LOCALLY CONFORMALLY FLAT WARPED MANIFOLDS

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We show that a locally conformally flat Riemannian manifold which is a warped product on an interval of $\mathbb{R}$, has a warped factor with constant curvature.

Theorem 1. *Let $M = I \times N$, the dimension of M is larger than three and $g = dt^2 \otimes h^2(t,p)\sigma$ for some metric σ on N and a positive function $h : I \times M \to \mathbb{R}$.*
If (M, g) is locally conformally flat then there exists a constant curvature metric g^K on N and a positive function $f : I \to \mathbb{R}$ such that $g = dt^2 \otimes f^2(t)g^K$.

Proof. We compute easily,

$$g^{tt} = 1 \qquad g^{it} = 0 \qquad g^{ij} = h^{-2}\sigma^{ij}$$

$$\begin{aligned}
\Gamma^t_{tt} &= 0 \qquad \Gamma^k_{tt} = 0 \qquad \Gamma^t_{ti} = 0\\
\Gamma^t_{ij} &= -\sigma_{ij}h\partial_t h = -\frac{\partial_t h}{h}g_{ij}\\
\Gamma^k_{it} &= \frac{\partial_t h}{h}\delta^k_i\\
\Gamma^k_{ij} &= \sigma^k_{ij} + \frac{1}{h}(\delta^k_j\partial_i h + \delta^k_i\partial_j h - \sigma^{ks}\sigma_{ij}\partial_s h)
\end{aligned}$$

where we denoted with σ^k_{ij} the Christoffel symbols of the metric σ.
By the formula

$$\mathrm{R}_{abcd} = (\partial_a\Gamma^p_{bd} - \partial_b\Gamma^p_{ad} + \Gamma^q_{bd}\Gamma^p_{qa} - \Gamma^q_{ad}\Gamma^p_{qb})g_{pc}$$

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we get, supposing to be in normal coordinates with respect to the metric σ on N (that is, $\sigma^k_{ij}=0$),

$$
\begin{aligned}
\mathrm{R}_{tttt} &= \mathrm{R}_{ittt} = 0\\
\mathrm{R}_{itkt} &= (\partial_i\Gamma^p_{tt} - \partial_t\Gamma^p_{it} + \Gamma^q_{tt}\Gamma^p_{qi} - \Gamma^q_{it}\Gamma^p_{qt})g_{pk}\\
&= -\,(\partial_t\Gamma^p_{it} + \Gamma^q_{it}\Gamma^p_{qt})g_{pk}\\
&= -\left(\partial_t\frac{\partial_t h}{h}\delta^p_i + \frac{\partial_t h}{h}\delta^q_i\frac{\partial_t h}{h}\delta^p_q\right)g_{pk}\\
&= -\left(\partial^2_t\log h + \left(\frac{\partial_t h}{h}\right)^2\right)g_{ik}\\
&= -\,\frac{\partial^2_t h}{h}g_{ik} = -h\partial^2_t h\sigma_{ik}\\
\mathrm{R}_{ijkt} &= -\,\mathrm{R}_{ijtk}\\
&= -\,\partial_i\Gamma^t_{jk} + \partial_j\Gamma^t_{ik} - \Gamma^l_{jk}\Gamma^t_{il} + \Gamma^l_{ik}\Gamma^t_{jl}\\
&= h\partial^2_{it}h\sigma_{jk} - h\partial^2_{jt}h\sigma_{ik} + \partial_t h(\sigma_{ik}\partial_j h - \sigma_{jk}\partial_i h)\\
&= h(\sigma_{jk}\mathrm{Hess}_{it}h - \sigma_{ik}\mathrm{Hess}_{jt}h) = \frac{g_{jk}\mathrm{Hess}_{it}h - g_{ik}\mathrm{Hess}_{jt}h}{h}\\
\mathrm{R}_{ijkl} &= (\partial_i\Gamma^p_{jl} - \partial_j\Gamma^p_{il} + \Gamma^q_{jl}\Gamma^p_{qi} - \Gamma^q_{il}\Gamma^p_{qj})g_{pk} + (\Gamma^t_{jl}\Gamma^p_{ti} - \Gamma^t_{il}\Gamma^p_{tj})g_{pk}\\
&= h^2\mathrm{R}^\sigma_{ijkl} + \left(\partial_i\left[\frac{1}{h}(\delta^p_j\partial_l h + \delta^p_l\partial_j h - \sigma^{ps}\sigma_{jl}\partial_s h)\right] - \partial_j\left[\frac{1}{h}(\delta^p_i\partial_l h + \delta^p_l\partial_i h - \sigma^{ps}\sigma_{il}\partial_s h)\right]\right)g_{pk}\\
&\quad+\frac{1}{h^2}\left\{\left(\delta^q_j\partial_l h + \delta^q_l\partial_j h - \sigma^{qs}\sigma_{jl}\partial_s h\right)\left(\delta^p_q\partial_i h + \delta^p_i\partial_q h - \sigma^{ps}\sigma_{qi}\partial_s h\right)\right.\\
&\qquad\left.-\,(\delta^q_i\partial_l h + \delta^q_l\partial_i h - \sigma^{qs}\sigma_{il}\partial_s h)\left(\delta^p_q\partial_j h + \delta^p_j\partial_q h - \sigma^{ps}\sigma_{qj}\partial_s h\right)\right\}g_{pk}\\
&\quad-(\partial_t h)^2(\sigma_{jl}g_{ik} - \sigma_{il}g_{jk})\\
&= h^2\mathrm{R}^\sigma_{ijkl} - \frac{1}{h^2}\left(\partial_i h\left[\delta^p_j\partial_l h + \delta^p_l\partial_j h - \sigma^{ps}\sigma_{jl}\partial_s h\right] - \partial_j h\left[\delta^p_i\partial_l h + \delta^p_l\partial_i h - \sigma^{ps}\sigma_{il}\partial_s h\right]\right)g_{pk}\\
&\quad+\frac{1}{h}\left(\delta^p_j\partial^2_{il}h + \delta^p_l\partial^2_{ij}h - \sigma^{ps}\sigma_{jl}\partial^2_{is}h - \delta^p_i\partial^2_{jl}h - \delta^p_l\partial^2_{ji}h - \sigma^{ps}\sigma_{il}\partial^2_{js}h)\right)g_{pk}\\
&\quad+\frac{1}{h^2}\left(g_{ik}\partial_j h\partial_l h + g_{jl}\partial_i h\partial_k h - g_{il}\partial_j h\partial_k h - g_{jk}\partial_i h\partial_l h - |\widetilde{\nabla}h|^2_g(g_{jl}g_{ik} - g_{il}g_{jk})\right)\\
&\quad-h^2\frac{(\partial_t h)^2}{2}(\sigma\circ\sigma)_{ijkl}\\
&= h^2\mathrm{R}^\sigma_{ijkl} - \frac{1}{h^2}\left(g_{jk}\partial_i h\partial_l h - g_{jl}\partial_i h\partial_k h - g_{ik}\partial_j h\partial_l h + g_{il}\partial_j h\partial_k h\right)\\
&\quad+\frac{1}{h}\left(g_{jk}\partial^2_{il}h - g_{jl}\partial^2_{ik}h - g_{ik}\partial^2_{jl}h + g_{il}\partial^2_{jk}h\right)\\
&\quad+\frac{1}{h^2}\left(g_{ik}\partial_j h\partial_l h + g_{jl}\partial_i h\partial_k h - g_{il}\partial_j h\partial_k h - g_{jk}\partial_i h\partial_l h - |\widetilde{\nabla}h|^2_g(g_{jl}g_{ik} - g_{il}g_{jk})\right)\\
&\quad-h^2\frac{(\partial_t h)^2}{2}(\sigma\circ\sigma)_{ijkl}\\
&= h^2\mathrm{R}^\sigma_{ijkl} - \frac{|\widetilde{\nabla}h|^2_g + (\partial_t h)^2}{2h^2}(g\circ g)_{ijkl} + \frac{1}{h}\left(g_{jk}\partial^2_{il}h - g_{jl}\partial^2_{ik}h - g_{ik}\partial^2_{jl}h + g_{il}\partial^2_{jk}h\right)\\
&\quad+\frac{2}{h^2}\left(g_{ik}\partial_j h\partial_l h - g_{il}\partial_j h\partial_k h - g_{jk}\partial_i h\partial_l h + g_{jl}\partial_i h\partial_k h\right)
\end{aligned}
$$

where $\widetilde{\nabla} h$ is the gradient of h with respect to N only and $\circ$ denotes the Kulkarni–Nomizu product of a pair of symmetric bilinear forms.

Since we assumed to be in normal coordinates on (N,σ), the Christoffel symbols of σ are zero and we can rewrite

$$\mathrm{R}_{ijkl} = h^2\mathrm{R}^\sigma_{ijkl} - (|\widetilde{\nabla} h|^2_\sigma + h^2(\partial_t h)^2)(\sigma\circ\sigma)_{ijkl}/2 - h\left[\left(\mathrm{Hess}^\sigma h - 2\frac{dh^\sigma\otimes dh^\sigma}{h}\right)\circ\sigma\right]_{ijkl}.$$

As $\mathrm{W}=0$ we have that $\mathrm{Riem} = Z\circ g$ for some symmetric bilinear form Z on M, hence, "freezing" the variable t, we have

$$\mathrm{Riem}^\sigma = \left(Z + \frac{|\widetilde{\nabla} h|^2_\sigma}{2h^2}\sigma + \frac{(\partial_t h)^2}{2}\sigma + \frac{\mathrm{Hess}^\sigma h}{h} - 2\frac{dh^\sigma\otimes dh^\sigma}{h^2}\right)\circ\sigma$$

which implies, by the decomposition formula of the Riemann tensor, that $\mathrm{W}^\sigma = 0$, that is, also (N,σ) is LCF.

We now trace with g to get the Ricci tensor

$$\begin{aligned}
\mathrm{R}_{tt} &= g^{ik}\mathrm{R}_{itkt} = -g^{ik}\frac{\partial^2_t h}{h}g_{ik} = -(n-1)\frac{\partial^2_t h}{h}\\
\mathrm{R}_{it} &= g^{jl}\mathrm{R}_{ijtl} + g^{tt}\mathrm{R}_{ittt} = -g^{jl}\frac{g_{jl}\mathrm{Hess}_{it}h - g_{il}\mathrm{Hess}_{jt}h}{h} = -(n-2)\frac{\mathrm{Hess}_{it}}{h}\\
\mathrm{R}_{ik} &= \mathrm{R}_{itkt} + g^{jl}\mathrm{R}_{ijkl}\\
&= -\frac{\partial^2_t h}{h}g_{ik} + g^{jl}\left(h^2\mathrm{R}^\sigma_{ijkl} - (|\widetilde{\nabla} h|^2_\sigma + h^2(\partial_t h)^2)(\sigma\circ\sigma)_{ijkl}/2\right)\\
&\quad - hg^{jl}\left[\left(\mathrm{Hess}^\sigma h - 2\frac{dh^\sigma\otimes dh^\sigma}{h}\right)\circ\sigma\right]_{ijkl}\\
&= -\frac{\partial^2_t h}{h}g_{ik} + \mathrm{R}^\sigma_{ik} - (n-2)(\partial_t h)^2\sigma_{ik} - (n-2)\frac{|\widetilde{\nabla} h|^2_\sigma}{h^2}\sigma_{ik} - \frac{\Delta^\sigma h}{h}\sigma_{ik} + 2\frac{|\widetilde{\nabla} h|^2_\sigma}{h^2}\sigma_{ik}\\
&\quad - (n-3)\left(\frac{\mathrm{Hess}^\sigma_{ik}h}{h} - 2\frac{dh^\sigma_i dh^\sigma_k}{h^2}\right)\\
&= -\frac{\partial^2_t h}{h}g_{ik} + \mathrm{R}^\sigma_{ik} - (n-2)(\partial_t h)^2\sigma_{ik} - (n-4)\frac{|\widetilde{\nabla} h|^2_\sigma}{h^2}\sigma_{ik} - \frac{\Delta^\sigma h}{h}\sigma_{ik}\\
&\quad - (n-3)\left(\frac{\mathrm{Hess}^\sigma_{ik}h}{h} - 2\frac{dh^\sigma_i dh^\sigma_k}{h^2}\right).
\end{aligned}$$

Now, by Theorem 1.159 in [1], about the transformation rules under a conformal change of metric we have, passing from σ to σh^2 on N (freezing the t variable),

$$\begin{aligned}
\mathrm{R}^{\sigma h^2}_{ik} &= \mathrm{R}^\sigma_{ik} - (n-3)\left(\mathrm{Hess}^\sigma_{ik}\log h - \frac{dh^\sigma_i\, dh^\sigma_k}{h^2}\right) - (\Delta^\sigma\log h + (n-3)|\widetilde{\nabla}\log h|^2_\sigma)\sigma_{ik}\\
&= \mathrm{R}^\sigma_{ik} - (n-3)\left(\frac{\mathrm{Hess}^\sigma_{ik}h}{h} - 2\frac{dh^\sigma_i\, dh^\sigma_k}{h^2}\right) - \left(\frac{\Delta^\sigma h}{h} + (n-4)\frac{|\widetilde{\nabla} h|^2_\sigma}{h^2}\right)\sigma_{ik},
\end{aligned}$$

so we can conclude

$$\mathrm{R}_{ik} = -(h\partial_t^2 h + (n-2)(\partial_t h)^2)\sigma_{ik} + \mathrm{R}_{ik}^{\sigma h^2}\,.$$

Contracting again to get the scalar curvature,

$$\begin{aligned}\mathrm{R} =& \mathrm{R}_{tt} - g^{ik}((h\partial_t^2 h + (n-2)(\partial_t h)^2)\sigma_{ik} + \mathrm{R}_{ik}^{\sigma h^2}) \\ =& -2(n-1)\frac{\partial_t^2 h}{h} - (n-1)(n-2)\frac{(\partial_t h)^2}{h^2} + \mathrm{R}^{\sigma h^2}\,.\end{aligned} \tag{1}$$

If now we look at R_{itkt} and we consider that $\mathrm{W}_{itkt} = 0$, by the decomposition of the Riemann tensor we get

$$\begin{aligned}\mathrm{R}_{itkt} =& -h\partial_t^2 h\sigma_{ik} \\ =& \frac{1}{n-2}(\mathrm{R}_{ik} + \mathrm{R}_{tt}g_{ik}) - \frac{\mathrm{R}}{(n-1)(n-2)}g_{ij} \\ =& \frac{1}{n-2}(-(h\partial_t^2 h + (n-2)(\partial_t h)^2)\sigma_{ik} + \mathrm{R}_{ik}^{\sigma h^2} - (n-1)h\partial_t^2 h\sigma_{ik}) \\ &+ \frac{g_{ij}}{(n-1)(n-2)}\left(2(n-1)\frac{\partial_t^2 h}{h} + (n-1)(n-2)\frac{(\partial_t h)^2}{h^2} - \mathrm{R}^{\sigma h^2}\right) \\ =& -\frac{nh\partial_t^2 h}{n-2}\sigma_{ik} - (\partial_t h)^2\sigma_{ik} + \frac{\mathrm{R}_{ik}^{\sigma h^2}}{n-2} + \frac{2\partial_t^2 h}{n-2}\sigma_{ik} + (\partial_t h)^2\sigma_{ik} - \frac{\mathrm{R}^{\sigma h^2}}{(n-1)(n-2)}\sigma_{ij}h^2 \\ =& -h\partial_t^2 h\sigma_{ik} + \frac{\mathrm{R}_{ik}^{\sigma h^2}}{n-2} - \frac{\mathrm{R}^{\sigma h^2}}{(n-1)(n-2)}\sigma_{ij}h^2\end{aligned}$$

which clearly implies

$$\mathrm{R}_{ik}^{\sigma h^2} = \frac{\mathrm{R}^{\sigma h^2}}{n-1}\sigma_{ij}h^2$$

hence, as $(n-1)$ is at least three, by hypothesis, for every fixed $t \in I$, the Riemannian manifold (N, σ_t) with $\sigma_t = \sigma h^2(t, \cdot)$ is Einstein and being also LCF (as we have seen that (N, σ) is LCF) it must have constant curvature.

Suppose that at some time $t_0 \in I$ the manifold (N, σ_{t_0}) has positive (respectively negative) constant curvature $K(t_0)$, then, by formula (1), the curvature $K(t)$ of (N, σ_t) is continuous in I and positive (respectively negative) in a maximal open interval $I' \subset I$ around t_0 and $\sigma(p)h^2(t, p) = \frac{g^1(p)}{K^2(t)}$ where g^1 is a metric of constant curvature equal to one on N. It follows that the function $K^2(t)h^2(t, p)$ is independent of t in such interval and if at the borders of I' the curvature $K(t)$ tend to zero, the function h^2 cannot be bounded. This gives a contradiction with the fact that h is a smooth function if I' does not coincide with all the interval I. By this argument the curvature of (N, σ_t) either does not change sign in I or it is always zero.

The thesis then follows by taking $f = K^{-2}$ when the curvature is nonzero, choosing a fixed flat metric σ^0 on N and defining f by the equality $f^2(t)\sigma^0 = \sigma_t = \sigma h^2$ (for instance, comparing the volumes of σ^0 and σ_t), when the curvature is zero. □

Proposition 2. *Let (M, g) be LCF, the dimension of M larger than three and the Ricci tensor at every point has only two distinct eigenvalues and one of them has multiplicity one.*

Then, locally around every point in M, the manifold is a product $I \times N$ and the metric g can be expressed as

$$g(t,p) = \frac{dt^2 \otimes \sigma^K(p)}{[z(t) + \beta(p)]^2}$$

where σ^K is a metric on N of constant curvature K, $z : I \to \mathbb{R}$ and the function $\beta : N \to \mathbb{R}$ satisfies $\mathrm{Hess}^K_{ij}\beta = \delta(p)\sigma^K_{ij}$.

Conversely, for every N, σ^K, $z : I \to \mathbb{R}$ and $\beta : N \to \mathbb{R}$ satisfying $\mathrm{Hess}^K_{ij}\beta = \delta(p)\sigma^K_{ij}$, the above metric g on $I \times N$ has the above properties.

Proof. As $\mathrm{W}^g = 0$, the Schouten tensor $\mathrm{S} = \mathrm{Ric} - \frac{Rg}{2(n-1)}$ of (M, g) is a Codazzi tensor and, by the hypotheses, it has only two eigenvalues σ_1 and σ_2, with σ_2 of multiplicity $(n-1)$, with the same eigenspaces of the eigenvalues of the Ricci tensor of g.
By the results in [1, 2, 3, 4], the manifold can be written as a twisted product $I \times N$ with $g = e^{2f(t,p)}dt^2 \otimes b^2(t,p)d\sigma(p)$ where $T_pI \subset T_pM$ and $T_pN \subset T_pM$ are the eigenspaces associated to the two eigenvalues of the Ricci tensor and σ is a metric on N.
Moreover, the eigenvalue σ_2 of the Schouten tensor, of multiplicity $(n-1)$, is constant along N, for every fixed $t \in I$.

Let $h^2 = b^2e^{-2f}$ and $g_0 = dt^2 \otimes h^2(t,p)d\sigma(p)$, then $g = e^{2f}g_0$, moreover, being a conformal change of metric, also g_0 has a null Weyl tensor.
By Theorem 1, the metric g_0 has the form $g_0 = dt^2 \otimes h^2(t)\sigma^K$ for some constant curvature metric σ^K on N and a positive function $h : I \to \mathbb{R}$, and $g = e^{2f(t,p)}g_0$.
As the Ricci tensor of g "factorizes" along the fibers $\{t = \text{constant}\}$ we have $\mathrm{Ric}^g = \lambda e^{2f}dt^2 \otimes \mu e^{2f}h^2\sigma^K$.

We compute,

$$\begin{aligned}\mathrm{Ric}^g =& \mathrm{Ric}^0 - (n-2)(\mathrm{Hess}^0 f - df^0 \otimes df^0) - (\Delta^0 f + (n-2)|\nabla f|^2_0)g_0 \\ =& -(n-1)\,h''/h\,dt^2 + \big((n-2)K - h\,h'' - (n-2)(h')^2\big)\sigma^K \\ & - (n-2)(\mathrm{Hess}^0 f - df^0 \otimes df^0) - (\Delta^0 f + (n-2)|\nabla f|^2_0)g_0\end{aligned} \tag{2}$$

and

$$\begin{aligned}\mathrm{R}^g =& e^{-2f}\big(-2(n-1)\,h''/h + (n-1)(n-2)Kh^{-2} - (n-1)(n-2)(h')^2h^{-2} \\ & - 2(n-1)\Delta^0 f - (n-1)(n-2)|\nabla f|^2_0\big)\,.\end{aligned}$$

Hence, the Schouten tensor is given by

$$
\begin{aligned}
\mathrm{S}^g &= \mathrm{Ric}^g - \frac{\mathrm{R}}{2(n-1)} g = \mathrm{Ric}^g - \frac{\mathrm{R}}{2(n-1)} e^{2f} g_0 \\
&= -(n-1)h''/h\, dt^2 + \big((n-2)K - h\, h'' - (n-2)(h')^2\big)\sigma^K \\
&\quad - (n-2)(\mathrm{Hess}^0 f - df^0 \otimes df^0) - (\Delta^0 f + (n-2)|\nabla f|_0^2) g_0 \\
&\quad + \frac{2(n-1)h''/h - (n-1)(n-2)Kh^{-2} + (n-1)(n-2)(h')^2 h^{-2}}{2(n-1)} g_0 \\
&\quad + \frac{2(n-1)\Delta^0 f + (n-1)(n-2)|\nabla f|_0^2}{2(n-1)} g_0 \\
&= -(n-1)h''/h\, dt^2 + \big((n-2)K - h\, h'' - (n-2)(h')^2\big)\sigma^K \\
&\quad - (n-2)(\mathrm{Hess}^0 f - df^0 \otimes df^0) - (\Delta^0 f + (n-2)|\nabla f|_0^2) g_0 \\
&\quad + \big(h''/h - (n-2)Kh^{-2}/2 + (n-2)(h')^2 h^{-2}/2\big) g_0 \\
&\quad + \big(\Delta^0 f + (n-2)|\nabla f|_0^2/2\big) g_0 \\
&= -\frac{n-2}{2h^2}\big(2hh'' + K - (h')^2\big)\, dt^2 \\
&\quad + \frac{n-2}{2}\big(K - (h')^2\big)\sigma^K \\
&\quad - (n-2)(\mathrm{Hess}^0 f - df^0 \otimes df^0) \\
&\quad - \frac{n-2}{2}|\nabla f|_0^2 (dt^2 \otimes h^2 \sigma^K) \\
&= -\frac{n-2}{2h^2}\big(2hh'' + K - (h')^2 - |\nabla f|_K^2 - (\partial_t f)^2 h^2\big)\, dt^2 \\
&\quad + \frac{n-2}{2}\big(K - (h')^2 - |\nabla f|_K^2 - (\partial_t f)^2 h^2\big)\sigma^K \\
&\quad - (n-2)(\mathrm{Hess}^0 f - df^0 \otimes df^0)\,.
\end{aligned}
\tag{3}
$$

In particular,

$$
\mathrm{S}^g_{ij} = \frac{n-2}{2}\big(K - (h')^2 - |\nabla f|_K^2 - (\partial_t f)^2 h^2\big)\sigma^K_{ij} - (n-2)(\mathrm{Hess}^0_{ij} f - df^0_i df^0_j)
$$

Now, we know that $\mathrm{S}^g_{ij} = \sigma_2 g_{ij} = \sigma_2 e^{2f} h^2 \sigma^K_{ij}$, with σ_2 depending only on t, hence,

$$\begin{aligned}
\frac{2\sigma_2 h^2}{n-2}\sigma^K_{ij} &= \big(K-(h')^2\big)e^{-2f}\sigma^K_{ij} - \big(|\nabla f|^2_K e^{-2f} + (\partial_t f)^2 h^2 e^{-2f}\big)\sigma^K_{ij}\\
&\quad - 2e^{-2f}(\mathrm{Hess}^0_{ij} f - df^0_i df^0_j)\\
&= \big(K-(h')^2\big)e^{-2f}\sigma^K_{ij} - \big(|\nabla e^{-f}|^2_K + (\partial_t e^{-f})^2 h^2\big)\sigma^K_{ij}\\
&\quad + 2e^{-f}\mathrm{Hess}^0_{ij} e^{-f}\\
&= \big(K-(h')^2\big)e^{-2f}\sigma^K_{ij}\\
&\quad - \big(|\nabla e^{-f}|^2_K + (\partial_t e^{-f})^2 h^2 - 2e^{-f} hh'\partial_t e^{-f}\big)\sigma^K_{ij}\\
&\quad + 2e^{-f}\mathrm{Hess}^K_{ij} e^{-f}\\
&= \big(K-(h')^2\big)e^{-2f}\sigma^K_{ij}\\
&\quad - \big(|\nabla e^{-f}|^2_K + (h\partial_t e^{-f} - h' e^{-f})^2 - (h')^2 e^{-2f}\big)\sigma^K_{ij}\\
&\quad + 2e^{-f}\mathrm{Hess}^K_{ij} e^{-f}\\
&= K e^{-2f}\sigma^K_{ij}\\
&\quad - |\nabla e^{-f}|^2_K \sigma^K_{ij} - (h\partial_t e^{-f} - h' e^{-f})^2\sigma^K_{ij}\\
&\quad + 2e^{-f}\mathrm{Hess}^K_{ij} e^{-f}\,,
\end{aligned}$$

where we used the equality

$$\begin{aligned}
\mathrm{Hess}^{\sigma^K}_{ik} e^{-f} &= \mathrm{Hess}^0_{ik} e^{-f} + \Gamma(g_0)^t_{ik}\partial_t e^{-f}\\
&= \mathrm{Hess}^0_{ik} e^{-f} - hh'\partial_t e^{-f}\sigma^K_{ik}\,.
\end{aligned}$$

The above equation implies that (freezing the variable t), the Hessian of the function e^{-f} is proportional to the metric in the constant curvature space (N, σ^K), that is, $\mathrm{Hess}^K_{ij} e^{-f} = \theta\sigma^K_{ij}$ for some function $\theta : I \times N \to \mathbb{R}$.

Moreover, again by equation (3), we have also $\mathrm{Hess}^0_{it} e^{-f} = 0$ which implies

$$\partial^2_{it} e^{-f} = h'/h\,\partial_i e^{-f}\,.$$

It follows that

$$\partial_i e^{-f(t,p)} = \alpha_i(p)h(t)$$

and $e^{-f(t,p)} = (z(t) + \beta(p))h(t)$ for some functions $\alpha : N \to \mathbb{R}$, $\beta : N \to \mathbb{R}$ and $z : I \to \mathbb{R}$. Hence,

$$\mathrm{Hess}^K_{ij}\beta(p) = h^{-1}(t)\theta(t,p)\sigma^K_{ij} = \delta(p)\sigma^K_{ij}\,. \tag{4}$$

and we can write

$$\begin{aligned}
\frac{2\sigma_2 h^2}{n-2}\sigma^K_{ij} &= K(z+\beta)^2 h^2\sigma^K_{ij}\\
&\quad - |\nabla\beta|^2_K h^2\sigma^K_{ij} - (z')^2 h^4\sigma^K_{ij}\\
&\quad + 2h^2(z+\beta)\mathrm{Hess}^K_{ij}\beta
\end{aligned}$$

and

$$\frac{2\sigma_2}{n-2}\sigma^K_{ij} = \big(K(z+\beta)^2 - |\nabla\beta|^2_K - (z'h)^2\big)\sigma^K_{ij} + 2(z+\beta)\mathrm{Hess}^K_{ij}\beta\,.$$

Clearly we have $\Delta^K\beta = (n-1)\delta$ and taking the divergence of both sides of the equation (4),

$$\nabla^K_i\delta = \Delta^K\nabla^K_i\beta = (\mathrm{Ric}^K)^j_i\nabla^K_j\beta + \nabla^K_i\Delta^K\beta = K(n-2)\nabla^K_i\beta + (n-1)\nabla^K_i\delta \tag{5}$$

that is, if $n \neq 2$,

$$K\nabla^K_i\beta = -\nabla^K_i\delta\,. \tag{6}$$

We differentiate the previous equality

$$\frac{2\sigma_2}{n-2} = K(z+\beta)^2 - |\nabla\beta|^2_K - (z'h)^2 + 2(z+\beta)\delta$$

using this last information,

$$\begin{aligned}0 &= 2K(z+\beta)\nabla_i\beta - 2\mathrm{Hess}^K_{ij}\beta\nabla^j\beta + 2\nabla_i\beta\delta + 2(z+\beta)\nabla_i\delta\\ &= 2K(z+\beta)\nabla_i\beta + 2(z+\beta)\nabla_i\delta\end{aligned}$$

which is an equality.
Hence, every pair of functions β and $z : I \to \mathbb{R}$, satisfying $\mathrm{Hess}^K_{ij}\beta = \delta(p)\sigma^K_{ij}$, give rise to a function $f(t,p) = -\log(z(t)+\beta(p)) - \log h(t)$ and a metric

$$g(t,p) = e^{2f(t,p)}g_0(t,p) = \frac{dt^2 \otimes h^2(t)\sigma^K(p)}{[(z(t)+\beta(p))h(t)]^2}$$

with the desired properties.
In conclusion

$$g(t,p) = \frac{dt^2/h^2(t) \otimes \sigma^K(p)}{[z(t)+\beta(p)]^2}$$

and, reparametrizing in t we can write

$$g(t,p) = \frac{dt^2 \otimes \sigma^K(p)}{[z(t)+\beta(p)]^2}\,. \tag{7}$$

□

References

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