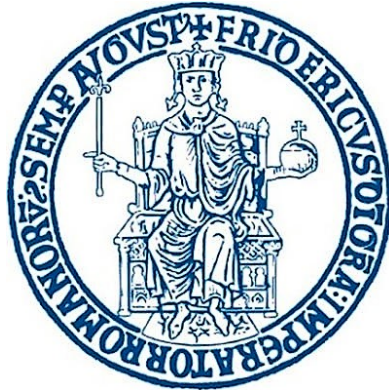


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DIPARTIMENTO DI MATEMATICA E APPLICAZIONI
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The Yamabe Problem

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Introduction

The *Yamabe problem* asks whether every conformal class on a closed Riemannian manifold of dimension $n \geq 3$ contains a metric of constant scalar curvature. It may be viewed as a higher-dimensional counterpart of the uniformization theorem for surfaces (see [15]).

Throughout the thesis, all manifolds are assumed to be connected unless explicitly stated otherwise.

Let (M^n, g) be a smooth closed Riemannian manifold of dimension $n \geq 3$. A metric conformal to g can be written as

$$\tilde{g} = u^{\frac{4}{n-2}} g,$$

where u is a smooth positive function. The scalar curvatures of these metrics are related by

$$-\frac{4(n-1)}{n-2} \Delta u + Ru = \tilde{R} u^{\frac{n+2}{n-2}}.$$

Thus, the geometric problem becomes a nonlinear elliptic problem: one seeks a positive function u for which the scalar curvature $\tilde{R}$ on the right-hand side is a constant λ .

This equation has a natural variational structure. Writing $2^* = 2n/(n-2)$, one introduces the *Yamabe functional*

$$J_{2^*}(u) = \frac{\int_M \left(\frac{4(n-1)}{n-2} |\nabla u|^2 + Ru^2 \right) d\text{Vol}}{\left(\int_M |u|^{2^*} d\text{Vol} \right)^{2/2^*}}.$$

Its infimum $Y(M, [g])$ over all nonzero smooth functions is called *Yamabe constant* and is independent of the metric in the conformal class $[g]$ (see [21]). A smooth positive minimizer satisfies the corresponding Euler–Lagrange equation and therefore defines a conformal metric of constant scalar curvature.

The principal analytic obstacle is the failure of compactness at the Sobolev exponent 2^* . Although the embedding

$$W^{1,2}(M) \hookrightarrow L^{2^*}(M)$$

is continuous, it is not compact. A minimizing sequence for the Yamabe functional may therefore fail to converge strongly, concentrating instead near a point of the manifold. After rescaling, such concentration is modeled on the extremals of the sharp Euclidean Sobolev inequality (see [30]). The concentration–compactness principle of P.-L. Lions provides a precise way to analyze this phenomenon (see [22, 23, 29]).

The history of the problem reflects this difficulty. Yamabe introduced it in 1960, motivated in part by possible connections with the Poincaré conjecture (see [1, 33]), but his original argument contained a gap in the passage to the limiting Sobolev exponent. Trudinger subsequently obtained a solution under an additional assumption [31]. Aubin then developed

the variational and geometric methods that led to the fundamental universal bound

$$Y(M, [g]) \leq Y(\mathbb{S}^n, [g_{\text{std}}]),$$

where g_{std} is the round metric on $\mathbb{S}^n$, and proved that the inequality is strict when $n \geq 6$ and (M^n, g) is not locally conformally flat [3].

The significance of the spherical value is that *strict inequality rules out concentration*: whenever the Yamabe constant lies below this threshold, a minimizing sequence yields a minimizer. To handle the remaining cases, Schoen introduced a global argument based on the Green function of the conformal Laplacian. The corresponding conformal blow-up produces a complete asymptotically flat manifold, whose ADM mass is linked to the relevant term in the Green function expansion. The positive mass theorem then supplies the positivity and rigidity needed to establish the strict inequality, except for the conformal class of the standard sphere (see [21,25,26,27]).

The thesis is divided into two complementary parts, which we shall call the “analytic” and the “geometric” parts. This division reflects the two main stages in the approach to the Yamabe problem developed here. The first is concerned with the variational formulation, the existence and regularity of minimizers, and the role of compactness; the second turns to the local and global geometry of the manifold in order to understand the strict inequality that makes the existence results applicable. The two terms indicate the prevailing viewpoint in each part rather than a sharp separation of methods, since analytic and geometric ideas are closely intertwined throughout the argument.

The first, “analytic”, part is developed in Chapter 1. We introduce the Yamabe functional, compute the Yamabe constant of the standard sphere and prove Aubin’s universal upper bound. After establishing the regularity of minimizers, we treat the case $Y(M, [g]) \leq 0$ and characterize the sign of a conformal class. The chapter concludes with an existence criterion below the spherical threshold.

The second, “geometric”, part is developed in Chapter 2 and is devoted to the methods used to establish strict inequalities for the Yamabe constant. We first present Aubin’s local approach, based on conformal normal coordinates, truncated Talenti bubbles and the Weyl tensor in dimensions $n \geq 6$, and then turn to Schoen’s global approach, involving the Green function, asymptotically flat geometry and the positive mass theorem (see [3,21,25]).

Finally, Appendix A collects some analytic results used in the main text.

We mention that there is also an alternative approach to the existence problem, based on replacing the critical exponent 2^* in the variational problem by exponents $2 < q < 2^*$. For each such q , the compact embedding $W^{1,2}(M) \hookrightarrow L^q(M)$ allows one to find a minimizer of the corresponding functional. One then studies whether these minimizers converge to a minimizer of J_{2^*} , as q approaches 2^* . This approach actually goes back to Yamabe and was developed by Trudinger and Aubin (see [3,21,31,33]).

CHAPTER 1

The Yamabe problem

The *uniformization theorem* states that every simply connected Riemann surface is conformally equivalent to exactly one of the Riemann sphere $\mathbb{S}^2$, the complex plane $\mathbb{C}$, or the unit disk $\mathbb{D}$ (see [15]). There is no analogous uniformization theorem in higher dimensions. One may nevertheless ask whether every closed Riemannian manifold of dimension $n \geq 3$ admits a conformal metric with constant scalar curvature. This is the Yamabe problem. Its affirmative solution is stated in the following theorem (see [21]).

THEOREM 1.1. *Let (M^n, g) be a closed Riemannian manifold of dimension $n \geq 3$. Then there exists a conformal change of the metric g for which the scalar curvature is constant.*

We begin by recalling the conformal transformation formulas (see [7, Chapter 1, Section J, Theorem 1.159 and Corollary 1.161(a)]). We consider the following *conformal change* of the metric,

$$\tilde{g} = u^{\frac{4}{n-2}} g = e^{2\psi} g, \quad u \in C^\infty(M), \quad u > 0, \quad \psi = \frac{2}{n-2} \log u,$$

moreover, we denote by $[g]$ the *conformal class* of g , which is the set of all smooth metrics conformal to g , that is,

$$[g] = \{ u^{\frac{4}{n-2}} g \mid u \in C^\infty(M), \quad u > 0 \}.$$

From this we get

$$\tilde{\nabla}_X Y = \nabla_X Y + X(\psi)Y + Y(\psi)X - g(X, Y)\nabla\psi$$

and

$$e^{-2\psi} \widetilde{\text{Riem}} = \text{Riem} - 2(\text{Hess } \psi - d\psi \otimes d\psi) \otimes g - |d\psi|^2 g \otimes g,$$

where $\otimes$ denotes the so-called *Kulkarni–Nomizu product* defined for any two symmetric $(0, 2)$ tensors h and k as follows:

$$\begin{aligned} h \otimes k(v_1, v_2, v_3, v_4) &= \frac{1}{2}h(v_1, v_3)k(v_2, v_4) - \frac{1}{2}h(v_1, v_4)k(v_2, v_3) \\ &\quad + \frac{1}{2}h(v_2, v_4)k(v_1, v_3) - \frac{1}{2}h(v_2, v_3)k(v_1, v_4). \end{aligned}$$

Taking the appropriate trace of the Riemann tensor, we get

$$\widetilde{\text{Ric}} = \text{Ric} - (n-2)(\text{Hess } \psi - d\psi \otimes d\psi) - (\Delta_g \psi + (n-2)|d\psi|^2)g$$

and taking one further trace, we have the following scalar curvature transformation formula:

$$e^{2\psi} \tilde{R} = R - 2(n-1)\Delta\psi - (n-1)(n-2)|d\psi|^2. \quad (1.1)$$

If we write the equation (1.1) in terms of u , we then obtain

$$-c(n)\Delta u + Ru = \tilde{R}u^{(n+2)/(n-2)} \quad \text{where} \quad c(n) = \frac{4(n-1)}{(n-2)}$$

and the operator on the left is called the *conformal Laplacian*, which we denote by

$$L = -c(n)\Delta + R.$$

Moreover, we notice that the exponent on the right is exactly $2^* - 1$ where $2^* = 2n/(n-2)$ is the critical exponent of the Sobolev embedding.

Then, the resolution of the Yamabe problem is equivalent to solving

$$-c(n)\Delta u + Ru = \lambda u^{2^*-1} \quad (1.2)$$

with a positive function $u \in C^\infty(M)$, for some constant $\lambda \in \mathbb{R}$.

This equation provides the starting point for the first, “analytic”, part of the thesis, developed in this chapter. We shall study the variational formulation of the problem and establish existence results which, in particular, reduce the remaining question to a strict inequality involving the Yamabe constant of the standard sphere. The second, “geometric”, part, developed in Chapter 2, will be devoted to the geometric arguments needed to establish this inequality.

1.1. The variational approach – The Yamabe functional and the Yamabe constant

Fix $n \geq 3$. On a closed Riemannian manifold (M^n, g) consider the *Einstein–Hilbert functional*

$$\mathcal{R}(g) = \int_M R \, d\text{Vol},$$

giving the so-called *total scalar curvature* of the manifold (M^n, g) .

For a positive function $u \in C^\infty(M)$, we set $\tilde{g} = u^{4/(n-2)}g$. Then, by the conformal change formula for the scalar curvature and by integration by parts, we have

$$\mathcal{R}(\tilde{g}) = \mathcal{R}(u^{4/(n-2)}g) = \mathcal{J}(u) = \int_M (c(n)|\nabla u|_g^2 + Ru^2) \, d\text{Vol},$$

while

$$\widetilde{\text{Vol}}(M) = \mathcal{V}(u) = \int_M |u|^{2^*} \, d\text{Vol}.$$

These integrals define $\mathcal{J}$ and $\mathcal{V}$ for every $u \in C^\infty(M)$, without a sign restriction. Their geometric interpretation applies when $u > 0$. For such functions, the unit volume condition is equivalent to $\|u\|_{2^*} = 1$.

We therefore compute the first variation of $\mathcal{J}$ under the constraint $\mathcal{V}(u) = 1$. For every $v \in C^\infty(M)$, compactness of M ensures that $u + tv$ remains positive for sufficiently small $|t|$, and we have

$$\left. \frac{d}{dt} \mathcal{J}(u + tv) \right|_{t=0} = 2 \int_M (-c(n)\Delta u + Ru)v \, d\text{Vol},$$

whereas

$$\left. \frac{d}{dt} \mathcal{V}(u + tv) \right|_{t=0} = 2^* \int_M u^{2^*-1} v \, d\text{Vol}.$$

Hence, the Lagrange multiplier method shows that a positive constrained critical point $u \in C^\infty(M)$ of $\mathcal{J}$ satisfies

$$-c(n)\Delta u + Ru = \lambda u^{2^*-1}$$

for some constant $\lambda \in \mathbb{R}$. By equation (1.2), this is exactly the condition that the conformal metric $\tilde{g} = u^{4/(n-2)}g$ have constant scalar curvature. Therefore the Yamabe problem is

equivalent to finding a critical point of the total scalar curvature in the conformal class $[g]$ under the above volume constraint or, equivalently, to finding a smooth positive constrained critical point of $\mathcal{J}$.

In order to get such a critical point, we can try to minimize the functional $\mathcal{J}$ among all $u \in C^\infty(M)$ (without imposing a sign condition) under the constraint $\mathcal{V}(u) = 1$, hoping that a minimizer exists (hence it would be a critical point of $\mathcal{J}$ under the volume constraint) and it is positive. Equivalently, we can “remove” the constraint (as it is usual), by introducing the quotient

$$J_{2^*}(u) = \mathcal{J}(u) / \|u\|_{2^*}^2 = \frac{c(n) \int_M |\nabla u|_g^2 d\text{Vol} + \int_M R u^2 d\text{Vol}}{\|u\|_{2^*}^2},$$

called *Yamabe functional* and minimize it on $C^\infty(M) \setminus \{0\}$. Indeed, since both the numerator and the denominator are homogeneous of degree two, there holds $J_{2^*}(u) = J_{2^*}(u / \|u\|_{2^*})$ and if $u > 0$, the metric $(u / \|u\|_{2^*})^{4/(n-2)} g$ has volume one and total scalar curvature $J_{2^*}(u)$. Then, in order to apply the variational methods, we extend J_{2^*} to the space $W^{1,2}(M) \setminus \{0\}$. This extension does not change the infimum of such a functional, since $J_{2^*}(|u|) = J_{2^*}(u)$ and every nonzero nonnegative Sobolev function can be approximated in $W^{1,2}(M)$ by smooth nonnegative functions, which become strictly positive after adding positive constants tending to zero. By the continuous embedding into $L^{2^*}(M)$, the corresponding quotients converge (see [20]). Hence the infimum over all nonzero Sobolev functions equals the infimum over smooth positive functions.

By its clear relevance to this minimization problem, we then define the *Yamabe constant* of the conformal class $[g]$ of g as

$$Y(M, [g]) = \inf \{ J_{2^*}(u) \mid u \in W^{1,2}(M) \setminus \{0\} \},$$

or equivalently, by the analysis above,

$$Y(M, [g]) = \inf \{ \mathcal{R}(\tilde{g}) \mid \tilde{g} \in [g], \tilde{\text{Vol}}(M) = 1 \}.$$

Notice that, since R is bounded below on M and $\|u\|_2^2 \leq \text{Vol}(M)^{2/n} \|u\|_{2^*}^2$, the Yamabe constant is finite.

More generally, on a Riemannian manifold (M^n, g) without boundary, possibly noncompact, we define

$$Y_c(M, [g]) = \inf \{ J_{2^*}(v) \mid v \in C_c^\infty(M) \setminus \{0\} \},$$

where the Yamabe functional is computed with respect to g . This quantity depends only on the conformal class $[g]$, since under a conformal change $\tilde{g} = u^{4/(n-2)} g$, with $u > 0$ smooth, the Yamabe functional relative to $\tilde{g}$ evaluated on v equals the one relative to g evaluated on uv and multiplication by u is a bijection of $C_c^\infty(M)$ in itself. Finally, when M is closed, the density of the smooth functions in $W^{1,2}(M)$ implies $Y_c(M, [g]) = Y(M, [g])$.

We now compute the Yamabe constant of the round sphere, which will provide a universal upper bound for that of any closed Riemannian manifold.

Let $(\mathbb{S}^n, g_{\text{std}})$ be the unit sphere in $(\mathbb{R}^{n+1}, g_{\text{eucl}})$, endowed with the metric induced by the Euclidean metric g_{eucl} . Denoting by N the “north pole” of $\mathbb{S}^n$ and by F the *stereographic projection* from N , we have

$$F: \mathbb{S}^n \setminus \{N\} \rightarrow \mathbb{R}^n.$$

The inverse stereographic projection pulls back the round metric on $\mathbb{S}^n \setminus \{N\}$ to the conformally Euclidean metric

$$\widehat{g} = (F^{-1})^* g_{\text{std}} = \frac{4}{(1 + |x|^2)^2} \sum_{i=1}^n dx_i^2 = u^{4/(n-2)} g_{\text{eucl}} \quad \text{with} \quad u(x) = \left[\frac{2}{1 + |x|^2} \right]^{(n-2)/2}.$$

Since the scalar curvature of the round metric is $R_{\text{std}} = n(n-1)$, the conformal identification above makes it possible to express its Yamabe constant through the sharp *Euclidean Sobolev inequality*. We denote its optimal constant by σ_n , meaning that

$$\sigma_n \|v\|_{L^{2^*}(\mathbb{R}^n)} \leq \|\nabla v\|_{L^2(\mathbb{R}^n)} \quad (1.3)$$

for every $v \in C_c^\infty(\mathbb{R}^n)$. By the Aubin–Talenti characterization of the equality cases (see [4, 30]), the *stereographic conformal factor* u attains equality in the corresponding inequality on the Sobolev completion.

THEOREM 1.2. *For the standard sphere $(\mathbb{S}^n, g_{\text{std}})$, one has*

$$Y(\mathbb{S}^n, [g_{\text{std}}]) = c(n)\sigma_n^2 = n(n-1)\text{Vol}_{\text{std}}(\mathbb{S}^n)^{2/n}.$$

PROOF. We set $\mathbb{S}_0^n = \mathbb{S}^n \setminus \{N\}$, so that stereographic projection identifies $(\mathbb{S}_0^n, g_{\text{std}})$ isometrically with Euclidean space equipped with the metric $\widehat{g}$

$$F: (\mathbb{S}_0^n, g_{\text{std}}) \rightarrow (\mathbb{R}^n, \widehat{g}).$$

Since $\widehat{g} \in [g_{\text{eucl}}]$, the isometry and conformal invariance of J_{2^*} described above give

$$Y_c(\mathbb{S}_0^n, [g_{\text{std}}]) = Y_c(\mathbb{R}^n, [\widehat{g}]) = Y_c(\mathbb{R}^n, [g_{\text{eucl}}]) = c(n)\sigma_n^2.$$

To identify the Yamabe constant of the punctured sphere with that of the whole sphere, we consider the map

$$G_v(x) = u(x)v(F^{-1}(x)),$$

for every $v \in C^\infty(\mathbb{S}^n)$. Since v is smooth on $\mathbb{S}^n$, the stereographic conformal factor gives

$$G_v(x) = O(|x|^{2-n}), \quad |\nabla G_v(x)| = O(|x|^{1-n}) \quad \text{as } |x| \rightarrow +\infty.$$

Consequently, the conformal change identities, initially established for compactly supported functions, extend to G_v by a cutoff argument at infinity. In particular, for every nonzero $v \in C^\infty(\mathbb{S}^n)$, we have

$$\int_{\mathbb{S}^n} |v|^{2^*} d\text{Vol}_{\text{std}} = \int_{\mathbb{R}^n} |G_v|^{2^*} dx$$

and

$$\int_{\mathbb{S}^n} (c(n)|\nabla v|_{\text{std}}^2 + n(n-1)v^2) d\text{Vol}_{\text{std}} = c(n) \int_{\mathbb{R}^n} |\nabla G_v|^2 dx.$$

The sharp Euclidean Sobolev inequality therefore gives $J_{2^*}(v) \geq c(n)\sigma_n^2$, and taking the infimum over smooth functions yields

$$Y(\mathbb{S}^n, [g_{\text{std}}]) \geq c(n)\sigma_n^2.$$

Conversely, every smooth function compactly supported in $\mathbb{S}_0^n$ extends by zero to a smooth function on $\mathbb{S}^n$, so that

$$Y(\mathbb{S}^n, [g_{\text{std}}]) \leq Y_c(\mathbb{S}_0^n, [g_{\text{std}}]) = c(n)\sigma_n^2.$$

Moreover, as observed above, the stereographic conformal factor u attains equality in the sharp Euclidean Sobolev inequality. Since $G_1 = u$, conformal invariance of the Yamabe

functional shows that the constant function 1 realizes the Yamabe constant of the round sphere, and therefore

$$Y(\mathbb{S}^n, [g_{\text{std}}]) = J_{2^*}(1) = n(n-1)\text{Vol}_{\text{std}}(\mathbb{S}^n)^{2/n}.$$

□

This sphere calculation leads to Aubin's classical *universal upper bound* (see [3,21]).

THEOREM 1.3. *For every closed Riemannian manifold (M^n, g) , one has*

$$Y(M, [g]) \leq Y(\mathbb{S}^n, [g_{\text{std}}]).$$

In other words, the standard sphere has the largest Yamabe constant among all closed Riemannian manifolds of dimension n .

REMARK 1.4. This universal upper bound actually does not require the compactness. Indeed, for any Riemannian manifold (M^n, g) , with $n \geq 3$, possibly noncompact or noncomplete, the same local argument gives

$$Y_c(M, [g]) \leq Y(\mathbb{S}^n, [g_{\text{std}}]).$$

For closed manifolds, however, the results of Aubin and Schoen show that equality holds if and only if (M^n, g) is conformally diffeomorphic to the standard sphere (see [3,25]). This characterization, which will follow from the strict inequalities established in Theorem 2.2 and Theorem 2.6 in Chapter 2, fails in the noncompact setting. For instance, the sharp Euclidean Sobolev inequality implies

$$Y_c(\mathbb{R}^n, [g_{\text{eucl}}]) = Y(\mathbb{S}^n, [g_{\text{std}}]).$$

PROOF OF THEOREM 1.3. We fix $p \in M$ and identify $T_p M$ with $\mathbb{R}^n$ by choosing a g -orthonormal basis. In the corresponding normal coordinates, defined on a sufficiently small ball around p , we have

$$g_{ij}(x) = \delta_{ij} + O(|x|^2), \quad d\text{Vol} = (1 + O(|x|^2)) dx.$$

For an arbitrary nonzero function $\varphi \in C_c^\infty(\mathbb{R}^n)$, we choose $R > 0$ such that $\text{supp } \varphi \subseteq B_R$ and, for $\varepsilon > 0$ sufficiently small, define a smooth function on M by

$$u_\varepsilon(\exp_p(\varepsilon y)) = \varepsilon^{-(n-2)/2} \varphi(y) \quad \text{for } y \in B_R,$$

extending it by zero outside $\exp_p(\varepsilon B_R)$. The critical scaling cancels the Jacobian factor in the Dirichlet energy and in the L^{2^*} norm, while the scalar-curvature term acquires a factor ε^2 . Since $g_{ij}(\varepsilon y) \rightarrow \delta_{ij}$ uniformly on the support of φ , it follows that

$$\begin{aligned} \int_M |\nabla u_\varepsilon|_g^2 d\text{Vol} &= \int_{\mathbb{R}^n} |\nabla \varphi|^2 dy + o(1), \\ \int_M |u_\varepsilon|^{2^*} d\text{Vol} &= \int_{\mathbb{R}^n} |\varphi|^{2^*} dy + o(1), \\ \int_M R u_\varepsilon^2 d\text{Vol} &= O(\varepsilon^2). \end{aligned}$$

Consequently,

$$\lim_{\varepsilon \rightarrow 0} J_{2^*}(u_\varepsilon) = c(n) \frac{\int_{\mathbb{R}^n} |\nabla \varphi|^2 dy}{\left(\int_{\mathbb{R}^n} |\varphi|^{2^*} dy \right)^{2/2^*}}.$$

Since $Y(M, [g]) \leq J_{2^*}(u_\varepsilon)$ and φ is arbitrary, taking the infimum over $C_c^\infty(\mathbb{R}^n) \setminus \{0\}$ gives

$$Y(M, [g]) \leq c(n)\sigma_n^2 = Y(\mathbb{S}^n, [g_{\text{std}}]),$$

as claimed. $\square$

Having established Aubin's universal upper bound, we now turn to the regularity of minimizers of the Yamabe functional, postponing the question of their existence. Following the classical regularity argument of Trudinger (see [31, Theorem 3] and [21]), we shall prove that every minimizer, whenever it exists, is smooth and, up to a change of sign, strictly positive, so that the Yamabe problem ultimately reduces to proving that the infimum defining $Y(M, [g])$ is attained.

1.2. Regularity of Yamabe minimizers

The main difficulty in proving the regularity of a minimizer of the Yamabe functional comes from the borderline growth of the nonlinear term in its Euler–Lagrange equation. Indeed, starting from $u \in W^{1,2}(M)$, the Sobolev embedding gives $u \in L^{2^*}(M)$, so that $|u|^{2^*-2}u \in L^{2n/(n+2)}(M)$. Elliptic regularity then yields $u \in W^{2,2n/(n+2)}(M)$, but the corresponding Sobolev embedding only recovers $u \in L^{2^*}(M)$, since

$$\frac{2^* - 1}{2^*} - \frac{2}{n} = \frac{1}{2^*}.$$

Thus, the usual bootstrap does not improve the integrability of u . The argument below overcomes this difficulty by splitting the coefficient of u in the Euler–Lagrange equation into a bounded part and a remainder with arbitrarily small $L^{n/2}(M)$ norm.

THEOREM 1.5 (Regularity of minimizers). *If $u \in W^{1,2}(M)$ is a minimizer of J_{2^*} , then $u \in C^\infty(M)$.*

PROOF. By homogeneity, we may assume $\|u\|_{2^*} = 1$, so the Euler–Lagrange equation

$$L_g u = Y(M, [g])|u|^{2^*-2}u$$

has its right-hand side in $L^{2n/(n+2)}(M)$ and Corollary A.2 gives $u \in W^{2,2n/(n+2)}(M)$. To improve this regularity, we fix

$$\frac{2n}{n+2} < p < \min\left\{\frac{n}{2}, 2^*\right\}$$

and consider the operator

$$L: W^{2,p}(M) \longrightarrow L^p(M), \quad L(v) = -c(n)\Delta v + (R + \gamma)v,$$

where $\gamma > 0$ is chosen sufficiently large that the associated bilinear form is coercive on $W^{1,2}(M)$. The embeddings $L^p(M) \hookrightarrow [W^{1,2}(M)]^*$ and $W^{2,p}(M) \hookrightarrow W^{1,2}(M)$, together with the Lax–Milgram theorem, ensure that every forcing term in $L^p(M)$ determines a unique weak solution in $W^{1,2}(M)$, which also belongs to $L^p(M)$ because $p < 2^*$ and therefore lies in $W^{2,p}(M)$ by Corollary A.2. Thus L is an isomorphism with bounded inverse.

Writing the equation for u as

$$Lu = au, \quad a = Y(M, [g])|u|^{2^*-2} + \gamma \in L^{n/2}(M),$$

we decompose, for $C > 0$,

$$a = b_C + h_C, \quad b_C = a\chi_{\{|a| \leq C\}}, \quad h_C = a\chi_{\{|a| > C\}},$$

so that $b_C \in L^\infty(M)$ and $\|h_C\|_{n/2} \rightarrow 0$ as $C \rightarrow +\infty$. If H_C denotes multiplication by h_C , Hölder's inequality and the Sobolev embedding give

$$\|H_C(v)\|_p \leq \|h_C\|_{n/2} \|v\|_{p^*} \leq B \|h_C\|_{n/2} \|v\|_{W^{2,p}}, \quad p^* = \frac{np}{n-2p},$$

with B independent of C , hence $H_C: W^{2,p}(M) \rightarrow L^p(M)$ has arbitrarily small operator norm for large C . Choosing C so that $\|L^{-1}\|_{op} \|H_C\|_{op} < 1$, the Neumann series shows that $L_C = L - H_C$ is invertible, and since $b_C u \in L^p(M)$ there exists a unique $v \in W^{2,p}(M)$ such that

$$L_C v = b_C u.$$

Although u satisfies the same equation weakly, we cannot yet use uniqueness in $W^{2,p}(M)$, since this is precisely the regularity we want to establish; instead, we compare u and v in $W^{1,2}(M)$, to which both belong. Indeed, setting $w = u - v$ and introducing the coercive bilinear form

$$B(w, \eta) = \int_M [c(n)g(\nabla w, \nabla \eta) + (R + \gamma)w\eta] d\text{Vol},$$

we have

$$B(w, \eta) = \int_M h_C w \eta d\text{Vol} \quad \text{for every } \eta \in W^{1,2}(M),$$

where the right-hand side is well defined by Hölder's inequality and the critical Sobolev embedding. Testing with $\eta = w$ therefore gives

$$\begin{aligned} \alpha \|w\|_{W^{1,2}}^2 &\leq B(w, w) = \int_M h_C w^2 d\text{Vol} \\ &\leq \|h_C\|_{n/2} \|w\|_{2^*}^2 \leq K \|h_C\|_{n/2} \|w\|_{W^{1,2}}^2, \end{aligned}$$

where $\alpha > 0$ depends only on L and K is a Sobolev embedding constant, so by taking C sufficiently large that $K \|h_C\|_{n/2} < \alpha$ we conclude that $w = 0$ and hence $u = v \in W^{2,p}(M)$. Once this first improvement has been obtained, the remaining integrability follows by iteration, since, whenever $2p < n$, the Sobolev embedding and elliptic regularity yield

$$u \in W^{2,p}(M) \implies u \in L^{p^*}(M) \implies |u|^{2^*-2}u \in L^{p^*/(2^*-1)}(M) \implies u \in W^{2,p^*/(2^*-1)}(M).$$

Starting with $p_0 = p$ and defining

$$p_{k+1} = \frac{np_k}{(n-2p_k)(2^*-1)}$$

as long as $2p_k < n$, we find

$$\frac{p_{k+1}}{p_k} = \frac{n(n-2)}{(n-2p_k)(n+2)} \geq \frac{n(n-2)}{(n-2p_0)(n+2)} = \lambda > 1,$$

where the last inequality follows from the fact that the exponents are increasing and the strict inequality $\lambda > 1$ from $p_0 > 2n/(n+2)$. Consequently $p_k \geq \lambda^k p_0$ throughout the iteration, which must therefore reach an exponent $p_k \geq n/2$ after finitely many steps; if equality occurs, compactness allows us to replace p_k by an exponent $q < n/2$ sufficiently close to $n/2$ that

$$\frac{nq}{(n-2q)(2^*-1)} > \frac{n}{2},$$

and one further application of elliptic regularity then gives $u \in W^{2,s}(M)$ for some $s > n/2$. The Sobolev–Morrey embedding now implies $u \in C^{0,\theta}(M)$ for some $0 < \theta < 1$, and since

the map $t \mapsto |t|^{2^*-2}t$ is locally Lipschitz, the right-hand side of the Euler–Lagrange equation is Hölder continuous, so Schauder regularity yields $u \in C^{2,\theta}(M)$ (see [18, Theorem 9.19]). Moreover, $|u|$ is also a minimizer and hence enjoys the same regularity and satisfies the same equation; choosing

$$A \geq \max \left\{ 0, \sup_M c(n)^{-1} (R - Y(M, [g])|u|^{2^*-2}) \right\},$$

we obtain

$$(-\Delta + A)|u| \geq 0,$$

so Theorem A.3 implies that $|u|$ is either identically zero or strictly positive. The first alternative is excluded by $\|u\|_{2^*} = 1$, and therefore $|u| > 0$ everywhere; by continuity and connectedness u has constant sign, while compactness gives $\min_M |u| > 0$, making the nonlinearity smooth on the range of u and allowing the Schauder bootstrap to conclude that $u \in C^\infty(M)$. $\square$

By this theorem, solving the Yamabe problem reduces to showing that the infimum defining $Y(M, [g])$ is attained by a nonzero function in $W^{1,2}(M)$. We first establish this when $Y(M, [g]) \leq 0$, using the direct method of the calculus of variations (see [21,29,31]). The case of positive Yamabe constant, where the possible loss of compactness is a central issue, will be approached through the *concentration–compactness principle* (see [22,23,29]) in the last section of this chapter and through the geometric inequalities of Chapter 2.

1.3. The case of nonpositive Yamabe constant

We now show that the Yamabe constant is attained whenever $Y(M, [g]) \leq 0$, thereby solving the Yamabe problem in the nonpositive case and establishing Theorem 1.1 under this assumption.

THEOREM 1.6 (Solution in the case $Y(M, [g]) \leq 0$). *Let $Y(M, [g]) \leq 0$. Then $u \in W^{1,2}(M) \setminus \{0\}$ is a minimizer of J_{2^*} if and only if $E(u) = 0$ where E is the following nonnegative functional*

$$E(u) = c(n) \int_M |\nabla u|^2 d\text{Vol} + \int_M R u^2 d\text{Vol} - Y(M, [g]) \|u\|_{2^*}^2.$$

Moreover, there exists a strictly positive minimizer $u \in C^\infty(M)$.

PROOF. The nonnegativity of E comes from its definition, since $E(0) = 0$ and for every $u \neq 0$,

$$E(u) = (J_{2^*}(u) - Y(M, [g])) \|u\|_{2^*}^2 \geq 0,$$

moreover, the first claim clearly follows. To prove existence, we use the direct method, starting with a minimizing sequence u_k for J_{2^*} , normalized by $\|u_k\|_{2^*} = 1$. Since $J_{2^*}(u_k)$ is bounded, R is bounded and $\|u_k\|_2$ is controlled by $\|u_k\|_{2^*} = 1$, the definition of J_{2^*} gives

$$\sup_{k \in \mathbb{N}} \|u_k\|_{W^{1,2}(M)} \leq C.$$

Now we extract a subsequence such that

- (1) $u_k \rightharpoonup u$ in $W^{1,2}(M)$ for some $u \in W^{1,2}(M)$,
- (2) $u_k \rightarrow u$ strongly in $L^2(M)$, by the Rellich–Kondrachov embedding theorem,
- (3) $u_k \rightharpoonup u$ in $L^{2^*}(M)$.

Since $-Y(M, [g]) \geq 0$, the weak lower semicontinuity of the Dirichlet integral and of the L^{2^*} norm, together with the strong L^2 convergence, imply

$$0 = \liminf_{k \rightarrow +\infty} E(u_k) \geq E(u) \geq 0,$$

thus, we need to show that $u \neq 0$. Suppose by contradiction that $u = 0$ and first notice that, by strong convergence in $L^2(M)$, we have

$$\lim_{k \rightarrow +\infty} \int_M R u_k^2 d\text{Vol} = 0.$$

If $Y(M, [g]) < 0$, then

$$0 = \lim_{k \rightarrow +\infty} E(u_k) = \lim_{k \rightarrow +\infty} c(n) \int_M |\nabla u_k|^2 d\text{Vol} - Y(M, [g]),$$

which is absurd. If $Y(M, [g]) = 0$, then

$$\lim_{k \rightarrow +\infty} \int_M |\nabla u_k|^2 d\text{Vol} = 0,$$

hence, by using the Sobolev inequality, we obtain a contradiction: for all $k \in \mathbb{N}$ there holds

$$1 = \|u_k\|_{2^*}^2 \leq C(\|u_k\|_2^2 + \|\nabla u_k\|_2^2).$$

Hence, we have shown that u is a nontrivial minimizer. If now we consider $|u|$, this is still a minimizer and from Theorem 1.5, we know that $|u| \in C^\infty(M)$. The argument in the proof of Theorem 1.5 then shows that $|u|$ is strictly positive on M , concluding the proof. $\square$

COROLLARY 1.7. *If $Y(M, [g]) \leq 0$, the conformal class $[g]$ contains a smooth metric $\tilde{g}$ of unit volume and constant scalar curvature*

$$\tilde{R} = Y(M, [g]).$$

PROOF. By Theorem 1.6, there exists a smooth positive minimizer u of J_{2^*} , which we may normalize by $\|u\|_{2^*} = 1$. Its Euler–Lagrange equation reads

$$-c(n)\Delta u + Ru = Y(M, [g])u^{2^*-1}.$$

The conformal change formulas then show that $\tilde{g} = u^{4/(n-2)}g$ has unit volume and constant scalar curvature equal to $Y(M, [g])$. $\square$

1.4. The sign of the conformal class

Having solved the Yamabe problem when $Y(M, [g]) \leq 0$, we now consider a related, but weaker, question: whether a conformal class contains a metric whose scalar curvature has a prescribed sign, without necessarily being constant. The following classical result shows that this property is completely characterized by the sign of the Yamabe constant, which also coincides with that of the first eigenvalue of the conformal Laplacian (see [5, 21]).

THEOREM 1.8. *Let (M^n, g) be a closed Riemannian manifold. Then, the following equivalences hold:*

- (1) $Y(M, [g]) > 0$ if and only if there exists $\tilde{g} \in [g]$ with $\tilde{R} > 0$ on M ,
- (2) $Y(M, [g]) = 0$ if and only if there exists $\tilde{g} \in [g]$ with $\tilde{R} = 0$ on M ,
- (3) $Y(M, [g]) < 0$ if and only if there exists $\tilde{g} \in [g]$ with $\tilde{R} < 0$ on M .

In particular, for any given conformal class $[g]$, exactly one of these three sign alternatives occurs.

PROOF. If $Y(M, [g]) \leq 0$, the existence of a conformal metric with scalar curvature of the prescribed sign follows from Corollary 1.7. Conversely, suppose first that there exists $\tilde{g} \in [g]$ with $\tilde{R} < 0$ on M . Since the infimum of the Yamabe functional is the same for all the metrics in the same conformal class, we may replace g by $\tilde{g}$. Using $u = 1$ as a test function, we obtain

$$Y(M, [g]) \leq J_{2^*}(1) = [\text{Vol}_g(M)]^{-2/2^*} \int_M R d\text{Vol} < 0.$$

If instead there exists $\tilde{g} \in [g]$ with $\tilde{R} = 0$, the same argument allows us to assume $R = 0$. Then $Y(M, [g]) \leq 0$, while $J_{2^*}(u) \geq 0$ for every $u \in W^{1,2}(M) \setminus \{0\}$; hence $Y(M, [g]) = 0$. Then, only the positive case remains. Consider $Y(M, [g]) > 0$ and denote by λ_1 the first eigenvalue of the conformal Laplacian; we will show that

$$Y(M, [g]) > 0 \implies \lambda_1 > 0.$$

Since $Y(M, [g]) > 0$, the numerator of the Yamabe functional is positive for every $u \neq 0$, indeed

$$c(n) \|\nabla u\|_2^2 + \int_M Ru^2 d\text{Vol} \geq Y(M, [g]) \|u\|_{2^*}^2 > 0,$$

hence, Hölder inequality gives

$$\|u\|_2^2 \leq [\text{Vol}(M)]^{1-2/2^*} \|u\|_{2^*}^2.$$

Therefore, for every $u \in W^{1,2}(M) \setminus \{0\}$, there holds

$$J_{2^*}(u) [\text{Vol}(M)]^{-2/n} \leq \frac{c(n) \|\nabla u\|_2^2 + \int_M Ru^2 d\text{Vol}}{\|u\|_2^2},$$

and taking the infimum over such functions, by the Rayleigh–Ritz characterization of the first eigenvalue,

$$Y(M, [g]) [\text{Vol}(M)]^{-2/n} \leq \lambda_1,$$

so $\lambda_1 > 0$.

Let u be a first eigenfunction of L . Since replacing u by $|u|$ leaves the Rayleigh quotient unchanged, $|u|$ is again a minimizer, hence a smooth eigenfunction and the strong maximum principle shows that it is strictly positive; we may therefore assume $u > 0$ and write

$$-c(n)\Delta u + Ru = \lambda_1 u = \lambda_1 u^{2-2^*} u^{2^*-1}.$$

Then, after the conformal change using u , we get the metric $\tilde{g}$ with the following scalar curvature

$$\tilde{R} = \lambda_1 u^{2-2^*} > 0.$$

Conversely, suppose there exists $\tilde{g} \in [g]$ with $\tilde{R} > 0$ on M . Without loss of generality, after doing a conformal change of the metric and using once again the fact that the infimum of the Yamabe functional is invariant under such change, we can assume that g is that metric. We then obtain the following estimate

$$\|u\|_{2^*}^{-2} \left[c(n) \int_M |\nabla u|^2 d\text{Vol} + \int_M Ru^2 d\text{Vol} \right] \geq \min \{ c(n), \min_M R \} \|u\|_{W^{1,2}}^2 \|u\|_{2^*}^{-2}$$

and the Sobolev inequality gives a uniform lower bound for the quotient of norms,

$$\frac{\|u\|_{W^{1,2}}^2}{\|u\|_{2^*}^2} \geq C > 0.$$

This gives a strictly positive lower bound for the Yamabe functional and hence $Y(M, [g]) > 0$. Finally, we observe that the sign of the Yamabe constant also coincides with that of the first eigenvalue of the conformal Laplacian. Indeed, if $u > 0$ is a first eigenfunction of L , with eigenvalue λ_1 , the conformal metric $\tilde{g} = u^{4/(n-2)}g$ has scalar curvature

$$\tilde{R} = \lambda_1 u^{2-2^*}.$$

Since u is strictly positive, the scalar curvature of $\tilde{g}$ has everywhere the same sign as λ_1 , and the equivalences just proved show that this is precisely the sign of $Y(M, [g])$. $\square$

In the proof of this theorem we also showed the following fact.

COROLLARY 1.9. *The first eigenvalue of the conformal Laplacian L_g of (M^n, g) has the same sign as the Yamabe constant $Y(M, [g])$.*

1.5. Concentration–compactness and existence below the spherical threshold

We conclude the “analytic part” of the thesis by proving that the strict inequality

$$Y(M, [g]) < Y(\mathbb{S}^n, [g_{\text{std}}])$$

guarantees that the infimum defining $Y(M, [g])$ is attained by a nonzero Sobolev function $u \in W^{1,2}(M)$. The proof relies on the concentration–compactness principle of P.-L. Lions (see [22,23,29]). Establishing this strict inequality for every closed manifold not conformally diffeomorphic to the standard sphere will be the geometric task of Chapter 2.

We begin by examining how “concentration” may prevent the strong convergence at the Sobolev exponent. Given a bounded sequence u_i in $W^{1,2}(M)$, the possible loss of compactness can be described through the measures

$$\nu_i = |u_i|^{2^*} d\text{Vol}, \quad \mu_i = |\nabla u_i|_g^2 d\text{Vol}.$$

Since M is closed, the “mass” cannot escape to infinity, and the Rellich–Kondrachov theorem gives, after passing to a subsequence, strong convergence in $L^2(M)$. Nevertheless, strong convergence in $L^{2^*}(M)$ may fail because of concentration, as described by the following lemma of P.-L. Lions (see [22,23,29]).

LEMMA 1.10 (Concentration–compactness principle). *Let $n > 2$ and let u_i be a sequence in $W^{1,2}(\mathbb{R}^n)$ satisfying the following assumptions.*

- *There exists $u \in W^{1,2}(\mathbb{R}^n)$ such that $u_i \rightharpoonup u$ in $W^{1,2}(\mathbb{R}^n)$.*
- *$u_i \rightarrow u$ almost everywhere.*
- *$\mu_i = |\nabla u_i|^2 dx \xrightarrow{*} \mu$ where μ is a bounded Radon measure.*
- *$\nu_i = |u_i|^{2^*} dx \xrightarrow{*} \nu$ where ν is a bounded Radon measure.*

Then the following conclusions hold.

- (1) *There is an at most countable index set J , with distinct points $x_j \in \mathbb{R}^n$ and positive numbers ν_j , for $j \in J$, such that*

$$\nu = |u|^{2^*} dx + \sum_{j \in J} \nu_j \delta_{x_j}.$$

- (2) *The weak form of the Sobolev inequality gives*

$$\mu \geq |\nabla u|^2 dx + \sum_{j \in J} \mu_j \delta_{x_j},$$

for a family of positive numbers $\{\mu_j \mid j \in J\}$ satisfying

$$\sigma_n \nu_j^{1/2^*} \leq \mu_j^{1/2} \quad \text{for all } j \in J.$$

In particular, $\sum_{j \in J} \nu_j^{2/2^*} < +\infty$.

A proof of the Euclidean statement is given in Appendix A, Lemma A.5, while its extension to a closed manifold follows by localization (see also [23, 29]).

COROLLARY 1.11 (Concentration–compactness on a closed manifold). *Let (M^n, g) be a closed Riemannian manifold with $n > 2$, and let $u_i \rightharpoonup u$ in $W^{1,2}(M)$, with $u_i \rightarrow u$ almost everywhere after passing to a subsequence. Suppose also that*

$$|u_i|^{2^*} d\text{Vol} \xrightarrow{*} \nu, \quad |\nabla u_i|_g^2 d\text{Vol} \xrightarrow{*} \mu.$$

Then there exist at most countably many points $x_j \in M$ and positive numbers ν_j, μ_j such that

$$\nu = |u|^{2^*} d\text{Vol} + \sum_j \nu_j \delta_{x_j}, \quad \mu \geq |\nabla u|_g^2 d\text{Vol} + \sum_j \mu_j \delta_{x_j},$$

and

$$\sigma_n^2 \nu_j^{2/2^*} \leq \mu_j \quad \text{for every } j.$$

To obtain the manifold version, we localize the Euclidean Sobolev inequality in normal coordinates on balls of radius r . Since $g_{ij} = \delta_{ij} + O(r^2)$ on such balls, its sharp constant is $\sigma_n^2 - o(1)$ as $r \rightarrow 0$; applying the resulting inequality to cutoffs of $u_i - u$, passing to the limit in i and then letting $r \rightarrow 0$ yields the stated atomic inequality and measure decomposition. In particular, a strict inequality between the Yamabe constant of M and that of the sphere excludes concentration of a minimizing sequence.

THEOREM 1.12. *On every closed Riemannian manifold (M^n, g) with $Y(M, [g]) < Y(\mathbb{S}^n, [g_{\text{std}}])$, the functional J_{2^*} admits minimizers.*

PROOF. We consider a minimizing sequence u_i in $W^{1,2}(M)$ normalized by

$$\|u_i\|_{2^*} = 1, \quad J_{2^*}(u_i) \rightarrow Y(M, [g]).$$

The normalization and Hölder's inequality give a uniform bound for $\|u_i\|_2$, while the boundedness of the Yamabe functionals then yields

$$c(n) \|\nabla u_i\|_2^2 = J_{2^*}(u_i) - \int_M R_g u_i^2 d\text{Vol} \leq C + \|R\|_\infty \|u_i\|_2^2 \leq C.$$

We may therefore pass to a subsequence for which

- (1) $u_i \rightarrow u$ almost everywhere,
- (2) $u_i \rightharpoonup u$ in $W^{1,2}(M)$,
- (3) $u_i \rightarrow u$ in $L^2(M)$,
- (4) $u_i \rightharpoonup u$ in $L^{2^*}(M)$.

For this subsequence, we consider the measures

$$\nu_i = |u_i|^{2^*} d\text{Vol}, \quad \mu_i = |\nabla u_i|_g^2 d\text{Vol}.$$

Their total masses are uniformly bounded, so that, after passing to a further subsequence,

$$\mu_i \xrightarrow{*} \mu, \quad \nu_i \xrightarrow{*} \nu.$$

By Corollary 1.11,

$$\nu = |u|^{2^*} d\text{Vol} + \sum_{j \in J} \nu_j \delta_{x_j}, \quad \mu \geq |\nabla u|_g^2 d\text{Vol} + \sum_{j \in J} \mu_j \delta_{x_j},$$

with $\sigma_n^2 \nu_j^{2/2^*} \leq \mu_j$.

We denote the strict gap between the two Yamabe constants by

$$\delta = Y(\mathbb{S}^n, [g_{\text{std}}]) - Y(M, [g]) > 0.$$

Since M is compact, the weak–star convergence of the gradient measures, together with the strong convergence in $L^2(M)$ and $Y(\mathbb{S}^n, [g_{\text{std}}]) = c(n)\sigma_n^2$ gives

$$\begin{aligned} c(n)\sigma_n^2 - \delta &= Y(M, [g]) \\ &\geq c(n) \int_M |\nabla u|^2 d\text{Vol} + \int_M Ru^2 d\text{Vol} + c(n) \sum_{j \in J} \mu_j \\ &\geq c(n) \int_M |\nabla u|^2 d\text{Vol} + \int_M Ru^2 d\text{Vol} + c(n)\sigma_n^2 \sum_{j \in J} \nu_j^{2/2^*}. \end{aligned}$$

Moreover,

$$1 = \|u\|_{2^*}^{2^*} + \sum_{j \in J} \nu_j.$$

We distinguish the possible values of $\|u\|_{2^*}$. If $\|u\|_{2^*} = 1$, the measure identity rules out concentration and

$$J_{2^*}(u) \leq \liminf_{i \rightarrow +\infty} J_{2^*}(u_i) = Y(M, [g]).$$

The reverse inequality follows from the definition of $Y(M, [g])$ and hence u is a minimizer.

If instead $\|u\|_{2^*} = 0$, all the mass is concentrated, so $\sum_j \nu_j = 1$ and because $2/2^* < 1$,

$$\sum_j \nu_j^{2/2^*} \geq \left(\sum_j \nu_j \right)^{2/2^*} = 1,$$

contradicting $c(n)\sigma_n^2 - \delta \geq c(n)\sigma_n^2$.

In the remaining case, we write $s = \|u\|_{2^*} \in (0, 1)$ and combine the preceding inequality with $J_{2^*}(u) \geq Y(M, [g])$ to obtain

$$Y(M, [g]) \geq Y(M, [g])s^2 + Y(\mathbb{S}^n, [g_{\text{std}}])(1 - s^{2^*})^{2/2^*}.$$

After rearranging, this becomes

$$Y(M, [g]) \geq \frac{(1 - s^{2^*})^{2/2^*}}{1 - s^2} Y(\mathbb{S}^n, [g_{\text{std}}]).$$

For $0 < s < 1$ and $2^* > 2$ we have

$$(1 - s^{2^*})^{2/2^*} > 1 - s^2,$$

so the factor on the right is strictly larger than 1, contradicting the assumed strict inequality. Hence $\|u\|_{2^*} = 1$, and the limit is a minimizer. $\square$

CHAPTER 2

Aubin's and Schoen's strict inequalities

Chapter 1 provided the first, “analytic”, part of our treatment of the Yamabe problem, culminating in the existence criterion that the Yamabe constant is attained whenever

$$Y(M, [g]) < Y(\mathbb{S}^n, [g_{\text{std}}]) .$$

We now turn to the second, “geometric”, part, whose central question is how to establish this strict inequality when (M^n, g) is not conformally diffeomorphic to the standard sphere. This is clearly equivalent to exhibiting a nonzero function $u \in W^{1,2}(M)$ such that

$$J_{2^*}(u) < Y(\mathbb{S}^n, [g_{\text{std}}]) = n(n-1)\text{Vol}(\mathbb{S}^n)^{2/n} .$$

Hence, the emphasis shifts from the compactness and the existence of minimizers to the geometric properties of (M^n, g) allowing the construction of such a function.

In the next two sections, we present two complementary approaches: Aubin's method, which exploits local conformal geometry through the Weyl tensor, and Schoen's method, which involves the Green function of the conformal Laplacian and the positive mass theorem (see [3,21,25]). The first applies to non-locally conformally flat manifolds of dimension $n \geq 6$, whereas the second addresses the remaining cases.

2.1. Aubin's strict inequality in the non-locally conformally flat case

We recall that, for $n \geq 4$, the *Weyl tensor* W of (M^n, g) is defined by the following orthogonal decomposition formula for the Riemann curvature tensor (see [7, Section 1.G])

$$\text{Riem} = \frac{R}{n(n-1)}g \otimes g + \frac{2}{n-2}\left(\text{Ric} - \frac{R}{n}g\right) \otimes g + W .$$

It follows that W is a *completely trace-free* $(0,4)$ -tensor (all its traces are zero) with the same algebraic symmetries as the Riemann tensor. Moreover, it transforms under a conformal change of the metric as follows: if $\tilde{g} = e^{2f}g$, with $f \in C^\infty(M)$, then

$$\widetilde{W} = e^{2f}W ,$$

where W and $\widetilde{W}$ are the Weyl tensors of g and $\tilde{g}$, respectively (see [7, Theorem 1.159]). Finally, the *Weyl-Schouten theorem* states that for $n \geq 4$, the metric g is locally conformally flat if and only if $W = 0$ (see [7, Theorem 1.165]).

REMARK 2.1. In dimension $n = 3$, the local conformal flatness is characterized by the vanishing of the *Cotton tensor*, of type $(0,3)$, defined in local coordinates by

$$C_{ijk} = \nabla_i \left(\text{Ric}_{jk} - \frac{R}{4}g_{jk} \right) - \nabla_j \left(\text{Ric}_{ik} - \frac{R}{4}g_{ik} \right)$$

(see [7, p. 435]).

By the Weyl–Schouten theorem, it clearly follows that if $n \geq 4$ and (M^n, g) is not locally conformally flat, then $W(p) \neq 0$ for some $p \in M$. This is the geometric condition used in the proof of the following theorem of Aubin (see [3,21]).

THEOREM 2.2. *For every closed Riemannian manifold (M^n, g) with $n \geq 6$ which is not locally conformally flat, we have*

$$Y(M, [g]) < Y(\mathbb{S}^n, [g_{\text{std}}]) .$$

The proof relies on the use of the *conformal normal coordinates*, described in the following theorem (see [21]).

THEOREM 2.3 (Conformal normal coordinates). *Let (M^n, g) be a Riemannian manifold with $n \geq 2$ and let $p \in M$. For every integer $N \geq 2$, there exists a smooth function f on M such that, setting $\tilde{g} = e^{2f}g$ and choosing normal coordinates for $\tilde{g}$ centered at p , we have*

$$\det \tilde{g}_{ij}(x) = 1 + O(|x|^N) ,$$

as $|x| \rightarrow 0$, where $|x|$ denotes the Euclidean norm of the coordinate vector. Moreover, if $N \geq 5$, then

$$\tilde{R}(x) = O(|x|^2) \quad \text{and} \quad \tilde{\Delta} \tilde{R}(p) = -\frac{1}{6} |\tilde{W}(p)|_{\tilde{g}}^2 ,$$

where $\tilde{W}$ is the Weyl tensor of $\tilde{g}$ (conventionally set to zero, if $n = 2, 3$).

REMARK 2.4. If g is locally conformally flat near p , the obvious choice for $\tilde{g}$ in this theorem is the flat metric in a neighbourhood around p .

REMARK 2.5. This result was subsequently strengthened by Cao [11] and Günther [19], who proved that the conformal metric $\tilde{g}$ can be chosen so that the determinant of $\tilde{g}_{ij}$ is identically equal to one in its normal coordinates near p .

Since the infimum of the Yamabe functional is the same for all the metrics in the conformal class of g , after possibly replacing g by the conformal metric $\tilde{g}$ provided by Theorem 2.3 and putting ourselves in normal coordinates centered at p , we shall construct a family of functions $u_\lambda \in W^{1,2}(M)$ (called “test functions”) and show that, for all sufficiently large $\lambda > 0$, we have

$$J_{2^*}(u_\lambda) < Y(\mathbb{S}^n, [g_{\text{std}}]) ,$$

hence proving Theorem 2.2, as $Y(M, [g]) \leq J_{2^*}(u_\lambda)$.

We give an idea of how these test functions u_λ are built. We start from the *Talenti bubbles* in $\mathbb{R}^n$,

$$\mu_\lambda(x) = \left(\frac{\lambda}{1 + \lambda^2 |x|^2} \right)^{(n-2)/2} \quad \text{for every } \lambda > 0 , \quad (2.1)$$

which attain equality in the sharp Euclidean Sobolev inequality (see [30]) and become increasingly concentrated near the origin as $\lambda \rightarrow +\infty$. We choose a point $p \in M$ with $W(p) \neq 0$ and use normal coordinates centered at p to identify an open neighborhood U of p with a ball in $\mathbb{R}^n$; then we multiply μ_λ by a smooth cutoff function supported in the ball; finally, we define u_λ at each point of U as the value of this product at the corresponding point of the ball, and set $u_\lambda = 0$ outside U .

PROOF OF THEOREM 2.2. As we observed above, by the Weyl–Schouten theorem, there exists a point $p \in M$ such that $W(p) \neq 0$. Since the infimum of J_{2^*} , depends only on the conformal class of the metric (and gives the Yamabe constant of such class) and the condition $W(p) \neq 0$ is preserved under conformal changes, we may assume that g is the metric given

by Theorem 2.3 at p , choosing an integer $N \geq n+4$. In the corresponding normal coordinates centered at such point, we then have

$$\det g_{ij}(x) = 1 + O(|x|^N), \quad R(x) = O(|x|^2), \quad \Delta R(p) = -\frac{1}{6}|W(p)|^2 < 0,$$

where $|x|$ is the Euclidean distance from the origin. In particular,

$$d\text{Vol} = (1 + O(|x|^N)) dx \quad (2.2)$$

in these coordinates.

Choose a g -orthonormal basis of $T_p M$, identify $T_p M$ with $\mathbb{R}^n$, and let B be a ball centered at the origin on which the exponential map $\exp_p$ is a diffeomorphism onto its image. As usual, we denote the pullback metric on B by g as well. Fix radii $0 < \rho_1 < \rho_2$ with $\overline{B}_{\rho_2} \subseteq B$, and choose a smooth radial *cutoff* function $\varphi \in C_c^\infty(\mathbb{R}^n)$ such that $0 \leq \varphi \leq 1$, $\varphi = 1$ on B_{ρ_1} and $\text{supp } \varphi \subseteq B_{\rho_2}$. On B we set $u_\lambda = \varphi \mu_\lambda$, using the functions μ_λ introduced above. We then regard u_λ as a function on $\exp_p(B)$ and extend it by zero to the rest of M . The resulting functions are smooth and are supported in the fixed neighborhood $\exp_p(B_{\rho_2})$ of p .

We first estimate the scalar-curvature term in

$$J_{2^*}(u_\lambda) = \frac{c(n) \int_M |\nabla u_\lambda|^2 d\text{Vol} + \int_M R u_\lambda^2 d\text{Vol}}{\|u_\lambda\|_{2^*}^2}.$$

Since $R(x) = O(|x|^2)$, both $R(0)$ and its first derivatives vanish and Taylor's formula in these coordinates gives

$$R(x) = \frac{1}{2} \sum_{i,j=1}^n \partial_{ij}^2 R(0) x^i x^j + \frac{1}{6} \sum_{i,j,k=1}^n \partial_{ijk}^3 R(0) x^i x^j x^k + O(|x|^4).$$

Then, substituting this expansion of R and the one of the volume given by formula (2.2) in the integral of $R u_\lambda^2$ and observing that the integral of the cubic polynomial against u_λ^2 with respect to dx vanishes, being u_λ radially symmetric, we get

$$\int_M R u_\lambda^2 d\text{Vol} = \frac{1}{2} \sum_{i,j=1}^n \partial_{ij}^2 R(0) \int_B x^i x^j u_\lambda^2 dx + E_\lambda, \quad (2.3)$$

where

$$|E_\lambda| \leq C \int_B (|x|^4 + |x|^{N+2} + |x|^{N+3} + |x|^{N+4}) u_\lambda^2 dx \leq C \int_B |x|^4 u_\lambda^2 dx.$$

We now determine the asymptotic order of the last integral, as $\lambda \rightarrow +\infty$. We recall that $u_\lambda = \varphi \mu_\lambda$, with μ_λ defined in formula (2.1). Using the change of variables $y = \lambda x$, we obtain

$$|E_\lambda| \leq C \int_B |x|^4 u_\lambda^2 dx \leq C \lambda^{-6} \int_{|y| < \rho_2 \lambda} \frac{|y|^4 \varphi^2(y/\lambda)}{(1 + |y|^2)^{n-2}} dy \leq C \lambda^{-6} \int_{|y| < \rho_2 \lambda} \frac{|y|^4}{(1 + |y|^2)^{n-2}} dy,$$

as $\varphi \leq 1$ and since $|y|^4/(1 + |y|^2)^{n-2}$ is clearly bounded by $|y|^{8-2n}$, for $|y| \geq 1$, after some straightforward computation, we have

$$\int_B |x|^4 u_\lambda^2 dx = \begin{cases} O(\lambda^{-4}) & n = 6 \\ O(\lambda^{-5}) & n = 7 \\ O(\lambda^{-6} \log \lambda) & n = 8 \\ O(\lambda^{-6}) & n \geq 9 \end{cases}$$

Combining these estimates, we conclude that

$$E_\lambda = \begin{cases} O(\lambda^{-4}) & n = 6 \\ o(\lambda^{-4}) & n > 6 \end{cases}$$

We now compute the quadratic contribution to the scalar-curvature integral in formula (2.3), using the radial symmetry of u_λ to evaluate the integrals of $x^i x^j u_\lambda^2$ over B . We shall express this contribution in terms of $\Delta R(p)$ and then use the identity $\Delta R(p) = -|W(p)|^2/6$. Since $u_\lambda = \varphi\mu_\lambda$ is radial and B is a ball centered at the origin, we have

$$\int_B x^i x^j u_\lambda^2 dx = 0,$$

for every pair $i \neq j$, by the change of variables $x^i \mapsto -x^i$, while the invariance under permutations of the coordinates gives

$$\int_B (x^i)^2 u_\lambda^2 dx = \frac{1}{n} \sum_{k=1}^n \int_B (x^k)^2 u_\lambda^2 dx = \frac{1}{n} \int_B |x|^2 u_\lambda^2 dx.$$

Consequently,

$$\int_B x^i x^j u_\lambda^2 dx = \frac{\delta^{ij}}{n} \int_B |x|^2 u_\lambda^2 dx$$

and using the identity

$$\sum_{i=1}^n \partial_{ii}^2 R(0) = \Delta R(p) = -|W(p)|^2/6,$$

we conclude that

$$\frac{1}{2} \sum_{i,j=1}^n \partial_{ij}^2 R(0) \int_B x^i x^j u_\lambda^2 dx = -\frac{|W(p)|^2}{12n} \int_B |x|^2 u_\lambda^2 dx.$$

We now estimate the integral of $|x|^2 u_\lambda^2$: recalling that $u_\lambda = \varphi\mu_\lambda$, making the change of variables $y = \lambda x$, we obtain

$$\int_B |x|^2 u_\lambda^2 dx = \lambda^{-4} \int_{|y| < \rho_2 \lambda} \frac{|y|^2 \varphi^2(y/\lambda)}{(1 + |y|^2)^{n-2}} dy$$

and we distinguish the cases $n = 6$ and $n > 6$, since the behavior of the last integral as $\lambda \rightarrow +\infty$ is different in the two cases.

For $n = 6$, we have

$$\frac{|y|^2}{(1 + |y|^2)^4} = |y|^{-6} + O(|y|^{-8})$$

for $|y| \geq 1$. Denoting by ω_n the volume of the unit ball in $\mathbb{R}^n$, for $i = 1, 2$ and λ sufficiently large we obtain

$$\int_{1 < |y| < \rho_i \lambda} |y|^{-6} dy = 6\omega_6 \int_1^{\rho_i \lambda} \frac{dt}{t} = 6\omega_6 \log(\rho_i \lambda),$$

whereas

$$\int_{1 < |y| < \rho_i \lambda} |y|^{-8} dy \leq 6\omega_6 \int_1^{+\infty} t^{-3} dt = 3\omega_6.$$

Moreover, the function $|y|^2/(1+|y|^2)^4$ is bounded on the unit ball, so its integral over this ball is finite. Combining these estimates, we conclude that

$$\int_{|y| < \rho_i \lambda} \frac{|y|^2}{(1+|y|^2)^4} dy = 6\omega_6 \log(\rho_i \lambda) + O(1) = 6\omega_6 \log \lambda + O(1),$$

since $\rho_i > 0$ is fixed. Finally, recalling that $0 \leq \varphi \leq 1$, $\varphi = 1$ on B_{ρ_1} and $\varphi = 0$ outside B_{ρ_2} , we have

$$\int_{|y| < \rho_1 \lambda} \frac{|y|^2}{(1+|y|^2)^4} dy \leq \int_{|y| < \rho_2 \lambda} \frac{|y|^2 \varphi^2(y/\lambda)}{(1+|y|^2)^4} dy \leq \int_{|y| < \rho_2 \lambda} \frac{|y|^2}{(1+|y|^2)^4} dy.$$

As the preceding asymptotic estimate applies to both the first and the last integral, we get

$$\int_{|y| < \rho_2 \lambda} \frac{|y|^2 \varphi^2(y/\lambda)}{(1+|y|^2)^4} dy = 6\omega_6 \log \lambda + O(1)$$

as $\lambda \rightarrow +\infty$.

For $n > 6$, the function $|y|^2/(1+|y|^2)^{n-2}$ is integrable on $\mathbb{R}^n$, as it is bounded near the origin and decays as $O(|y|^{6-2n})$ at infinity. Recalling that $\varphi = 1$ on B_{ρ_1} , $0 \leq \varphi \leq 1$ and $\varphi = 0$ outside B_{ρ_2} , we obtain

$$\begin{aligned} 0 &\leq \int_{\mathbb{R}^n} \frac{|y|^2}{(1+|y|^2)^{n-2}} dy - \int_{|y| < \rho_2 \lambda} \frac{|y|^2 \varphi^2(y/\lambda)}{(1+|y|^2)^{n-2}} dy = \int_{\mathbb{R}^n} \frac{|y|^2(1-\varphi^2(y/\lambda))}{(1+|y|^2)^{n-2}} dy \\ &\leq \int_{|y| \geq \rho_1 \lambda} \frac{|y|^2}{(1+|y|^2)^{n-2}} dy \leq \int_{|y| \geq \rho_1 \lambda} |y|^{6-2n} dy = \frac{n\omega_n}{n-6} (\rho_1 \lambda)^{6-n} = O(\lambda^{6-n}), \end{aligned}$$

hence, as $\lambda \rightarrow +\infty$,

$$\int_{|y| < \rho_2 \lambda} \frac{|y|^2 \varphi^2(y/\lambda)}{(1+|y|^2)^{n-2}} dy = \int_{\mathbb{R}^n} \frac{|y|^2}{(1+|y|^2)^{n-2}} dy + O(\lambda^{6-n}) = C_n + O(\lambda^{6-n}).$$

with $C_n = \int_{\mathbb{R}^n} \frac{|y|^2}{(1+|y|^2)^{n-2}} dy > 0$.

Combining the estimates above with formula (2.3), we conclude that for all sufficiently large λ ,

$$\int_M Ru_\lambda^2 d\text{Vol} = -\frac{\omega_6 |W(p)|^2}{12} \lambda^{-4} \log \lambda + O(\lambda^{-4}) < 0, \quad (2.4)$$

when $n = 6$, while for $n > 6$,

$$\int_M Ru_\lambda^2 d\text{Vol} = -\frac{C_n}{12n} |W(p)|^2 \lambda^{-4} + o(\lambda^{-4}) < 0. \quad (2.5)$$

We now estimate the gradient term and the L^{2^*} norm of u_λ to complete the asymptotic expansion of $J_{2^*}(u_\lambda)$. As $u_\lambda = \varphi \mu_\lambda$ is radial in the chosen normal coordinates, the Gauss lemma gives $|\nabla u_\lambda|^2 = |\nabla^{\mathbb{R}^n} u_\lambda|_{\mathbb{R}^n}^2$, where $\nabla^{\mathbb{R}^n}$ and $|\cdot|_{\mathbb{R}^n}$ denote, respectively, the gradient and the norm with respect to the Euclidean metric on $\mathbb{R}^n$. Then, formula (2.2) yields

$$\int_M |\nabla u_\lambda|^2 d\text{Vol} = \int_B |\varphi \nabla^{\mathbb{R}^n} \mu_\lambda + \mu_\lambda \nabla^{\mathbb{R}^n} \varphi|_{\mathbb{R}^n}^2 (1 + O(|x|^N)) dx.$$

Differentiating the expression for μ_λ in formula (2.1), we find

$$\nabla^{\mathbb{R}^n} \mu_\lambda(x) = -(n-2) \frac{\lambda^{(n+2)/2} x}{(1 + \lambda^2 |x|^2)^{n/2}},$$

hence, for $x \neq 0$, there holds

$$\mu_\lambda(x) \leq \lambda^{(2-n)/2} |x|^{2-n} \quad \text{and} \quad |\nabla^{\mathbb{R}^n} \mu_\lambda(x)|_{\mathbb{R}^n} \leq (n-2) \lambda^{(2-n)/2} |x|^{1-n}.$$

We now compare the integral of the squared Euclidean gradient of $u_\lambda = \varphi \mu_\lambda$ with the corresponding integral for μ_λ . Expanding the square in the preceding formula, we have

$$|\nabla^{\mathbb{R}^n} u_\lambda|_{\mathbb{R}^n}^2 = \varphi^2 |\nabla^{\mathbb{R}^n} \mu_\lambda|_{\mathbb{R}^n}^2 + 2\varphi \mu_\lambda \nabla^{\mathbb{R}^n} \varphi \cdot \nabla^{\mathbb{R}^n} \mu_\lambda + \mu_\lambda^2 |\nabla^{\mathbb{R}^n} \varphi|_{\mathbb{R}^n}^2.$$

The last two terms vanish outside $B_{\rho_2} \setminus B_{\rho_1}$, where both functions μ_λ and $|\nabla^{\mathbb{R}^n} \mu_\lambda|_{\mathbb{R}^n}$ are $O(\lambda^{(2-n)/2})$, so their integrals over B are $O(\lambda^{2-n})$, whereas the difference between the integral of $\varphi^2 |\nabla^{\mathbb{R}^n} \mu_\lambda|_{\mathbb{R}^n}^2$ over B and that of $|\nabla^{\mathbb{R}^n} \mu_\lambda|_{\mathbb{R}^n}^2$ over $\mathbb{R}^n$ is controlled by

$$\int_{\mathbb{R}^n \setminus B_{\rho_1}} |\nabla^{\mathbb{R}^n} \mu_\lambda|_{\mathbb{R}^n}^2 dx \leq C \lambda^{2-n} \int_{\mathbb{R}^n \setminus B_{\rho_1}} |x|^{2-2n} dx = O(\lambda^{2-n}),$$

thus,

$$\int_B |\nabla^{\mathbb{R}^n} u_\lambda|_{\mathbb{R}^n}^2 dx = \int_{\mathbb{R}^n} |\nabla^{\mathbb{R}^n} \mu_\lambda|_{\mathbb{R}^n}^2 dx + O(\lambda^{2-n}). \quad (2.6)$$

We then estimate the difference between the integral of $|\nabla u_\lambda|^2$ over M with respect to $d\text{Vol}$ and the integral of $|\nabla^{\mathbb{R}^n} u_\lambda|_{\mathbb{R}^n}^2$ over B with respect to dx . Recalling that $|\nabla u_\lambda|^2 = |\nabla^{\mathbb{R}^n} u_\lambda|_{\mathbb{R}^n}^2$ and using $d\text{Vol} = (1 + O(|x|^N)) dx$ together with the estimates obtained above, we get

$$\begin{aligned} \left| \int_M |\nabla u_\lambda|^2 d\text{Vol} - \int_B |\nabla^{\mathbb{R}^n} u_\lambda|_{\mathbb{R}^n}^2 dx \right| &\leq C \int_B |x|^N |\nabla^{\mathbb{R}^n} u_\lambda|_{\mathbb{R}^n}^2 dx \\ &\leq C \lambda^{2-n} \left(\int_B |x|^{N+2-2n} dx + \int_{B_{\rho_2} \setminus B_{\rho_1}} |x|^{N+4-2n} dx \right) = O(\lambda^{2-n}), \end{aligned}$$

where the first integral in parentheses is finite because $N > n - 2$, while the second is taken over a fixed annulus bounded away from the origin.

Combining this inequality with equation (2.6), we obtain

$$\int_M |\nabla u_\lambda|^2 d\text{Vol} = \int_{\mathbb{R}^n} |\nabla^{\mathbb{R}^n} \mu_\lambda|_{\mathbb{R}^n}^2 dx + O(\lambda^{2-n}). \quad (2.7)$$

Finally, to estimate the denominator in the definition of $J_{2^*}(u_\lambda)$, we use the expression for μ_λ in formula (2.1) and the identity $2^* = 2n/(n-2)$, obtaining

$$\mu_\lambda^{2^*}(x) = \frac{\lambda^n}{(1 + \lambda^2 |x|^2)^n} \leq \lambda^{-n} |x|^{-2n},$$

for $x \neq 0$. Then, since $u_\lambda = \mu_\lambda$ on B_{ρ_1} and $0 \leq u_\lambda \leq \mu_\lambda$ elsewhere, using $d\text{Vol} = (1 + O(|x|^N)) dx$, we obtain

$$\left| \int_M u_\lambda^{2^*} d\text{Vol} - \int_{\mathbb{R}^n} \mu_\lambda^{2^*} dx \right| \leq C \int_B |x|^N u_\lambda^{2^*} dx + \int_{\mathbb{R}^n \setminus B_{\rho_1}} \mu_\lambda^{2^*} dx$$

and the two integrals on the right-hand side satisfy

$$\int_B |x|^N u_\lambda^{2^*} dx \leq \lambda^{-n} \int_B |x|^{N-2n} dx = O(\lambda^{-n}) \quad \text{and} \quad \int_{\mathbb{R}^n \setminus B_{\rho_1}} \mu_\lambda^{2^*} dx = O(\lambda^{-n}),$$

where the first estimate uses $N > n$ to ensure that the integral of $|x|^{N-2n}$ is finite. Consequently,

$$\int_M u_\lambda^{2^*} d\text{Vol} = \int_{\mathbb{R}^n} \mu_\lambda^{2^*} dx + O(\lambda^{-n}). \quad (2.8)$$

To evaluate the leading terms in formulas (2.7) and (2.8), we apply the change of variables $y = \lambda x$ to the integrals over $\mathbb{R}^n$ appearing on their right-hand sides and obtain

$$\int_{\mathbb{R}^n} |\nabla^{\mathbb{R}^n} \mu_\lambda|_{\mathbb{R}^n}^2 dx = \int_{\mathbb{R}^n} |\nabla^{\mathbb{R}^n} \mu_1|_{\mathbb{R}^n}^2 dy \quad \text{and} \quad \int_{\mathbb{R}^n} \mu_\lambda^{2^*} dx = \int_{\mathbb{R}^n} \mu_1^{2^*} dy > 0.$$

We now use the estimates for the gradient integral and the L^{2^*} integral obtained in formulas (2.7) and (2.8) to evaluate $J_{2^*}(u_\lambda)$. By means of formula (2.8) and of the second equality above, the estimate for the L^{2^*} integral yields

$$\|u_\lambda\|_{2^*}^2 = \left(\int_M u_\lambda^{2^*} d\text{Vol} \right)^{2/2^*} = \left(\int_{\mathbb{R}^n} \mu_1^{2^*} dy + O(\lambda^{-n}) \right)^{2/2^*} = \|\mu_1\|_{L^{2^*}(\mathbb{R}^n)}^2 + O(\lambda^{-n}),$$

therefore, since $\|\mu_1\|_{L^{2^*}(\mathbb{R}^n)} > 0$, we have

$$\frac{1}{\|u_\lambda\|_{2^*}^2} = \frac{1}{\|\mu_1\|_{L^{2^*}(\mathbb{R}^n)}^2} + O(\lambda^{-n}).$$

Substituting this expansion and the estimate for the gradient integral into the definition of $J_{2^*}(u_\lambda)$, we obtain

$$\begin{aligned} J_{2^*}(u_\lambda) &= \left(c(n) \int_{\mathbb{R}^n} |\nabla^{\mathbb{R}^n} \mu_1|_{\mathbb{R}^n}^2 dy + \int_M Ru_\lambda^2 d\text{Vol} + O(\lambda^{2-n}) \right) \left(\frac{1}{\|\mu_1\|_{L^{2^*}(\mathbb{R}^n)}^2} + O(\lambda^{-n}) \right) \\ &= \frac{c(n) \int_{\mathbb{R}^n} |\nabla^{\mathbb{R}^n} \mu_1|_{\mathbb{R}^n}^2 dy}{\|\mu_1\|_{L^{2^*}(\mathbb{R}^n)}^2} + \frac{\int_M Ru_\lambda^2 d\text{Vol}}{\|\mu_1\|_{L^{2^*}(\mathbb{R}^n)}^2} + O(\lambda^{2-n}). \end{aligned}$$

Indeed, formula (2.7) and the estimate $\int_M Ru_\lambda^2 d\text{Vol} = O(\lambda^{-4} \log \lambda)$ obtained above imply that $c(n) \int_M |\nabla u_\lambda|^2 d\text{Vol} + \int_M Ru_\lambda^2 d\text{Vol}$ is bounded; thus, multiplying this sum by the $O(\lambda^{-n})$ term in the expansion of $1/\|u_\lambda\|_{2^*}^2$ gives $O(\lambda^{-n})$, which is clearly also $O(\lambda^{2-n})$.

Finally, since μ_1 attains equality in the sharp Euclidean Sobolev inequality (1.3), we have

$$\frac{\int_{\mathbb{R}^n} |\nabla^{\mathbb{R}^n} \mu_1|_{\mathbb{R}^n}^2 dy}{\|\mu_1\|_{L^{2^*}(\mathbb{R}^n)}^2} = \sigma_n^2,$$

where σ_n is the optimal constant in such inequality. Then, recalling that there holds $c(n)\sigma_n^2 = Y(\mathbb{S}^n, [g_{\text{std}}])$ by Theorem 1.2, we conclude that

$$J_{2^*}(u_\lambda) = Y(\mathbb{S}^n, [g_{\text{std}}]) + \frac{\int_M Ru_\lambda^2 d\text{Vol}}{\|\mu_1\|_{L^{2^*}(\mathbb{R}^n)}^2} + O(\lambda^{2-n}).$$

Since $O(\lambda^{2-n}) = O(\lambda^{-4})$ for $n = 6$ and $O(\lambda^{2-n}) = o(\lambda^{-4})$ for $n > 6$, substituting formulas (2.4) and (2.5) into the preceding expression for $J_{2^*}(u_\lambda)$ gives, for $n = 6$,

$$J_{2^*}(u_\lambda) = Y(\mathbb{S}^6, [g_{\text{std}}]) - \frac{\omega_6 |W(p)|^2}{12 \|\mu_1\|_{L^{2^*}(\mathbb{R}^6)}^2} \lambda^{-4} \log \lambda + O(\lambda^{-4}),$$

while for $n > 6$ we obtain

$$J_{2^*}(u_\lambda) = Y(\mathbb{S}^n, [g_{\text{std}}]) - \frac{C_n |W(p)|^2}{12n \|\mu_1\|_{L^{2^*}(\mathbb{R}^n)}^2} \lambda^{-4} + o(\lambda^{-4}).$$

Hence, for $n = 6$, we have

$$\lim_{\lambda \rightarrow +\infty} \frac{Y(\mathbb{S}^6, [g_{\text{std}}]) - J_{2^*}(u_\lambda)}{\lambda^{-4} \log \lambda} = \frac{\omega_6 |W(p)|^2}{12 \|\mu_1\|_{L^{2^*}(\mathbb{R}^6)}^2} > 0,$$

whereas for $n > 6$,

$$\lim_{\lambda \rightarrow +\infty} \frac{Y(\mathbb{S}^n, [g_{\text{std}}]) - J_{2^*}(u_\lambda)}{\lambda^{-4}} = \frac{C_n |W(p)|^2}{12n \|\mu_1\|_{L^{2^*}(\mathbb{R}^n)}^2} > 0.$$

The strict positivity follows from $W(p) \neq 0$. Thus, in either case, $J_{2^*}(u_\lambda) < Y(\mathbb{S}^n, [g_{\text{std}}])$ for all sufficiently large λ , then by the definition of the Yamabe constant,

$$Y(M, [g]) \leq J_{2^*}(u_\lambda) < Y(\mathbb{S}^n, [g_{\text{std}}]),$$

which proves the theorem. $\square$

2.2. Schoen's approach to the remaining cases

The method of the preceding section uses the Weyl tensor to prove that the Yamabe constant is strictly smaller than that of the standard sphere when $n \geq 6$ and the metric is not locally conformally flat. The remaining cases were settled by Schoen in the following theorem (see [21, 25]), which, together with the result of the preceding section, completes the proof of Theorem 1.1.

THEOREM 2.6. *Let (M^n, g) be a closed Riemannian manifold of dimension $n \geq 3$ that is not conformally diffeomorphic to the standard sphere. If $3 \leq n \leq 5$, or if g is locally conformally flat, then*

$$Y(M, [g]) < Y(\mathbb{S}^n, [g_{\text{std}}]).$$

Schoen's approach is more "global" than Aubin's one, which is instead essentially "local". This is necessary since, for instance, in the locally conformally flat case there is no hope of finding a function $u \in W^{1,2}(M)$ such that $J_{2^*}(u) < Y(\mathbb{S}^n, [g_{\text{std}}])$ by considering only the geometry in a small neighborhood of a point of M . Indeed, such a neighborhood is conformally equivalent to a domain in $\mathbb{R}^n$, and the sharp Euclidean Sobolev inequality, given in formula (1.3), implies that every nonzero function supported in that neighborhood satisfies

$$J_{2^*}(u) \geq c(n) \sigma_n^2 = Y(\mathbb{S}^n, [g_{\text{std}}]),$$

by the conformal invariance of J_{2^*} . Thus, the difficulty is not simply that the Weyl tensor vanishes identically, so that Aubin's curvature expansion cannot provide a negative contribution to $J_{2^*}(u)$, but that the geometry of a sufficiently small neighborhood cannot distinguish the manifold from the standard sphere for the purpose of this estimate.

We now explain the main steps of the proof. As established in Corollary 1.9, the first eigenvalue $\lambda_1(L_g)$ of the conformal Laplacian L_g and the Yamabe constant $Y(M, [g])$ have the same sign. Thus, if $\lambda_1(L_g) \leq 0$, the desired strict inequality is immediate, and we may assume $\lambda_1(L_g) > 0$. Under this assumption, we shall first recall the existence of a positive *Green function* of L_g and its expansion near a fixed point $p \in M$, after possibly replacing (as in Aubin's proof) the metric g by the conformal metric $\tilde{g}$ provided by Theorem 2.3 and putting

ourselves in normal coordinates centered at p . With the normalization adopted below, this expansion is given by

$$G_p(x) = r^{2-n} + A + O(r),$$

where $r = d_g(x, p)$ and A is a constant. We shall then use G_p to define the conformal metric

$$\widehat{g} = G_p^{4/(n-2)} g$$

on $\widehat{M} = M \setminus \{p\}$. Since $L_g G_p = 0$ on $\widehat{M}$, the scalar curvature of $\widehat{g}$ vanishes. We shall prove that $(\widehat{M}, \widehat{g})$ is complete and asymptotically flat, using the expansion of G_p and the inversion of the normal coordinates centered at p .

In the second subsection, we shall compute the ADM mass of $(\widehat{M}, \widehat{g})$ and prove that $m_{\text{ADM}} = 2A$. The positive mass theorem will then give $A \geq 0$, while its rigidity statement will imply that $A = 0$ is possible only if (M^n, g) is conformally diffeomorphic to the standard sphere. Under the hypotheses of Theorem 2.6, we shall therefore have $A > 0$.

In the last subsection, we shall construct a family of test functions $\widehat{\mu}_\lambda$ directly on $(\widehat{M}, \widehat{g})$, using Euclidean Talenti bubbles in the asymptotically flat end and extending them as constants on the remaining compact region. We shall estimate the integrals of $|\nabla \widehat{\mu}_\lambda|_{\widehat{g}}^2$ and $\widehat{\mu}_\lambda^{2^*}$ with respect to $d\widehat{\text{Vol}}$, obtaining

$$c(n) \int_{\widehat{M}} |\nabla \widehat{\mu}_\lambda|_{\widehat{g}}^2 d\widehat{\text{Vol}} = Y(\mathbb{S}^n, [g_{\text{std}}]) \|\widehat{\mu}_\lambda\|_{2^*}^2 - 4n(n-1)\omega_n A \lambda^{2-n} + O(\lambda^{1-n}).$$

Since $A > 0$, this estimate shows that the Yamabe functional relative to $\widehat{g}$ evaluated $\widehat{\mu}_\lambda$, is strictly smaller than $Y(\mathbb{S}^n, [g_{\text{std}}])$ for all sufficiently large λ . Finally, we shall show that $G_p \widehat{\mu}_\lambda$ extends across p to a function in $W^{1,2}(M)$ and use the conformal invariance of the Yamabe functional to conclude that

$$Y(M, [g]) < Y(\mathbb{S}^n, [g_{\text{std}}]),$$

completing the proof of Theorem 2.6.

2.2.1. The Green function and the conformal blow-up. We first recall the existence and the local expansion of the Green function of the conformal Laplacian L_g on a closed Riemannian manifold (M^n, g) (see [5, 13]).

THEOREM 2.7 (Existence of the Green function). *Let (M^n, g) be a closed Riemannian manifold of dimension $n \geq 3$. If $\lambda_1(L_g) > 0$ then there exists a continuous positive Green function G , satisfying $L_g G(x, \cdot) = \delta_x$ in the sense of distributions,*

$$G: (M \times M) \setminus \Delta \longrightarrow \mathbb{R} \quad \text{where} \quad \Delta = \{(x, x) \mid x \in M\}.$$

The Green function G has the following properties:

- For any x, y in M with $x \neq y$ there holds

$$G(x, y) = G(y, x)$$

and writing $G_x(y) = G(x, y)$, we have $G_x \in C^\infty(M \setminus \{x\})$.

- For any x, y in M with $x \neq y$, we have

$$d_g(x, y)^{n-2} G(x, y) \leq C,$$

where the constant $C > 0$ depends only on (M^n, g) .

- There exists $\delta > 0$ that depends only on (M^n, g) such that for any $x, y \in M, x \neq y$, if $d_g(x, y) < \delta$, then

$$\left| d_g(x, y)^{n-2} G(x, y) - \frac{1}{c(n)n(n-2)\omega_n} \right| \leq C d_g(x, y)$$

and

$$\left| d_g(x, y)^{n-1} |\nabla_y G(x, y)| - \frac{1}{c(n)n\omega_n} \right| \leq C d_g(x, y),$$

where the constant $C > 0$ depends only on (M^n, g) .

Now we assume that the metric g is the one in the conformal class $[g]$ provided by Theorem 2.3 and we put ourselves in normal coordinates centered at p in an open neighborhood $U \subseteq M$. In the locally conformally flat case, we choose g to be flat near p , as allowed by Remark 2.4. Since we are considering the case $Y(M, [g]) > 0$, we have $\lambda_1(L_g) > 0$ (Corollary 1.9), hence the Green function of L_g exists. We want to describe its behavior near its pole $p \in M$.

We denote by $\mathcal{P}_k(U)$ the space of functions on U whose expressions in these coordinates are homogeneous polynomials of degree k on $\mathbb{R}^n$. Moreover, in the next theorem, $f(x) = O_2(r^k)$ means that

$$|D^j f(x)| \leq C r^{k-j},$$

for $j = 0, 1, 2$, where D^j denotes the Euclidean derivatives in such coordinates and $r = d_g(x, p)$. The following expansion of the Green function is given in [21, Lemma 6.4] (see also Theorem A.6 in Appendix A).

THEOREM 2.8. *Let (M^n, g) be a closed Riemannian manifold with $n \geq 3$ and g chosen as in Theorem 2.3, with N large enough and $\lambda_1(L_g) > 0$. Letting $p \in M$ and let G be the positive Green function of L_g with pole at p , normalized by $L_g G(p, \cdot) = \delta_p$ and set*

$$G_p = c(n)n(n-2)\omega_n G(p, \cdot).$$

Then,

$$L_g G_p = c(n)n(n-2)\omega_n \delta_p$$

and in normal coordinates on $U \setminus \{p\}$,

$$G_p(x) = r^{2-n} \left(1 + \sum_{k=4}^n \psi_k(x) \right) + C \log r + O_2(1),$$

where $\psi_k \in \mathcal{P}_k(U)$ and C is a constant that can be nonzero only if n is even.

In particular, if $3 \leq n \leq 5$ or if g is locally conformally flat near p , there exists a constant A such that

$$G_p(x) = r^{2-n} + A + O_2(r).$$

DEFINITION 2.9. Let (M^n, g) be a closed Riemannian manifold with $\lambda_1(L_g) > 0$, fix $p \in M$, and let G_p be the positive Green function of L_g with pole at p , normalized as in Theorem 2.8.

Set $\widehat{M} = M \setminus \{p\}$ and $\widehat{g} = G_p^{\frac{4}{n-2}} g$. We call $(\widehat{M}, \widehat{g})$ the *generalized stereographic projection* or the *conformal blow-up* of (M^n, g) at p . The associated map is the natural inclusion

$$S: \widehat{M} \hookrightarrow M.$$

Denoting by $\widehat{R}$ the scalar curvature of $(\widehat{M}, \widehat{g})$, the conformal transformation law gives

$$\widehat{R} = G_p^{-\frac{n+2}{n-2}} L_g G_p = 0,$$

since $L_g G_p = 0$ on $\widehat{M}$.

REMARK 2.10. The sphere provides the model for this construction. Let g_0 be the standard metric on $\mathbb{S}^n$, let N be its north pole, and let $\pi_N: \mathbb{S}^n \setminus \{N\} \rightarrow \mathbb{R}^n$ be the standard stereographic projection from N . The pullback metric $\pi_N^* \delta$ is flat and conformal to g_0 . Its conformal factor is a positive multiple of the Green function G_N of L_{g_0} with pole at N . With the normalization of Theorem 2.8, we have

$$\pi_N^* \delta = 4G_N^{4/(n-2)} g_0.$$

To study the geometry of $\widehat{g}$ near the removed point, we recall the definition of asymptotic flatness, following [6, 21].

DEFINITION 2.11 (Asymptotically flat manifold). Let (M^n, g) , $n \geq 3$, be a complete Riemannian manifold. Given $\tau > 0$ and $\alpha \in (0, 1)$, we say that (M^n, g) is $C_\tau^{1, \alpha}$ -asymptotically flat of order τ if there exists a compact set $K \subseteq M$ such that $M \setminus K$ is the disjoint union of finitely many ends $E_1, \dots, E_N$ and, for every $k \in \{1, \dots, N\}$, there exists a diffeomorphism

$$\Phi_k: E_k \rightarrow \mathbb{R}^n \setminus \overline{B}_{\rho_0}(0),$$

for some $\rho_0 > 0$. The coordinates $(x^1, \dots, x^n)$ induced by Φ_k are called *asymptotically flat coordinates*. In each such coordinate system we write

$$g = g_{ij} dx^i \otimes dx^j, \quad g_{ij} = \delta_{ij} + \gamma_{ij}.$$

For $\rho > 0$, set $A_\rho = \{x \in \mathbb{R}^n \mid \rho/2 < |x| < 2\rho\}$ and require

$$\sup_{A_\rho} (\rho^\tau |\gamma| + \rho^{1+\tau} |D\gamma|) + \rho^{1+\tau+\alpha} [D\gamma]_{\alpha; A_\rho} \leq C$$

for all sufficiently large ρ , with a constant C independent of ρ . Here D is the Euclidean differential and the tensor norms are Euclidean, and the Hölder seminorm is defined by

$$[D\gamma]_{\alpha; A_\rho} = \sup_{\substack{x, y \in A_\rho \\ x \neq y}} \frac{|D\gamma(x) - D\gamma(y)|}{|x - y|^\alpha}.$$

For the expansions used below, we also need the stronger C_τ^2 decay, meaning that in each asymptotically flat coordinate system, there holds

$$\sum_{j=0}^2 |x|^j |D^j \gamma| \leq C |x|^{-\tau}$$

for sufficiently large $|x|$. This is the meaning of $g_{ij} - \delta_{ij} = O_2(|x|^{-\tau})$ in the formulas below.

We now prove the asymptotic flatness and completeness of the conformal blow-up. In the normal coordinates centered at p , we introduce the “inverted” coordinates

$$y^i = r^{-2} x^i.$$

We set $\rho = |y| = 1/r$ and notice that this is the Euclidean inversion $x = y/|y|^2$, whose differential is $|y|^{-2}$ times a reflection in the radial direction.

THEOREM 2.12. *Let (M^n, g) be closed with g as in Theorem 2.8 and let $(\widehat{M}, \widehat{g})$ be its generalized stereographic projection from $p \in M$. Then $(\widehat{M}, \widehat{g})$ is complete and C_τ^2 -asymptotically flat, with $\tau = n - 2$ if g is locally conformally flat or $n = 3, 4$, and $\tau = 2$ if $n = 5$ and g is not locally conformally flat.*

Explicitly, when $n = 3, 4$ or if g is locally conformally flat, the expansion becomes

$$\widehat{g}_{ij} = \left(1 + \frac{4A}{n-2}\rho^{2-n} + O_2(\rho^{1-n})\right)(\delta_{ij} + O_2(\rho^{-2})).$$

If the "original" metric was locally conformally flat near p , by Remark 2.4, the metric g here is actually flat near p . Then, in normal coordinates, $g_{ij} = \delta_{ij}$, so the second factor in the expansion above is simply δ_{ij} .

PROOF. We consider a neighborhood U of p with the associated normal coordinates x^i . The exponential map is a diffeomorphism

$$\exp_p: B_\varepsilon(0) \subseteq T_p M \longrightarrow U$$

and we set $\widehat{U} = U \setminus \{p\}$. Then, in the inverted coordinates defined above, the corresponding diffeomorphism is

$$\Phi: \{y \in \mathbb{R}^n \mid |y| > \varepsilon^{-1}\} \longrightarrow \widehat{U}, \quad \Phi(y) = \exp_p(y/|y|^2),$$

where $y^i = \rho^2 x^i$ are the inverted normal coordinates and $\rho = |y| = 1/r$. In the normal coordinates centered at p , the Taylor expansion of the metric is given by

$$g_{kl}(x) = \delta_{kl} - \frac{1}{3} \text{Riem}_{ikjl}(p) x^i x^j + O(|x|^3)$$

(see [21, Section 5, equation (5.4)]).

Since $x = y/\rho^2$, we have

$$\frac{\partial x^k}{\partial y^i} = \rho^{-2} \left(\delta_i^k - 2 \frac{y^k y^i}{\rho^2} \right)$$

and by the Gauss lemma, in normal coordinates, there holds

$$g_{kl}(x) y^l = y^k \quad \text{and} \quad g_{kl}(x) y^k y^l = \rho^2.$$

Therefore,

$$g_{kl}(x) \frac{\partial x^k}{\partial y^i} \frac{\partial x^l}{\partial y^j} = \rho^{-4} \left(g_{ij}(x) - \frac{4y^i y^j}{\rho^2} + \frac{4y^i y^j}{\rho^2} \right) = \rho^{-4} g_{ij}(x),$$

hence, the components of $\widehat{g}$ in the inverted coordinates are given by

$$\widehat{g}_{ij}(y) = \rho^{-4} G_p(y/\rho^2)^{4/(n-2)} g_{ij}(y/\rho^2). \quad (2.9)$$

Then, from the expansion given by Theorem 2.8, we have

$$G_p(y/\rho^2) = \rho^{n-2} (1 + A\rho^{2-n} + O_2(\rho^{1-n})), \quad (2.10)$$

combining all together, we conclude

$$\widehat{g}_{ij}(y) = (1 + A\rho^{2-n} + O_2(\rho^{1-n}))^{\frac{4}{n-2}} (\delta_{ij} + O_2(\rho^{-2})).$$

Then, for $n = 3$, we obtain

$$\widehat{g}_{ij}(y) = \delta_{ij} + 4A\rho^{-1}\delta_{ij} + O_2(\rho^{-2}),$$

that is, $(\widehat{M}, \widehat{g})$ is C_1^2 -asymptotically flat. Analogously, for $n = 4$,

$$\widehat{g}_{ij}(y) = \delta_{ij} + 2A\rho^{-2}\delta_{ij} + O_2(\rho^{-2}),$$

hence, $(\widehat{M}, \widehat{g})$ is C_2^2 -asymptotically flat.

In the other cases we expand the function $(1+t)^{4/(n-2)}$ as follows:

$$(1+t)^{4/(n-2)} = 1 + \frac{4}{n-2}t + O(t^2),$$

yielding

$$(1 + A\rho^{2-n} + O_2(\rho^{1-n}))^{\frac{4}{n-2}} = 1 + \frac{4A}{n-2}\rho^{2-n} + O_2(\rho^{1-n}).$$

For $n = 5$, the coefficients $g_{ij}(y/\rho^2)$ differ from δ_{ij} by $O_2(\rho^{-2})$, while the factor involving G_p differs from 1 by $O_2(\rho^{-3})$, therefore

$$\widehat{g}_{ij}(y) = \delta_{ij} + O_2(\rho^{-2}),$$

that is, $(\widehat{M}, \widehat{g})$ is asymptotically flat of order $\tau = 2$.

If g is locally conformally flat, Remark 2.4 ensures that g is flat in a neighborhood of p . Therefore, its coefficients in normal coordinates centered at p satisfy $g_{ij}(x) = \delta_{ij}$. We substitute these coefficients into the formula for $\widehat{g}_{ij}$ in equation (2.9) and use the expansion of the Green function G_p in equation (2.10) to obtain

$$\widehat{g}_{ij}(y) = (1 + A\rho^{2-n} + O_2(\rho^{1-n}))^{4/(n-2)}\delta_{ij}.$$

The Taylor expansion of $(1+t)^{4/(n-2)}$ then gives

$$\widehat{g}_{ij}(y) = \delta_{ij} + \frac{4A}{n-2}\rho^{2-n}\delta_{ij} + O_2(\rho^{1-n}),$$

which shows that $(\widehat{M}, \widehat{g})$ is C_{n-2}^2 -asymptotically flat.

It remains to prove that $(\widehat{M}, \widehat{g})$ is complete. Suppose, by contradiction, that $(\widehat{M}, \widehat{g})$ is not complete. By the Hopf–Rinow theorem, there exist $T < +\infty$ and a unit-speed geodesic $\gamma : [0, T) \rightarrow \widehat{M}$ that cannot be extended beyond T . We first show that γ eventually leaves every compact subset of $\widehat{M}$. Indeed, suppose that there exist a compact set $K \subseteq \widehat{M}$ and a sequence $t_j \rightarrow T$ such that $\gamma(t_j) \in K$. Since γ has unit speed, the vectors $\dot{\gamma}(t_j)$ belong to the unit tangent bundle over K , which is compact. The local existence theorem for the geodesic equation then provides a positive time $\eta > 0$, independent of j , for which the geodesics with initial data $(\gamma(t_j), \dot{\gamma}(t_j))$ are defined. For sufficiently large j , we have $t_j + \eta > T$, and the uniqueness theorem for the geodesic equation allows us to extend γ beyond T . This contradicts the choice of γ .

Since M is compact and $\widehat{M} = M \setminus \{p\}$, the geodesic γ must approach p as $t \rightarrow T$. Consequently, γ eventually lies in $\widehat{U}$, and its inverted normal coordinates satisfy

$$\lim_{t \rightarrow T} \rho(\gamma(t)) = +\infty.$$

Since the coefficients of $\widehat{g}$ converge to those of the Euclidean metric as $\rho \rightarrow +\infty$, there exist constants $C > 0$ and $\bar{\rho} > 0$ such that $\widehat{g} \geq C\delta$ whenever $\rho \geq \bar{\rho}$, where δ denotes the Euclidean metric in the inverted normal coordinates.

We choose $t_0 < T$ such that $\rho(\gamma(t)) \geq \bar{\rho}$ for every $t \in [t_0, T)$. Since γ has unit speed with respect to $\hat{g}$, we obtain

$$t - t_0 = \int_{t_0}^t \sqrt{\hat{g}(\dot{\gamma}(s), \dot{\gamma}(s))} ds \geq \sqrt{C} \int_{t_0}^t |\dot{\gamma}(s)| ds \geq \sqrt{C} |\rho(\gamma(t)) - \rho(\gamma(t_0))|,$$

where $|\dot{\gamma}(s)|$ denotes the Euclidean norm of the velocity in the inverted coordinates. Since $t - t_0 < T - t_0$, this inequality implies

$$\rho(\gamma(t)) \leq \rho(\gamma(t_0)) + \frac{T - t_0}{\sqrt{C}}$$

for every $t \in [t_0, T)$. This contradicts the fact that $\rho(\gamma(t))$ tends to $+\infty$ as $t \rightarrow T$. Therefore, $(\widehat{M}, \widehat{g})$ is complete. $\square$

2.2.2. The ADM mass and the positive mass theorem. Let E be an end of the asymptotically flat manifold (M^n, g) , with asymptotically flat coordinates $(x^1, \dots, x^n)$ as in Definition 2.11. Following Arnowitt, Deser and Misner [2], we define the *ADM mass* of E by the formula

$$m_{\text{ADM}}(E, g) = \frac{1}{2n(n-1)\omega_n} \lim_{\rho \rightarrow +\infty} \int_{\{|x|=\rho\}} \sum_{i,j=1}^n (\partial_j g_{ij} - \partial_i g_{jj}) \frac{x^i}{|x|} d\mathcal{H}_0^{n-1},$$

where ω_n denotes the volume of the Euclidean unit ball in $\mathbb{R}^n$ and $\mathcal{H}_0^{n-1}$ denotes the Euclidean hypersurface measure in these coordinates. When $n = 3$, the normalization constant is $1/(16\pi)$. If the metric satisfies the C_τ^2 decay estimates with $\tau > (n-2)/2$ and its scalar curvature is integrable, this limit exists, is finite and is independent of the choice of asymptotically flat coordinates (see [6, 12]).

The conformal blow-up constructed in the previous section is asymptotically flat with a single end, whose ADM mass can be computed explicitly in terms of the constant A in the Green-function expansion.

THEOREM 2.13. *Let (M^n, g) be closed with g as in Theorem 2.8 and suppose (M^n, g) has dimension 3, 4, 5 or it is locally conformally flat. Then, the ADM mass of the stereographic projection is $m_{\text{ADM}} = 2A$.*

PROOF. In the locally conformally flat case, Theorem 2.12 gives the following expansion of the coefficients of $\hat{g}$ in the inverted normal coordinates $(y^1, \dots, y^n)$ on the end $\widehat{U}$,

$$\hat{g}_{ij}(y) = \delta_{ij} + \frac{4A}{n-2} \rho^{2-n} \delta_{ij} + O_2(\rho^{1-n}),$$

with $\rho = |y|$. Differentiating the coefficients of $\hat{g}$, we have

$$\sum_{j=1}^n (\partial_j \hat{g}_{ij} - \partial_i \hat{g}_{jj}) = 4A(n-1) \rho^{-n} y^i + O(\rho^{-n}),$$

then multiplying this identity by the components y^i/ρ of the Euclidean outward unit normal to the sphere $\{|y| = \rho\}$ and summing over i , we obtain

$$\sum_{i,j=1}^n (\partial_j \hat{g}_{ij} - \partial_i \hat{g}_{jj}) \frac{y^i}{\rho} = 4A(n-1) \rho^{1-n} + O(\rho^{-n}).$$

Since the Euclidean volume of the sphere $\{|y| = \rho\}$ is $n\omega_n\rho^{n-1}$, recalling the definition of the ADM mass, we get

$$m_{\text{ADM}}(\widehat{U}, \widehat{g}) = \frac{1}{2n(n-1)\omega_n} \lim_{\rho \rightarrow +\infty} (4A(n-1)n\omega_n + O(\rho^{-1})) = 2A.$$

For $n = 3$, the expansion of the coefficients of $\widehat{g}$ is

$$\widehat{g}_{ij}(y) = \delta_{ij} + 4A\rho^{-1}\delta_{ij} + O_2(\rho^{-2}).$$

The first derivatives of the term $O_2(\rho^{-2})$ in the expansion of the coefficients of $\widehat{g}$ are $O(\rho^{-3})$ and the Euclidean area of the sphere $\{|y| = \rho\}$ is $4\pi\rho^2$, this term contributes as $O(\rho^{-1})$ to the integral defining the ADM mass. Then, differentiating the term $4A\rho^{-1}\delta_{ij}$, we obtain

$$\sum_{i,j=1}^3 (\partial_j \widehat{g}_{ij} - \partial_i \widehat{g}_{jj}) \frac{y^i}{\rho} = 8A\rho^{-2} + O(\rho^{-3}).$$

Therefore,

$$m_{\text{ADM}}(\widehat{U}, \widehat{g}) = \frac{1}{16\pi} \lim_{\rho \rightarrow +\infty} \int_{\{|y|=\rho\}} (8A\rho^{-2} + O(\rho^{-3})) d\mathcal{H}_0^2 = 2A.$$

For $n = 4, 5$, recalling the identity for the metric coefficients in equation (2.9), the expansion of the Green function in equation (2.10) and the following Taylor expansion up to the third order of g_{ij} in normal coordinates centered at p (see [21, Section 5, equation (5.4)]),

$$g_{kl}(x) = \delta_{kl} - \frac{1}{3} \text{Riem}_{ikjl}(p)x^i x^j - \frac{1}{6} \nabla_m \text{Riem}_{ikjl}(p)x^i x^j x^m + O(|x|^4), \quad (2.11)$$

we have

$$\widehat{g}_{ij}(y) = \delta_{ij} + \frac{4A}{n-2} \rho^{2-n} \delta_{ij} - \frac{1}{3} \text{Riem}_{ikjs}(p) \rho^{-4} y^k y^s + O_2(\rho^{-3}).$$

We first examine the contribution of the curvature term

$$h_{ij}(y) = -\frac{1}{3} \text{Riem}_{ikjs}(p) \rho^{-4} y^k y^s,$$

which comes from the second-order Taylor expansion of the coefficients of g at p . Taking the determinant in formula (2.11), we obtain

$$\det g_{ij}(x) = 1 - \frac{1}{3} \text{Ric}_{k\ell}(p) x^k x^\ell + O(|x|^3)$$

and since $\det g_{ij}(x) = 1 + O(|x|^N)$, with $N \geq 3$, by Theorem 2.3, we conclude that $\text{Ric}(p) = 0$. Then,

$$\sum_{j=1}^n h_{jj} = -\frac{1}{3} \rho^{-4} \text{Ric}_{ks}(p) y^k y^s = 0.$$

Differentiating the coefficients h_{ij} , we observe that the terms arising from the derivatives of ρ^{-4} and y^s vanish by the antisymmetry of the Riemann tensor in its last two indices, moreover, as $\text{Ric}(p) = 0$, we obtain

$$\sum_{j=1}^n \partial_j h_{ij} = -\frac{1}{3} \rho^{-4} \sum_{j=1}^n \text{Riem}_{ijjs}(p) y^s = \frac{1}{3} \rho^{-4} \text{Ric}_{is}(p) y^s = 0,$$

therefore,

$$\sum_{j=1}^n (\partial_j h_{ij} - \partial_i h_{jj}) = 0,$$

so this curvature term gives no contribution to the integral defining the ADM mass.

For $n = 4$, the remaining terms in the expansion of the metric coefficients are $O_2(\rho^{-3})$, their derivatives are $O(\rho^{-4})$ and the Euclidean area of the sphere of radius ρ is proportional to ρ^3 . Hence, their contribution is $O(\rho^{-1})$ and tends to zero as $\rho \rightarrow +\infty$.

For $n = 5$, we must also consider the third-order term in the Taylor expansion of the metric coefficients in formula (2.11). Let $P_{ij}(x)$ denote the homogeneous polynomial of degree three appearing in such expansion, that is,

$$P_{ij}(x) = -\frac{1}{6} \nabla_m \text{Riem}_{ikjs}(p) x^k x^s x^m,$$

then, by formula (2.9) together with the Green-function expansion (2.10), we obtain

$$\widehat{g}_{ij}(y) = \delta_{ij} + \frac{4A}{3} \rho^{-3} \delta_{ij} + h_{ij}(y) + \rho^{-6} P_{ij}(y) + O_2(\rho^{-4}).$$

Since P_{ij} is homogeneous of degree three, the term $\rho^{-6} P_{ij}(y)$ is odd with respect to y , so its first derivatives are even and since the components y^i/ρ of the Euclidean outward unit normal are odd, the function

$$\sum_{i,j=1}^5 (\partial_j (\rho^{-6} P_{ij}(y)) - \partial_i (\rho^{-6} P_{jj}(y))) \frac{y^i}{\rho}$$

is odd. Its integral over the sphere $\{|y| = \rho\}$ therefore vanishes.

Finally, the first derivatives of the remaining terms $O_2(\rho^{-4})$ in the expansion of $\widehat{g}$ are $O(\rho^{-5})$ and since the Euclidean area of $\{|y| = \rho\}$ is $5\omega_5 \rho^4$, their contribution to the integral defining the ADM mass is $O(\rho^{-1})$, which tends to zero as $\rho \rightarrow +\infty$.

If g is locally conformally flat, Remark 2.4 ensures that g is flat near p . The expansion of $\widehat{g}$ then shows that the terms other than δ_{ij} and $\frac{4A}{n-2} \rho^{2-n} \delta_{ij}$ contribute $O(\rho^{-1})$, in every dimension. Thus, we conclude that in all the cases we considered, only the term

$$\frac{4A}{n-2} \rho^{2-n} \delta_{ij}$$

contributes to the ADM mass. Differentiating its coefficients, we obtain

$$\sum_{i,j=1}^n \left(\partial_j \left(\frac{4A}{n-2} \rho^{2-n} \delta_{ij} \right) - \partial_i \left(\frac{4A}{n-2} \rho^{2-n} \delta_{jj} \right) \right) \frac{y^i}{\rho} = 4A(n-1) \rho^{1-n}.$$

Since the Euclidean area of the sphere $\{|y| = \rho\}$ is $n\omega_n \rho^{n-1}$, by the definition of the ADM mass, we finally get

$$m_{\text{ADM}} = \frac{4A(n-1)n\omega_n}{2n(n-1)\omega_n} = 2A.$$

□

THEOREM 2.14 (Riemannian positive mass theorem). *Let (M^n, g) , $n \geq 3$, be a connected, complete Riemannian manifold without boundary, with finitely many asymptotically flat ends. Assume that the metric is C_τ^2 -asymptotically flat in each end for some $\tau > (n-2)/2$, that $R \in L^1(M)$, and that $R \geq 0$ on M . Then, the ADM mass of every end E_k satisfies*

$$m_{\text{ADM}}(E_k, g) \geq 0.$$

Moreover, if $m_{\text{ADM}}(E_k, g) = 0$ for some end E_k , then (M^n, g) is isometric to Euclidean space $(\mathbb{R}^n, \delta)$.

The cases relevant to Schoen's argument are the dimensions $3 \leq n \leq 5$ and locally conformally flat metrics in arbitrary dimension. By Theorem 2.12, the conformal blow-up $(\widehat{M}, \widehat{g})$ constructed above is complete and C_τ^2 -asymptotically flat, with $\tau = n-2$ if $n = 3, 4$ or g is locally conformally flat, and $\tau = 2$ in the remaining case $n = 5$. In all these cases, $\tau > (n-2)/2$. Since $\widehat{R} = 0$ on $\widehat{M}$, the positive mass theorem 2.14, together with the identity $m_{\text{ADM}} = 2A$, yields $A \geq 0$. If $A = 0$, the rigidity statement implies that $(\widehat{M}, \widehat{g})$ is isometric to Euclidean space, then, since (M^n, g) is the one-point conformal compactification of $(\widehat{M}, \widehat{g})$, it follows that (M^n, g) is conformally diffeomorphic to the standard sphere (see [21, 25]). Consequently, under the hypotheses of Theorem 2.6, one has $A > 0$, as needed in the test-function arguments of the next section.

Theorem 2.14 arose from the study of gravitational energy in general relativity, but its Riemannian formulation concerns only the scalar curvature and the geometry at infinity. Schoen and Yau first proved it in dimension three, including the rigidity statement that the vanishing of the ADM mass implies that the manifold is isometric to Euclidean space [26]. Their proof, based on minimal surfaces, extends to all dimensions $3 \leq n \leq 7$ (see [21]). Witten subsequently introduced a different approach, based on spinors, which yields the positive mass theorem for spin manifolds in arbitrary dimension [6, 24, 32]. In dimensions $n \geq 8$, minimal hypersurfaces may have singularities, preventing a direct extension of the original argument of Schoen and Yau. They later developed a new proof of the theorem in arbitrary dimension, without the spin assumption. Their work first appeared as a preprint in 2017 and was subsequently published [28]. More recently, Bi, Hao, He, Shi and Zhu obtained another proof in dimensions up to 19 [8], while Brendle and Wang developed a different argument covering arbitrary dimensions [9].

2.2.3. Schoen's test functions. Following the construction in [21, 25], we define a family of test functions on the asymptotically flat manifold $(\widehat{M}, \widehat{g})$, using the Euclidean Talenti bubbles introduced in formula (2.1). Recall that $\widehat{U}$ is the asymptotic end, with inverted normal coordinates y and $\rho = |y|$. Fix $\rho_0 > 0$ sufficiently large that the set $\{\rho > \rho_0\}$ is contained in this chart, and let $K_{\rho_0} = \widehat{M} \setminus \{\rho > \rho_0\}$, which is a compact set. We set

$$\widehat{\mu}_\lambda = \begin{cases} \left(\frac{\lambda}{\lambda^2 + \rho^2} \right)^{(n-2)/2} & \text{if } y \in \widehat{U} \text{ and } \rho > \rho_0, \\ \left(\frac{\lambda}{\lambda^2 + \rho_0^2} \right)^{(n-2)/2} & \text{if } y \in K_{\rho_0}. \end{cases}$$

On $\{\rho > \rho_0\}$, its radial derivative is

$$\partial_\rho \widehat{\mu}_\lambda = -(n-2) \lambda^{(n-2)/2} \rho (\lambda^2 + \rho^2)^{-n/2}. \quad (2.12)$$

The function $\widehat{\mu}_\lambda$ is constant on K_{ρ_0} , hence its derivative vanishes in the interior of K_{ρ_0} . Although its gradient may develop a discontinuity across $\{\rho = \rho_0\}$, we can still prove that $\widehat{\mu}_\lambda$ is locally Lipschitz. Indeed, it is sufficient to consider points x and y in a coordinate neighborhood of $\{\rho = \rho_0\}$, with $|x| > \rho_0$ and $|y| \leq \rho_0$:

$$|\widehat{\mu}_\lambda(x) - \widehat{\mu}_\lambda(y)| \leq \frac{\lambda^{\frac{3}{2}(n-2)}}{(\lambda^2 + |x|^2)^{\frac{n-2}{2}}(\lambda^2 + \rho_0^2)^{\frac{n-2}{2}}} \left| \left(1 + \frac{|x|^2}{\lambda^2}\right)^{\frac{n-2}{2}} - \left(1 + \frac{\rho_0^2}{\lambda^2}\right)^{\frac{n-2}{2}} \right|.$$

The first factor on the right is uniformly bounded by $\lambda^{\frac{n-2}{2}}(\lambda^2 + \rho_0^2)^{\frac{2-n}{2}}$, hence

$$|\widehat{\mu}_\lambda(x) - \widehat{\mu}_\lambda(y)| \leq \lambda^{\frac{n-2}{2}}(\lambda^2 + \rho_0^2)^{\frac{2-n}{2}} \left| \left(1 + \frac{|x|^2}{\lambda^2}\right)^{\frac{n-2}{2}} - \left(1 + \frac{\rho_0^2}{\lambda^2}\right)^{\frac{n-2}{2}} \right|.$$

Using the condition $|y| \leq \rho_0$, and since $(1+x)^{\frac{n-2}{2}}$ is locally Lipschitz, the claim then follows. This implies that formula (2.12), extended to zero in K_{ρ_0} , also determines the weak gradient almost everywhere on $\widehat{M}$.

To estimate the integrals with respect to $\widehat{g}$, we use the expansion of G_p in Theorem 2.8 and the normalization of the determinant in conformal normal coordinates. Since $\widehat{g} = G_p^{4/(n-2)}g$, in the inverted normal coordinates we obtain

$$d\widehat{\text{Vol}} = (1 + O(\rho^{-N}))(1 + A\rho^{2-n} + O_2(\rho^{1-n}))^{2n/(n-2)} dy = \left(1 + \frac{2n}{n-2}A\rho^{2-n} + O(\rho^{1-n})\right) dy,$$

where N is chosen sufficiently large. In particular, the gradient of $\widehat{\mu}_\lambda$ belongs to $L^2(\widehat{M})$, as follows from formula (2.12) and the estimates below.

THEOREM 2.15. *Let (M^n, g) be a closed Riemannian manifold with $\lambda_1(L_g) > 0$, and assume that $3 \leq n \leq 5$ or that g is locally conformally flat. Consider also g and G_p as in Theorem 2.8, let A be the constant in the expansion of G_p , and let $\widehat{\mu}_\lambda$ be defined above. Then, for fixed ρ_0 and $\lambda \rightarrow +\infty$,*

$$c(n) \int_{\widehat{M}} |\nabla \widehat{\mu}_\lambda|_{\widehat{g}}^2 d\widehat{\text{Vol}} = Y(\mathbb{S}^n, [g_{\text{std}}]) \|\widehat{\mu}_\lambda\|_{2^*}^2 - 4n(n-1)\omega_n A \lambda^{2-n} + O(\lambda^{1-n}).$$

Here the L^{2^*} norm is computed with respect to $d\widehat{\text{Vol}}$.

PROOF. Since $\widehat{\mu}_\lambda$ is constant on K_{ρ_0} , the integral of its squared gradient is taken over $\{\rho > \rho_0\}$. Taking the inverse of formula (2.9), we obtain

$$\widehat{g}^{ij}(y) = \rho^4 G_p^{-\frac{4}{n-2}}(y/\rho^2) g^{ij}(y/\rho^2).$$

Using the Gauss lemma in normal coordinates, the inversion $x = y/\rho^2$ and equation (2.10), we get

$$\widehat{g}^{ij}(y) y_i y_j = G_p^{-4/(n-2)} \rho^8 g^{ij}(x) x_i x_j = \rho^2 \left(1 - \frac{4A}{n-2} \rho^{2-n} + O(\rho^{1-n})\right)$$

and combining this identity with the expansion of $d\widehat{\text{Vol}}$, we obtain

$$\begin{aligned}
\int_{\widehat{M}} |\nabla \widehat{\mu}_\lambda|_{\widehat{g}}^2 d\widehat{\text{Vol}} &= \int_{\{\rho > \rho_0\}} \rho^{-2} \widehat{g}^{ij}(y) y_i y_j (\partial_\rho \widehat{\mu}_\lambda)^2 \left(1 + \frac{2n}{n-2} A \rho^{2-n} + O(\rho^{1-n}) \right) dy \\
&= \int_{\{\rho > \rho_0\}} (\partial_\rho \widehat{\mu}_\lambda)^2 (1 + 2A \rho^{2-n} + O(\rho^{1-n})) dy \\
&= \underbrace{\int_{\{\rho > \rho_0\}} |\nabla^{\mathbb{R}^n} \mu_{1/\lambda}|_{\mathbb{R}^n}^2 dy}_{I_1} + \underbrace{2A \int_{\{\rho > \rho_0\}} (\partial_\rho \widehat{\mu}_\lambda)^2 \rho^{2-n} dy}_{I_2} \\
&\quad + \underbrace{\int_{\{\rho > \rho_0\}} (\partial_\rho \widehat{\mu}_\lambda)^2 O(\rho^{1-n}) dy}_{I_3}.
\end{aligned}$$

Formula (2.12), the change of variables $s = \rho/\lambda$ and polar coordinates give

$$|I_3| \leq C \lambda^{1-n} \int_{\rho_0/\lambda}^{+\infty} s^2 (1+s^2)^{-n} ds = O(\lambda^{1-n}),$$

while

$$\begin{aligned}
I_2 &= 2n(n-2)^2 \omega_n A \lambda^{2-n} \int_{\rho_0/\lambda}^{+\infty} s^3 (1+s^2)^{-n} ds \\
&= 2n(n-2)^2 \omega_n A \lambda^{2-n} \int_0^{+\infty} s^3 (1+s^2)^{-n} ds + O(\lambda^{-n-2}).
\end{aligned}$$

The integral in the last expression equals

$$\int_0^{+\infty} s^3 (1+s^2)^{-n} ds = \frac{1}{2(n-1)(n-2)}.$$

For I_1 , using that the Euclidean bubble $\mu_{1/\lambda}$ attains equality in the sharp Sobolev inequality, formula (1.3), we have

$$I_1 = \int_{\mathbb{R}^n} |\nabla^{\mathbb{R}^n} \mu_{1/\lambda}|_{\mathbb{R}^n}^2 dy - \int_{\{\rho \leq \rho_0\}} |\nabla^{\mathbb{R}^n} \mu_{1/\lambda}|_{\mathbb{R}^n}^2 dy = \sigma_n^2 \|\mu_{1/\lambda}\|_{L^{2^*}(\mathbb{R}^n)}^2 + O(\lambda^{-n-2}),$$

since

$$\int_{\{\rho \leq \rho_0\}} |\nabla^{\mathbb{R}^n} \mu_{1/\lambda}|_{\mathbb{R}^n}^2 dy = n \omega_n (n-2)^2 \int_0^{\rho_0/\lambda} s^{n+1} (1+s^2)^{-n} ds = O(\lambda^{-n-2}).$$

The estimates for I_1 , I_2 and I_3 , together with $c(n) = 4(n-1)/(n-2)$, now give

$$c(n) \int_{\widehat{M}} |\nabla \widehat{\mu}_\lambda|_{\widehat{g}}^2 d\widehat{\text{Vol}} = Y(\mathbb{S}^n, [g_{\text{std}}]) \|\mu_{1/\lambda}\|_{L^{2^*}(\mathbb{R}^n)}^2 + 4n \omega_n A \lambda^{2-n} + O(\lambda^{1-n}). \quad (2.13)$$

We now compare the L^{2^*} integral of $\widehat{\mu}_\lambda$ on $\widehat{M}$ with that of $\mu_{1/\lambda}$ on $\mathbb{R}^n$. By the definition of K_{ρ_0} and the expansion of $d\widehat{\text{Vol}}$, we obtain

$$\begin{aligned} \int_{\widehat{M}} \widehat{\mu}_\lambda^{2^*} d\widehat{\text{Vol}} &= \lambda^n \int_{\{\rho > \rho_0\}} (\lambda^2 + \rho^2)^{-n} \left(1 + \frac{2n}{n-2} A \rho^{2-n} + O(\rho^{1-n}) \right) dy \\ &\quad + \lambda^n (\lambda^2 + \rho_0^2)^{-n} \widehat{\text{Vol}}(K_{\rho_0}) \\ &= \underbrace{\int_{\{\rho > \rho_0\}} \mu_{1/\lambda}^{2^*} dy}_{E_1} + \underbrace{\frac{2n\lambda^n A}{n-2} \int_{\{\rho > \rho_0\}} \rho^{2-n} (\lambda^2 + \rho^2)^{-n} dy}_{E_2} \\ &\quad + \underbrace{\lambda^n \int_{\{\rho > \rho_0\}} (\lambda^2 + \rho^2)^{-n} O(\rho^{1-n}) dy}_{E_3} + \underbrace{\lambda^n (\lambda^2 + \rho_0^2)^{-n} \widehat{\text{Vol}}(K_{\rho_0})}_{E_4}. \end{aligned}$$

Since K_{ρ_0} is compact and G_p is positive and continuous on it, we have

$$\widehat{\text{Vol}}(K_{\rho_0}) \leq \text{Vol}(M) \max_{K_{\rho_0}} G_p^{2n/(n-2)} < +\infty,$$

and hence $E_4 = O(\lambda^{-n})$. By polar coordinates and $s = \rho/\lambda$,

$$E_2 = \frac{2n^2 \omega_n A}{n-2} \lambda^{2-n} \int_{\rho_0/\lambda}^{+\infty} s(1+s^2)^{-n} ds = \frac{n^2 \omega_n A}{(n-2)(n-1)} \lambda^{2-n} + O(\lambda^{-n}),$$

where we used $\int_0^{+\infty} s(1+s^2)^{-n} ds = 1/[2(n-1)]$. The remaining integrals satisfy

$$E_3 = O(\lambda^{1-n}), \quad E_1 = \|\mu_{1/\lambda}\|_{L^{2^*}(\mathbb{R}^n)}^{2^*} - \int_{\{\rho \leq \rho_0\}} \mu_{1/\lambda}^{2^*} dy,$$

with

$$\int_{\{\rho \leq \rho_0\}} \mu_{1/\lambda}^{2^*} dy = n\omega_n \int_0^{\rho_0/\lambda} s^{n-1} (1+s^2)^{-n} ds = O(\lambda^{-n}).$$

Combining these estimates yields

$$\int_{\widehat{M}} \widehat{\mu}_\lambda^{2^*} d\widehat{\text{Vol}} = \|\mu_{1/\lambda}\|_{L^{2^*}(\mathbb{R}^n)}^{2^*} + \frac{n^2 \omega_n A}{(n-2)(n-1)} \lambda^{2-n} + O(\lambda^{1-n}).$$

The Euclidean norm $\|\mu_{1/\lambda}\|_{L^{2^*}(\mathbb{R}^n)}$ is positive and independent of λ by the change of variables $y = \lambda z$. Taking the power $2/2^* = (n-2)/n$ and expanding at this positive constant, we obtain

$$\|\widehat{\mu}_\lambda\|_{2^*}^2 = \|\mu_{1/\lambda}\|_{L^{2^*}(\mathbb{R}^n)}^2 + \frac{n\omega_n A}{n-1} \|\mu_{1/\lambda}\|_{L^{2^*}(\mathbb{R}^n)}^{-4/(n-2)} \lambda^{2-n} + O(\lambda^{1-n}). \quad (2.14)$$

Combining formulas (2.13) and (2.14), and recalling that $Y(\mathbb{S}^n, [g_{\text{std}}]) = c(n)\sigma_n^2$, we get

$$c(n) \int_{\widehat{M}} |\nabla \widehat{\mu}_\lambda|_g^2 d\widehat{\text{Vol}} = Y(\mathbb{S}^n, [g_{\text{std}}]) \|\widehat{\mu}_\lambda\|_{2^*}^2 + 4n\omega_n \left(1 - \frac{\sigma_n^2}{n-2} \|\mu_{1/\lambda}\|_{L^{2^*}(\mathbb{R}^n)}^{-4/(n-2)} \right) A \lambda^{2-n} + O(\lambda^{1-n})$$

and to evaluate the constant in parentheses we combine the equation $-\Delta \mu_{1/\lambda} = n(n-2)\mu_{1/\lambda}^{2^*-1}$ with the fact that $\mu_{1/\lambda}$ attains equality in the sharp Sobolev inequality, formula (1.3).

Integration by parts gives

$$\sigma_n^2 \|\mu_{1/\lambda}\|_{L^{2^*}(\mathbb{R}^n)}^2 = \int_{\mathbb{R}^n} |\nabla^{\mathbb{R}^n} \mu_{1/\lambda}|_{\mathbb{R}^n}^2 dy = n(n-2) \|\mu_{1/\lambda}\|_{L^{2^*}(\mathbb{R}^n)}^{2^*},$$

and therefore

$$\sigma_n^2 \|\mu_{1/\lambda}\|_{L^{2^*}(\mathbb{R}^n)}^{-4/(n-2)} = n(n-2).$$

Consequently,

$$c(n) \int_{\widehat{M}} |\nabla \widehat{\mu}_\lambda|_{\widehat{g}}^2 d\widehat{\text{Vol}} = Y(\mathbb{S}^n, [g_{\text{std}}]) \|\widehat{\mu}_\lambda\|_{2^*}^2 - 4n(n-1)\omega_n A \lambda^{2-n} + O(\lambda^{1-n}),$$

which proves the theorem. $\square$

Since $\widehat{M} = M \setminus \{p\}$ and $\widehat{g} = G_p^{4/(n-2)} g$, we define a function on $M \setminus \{p\}$ by

$$u_\lambda(x) = G_p(x) \widehat{\mu}_\lambda(x).$$

In the conformal normal coordinates on $U \setminus \{p\}$, the expansions of G_p and $\widehat{\mu}_\lambda$ give

$$u_\lambda(x) = (1 + A|x|^{n-2} + O_2(|x|^{n-1})) \left(\frac{\lambda}{1 + \lambda^2|x|^2} \right)^{(n-2)/2}.$$

Setting $u_\lambda(p) = \lambda^{(n-2)/2}$ gives a Lipschitz extension of u_λ to M , so $u_\lambda \in W^{1,2}(M)$. By conformal invariance, the Yamabe functional of u_λ with respect to g is

$$J_{2^*}(u_\lambda) = c(n) \left(\int_{\widehat{M}} |\nabla \widehat{\mu}_\lambda|_{\widehat{g}}^2 d\widehat{\text{Vol}} \right) \left(\int_{\widehat{M}} \widehat{\mu}_\lambda^{2^*} d\widehat{\text{Vol}} \right)^{-2/2^*}.$$

Since $A > 0$ by the positive mass theorem and its rigidity statement, Theorem 2.15 yields $J_{2^*}(u_\lambda) < Y(\mathbb{S}^n, [g_{\text{std}}])$ for all sufficiently large λ . By the definition of the Yamabe constant,

$$Y(M, [g]) \leq J_{2^*}(u_\lambda) < Y(\mathbb{S}^n, [g_{\text{std}}]).$$

Together with the result of the preceding section, this completes the proof of Theorem 1.1.

APPENDIX A

Analytic results

A.1. Some Results about elliptic partial differential equations

THEOREM A.1 ([17,18]). *Let $V \in C^\infty(\Omega)$, $A \in C^\infty(\Omega, \mathbb{R}^{n \times n})$ satisfying:*

$$\langle A(x)\xi, \xi \rangle \geq \lambda |\xi|^2$$

for some constant $\lambda > 0$ and for every $x \in \Omega$ and $\xi \in \mathbb{R}^n$. Consider $f \in L^p(\Omega)$, with $1 < p < +\infty$. Suppose that $u \in W^{1,2}(\Omega) \cap L^p(\Omega)$ is a weak solution of

$$-\operatorname{div}(A\nabla u) + Vu = f \quad \text{in } \Omega.$$

Then $u \in W_{\text{loc}}^{2,p}(\Omega)$ and for every open set $\Omega' \Subset \Omega$,

$$\|u\|_{W^{2,p}(\Omega')} \leq C(\Omega') (\|f\|_p + \|u\|_p).$$

COROLLARY A.2. *Let (M^n, g) be a closed Riemannian manifold, let*

$$L = -\operatorname{div}_g(A\nabla \cdot) + V,$$

where A is a smooth uniformly elliptic $(1,1)$ -tensor field on M and $V \in C^\infty(M)$, and let $1 < p < +\infty$. Assume that $f \in L^p(M)$ and that $u \in W^{1,2}(M) \cap L^p(M)$ is a weak solution of

$$Lu = f.$$

Then $u \in W^{2,p}(M)$ and

$$\|u\|_{W^{2,p}(M)} \leq C (\|f\|_{L^p(M)} + \|u\|_{L^p(M)}),$$

where C depends on M, g, L , and p .

PROOF. Cover M by finitely many coordinate charts Ω_α and choose relatively compact subcharts $\Omega'_\alpha \Subset \Omega_\alpha$ that still cover M . In each chart, multiplying the equation by the smooth positive density $\sqrt{\det g_{ij}}$, we have a divergence-form equation with right-hand side $\sqrt{\det g_{ij}}(f - Vu) \in L^p(M)$. Then, applying the preceding theorem in the relatively compact subcharts and summing over the finite cover, we obtain the asserted global estimate. $\square$

THEOREM A.3 (Strong maximum principle – See [14]). *Let Ω be a connected domain and let*

$$Lu(x) = -a^{ij}(x)\partial_{ij}^2 u(x) + b^i(x)\partial_i u(x) + c(x)u(x)$$

be uniformly elliptic, with $a^{ij}, b^i, c \in C(\Omega)$ and $c \geq 0$. Let $u \in C^2(\Omega) \cap C(\overline{\Omega})$. If $Lu \leq 0$ in Ω and u attains a nonnegative maximum at an interior point, then u is constant. Likewise, if $Lu \geq 0$ in Ω and u attains a nonpositive minimum at an interior point, then u is constant.

A.2. The concentration–compactness lemma

We recall the Brezis–Lieb lemma (see [10]), following the proof in [34].

LEMMA A.4 (Brezis–Lieb). *Let $1 \leq p < +\infty$ and let u_i be a sequence in $L^p(\mathbb{R}^n)$ converging almost everywhere to u . If u_i is uniformly bounded in $L^p(\mathbb{R}^n)$, then*

$$\lim_{i \rightarrow +\infty} \| |u_i - u|^p - |u_i|^p + |u|^p \|_1 = 0$$

PROOF. By Fatou’s lemma and the uniform boundedness assumption, we have:

$$u \in L^p(\mathbb{R}^n)$$

For every $\varepsilon > 0$ we consider the following continuous function:

$$h_\varepsilon(t) = ||1 + t|^p - |t|^p| - \varepsilon|t|^p$$

Moreover:

$$\lim_{|t| \rightarrow +\infty} h_\varepsilon(t) = -\infty$$

So there exists $A_\varepsilon > 0$ such that $h_\varepsilon(t) < A_\varepsilon$. From this, after fixing ε , we get the following estimate.

$$||a + b|^p - |a|^p| \leq \varepsilon|a|^p + A_\varepsilon|b|^p$$

From which it follows:

$$||u_i - u|^p - |u_i|^p + |u|^p| \leq \varepsilon|u_i - u|^p + C_\varepsilon|u|^p$$

If we define the auxiliary sequence g_i^ε as:

$$g_i^\varepsilon = [||u_i - u|^p - |u_i|^p + |u|^p| - \varepsilon|u_i - u|^p]^+$$

Then by Lebesgue dominated convergence theorem we have $g_i^\varepsilon \rightarrow 0$ in L^1 . Combining all together:

$$||u_i - u|^p - |u_i|^p + |u|^p| \leq g_i^\varepsilon + \varepsilon|u_i - u|^p$$

Using the L^1 strong convergence of g_i^ε and the uniform bound on the norms of the sequence gets:

$$\limsup_{i \rightarrow +\infty} \int_{\mathbb{R}^n} ||u_i - u|^p - |u_i|^p + |u|^p| dx \leq \varepsilon \sup_{i \in \mathbb{N}} \|u_i - u\|_p^p.$$

Since ε is arbitrary the assertion follows. $\square$

We are going to use the concentration–compactness principle of P.-L. Lions in the form presented in [29]; see also the original papers [22,23].

LEMMA A.5 (Concentration–compactness principle). *Let $n > 2$ and let u_i be a sequence in $W^{1,2}(\mathbb{R}^n)$ satisfying*

- $u_i \rightharpoonup u$ in $W^{1,2}(\mathbb{R}^n)$ for some $u \in W^{1,2}(\mathbb{R}^n)$;
- $u_i \rightarrow u$ almost everywhere;
- $|\nabla u_i|^2 dx \xrightarrow{*} \mu$ and $|u_i|^{2^*} dx \xrightarrow{*} \nu$, where μ and ν are bounded Radon measures.

Then there exist an at most countable set J , distinct points $\{x_j \mid j \in J\} \subseteq \mathbb{R}^n$, and positive numbers ν_j, μ_j such that

$$\nu = |u|^{2^*} dx + \sum_{j \in J} \nu_j \delta_{x_j}, \quad \mu \geq |\nabla u|^2 dx + \sum_{j \in J} \mu_j \delta_{x_j} \quad \text{and} \quad \sigma_n \nu_j^{1/2^*} \leq \mu_j^{1/2},$$

for every $j \in J$, where σ_n is the sharp Euclidean Sobolev constant. In particular,

$$\sum_{j \in J} \nu_j^{2/2^*} < +\infty.$$

PROOF. Set $v_i = u_i - u$. After passing to a subsequence, the bounded measures

$$\omega_i = |v_i|^{2^*} dx \quad \text{and} \quad \lambda_i = |\nabla v_i|^2 dx$$

converge weakly star to bounded Radon measures ω and λ . By the Brezis–Lieb lemma,

$$\lim_{i \rightarrow +\infty} \int_{\mathbb{R}^n} ||v_i|^{2^*} - |u_i|^{2^*} + |u|^{2^*}| dx = 0$$

Writing $\nu_i = |u_i|^{2^*} dx$ and $h_i = |v_i|^{2^*} - |u_i|^{2^*} + |u|^{2^*}$, we get

$$\omega_i = \nu_i - |u|^{2^*} dx + h_i dx.$$

Passing to the limit gives:

$$\omega = \lim_{i \rightarrow +\infty} \omega_i = \nu - |u|^{2^*} dx$$

Moreover, since $v_i \rightharpoonup 0$ in $W^{1,2}(\mathbb{R}^n)$, Rellich–Kondrachov gives $v_i \rightarrow 0$ strongly in $L^2_{\text{loc}}(\mathbb{R}^n)$. Hence, for every $\varphi \in C_c^\infty(\mathbb{R}^n)$, the sharp Sobolev inequality applied to φv_i yields

$$\sigma_n^2 \left(\int_{\mathbb{R}^n} |\varphi|^{2^*} d\omega \right)^{2/2^*} \leq \int_{\mathbb{R}^n} |\varphi|^2 d\lambda.$$

Indeed, after expanding $\nabla(\varphi v_i)$, all terms containing v_i converge to zero, while the principal term converges to the right-hand side above.

By approximation of characteristic functions and regularity of Radon measures, the preceding inequality implies, for every Borel set E ,

$$\sigma_n^2 \omega(E)^{2/2^*} \leq \lambda(E).$$

Thus $\omega \ll \lambda$. Let $f = d\omega/d\lambda$. At λ -almost every point x which is not an atom of λ , differentiation of measures gives

$$\sigma_n^2 f(x)^{2/2^*} \leq \lim_{r \rightarrow 0^+} \lambda(B_r(x))^{1-2/2^*} = 0.$$

Therefore $f = 0$ away from the atoms of λ . Since a finite Radon measure has at most countably many atoms, there are distinct points $\{x_j \mid j \in J\}$ and numbers $\nu_j > 0$ such that

$$\omega = \sum_{j \in J} \nu_j \delta_{x_j}.$$

Applying the measure inequality to the singleton $\{x_j\}$ gives

$$\sigma_n^2 \nu_j^{2/2^*} \leq \lambda(\{x_j\}).$$

Finally, from $\nabla u_i = \nabla v_i + \nabla u$ we have, for every $\varphi \in C_c^\infty(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} \varphi |\nabla u_i|^2 dx = \int_{\mathbb{R}^n} \varphi |\nabla v_i|^2 dx + 2 \int_{\mathbb{R}^n} \varphi \nabla v_i \cdot \nabla u dx + \int_{\mathbb{R}^n} \varphi |\nabla u|^2 dx.$$

The middle term tends to zero because $\nabla v_i \rightharpoonup 0$ in L^2 . Passing to the limit therefore gives the identity of measures

$$\mu = \lambda + |\nabla u|^2 dx.$$

Since $|\nabla u|^2 dx$ has no atoms, put $\mu_j = \lambda(\{x_j\}) = \mu(\{x_j\})$. We then have

$$\mu \geq |\nabla u|^2 dx + \sum_{j \in J} \mu_j \delta_{x_j} \quad \text{and} \quad \sigma_n \nu_j^{1/2^*} \leq \mu_j^{1/2}.$$

Summing the squared inequality over j gives

$$\sigma_n^2 \sum_{j \in J} \nu_j^{2/2^*} \leq \sum_{j \in J} \mu_j \leq \mu(\mathbb{R}^n) < +\infty,$$

which proves the last assertion. $\square$

A.3. The Green function in conformal normal coordinates

The following discussion of the Green function in conformal normal coordinates is based on [21, Section 6].

Let $U \subseteq \mathbb{R}^n$. We introduce the spaces of functions

$$\begin{aligned} \mathcal{P}_k(U) &= \{\psi \mid \psi : U \rightarrow \mathbb{R}, \psi \text{ is a homogeneous polynomial of degree } k\}, \\ \mathcal{C}_k(U) &= \{\psi \mid \psi \in C^\infty(U), \nabla^i \psi(p) = 0 \text{ for } 0 \leq i < k\}. \end{aligned}$$

We equip $\mathcal{P}_k(U)$ with the following inner product:

$$\langle \psi(x), \varphi(x) \rangle = \int_{\mathbb{S}^{n-1}} \psi(x) \varphi(x) d\mathcal{H}^{n-1} \quad (\text{A.1})$$

This section is devoted to the proof of:

THEOREM A.6. *Let $G(p, \cdot)$ be the positive Green function of the conformal Laplacian with pole at p , normalized by $LG(p, \cdot) = \delta_p$ and set:*

$$G_p = c(n)n(n-2)\omega_n G(p, \cdot)$$

Then,

$$LG_p = c(n)n(n-2)\omega_n \delta_p$$

and in a punctured neighborhood $U \setminus \{p\}$, using conformal normal coordinates, the function G_p has the following expansion:

$$G_p(x) = r^{2-n} \left(1 + \sum_{k=4}^n \psi_k \right) + c \log r + O_2(1)$$

Where $r = d_g(x, p)$, $\psi_k \in \mathcal{P}_k(U)$ and the log term appears only if n is even. In particular if $3 \leq n \leq 5$ or if M is locally-conformally flat we have:

$$G_p(x) = r^{2-n} + A + O_2(r)$$

Where A is a constant.

For the proof we will need the following lemma:

LEMMA A.7. *The eigenvalues of $r^2 \Delta_0$ on $\mathcal{P}_m(\mathbb{R}^n)$ are:*

$$\lambda_j = 2j(n-2+2m-2j), \quad 0 \leq j \leq \left\lfloor \frac{m}{2} \right\rfloor$$

The eigenfunctions of λ_j are of the form $r^{2j}u$, where $u \in \mathcal{P}_{m-2j}(\mathbb{R}^n)$ is harmonic.

PROOF. For the proof see [21]. $\square$

An immediate corollary is the following:

COROLLARY A.8. *Let T be the following operator:*

$$T\psi = -r^2\Delta_0\psi + 2(n-2)x \cdot \nabla\psi$$

Then T , on $\mathcal{P}_m(\mathbb{R}^n)$, has eigenvalues of the form:

$$\lambda_j = 2(m-j)(n-2-2j), \quad 0 \leq j \leq \left\lfloor \frac{m}{2} \right\rfloor$$

The eigenfunctions of λ_j are $r^{2j}u$, where $u \in \mathcal{P}_{m-2j}(\mathbb{R}^n)$ is harmonic.

PROOF. Use Lemma A.7 and Euler's identity for homogeneous functions. $\square$

LEMMA A.9. *Let $G(p, \cdot)$ be the positive Green function of the elliptic operator L . Then, for every open set U containing p , we have $G(p, \cdot) \in C^\infty(U \setminus \{p\})$.*

PROOF. Let U be an open set with $p \in U$. By definition of Green function the following equation holds in the sense of distribution in U :

$$LG(p, x) = \delta_p$$

Therefore,

$$LG(p, x) = 0, \quad x \in U \setminus \{p\}$$

thanks to Theorem 2.7 we know that $G(p, \cdot) \in C_{loc}^{2,\alpha}(U \setminus \{p\})$, with this regularity assumption the equation $LG(p, x) = 0$ also holds in the weak sense in $U \setminus \{p\}$. Applying elliptic regularity to any $V \subseteq U \setminus \{p\}$ the claim follows. $\square$

PROOF OF THEOREM A.6. Using Theorem 2.7 we can write G_p as:

$$G_p = r^{2-n}(1 + \psi), \quad \psi \in C^\infty(U \setminus \{p\}) \cap C(U), \quad \psi = O(r), \quad |r\nabla\psi| = O(1)$$

Where the regularity and the decay of ψ and its gradient follows from Theorem 2.7 and Lemma A.9. Since we are working in conformal normal coordinates, the determinant of the metric admits the following expansion:

$$\sqrt{\det g_{ij}} = 1 + \eta(x), \quad \eta \in C^\infty(U), \quad \eta = O(r^N)$$

A Taylor expansion of η , together with the estimate $\eta = O(r^N)$, yields $\eta(x) \in \mathcal{C}_N(U)$. To express Δ in terms of the Euclidean Laplacian, we consider its coordinate expression applied to an arbitrary smooth function u :

$$\Delta u = \frac{1}{\sqrt{\det g_{ij}}} \partial_k (\sqrt{\det g_{ij}} g^{kl} \partial_l u) = \partial_k (g^{kl} \partial_l u) + \frac{\partial_k \eta(x)}{1 + \eta(x)} g^{kl} \partial_l u.$$

By expanding the function $1/(1 + \eta)$ and since $\eta \in O(r^N)$ we get:

$$\Delta u = \partial_k (g^{kl} \partial_l u) + O_u(r^{N-1})$$

Where $O_u(r^{N-1})$ contains terms in the derivatives of u . By defining the operator K as:

$$Ku = \partial_k ((g^{kl} - \delta^{kl}) \partial_l u)$$

We obtain the following formula:

$$\Delta u = \Delta_0 u + Ku + O_u(r^{N-1}) \tag{A.2}$$

While for a radial function f , in geodesic polar coordinates, holds:

$$\Delta f = \partial_r^2 f + \frac{n-1}{r} \partial_r f + O(r^{N-1}) \partial_r f = \Delta_0 f + O(r^{N-1}) \partial_r f \quad (\text{A.3})$$

If we apply equation (A.3) to r^{2-n} and equation (A.2) to ψ in $U \setminus \{p\}$ we get:

$$\Delta r^{2-n} = O(r^{N-n}), \quad \Delta \psi = \Delta_0 \psi + K\psi + O_\psi(r^{N-2}) \quad (\text{A.4})$$

Where the condition $|r \nabla \psi| = O(1)$ implies that the remainder is $O_\psi(r^{N-2})$. The decomposition of G_p and the identities (A.4) yield:

$$LG_p(x) = L(r^{2-n}\psi) + Rr^{2-n} + O_\psi(r^{N-n}) = 0 \quad x \in U \setminus \{p\}$$

Equivalently, multiplying by $r^n/c(n)$:

$$-r^2 \Delta \psi + 2(n-2)r \partial_r \psi + c(n)^{-1} r^2 R(1 + \psi) + O_\psi(r^N) = 0$$

Defining the operator $T = -r^2 \Delta_0 + 2(n-2)r \partial_r$, we can rewrite the equation as follows:

$$T\psi + c(n)^{-1} r^2 (R + R\psi - c(n)K\psi) + O_\psi(r^N) = 0$$

We first construct a formal approximation by successively canceling the homogeneous terms in the equation, logarithmic corrections will be introduced when necessary. Since we are interested in a polynomial decomposition of fixed degree n of ψ we write the term R as:

$$R(x) = Q_N(x) + t(x), \quad Q_N = \sum_{k=2}^N R_k, \quad R_k \in \mathcal{P}_k(U), \quad t \in \mathcal{C}_N(U)$$

The sum starts from R_2 because $R \in O(r^2)$. With this adjustment the equation becomes:

$$T\psi + c(n)^{-1} r^2 (Q_N + Q_N\psi - c(n)K\psi) + O_\psi(r^N) = 0 \quad (\text{A.5})$$

Where the term in t has been included in $O_\psi(r^N)$. The inductive strategy is the following, consider the finite sum:

$$S_{m-1} = \sum_{k=1}^{m-1} \psi_k$$

Suppose inductively that applying equation (A.5) to S_{m-1} yields:

$$TS_{m-1} + c(n)^{-1} r^2 (Q_N + Q_N S_{m-1} - c(n)K S_{m-1}) + O_{S_{m-1}}(r^N) = b_m + E_{m+1} + F_{m+1} \log r$$

Where:

$$F_{m+1} \in \mathcal{C}_{n+1}(U), \quad E_{m+1} \in \mathcal{C}_{m+1}(U), \quad b_m \in \mathcal{P}_m(U)$$

And also $F_{m+1} = 0$ for n odd. Notice that, when ψ_k is a polynomial, we have:

$$r^2(Q_N + Q_N \psi_k - c(n)K \psi_k) = O_2(r^4) + O(r^{k+4}) + O(r^{k+2}), \quad T\psi_k \in \mathcal{P}_k(U)$$

To compensate for b_m we need to solve, if possible, for:

$$T\psi_m = -b_m$$

Since the error in the equation starts from $O_2(r^4)$ we can impose:

$$\psi_1 = \psi_2 = \psi_3 = 0$$

To start the induction, note that $S_3 = 0$. Since the term of lowest order in the equation is $O(r^4)$, we get:

$$TS_3 + c(n)^{-1} r^2 (Q_N + Q_N S_3 - c(n)K S_3) + O_{S_3}(r^N) = c(n)^{-1} r^2 R_2 + E_5$$

The problem thus reduces to determining when $T\psi_m = -b_m$ is solvable in $\mathcal{P}_m(U)$. We distinguish two cases according to whether the dimension is odd or even. If n is odd the operator T is invertible on $\mathcal{P}_m(U)$ thanks to Corollary A.8. By induction there exists:

$$\bar{\psi} = \sum_{k=4}^n \psi_k$$

Such that:

$$T\bar{\psi} + c(n)^{-1}r^2(Q_N + Q_N\bar{\psi} - c(n)K\bar{\psi}) + O_{\bar{\psi}}(r^N) = h, \quad h \in \mathcal{C}_{n+1}(U)$$

The remainder $O_{\bar{\psi}}(r^N)$ depends only on η , t , and $\bar{\psi}$ which is a polynomial and can therefore be regarded as smooth.

For n even we need to consider two cases. If $m < n - 2$ then the induction still works because the operator is invertible thanks to Corollary A.8. For $m \geq n - 2$ the operator T is no longer invertible. However, one can show that T is self-adjoint with respect to the inner product defined previously on $\mathcal{P}_m(U)$, equation (A.1), yielding the following orthogonal decomposition:

$$\mathcal{P}_m(U) = \text{Im } T \oplus \ker T$$

We can write $-b_m = Tp_m + \tilde{q}_m$ where:

$$\tilde{q}_m \in \ker T$$

Consider then the term:

$$\psi_m = p_m + q_m \log r, \quad q_m \in \ker T$$

By direct computations we see:

$$T\psi_m = Tp_m + (n - 2 - 2m)q_m$$

Choose then p_m as in the decomposition of $-b_m$ and q_m as:

$$(n - 2 - 2m)q_m = \tilde{q}_m$$

It remains to verify that the term

$$c(n)^{-1}r^2(Q_N\psi_m - c(n)K\psi_m)$$

does not introduce logarithmic terms at the orders relevant to the induction. Such terms would prevent us from continuing the argument, since each step requires the term to be canceled to be a homogeneous polynomial. Starting from $m = n - 2$ we know, thanks to Corollary A.8, that the kernel of T is spanned by r^{n-2} , this implies that:

$$\psi_{n-2} = p_{n-2} + cr^{n-2} \log r$$

The term Q_N gives no problem since $r^2Q_N\psi_{n-2} = O(r^{n+2} \log r)$. For the term involving K , however, a crude estimate gives:

$$r^2K\psi_{n-2} = O(r^n \log r)$$

A more refined estimate is therefore needed. By Gauss lemma, for any radial function f on $U \setminus \{p\}$, we have:

$$Kf = \partial_k \left((g^{kl} - \delta^{kl}) x_l \frac{\partial_r f}{r} \right) = 0$$

In particular, for a generic function of the form $q_m(x) \log r$ and for the logarithmic part of ψ_{n-2} , we get:

$$K(q_m(x) \log r) = K(q_m(x)) \log r, \quad K(r^{n-2} \log r) = 0$$

For $k = n - 1$, n this yields:

$$r^2 K(\psi_k) = r^2 K(p_k) + K(q_k) r^2 \log r$$

The order is big enough to continue the induction. In this case, the remainder may also contain logarithmic terms. Nevertheless, by choosing the truncation order N sufficiently large, we can ensure that it extends across p with any prescribed finite order of differentiability. By choosing $N > n + 2$, these contributions can be absorbed into a function $z(x) \in C^{n+2}(U)$. We collect below the equations obtained together with the corresponding approximations:

$$T\bar{\psi} + c(n)^{-1} r^2 (Q_N + Q_N \bar{\psi} - c(n) K \bar{\psi}) = h + f \log r + z, \quad h, f \in \mathcal{C}_{n+1}(U) \quad (\text{A.6})$$

$$\bar{\psi} = \sum_{k=4}^n \tilde{\psi}_k + (c r^{n-2} + q_{n-1} + q_n) \log r$$

Where $\tilde{\psi}_k = \psi_k$ for odd n or when $k < n - 2$, otherwise they coincide with the polynomial component of ψ_k , also q_k, f, c are zero for odd n . Denote by G_p^{app} and w the following:

$$G_p^{\text{app}} = r^{2-n} (1 + \bar{\psi}), \quad w = G_p^{\text{app}} - G_p$$

Using equation (A.6), after multiplying by $c(n)r^{-n}$, and the divergence theorem we get:

$$L G_p^{\text{app}} = c(n)n(n-2)\omega_n \delta_p + c(n)r^{-n}(h + f \log r + z)$$

Where the equation holds in the sense of distributions. Since $h \in \mathcal{C}_{n+1}(U)$, we have:

$$r^{-n}h \in O(r), \quad \nabla(r^{-n}h) = O(1)$$

Thus, extending $r^{-n}h$ by zero at p , we obtain $r^{-n}h \in \text{Lip}(U)$. For the logarithmic term, since $f \in \mathcal{C}_{n+1}(U)$, we have in a sufficiently small neighborhood of p :

$$|r_x^{-n} \log r_x f(x) - r_y^{-n} \log r_y f(y)| \leq C |1 + \log |x - y|| |x - y| \leq \tilde{C} |x - y|^\alpha$$

For every $0 < \alpha < 1$, where $r_x = d_g(x, p)$, $r_y = d_g(y, p)$, and $|x - y|$ is the Euclidean distance. Therefore, this term also extends by zero at p to a function in $C_{loc}^{0,\alpha}(U)$. Finally, since $z \in C^{n+2}(U)$ and its derivatives up to order $n + 2$ vanish at p , the last term $r^{-n}z$ extends as a $C^2(U)$ function by setting its value at p to zero. The function w solves, in the sense of distributions, the following:

$$Lw = c(n)r^{-n}(h + f \log r + z) \in C_{loc}^{0,\alpha}(U)$$

By interior elliptic regularity for distributional equations, see [16], we get $w \in C_{loc}^{2,\alpha}(U)$ and the first decomposition of G_p follows:

$$G_p = G_p^{\text{app}} - w = r^{2-n} \left(1 + \sum_{k=4}^n \tilde{\psi}_k \right) + c \log r + r^{2-n} (q_{n-1} + q_n) \log r - w$$

Notice that $w - w(p) = O_2(r)$ and for the non-constant logarithmic terms we have:

$$r^{2-n} q_{n-1} \log r = O_2(r \log r), \quad r^{2-n} q_n \log r = O_2(r^2 \log r)$$

If we denote by $A = -w(p)$, and by considering the estimate above, we can rewrite the expression as:

$$G_p = r^{2-n} \left(1 + \sum_{k=4}^n \tilde{\psi}_k \right) + c \log r + A + O_2(1)$$

For $n = 3, 5$ the expansion becomes:

$$G_p = r^{2-n} + A + O_2(r)$$

For $n = 4$ the expression still holds, indeed $c = 0$ and $q_3 = 0$ since the expansion has no second order terms. For the term of order four we have:

$$r^{-2} q_4 \log r = O_2(r^2 \log r) \subseteq O_2(r)$$

For a locally-conformally flat manifold choose, after a conformal change, the flat metric in a neighborhood of p . The conformal Laplacian then becomes the Euclidean Laplacian in normal coordinates. In this setting the equation is:

$$LG_p = -c(n)\Delta_0 G_p = c(n)n(n-2)\omega_n \delta_p$$

This means that $u = G_p - r^{2-n}$ is weakly harmonic in U :

$$Lu = -c(n)\Delta_0 u = 0$$

Thanks to Weyl lemma weakly harmonic functions are smooth and so:

$$G_p = r^{2-n} + u$$

From this it is immediate to get the decomposition. □

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Ringraziamenti