F}_q)} L_{\mathbf{0},[\mathbf{v}]}, \qquad F := \mathbf{1}_S.$$

Distinct horizontal lines through $\mathbf{p}$ intersect only at $\mathbf{p}$. Since each such line has q points and there are $|\mathbb{P}^{2n-1}(\mathbb{F}_q)|$ directions, together with (33), we obtain

$$|S| = 1 + \left|\mathbb{P}^{2n-1}(\mathbb{F}_q)\right|(q-1) = q^{2n}.$$

Therefore, $\|F\|_{\ell^u(\mathbb{H}_n(\mathbb{F}_q))} = |S|^{\frac{1}{u}} = q^{\frac{2n}{u}}$. For every direction $[\mathbf{v}]$ we have $L_{\mathbf{p},[\mathbf{v}]} \subset S$, and hence $\mathcal{M}_{\mathbb{H}_n}F([\mathbf{v}]) \geq q$. Thus,

$$\|\mathcal{M}_{\mathbb{H}_n}F\|_{\ell^v(\mathbb{P}^{2n-1}(\mathbb{F}_q))} \geq q\left|\mathbb{P}^{2n-1}(\mathbb{F}_q)\right|^{\frac{1}{v}} \geq q^{1+\frac{2n-1}{v}}$$

by (34).

This forces $A_n(u,v) \geq 1 + \frac{2n-1}{v} - \frac{2n}{u}$. □

7.7 Proof of Theorem 1.5

Proof of Theorem 1.5. Lemmas 7.9, 7.10, and 7.11 give

$$A_n(u,v) \geq \max\Big(\frac{2n-1}{v},\, 1-\frac{1}{u},\, 1+\frac{2n-1}{v}-\frac{2n}{u}\Big).$$

For the matching upper bound, combine Lemmas 7.4 through 7.8. These lemmas cover all pairs (u,v). Therefore,

$$A_n(u,v) \leq \max\Big(\frac{2n-1}{v},\, 1-\frac{1}{u},\, 1+\frac{2n-1}{v}-\frac{2n}{u}\Big),$$

and the theorem follows. □

8 Proof of Theorem 1.6

For any $\omega = [a:b:c] \in \mathcal{D}_1$, we take any fixed $(x_\omega, y_\omega) \in \mathbb{F}_q^2$ with $x_\omega b - y_\omega a = c$. Then

$$\mathcal{L}(\omega) = \Big\{L_{(x_\omega,y_\omega,t),[a:b]} = \{(x_\omega + sa, y_\omega + sb, t + sc) :\ s \in \mathbb{F}_q\} :\ t \in \mathbb{F}_q\Big\}. \tag{37}$$

Let $\pi : \mathbb{H}_1(\mathbb{F}_q) \to \mathbb{F}_q^2$ be the projection defined by $\pi(x,y,t) = (x,y)$. Let ℓ_ω be the affine line on $\mathbb{F}_q^2$ given by

$$\ell_\omega = \{(x_\omega + sa, y_\omega + sb) :\ s \in \mathbb{F}_q\} = \{(x,y) \in \mathbb{F}_q :\ bx - ay = c\}.$$

Then $\pi(L) = \ell_\omega$ for any $L \in \mathcal{L}_\omega$. Indeed, the map $\omega \mapsto \ell_\omega$ is bijective from $\mathcal{D}_1$ to the set of all affine lines in $\mathbb{F}_q^2$. Moreover, the line $l_{[a:b:c]}$, with $[a:b:c] \in \mathcal{D}_1$, is parallel to $[a:b]$. For each $[\mathbf{v}] \in \mathbb{P}^1(\mathbb{F}_q)$, there are exactly q elements ω of $\mathcal{D}_1$ such that ℓ_ω is parallel to $[\mathbf{v}]$.

For a given choice of the family $\{L_\omega\}_{\omega\in\mathcal{D}_1}$ with $L_\omega \in \mathcal{L}(\omega)$, we will use the explicit parametrizations as follows. Consider the disjoint union $\mathcal{D}_1 = \mathcal{D}_1' \cup \mathcal{D}_1''$, with

$$\mathcal{D}_1' = \{[1:m:\gamma] :\ m,\gamma \in \mathbb{F}_q\}, \quad \mathcal{D}_1'' = \{[0:1:\gamma] :\ \gamma \in \mathbb{F}_q\}.$$

For $\omega = [1:m:\gamma] \in \mathcal{D}_1$, we rewrite $x = x_\omega + s$ and $\tau = t - x_\omega\gamma$ in (37). Then

$$L_\omega = \{(x, mx - \gamma, \tau_\omega + \gamma x) :\ x \in \mathbb{F}_q\} \tag{38}$$

for some $\tau_\omega \in \mathbb{F}_q$. For $\omega = [0:1:\gamma]$, we take $(x_\omega, y_\omega) = (\gamma, 0)$ and rewrite $y = s$, $\tau_\omega = t$ in (37). Then

$$L_\omega = \{(\gamma, y, \tau_\omega + \gamma y) : y \in \mathbb{F}_q\} \tag{39}$$

for some $\tau_\omega \in \mathbb{F}_q$.

Proof of Theorem 1.6. It suffices to prove the claim for non-negative F. For each $\omega \in \mathcal{D}_1$, we choose an $L_\omega \in \mathcal{L}(\omega)$ such that

$$(\mathcal{M}^{\mathrm{rd}}_{\mathbb{H}_1} F)(\omega) = \sum_{\mathbf{p} \in L_\omega} F(\mathbf{p}).$$

For this family $\{L_\omega\}_{\omega \in \mathcal{D}_1}$, we will use the parametrizations in (38) and (39) later.

Let $T : \ell^2(\mathbb{H}_1(\mathbb{F}_q)) \to \ell^2(\mathcal{D}_1)$ be the linear transformation given by

$$(Tf)(\omega) := \sum_{\mathbf{p} \in L_\omega} f(\mathbf{p}), \qquad \omega \in \mathcal{D}_1.$$

Theorem 1.6 will follow if we can prove

$$\|Tf\|_{\ell^2(\mathcal{D}_1)} \le 5q^{\frac{1}{2}} \|f\|_{\ell^2(\mathbb{H}_1(\mathbb{F}_q))} \tag{40}$$

for any $f \in \ell^2(\mathbb{H}_1(\mathbb{F}_q))$.

For $(x, y) \in \mathbb{F}_q^2$, we define the Fourier transform in the central variable t by

$$\widehat{f}(x, y; \xi) := \sum_{t \in \mathbb{F}_q} f(x, y, t)\, \chi(-\xi t), \qquad \xi \in \mathbb{F}_q.$$

The Fourier inversion gives

$$f(x, y, t) = \frac{1}{q} \sum_{\xi \in \mathbb{F}_q} \widehat{f}(x, y; \xi)\, \chi(\xi t),$$

with χ a given non-trivial additive character. Applying the Plancherel theorem in t and summing over all (x, y) implies

$$\sum_{x,y \in \mathbb{F}_q} \sum_{\xi \in \mathbb{F}_q} \left|\widehat{f}(x, y; \xi)\right|^2 = q\, \|f\|^2_{\ell^2(\mathbb{H}_1(\mathbb{F}_q))}. \tag{41}$$

Now for any $\omega \in \mathcal{D}_1$,

$$(Tf)(\omega) = \sum_{(x,y,t) \in L_\omega} f(x, y, t) = \frac{1}{q} \sum_{\xi \in \mathbb{F}_q} \sum_{(x,y,t) \in L_\omega} \widehat{f}(x, y; \xi)\, \chi(\xi t) = \sum_{\xi \in \mathbb{F}_q} (T_\xi f)(\omega),$$

where T_ξ are linear transformations given by

$$(T_\xi f)(\omega) := \frac{1}{q} \sum_{(x,y,t) \in L_\omega} \widehat{f}(x, y; \xi)\, \chi(\xi t). \tag{42}$$

In the following, we consider

$$Tf = T_0 f + T_{\neq 0} f, \quad T_{\neq 0} f = \sum_{\xi \in \mathbb{F}_q^*} T_\xi f.$$

Our aim is to prove

$$\|T_0 f\|_{\ell^2(\mathcal{D}_1)} \le \sqrt{2} q^{\frac{1}{2}} \|f\|_{\ell^2(\mathbb{H}_1(\mathbb{F}_q))}, \tag{43}$$

and

$$\|T_{\neq 0} f\|_{\ell^2(\mathcal{D}_1)} \le \sqrt{5} q^{\frac{1}{2}} \|f\|_{\ell^2(\mathbb{H}_1(\mathbb{F}_q))}. \tag{44}$$

Then

$$\|Tf\|_{\ell^2(\mathcal{D}_1)} \le \|T_0 f\|_{\ell^2(\mathcal{D}_1)} + \|T_{\neq 0} f\|_{\ell^2(\mathcal{D}_1)} \le (\sqrt{2}+\sqrt{5})\, q^{\frac{1}{2}} \|f\|_{\ell^2(\mathbb{H}_1(\mathbb{F}_q))} \le 5\, q^{\frac{1}{2}} \|f\|_{\ell^2(\mathbb{H}_1(\mathbb{F}_q))}$$

and (40) holds.

Case 1: $\xi = 0$. Take $g : \mathbb{F}_q^2 \to \mathbb{C}$ so that

$$g_0(x,y) := \widehat{f}(x,y;0) = \sum_{t \in \mathbb{F}_q} f(x,y,t). \tag{45}$$

Then, with $\pi(L_\omega) = \ell_\omega$, one has

$$(T_0 f)(\omega) = \frac{1}{q} \sum_{(x,y,t) \in L_\omega} \widehat{f}(x,y;0) = \frac{1}{q} \sum_{(x,y) \in \ell_\omega} g_0(x,y). \tag{46}$$

For each slope $[\mathbf{v}] \in \mathbb{P}^1(\mathbb{F}_q)$, let $\mathcal{L}([\mathbf{v}])$ be the family of all affine lines in $\mathbb{F}_q^2$ with direction $[\mathbf{v}]$. Then $|\mathcal{L}([\mathbf{v}])| = q$, and these families partition the set of all affine lines in $\mathbb{F}_q^2$. Moreover, by definition (13), for every $\ell \in \mathcal{L}([\mathbf{v}])$ one has

$$\sum_{(x,y) \in \ell} |g_0(x,y)| \le \mathcal{M}_2(|g_0|)([\mathbf{v}]). \tag{47}$$

Therefore, grouping the $\ell^2(\mathcal{D}_1)$-sum by slopes, together with inserting (46) and (47), gives

$$\begin{aligned} \|T_0 f\|_{\ell^2(\mathcal{D}_1)}^2 &= \sum_{\omega \in \mathcal{D}_1} \Big(\frac{1}{q} \sum_{(x,y) \in \ell_\omega} |g_0(x,y)| \Big)^2 \le \frac{1}{q^2} \sum_{[\mathbf{v}] \in \mathbb{P}^1(\mathbb{F}_q)} \sum_{\ell \in \mathcal{L}([\mathbf{v}])} \Big(\sum_{(x,y) \in \ell} |g_0(x,y)| \Big)^2 \\ &\le \frac{1}{q^2} \sum_{[\mathbf{v}] \in \mathbb{P}^1(\mathbb{F}_q)} q \cdot \big(\mathcal{M}_2(|g_0|)([\mathbf{v}]) \big)^2 = \frac{1}{q} \|\mathcal{M}_2(|g_0|)\|_{\ell^2(\mathbb{P}^1(\mathbb{F}_q))}^2. \end{aligned}$$

Using Lemma 2.5, (45) and the Cauchy-Schwarz inequality sequentially, we obtain that

$$\begin{aligned} \|\mathcal{M}_2(|g_0|)\|_{\ell^2(\mathbb{P}^1(\mathbb{F}_q))}^2 &\le 2q \|g_0\|_{\ell^2(\mathbb{F}_q^2)}^2 = 2q \sum_{(x,y)} \Big| \sum_t f(x,y,t) \Big|^2 \\ &\le 2q \sum_{(x,y)} q \sum_t |f(x,y,t)|^2 = 2q^2 \|f\|_{\ell^2(\mathbb{H}_1(\mathbb{F}_q))}^2. \end{aligned}$$

Therefore (43) holds.

Case 2: $\xi \neq 0$. When $\xi \in \mathbb{F}_q^*$, define, for $\omega = [1 : m : \gamma]$,

$$U_\xi(m,\gamma) := \sum_{x \in \mathbb{F}_q} \widehat{f}(x, mx - \gamma; \xi)\, \chi(\xi \gamma x),$$

and for $\omega = [0:1:\gamma]$,

$$U_\xi^\infty(\gamma) := \sum_{y\in\mathbb{F}_q} \widehat{f}(\gamma, y; \xi)\, \chi(\xi\gamma y).$$

We assume at the moment that for every $\xi \in \mathbb{F}_q^*$,

$$\sum_{m\in\mathbb{F}_q}\sum_{\gamma\in\mathbb{F}_q} |U_\xi(m,\gamma)|^2 \le 2q \sum_{x,y\in\mathbb{F}_q} |\widehat{f}(x,y;\xi)|^2, \tag{48}$$

and

$$\sum_{\gamma\in\mathbb{F}_q} |U_\xi^\infty(\gamma)|^2 \le q \sum_{x,y\in\mathbb{F}_q} |\widehat{f}(x,y;\xi)|^2. \tag{49}$$

Inserting the explicit parametrizations in (38) and (39) into (42), we obtain

$$(T_\xi f)(\omega) = \frac{1}{q}\,\chi(\xi\tau_\omega) \sum_{x\in\mathbb{F}_q} \widehat{f}(x, mx-\gamma;\xi)\,\chi(\xi\gamma x) = \frac{1}{q}\,\chi(\xi\tau_\omega)\, U_\xi(m,\gamma), \quad \omega = [1:m:\gamma],$$

$$(T_\xi f)(\omega) = \frac{1}{q}\,\chi(\xi\tau_\omega) \sum_{y\in\mathbb{F}_q} \widehat{f}(\gamma, y;\xi)\,\chi(\xi\gamma y) = \frac{1}{q}\,\chi(\xi\tau_\omega)\, U_\xi^\infty(\gamma), \quad \omega = [0:1:\gamma].$$

Since $|\chi(\xi\tau(\omega))| = 1$, we obtain from (48)–(49) that

$$\|T_\xi f\|^2_{\ell^2(\mathcal{D}_1)} = \sum_{m,\gamma}\Big|\frac{1}{q}U_\xi(m,\gamma)\Big|^2 + \sum_{\gamma}\Big|\frac{1}{q}U_\xi^\infty(\gamma)\Big|^2 \le \frac{5}{q}\sum_{x,y\in\mathbb{F}_q}|\widehat{f}(x,y;\xi)|^2. \tag{50}$$

It follows by triangle inequality, Cauchy–Schwarz inequality, (50) and (41) that

$$\begin{aligned}\Big\|\sum_{\xi\in\mathbb{F}_q^*} T_\xi f\Big\|^2_{\ell^2(\mathcal{D}_1)} &\le \Big(\sum_{\xi\in\mathbb{F}_q^*}\|T_\xi f\|_{\ell^2(\mathcal{D}_1)}\Big)^2 \le q\sum_{\xi\in\mathbb{F}_q^*}\|T_\xi f\|^2_{\ell^2(\mathcal{D}_1)}\\ &\le 5\sum_{\xi\in\mathbb{F}_q^*}\sum_{x,y\in\mathbb{F}_q}|\widehat{f}(x,y;\xi)|^2 \le 5q\,\|f\|^2_{\ell^2(\mathbb{H}_1(\mathbb{F}_q))}.\end{aligned}$$

Then (44) follows, and Theorem 1.6 holds.

In the rest of the proof, we shall prove (48) and (49). Fix $\xi \neq 0$, and write

$$h_m(\gamma) := U_\xi(m,\gamma) = \sum_{x\in\mathbb{F}_q} \widehat{f}(x, mx-\gamma;\xi)\,\chi(\xi\gamma x).$$

With the standard Fourier transform in γ, i.e.,

$$\widehat{h}_m(\rho) = \sum_\gamma h_m(\gamma)\chi(-\rho\gamma), \qquad (\rho\in\mathbb{F}_q),$$

the Plancherel theorem gives

$$\sum_\gamma |h_m(\gamma)|^2 = \frac{1}{q}\sum_{\rho\in\mathbb{F}_q}|\widehat{h}_m(\rho)|^2. \tag{51}$$

By making the change of variables $y = mx-\gamma$, we obtain

$$\widehat{h}_m(\rho) = \sum_{\gamma,x}\widehat{f}(x,mx-\gamma;\xi)\,\chi((\xi x-\rho)\gamma) = \sum_{x,y}\widehat{f}(x,y;\xi)\,\chi((\xi x-\rho)(mx-y))\,.$$

Define

$$G_\rho(x) := \sum_y \widehat{f}(x,y;\xi)\,\chi(-(\xi x-\rho)y)\,, \qquad Q_\rho(x) := (\xi x-\rho)x.$$

Then

$$\widehat{h}_m(\rho) = \sum_x G_\rho(x)\,\chi\big(mQ_\rho(x)\big).$$

Summing in m, we have

$$\begin{aligned}\sum_m |\widehat{h}_m(\rho)|^2 &= \sum_{x,x'} G_\rho(x)\overline{G_\rho(x')}\sum_m \chi(m(Q_\rho(x)-Q_\rho(x'))) \\ &= q\sum_{\substack{x,x'\\ Q_\rho(x)=Q_\rho(x')}} G_\rho(x)\overline{G_\rho(x')} = q\sum_{t\in\mathbb{F}_q}\Big|\sum_{x:\,Q_\rho(x)=t} G_\rho(x)\Big|^2.\end{aligned}$$

Since $\xi\neq 0$, for any given t the equation $Q_\rho(x)=t$ has at most 2 solutions. Therefore,

$$\sum_m |\widehat{h}_m(\rho)|^2 \le 2q\sum_x |G_\rho(x)|^2. \tag{52}$$

Moreover, for any given x, we have

$$\sum_\rho |G_\rho(x)|^2 = \sum_{y,y'} \widehat{f}(x,y;\xi)\overline{\widehat{f}(x,y';\xi)}\chi(-\xi x(y-y'))\sum_\rho \chi(\rho(y-y')) = q\sum_y |\widehat{f}(x,y;\xi)|^2. \tag{53}$$

Now, combining (51)-(53) implies

$$\sum_{m,\gamma}|U_\xi(m,\gamma)|^2 = \sum_{m,\gamma}|h_m(\gamma)|^2 = \frac{1}{q}\sum_{m,\rho\in\mathbb{F}_q}|\widehat{h}_m(\rho)|^2 \le 2\sum_{\rho,x}|G_\rho(x)|^2 = 2q\sum_{x,y}|\widehat{f}(x,y;\xi)|^2,$$

which proves (48). The bound (49) follows immediately by using the Cauchy-Schwarz inequality in y. This completes the proof. □

Regarding the sharpness of Theorem 1.6, let $\mathbf{0} := (0,0,0) \in \mathbb{H}_1(\mathbb{F}_q)$ and let $F := \delta_{\mathbf{0}}$. Then, $\|F\|_{\ell^2(\mathbb{H}_1(\mathbb{F}_q))} = 1$.

For each direction $[\mathbf{v}] = [(a,b)] \in \mathbb{P}^1(\mathbb{F}_q)$, consider the horizontal line $L_{\mathbf{0},[(a,b)]}$ through $\mathbf{0}$ with direction $[(a,b)]$. Since $\mathbf{0}\in L_{\mathbf{0},[(a,b)]}$, we have

$$\sum_{\mathbf{p}\in L_{\mathbf{0},[(a,b)]}} |F(\mathbf{p})| = 1.$$

Moreover, by the definition of the t-slope,

$$c\big(L_{\mathbf{0},[(a,b)]}\big) \;=\; x_0b - y_0a \;=\; 0,$$

and hence

$$\mathrm{Dir}\big(L_{\mathbf{0},[(a,b)]}\big) = [a:b:0] \in \mathcal{D}_1.$$

Therefore, for $\omega\in\mathcal{D}_1$,

$$\mathcal{M}^{\mathrm{rd}}_{\mathbb{H}_1}F(\omega) = \begin{cases}1, & \text{if } \omega=[a:b:0] \text{ for some } [(a,b)]\in\mathbb{P}^1(\mathbb{F}_q),\\ 0, & \text{otherwise.}\end{cases}$$

Since the set $\{[a:b:0] : [(a,b)] \in \mathbb{P}^1(\mathbb{F}_q)\}$ has cardinality $|\mathbb{P}^1(\mathbb{F}_q)| = q+1$, it follows that

$$\|\mathcal{M}^{\mathrm{rd}}_{\mathbb{H}_1} F\|^2_{\ell^2(\mathcal{D}_1)} = \sum_{\omega \in \mathcal{D}_1} \left|\mathcal{M}^{\mathrm{rd}}_{\mathbb{H}_1} F(\omega)\right|^2 = q+1,$$

and hence

$$\|\mathcal{M}^{\mathrm{rd}}_{\mathbb{H}_1} F\|_{\ell^2(\mathcal{D}_1)} = (q+1)^{\frac{1}{2}} \sim q^{\frac{1}{2}}.$$

Consequently, any inequality of the form $\|\mathcal{M}^{\mathrm{rd}}_{\mathbb{H}_1} F\|_{\ell^2(\mathcal{D}_1)} \le C q^{\alpha} \|F\|_{\ell^2(\mathbb{H}_1(\mathbb{F}_q))}$ forces $\alpha \ge \frac{1}{2}$, proving that the exponent $q^{\frac{1}{2}}$ in Theorem 1.6 is sharp.

9 Proof of Theorem 1.7

For each $\omega \in \mathcal{D}_1$, fix a horizontal line $L_\omega \in \mathcal{L}_\omega$, and define the associated linear operator

$$(Tf)(\omega) := \sum_{\mathbf{p} \in L_\omega} f(\mathbf{p}), \qquad \omega \in \mathcal{D}_1.$$

By the standard linearization argument used previously, for each given F one can choose $\{L_\omega\}_\omega$ so that $\mathcal{M}^{\mathrm{rd}}_{\mathbb{H}_1} F = T(|F|)$ pointwise on $\mathcal{D}_1$. Consequently, it suffices to prove operator norm bounds for T that are uniform over all choices of $\{L_\omega\}$.

We begin by recording the basic endpoint bounds for any linearization T of $\mathcal{M}^{\mathrm{rd}}_{\mathbb{H}_1}$. These follow directly from the fact that each line L_ω has exactly q points and is contained in $\mathbb{H}_1(\mathbb{F}_q)$, together with a counting argument on the number of refined directions passing through a given point.

Lemma 9.1. *Let T be any linearization as above. Then for every $f : \mathbb{H}_1(\mathbb{F}_q) \to \mathbb{C}$ one has*

$$\|Tf\|_{\ell^\infty(\mathcal{D}_1)} \le \|f\|_{\ell^1(\mathbb{H}_1(\mathbb{F}_q))}, \tag{54}$$

$$\|Tf\|_{\ell^\infty(\mathcal{D}_1)} \le q\, \|f\|_{\ell^\infty(\mathbb{H}_1(\mathbb{F}_q))}, \tag{55}$$

$$\|Tf\|_{\ell^1(\mathcal{D}_1)} \le (q+1)\, \|f\|_{\ell^1(\mathbb{H}_1(\mathbb{F}_q))}. \tag{56}$$

Proof. The bounds (54) and (55) follow from $L_\omega \subset \mathbb{H}_1(\mathbb{F}_q)$ and $|L_\omega| = q$.

For (56), we may assume $f \ge 0$. Then

$$\|Tf\|_{\ell^1(\mathcal{D}_1)} = \sum_{\omega \in \mathcal{D}_1} \sum_{\mathbf{p} \in L_\omega} f(\mathbf{p}) = \sum_{\mathbf{p} \in \mathbb{H}_1(\mathbb{F}_q)} f(\mathbf{p})\, N(\mathbf{p}),$$

where $N(\mathbf{p}) := \#\{\omega \in \mathcal{D}_1 : \ \mathbf{p} \in L_\omega\}$. If $\mathbf{p} = (x,y,t) \in L_\omega$, then $(x,y) \in \pi(L_\omega) = \ell_\omega$. Distinct refined directions ω correspond to distinct affine lines $\ell_\omega \subset \mathbb{F}_q^2$, and there are exactly $q+1$ affine lines through a given point $(x,y) \in \mathbb{F}_q^2$. Hence $N(\mathbf{p}) \le q+1$, proving (56). □

From the endpoint bounds in Lemma 9.1 and the sharp ℓ^2 estimate of Theorem 1.6, we can now derive diagonal bounds for $\mathcal{M}^{\mathrm{rd}}_{\mathbb{H}_1}$ across the full range of exponents $1 \le u \le \infty$ by interpolation. The exponent $\sigma(u)$ reflects the two natural ranges: for $1 \le u \le 2$ one interpolates between the $\ell^1 \to \ell^1$ and $\ell^2 \to \ell^2$ estimates, while for $2 \le u \le \infty$ one interpolates between the $\ell^2 \to \ell^2$ and $\ell^\infty \to \ell^\infty$ estimates.

Lemma 9.2. *For every $1 \le u \le \infty$ there exists a constant C_u (independent of q) such that for all $F : \mathbb{H}_1(\mathbb{F}_q) \to \mathbb{C}$,*

$$\|\mathcal{M}^{\mathrm{rd}}_{\mathbb{H}_1} F\|_{\ell^u(\mathcal{D}_1)} \le C_u\, q^{\sigma(u)}\, \|F\|_{\ell^u(\mathbb{H}_1(\mathbb{F}_q))}, \tag{57}$$

where

$$\sigma(u) := \begin{cases} \frac{1}{u}, & 1 \le u \le 2, \\ 1 - \frac{1}{u}, & 2 \le u \le \infty, \end{cases}$$

and

$$C_u := \begin{cases} 2^{1-\theta(u)} \cdot 5^{\theta(u)}, & 1 \le u \le 2, \\ 5^{\frac{2}{u}}, & 2 \le u \le \infty, \end{cases} \qquad \theta(u) = 2\big(1 - \frac{1}{u}\big).$$

Proof. By linearization, it suffices to prove (57) for T uniformly.

If $1 \le u \le 2$, interpolate between $\|T\|_{\ell^1 \to \ell^1} \le q + 1 \le 2q$ from (56) and $\|T\|_{\ell^2 \to \ell^2} \le 5q^{\frac{1}{2}}$ from Theorem 1.6. Choose $\theta \in [0, 1]$ so that

$$\frac{1}{u} = (1 - \theta) \cdot 1 + \theta \cdot \frac{1}{2}, \qquad \text{equivalently} \quad \theta = 2\Big(1 - \frac{1}{u}\Big).$$

The Riesz–Thorin theorem gives

$$\|T\|_{\ell^u \to \ell^u} \le 2^{1-\theta} \cdot 5^{\theta}\, q^{1 - \frac{\theta}{2}} = 2^{1-\theta} \cdot 5^{\theta}\, q^{\frac{1}{u}}.$$

If $2 \le u \le \infty$, interpolate between $\|T\|_{\ell^2 \to \ell^2} \le 5q^{\frac{1}{2}}$ and $\|T\|_{\ell^\infty \to \ell^\infty} \le q$ from (55). Choose $\theta \in [0, 1]$ so that

$$\frac{1}{u} = (1 - \theta) \cdot \frac{1}{2} + \theta \cdot 0, \qquad \text{equivalently} \quad \theta = 1 - \frac{2}{u}.$$

Then,

$$\|T\|_{\ell^u \to \ell^u} \le 5^{1-\theta} q^{\frac{1-\theta}{2} + \theta} = 5^{\frac{2}{u}}\, q^{1 - \frac{1}{u}}.$$

This proves (57). □

9.1 The region $1 \le u \le 2$ – upper bounds

Lemma 9.3. *Let $1 \le u \le 2$ and C_u be given in Lemma 9.2.*

(a) *If $1 \le v \le u$, then*

$$\|\mathcal{M}^{\mathrm{rd}}_{\mathbb{H}_1} F\|_{\ell^v(\mathcal{D}_1)} \le C_u \cdot 2^{\frac{1}{v} - \frac{1}{u}}\, q^{\frac{2}{v} - \frac{1}{u}}\, \|F\|_{\ell^u(\mathbb{H}_1(\mathbb{F}_q))}.$$

(b) *If $u \le v \le \frac{u}{u-1}$ (with the convention $\frac{u}{u-1} = \infty$ when $u = 1$), then*

$$\|\mathcal{M}^{\mathrm{rd}}_{\mathbb{H}_1} F\|_{\ell^v(\mathcal{D}_1)} \le 2^{\frac{1}{v} - 1 + \frac{1}{u}} \cdot 5^{2(1 - \frac{1}{u})}\, q^{\frac{1}{v}}\, \|F\|_{\ell^u(\mathbb{H}_1(\mathbb{F}_q))}.$$

(c) *If $v \ge \frac{u}{u-1}$, then*

$$\|\mathcal{M}^{\mathrm{rd}}_{\mathbb{H}_1} F\|_{\ell^v(\mathcal{D}_1)} \le 5^{2(1 - \frac{1}{u})}\, q^{1 - \frac{1}{u}}\, \|F\|_{\ell^u(\mathbb{H}_1(\mathbb{F}_q))}.$$

Proof. By linearization it suffices to prove the bounds for T uniformly.

(a) Since $v \le u$, Lemma 2.1 gives

$$\|Tf\|_{\ell^v(\mathcal{D}_1)} \le |\mathcal{D}_1|^{\frac{1}{v} - \frac{1}{u}}\, \|Tf\|_{\ell^u(\mathcal{D}_1)} \le 2^{\frac{1}{v} - \frac{1}{u}} q^{2(\frac{1}{v} - \frac{1}{u})}\, \|Tf\|_{\ell^u(\mathcal{D}_1)}.$$

Apply Lemma 9.2 with $\sigma(u) = \frac{1}{u}$ to obtain

$$\|Tf\|_{\ell^v(\mathcal{D}_1)} \le C_u \cdot 2^{\frac{1}{v}-\frac{1}{u}}\, q^{2(\frac{1}{v}-\frac{1}{u})}\, q^{\frac{1}{u}}\, \|f\|_{\ell^u(\mathbb{H}_1(\mathbb{F}_q))} = C_u \cdot 2^{\frac{1}{v}-\frac{1}{u}}\, q^{\frac{2}{v}-\frac{1}{u}}\, \|f\|_{\ell^u(\mathbb{H}_1(\mathbb{F}_q))}.$$

(b) Interpolating (56) and (54) implies, for every $1 \le r \le \infty$,

$$\|Tf\|_{\ell^r(\mathcal{D}_1)} \le (q+1)^{\frac{1}{r}}\, \|f\|_{\ell^1(\mathbb{H}_1(\mathbb{F}_q))} \le 2^{\frac{1}{r}} q^{\frac{1}{r}}\, \|f\|_{\ell^1(\mathbb{H}_1(\mathbb{F}_q))}. \tag{58}$$

Assume first that $u \in (1,2]$. Choose $\theta \in [0,1]$ so that

$$\frac{1}{u} = (1-\theta)\cdot 1 + \theta \cdot \frac{1}{2}, \qquad \text{so } \ \theta = 2\Big(1-\frac{1}{u}\Big).$$

The condition $u \le v \le \frac{u}{u-1}$ is equivalent to

$$\frac{1}{u} \ \ge\ \frac{1}{v} \ \ge\ 1-\frac{1}{u} \ =\ \frac{\theta}{2}.$$

Hence we may choose $r \in [1,\infty]$ such that

$$\frac{1}{v} = (1-\theta)\cdot\frac{1}{r} + \theta\cdot\frac{1}{2}.$$

Interpolating between (58) and the $\ell^2 \to \ell^2$ estimate from Theorem 1.6 gives

$$\begin{aligned}\|Tf\|_{\ell^v(\mathcal{D}_1)} \le& (2^{\frac{1}{r}} q^{\frac{1}{r}})^{1-\theta}\, (5q^{\frac{1}{2}})^{\theta}\, \|f\|_{\ell^u(\mathbb{H}_1(\mathbb{F}_q))} = 2^{\frac{1-\theta}{r}} \cdot 5^{\theta}\, q^{\frac{1-\theta}{r}+\frac{\theta}{2}}\, \|f\|_{\ell^u(\mathbb{H}_1(\mathbb{F}_q))} \\ =& 2^{\frac{1}{v}-1+\frac{1}{u}} \cdot 5^{2(1-\frac{1}{u})}\, q^{\frac{1}{v}}\, \|f\|_{\ell^u(\mathbb{H}_1(\mathbb{F}_q))}.\end{aligned}$$

The case $u = 1$ follows directly from (58) with $r = v$.

(c) Set $v_0 := \frac{u}{u-1}$, so that $\frac{1}{v_0} = 1 - \frac{1}{u}$. Interpolating between $\ell^1 \to \ell^\infty$ (norm ≤ 1, (54)) and $\ell^2 \to \ell^2$ (norm $\le 5q^{\frac{1}{2}}$, Theorem 1.6) with the same $\theta = 2(1-\frac{1}{u})$ implies

$$\|Tf\|_{\ell^{v_0}(\mathcal{D}_1)} \le 5^{\theta} q^{\frac{\theta}{2}}\, \|f\|_{\ell^u(\mathbb{H}_1(\mathbb{F}_q))} = 5^{2(1-\frac{1}{u})}\, q^{1-\frac{1}{u}}\, \|f\|_{\ell^u(\mathbb{H}_1(\mathbb{F}_q))}.$$

If $v \ge v_0$, then Lemma 2.1 gives $\|Tf\|_{\ell^v(\mathcal{D}_1)} \le \|Tf\|_{\ell^{v_0}(\mathcal{D}_1)}$, proving (c). □

9.2 The region $2 \le u \le \infty$ – upper bounds

Lemma 9.4. *Let $2 \le u \le \infty$.*

(a) *If $u \le v \le \infty$, then*

$$\|\mathcal{M}^{\mathrm{rd}}_{\mathbb{H}_1} F\|_{\ell^v(\mathcal{D}_1)} \le 5^{\frac{2}{u}}\, q^{1-\frac{1}{u}}\, \|F\|_{\ell^u(\mathbb{H}_1(\mathbb{F}_q))}.$$

(b) *If $1 \le v \le u$, then*

$$\|\mathcal{M}^{\mathrm{rd}}_{\mathbb{H}_1} F\|_{\ell^v(\mathcal{D}_1)} \le 2^{\frac{1}{v}-\frac{1}{u}} \cdot 5^{\frac{2}{u}}\, q^{1+\frac{2}{v}-\frac{3}{u}}\, \|F\|_{\ell^u(\mathbb{H}_1(\mathbb{F}_q))}.$$

Proof. By Lemma 9.2 (since $u \ge 2$),

$$\|\mathcal{M}^{\mathrm{rd}}_{\mathbb{H}_1} F\|_{\ell^u(\mathcal{D}_1)} \le 5^{\frac{2}{u}}\, q^{1-\frac{1}{u}}\, \|F\|_{\ell^u(\mathbb{H}_1(\mathbb{F}_q))}.$$

(a) If $v \geq u$, then Lemma 2.1 gives $\|g\|_{\ell^v(\mathcal{D}_1)} \leq \|g\|_{\ell^u(\mathcal{D}_1)}$ for all g on $\mathcal{D}_1$. Apply this to $g = \mathcal{M}^{\mathrm{rd}}_{\mathbb{H}_1} F$ to obtain (a).

(b) If $v \leq u$, then Lemma 2.1 gives

$$\|g\|_{\ell^v(\mathcal{D}_1)} \leq |\mathcal{D}_1|^{\frac{1}{v}-\frac{1}{u}} \, \|g\|_{\ell^u(\mathcal{D}_1)} \leq 2^{\frac{1}{v}-\frac{1}{u}} \, q^{2(\frac{1}{v}-\frac{1}{u})} \, \|g\|_{\ell^u(\mathcal{D}_1)}.$$

Apply this to $g = \mathcal{M}^{\mathrm{rd}}_{\mathbb{H}_1} F$ and combine with the diagonal bound above:

$$\|\mathcal{M}^{\mathrm{rd}}_{\mathbb{H}_1} F\|_{\ell^v(\mathcal{D}_1)} \leq 2^{\frac{1}{v}-\frac{1}{u}} \cdot 5^{\frac{2}{u}} \, q^{2(\frac{1}{v}-\frac{1}{u})} \, q^{1-\frac{1}{u}} \, \|F\|_{\ell^u(\mathbb{H}_1(\mathbb{F}_q))} = 2^{\frac{1}{v}-\frac{1}{u}} \cdot 5^{\frac{2}{u}} \, q^{1+\frac{2}{v}-\frac{3}{u}} \, \|F\|_{\ell^u(\mathbb{H}_1(\mathbb{F}_q))}.$$

This completes the proof. □

We now record four test functions forcing the four terms in (10).

9.3 The case $A_1^{\mathrm{rd}}(u,v) \;\geq\; \frac{1}{v}$

Lemma 9.5. *For every $1 \leq u, v \leq \infty$, one has*

$$A_1^{\mathrm{rd}}(u,v) \;\geq\; \frac{1}{v}.$$

Proof. Let $\mathbf{p}_*$ be any fixed point in $\mathbb{H}_1(\mathbb{F}_q)$ and take $F = \delta_{\mathbf{p}_*}$. Then $\|F\|_{\ell^u(\mathbb{H}_1(\mathbb{F}_q))} = 1$. Note that there are exactly $q+1$ different refined directions $\omega \in \mathcal{D}_1$ for the horizontal lines through $\mathbf{p}_*$. For such ω, one has $(\mathcal{M}^{\mathrm{rd}}_{\mathbb{H}_1} F)(\omega) \geq 1$. Therefore,

$$\|\mathcal{M}^{\mathrm{rd}}_{\mathbb{H}_1} F\|_{\ell^v(\mathcal{D}_1)} \;\geq\; (q+1)^{\frac{1}{v}} \;\geq\; q^{\frac{1}{v}}.$$

This forces $q^{A_1^{\mathrm{rd}}(u,v)} \gtrsim q^{\frac{1}{v}}$, and hence $A_1^{\mathrm{rd}}(u,v) \geq \frac{1}{v}$. □

9.4 The case $A_1^{\mathrm{rd}}(u,v) \;\geq\; 1 - \frac{1}{u}$

Lemma 9.6. *For every $1 \leq u, v \leq \infty$, one has*

$$A_1^{\mathrm{rd}}(u,v) \;\geq\; 1 - \frac{1}{u}.$$

Proof. Let $L \subset \mathbb{H}_1(\mathbb{F}_q)$ be a fixed horizontal line and set $F := \mathbf{1}_L$. Then $\|F\|_{\ell^u(\mathbb{H}_1(\mathbb{F}_q))} = |L|^{\frac{1}{u}} = q^{\frac{1}{u}}$. Let $\omega_0 := \mathrm{Dir}(L)$. Since the maximum in $\mathcal{M}^{\mathrm{rd}}_{\mathbb{H}_1}$ ranges over all horizontal lines of refined direction ω_0, we have

$$(\mathcal{M}^{\mathrm{rd}}_{\mathbb{H}_1} F)(\omega_0) \;\geq\; \sum_{\mathbf{p} \in L} \mathbf{1}_L(\mathbf{p}) \;=\; q.$$

Hence, $\|\mathcal{M}^{\mathrm{rd}}_{\mathbb{H}_1} F\|_{\ell^v(\mathcal{D}_1)} \geq q$ for every v. Therefore $q^{A_1^{\mathrm{rd}}(u,v)} \gtrsim q/q^{\frac{1}{u}}$, and the claim follows. □

9.5 The case $A_1^{\mathrm{rd}}(u,v) \;\geq\; \frac{2}{v} - \frac{1}{u}$

Lemma 9.7. *For every $1 \leq u, v \leq \infty$, one has*

$$A_1^{\mathrm{rd}}(u,v) \;\geq\; \frac{2}{v} - \frac{1}{u}.$$

Proof. Let $\ell_1, \ell_2 \subset \mathbb{F}_q^2$ be two non-parallel affine lines and set $B := \ell_1 \cup \ell_2$. Then $|B| = 2q - 1 \sim q$. Define

$$E := \{(x, y, 0) \in \mathbb{H}_1(\mathbb{F}_q) : (x, y) \in B\}, \qquad F := \mathbf{1}_E.$$

Then $\|F\|_{\ell^u(\mathbb{H}_1(\mathbb{F}_q))} = |E|^{\frac{1}{u}} \sim q^{\frac{1}{u}}$.

We claim that $(\mathcal{M}^{\mathrm{rd}}_{\mathbb{H}_1} F)(\omega) \geq 1$ for every $\omega \in \mathcal{D}_1$. Fix $\omega = [a : b : c] \in \mathcal{D}_1$, and let $\ell_\omega \subset \mathbb{F}_q^2$ be the corresponding affine line. If ℓ_ω is not parallel to ℓ_1, then $\ell_\omega \cap \ell_1 \neq \emptyset$; otherwise ℓ_ω intersects ℓ_2 since ℓ_2 is not parallel to ℓ_1. In either case we may choose $(x_\omega, y_\omega) \in \ell_\omega \cap B$, so $(x_\omega, y_\omega, 0) \in E$. By (37), the horizontal line $L_\omega := L_{(x_\omega, y_\omega, 0), [a:b]}$ has refined direction ω. Hence

$$(\mathcal{M}^{\mathrm{rd}}_{\mathbb{H}_1} F)(\omega) \;\geq\; \sum_{\mathbf{p} \in L_\omega} \mathbf{1}_E(\mathbf{p}) \;\geq\; 1.$$

Therefore,

$$\|\mathcal{M}^{\mathrm{rd}}_{\mathbb{H}_1} F\|_{\ell^v(\mathcal{D}_1)} \;\geq\; |\mathcal{D}_1|^{\frac{1}{v}} \;\sim\; q^{\frac{2}{v}},$$

using $|\mathcal{D}_1| = q^2 + q$.

This forces $q^{A_1^{\mathrm{rd}}(u,v)} \gtrsim q^{\frac{2}{v}} / q^{\frac{1}{u}}$, and hence $A_1^{\mathrm{rd}}(u, v) \geq \frac{2}{v} - \frac{1}{u}$. □

9.6 The case $A_1^{\mathrm{rd}}(u, v) \;\geq\; 1 + \frac{2}{v} - \frac{3}{u}$

Lemma 9.8. *For every $1 \leq u, v \leq \infty$, one has*

$$A_1^{\mathrm{rd}}(u, v) \;\geq\; 1 + \frac{2}{v} - \frac{3}{u}.$$

Proof. Let $F \equiv 1$ on $\mathbb{H}_1(\mathbb{F}_q)$. Then $\|F\|_{\ell^u(\mathbb{H}_1(\mathbb{F}_q))} = |\mathbb{H}_1(\mathbb{F}_q)|^{\frac{1}{u}} = q^{\frac{3}{u}}$. For each $\omega \in \mathcal{D}_1$, choose any horizontal line $L_\omega \in \mathcal{L}_\omega$. Then $\sum_{\mathbf{p} \in L_\omega} |F(\mathbf{p})| = |L_\omega| = q$, so $(\mathcal{M}^{\mathrm{rd}}_{\mathbb{H}_1} F)(\omega) = q$ for all $\omega \in \mathcal{D}_1$. Hence

$$\|\mathcal{M}^{\mathrm{rd}}_{\mathbb{H}_1} F\|_{\ell^v(\mathcal{D}_1)} = q\, |\mathcal{D}_1|^{\frac{1}{v}} \sim q \cdot q^{\frac{2}{v}} = q^{1 + \frac{2}{v}}.$$

Therefore $q^{A_1^{\mathrm{rd}}(u,v)} \gtrsim q^{1+\frac{2}{v}} / q^{\frac{3}{u}}$, giving the claim.

□

Corollary 9.9. *For every $1 \leq u, v \leq \infty$, one has*

$$A_1^{\mathrm{rd}}(u, v) \;\geq\; \max\Big(\frac{1}{v},\; 1 - \frac{1}{u},\; \frac{2}{v} - \frac{1}{u},\; 1 + \frac{2}{v} - \frac{3}{u}\Big).$$

Proof. This follows immediately from Lemmas 9.5–9.8. □

9.7 Proof of Theorem 1.7

Proof of Theorem 1.7. For every $1 \leq u, v \leq \infty$, it follows from Lemmas 9.5–9.8 that

$$A_1^{\mathrm{rd}}(u, v) \;\geq\; \max\Big(\frac{1}{v},\; 1 - \frac{1}{u},\; \frac{2}{v} - \frac{1}{u},\; 1 + \frac{2}{v} - \frac{3}{u}\Big).$$

For the matching upper bound, Lemmas 9.3 and 9.4 give

$$A_1^{\mathrm{rd}}(u, v) \;\leq\; \max\Big(\frac{1}{v},\; 1 - \frac{1}{u},\; \frac{2}{v} - \frac{1}{u},\; 1 + \frac{2}{v} - \frac{3}{u}\Big),$$

and the theorem follows. □

10 Proof of Theorems 1.8 and 1.9

Proof of Theorem 1.8. Let $F := \mathbf{1}_E$. For each $\omega \in \Omega$, the hypothesis gives a horizontal line L with $\mathrm{Dir}(L) = \omega$ and $|E \cap L| \geq m$, hence

$$(\mathcal{M}^{\mathrm{rd}}_{\mathbb{H}_1} F)(\omega) \geq m.$$

Therefore,

$$\|\mathcal{M}^{\mathrm{rd}}_{\mathbb{H}_1} F\|_{\ell^v(\mathcal{D}_1)} \geq \|\mathcal{M}^{\mathrm{rd}}_{\mathbb{H}_1} F\|_{\ell^v(\Omega)} \geq m \, \|\mathbf{1}_\Omega\|_{\ell^v(\Omega)} = m \, |\Omega|^{1/v},$$

with the usual convention $|\Omega|^{1/\infty} = 1$.

On the other hand, by (9),

$$\|\mathcal{M}^{\mathrm{rd}}_{\mathbb{H}_1} F\|_{\ell^v(\mathcal{D}_1)} \leq C_{u,v} \, q^{A_1^{\mathrm{rd}}(u,v)} \, \|F\|_{\ell^u(\mathbb{H}_1(\mathbb{F}_q))}.$$

Since $\|F\|_{\ell^u(\mathbb{H}_1(\mathbb{F}_q))} = |E|^{\frac{1}{u}}$, we obtain

$$m \, |\Omega|^{1/v} \leq C_{u,v} \, q^{A_1^{\mathrm{rd}}(u,v)} \, |E|^{\frac{1}{u}}.$$

Rearranging gives (11). This completes the proof. □

Proof of Theorem 1.9. By Theorem 1.7,

$$A_1^{\mathrm{rd}}\left(\frac{s}{s-1}, s\right) = \frac{1}{s}.$$

Apply (9) with

$$u = \frac{s}{s-1}, \qquad v = s,$$

we have

$$\|M_E\|_{\ell^s(\mathcal{D}_1)} \leq C_{\frac{s}{s-1}, s} \, q^{1/s} \, |E|^{1-1/s},$$

and raising to the sth power gives the desired result. □

11 Discussions and examples

11.1 Examples distinguishing affine Kakeya and refined-direction horizontal Kakeya

In this subsection, we record examples showing that the usual affine Kakeya in $\mathbb{F}_q^3$ and the full refined-direction horizontal Heisenberg Kakeya in $\mathbb{H}_1(\mathbb{F}_q)$ are incomparable: neither condition implies the other.

We use the standard coordinate identification of sets

$$\mathbb{H}_1(\mathbb{F}_q) \cong \mathbb{F}_q^3, \qquad \mathbf{p} = (x, y, t).$$

Recall that the set of refined directions is

$$\mathcal{D}_1 = \{[a : b : c] \in \mathbb{P}^2(\mathbb{F}_q) : (a, b) \neq (0, 0)\}.$$

If $L = L_{\mathbf{p},[(a,b)]}$ is a horizontal line (with horizontal direction $(a,b) \neq (0,0)$), we write

$$\mathrm{Dir}(L) = [a : b : c(L)] \in \mathcal{D}_1, \qquad c(L) := x_0 b - y_0 a \quad \text{for } \mathbf{p} = (x_0, y_0, t_0) \in L.$$

As discussed at the beginning of Section 8, every $\omega \in \mathcal{D}_1$ admits a unique normal form

$$\omega = [1 : m : \gamma] \quad (m, \gamma \in \mathbb{F}_q), \qquad \text{or} \qquad \omega = [0 : 1 : \gamma] \quad (\gamma \in \mathbb{F}_q),$$

and the family of horizontal lines with refined direction ω is precisely

$$\begin{aligned} L_{\omega,\tau} &= \{(x,\ mx - \gamma,\ \tau + \gamma x) : x \in \mathbb{F}_q\}, && \text{if } \omega = [1 : m : \gamma], \\ L_{\omega,\tau} &= \{(\gamma,\ y,\ \tau + \gamma y) : y \in \mathbb{F}_q\}, && \text{if } \omega = [0 : 1 : \gamma], \end{aligned}$$

where $\tau \in \mathbb{F}_q$ parametrizes the q distinct horizontal lines of refined direction ω.

Example 11.1 (Affine Kakeya but not full refined-direction horizontal Kakeya)**.** *Fix $\omega_0 \in \mathcal{D}_1$. Write ω_0 in normal form as $\omega_0 = [1 : m_0 : \gamma_0]$ or $\omega_0 = [0 : 1 : \gamma_0]$. Define a subset $S_{\omega_0} \subset \mathbb{H}_1(\mathbb{F}_q)$ by*

$$S_{\omega_0} := \begin{cases} \{(0,\ -\gamma_0, t) : t \in \mathbb{F}_q\}, & \text{if } \omega_0 = [1 : m_0 : \gamma_0], \\ \{(\gamma_0, 0, t) : t \in \mathbb{F}_q\}, & \text{if } \omega_0 = [0 : 1 : \gamma_0]. \end{cases}$$

Let $E := \mathbb{H}_1(\mathbb{F}_q) \setminus S_{\omega_0}$. Then, viewed as a subset of $\mathbb{F}_q^3$, the set E is an affine Kakeya set, but E is not a full refined-direction horizontal Heisenberg Kakeya set in the sense of Definition 1.3.

Proof. We first verify that E is not full refined-direction horizontal Heisenberg Kakeya. Consider the family $\{L_{\omega_0,\tau}\}_{\tau \in \mathbb{F}_q}$ of all horizontal lines with refined direction ω_0. If $\omega_0 = [1 : m_0 : \gamma_0]$, then for every $\tau \in \mathbb{F}_q$ one has

$$(0,\ -\gamma_0, \tau) \in L_{\omega_0,\tau} \cap S_{\omega_0}.$$

If $\omega_0 = [0 : 1 : \gamma_0]$, then for every $\tau \in \mathbb{F}_q$ one has

$$(\gamma_0, 0, \tau) \in L_{\omega_0,\tau} \cap S_{\omega_0}.$$

In either case, every horizontal line L with $\mathrm{Dir}(L) = \omega_0$ meets S_{ω_0}, and hence no such line can be contained in E. Therefore, E fails the defining condition in Definition 1.3 at the refined direction ω_0.

We next show that E is affine Kakeya in $\mathbb{F}_q^3$. Fix an ambient direction $[\mathbf{v}] \in \mathbb{P}^2(\mathbb{F}_q)$ and choose a representative $\mathbf{v} \in \mathbb{F}_q^3 \setminus \{0\}$. The affine lines in direction $[\mathbf{v}]$ are the cosets of the 1-dimensional subspace $\langle \mathbf{v} \rangle$, and these q^2 lines form a partition of $\mathbb{F}_q^3$. Since $|S_{\omega_0}| = q$, at most q of these q^2 lines meet S_{ω_0} (each point lies on exactly one line of direction $[\mathbf{v}]$). Hence, there exists an affine line ℓ of ambient direction $[\mathbf{v}]$ such that $\ell \cap S_{\omega_0} = \varnothing$, and therefore $\ell \subset E$. As $[\mathbf{v}]$ was arbitrary, E contains an affine line in every ambient direction, i.e. E is an affine Kakeya set. □

Example 11.2 (Full refined-direction horizontal Kakeya but not affine Kakeya)**.** *Assume q is odd and $q > 3$. For each $\omega \in \mathcal{D}_1$ in normal form, define a horizontal line L_ω by*

$$L_{[1:m:\gamma]} := \{(x,\ mx - \gamma,\ m^2 + \gamma x) : x \in \mathbb{F}_q\}, \qquad L_{[0:1:\gamma]} := \{(\gamma,\ y,\ \gamma y) : y \in \mathbb{F}_q\}.$$

Let

$$E := \bigcup_{\omega \in \mathcal{D}_1} L_\omega \ \subset \ \mathbb{H}_1(\mathbb{F}_q).$$

Then, E is a full refined-direction horizontal Heisenberg Kakeya set (Definition 1.3), but E is not an affine Kakeya set in $\mathbb{F}_q^3$.

Proof. It is clear that E is full refined-direction horizontal Heisenberg Kakeya.

We now show that E is not affine Kakeya by proving that E contains no vertical affine line, i.e. no affine line of direction $[0:0:1] \in \mathbb{P}^2(\mathbb{F}_q)$. Fix $(x_0, y_0) \in \mathbb{F}_q^2$ and consider the vertical fiber

$$V_{(x_0,y_0)} := \{(x_0, y_0, t) : t \in \mathbb{F}_q\}.$$

It suffices to show that $V_{(x_0,y_0)} \nsubseteq E$ for every (x_0, y_0).

Define

$$T(x_0, y_0) := \{t \in \mathbb{F}_q : \ (x_0, y_0, t) \in E\}.$$

We bound $|T(x_0, y_0)|$. First consider the contribution from directions of the form $\omega = [1 : m : \gamma]$. For a fixed $m \in \mathbb{F}_q$, the point (x_0, y_0, t) can lie on $L_{[1:m:\gamma]}$ only if $y_0 = mx_0 - \gamma$, i.e. $\gamma = mx_0 - y_0$. For this value of γ, the point of $L_{[1:m:\gamma]}$ with $x = x_0$ has

$$t = m^2 + \gamma x_0 = m^2 + (mx_0 - y_0)x_0 = m^2 + x_0^2 m - x_0 y_0.$$

Hence, the set of t-values arising from all such ω is contained in

$$A(x_0, y_0) := \left\{m^2 + x_0^2 m - x_0 y_0 : \ m \in \mathbb{F}_q\right\}.$$

Since q is odd, completing the square shows that $A(x_0, y_0)$ is a translate of the set of squares in $\mathbb{F}_q$, and therefore

$$|A(x_0, y_0)| \leq \frac{q+1}{2}.$$

Next consider directions of the form $\omega = [0 : 1 : \gamma]$. For any $x_0 \in \mathbb{F}_q$ there is a unique $\gamma = x_0$ such that $L_{[0:1:\gamma]}$ has first coordinate equal to x_0. Intersecting

$$L_{[0:1:x_0]} = \left\{\left(x_0, \ y, \ x_0 y\right) : y \in \mathbb{F}_q\right\}$$

with $y = y_0$ yields at most one additional value $t = x_0 y_0$.

Consequently,

$$|T(x_0, y_0)| \leq |A(x_0, y_0)| + 1 \leq \frac{q+1}{2} + 1 = \frac{q+3}{2} < q \qquad \text{since } q > 3.$$

Thus, $V_{(x_0,y_0)} \nsubseteq E$ for all (x_0, y_0), so E contains no vertical affine line. In particular, E does not contain an affine line in the ambient direction $[0:0:1]$, and hence E is not an affine Kakeya set in $\mathbb{F}_q^3$. □

The preceding examples show that, although $\mathbb{H}_1(\mathbb{F}_q)$ and $\mathbb{F}_q^3$ coincide as sets, the affine Kakeya property in $\mathbb{F}_q^3$ and the full refined-direction horizontal Heisenberg Kakeya property in $\mathbb{H}_1(\mathbb{F}_q)$ are incomparable.

Example 11.3 (Sharpness of the $\ell^3 \to \ell^3$ bound)**.** *Let $F \equiv 1$ on $\mathbb{H}_1(\mathbb{F}_q)$, i.e. $F = \mathbf{1}_{\mathbb{H}_1(\mathbb{F}_q)}$. Then*

$$\|\mathcal{M}^{\mathrm{rd}}_{\mathbb{H}_1} F\|_{\ell^3(\mathcal{D}_1)} = q\,|\mathcal{D}_1|^{1/3} \sim q^{5/3}, \qquad \|F\|_{\ell^3(\mathbb{H}_1(\mathbb{F}_q))} = |\mathbb{H}_1(\mathbb{F}_q)|^{1/3} = q.$$

Consequently,

$$\frac{\|\mathcal{M}^{\mathrm{rd}}_{\mathbb{H}_1} F\|_{\ell^3(\mathcal{D}_1)}}{\|F\|_{\ell^3(\mathbb{H}_1(\mathbb{F}_q))}} = |\mathcal{D}_1|^{\frac{1}{3}} = (q^2+q)^{\frac{1}{3}} \geq q^{\frac{2}{3}}.$$

In particular, any estimate of the form

$$\|\mathcal{M}^{\mathrm{rd}}_{\mathbb{H}_1} F\|_{\ell^3(\mathcal{D}_1)} \leq C\,q^{\alpha}\,\|F\|_{\ell^3(\mathbb{H}_1(\mathbb{F}_q))}$$

(valid uniformly in q) forces $\alpha \geq \frac{2}{3}$. Hence, the exponent $q^{2/3}$ in the $\ell^3 \to \ell^3$ bound is sharp up to absolute constants.

Proof. Fix $\omega \in \mathcal{D}_1$. As above, there exists a horizontal line L with $\mathrm{Dir}(L) = \omega$, and $|L| = q$. Since $F \equiv 1$, it follows that

$$(\mathcal{M}^{\mathrm{rd}}_{\mathbb{H}_1} F)(\omega) = \max_{\substack{L \text{ horizontal} \\ \mathrm{Dir}(L)=\omega}} \sum_{\mathbf{p}\in L} |F(\mathbf{p})| = q.$$

Thus, $\mathcal{M}^{\mathrm{rd}}_{\mathbb{H}_1} F \equiv q$ on $\mathcal{D}_1$, and therefore

$$\|\mathcal{M}^{\mathrm{rd}}_{\mathbb{H}_1} F\|_{\ell^3(\mathcal{D}_1)} = \Big(\sum_{\omega\in\mathcal{D}_1} q^3\Big)^{1/3} = q\,|\mathcal{D}_1|^{1/3}.$$

Also $|\mathbb{H}_1(\mathbb{F}_q)| = q^3$, so $\|F\|_{\ell^3(\mathbb{H}_1(\mathbb{F}_q))} = q$.

Finally, $|\mathcal{D}_1| = q^2 + q$, hence $|\mathcal{D}_1|^{1/3} = (q^2+q)^{1/3} \geq q^{2/3}$, which implies the stated sharpness. □

11.2 An outlook toward a new approach of the affine Kakeya problem in $\mathbb{F}_q^3$

Let $E \subset \mathbb{F}_q^3$ be an affine Kakeya set, and identify $\mathbb{F}_q^3$ with the underlying set of $\mathbb{H}_1(\mathbb{F}_q)$. Write $\mathcal{D}_1 = \mathbb{P}^2(\mathbb{F}_q) \setminus \{[0:0:1]\}$ for the set of non-vertical directions, and decompose $\mathcal{D}_1 = \Omega_1 \sqcup \Omega_2$ as follows: Ω_1 consists of those $\omega \in \mathcal{D}_1$ for which E contains a horizontal line $L \subset \mathbb{H}_1(\mathbb{F}_q)$ with $\mathrm{Dir}(L) = \omega$, and $\Omega_2 := \mathcal{D}_1 \setminus \Omega_1$.

Theorem 1.8 immediately controls the Ω_1 contribution. More precisely,

$$|E| \;\geq\; C^{-2}\,q\,|\Omega_1|.$$

In particular, $|\Omega_1| \gtrsim q^2$ already forces $|E| \gtrsim q^3$. Since $|\mathcal{D}_1| = q^2 + q$, at least one of Ω_1 or Ω_2 has size $\gtrsim q^2$; hence, for the purpose of proving a lower bound of order q^3, it suffices to treat the complementary range

$$|\Omega_2| \;\gtrsim\; q^2. \tag{59}$$

In this sense, the affine Kakeya problem in $\mathbb{F}_q^3$ is reduced to understanding an Ω_2-affine Kakeya configuration, in which a large proportion of directions is realized only by non-horizontal lines in $\mathbb{H}_1(\mathbb{F}_q)$.

The advantage of this viewpoint is that the Ω_2 lines satisfy a concrete structural constraint. Fix $\omega = [a:b:c] \in \Omega_2$ with $(a,b) \neq (0,0)$, and let

$$\ell_\omega = \{(x_0,y_0,t_0) + s(a,b,c) : s \in \mathbb{F}_q\} \subset E$$

be a Kakeya line of ambient direction ω contained in E. By definition of Ω_2, the line ℓ_ω is *not* horizontal in $\mathbb{H}_1(\mathbb{F}_q)$. Equivalently, its basepoint fails the horizontality relation

$$bx_0 - ay_0 \;\neq\; c.$$

Thus, the Kakeya line in direction $[a : b : c]$ is forced to choose its basepoint away from the affine line $\{(x, y) \in \mathbb{F}_q^2 : \; bx - ay = c\}$ in the (x, y) plane. We refer to this as a *basepoint-avoidance* constraint.

To quantify this failure of horizontality in a way that is intrinsic to the affine line (independent of the chosen point on the line and independent of scaling of (a, b, c)), we define a scalar parameter $\mu(\ell) \in \mathbb{F}_q$ using the normal-form coordinates on $\mathcal{D}_1$. Write the direction of ℓ in the unique normal form

$$\omega = [1 : m : \gamma] \quad (m, \gamma \in \mathbb{F}_q), \qquad \text{or} \qquad \omega = [0 : 1 : \gamma] \quad (\gamma \in \mathbb{F}_q).$$

If ℓ has direction $\omega = [1 : m : \gamma]$, pick any point $(x_0, y_0, t_0) \in \ell$, and set

$$\mu(\ell) \;:=\; \gamma - (mx_0 - y_0) \in \mathbb{F}_q.$$

If ℓ has direction $\omega = [0 : 1 : \gamma]$, pick any point $(x_0, y_0, t_0) \in \ell$, and set

$$\mu(\ell) \;:=\; \gamma - x_0 \in \mathbb{F}_q.$$

A direct check shows that $\mu(\ell)$ does not depend on the choice of $(x_0, y_0, t_0) \in \ell$. Moreover, $\mu(\ell) = 0$ if and only if ℓ is horizontal in $\mathbb{H}_1(\mathbb{F}_q)$. In particular, for $\omega \in \Omega_2$ we necessarily have $\mu(\ell_\omega) \in \mathbb{F}_q^*$.

Now fix a choice of the Kakeya line $\ell_\omega \subset E$ for each $\omega \in \Omega_2$, and define

$$k(\omega) := \mu(\ell_\omega) \in \mathbb{F}_q^*, \qquad \Lambda_k := \{\omega \in \Omega_2 : \; k(\omega) = k\}.$$

Then, $\Omega_2 = \bigsqcup_{k \in \mathbb{F}_q^*} \Lambda_k$ is a partition.

For a fixed $k \in \mathbb{F}_q^*$, lines with $\mu(\ell) = k$ can be *straightened* into horizontal lines by an explicit mapping: in the chart $\omega = [1 : m : \gamma]$, one uses $(x, y, t) \mapsto (x, y, t - kx)$, while in the chart $\omega = [0 : 1 : \gamma]$, one uses $(x, y, t) \mapsto (x, y, t - ky)$. Thus, each slice Λ_k may be viewed (after such a mapping) as contributing a horizontal Kakeya configuration, to which Theorem 1.6 applies. In particular, if $|\Lambda_k| \gtrsim q^2$ for some k, then this immediately forces $|E| \gtrsim q^3$.

In the remaining range, where $|\Omega_2| \gtrsim q^2$, but each individual slice $|\Lambda_k|$ is much smaller than q^2, one must exploit how the different μ-slices overlap inside E. Equivalently, one needs a mechanism that controls cross-incidences between the families of lines with $\mu = k$ and $\mu = k'$ for $k \neq k'$. This is the point at which the basepoint-avoidance constraint may provide additional structure not visible in the usual affine formulation. We will expand the discussion in the next arXiv revision.

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